Contents

1 Review

It is absolutely imperative that you have a fairly good grasp of the mathematics covered in prior ECC mathematics courses including Algebra (MTH-100) Pre-Calculus I and II (MTH 119-120). You may visit

MTH 100: http://mth100.mathography.org/

MTH 119: http://mth119.mathography.org/

MTH 120: http://mth120.mathography.org/

to see what materials are covered in these three courses. Although your teacher may spend some time in class discussing material covered in prior mathematics courses, you can not expect that this pre-requisite materials will be covered sufficiently well enough to remediate inadequate preparation. Being prepared is the first step!

2 Course Web Page

Please visit http://mth121.mathography.org/ to learn more about how this course will be run. On this website you will be able to view the course outline, section specific syllabi, this guide,[1] and information about accessing online homework. Visit this page often!

[1]This is what I'd like my notes to look like If I were a student in this class. You should make some attempt to write your own notes, or at least annotate these. Please try to follow what is being done in class, and do not do the homework until you understand what was done in class.

3 This Guide

This guide is being provided as a way to structure MTH 121 across sections and semesters. Section headings that follow correspond to actual WebAssign assignments, and the material in these sections address materials in these assignments. This guide is not a textbook and it should not be used as such—it just provides *structure* for teachers and students.

1. Please come to all classes, these notes are not to be used as an excuse not to attend.

2. These notes will not be read as a script, but teachers/students are encouraged to follow the content of these notes. However, your teacher may or may not do all the problems that are in the notes. You should make every attempt to follow what your teacher is doing in class.

4 mth.121.01.00

4.1 Diagnostic Review

Your eventual success in future mathematics courses, including this course, depends to a large extent on knowledge learned in prior mathematics courses. The following example questions are intended to self-diagnose weaknesses that you might have in the following areas: algebra, trigonometry, logarithms, analytic geometry, and functions. This is not an all-inclusive diagnostic, but it should nonetheless give you an idea of what you *should be able to do* at this point in time.

This material is not to be done during class time. It is presented here for self-review purposes only. Every student needs to look over this section's material and determine if they are ready for calculus. If what follows appears to be foreign, you are advise to discuss this with your teacher as soon as possible—you may need to reconsider your current registration status.

4.2 Examples

1. Find the complement of $\dfrac{\pi}{6}$.

Solution:

$$\frac{\pi}{2} - \frac{\pi}{6} = \boxed{\frac{\pi}{3}}$$

2. Find the first five terms of the recursively defined sequence.

$$a_1 = 15, \quad a_{k+1} = 3a_k - 2.$$

Solution:

$$a_1 = \boxed{15},\ a_2 = 3a_1 - 2 = 3 \cdot 15 - 2 = \boxed{43},\ a_3 = 3a_2 - 2 = 3 \cdot 43 - 2 = \boxed{127},$$

$$a_4 = 3a_3 - 2 = 3 \cdot 127 - 2 = \boxed{379},\ a_5 = 3a_4 - 2 = 3 \cdot 379 - 2 = \boxed{1135}.$$

3. Using your calculator to make relevant graphs, determine which of the following is the equation of the line tangent to the curve $y = x \sin(\pi x)$ when $x = 1/2$.

$$y = 2x - \frac{1}{2} \qquad \text{or} \qquad y = x \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad y = \frac{1}{2} \qquad \text{or} \qquad y = \frac{x}{2}$$

Solution: $\boxed{y = x}$

4. Use the properties of inverse functions to find the exact value of

$$\arccos\left(\cos\frac{7\pi}{2}\right).$$

Solution:

$$\arccos\left(\cos\frac{7\pi}{2}\right) = \arccos(0) = \boxed{\frac{\pi}{2}}$$

5. Solve the triangle, given $a = 13.25$ inches $b = 7.28$ inches, and $c = 9.02$ inches.

Solution: You should, of course, draw an approximating triangle first.
$\boxed{A = 108.28°,\ B = 31.45° \text{ and } C = 40.27°.}$

6. Evaluate the given infinite geometric sum.

$$\sum_{n=1}^{\infty} 0.9^n$$

Solution: Here's one possible answer.

$$\sum_{n=1}^{\infty} 0.9^n = 0.9 \cdot \frac{1}{1-0.9} = \frac{0.9}{0.1} = \boxed{9}$$

7. Let

$$f(x) = \frac{1}{x^2+1}.$$

What are the domain and range of f?

Solution: $\boxed{\text{Domain: } \mathbb{R}\text{; Range: } (0,\ 1]}$.

8. Which of the following functions are *increasing* for all x?

$$f(x) = \sqrt[3]{x} - 2 \qquad \text{or} \qquad g(x) = \sin x + 3 \qquad \text{or} \qquad h(x) = \ln x \qquad \text{or} \qquad i(x) = e^{-x}$$

Solution: $\boxed{f(x) \text{ is always increasing. } h(x) \text{ is increasing on its domain.}}$

9. Which are *decreasing* for all x?

$$f(x) = \sqrt[3]{x} - 2 \quad \text{or} \quad g(x) = \sin x + 3 \quad \text{or} \quad h(x) = \ln x \quad \text{or} \quad i(x) = e^{-x}$$

Solution: $\boxed{i(x) \text{ is always decreasing}}$.

10. Let

$$f(x) = x^3$$

and

$$g(x) = \frac{f(x+h) - f(x)}{h}.$$

What is $g(2)$ when $h = 0.1$? What is $g(x)$ when $h = 0.1$?

Solution: $\boxed{g(2) = 12.61; \text{ when } h = 0.1,\ g(x) = 3x^2 + 0.3x + 0.01}$

11. Find the domain of the function.

$$h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$$

Solution: The domain of $\sqrt{4-x}$ is $(-\infty,\ 4]$; and the domain of $\sqrt{x^2 - 1}$ is $(-\infty,\ -1] \cup [1,\ \infty)$. The domain if $h(x)$ is the intersection of these two domains. Hence:

$$\boxed{(-\infty,\ -1]\ \cup\ [1,\ 4]}$$

12. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following.

(a) $f \circ g$

Solution:

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(2x - 3) \\ &= (2x - 3)^2 + 2(2x - 3) - 1 \\ &= \boxed{4x^2 - 8x + 2} \end{aligned}$$

(b) $g \circ f$

Solution:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(x^2 + 2x - 1\right) \\ &= 2\left(x^2 + 2x - 1\right) - 3 \\ &= \boxed{2x^2 + 4x - 5}\end{aligned}$$

(c) $g \circ g \circ g$

Solution:

$$\begin{aligned}(g \circ g \circ g)(x) &= g(g(g(x))) \\ &= g(g(2x-3)) \\ &= g(2(2x-3)-3) \\ &= g(4x-9) \\ &= 2(4x-9)-3 \\ &= \boxed{8x-21}\end{aligned}$$

13. Solve each inequality, and use interval notation to express the solution.

(a) $\dfrac{1}{x} \geq \dfrac{1}{x^3}$

Solution:

$$\begin{aligned}\frac{1}{x} &\geq \frac{1}{x^3} \\ \frac{1}{x} - \frac{1}{x^3} &\geq 0 \\ \frac{x^2-1}{x^3} &\geq 0 \\ \frac{(x-1)(x+1)}{x^3} &\geq 0\end{aligned}$$

Using a number line, you'll get $\boxed{[-1,\ 0) \cup [1,\ \infty)}$.

(b) $(1-x)^2 \leq 10 - 2x$

Solution:

$$\begin{aligned}(1-x)^2 &\leq 10-2x \\ 1-2x+x^2 &\leq 10-2x \\ x^2-9 &\leq 0 \\ (x-3)(x+3) &\leq 0\end{aligned}$$

Using a number line, you'll get $\boxed{[-3,\ 3]}$.

14. Solve for x.

(a) $\log_5 x = 2$

Solution:

$$\log_5 x = 2 \qquad \Leftrightarrow \qquad x = 5^2 \qquad \Rightarrow \qquad \boxed{x = 25}$$

(b) $\log_x 64 = 3$

Solution:

$$\log_x 64 = 3 \qquad \Leftrightarrow \qquad x^3 = 64 \qquad \Rightarrow \qquad \boxed{x = 4}$$

(c) $\log_3 \dfrac{1}{9} = x$

Solution:

$$\log_3 \frac{1}{9} = x \qquad \Leftrightarrow \qquad 3^x = \frac{1}{9} \qquad \Rightarrow \qquad \boxed{x = -2}$$

15. Find the equation of the line that passes through the point $(2,\ -5)$ and is perpendicular to the line $2x - 4y = 5$.

Solution: $\boxed{y + 5 = -2\,(x - 2)}$

16. Find the equation of the circle that has center $(-1,\ 4)$ and passes through the point $(3,\ -2)$.

Solution: Using $(x - h)^2 + (y - k)^2 = r^2$ to find the radius, where $(h,\ k)$ is the center, r is the radius, and $(x,\ y)$ is any point on the circle.

$$\begin{aligned} (3+1)^2 + (-2-4)^2 &= r^2 \\ 16 + 36 &= r^2 \\ 52 &= r^2 \end{aligned}$$

$$\boxed{(x+1)^2 + (y-4)^2 = 52 \quad \Rightarrow \quad x^2 + y^2 + 2x - 8y - 35 = 0}$$

17. Graph

$$h\,(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$$

Solution: You may have a tough time with this one, and an electronic aid may in fact be necessary to graph (Figure 1, page 16) many functions.

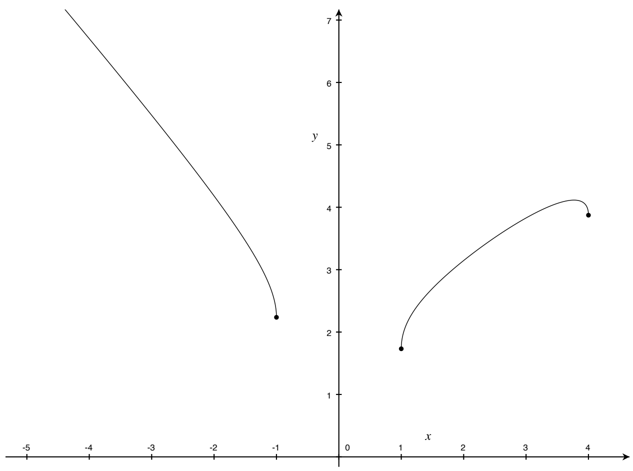

Figure 1: Partial graph of $h(x)$.

18. Find the radius and center of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.

Solution:

$$\begin{aligned} x^2 + y^2 - 6x + 10y + 9 &= 0 \\ x^2 - 6x + y^2 + 10y &= -9 \\ x^2 - 6x + 9 + y^2 + 10y + 25 &= -9 + 9 + 25 \\ (x-3)^2 + (y+5)^2 &= 5^2 \end{aligned}$$

So the $\boxed{\text{center is } (3,\ -5)}$ and the $\boxed{\text{radius is } 5}$.

19. If $f(x) = x^3$, evaluate the difference quotient

$$\frac{f(2+h) - f(2)}{h}, \quad h \neq 0$$

Solution: Do I need to say it? Yes, the following is true if and only if $h \neq 0$.

$$\begin{aligned} \frac{f(2+h) - f(2)}{h} &= \frac{(2+h)^3 - (2)^3}{h} \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \boxed{12 + 6h + h^2} \end{aligned}$$

20. Solve for x.

$$\frac{2x}{x+1} = \frac{2x-1}{x}$$

Solution:

$$\begin{aligned}
\frac{2x}{x+1} &= \frac{2x-1}{x} \\
2x \cdot x &= (x+1)(2x-1) \\
2x^2 &= 2x^2 - x + 2x - 1 \\
0 &= x - 1 \\
1 &= x
\end{aligned}$$

So, it appears that $\boxed{x=1}$ is a solution. Checking in the original equation, we have,

$$\frac{2}{2} = \frac{1}{1},$$

which is clearly true.

21. Verify the identity.

$$\sin^2\theta = \frac{\sec^2\theta - 1}{\sec^2\theta}$$

Solution: Select the right side.

$$\begin{aligned}
\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta} \\
&= 1 - \cos^2\theta \\
&= \sin^2\theta
\end{aligned}$$

Q.E.D.

22. Solve for x.

$$\frac{2x}{\sqrt{4-x}} - 3\sqrt{4-x} = 0$$

Solution:

$$\begin{aligned}
\frac{2x}{\sqrt{4-x}} - 3\sqrt{4-x} &= 0 \\
\frac{2x}{\sqrt{4-x}} &= 3\sqrt{4-x} \\
2x &= 3\left(4-x\right) \\
2x &= 12-3x \\
5x &= 12 \\
x &= \frac{12}{5}
\end{aligned}$$

The solution (please check) is $\boxed{x = \frac{12}{5}}$.

23. Find the domain of

$$f\left(x\right) = \frac{x^2-49}{\sqrt{x^2+9}-5}.$$

Solution: x^2+9 is always positive so we don't have to worry about the square root. However, to find the domain we need to solve $\sqrt{x^2+9}-5=0$.

$$\begin{aligned}
\sqrt{x^2+9}-5 &= 0 \\
\sqrt{x^2+9} &= 5 \\
x^2+9 &= 25 \\
x^2 &= 16 \\
x &= \pm 4
\end{aligned}$$

So, the domain is $\boxed{\mathbb{R}, x \neq \pm 4}$.

24. Solve the inequality.

$$|14-x|-3 < 17$$

Solution:

$$\begin{aligned}
|14-x|-3 &< 17 \\
|14-x| &< 20
\end{aligned}$$

We know that for $a > 0$, that $|x| < a$ is equivalent to $-a < x < a$, so

$$\begin{aligned} |14 - x| &< 20 \\ -20 < 14 - x &< 20 \\ -34 < -x &< 6 \\ 34 > x &> -6 \\ -6 < x &< 34 \end{aligned}$$

Here the proper interval is: $\boxed{(-6,\ 34)}$.

25. Solve by using an augmented matrix and elementary row operations.

$$\begin{cases} 2x + 3y - z = -7 \\ 3x - 3y + z = 12 \\ 2x + 4y + z = -3 \end{cases}$$

Solution: Augmented matrix form of the system:

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & -7 \\ 3 & -3 & 1 & 12 \\ 2 & 4 & 1 & -3 \end{array}\right]$$

Elementary row operations, in order given:

$$R_1 + R_2 \to R_2$$
$$R_1 + R_3 \to R_3$$

Produces:

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & -7 \\ 3 & -3 & 1 & 12 \\ 2 & 4 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 2 & 3 & -1 & -7 \\ 5 & 0 & 0 & 5 \\ 4 & 7 & 0 & -10 \end{array}\right]$$

The second row gives $\boxed{x = 1}$; using $x = 1$ in row three, gives $\boxed{y = -2}$; finally, using $x = 1$ and $y = -2$ in row one, gives $\boxed{z = 3}$.

26. Use long division to find the quotient and remainder when $x^4 - 4x^2 + 2x + 5$ is divided by $x - 2$.

Solution: Doing the division gives a $\boxed{\text{remainder of } 9}$ and a $\boxed{\text{quotient of } x^3 + 2x^2 + 2}$.

27. Is $x = -1$ a root of the polynomial function $f(x) = 2x^3 - 5x^2 - 4x + 3$?

Solution: $\boxed{\text{Yes}}$, because $f(-1) = 0$.

28. Factor[2] $f(x) = 2x^3 - 5x^2 - 4x + 3$.

Solution: Using the fact that $x = -1$ is a root, we can divide $f(x)$ by $x + 1$, getting

$$\frac{2x^3 - 5x^2 - 4x + 3}{x + 1} = 2x^2 - 7x + 3 = (2x - 1)(x - 3),$$

so the complete factorization of $f(x)$ is

$$\boxed{(x + 1)(2x - 1)(x - 3)}.$$

29. Solve the inequality.

$$\frac{3 - 2x}{x - 1} + 2 \geq 0$$

Solution:

$$\begin{aligned}
\frac{3 - 2x}{x - 1} + 2 &\geq 0 \\
\frac{3 - 2x + 2(x - 1)}{x - 1} &\geq 0 \\
\frac{3 - 2x + 2x - 2}{x - 1} &\geq 0 \\
\frac{1}{x - 1} &\geq 0
\end{aligned}$$

Using simple sign analysis, the proper interval is: $\boxed{(1,\ \infty)}$.

30. Solve for x.

$$\log_3(2x + 1) + \log_3(2x - 1) = 1$$

[2] Previous problem may be helpful.

Solution:

$$\begin{aligned}
\log_3(2x+1) + \log_3(2x-1) &= 1 \\
\log_3(2x+1)(2x-1) &= 1 \\
\log_3\left(4x^2-1\right) &= 1 \\
4x^2-1 &= 3 \\
4x^2-4 &= 0 \\
x^2-1 &= 0 \\
x &= \pm 1
\end{aligned}$$

The solutionis $\boxed{x=1}$ and you should check this.

Note: If you're limited to using real numbers only, whereas complex numbers can look mighty strange, such as the famous—or infamous—identity $e^{i\pi}+1=0$.

31. Given

$$f(x) = \frac{2x-1}{2-3x},$$

find and simplify the difference quotient

$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0.$$

Solution: Do I need to say it? Yes, the following is true if and only if $h \neq 0$.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\dfrac{2x+2h-1}{2-3x-3h} + \dfrac{2x-1}{2-3x}}{h} \\
&= \frac{(2x+2h-1)(2-3x)-(2x-1)(2-3x-3h)}{h(2-3x-3h)(2-3x)} \\
&= \frac{4x+4h-2-6x^2-6xh+3x-4x+2+6x^2-3x+6xh-3h}{h(2-3x-3h)(2-3x)} \\
&= \frac{h}{h(2-3x-3h)(2-3x)} \\
&= \boxed{\frac{1}{(2-3x-3h)(2-3x)}}
\end{aligned}$$

32. Find the domain of

$$f(x) = \frac{x+2}{\sqrt{2x^2 - 18}}.$$

Solution: Finding the domain requires solving $2x^2 - 18 > 0$ for x.

$$\begin{aligned} 2x^2 - 18 &> 0 \\ x^2 - 9 &> 0 \\ (x-3)(x+3) &> 0 \end{aligned}$$

Simple sign analysis gives $\boxed{(-\infty,\ -3) \cup (3,\ \infty)}$.

33. Given

$$f(x) = (x+3)^2, \quad x \geq -3,$$

find $f^{-1}(x)$. Graphing may be helpful, but is not required.

Solution: The domain of $f(x)$ is $[-3, \infty)$ and the range is $[0, \infty)$, so the domain of $f^{-1}(x)$ is $[0, \infty)$ and the range is $[-3, \infty)$.

$$\begin{aligned} f(x) &= (x+3)^2 \\ y &= (x+3)^2 \\ x &= (y+3)^2 \\ \pm\sqrt{x} &= y+3 \\ \pm\sqrt{x} - 3 &= y \end{aligned}$$

However, since $y \geq -3$ we have

$$\boxed{f^{-1}(x) = \sqrt{x} - 3, \quad x \geq 0}.$$

34. Given

$$f(x) = \sqrt{x+6} \quad \text{and} \quad g(x) = x^2 - 5,$$

answer each of the following questions.

(a) Find $(f \circ g)(x)$

Solution:

$$(f \circ g)(x) = f(g(x)) = f\left(x^2 - 5\right) = \sqrt{x^2 - 5 + 6} = \boxed{\sqrt{x^2 + 1}}$$

(b) The domain of $(f \circ g)(x)$

Solution: $\boxed{\mathbb{R}}$.

(c) Find $(g \circ f)(x)$

Solution:

$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x+6}\right) = \left(\sqrt{x+6}\right)^2 - 5 = x + 6 - 5 = \boxed{x+1}$$

(d) The domain of $(g \circ f)(x)$

Solution: $\boxed{x \geq -6}$.

35. Given

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18, \quad \text{and} \quad f(2) = f(-3) = 0$$

answer the following questions.

(a) Use the given roots, and long division, to completely factor $f(x)$.

Solution: Since we are given two roots, we know two factors of $f(x)$ are $(x-2)$ and $(x+3)$. Dividing $f(x)$ by the product of these two factors gives

$$2x^2 + 5x + 3 = (2x+3)(x+1).$$

So the complete factorization of $f(x)$ is

$$\boxed{(2x+3)(x+1)(x-2)(x+3)}$$

(b) Graph $f(x)$ using the roots, y-intercept and sign-analyses.

Solution: Your graph (Figure 2, page 24) does not need to have such precise detail as mine, but it should still reflect the key pre-calculus analysis.

36. Given

$$f(x) = \frac{x^3 + 2x^2}{x^2 + 1},$$

answer the following questions.

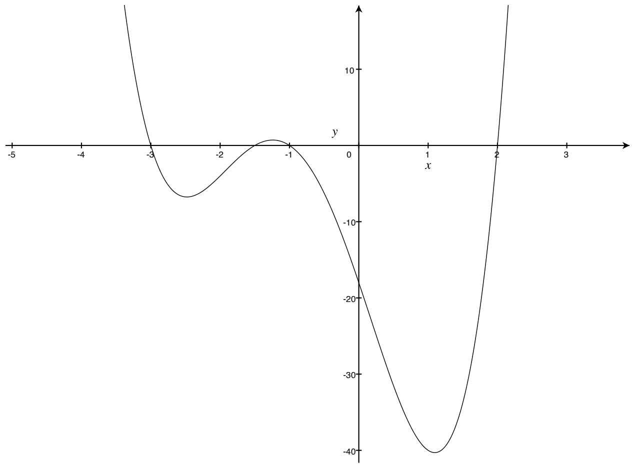

Figure 2: Graph of $f(x)$.

(a) x-intercept(s) in point form.

Solution: Set $f(x) = 0$ and solve for x.

$$0 = \frac{x^3 + 2x^2}{x^2 + 1}$$

$$0 = \frac{x^2(x+2)}{x^2 + 1}$$

Clearly, $\boxed{(-2,\ 0) \text{ and } (0,\ 0)}$.

(b) y-intercept in point form.

Solution: Set $x = 0$ and evaluate. Clearly, $\boxed{(0,\ 0)}$.

(c) All linear asymptotes in equation form.

Solution: Since the degree of the numerator is one more than the denominator, we have the possibility of getting a slant asymptote. The long division gives

$$f(x) = \frac{x^3 + 2x^2}{x^2 + 1} = x + 2 - \frac{x+2}{x^2+1},$$

so the slant asymptote is $\boxed{y = x + 2}$.

(d) Graph $f(x)$ using the information above and sign-analyses.

Solution: Your graph (Figure 3, page 25) does not need to have such precise detail as mine, but it should still reflect the key pre-calculus analysis.

37. Given

$$f(x) = x^2 - x + 1,$$

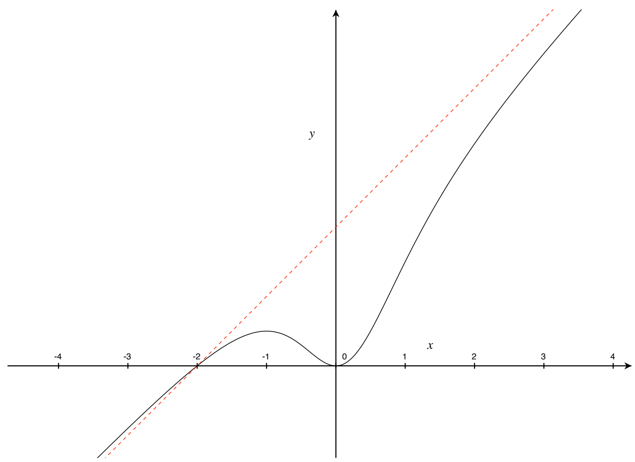

Figure 3: Graph of $f(x)$.

find and simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0.$$

Solution: Do I need to say it? Yes, the following is true if and only if $h \neq 0$.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - (x+h) + 1 - (x^2 - x + 1)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h + 1 - x^2 + x - 1}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \boxed{2x + h - 1} \end{aligned}$$

38. Solve the inequality.

$$|14 - 3x| - 5 > -2$$

Solution:

$$\begin{aligned} |14 - 3x| - 5 &> -2 \\ |14 - 3x| &> 3 \end{aligned}$$

We know that $|x| > a$ is equivalent to $x > a$ or $x < -a$, so

$$\begin{array}{rcccc} |14-3x| > 3 & \Rightarrow & 14-3x > 3 & \text{or} & 14-3x < -3 \\ & & x < 11/3 & \text{or} & 17/3 < x \end{array}$$

Here the proper intervals are: $\boxed{(-\infty,\ 11/3) \cup (17/3,\ \infty)}$.

39. Write the system of linear equations represented by the augmented matrix. Then use back-substitution to find the solution. (Use the variables x, y, and z.)

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Solution:

$$\left\{\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 4 \\ & & y & - & z & = & 2 \\ & & & & z & = & -2 \end{array}\right.$$

The last line gives $\boxed{z=-2}$, then using the value for z in line two, we get $\boxed{y=0}$, finally using these two values in line one we get $\boxed{x=8}$.

40. Condense the expression to the logarithm of a single quantity.

$$\frac{1}{3}\left[2\ln(x+3) + \ln x - \ln\left(x^2-1\right)\right]$$

Solution:

$$\begin{aligned} \frac{1}{3}\left[2\ln(x+3) + \ln x - \ln\left(x^2-1\right)\right] &= \frac{1}{3}\left[\ln(x+3)^2 + \ln x - \ln\left(x^2-1\right)\right] \\ &= \frac{1}{3}\left[\ln\frac{x(x+3)^2}{x^2-1}\right] \\ &= \boxed{\ln\sqrt[3]{\frac{x(x+3)^2}{x^2-1}}} \end{aligned}$$

41. Solve for x.

$$\left(\frac{2}{3}\right)^x = \frac{81}{16}$$

Solution:

$$\begin{aligned} \left(\frac{2}{3}\right)^x &= \frac{81}{16} \\ \left(\frac{2}{3}\right)^x &= \left(\frac{3}{2}\right)^4 \\ \left(\frac{2}{3}\right)^x &= \left(\frac{2}{3}\right)^{-4} \end{aligned}$$

So, $\boxed{x = -4}$.

42. Solve the inequality.

$$\frac{x+6}{x+1} - 2 < 0$$

Solution:

$$\begin{aligned} \frac{x+6}{x+1} - 2 &< 0 \\ \frac{x+6}{x+1} - \frac{2(x+1)}{x+1} &< 0 \\ \frac{4-x}{x+1} &< 0 \end{aligned}$$

Using simple sign analysis, the proper intervals are: $\boxed{(-\infty,\ -1) \cup (4,\ \infty)}$.

43. Solve algebraically (exact answer) and then approximate to three decimal places.

$$-2 + 2\ln 3x = 17$$

Solution:

$$\begin{aligned}
-2 + 2\ln 3x &= 17 \\
2\ln 3x &= 19 \\
\ln 3x &= \frac{19}{2} \\
3x &= e^{19/2} \\
x &= \boxed{\frac{e^{19/2}}{3} \approx 4,453.242}
\end{aligned}$$

44. Given

$$f(x) = \frac{x^2 - 3x + 2}{2x^2 + 5x + 3} = \frac{(x-1)(x-2)}{(2x+3)(x+1)},$$

answer the following questions:

(a) x-intercepts in point form.

Solution: $\boxed{(1,\ 0)\,;(2,\ 0)}$

(b) y-intercept in point form.

Solution: $\boxed{(0,\ 2/3)}$

(c) Equation of the horizontal asymptote.

Solution: $\boxed{y = 1/2}$

(d) Equation of the vertical asymptotes.

Solution: $\boxed{x = -1; x = -3/2}$

(e) Graph $f(x)$ using the above information and sign analysis.

Solution: Your graph (Figure 4, page 29) should roughy look like the one below.

45. Use summation notation to write the given sum.

$$3 + 9 + 27 + 81 + \cdots + 729$$

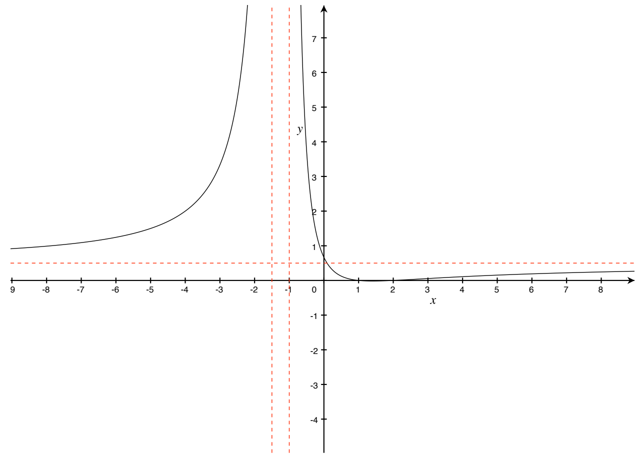

Figure 4: Graph of $f(x)$.

Solution: Here's one possible answer.

$$3 + 9 + 27 + 81 + \cdots + 729 = \boxed{\sum_{n=1}^{6} 3^n}$$

46. Find the rational number representation of the given repeating decimal.

$$1.2\overline{87}$$

Solution: Let $x = 1.2\overline{87}$.

$$\begin{aligned}
10x &= 12.\overline{87} \\
1000x &= 1287.\overline{87} \\
1000x - 10x &= 1287.\overline{87} - 12.\overline{87} \\
990x &= 1275.00\overline{00} \\
x &= \boxed{\frac{1275}{990}} = \boxed{\frac{85}{66}}
\end{aligned}$$

47. Use your calculator to approximate each of the following to three decimal places.
   (a) $\arcsin(-0.987)$

Solution: $\arcsin(-0.987) \approx \boxed{-1.409 \quad \text{or} \quad -80.751^\circ}$

(b) $\arctan(-89.456)$

Solution: $\arctan(-89.456) \approx \boxed{-1.560 \quad \text{or} \quad -89.360^\circ}$

(c) $\arccos(-0.009)$

Solution: $\arccos(-0.009) \approx \boxed{1.580 \quad \text{or} \quad 90.516^\circ}$

48. Find the exact value of the following.

(a) $\arccos\left(\dfrac{1}{\sqrt{2}}\right)$

Solution: $\arccos\left(\dfrac{1}{\sqrt{2}}\right) = \boxed{\dfrac{\pi}{4} \quad \text{or} \quad 45^\circ}$

(b) $\arccos\left[\sin\left(\dfrac{5\pi}{3}\right)\right]$

Solution: $\arccos\left[\sin\left(\dfrac{5\pi}{3}\right)\right] = \arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \boxed{\dfrac{5\pi}{6} \quad \text{or} \quad 150^\circ}$

49. Simplify the factorial expression.

$$\frac{(2n+2)!}{(2n)!}$$

Solution:

$$\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = \boxed{(2n+2)(2n+1) = 4n^2 + 6n + 2}$$

50. A ramp 21 feet in length rises to a loading platform that is 5 feet off the ground. Assuming that the ground is level, what is the angle (to the nearest whole degree) between the ramp and the ground?

Solution: You should draw a diagram first.

$$\sin\theta = \frac{5}{21}, \quad \Rightarrow \quad \arcsin\left(\frac{5}{21}\right) \approx \boxed{14^\circ}$$

51. Write an algebraic expression that is equivalent to

$$\sec\left[\arcsin\left(x-1\right)\right].$$

Solution: You should draw a triangle first and use the Pythagorean Theorem to determine the missing side.

$$\sec\left[\arcsin\left(x-1\right)\right] = \boxed{\frac{1}{\sqrt{2x-x^2}}}$$

52. Find the *numerical coefficient* of the term whose variable part is x^6y^3 in the expansion of $(x-2y)^9$.

Solution:

$$\binom{9}{6}(x)^6(-2y)^3 = 84\cdot(-8)\,x^6y^3 = -672x^6y^3$$

So, the numerical coefficient is $\boxed{-672}$.

53. Solve for x.

(a) $\sin x + \sqrt{2} = -\sin x$ in the interval $[0,\ 2\pi)$.

Solution:

$$\begin{aligned}\sin x + \sqrt{2} &= -\sin x\\ 2\sin x &= -\sqrt{2}\\ \sin x &= -\frac{\sqrt{2}}{2}.\end{aligned}$$

The reference is $45°$ and the solutions occur in the third and fourth quadrant, so

$$x = 45° + 180° = 225° = \boxed{\frac{5\pi}{4}} \qquad \text{and} \qquad x = 360° - 45° = 315° = \boxed{\frac{7\pi}{4}}$$

(b) $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0,\ 2\pi)$.

Solution:

$$\begin{aligned}2\sin^2 x - \sin x - 1 &= 0\\ (2\sin x + 1)(\sin x - 1) &= 0.\end{aligned}$$

Which gives two equations to solve.

$$\sin x = -\frac{1}{2} \qquad \text{and} \qquad \sin x = 1.$$

The reference for the first equation is 30° and the solutions occur in the third and fourth quadrant, so

$$x = 30° + 180° = 210° = \boxed{\frac{7\pi}{6}} \qquad \text{and} \qquad x = 360° - 30° = 330° = \boxed{\frac{11\pi}{6}},$$

and the solution to the second equation is $\boxed{x = \frac{\pi}{2}}$

54. Write an expression for the n^{th} term.[3]

$$1,\ \frac{5}{2}, \frac{25}{6},\ \frac{125}{24},\ \frac{625}{120}, \ldots$$

Solution:

$$\boxed{a_n = \frac{5^{n-1}}{n!}}$$

55. Convert each of the following angle measures to radian measure.

(a) 60°

Solution:

$$60° = 60° \cdot \left(\frac{\pi}{180°}\right) = \boxed{\frac{\pi}{3}}$$

(b) 90°

Solution:

$$90° = 90° \cdot \left(\frac{\pi}{180°}\right) = \boxed{\frac{\pi}{2}}$$

(c) 50°

[3]Let n start at 1, that is $a_1 = 1$.

Solution:

$$50^\circ = 50^\circ \cdot \left(\frac{\pi}{180^\circ}\right) = \boxed{\frac{5\pi}{18}}$$

56. Use the Binomial Theorem to expand and simplify

$$\left(3\sqrt[3]{x^2} - 2\sqrt[3]{y}\right)^3.$$

Solution: First the expansion

$$\boxed{\left(3\sqrt[3]{x^2}\right)^3 + 3\left(3\sqrt[3]{x^2}\right)^2(-2\sqrt[3]{y}) + 3\left(3\sqrt[3]{x^2}\right)(-2\sqrt[3]{y})^2 + (-2\sqrt[3]{y})^3},$$

then the simplification

$$\boxed{27x^2 - 54\sqrt[3]{x^4y} + 36\sqrt[3]{x^2y^2} - 8y}.$$

57. Evaluate (exact values) all six trigonometric functions for $x = -120^\circ$.

(a) $\sin x$

Solution: $\sin x = \boxed{-\frac{\sqrt{3}}{2}}$

(b) $\cos x$

Solution: $\cos x = \boxed{-\frac{1}{2}}$

(c) $\tan x$

Solution: $\tan x = \boxed{\sqrt{3}}$

(d) $\cot x$

Solution: $\cot x = \boxed{\frac{1}{\sqrt{3}}}$

(e) $\sec x$

Solution: $\sec x = \boxed{-2}$

(f) $\csc x$

Solution: $\csc x = \boxed{-\dfrac{2}{\sqrt{3}}}$

58. Find the sum.

$$\sum_{n=48}^{70} (2n-1)$$

Solution: Expand first. You should also notice that there are $70 - 48 + 1 = 23$ terms.

$$\begin{aligned}
\sum_{n=48}^{70} (2n-1) &= (2\cdot 48 - 1) + (2\cdot 49 - 1) + (2\cdot 50 - 1) + \cdots + (2\cdot 70 - 1) \\
&= 2\cdot(48 + 49 + 50 + \cdots + 70) - (1 + 1 + 1 + \cdots + 1) \\
&= 2\cdot\frac{23}{2}\cdot(70 + 48) - 23 = \boxed{2691}
\end{aligned}$$

59. Use mathematical induction to prove the formula for every positive integer n.

$$2\left(1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}\right) = 3^n - 1$$

Solution: Verify for $n = 1$.

$$\begin{aligned}
P_1:\quad 2(1) &= 3 - 1 \\
2 &= 2
\end{aligned}$$

Assume P_k and show that $P_k \to P_{k+1}$.

$$P_k:\quad 2\left(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}\right) = 3^k - 1$$

Add $2\cdot 3^k$ to both sides.

$$\begin{aligned}
P_k:\quad 2\left(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}\right) &= 3^k - 1 \\
2\left(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}\right) + 2\cdot 3^k &= 3^k - 1 + 2\cdot 3^k \\
2\left(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1} + 3^k\right) &= 3\cdot 3^k - 1 \\
2\left(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1} + 3^k\right) &= 3^{k+1} - 1
\end{aligned}$$

This last line is exactly what we wanted, P_{k+1}. *Q.E.D.*

60. When an airplane leaves the runway, its angle of climb is $19°$ and its speed is 300 feet per second. Find the plane's altitude after 30 seconds.

Solution: The plane will travel $30 \cdot 300 = 9,000$ feet (this is the hypotenuse of a right triangle), so the plane's altitude (this is the side opposite the $19°$ angle) is $9000 \cdot \sin 19° \approx \boxed{2930}$ feet.

61. Use your calculator to evaluate the trigonometric function. Round your answers to five decimal places.

(a) $\sin 11.67°$

Solution: $\sin 11.67° \approx \boxed{0.20227}$

(b) $\cos 0.345$

Solution: $\cos 0.345 \approx \boxed{0.94108}$

(c) $\tan\left(-\dfrac{8\pi}{9}\right)$

Solution: $\tan\left(-\dfrac{8\pi}{9}\right) \approx \boxed{0.36397}$

(d) $\cot 2.379$

Solution: $\cot 2.379 = \frac{1}{\tan 2.379} \approx \boxed{-1.04668}$

(e) $\csc(-2.689)$

Solution: $\csc(-2.689) = \frac{1}{\sin(-2.689)} \approx \boxed{-2.28677}$

(f) $\sec 45$

Solution: $\sec 45 = \frac{1}{\cos 45} \approx \boxed{1.90359}$

(g) $\arcsin 0.564$

Solution: $\arcsin 0.564 \approx \boxed{0.59922}$ or $\arcsin 0.564 \approx \boxed{34.33288°}$

(h) $\arccos(-0.367)$

Solution: $\arccos(-0.367) \approx \boxed{1.94658}$ or $\arccos(-0.367) \approx \boxed{111.53072°}$

(i) $\arctan 1.113$

Solution: $\arctan 1.113 \approx \boxed{0.83883}$ or $\arctan 1.113 \approx \boxed{48.06117^\circ}$

62. Find the infinite geometric sum.

$$8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$$

Solution: $a_1 = 8$ and $r = \frac{6}{8} = \frac{3}{4}$.

$$S_\infty = 8 \cdot \frac{1}{1 - 3/4} = \boxed{32}$$

63. Let

$$a_n = n2^n$$

(a) Write the first five terms of the sequence, starting with a_1.

Solution:

$$\begin{aligned} a_1 &= 1 \cdot 2^1 = \boxed{2} \\ a_2 &= 2 \cdot 2^2 = \boxed{8} \\ a_3 &= 3 \cdot 2^3 = \boxed{24} \\ a_4 &= 4 \cdot 2^4 = \boxed{64} \\ a_5 &= 5 \cdot 2^5 = \boxed{160} \end{aligned}$$

(b) Is this sequence arithmetic, geometric, or neither.

Solution: $\boxed{\text{Neither}}$.

64. Find θ in degrees $(0^\circ < \theta < 90^\circ)$ and radians $(0 < \theta < \frac{\pi}{2})$, if

$$\cot \theta = \frac{1}{\sqrt{3}}.$$

Solution: Draw a triangle first, clearly $\theta = \boxed{60^\circ = \frac{\pi}{3}}$.

65. Find the sum.

$$\sum_{n=51}^{100} 6n$$

Solution: Expand first.

$$\begin{aligned}\sum_{n=51}^{100} 6n &= 6\cdot 51 + 6\cdot 52 + 6\cdot 53 + \cdots + 6\cdot 100\\ &= 6\cdot(51+52+53+\cdots+100)\\ &= 6\cdot\frac{50}{2}\cdot(100+51) = \boxed{22650}\end{aligned}$$

66. Use $\csc\theta = 3$ and $\sec\theta = \dfrac{3\sqrt{2}}{4}$ to find the exact value of each of the following.

(a) The quadrant that θ is in.

Solution: $\boxed{\text{First (I) quadrant}}$.

(b) $\sin\theta$

Solution: $\sin\theta = \boxed{\dfrac{1}{3}}$

(c) $\tan\theta$

Solution: $\tan\theta = \boxed{\dfrac{\sqrt{2}}{4}}$

(d) $\cos\theta$

Solution: $\cos\theta = \boxed{\dfrac{4}{3\sqrt{2}}}$

(e) $\sec(90^\circ - \theta)$

Solution: $\sec(90^\circ - \theta) = \boxed{3}$

67. Sketch one period of the graph of the function.

$$f(x) = \sin(\pi x + 2\pi) + 1$$

Solution: I'm mainly looking for five points.

$$\boxed{(-2,\ 1)\,,\quad \left(-\frac{3}{2},\ 2\right),\quad (-1,\ 1)\,,\quad \left(-\frac{1}{2},\ 0\right),\quad (0,\ 1)}.$$

They should be plotted (Figure 5, page 38) and then connected using a sine wave.

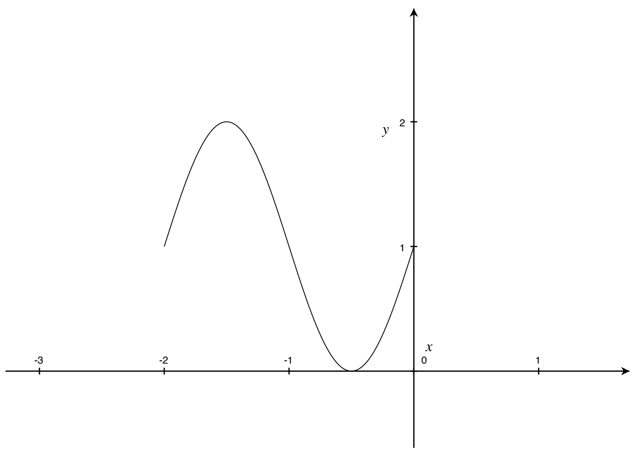

Figure 5: Graph of $f(x) = \sin(\pi x + 2\pi) + 1$.

68. Perform the indicated addition and use the fundamental identities to simplify.

$$\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x}$$

Solution:

$$\begin{aligned}
\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} &= \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} + \frac{1}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x} \\
&= \frac{1 + \sin x}{1 - \sin^2 x} + \frac{1 - \sin x}{1 - \sin^2 x} \\
&= \frac{1 + \sin x + 1 - \sin x}{1 - \sin^2 x} \\
&= \frac{2}{1 - \sin^2 x} \\
&= \boxed{\frac{2}{\cos^2 x}} = \boxed{2 \sec^2 x}
\end{aligned}$$

69. Write an expression for the n^{th} term.

$$2,\ 1, \frac{8}{9},\ 1,\ \frac{32}{25}, \ldots$$

Solution: By inspection.

$$\boxed{a_n = \frac{2^n}{n^2}}$$

70. Find the sum.

$$\frac{2\pi}{3} + \frac{\pi}{4} + \frac{\pi}{12} =$$

Solution:

$$\frac{2\pi}{3} + \frac{\pi}{4} + \frac{\pi}{12} = \frac{8\pi + 3\pi + \pi}{12} = \boxed{\pi}$$

71. If

$$\sin\alpha = -\frac{5}{13} \qquad \text{and} \qquad \cos\beta = \frac{3}{5},$$

with both α and β are in the fourth quadrant. Find the exact values of the following.

(a) $\sin(\alpha - \beta)$

Solution:

$$\sin\alpha = -\frac{5}{13} \qquad \text{and} \qquad \cos\alpha = \frac{12}{13}$$
$$\sin\beta = -\frac{4}{5} \qquad \text{and} \qquad \cos\beta = \frac{3}{5}$$

$$\begin{aligned}
\sin(\alpha - \beta) &= \sin\alpha\ \cos\beta - \sin\beta\ \cos\alpha \\
&= \left(-\frac{5}{13}\right)\left(\frac{3}{5}\right) - \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) \\
&= \boxed{\frac{33}{65}}
\end{aligned}$$

(b) $\cos(\alpha - \beta)$

Solution:

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta \\ &= \left(\frac{12}{13}\right)\left(\frac{3}{5}\right) + \left(-\frac{5}{13}\right)\left(-\frac{4}{5}\right) \\ &= \boxed{\frac{56}{65}}\end{aligned}$$

(c) $\tan(\alpha-\beta)$

Solution:

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} \\ &= \frac{\frac{33}{65}}{\frac{56}{65}} \\ &= \boxed{\frac{33}{56}}\end{aligned}$$

(d) $\sin\left(\frac{\alpha}{2}\right)$

Solution: Since α is in the fourth quadrant $\alpha/2$ will be in the second quadrant, which will determine the signs of the half-angle formulas. $\frac{\alpha}{2}$ is in the second quadrant, so the sine will be positive.

$$\begin{aligned}\sin\left(\frac{\alpha}{2}\right) &= \sqrt{\frac{1-\cos\alpha}{2}} \\ &= \sqrt{\frac{1-12/13}{2}} \\ &= \boxed{\frac{1}{\sqrt{26}}}\end{aligned}$$

(e) $\cos\left(\frac{\alpha}{2}\right)$

Solution: $\frac{\alpha}{2}$ is in the second quadrant, so the cosine will be negative.

$$\begin{aligned}\cos\left(\frac{\alpha}{2}\right) &= -\sqrt{\frac{1+\cos\alpha}{2}} \\ &= -\sqrt{\frac{1+12/13}{2}} \\ &= \boxed{-\frac{5}{\sqrt{26}}}\end{aligned}$$

(f) $\tan\left(\frac{\alpha}{2}\right)$

Solution: Just take the ratio of sine over cosine.

$$\begin{aligned}\tan\left(\frac{\alpha}{2}\right) &= \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} \\ &= \boxed{-\frac{1}{5}}\end{aligned}$$

72. Find all solutions.

$$2\sin^2 x + 3\cos x - 3 = 0$$

Solution: You'll need to write it in terms of cosines first.

$$\begin{aligned}2\sin^2 x + 3\cos x - 3 &= 0 \\ 2\left(1 - \cos^2 x\right) + 3\cos x - 3 &= 0 \\ -2\cos^2 x + 3\cos x - 1 &= 0\end{aligned}$$

Now multiply both sides of this equation by -1 and factor.

$$\begin{aligned}2\cos^2 x - 3\cos x + 1 &= 0 \\ \left(2\cos x - 1\right)\left(\cos x - 1\right) &= 0\end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned}\cos x = \frac{1}{2} \quad &\Rightarrow \quad x = \boxed{\frac{\pi}{3} + 2\pi k, \quad k \in \mathbb{Z}} \\ & = \boxed{\frac{5\pi}{3} + 2\pi k, \quad k \in \mathbb{Z}} \\ \cos x = 1 \quad &\Rightarrow \quad x = \boxed{2\pi k, \quad k \in \mathbb{Z}}.\end{aligned}$$

73. Verify the identity.[4]

$$\sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{1+\sin x}{|\cos x|}$$

[4]Here it might be nice to mention that $1 + \sin x \geq 0$ so $\sqrt{(1+\sin x)^2} = 1 + \sin x$. However, since $-1 \leq \cos x \leq 1$, the $\sqrt{\cos^2 x} = |\cos x|$.

Solution: Select the left side.

$$\begin{aligned}
\sqrt{\frac{1+\sin x}{1-\sin x}} &= \sqrt{\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}}\\
&= \sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}}\\
&= \sqrt{\frac{(1+\sin x)^2}{\cos^2 x}}\\
&= \frac{1+\sin x}{|\cos x|}
\end{aligned}$$

Q.E.D.

74. Rewrite the expression so that it is not in *fractional* form

$$\frac{\tan^2 x}{\csc x+1}.$$

Solution:

$$\begin{aligned}
\frac{\tan^2 x}{\csc x+1} &= \frac{\tan^2 x}{\dfrac{1}{\sin x}+1}\cdot\frac{\sin x}{\sin x}\\
&= \frac{\tan^2 x\ \sin x}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\\
&= \frac{\tan^2 x\ \sin x\cdot(1-\sin x)}{1-\sin^2 x}\\
&= \frac{\tan^2 x\ \sin x\cdot(1-\sin x)}{\cos^2 x}\\
&= \boxed{\tan^2 x\ \sin x\cdot(1-\sin x)\cdot\sec^2 x}\\
&= \boxed{\sin^3 x\cdot(1-\sin x)\cdot\sec^4 x}
\end{aligned}$$

75. If $z=-1-1i$, find the trigonometric form of z and z^9, also find z^9 in standard form.

Solution: Here $\theta = 225°$ or $\theta = \dfrac{5\pi}{4}$, and $r = \sqrt{2}$.

$$\begin{aligned} z &= \boxed{\sqrt{2}\cos 225° + \left(\sqrt{2}\sin 225°\right) i} \\ z^9 &= \boxed{16\sqrt{2}\cos 2025° + \left(16\sqrt{2}\sin 2025°\right) i} \\ z^9 &= \boxed{-16 - 16i} \end{aligned}$$

76. Find all solutions.

$$2\cos^2 x + 3\sin x - 3 = 0$$

Solution: You'll need to write it in terms of cosines first.

$$\begin{aligned} 2\cos^2 x + 3\sin x - 3 &= 0 \\ 2\left(1 - \sin^2 x\right) + 3\sin x - 3 &= 0 \\ -2\sin^2 x + 3\sin x - 1 &= 0 \end{aligned}$$

Now multiply both sides of this equation by -1 and factor.

$$\begin{aligned} 2\sin^2 x - 3\sin x + 1 &= 0 \\ (2\sin x - 1)(\sin x - 1) &= 0 \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} \sin x = \frac{1}{2} \qquad &\Rightarrow \qquad x = \boxed{\frac{\pi}{6} + 2\pi k, \quad k \in \mathbb{Z}} \\ & \qquad\qquad\;\; = \boxed{\frac{5\pi}{6} + 2\pi k, \quad k \in \mathbb{Z}} \\ \sin x = 1 \qquad &\Rightarrow \qquad x = \boxed{\frac{\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}}. \end{aligned}$$

77. Find the supplement of $83°$.

Solution:

$$180° - 83° = \boxed{97°}$$

78. Solve the triangle, given $a = 11.23$ inches $b = 8.24$ inches, and $B = 29.84°$.

Solution: You should, of course, draw a triangle first. This one actually gives two distinct answers. I'll give you credit for either one.

$\boxed{A = 42.70°,\ C = 107.46° \text{ and } c = 15.80 \text{ inches.}}$

or

$\boxed{A = 137.30°,\ C = 12.86° \text{ and } c = 3.69 \text{ inches.}}$

79. Determine two coterminal angles in radian measure (one positive, one negative) for $\theta = \frac{\pi}{3}$.

Solution:

$$\frac{\pi}{3} + 2\pi = \boxed{\frac{7\pi}{3}} \quad \text{and} \quad \frac{\pi}{3} - 2\pi = \boxed{-\frac{5\pi}{3}}$$

80. Find the length of arc on a circle of radius 14 inches and a central angle of 60°.

Solution:

$$S = 14 \cdot \frac{\pi}{3}$$

So, the length is $\boxed{\frac{14\pi}{3} \text{ inches}}$.

81. If the $\sec\theta = 7$ and $270° < \theta < 360°$, find the following. So, $y = -4\sqrt{3}$

   (a) $\sin\theta$

   Solution: From the information given we can conclude that $x = 1$, $r = 7$, and that $y < 0$. To find the value of y, solve

   $$7^2 = 1^2 + y^2 \quad \Rightarrow \quad y = \pm\sqrt{48} = \pm 4\sqrt{3}.$$

   $\sin\theta = \boxed{\frac{-4\sqrt{3}}{7}}$

(b) $\cos\theta$

Solution: $\cos\theta = \boxed{\dfrac{1}{7}}$

(c) $\tan\theta$

Solution: $\tan\theta = \boxed{-4\sqrt{3}}$

(d) $\cot\theta$

Solution: $\cot\theta = \boxed{\dfrac{-1}{4\sqrt{3}}}$

(e) $\csc\theta$

Solution: $\csc\theta = \boxed{\dfrac{-7}{4\sqrt{3}}}$

82. Given that

$$P_k = \frac{k}{2} \cdot [5k - 3],$$

find P_{k+1}.

Solution: Just replace k with $(k + 1)$.

$$\boxed{P_{k+1} = \frac{(k+1)}{2} \cdot [5(k+1) - 3]}.$$

4.3 Assignment

You should review, as necessary, the materials presented in Chapter 1 and do the WebAssign assignment mth.121.01.00.

5 mth.121.02.01

5.1 Tangents and Secants

The idea of a limit is central to calculus and an intuitive grasp of this concept is essential to further study. Initially we will examine two special types of limits: tangents and velocities. These two limits gives rise to the central idea in differential calculus—the derivative.

The word tangent is derived from the Latin word *tangens*, which means 'touching.' Thus, a tangent to a curve is a line that touches the curve and has the same direction as the curve at the point of contact.

Let's take a look at the curve[5] (Figure 6, page 46) $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + 5$ and a line $y = -2x + 5$, which is tangent to this curve at the point $(0,\ 5)$.

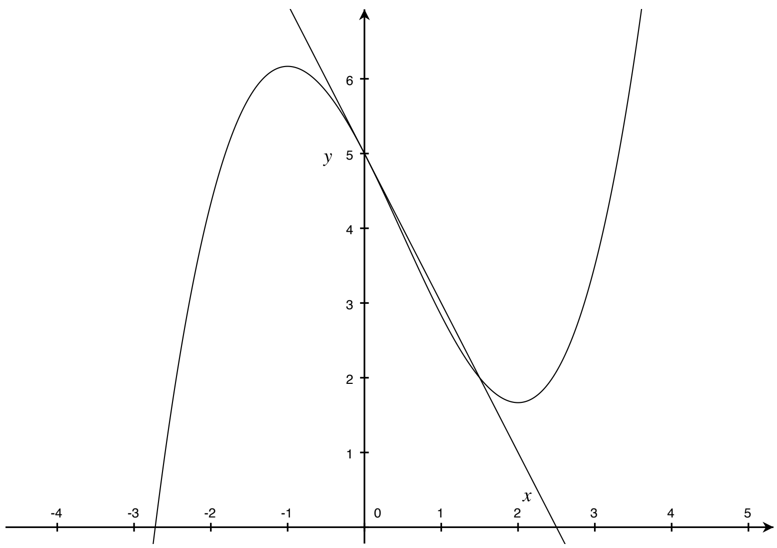

Figure 6: $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + 5$ and $y = -2x + 5$

This particular tangent line also happens to be a secant line,[6] because it touches this particular curve twice.[7] Again, take a look!

Example: Let's take a look at a very simple example. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P\,(1,\ 1)$.

[5]Make sure you can graph simple functions by hand, however, more difficult graphs may require the use of technology. Make sure you know how to graph both simple and more difficult functions.

[6]A secant is a straight line that cuts a curve in two or more parts. A tangent is a straight line that touches a curve at a point, but does not cross (some say puncture) it at that point. There's *subtlety* here, because you can have a line that touches and does not puncture a curve at a point, but is not tangent. We'll more precisely define tangent later.

[7]You should be able to visually note that there are at least two points of intersection, but can you show that there's only two? What are the two points?

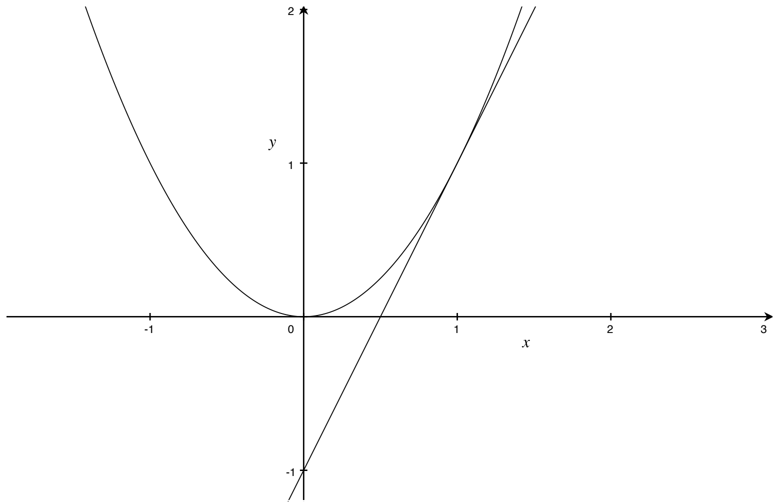

Figure 7: $y = x^2$ and a line tangent at $P(1,\ 1)$.

Solution: First we will make a rough sketch (Figure 7, page 47) and then try to make a guess at what this tangent line is. A good guess (slope *appears* to be $m = 2$ and the y-intercept *appears* to be $b = -1$) for the equation of the line tangent is:

$$y = 2x - 1.$$

Your answer may vary depending on how accurate your graph is.

Next we will take a sequence of secant lines that approach the tangent line, compute their slopes—this is often referred to as *rate of change*, or simply ROC—and make a prediction where these slopes are going. Since we want to move towards the point $P(1,\ 1)$, along the curve $y = x^2$, we can select points from both the right and left of P and compute the slope of the secant line using the slope formula. Let's not be timid (stay close) when choosing values to the right and left of $x = 1$. Here's a nice pattern for the right of $x = 1$: $1.1, 1.01, 1.001$; now lets compute the slope of the secant using $P(1,\ 1)$ for these three values for x.

From the *right* of $x = 1$.

$$\begin{aligned} m_{\text{secant}}(x) &= \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1 \\ m_{\text{secant}}(1.1) &= 2.1 \\ m_{\text{secant}}(1.01) &= 2.01 \\ m_{\text{secant}}(1.001) &= 2.001 \end{aligned}$$

Looks like this sequence is going towards 2. Now from the *left* of $x = 1$.

$$\begin{aligned} m_{\text{secant}}(x) &= \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1 \\ m_{\text{secant}}(0.9) &= 1.9 \\ m_{\text{secant}}(0.99) &= 1.99 \\ m_{\text{secant}}(0.999) &= 1.999 \end{aligned}$$

Again, it looks like this sequence is going towards 2. So now we have a point $P(1,\ 1)$ and an apparent slope $m = 2$ of the tangent.

$$\begin{aligned} y - 1 &= 2(x-1) \\ y &= 2x - 1 \end{aligned}$$

Just as we expected!

Example: This time we will look at a function that has units. Suppose that an object is dropped from a platform that is 400 meters above the ground, and its position (s in meters) above the ground is a function of time (t is seconds), where

$$s = s(t) = -4.9t^2 + 400.$$

Clearly this is a parabola, and if we draw (Figure 8, page 48) a tangent line at the point where $t = 5$, we will be able to approximate the slope.[8]

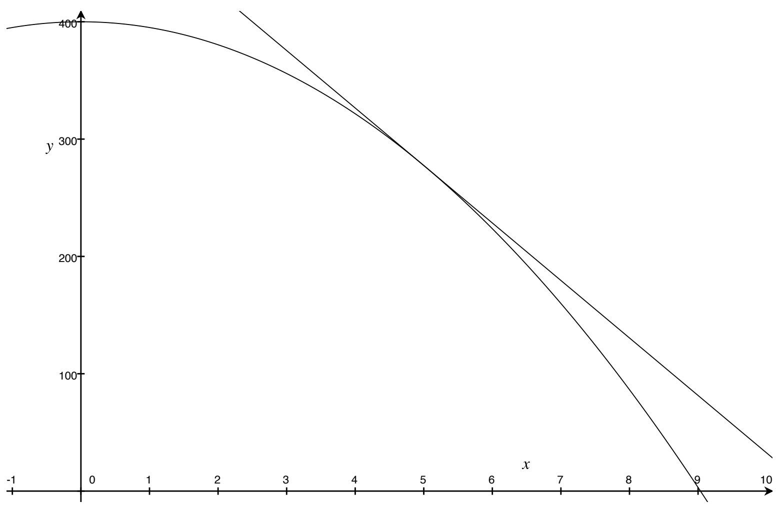

Figure 8: $s = s(t) = -4.9t^2 + 400$ and a line tangent at $t = 5$.

1. What is the unit of this slope?

   Solution: Meters per second.

2. What is the sign of this slope?

   Solution: Negative.

[8] I know, it's more difficult because of the scale and units.

3. Using $t = 5$, find values to the left and right of $t = 5$, compute the corresponding slope of the secant lines, and make a prediction about what the slope of the tangent line at this point.

Solution: From the *right* of $t = 5$.

$$\begin{aligned} m_{\text{secant}}(t) &= \frac{-4.9t^2 + 122.5}{t-5}, \quad t \neq 5 \\ m_{\text{secant}}(5.1) &= -49.49 \\ m_{\text{secant}}(5.01) &= -49.049 \\ m_{\text{secant}}(5.001) &= -49.0049 \end{aligned}$$

Looks like this is going towards -49.

From the *left* of $t = 5$.

$$\begin{aligned} m_{\text{secant}}(t) &= \frac{-4.9t^2 + 122.5}{t-5}, \quad t \neq 5 \\ m_{\text{secant}}(4.9) &= -48.51 \\ m_{\text{secant}}(4.99) &= -48.951 \\ m_{\text{secant}}(4.999) &= -48.9951 \end{aligned}$$

Looks like this is also going towards -49.

Prediction is -49 meters per second.

4. What is the equation of this tangent line?

Solution:

$$\begin{aligned} s - s_1 &= m(t - t_1) \\ s - 277.5 &= -49(t - 5) \\ s &= -49t + 522.5 \end{aligned}$$

5. What is the *instantaneous velocity* of this object at $t = 5$ seconds?

Solution: The slope of the tangent line for this problem is the same as the *instantaneous velocity*. So the *instantaneous velocity* of this object at $t = 5$ seconds is -49 meters per second. Yes, the object has negative velocity because it is *falling* towards earth.

You're going to have to become comfortable with doing these types of computations, and please consider mastering the use of your calculator to facilitate this process.

5.2 Examples

1. Compute $\dfrac{\Delta y}{\Delta x}$ for the interval $[-1,\ 2]$, where $y(x) = 3x^2 - 2x + 5$. This is just the slope of the secant line through the points $(y(2),\ 2)$ and $(y(-1),\ -1)$.

   Solution: We'll discuss this in class.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{y(x_2) - y(x_1)}{x_2 - x_1} \\ &= \frac{y(2) - y(-1)}{2+1} \\ &= \frac{13-10}{3} \\ &= 1\end{aligned}$$

2. Estimate the instantaneous rate of change of the function $y(x) = 3x^2 - 2x + 5$ at the point $x = -1$.

   Solution: We'll discuss this in class.

   You need to make sure you understand the examples presented in class to do this problem. You will not be given credit unless you *know* the proper steps. That is, computing a sequence to secants from both the left and right of $x = -1$, and then making a prediction about what these slopes are converging to.

   Final answer: -8.

3. With an initial deposit of 100 dollars, the balance in a bank account after t years is $f(t) = 100(1.05)^t$ dollars.

   (a) What are the units of the ROC[9] of $f(t)$?

   Solution: We'll discuss this in class.

   Dollars per year.

   (b) Graph[10] (Figure 9, page 9) this functions and estimate the slope of the tangent at $t = 1$.

   Solution: We'll discuss this in class.

   About 5.1 dollars per year.

[9] ROC = rate of change

[10] Again, make sure you can graph on your own!

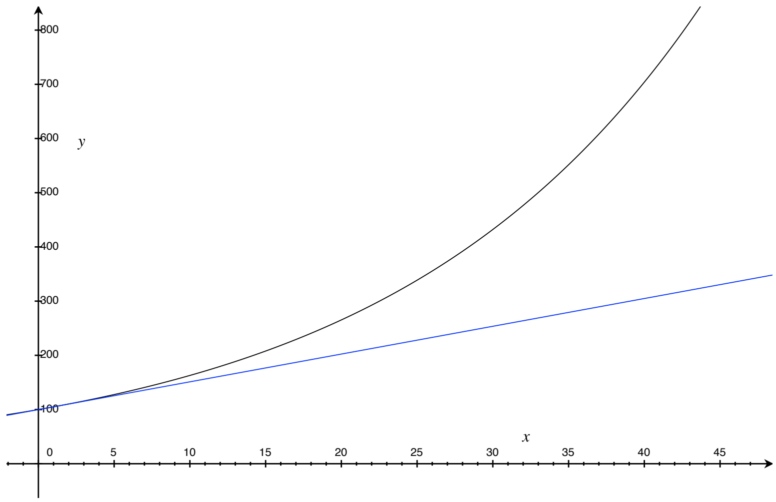

Figure 9: The growth of money with tangent at $t = 1$.

(c) Looking at the graph do the slopes of the tangents appear to be getting bigger as time goes on?

Solution: We'll discuss this in class.

Yes!

5.3 Assignment

You should read §2.1 and do the WebAssign assignment mth.121.02.01.

6 mth.121.02.02

6.1 Introductory Limits

Definition: We write

$$\lim_{x\to a} f(x) = L$$

and say, "the limit of $f(x)$, as x approaches a, equals L," if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

6.1.1 Calculating Limits

Although initially difficult to grasp,

$$\lim_{x\to a} f(x),$$

this notation basically tells us to get close to a (BUT DO NOT TOUCH a!). In a way, we are deifying a. Fact is, when you tell someone of a forbidden fruit, they want to touch it. But be warned, try not to touch it! If this notation says anything, it says, "do not touch a, but please try to get close to a. As close as you like!" Since a exist on a one dimensional number line, you should note that there are two ways to approach any finite a.

We write[11]

$$\lim_{x\to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a is equal to L. A left hand limit just means that we are approaching a from the left, or from values smaller than a. We can also write

$$\lim_{x\to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a is equal to L. A right hand limit just means that we are approaching a from the right, or from values larger than a. For the limit

$$\lim_{x\to a} f(x)$$

to exist, both the left and right limits must exist and be the same. That is

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

You should note that L is referring to a single finite number. If L is not finite, we say the limit does not exists. However, most mathematicians accept the following definitions.

[11] Some do not write the $a^{\pm}$, they write $a\pm$ instead.

$L \to \infty$: Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the value of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

$L \to -\infty$: Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the value of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

We can also allow x to approach $\pm\infty$, basically to analyze a function's behavior in the extremes of infinity. The definition is: let f be a function defined on some interval $(a,\ \infty)$. Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large.

We can also go in the other direction towards $-\infty$. The definition is: let f be a function defined on some interval $(-\infty,\ a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large negative.

Okay, so what are we expected to do with all this? Let's take a look at an example.

Example: Find the limit, if it exists, by using values close (don't touch) to $x = -1$, from both left and right, and also use a graph.[12]

$$\lim_{x \to -1} \frac{x+1}{x^2-1}$$

Solution: Typically we will start with a visual and then a sequence of numerical calculations. Here's a graph (Figure 10, page 54)

Looking at the graph you need to *visualize* getting close to $x = -1$, from both left and right, to see where $\dfrac{x+1}{x^2-1}$ is going. After looking at the graph (Figure 10, page 54), my guess is $-1/2$. I've placed a crosshair centered at $x = -1$ and you'll need to estimate what

[12] You'll need to use a graphic utility, either a handheld calculator/computer, or a web-based application such as Wolfram Alpha. You can access Wolfram Alpha for free at: http://www.wolframalpha.com.

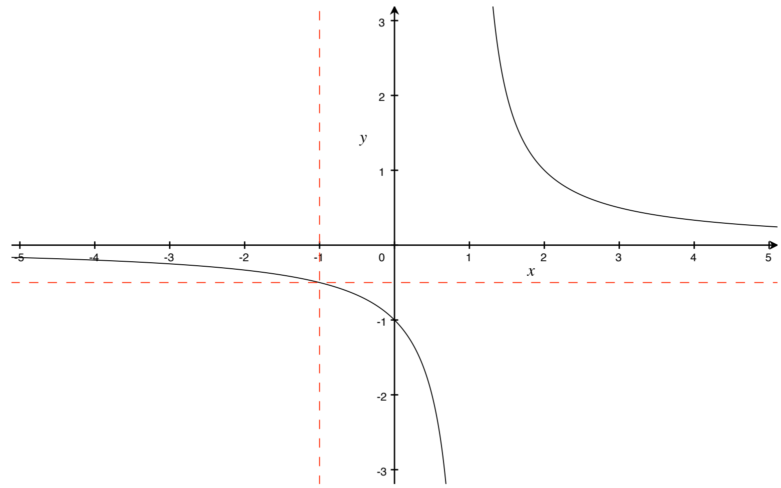

Figure 10: Graph of $y = \dfrac{x+1}{x^2-1}$ with *crosshair* at $x = -1$.

the y value is. You should note that the graph is undefined at $x = -1$. So I would be visually confident to write

$$\lim_{x \to -1} \frac{x+1}{x^2-1} = -\frac{1}{2},$$

but not certain. Again, we know where x is going, but we're not totally certain where $\dfrac{x+1}{x^2-1}$ is going as x gets closer to -1.

Now let's move on to numerical sequences. First from the right of $x = -1$.

$$\begin{aligned} y(x) &= \frac{x+1}{x^2-1} = \frac{1}{x-1}, \quad x \neq -1 \\ y(-0.9) &= -\frac{1}{1.9} \\ y(-0.99) &= -\frac{1}{1.99} \\ y(-0.999) &= -\frac{1}{1.999} \end{aligned}$$

Again, it looks like this sequence is going towards $-1/2$. So I write

$$\lim_{x \to -1^+} \frac{x+1}{x^2-1} = -\frac{1}{2}.$$

Now from the left of $x = -1$.

$$\begin{aligned} y(x) &= \frac{x+1}{x^2-1} = \frac{1}{x-1}, \quad x \neq -1 \\ y(-1.1) &= -\frac{1}{2.1} \\ y(-1.01) &= -\frac{1}{2.01} \\ y(-1.001) &= -\frac{1}{2.001} \end{aligned}$$

Again, it looks like this sequence is going towards $-1/2$. So I write

$$\lim_{x \to -1^-} \frac{x+1}{x^2-1} = -\frac{1}{2}.$$

Everything is in agreement, so I may conclude that

$$\lim_{x \to -1} \frac{x+1}{x^2-1} = -\frac{1}{2}.$$

For those of you interested in studying mathematics further, you should note that this is not a *proof* that this limit is $-1/2$.

6.2 Examples

1. Find the limit, if it exists.

$$\lim_{x \to 0} \frac{\sqrt{x^2+4}-2}{x^2}$$

Use values close to $x = 0$, from both left and right, and use a graph.[13]

Solution: The graph (Figure 11, page 56) indicates that the limit is $1/4$. Make sure you can graph this on your own!

Now let's move on to numerical sequences. First from the right of $x = 0$.

$$\begin{aligned} y(x) &= \frac{\sqrt{x^2+4}-2}{x^2}, \quad x \neq 0 \\ y(1) &\approx 0.236068 \\ y(0.5) &\approx 0.246211 \\ y(0.1) &\approx 0.249844 \\ y(0.05) &\approx 0.249961 \\ y(0.01) &\approx 0.249998 \end{aligned}$$

[13] Use: $\pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.01$.

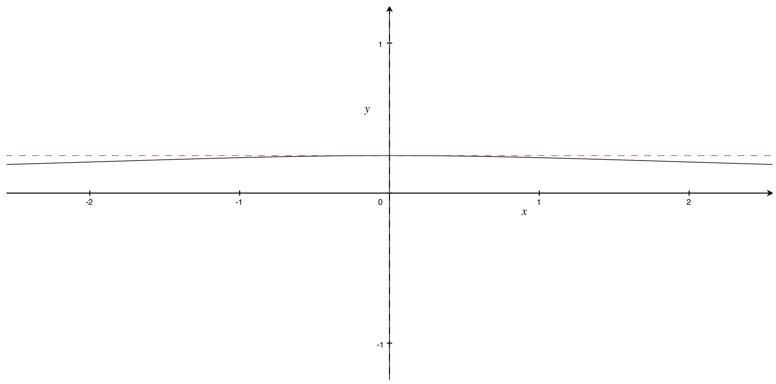

Figure 11: $y = \dfrac{\sqrt{x^2+4}-2}{x^2}$, and crosshair at $x=0$

Again, it looks like this sequence is going towards 0.25. So I write

$$\lim_{x\to 0^+} \frac{\sqrt{x^2+4}-2}{x^2} = \frac{1}{4}.$$

Now from the left of $x = 0$.

$$\begin{aligned}
y(x) &= \frac{\sqrt{x^2+4}-2}{x^2}, \quad x \neq 0 \\
y(-1) &\approx 0.236068 \\
y(-0.5) &\approx 0.246211 \\
y(-0.1) &\approx 0.249844 \\
y(-0.05) &\approx 0.249961 \\
y(-0.01) &\approx 0.249998
\end{aligned}$$

Again, it looks like this sequence is going towards $1/4$. So I write

$$\lim_{x\to 0^-} \frac{\sqrt{x^2+4}-2}{x^2} = \frac{1}{4}.$$

The graph (Figure 11, page 56) and the table of values are in full agreement so I feel comfortable in writing

$$\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} = \frac{1}{4}.$$

2. Find the limit, if it exists.

$$\lim_{x\to 0} \cos\frac{2\pi}{x}$$

Use values close to $x = 0$, from both left and right, and use a graph.[14]

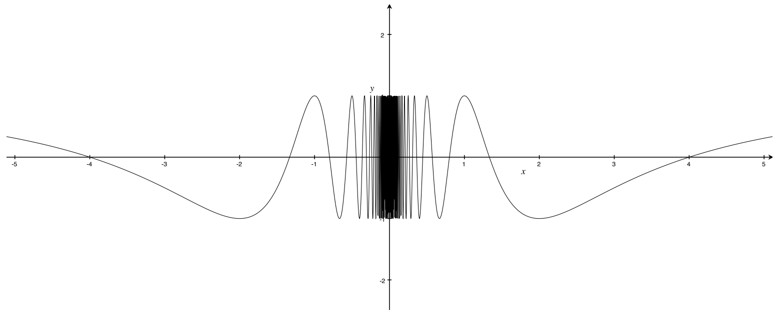

Figure 12: $y = \cos \dfrac{2\pi}{x}$

Solution: The graph (Figure 12, page 57) indicates that the limit is does not exist (DNE) because it does not go towards a single number. Make sure you can graph this on your own!

Now let's move on to numerical sequences. First from the right of $x = 0$.

$$
\begin{aligned}
y(x) &= \cos \frac{2\pi}{x}, \quad x \neq 0 \\
y(1) &= 1 \\
y(1/2) &= 1 \\
y(1/3) &= 1 \\
y(1/4) &= 1 \\
y(1/6) &= 1
\end{aligned}
$$

Again, it looks like this sequence is going towards 1. So I write

$$\lim_{x \to 0^+} \cos \frac{2\pi}{x} = 1. \quad \text{This is actually } \textit{wrong}!$$

Now from the left of $x = 0$.

$$
\begin{aligned}
y(x) &= \cos \frac{2\pi}{x}, \quad x \neq 0 \\
y(-1) &= 1 \\
y(-1/2) &= 1 \\
y(-1/3) &= 1 \\
y(-1/4) &= 1 \\
y(-1/6) &= 1
\end{aligned}
$$

[14] Use: $\pm 1, \pm 1/2, \pm 1/3, \pm 1/4, \pm 1/6$.

Again, it looks like this sequence is going towards 1. So I write

$$\lim_{x\to 0^-} \cos\frac{2\pi}{x} = 1. \quad \text{This is actually } \textit{wrong}!$$

Don't be fooled!

The graph (Figure 12, page 57) and the table of values are in disagreement! Yes, the values were chosen to illustrate that picking *easy* values may in fact lead to a **wrong** answer. Initially it is particularly important that you can both graph and use tables to feel your way through limit problems, It requires work, but once you master the use of your calculators you should be able to do this quickly.

The **correct** answer is

$$\lim_{x\to 0} \cos\frac{2\pi}{x} = DNE.$$

3. Find the limit, if it exists.

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

Use values close to $x = 0$, from both left and right, and use a graph.[15]

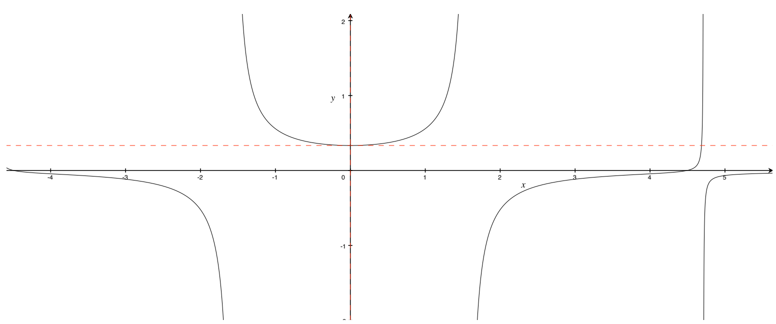

Figure 13: $y = \dfrac{\tan x - x}{x^3}$, and crosshair at $x = 0$

Solution: The graph (Figure 13, page 58) indicates that the limit is $1/3$. Make sure you can graph this on your own!

I am not providing tables here, but you should be capable of doing this on your own now. Be aware that you will need to be able to do this on assignments and exams!

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$$

[15]Use: $\pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.01, \pm 0.0005$.

4. Find the limit, if it exists.[16]

$$\lim_{x \to 2^+} \frac{1}{2-x}$$

Use values close to $x = 2$, from the right.

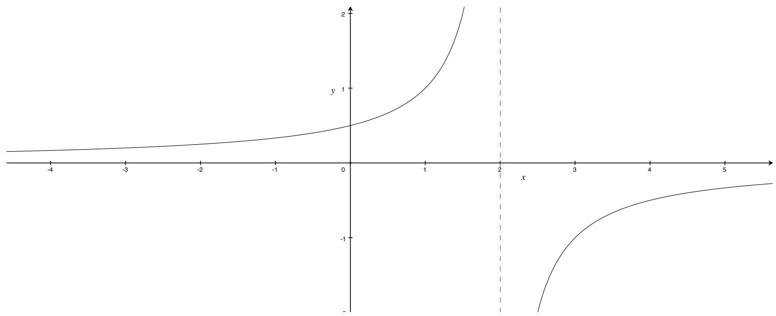

Figure 14: $y = \dfrac{1}{2-x}$, and asymptote at $x = 2$

Solution: The graph (Figure 14, page 59) indicates that the limit is infinite in the negative direction, and we indicate this using $-\infty$. Make sure you can graph this on your own!

Again, you should be able to create a reasonable table by now!

$$\lim_{x \to 2^+} \frac{1}{2-x} = -\infty$$

5. Find the limit, if it exists.

$$\lim_{x \to 2^-} \frac{1}{2-x}$$

Use values close to $x = 2$, from the left.

Solution: The graph (Figure 15, page 60) indicates that the limit is infinite in the positive direction, and we indicate this using ∞. Make sure you can graph this on your own!

Again, you should be able to create a reasonable table by now!

$$\lim_{x \to 2^-} \frac{1}{2-x} = \infty$$

[16] Exists means that the limit is a finite number. However, if the number gets "big" without bound we use this symbol ∞ indicating infinity. You should specify, using $\pm$, what direction the infinity is going in.

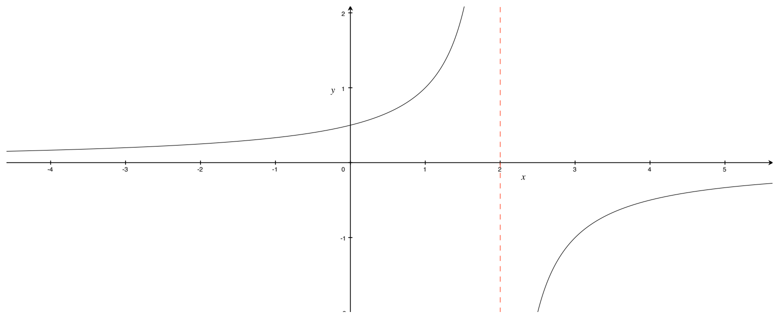

Figure 15: $y = \dfrac{1}{2-x}$, and asymptote at $x = 2$

6. Find the limit, if it exists.

$$\lim_{x\to 2} \frac{1}{2-x}$$

Solution:

$$\lim_{x\to 2} \frac{1}{2-x} = DNE$$

7. Without using a graph or tables, evaluate each of the following by trying to *visualize* in your head what the limit is.[17]

   (a) Find the limit, if it exists.

   $$\lim_{x\to 5} 3x - 2$$

   Solution: It's a line and you should be able to reason that the limit exists.

   $$\lim_{x\to 5} 3x - 2 = 13$$

   (b) Find the limit, if it exists.

   $$\lim_{x\to 1} x^2 - 2x + 3$$

[17]This may involve simple mental arithmetic as well.

Solution: It's a parabola and you should be able to reason that the limit exists.

$$\lim_{x \to 1} x^2 - 2x + 3 = 2$$

(c) Find the limit, if it exists.

$$\lim_{x \to -1} \pi$$

Solution: It's a horizontal line and you should be able to reason that the limit exists.

$$\lim_{x \to -1} \pi = \pi$$

8. Given the following graph (Figure 16, page 61) determine each of the limits.

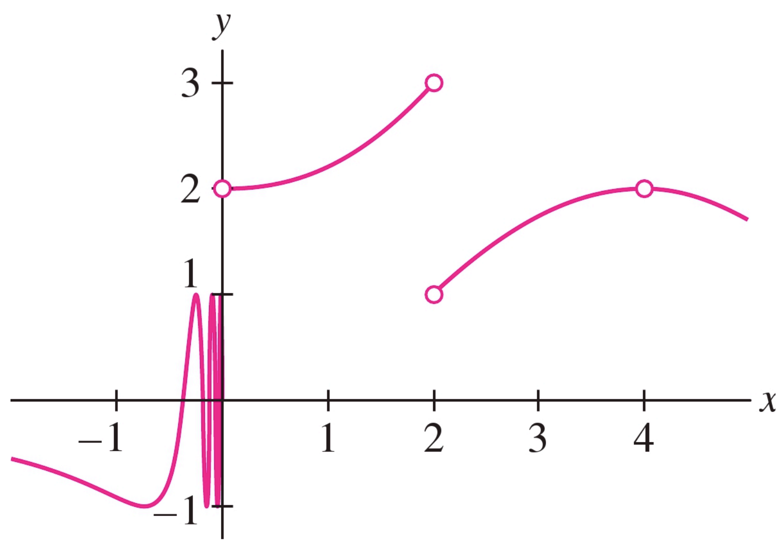

Figure 16: Partial graph of $f(x)$.

(a)

$$\lim_{x \to 0^-} f(x) =$$

Solution: We will discuss this in class.

(b)

$$\lim_{x \to 0^+} f(x) =$$

Solution: We will discuss this in class.

(c)

$$\lim_{x \to 0} f(x) =$$

Solution: We will discuss this in class.

(d)

$$\lim_{x \to 2^-} f(x) =$$

Solution: We will discuss this in class.

(e)

$$\lim_{x \to 2^+} f(x) =$$

Solution: We will discuss this in class.

(f)

$$\lim_{x \to 2} f(x) =$$

Solution: We will discuss this in class.

(g)

$$\lim_{x \to 4^-} f(x) =$$

Solution: We will discuss this in class.

(h)

$$\lim_{x \to 4^+} f(x) =$$

Solution: We will discuss this in class.

(i)

$$\lim_{x \to 4} f(x) =$$

Solution: We will discuss this in class.

6.3 Assignment

You should read §2.2 and do the WebAssign assignment mth.121.02.02.

7 mth.121.02.03

7.1 Calculating Limits

7.1.1 Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \lim_{x\to a} g(x)$$

exist. Then

$$\begin{aligned}
\lim_{x\to a}[f(x)+g(x)] &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\
\lim_{x\to a}[f(x)-g(x)] &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \\
\lim_{x\to a}[c\cdot f(x)] &= c\cdot \lim_{x\to a} f(x) \\
\lim_{x\to a}[f(x)\cdot g(x)] &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \\
\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0
\end{aligned}$$

These are the basic limit laws, but if you look in the book you'll see more. Bottom line, you'll need to carefully think about getting close to a number without actually touching it. Furthermore, any given number can be approached from two sides—and you must think through this carefully. Here are some additional, and useful, limit laws.

$$\begin{aligned}
\lim_{x\to a}[f(x)]^n &= \left[\lim_{x\to a} f(x)\right]^n \qquad n\in\mathbb{Z}^+ \\
\lim_{x\to a} c &= c \\
\lim_{x\to a} x &= a \\
\lim_{x\to a} x^n &= a^n \qquad n\in\mathbb{Z}^+ \\
\lim_{x\to a} \sqrt[n]{x} &= \sqrt[n]{a} \qquad n\in\mathbb{Z}^+ \quad \text{If } n \text{ is even, we must have } a>0. \\
\lim_{x\to a} \sqrt[n]{f(x)} &= \sqrt[n]{\lim_{x\to a} f(x)} \qquad n\in\mathbb{Z}^+ \quad \text{If } n \text{ is even, we must have } \lim_{x\to a} f(x) > 0.
\end{aligned}$$

Certainly, there are other useful limit laws. As we all know, laws can be quite confusing, and I am not trying to suggest that these laws are easily understood, let-alone provable. Again, using your head is important, and constantly ask yourself what's happening to the function as x gets close to a (don't touch a). Look from both the left and right—just like you're crossing the street! Furthermore, the information you receive from the left and right better agree. Understanding this takes time, and you may on occasion have to resort to tables and/or graphs to clarify your beliefs.

7.2 Examples

1. Evaluate the limit using the basic limit laws.

$$\lim_{x\to 1} \frac{1}{x^2}$$

Solution: This will be discussed in class.

2. Evaluate the limit using the basic limit laws.

$$\lim_{x\to 8} 3x^{2/3} - 16x^{-1}$$

Solution: This will be discussed in class.

3. Evaluate the limit using the basic limit laws.

$$\lim_{x\to 9} \frac{\sqrt{x}}{x-2}$$

Solution: This will be discussed in class.

4. Evaluate the limit using the basic limit laws.

$$\lim_{x\to 1/3} \left(18x^2 - 4\right)^4$$

Solution: This will be discussed in class.

5. Can the quotient law be applied to

$$\lim_{x\to 0} \frac{\sin x}{x}?$$

Explain.

Solution: This will be discussed in class.

6. Evaluate using a graph and/or table of values.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Solution: This will be discussed in class.

7.3 Assignment

You should read §2.3 and do the WebAssign assignment mth.121.02.03.

8 mth.121.02.04

8.1 Continuity

Quite simply, a continuous function has a graph that can be drawn without lifting the pencil from the paper. For example, all polynomial functions are continuous on any given interval. For example,

$$f(x) = 5x^7 + 6x^5 - 4x^4 - 3x^2 - 9x + 1,$$

is a polynomial of degree 7 and is continuous everywhere. The rational functions are a bit different, and you should note that your pencil remains on the paper for some intervals only. For example

$$f(x) = \frac{1}{x-1}$$

is continuous on any interval that does not contain $x = 1$ (division by zero). However, the rational function

$$f(x) = \frac{1}{x^2+1}$$

is continuous everywhere. Certainly there are many other functions that you studied in MTH-119 and MTH-120, and you basically need to think about tracing (drawing) the graph to determine where a function is continuous.

Here are some useful definitions on continuity.

Definition: A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Definition: A function f is continuous from the right of a number a if

$$\lim_{x \to a^+} f(x) = f(a),$$

and f is continuous from the left of a number a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Definition: A function f is continuous on an interval if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.

Here are some useful theorems on continuity.

Theorem Any polynomial function is continuous everywhere; that is, it is continuous on $\mathbb{R}$ (all real numbers).

Theorem Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem Any trigonometric function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem Any inverse trigonometric function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem Any root function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem Any exponential function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem Any logarithmic function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. $c \cdot f$
4. $f \cdot g$
5. f/g if $g(a) \neq 0$

Theorem Suppose that

$$\lim_{x \to a} g(x) = L,$$

and f is continuous at L, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L),$$

Example: Please don't believe that all functions are continuous on their domains. For example,

$$f(x) = [\![x]\!]$$

is the greatest integer function, it returns the largest integer less than of equal to x.[18]

[18] Computer scientist often refer to this as the floor function, and state "the greatest integer $\lfloor x \rfloor$ not larger than x." In the C language, the command is `floor()`.

Solution: Certainly its domain is $\mathbb{R}$, and its range is $\mathbb{Z}$. Just look at any integer to see why it is not continuous on its domain.

$$\lim_{x\to 3^-} [\![x]\!] = 2, \qquad \lim_{x\to 3^+} [\![x]\!] = 3, \qquad f(3) = 3$$

Optional Example: Another maddening example was discovered by Riemann, and this function is defined as follows:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational,} \\ \dfrac{1}{q} & \text{if} \quad x \text{ is rational, and } q \text{ is the denominator of } x \text{ in lowest terms.} \end{cases}$$

Solution: Although this function may prove difficult to analyze, with a little thought you should be able to convince yourself that its domain is $\mathbb{R}$ and its range is $\{1, 1/2, 1/3, 1/4, \ldots 0\}$. This function has the property that it is continuous where x is irrational, and not continuous where x is rational. This, of course, must have taken a considerable amount of mental fortitude to come up with. This may be worth a bit of thought, but don't fret this detail if it eludes you.

Example: Let's look at something more reasonable.

$$f(x) = \begin{cases} x^2 & \text{if} & x < 0 \\ \sin x & \text{if} & 0 \le x < \frac{\pi}{2} \\ \dfrac{2x}{\pi} & \text{if} & x \ge \frac{\pi}{2} \end{cases}$$

Solution: Here, you should note that each of these three functions are continuous by themselves, however we need to be concerned with how they are connected, that is, at the points where $x = 0$ and $x = \pi/2$.

$$\lim_{x\to 0^-} f(x) = 0, \qquad \lim_{x\to 0^+} f(x) = 0, \qquad f(0) = 0$$

Yes, by definition it's continuous at $x = 0$.

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = 1, \qquad \lim_{x\to \frac{\pi}{2}^+} f(x) = 1, \qquad f\left(\frac{\pi}{2}\right) = 1$$

Yes, by definition it's continuous at $x = \pi/2$.

Here's the partial graph (Figure 17, page 70) of $f(x)$.

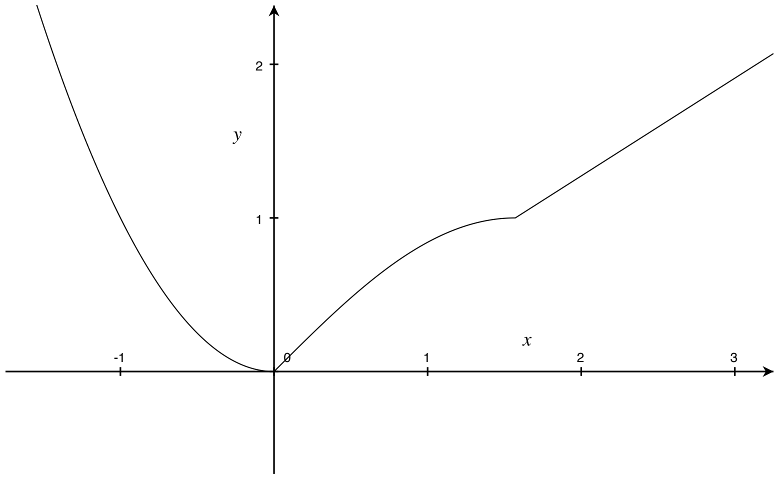

Figure 17: Partial graph of $f(x)$.

8.2 Examples

1. Examine the function

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 1 \\ 1 + \ln x & \text{if} \quad 1 \leq x < e \\ x & \text{if} \quad x \geq e \end{cases}$$

at the points where $x = 1$ and $x = e$.

Solution: This will be discussed in class.

2. Are there values of c and m that make

$$h(x) = \begin{cases} cx^2 & \text{if} \quad x < 1 \\ 4 & \text{if} \quad x = 1 \\ mx - x^3 & \text{if} \quad x > 1 \end{cases}$$

continuous at $x = 1$? If so. what are the values of c and m?

Solution: This will be discussed in class.

3. Show that

$$f(x) = \frac{x^4 - 1}{x - 1}$$

is discontinuous at $x = 1$. Rewrite this function (piecewise defined) so that it is continuous everywhere (remove the point discontinuity).

Solution: The function is not defined at $x = 1$, it has a removable discontinuity because

$$\lim_{x \to 1} f(x) = 4.$$

We need to plug the hole at $x = 1$.

$$h(x) = \begin{cases} \dfrac{x^4 - 1}{x - 1} & \text{if} \quad x \neq 1 \\ 4 & \text{if} \quad x = 1 \end{cases}$$

8.3 Assignment

You should read §2.4 and do the WebAssign assignment mth.121.02.04.

9 mth.121.02.05

9.1 Algebraic Manipulation and Reason!

Example: Supposed you're asked to find the following limit.

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$$

And you just plug the number in and you're then faced with $\infty-\infty$. You may believe you're looking at 0, but its's actually 1/2. So it looks like these infinities battle one another and when the dust settled you were left with 1/2, not 0 as some (mindless people) may conclude. Here, like many limit problems, you'll need to use your algebra skills to reason forward.

Solution:

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right) &= \lim_{x\to 1}\frac{(x-1)}{(x-1)(x+1)}\\ &= \lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}\end{aligned}$$

In the examples that follow, we will be faced with problems where thinking is *required*. I strongly suggest that you think before answering. And when you're faced with $0/0$, ∞/∞, $\infty\cdot 0$, and $\infty-\infty$ ... *you're really going to have to think.*[19] Yes, your numerical and algebraic skills will be needed.

9.2 Examples

1. Find the limit if it exist.

$$\lim_{x\to 0^+}\frac{|x|}{x}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x\to 0^+}\frac{|x|}{x}=1$$

[19] Indeterminate forms.

2. Find the limit if it exist.

$$\lim_{x \to 0^-} \frac{|x|}{x}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

3. Find the limit if it exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 0} \frac{|x|}{x} = DNE$$

4. Find the limit if it exist.

$$\lim_{x \to 8} \frac{x^3 - 64x}{x - 8}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 8} \frac{x^3 - 64x}{x - 8} = 128$$

5. Find the limit if it exist.

$$\lim_{x \to 0} \frac{(1 + x)^3 - 1}{x}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 0} \frac{(1+x)^3 - 1}{x} = 3$$

6. Find the limit if it exist.

$$\lim_{x \to 1} \frac{3^{2x} - 9}{3^x - 3}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 1} \frac{3^{2x} - 9}{3^x - 3} = 6$$

7. Find the limit if it exist.

$$\lim_{x \to 0^+} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2 + x}} \right)$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 0^+} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2 + x}} \right) = 0$$

8. Find the limit if it exist.

$$\lim_{x \to \pi/2} \frac{\cot x}{\csc x}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to \pi/2} \frac{\cot x}{\csc x} = 0$$

9. Given

$$f(x) = \begin{cases} 2x & \text{if} \quad x < 2 \\ x^2 & \text{if} \quad x \geq 2, \end{cases}$$

find each of the following if it exist.

(a)

$$\lim_{x \to 2^-} f(x)$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 2^-} f(x) = 4$$

(b)

$$\lim_{x \to 2^+} f(x)$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 2^+} f(x) = 4$$

(c)

$$\lim_{x \to 2} f(x)$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 2} f(x) = 4$$

(d)

$$f(2)$$

Solution: We'll discuss this in class

Final Answer.

$$f(2) = 4$$

10. Find the limit if it exist.

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = -2$$

11. Find the limit if it exist.

$$\lim_{x \to 0} \frac{x^2 + 4x}{\sqrt{x^3 + x^2}}$$

Solution:

Final Answer.

$$\lim_{x \to 0} \frac{x^2 + 4x}{\sqrt{x^3 + x^2}} = DNE$$

The following work would be expected if this were an exam question.

1. For $x > 0$ (from the right of 0) we have

$$\lim_{x \to 0^+} \frac{x^2 + 4x}{\sqrt{x^3 + x^2}} = \lim_{x \to 0^+} \frac{x\,(x + 4)}{x\sqrt{x + 1}} = \lim_{x \to 0^+} \frac{x + 4}{\sqrt{x + 1}} = 4$$

2. For $x < 0$ (from the left of 0) we have

$$\lim_{x \to 0^-} \frac{x^2 + 4x}{\sqrt{x^3 + x^2}} = \lim_{x \to 0^-} \frac{x\,(x + 4)}{-x\sqrt{x + 1}} = \lim_{x \to 0^-} \frac{-x - 4}{\sqrt{x + 1}} = -4$$

Knowing that $\sqrt{x^2} = |x|$ is certainly helpful, but so is a graph (Figure 18, page 77) of

$$f\,(x) = \frac{x^2 + 4x}{\sqrt{x^3 + x^2}}.$$

12. Find the limit if it exist.

$$\lim_{x \to 0} \frac{x e^{1-2x}}{x^2 + x}$$

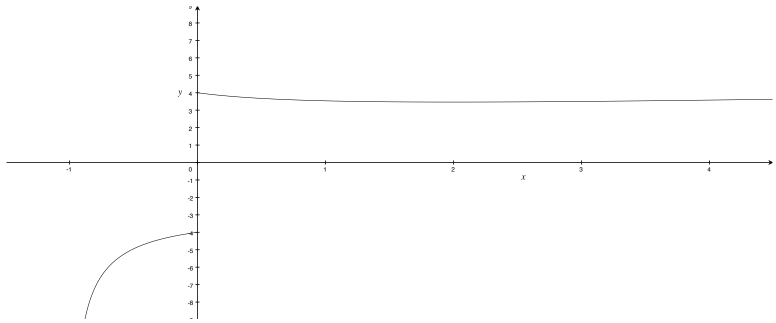

Figure 18: Convincing graph?

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x\to 0} \frac{xe^{1-2x}}{x^2+x} = e$$

13. Find the limit if it exist.

$$\lim_{x\to 0} \left(\frac{2}{x} - \frac{2}{|x|} \right)$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x\to 0} \left(\frac{2}{x} - \frac{2}{|x|} \right) = DNE$$

Again, on exams you will need to show two limits to reach this conclusion.

1. For $x > 0$ (from the right of 0) we have

$$\lim_{x\to 0^+} \left(\frac{2}{x} - \frac{2}{|x|} \right) = \lim_{x\to 0^+} \left(\frac{2}{x} - \frac{2}{x} \right) = 0$$

2. For $x < 0$ (from the left of 0) we have

$$\lim_{x\to 0^-} \left(\frac{2}{x} - \frac{2}{|x|} \right) = \lim_{x\to 0^-} \left(\frac{2}{x} + \frac{2}{x} \right) = \lim_{x\to 0-} \frac{4}{x} = -\infty$$

14. Find the limit if it exist.

$$\lim_{x\to 4^+} \sqrt{16-x^2}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 4^+} \sqrt{16 - x^2} = DNE$$

15. Find the limit if it exist.

$$\lim_{x \to 4^-} \sqrt{16 - x^2}$$

Solution: We'll discuss this in class

Final Answer.

$$\lim_{x \to 4^-} \sqrt{16 - x^2} = 0$$

16. Given that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, find

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2}.$$

Solution: We'll discuss this in class

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2} &= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \\
&= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} = 1
\end{aligned}$$

9.3 Assignment

You should read §2.5 and do the WebAssign assignment mth.121.02.05.

10 mth.121.02.06

10.1 Trigonometric Limits

10.1.1 Squeeze Theorem

On occasion we will be asked to evaluate a limit where our *thinking* will fail us, as will the laws. For example, let's say you're asked to evaluate

$$\lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right].$$

Although this may not seem like a fair question at this *point*, I want to use this example to illustrate a useful technique for evaluating limits. A graph (Figure 19, page 79) of

$$f(x) = \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right],$$

near $x = 0$ follows—you should note that $f(0)$ is undefined, that is, there's a hole at zero. Try to visualize the hole at $x = 0$—you actually can't see the hole, but please don't try to

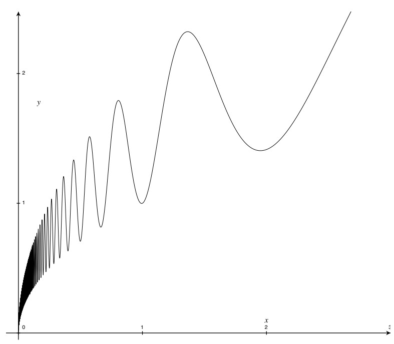

Figure 19: Partial graph of $f(x)$.

enter this abyss! This abyss is the dreaded division by zero. Yikes!

The main problem here—although easy to guess the limit from this graph—is actually proving that this limit exists and is 0. To prove this, we need the following theorem.

The Squeeze Theorem states, that if $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

Example: Now let's prove that

$$\lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] = 0$$

as our graph clearly indicates.

Solution: This will be discussed in class! Please try to follow what's being said.

$$-1 \leq \sin\left(\frac{2\pi}{x}\right) \leq 1 \quad \Rightarrow \quad 0 \leq \sin^2\left(\frac{2\pi}{x}\right) \leq 1 \quad \Rightarrow \quad 1 \leq 1+\sin^2\left(\frac{2\pi}{x}\right) \leq 2$$

Now multiply all sides of the last inequality by $\sqrt{x}$, given that $x > 0$ (Since x is from the right of zero, we know that $x > 0$.).

$$\sqrt{x} \leq \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] \leq 2\sqrt{x}$$

Since (The limit laws say-so. But, once again, please use your head.)

$$\lim_{x\to 0^+} \sqrt{x} = \lim_{x\to 0^+} 2\sqrt{x} = 0,$$

we have

$$\lim_{x\to 0^+} \sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right] = 0$$

by the Squeeze Theorem.

Illustration: The red curve (Figure 19, page 79) is $2\sqrt{x}$; the blue curve is $\sqrt{x}$; and the *squeezed* gray curve in the middle is

$$\sqrt{x}\left[1+\sin^2\left(\frac{2\pi}{x}\right)\right].$$

I think a graphing utility, whether a calculator or computer, is very helpful.

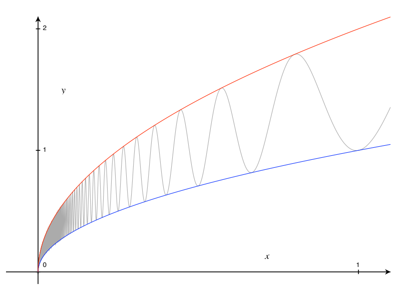

Figure 20: The Squeeze Theorem visualized!

10.1.2 An Important Case!

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$$

If you can recall, you were asked[20] to do this problem numerically and found that it equaled one. Now we need to show this. Here's one way to do so. This is a geometric argument and I am using a *unit* circle and simple area formulas as a basis for this argument. We will label

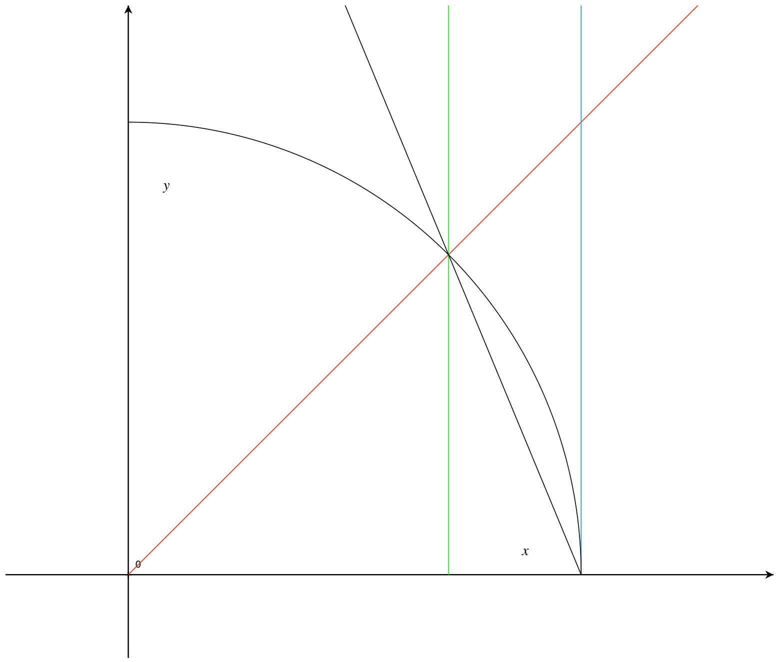

Figure 21: Will be discussed and labeled in class.

this diagram[21] in class where I discuss two triangles and one sector.[22] The inner triangle's area (smallest area right triangle—red hypotenuse, and green height) is

$$\frac{\sin\theta}{2};$$

the larger sector,[23]

$$\frac{\theta}{2};$$

and finally, a larger area (right triangle—base is 1 and the height is blue) is

$$\frac{\tan\theta}{2} = \frac{\sin\theta}{2\cos\theta}.$$

When $0 < \theta < \pi/2$ we get a very simple geometric (areas) inequality

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\sin\theta}{2\cos\theta},$$

[20]You've seen this limit before!
[21]The radius is 1, that angle is θ, and the point will be labeled using $\sin\theta$ and $\cos\theta$.
[22]You've got to listen to get this!
[23]I hope you can remember your simple trigonometry!

that can be easily manipulated to:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

This inequality also holds for $-\pi/2 < \theta < 0$.[24] Now just use the Squeeze Theorem to finally show that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

You'll need to remember this!

That was a lot of work and this limit will pop up often in calculus, so remember it!

Example: In fact this limit can be used to show that

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0.$$

Solution:

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} &= \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}\cdot\frac{\cos\theta + 1}{\cos\theta + 1}\\
&= \lim_{\theta\to 0} -\frac{\sin^2\theta}{\theta\left(\cos\theta + 1\right)}\\
&= \lim_{\theta\to 0} -\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\left(\cos\theta + 1\right)} = 0
\end{aligned}$$

10.2 Examples

1. Evaluate.

$$\lim_{x\to 0}\frac{\csc 8x}{\csc 4x}$$

Solution: We'll discuss this in class.

$$\begin{aligned}
\lim_{x\to 0}\frac{\csc 8x}{\csc 4x} &= \lim_{x\to 0}\frac{\sin 4x}{\sin 8x}\\
&= \lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{8x}{\sin 8x}\cdot\frac{4x}{8x}\\
&= \frac{1}{2}
\end{aligned}$$

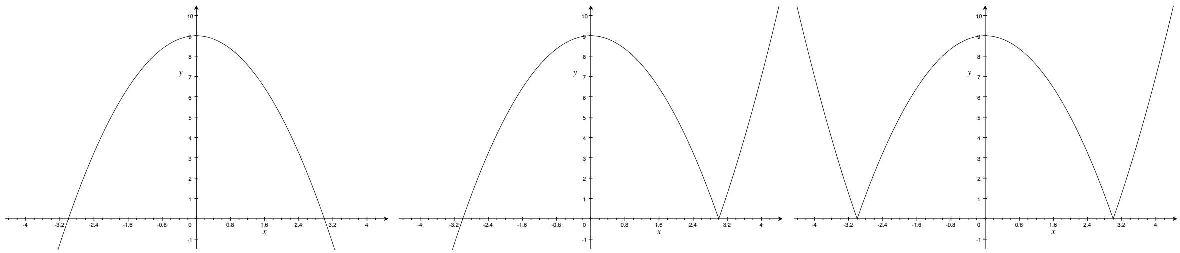

Figure 22: $y_1 = 9 - x^2$, $y_2 = (x^2 - 9)\,\dfrac{x-3}{|x-3|}$ and $y_3 = |x^2 - 9|$. Note that $y_1 \le y_2 \le y_3$.

2. Evaluate using the Squeeze Theorem.

$$\lim_{x\to 3} \left(x^2 - 9\right) \frac{x-3}{|x-3|}$$

Solution: We'll discuss this in class.

It's okay to use a calculator to make sure you're getting a squeeze. It's easy to get tricked!

$$9 - x^2 \le \left(x^2 - 9\right) \frac{x-3}{|x-3|} \le \left|x^2 - 9\right|$$

You should graph (Figure 22, page 83) these three functions to see that the inequality holds for all $x \ne 3$. Yes, it's tricky. Other squeezes exists, just be sure to graph them out.

$$\begin{aligned} \lim_{x\to 3} 9 - x^2 &= 0 \\ \lim_{x\to 3} \left|x^2 - 9\right| &= 0 \end{aligned}$$

By the ST,

$$\lim_{x\to 3} \left(x^2 - 9\right) \frac{x-3}{|x-3|} = 0.$$

3. Evaluate.

$$\lim_{x\to 0} \frac{\sin 4x}{3x}$$

[24]A good argument here can either be geometric, or noting the even nature of our relationship.

Solution: We'll discuss this in class.

$$\lim_{x\to 0}\frac{\sin 4x}{3x} = \lim_{x\to 0}\frac{4x}{3x}\cdot\frac{\sin 4x}{4x} = \frac{4}{3}$$

4. Evaluate using the Squeeze Theorem.

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

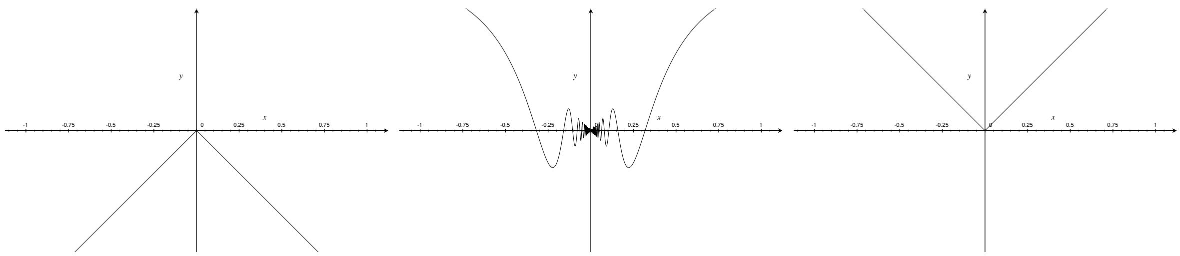

Figure 23: $y_1 = -|x|$, $y_2 = x\sin\left(\frac{1}{x}\right)$ and $y_3 = |x|$. Note that $y_1 \le y_2 \le y_3$.

Solution: We'll discuss this in class.

It's okay to use a calculator to make sure you're getting a squeeze. It's easy to get tricked!

$$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$$

You should graph (Figure 23, page 84) these three functions to see that the inequality holds for all $x \neq 0$. Yes, it's tricky. Other squeezes exists, just be sure to graph them out.

$$\begin{aligned}\lim_{x\to 0} -|x| &= 0\\ \lim_{x\to 0}|x| &= 0\end{aligned}$$

By the ST,

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

5. Evaluate.

$$\lim_{x \to 0^+} \frac{x^2}{\csc^2 x}$$

Solution: We'll discuss this in class.

$$\lim_{x \to 0^+} \frac{x^2}{\csc^2 x} = \lim_{x \to 0^+} x^2 \sin^2 x = 0$$

10.3 Assignment

You should read §2.6 and do the WebAssign assignment mth.122.06.06.

11 mth.121.02.07

11.1 Calculating Limits at Infinity

11.1.1 Review

Although initially difficult to grasp,

$$\lim_{x\to a} f(x),$$

this notation basically tells us to get close to a (BUT DO NOT TOUCH a!). In a way, we are deifying a. Fact is, when you tell someone of a forbidden fruit, they want to touch it. But be warned, try not to touch it! If this notation says anything, it says, "do not touch a, but please try to get close to a. As close as you like!" Since a exist on a one dimensional number line, you should note that there are two ways to approach any finite a.

We write

$$\lim_{x\to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a is equal to L. A left hand limit just means that we are approaching a from the left, or from values smaller than a. We can also write

$$\lim_{x\to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a is equal to L. A right hand limit just means that we are approaching a from the right, or from values larger than a. For the limit

$$\lim_{x\to a} f(x)$$

to exist, both the left and right limits must exist and be the same. That is

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

You should note that L is referring to a single finite number. If L is not finite, we say the limit does not exists. However, mathematicians accept the following definitions.

$L \to \infty$: Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the value of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

$L \to -\infty$: Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the value of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

We can also allow x to approach $\pm\infty$, basically to analyze a function's behavior in the extremes of infinity. The definition is: let f be a function defined on some interval $(a,\ \infty)$. Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large.

We can also go in the other direction towards $-\infty$. The definition is: let f be a function defined on some interval $(-\infty,\ a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large negative.

11.1.2 More Review

Limits are discussed[25] in MTH 119 (pre-calculus), I'll assume that you've had some practice with these limits and know that they are useful when graphing rational functions. Let's do an example to jog your memory.

Example: Given

$$f(x) = \frac{x^2 - 3x + 2}{2x^2 + 5x + 3} = \frac{(x-1)(x-2)}{(2x+3)(x+1)},$$

answer the following questions:

1. x-intercepts in point form.

Solution: By now this should seem very easy to do, especially considering how the function is presented in factored form.

$$(1,\ 0);(2,\ 0)$$

2. y-intercept in point form.

[25] Most teachers, and certainly most textbooks.

Solution: Again, this should be very easy to do.

$$\left(0,\ \frac{2}{3}\right)$$

3. Equation of the horizontal asymptote.

Solution: This will be discussed in class.

$$y = \frac{1}{2}$$

4. Equation of the vertical asymptotes.

Solution: This will be discussed in class.

$$x = -1; \quad x = -\frac{3}{2}$$

5. Graph $f(x)$ using the above information and sign analysis.

Solution: Your graph (Figure 24, page 89) should contain similar details.

Example: Now let's take a careful look at:

$$g(x) = \frac{5x^2(x-1)}{(x-1)(3x-2)(x+2)}$$

Here are some relevant questions.

1. $\lim\limits_{x\to-\infty} g(x) =$

Solution: This will be discussed in class.

$$\lim_{x\to+\infty} g(x) = \frac{5}{3}$$

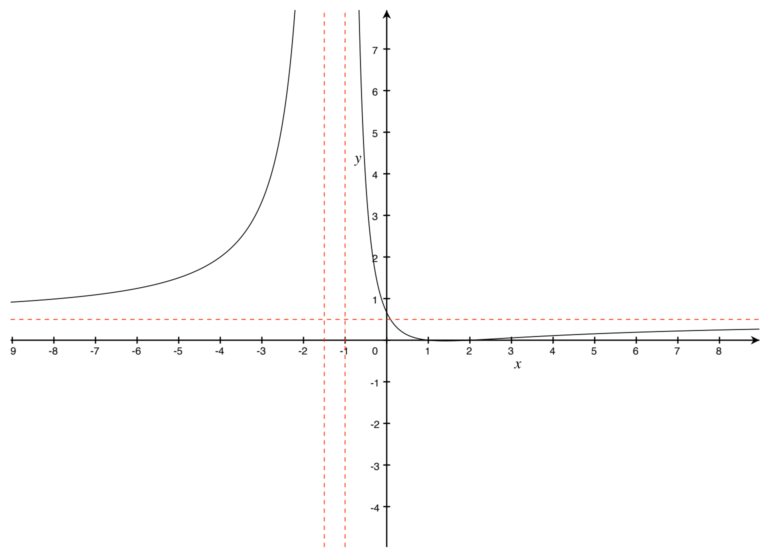

Figure 24: Partial graph of $f(x)$.

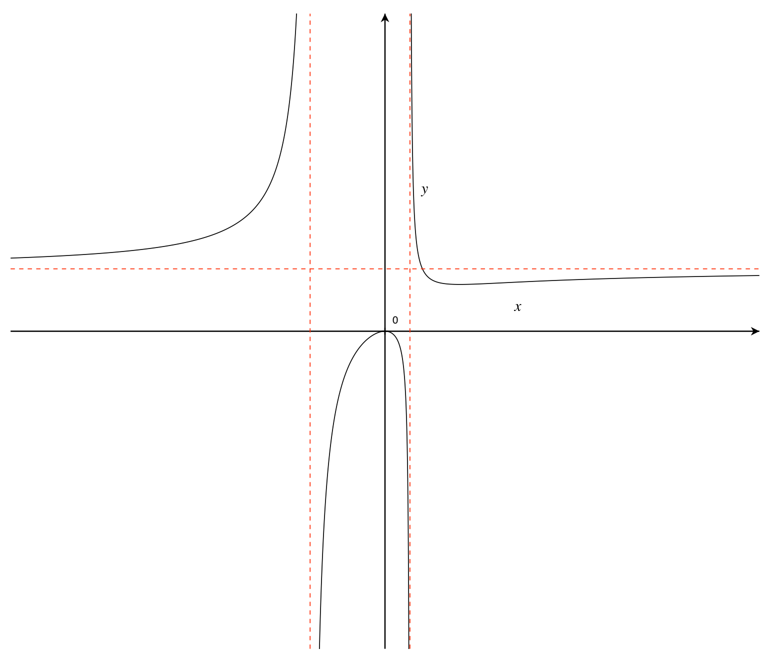

Figure 25: Partial graph of $g(x)$, properly scaled, but no units given.

2. $\lim_{x \to +\infty} g(x) =$

Solution: This will be discussed in class.

$$\lim_{x \to +\infty} g(x) = \frac{5}{3}$$

3. $\lim_{x \to 1^+} g(x) =$

Solution: This will be discussed in class.

$$\lim_{x \to 1^+} g(x) = \frac{5}{3}$$

4. $\lim_{x \to 1^-} g(x) =$

Solution: This will be discussed in class.

$$\lim_{x \to 1^+} g(x) = \frac{5}{3}$$

5. $\lim_{x \to 1} g(x) =$

Solution: This will be discussed in class.

$$\lim_{x \to 1^+} g(x) = \frac{5}{3}$$

6. $g(1) =$

Solution: This will be discussed in class.

Undefined.

7. Determine and evaluate the other limits that we should look at.

Solution: This will be discussed in class.

$$
\begin{aligned}
\lim_{x \to -2^+} g(x) &= -\infty \\
\lim_{x \to -2^-} g(x) &= \infty \\
\lim_{x \to \frac{2}{3}^+} g(x) &= \infty \\
\lim_{x \to \frac{2}{3}^-} g(x) &= -\infty
\end{aligned}
$$

11.2 Examples

1. Evaluate.

$$\lim_{x \to \infty} \frac{9x^2 - 3x + 4}{5 - 3x - 11x^2}$$

Solution: We'll discuss this in class.

$$\lim_{x \to \infty} \frac{9x^2 - 3x + 4}{5 - 3x - 11x^2} = -\frac{9}{11}$$

2. Evaluate.

$$\lim_{x \to \infty} \frac{4x - 3}{\sqrt{25x^2 + 4x}}$$

Solution: We'll discuss this in class.

$$\lim_{x \to \infty} \frac{4x - 3}{\sqrt{25x^2 + 4x}} = \frac{4}{5}$$

3. Evaluate.

$$\lim_{x \to -\infty} \frac{4x - 3}{\sqrt{25x^2 + 4x}}$$

Solution: We'll discuss this in class.

$$\lim_{x\to-\infty}\frac{4x-3}{\sqrt{25x^2+4x}}=-\frac{4}{5}$$

4. Evaluate.

$$\lim_{x\to\infty}\left(2\sqrt{x}-\sqrt{x+2}\right)$$

Solution: We'll discuss this in class.

$$\lim_{x\to\infty}\left(2\sqrt{x}-\sqrt{x+2}\right)=\infty$$

5. Evaluate.

$$\lim_{x\to\infty}\arctan\left(\frac{1+x}{1-x}\right)$$

Solution: We'll discuss this in class.

$$\lim_{x\to\infty}\arctan\left(\frac{1+x}{1-x}\right)=-\frac{\pi}{4}$$

6. Evaluate.

$$\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x+2}\right)$$

Solution: We'll discuss this in class.

$$\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x+2}\right)=0$$

7. Evaluate.

$$\lim_{x\to-\infty} \frac{|x|+x}{x+1}$$

Solution: We'll discuss this in class.

$$\lim_{x\to-\infty} \frac{|x|+x}{x+1} = 0$$

8. Evaluate.

$$\lim_{x\to\infty} \frac{|x|+x}{x+1}$$

Solution: We'll discuss this in class.

$$\lim_{x\to\infty} \frac{|x|+x}{x+1} = 2$$

11.3 Assignment

You should read §2.7 and do the WebAssign assignment mth.121.02.07.

12 mth.121.02.08

12.1 Intermediate Value Theorem

Karl Weierstrass was a German mathematician who proved the Intermediate Value Theorem (IVT), which was not an easy task. Some have said that he proved the obvious, but it nonetheless had to be proved before it could become a theorem. In MTH-119 you used this theorem to show the existence of roots. Here's what the theorem states:

> Suppose f is continuous on the closed interval $[a,\ b]$ and W is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there is a number $c \in (a,\ b)$ for which $f(c) = W$.

Example: Suppose you're asked to show that

$$f(x) = x^3 - 4x - 2,$$

has a zero on the interval $[-2,\ -1]$.

Solution: Keep in mind that you're not being asked for the root, but rather its existence. In short use the IVT and state the following.

- $f(x)$ is continuous on $[-2,\ -1]$.
- $f(-2) = -2 < 0 < 1 = f(-1)$
- By the IVT there is a number $c \in (-2,\ -1)$ for which $f(c) = 0 = W$

Here's a graph (Figure 26, page 94) *illustrating* the obvious.

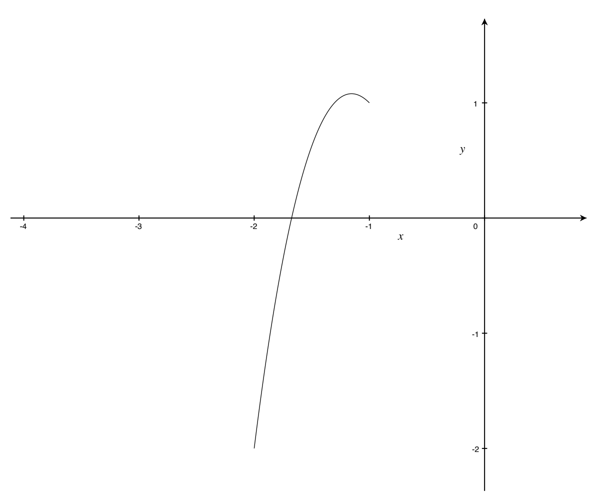

Figure 26: Graph of $f(x) = x^3 - 4x - 2$ on $[-2,\ -1]$.

What's not obvious is the value of c, but I guess you can make a pretty good guess at what it is. Furthermore, you can use software to get a good approximation, possibly the exact answer.[26]

12.2 Examples

1. Prove using the IVT, that

   $$\sqrt{x} + \sqrt{x+2} = 3,$$

   has a solution. You may also solve[27] this equation for x using methods learned in algebra.

   Solution: We'll discuss this in class.

   Let's do this solving the equation for zero first and then letting this be our function.

   $$\begin{aligned} \sqrt{x} + \sqrt{x+2} &= 3 \\ \sqrt{x} + \sqrt{x+2} - 3 &= 0 \\ f(x) &= \sqrt{x} + \sqrt{x+2} - 3 \end{aligned}$$

   - $f(x)$ is continuous on $[0,\ \infty)$.
   - $f(0) < 0 < f(7)$
   - By the IVT there is a number $c \in (0,\ 7)$ for which $f(c) = 0 = W$

2. Prove using the IVT, that

   $$\arctan x = \arccos x,$$

   has a solution. If you're so inclined, try to use software[28] to solve this equation.

   Solution: We'll discuss this in class.

   Let's do this solving the equation for zero first and then letting this be our function.

   $$\begin{aligned} \arctan x &= \arccos x \\ \arctan x - \arccos x &= 0 \\ f(x) &= \arctan x - \arccos x \end{aligned}$$

[26] My calculator returns -1.67513087057. Mathematica is able to compute three real roots exactly.

[27] $x = 49/36$

[28] Visit http://m10.mathography.org/ for more information about using software to solve difficult mathematical problems. Here, you'll see that

$$x = \sqrt{\frac{\sqrt{5}-1}{2}}$$

This can also be done using what you've learned in precalculus.

You'll probably need a calculator for this one.

- $f(x)$ is continuous on $[-1,\ 1]$.
- $f(-1) < 0 < f(1)$
- By the IVT there is a number $c \in (-1,\ 1)$ for which $f(c) = 0 = W$

12.3 Assignment

You should read §2.8 and do the WebAssign assignment mth.121.02.08.

13 mth.121.03.01

13.1 The Derivative

13.1.1 Slope of Tangent Lines

We will now return to the tangent line to the curve $y = f(x)$ at the point $(a,\ f(a))$. As I hope you recall, this is where we started our initial discussion of calculus.

Definition: The *tangent line* to the curve $y = f(x)$ at the point $(a,\ f(a))$ is the line through $(a,\ f(a))$ with slope

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

If $x \neq a$ you should note that

$$\frac{f(x) - f(a)}{x - a}$$

is just the slope of the secant line.

Example: Find the equation of the line tangent to the curve

$$y = \frac{2x}{(x+1)^2},$$

at the point where $x = 0$.[29]

Solution: The point is $(0,\ 0)$ and the slope is

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{2}{(x+1)^2} \\ &= 2 \end{aligned}$$

The equation is

$$y - 0 = 2(x - 0)$$

13.1.2 Alternative Method

Another way to write

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

[29] That means $a = 0$.

is to use the letter h, as follows

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

In fact, we have an exact equality:

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Example: Just for practice, try to find

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

for

$$y=\frac{2x}{(x+1)^2},$$

at the point where $a=0$. It should agree with our prior result.

Solution:

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{2}{(h+1)^2} \\ &= 2 \end{aligned}$$

13.1.3 The Derivative at a Point

Definition: The *derivative at a point* where $x=a$ is denoted

$$f'(a),$$

and is found by evaluating the limit:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

provided the limit exists.

This is often referred to as the instantaneous rate of change of $y=f(x)$ with respect to x, when $x=a$ and is called the slope of the tangent line at $x=a$. Furthermore the instantaneous rate of change of $s=s(t)$ with respect to t, when $t=a$ is called the instantaneous velocity. Both concepts are the same, but depending on units, their interpretation can differ.

13.1.4 The Derivative as a Function

All the examples so far were at a given at a particular point, that is, some fixed value, but we can generalize this to any point where the limit is defined. The definition of the *derivative as a function* simply follows from the above examples,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.[30]

Finding derivatives, then using them, will take up much of this course. Right now we will be using the definition above to find this derived function, but in time we will be developing rules that will allows us to quickly find derivatives. Once these derivatives are found, we will then need to use them. They have many real applications, but that's for later.

Example: Find $f'(x)$ for

$$f(x) = x^2 - 3x + 1.$$

State the domain for both f and f'.

Solution: The domain of f is $\mathbb{R}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) + 1 - (x^2 - 3x + 1)}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\ &= \lim_{h\to 0} 2x + h - 3 \\ &= 2x - 3 \end{aligned}$$

The domain of f' is also $\mathbb{R}$.

13.2 Examples

1. Find $f'(a)$.

$$f(x) = x^{-2}, \quad a = 3$$

[30] Now of course our new function will have a domain that may differ from the originating function. Yes, the derivative may not be defined at all points along the function.

Solution: We'll discuss this in class.

$$\begin{aligned}
f'(3) &= \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} \\
&= \lim_{x\to 3} \frac{1/x^2 - 1/9}{x-3} \cdot \frac{9x^2}{9x^2} \\
&= \lim_{x\to 3} \frac{9 - x^2}{9x^2(x-3)} \\
&= \lim_{x\to 3} -\frac{3+x}{9x^2} \\
&= -\frac{2}{27}
\end{aligned}$$

2. Find $f'(a)$ at $a = 1$, for

$$f(x) = \sqrt{x+1}.$$

Solution: We'll discuss this in class.

$$\begin{aligned}
f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} \\
&= \lim_{x\to 1} \frac{\sqrt{x+1} - \sqrt{2}}{x-1} \cdot \frac{\sqrt{x+1} + \sqrt{2}}{\sqrt{x+1} + \sqrt{2}} \\
&= \lim_{x\to 1} \frac{1}{\sqrt{x+1} + \sqrt{2}} \\
&= \frac{1}{2\sqrt{2}}
\end{aligned}$$

3. Find $f'(a)$.

$$f(x) = \frac{1}{\sqrt{2x+1}}, \quad a = 4$$

Solution: We'll discuss this in class.

$$\begin{aligned} f'(4) &= \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \ldots \quad \text{Work will be done in class.} \\ &= -\frac{1}{27} \end{aligned}$$

4. Given the following graph (Figure 27, page 101) estimate $f'(-2)$ and then estimate the equation of the tangent line at the point $(-2,\ f'(-2))$.

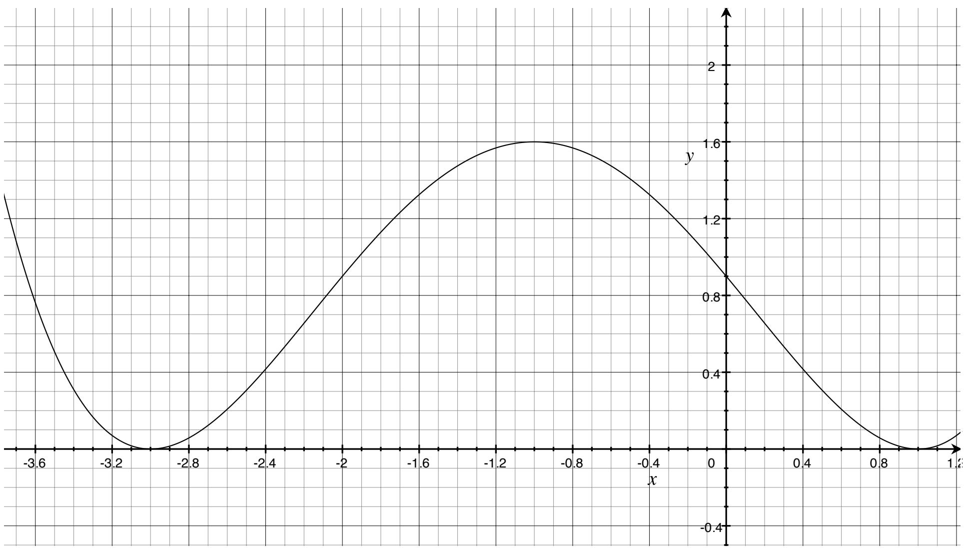

Figure 27: A partial graph of f.

Solution: We'll discuss this in class.

By inspection the point is approximately $(-2,\ 0.9)$ and the slope of the line tangent to the curve at this point is approximately $m = f'(-2) = 1.3$. So the equation of the tangent line is

$$y - 0.9 = 1.3\,(x + 2)\,.$$

5. Find $f'(x)$ for

$$f(x) = \frac{1 + x}{1 - x}.$$

State the domain for both f and f'.

Solution: We'll discuss this in class.

The domain of f is $\mathbb{R},\ x \neq 1$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \ldots \quad \text{Work will be done in class.} \\ &= \frac{2}{(1-x)^2} \end{aligned}$$

The domain of f' is also $\mathbb{R},\ x \neq 1$.

6. Find $f'(x)$ for

$$f(x) = \sqrt{1-x}.$$

State the domain for both f and f'.

Solution: We'll discuss this in class.

The domain of f is $(-\infty,\ 1]$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \ldots \quad \text{Work will be done in class.} \\ &= -\frac{1}{2\sqrt{1-x}} \end{aligned}$$

The domain of f' is $(-\infty,\ 1)$. They do differ!

13.3 Assignment

You should read §3.1 and do the WebAssign assignment mth.121.03.01.

14 mth.121.03.02

14.1 Finding Derivatives

The definition of the *derivative as a function* is,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.[31]

From here on out the following *new* notation will also be used.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,[f(x)] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is the definition that you are supposed to use if you are asked to find the derivative of a function by using the definition. However, we can also use this definition to develop simple rules for a colossal number of functions. That's good news, because if you were asked to find the derivative of

$$f(x) = \sqrt{\frac{(x^3 - 3x^2 + 5x - 9)^{200} - 1}{\sin x^2 - \cos^2 x}}$$

by definition, you would probably get stuck very quickly. In time you will be able to find the derivative of this particular function very quickly.

First, I will present some rules, then we will apply them to problems. Some rules will be proved, and you should be able to follow what's being presented in class.

14.1.1 Constants

Suppose $f(x) = k$, where k is any constant.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{k - k}{h} \\ &= \lim_{h \to 0} \frac{0}{h} = 0 \end{aligned}$$

Notational we will write this rule as follows. If $f(x) = k$, where k is any constant, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,[k] = \frac{\mathrm{d}}{\mathrm{d}x}\,[f(x)] = 0.$$

[31] Now of course our new function will have a domain that may differ from the originating function. Yes, the derivative may not be defined at all points along the function.

14.1.2 Power Rule

Suppose $f(x) = x^n$, where n is any positive integer.[32]

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \cdots + h^n - x^n}{h} \\ &= \lim_{h \to 0} nx^{n-1} + \text{ terms with a factor of } h \\ &= nx^{n-1} \end{aligned}$$

Notational we will write this rule as follows. If $f(x) = x^n$, where n is any positive integer, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,[x^n] = \frac{\mathrm{d}}{\mathrm{d}x}\,[f(x)] = nx^{n-1}.$$

14.1.3 Other Simple Rules

Suppose $f(x)$ and $g(x)$ are differentiable at x and c is any constant, then[33]

1. The derivative of a sum is the sum of the derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[f(x) + g(x)] = f'(x) + g'(x)$$

2. The derivative of a difference is the difference of the derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[f(x) - g(x)] = f'(x) - g'(x)$$

3. The derivative of a constant multiple of a function is the constant times the derivative of the function.

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[c \cdot f(x)] = c \cdot f'(x)$$

4. **Product Rule:** The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.[34]

$$\frac{\mathrm{d}}{\mathrm{d}x}\,[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

[32] The Binomial Theorem and its application to this problem will be discussed in class.
[33] This will be discussed in class.
[34] The product rule is being mentioned here, but will not be used until later.

5. **Quotient Rule:** The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator; all divided by the square of the denominator.[35]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

14.1.4 Exponential Functions

Material covered in MTH-119 is extremely important and exponential functions are no exception. Recall that the general exponential function is of the form,

$$f(x) = a^x,$$

where a is a positive constant, and $a \neq 1$. You should, of course, review the material that you learned in MTH-119. Specifically you should be able to solve exponential equations, and also graph simple exponential functions.

14.1.5 Derivatives of Exponential Functions

Recall that the derivative of a function is found by evaluating a limit. For example, if $f(x) = a^x$, then

$$\begin{aligned} f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= a^x\lim_{h\to 0}\frac{a^h-1}{h}. \end{aligned}$$

The tough part is evaluating the limit, but first observe that

$$f'(0) = \lim_{h\to 0}\frac{a^h-1}{h},$$

so

$$f'(x) = a^x f'(0).$$

Definition: Euler's number is indicated by e, where

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

This, of course leads to the following important result.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x$$

[35] The quotiet rule is being mentioned here, but will not be used until later.

14.1.6 Generalized Power Rule

If $f(x) = x^r$, where $r \in \mathbb{R}$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,[x^r] = rx^{r-1}.$$

As you may recall from MTH-120 you were presented with the Binomial Theorem for positive integers only, so the proof of this power rule is not so simple. Actually the proof involves using logarithms, and we will have to wait until we do logarithmic differentiation.

14.2 Examples

Show all work and box the final answer.

1. Compute $f'(x)$ using the limit definition and rules. $f(x) = e^{\pi}$

   Solution: We'll discuss this in class.

   e^{π} is a *constant*!

   $$f'(x) = 0$$

2. Differentiate the function. $f(x) = \dfrac{5x^8}{4}$

   Solution: We'll discuss this in class.

   $$f'(x) = 10x^7$$

3. Compute $f'(x)$ using the limit definition and rules. $f(x) = 5x^2 - 3x + 1$

   Solution: We'll discuss this in class.

   $$f'(x) = 10x - 3$$

4. Differentiate the function.[36] $f(x) = (x-2)\cdot(2x-5)$

[36] Do not use the product rule.

Solution: We'll discuss this in class.

You'll need to expand before differentiating.

$$\begin{aligned} f(x) &= (x-2)\cdot(2x-5) \\ &= 2x^2-9x+10 \\ f'(x) &= 4x-9 \end{aligned}$$

5. Compute $f'(x)$ using the limit definition and rules. $f(x)=\sqrt{x}$

Solution: We'll discuss this in class.

By *definition.*

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0}\frac{h}{h\cdot\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

By *rule.*

$$\begin{aligned} f(x) &= \sqrt{x} \\ &= x^{1/2} \\ f'(x) &= \frac{1}{2}x^{-1/2} \end{aligned}$$

Make sure you understand the difference between *rule* and *definition.* You should also note that the above two answers are identical.

$$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}=\frac{\sqrt{x}}{2x}$$

6. Differentiate the function.[37] $f(x)=\left(x^2-2x+3\right)^2$

[37] Do not use the product rule.

Solution: We'll discuss this in class.

You'll need to expand and then differentiate.

$$\begin{aligned} f(x) &= \left(x^2 - 2x + 3\right)^2 \\ &= x^4 - 4x^3 + 10x^2 - 12x + 9 \\ f'(x) &= 4x^3 - 12x^2 + 20x - 12 \end{aligned}$$

7. Differentiate the function.[38] $f(x) = \dfrac{x^2 - 1}{x + 1}$

Solution: We'll discuss this in class.

Simplify first!

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x + 1} \\ &= x - 1 \\ f'(x) &= 1, \quad x \neq 1 \end{aligned}$$

8. Differentiate the function. $f(x) = 3\sqrt{x} - \dfrac{5}{\sqrt[3]{x}}$

Solution: We'll discuss this in class.

$$\begin{aligned} f(x) &= 3\sqrt{x} - \frac{5}{\sqrt[3]{x}} \\ &= 3x^{1/2} - 5x^{-1/3} \\ f'(x) &= \frac{3}{2}x^{-1/2} + \frac{5}{3}x^{-4/3} \end{aligned}$$

9. Differentiate the function. $f(x) = x\sqrt{x}$

Solution: We'll discuss this in class.

$$f'(x) = \frac{3}{2}x^{1/2}$$

[38] Do not use the quotient rule.

10. Differentiate the function. $f(x) = \left(3\sqrt{x} - \dfrac{5}{\sqrt[3]{x}}\right)^2$

Solution: We'll discuss this in class.

Expand and then differentiate.

$$\begin{aligned} f(x) &= \left(3\sqrt{x} - \frac{5}{\sqrt[3]{x}}\right)^2 \\ &= 9x - 30x^{1/6} + 25x^{-2/3} \\ f'(x) &= 9 - 5x^{-5/6} - \frac{50}{3}x^{-5/3} \end{aligned}$$

11. Differentiate the function.[39] $f(x) = Ax^2 + Bx + C$

Solution: We'll discuss this in class.

$$f'(x) = 2Ax + B$$

12. Differentiate the function. $f(x) = e^{x+1}$

Solution: We'll discuss this in class.

$$f'(x) = e^{x+1}$$

13. Find the equation of the tangent line to the curve $f(x) = (e^x - 1)^2$, given that

$$f'(x) = 2e^x (e^x - 1),$$

at the point $x = 0$. A graph (Figure 28, page 110) is provided, please indicate both f and the line tangent to f at $x = 0$.

Solution: This will be discussed in class.

This is the horizontal line, point of tangency is $(0,\ 0)$, and slope of the tangent is $m = 0$.

$$y = 0$$

[39] This is a function of x, that is, the variable is x and A, B and C are constants.

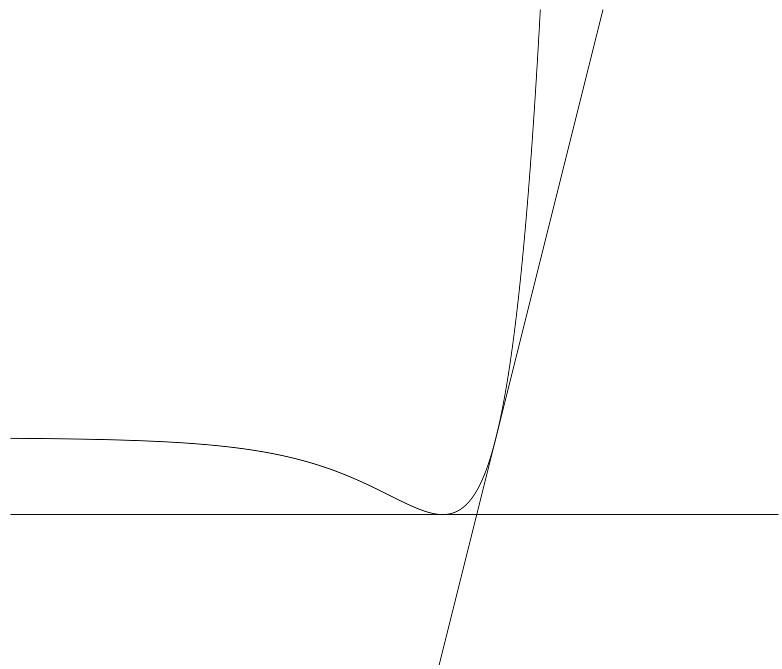

Figure 28: Partial graph of f and two tangent lines to f at $x = 0$ and $x = \ln 2$.

14. Find the equation of the tangent line to the curve $f(x) = (e^x - 1)^2$ at the point $x = \ln 2$. A graph (Figure 28, page 110) is provided, please indicate both f and the line tangent to f at $x = \ln 2$.

Solution: We'll discuss this in class.

This is the non-horizontal line, point of tangency is $(\ln 2,\ 1)$, and slope of the tangent is $m = 4$.

$$y - 1 = 4(x - \ln 2)$$

14.3 Assignment

You should read §3.2 and do the WebAssign assignment mth.121.03.02.

15 mth.121.03.03

15.1 Product and Quotient Rules

15.1.1 Product Rule

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.[40]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

15.1.2 Quotient Rule

The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator; all divided by the square of the denominator.[41]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2}$$

15.2 Examples

1. Find $f'(x)$ by using the product rule.

   $$f(x) = e^{2x}$$

   Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

   $$f'(x) = 2e^{2x}$$

2. Find $f'(x)$.[42]

   $$f(x) = \frac{1-x+x^2}{e^{2x}}$$

   Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

   $$f'(x) = -\frac{2x^2-4x+3}{e^{2x}}$$

[40] A proof will be given in class.
[41] A proof will not be given in class, but students are encouraged to try on their own.
[42] You should use the result from problem 1.

3. Find $f'(x)$.

$$f(x) = \left(4e^x - x^2\right)\left(x^3 - 5\right)$$

Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

$$f'(x) = 4x^3e^x + 12x^2e^x - 20e^x - 5x^4 + 10x$$

4. Find $f'(x)$.

$$f(x) = \frac{x}{3x^2 + 1}$$

Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

$$f'(x) = \frac{1 - 3x^2}{\left(3x^2 + 1\right)^2}$$

5. Find $f'(4)$.

$$f(x) = \frac{\sqrt{x} + 1}{\sqrt{x} - 1}$$

Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

$$f'(x) = -\frac{x\sqrt{x} + \sqrt{x} + 2x}{x\left(x - 1\right)^2}, \quad f'(4) = -\frac{1}{2}$$

6. Find $f'(x)$.

$$f(x) = x^2e^{-x}$$

Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

$$f'(x) = \frac{x(2-x)}{e^x}$$

7. Find

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}, \quad y = \frac{3x^2 + e^x - 2}{4x^3 + 1}$$

Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 1$$

8. Using the product rule to find $f'(x)$.

$$f(x) = e^{3x}$$

What do you think the derivative of $f(x) = e^{7x}$ is?

Solution: You need to be able to show work on exams. Make sure you can follow the steps presented in class. Although I am providing the final answer, you should be aware that simplifications are not always necessary. Final answer:

$$f'(x) = 3e^{3x}$$

The *guess* for the derivative of $f(x) = e^{7x}$ is $f'(x) = 7e^{7x}$.

15.3 Assignment

You should read §3.3 and do the WebAssign assignment mth.121.03.03.

16 mth.121.03.04

16.1 Rates of Change

16.1.1 Velocity

Recall that average velocity is

$$\frac{\Delta s}{\Delta t},$$

and when we let $\Delta t \to 0$ we get instantaneous velocity. It's really just the slope of the tangent line, so to get instantaneous velocity at time $t = a$, we need to evaluate the following limit.

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

Exapmle: Find the instantaneous velocity at $t = 3$ seconds, if the position (in meters) is related to time (in seconds) by

$$s(t) = t^2 - 5t + 3.$$

Solution: Take the derivative,

$$s'(t) = 2t - 5,$$

and evaluate this derivative at $t = 3$,

$$s'(3) = 1.$$

So the answer is 1 meter per second.

16.1.2 Rates in General

Rates of changes don't necessarily have to be velocities. Here, using similar notation for a function $y = f(x)$, average rate of change is

$$\frac{\Delta y}{\Delta x},$$

and when we let $\Delta x \to 0$ we get instantaneous rate of change. It's really just the slope of the tangent line, so to get instantaneous rate at $x = a$, we need to evaluate the following limit.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example: Find the instantaneous rate of change of the cube root of x with respect to x when $x = 8$.

Solution: Here, we are given that $f(x) = \sqrt[3]{x}$, and we're being asked to find $f'(8)$.

$$\begin{aligned} f(x) &= \sqrt[3]{x} \\ f(x) &= x^{1/3} \\ f'(x) &= \frac{x^{-2/3}}{3} \\ f'(8) &= \frac{8^{-2/3}}{3} = \frac{1}{12} \end{aligned}$$

16.2 Examples

1. Find the rate of change of the area of a square with respect to its side s when $s = 5$.

Solution: We'll discuss this in class.

$$\begin{aligned} A &= s^2 \\ \frac{\mathrm{d}A}{\mathrm{d}s} &= 2s \\ \left.\frac{\mathrm{d}A}{\mathrm{d}s}\right|_{s=5} &= 10 \end{aligned}$$

2. The position of a particle moving in a straight line during a five second trip is

$$s(t) = t^2 - t + 10$$

centimeters. Find the time t at which the instantaneous velocity is equal to the average velocity of the entire trip.

Solution: We'll discuss this in class.

$$\begin{aligned} s(0) &= 10 \\ s(5) &= 30 \\ \frac{\Delta s}{\Delta t} &= \frac{30 - 10}{5 - 0} = 4 \text{ centimeters per second} \\ s'(t) &= 2t - 1 \\ 4 &= 2t - 1 \\ t &= 2.5 \text{ seconds} \end{aligned}$$

3. A bullet is fired in the air vertically from two meters above ground level with an initial velocity of 200 meters per second. Find the bullet's maximum velocity and maximum height.[43]

Solution: We'll discuss this in class.

Final answer:

The maximum height is

$$s\left(\frac{1000}{49}\right) = 2042.82 \text{ meters.}$$

And the maximum velocity is:

$$s'(40.8263) = -200.098 \text{ meters per second.}$$

Realistically, this is really not true because we're neglecting many factors related to free falling bodies.

4. A particle moving along a line has position $s(t) = t^4 - 8t^2$ feet, at time $t \geq 0$ seconds. What is the farthest distance to the left of the origin attained by the particle?

Solution: We'll discuss this in class.

Final answer:

16 feet

16.3 Assignment

You should read §3.4 and do the WebAssign assignment mth.121.03.04.

[43] Here we have

$$s(t) = 2 + 200t - \frac{9.8t^2}{2}.$$

You may recognize this formula if you've taken high-school level physics.

17 mth.121.03.05

17.1 Higher Order Derivatives

The first derivative is often indicated using *pime* notation, for example if we have a function $f(x)$, we write $f'(x)$ to indicate

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)].$$

Continuing with this notation, we can take derivatives of derivatives. So the second derivative would be indicated by

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}[f'(x)].$$

And the third derivative,

$$f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}[f''(x)].$$

The *pime* notation usually stops at order three, but we can still continue, this time we use a superscript in parentheses to indicate the order. For example the forth derivative would be indicated by

$$f^{(4)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}[f'''(x)].$$

In general, if $y = f(x)$, we write the derivatives as follows.

$$\begin{aligned}
f'(x) = y' &= \frac{\mathrm{d}y}{\mathrm{d}x} && \text{first derivative} \\
f''(x) = y'' &= \frac{\mathrm{d}^2y}{\mathrm{d}x^2} && \text{second derivative} \\
f'''(x) = y''' &= \frac{\mathrm{d}^3y}{\mathrm{d}x^3} && \text{third derivative} \\
f^{(4)}(x) = y^{(4)} &= \frac{\mathrm{d}^4y}{\mathrm{d}x^4} && \text{fourth derivative} \\
\vdots &= \vdots \\
f^{(n)}(x) = y^{(n)} &= \frac{\mathrm{d}^ny}{\mathrm{d}x^n} && n^{\text{th}} \text{ derivative}
\end{aligned}$$

Since derivatives tell us how something changes, you can probably reason that the second derivative tells us how fast the first derivative is changing, and so forth and so on. As we start using derivatives, these relationships should become better understood.

17.2 Examples

1. Find y' and y''.

$$y = \frac{1}{1-x}$$

Solution: We'll discuss this in class.

$$\begin{aligned} y &= \frac{1}{1-x} \\ y' &= \frac{1}{(1-x)^2} = (1-x)^{-2} \\ y'' &= \frac{2}{(1-x)^3} = 2\,(1-x)^{-3} \end{aligned}$$

2. Find $h''(3)$.

$$h(t) = \frac{e^t}{t}$$

Solution: We'll discuss this in class.

$$\begin{aligned} h(t) &= \frac{e^t}{t} \\ h'(t) &= \frac{te^t - e^t}{t^2} \\ h''(t) &= \frac{t^2e^t - 2te^t + 2e^t}{t^3} \\ h''(3) &= \frac{5e^3}{27} \approx 3.71954 \end{aligned}$$

3. Find a formula[44] for $f^{(n)}(x)$.

$$f(x) = x^{-2}$$

Solution: We'll discuss this in class. Final answer:

$$f^{(n)}(x) = (-1)^n \frac{(n+1)!}{x^{n+2}}$$

4. The graph (Figure 29, page 119) below shows f, f', f''. Determine which is which.

[44] The factorial notation ($n!$) was introduced in MTH 120 and your answer should include a factorial.

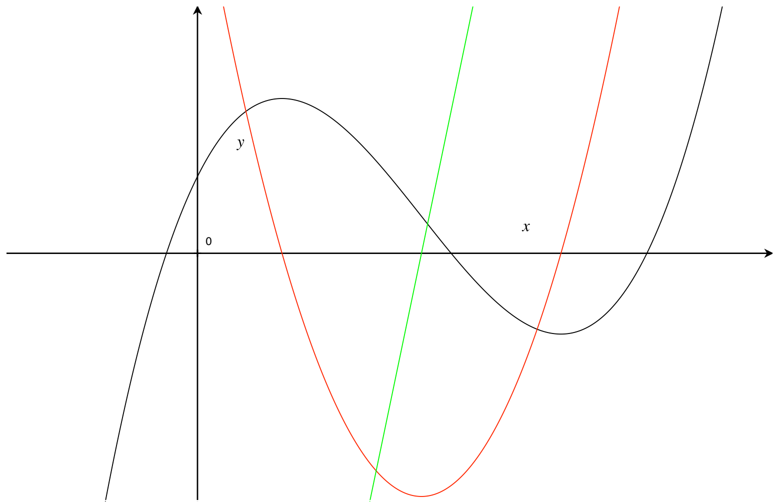

Figure 29: Partial graph of f, f', f''.

Solution: We'll discuss this in class. The black curve is f, the red curve is f' and green curve is f''.

17.3 Assignment

You should read §3.5 and do the WebAssign assignment mth.121.03.05.

18 mth.121.03.06

18.1 Trigonometric Derivatives

The definition of the *derivative as a function* is,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.[45] Okay, I'm being repetitious here, but it is nonetheless necessary to be reminded of this definition before proceeding forward.

18.1.1 The Derivative of Sine

I'd like to show that the derivative of the sine function is the cosine function. Showing that this is the case is not difficult. First, draw (Figure 30, page 120) one cycle of the sine function, and it's derivative[46] on the same graph.

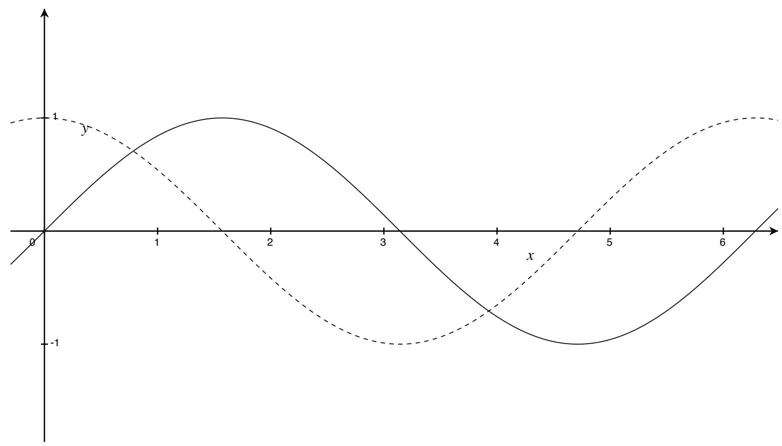

Figure 30: The sine function (solid) and its derivative (dashed).

You should observe that the dashed curve looks like the cosine curve, but this is certainly not a proof, but at least it's giving us a hint.

To find the derivative of the sine function we will need to use the definition of derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

[45] Now of course our new function will have a domain that may differ from the originating function. Yes, the derivative may not be defined at all points along the function.

[46] Same method that we're doing in class. That is, try to find the slopes at some points and then connect the dots.

This limit does not look easy, and you may wonder why its being rewritten in this form.[47]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h} \\
&= \lim_{h\to 0}\left[\frac{\sin x\cos h-\sin x}{h}+\frac{\sin h\cos x}{h}\right] \\
&= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\right]
\end{aligned}$$

Wow, I've seen[48] both of these limits before.

$$\lim_{h\to 0}\frac{\sin h}{h}=1, \qquad \lim_{h\to 0}\frac{\cos h-1}{h}=0$$

Continuing from above, we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\right] \\
&= \lim_{h\to 0}\sin x\cdot\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\frac{\sin h}{h} \\
&= \sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} \\
&= \sin x\cdot 0+\cos x\cdot 1 \\
&= \cos x
\end{aligned}$$

18.2 Examples

1. You *should* be able to prove using the definition of the derivative that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x,$$

Solution: This will be discussed in class.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\left[\frac{\cos x\cos h-\cos x}{h}-\frac{\sin x\sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\cos x\cdot\frac{\cos h-1}{h}-\sin x\cdot\frac{\sin h}{h}\right] \\
&= -\sin x
\end{aligned}$$

[47] The main reason is that this is what you did in MTH-120, and sum identities were extensively used.
[48] Look back over your notes if necessary.

2. Using rules you should be able to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x.$$

Solution: This will be discussed in class.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

3. Using rules you should be able to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\cot x \csc x.$$

Solution: You should be able to do this on your own now, especially after seeing the last example. And again, we'll do this in class.

4. Using rules you should be able to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \tan x \sec x.$$

Solution: You should be able to do this on your own now, especially after seeing the last example. And again, we'll do this in class.

5. Using rules you should be able to show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x.$$

Solution: You should be able to do this on your own now, especially after seeing the last example. And again, we'll do this in class.

6. Find an equation of the tangent line to the curve $y = \sec x - 2\cos x$ at the point $(\pi/3,\ 1)$.

Solution: We'll discuss this in class.

Final answer:

$$y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

7. Differentiate.

$$y = \frac{x + \sin x}{x^2 + \cos x}$$

Solution: We'll discuss this in class.

Final answer:

$$y'(x) = \frac{1 - x^2 - x\sin x + (x^2 + 1)\cos x}{(x^2 + \cos x)^2}$$

8. Find the limit.

$$\lim_{x \to 0} \frac{\sin^2 3x}{x^2}$$

Solution: We'll discuss this in class.

Final answer:

$$\lim_{x \to 0} \frac{\sin^2 3x}{x^2} = 9$$

9. Find the limit.

$$\lim_{x \to 0} \frac{1 - \tan x}{\sin x - \cos x}$$

Solution: We'll discuss this in class.

Final answer:

$$\lim_{x \to 0} \frac{1 - \tan x}{\sin x - \cos x} = -1$$

10. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x)\sin x$ and $h(x) = \cos x \div f(x)$. Find $g'(\pi/3)$ and $h'(\pi/3)$.

Solution: We'll discuss this in class.

Final answers:

$$\begin{aligned} g'(\pi/3) &= 2-\sqrt{3} \\ h'(\pi/3) &= \frac{1-2\sqrt{3}}{16} \end{aligned}$$

11. If $f(\beta) = \beta \cdot \sin\beta$, find $f'(\beta)$ and $f''(\beta)$.

Solution: We'll discuss this in class.

Final answers:

$$\begin{aligned} f'(\beta) &= \sin\beta + \beta\cos\beta \\ f''(\beta) &= 2\cos\beta - \beta\sin\beta \end{aligned}$$

18.3 Assignment

You should read §3.6 and do the WebAssign assignment mth.121.03.06.

19 mth.121.03.07

19.1 The Chain Rule

19.1.1 Local Linearization

In class I derived some of the simple rules for differentiation. The technique we used is exactly the same one that is used in the most textbooks. Now I'd like to use the concept of local linearization to derive the chain rule. We will still use the definition of the derivative, but now we will use the local linearization[49] as follows,

$$f(x+h) \approx f(x) + f'(x)\,h,$$

and

$$g(x+h) \approx g(x) + g'(x)\,h.$$

The smaller h gets, the better these approximations become. In fact if we take the limit of each side, as $h \to 0$, we get an equality. As a simple example, let's derive the product rule using these local linearization along with the definition of derivative.

$$\begin{aligned}
[f(x)\,g(x)]' &= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x) + f'(x)\,h]\,[g(x) + g'(x)\,h] - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x)\,g(x) + f(x)\,g'(x)\,h + f'(x)\,g(x)\,h + f'(x)\,g'(x)\,h^2 - f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x)\,g'(x)\,h + f'(x)\,g(x)\,h + f'(x)\,g'(x)\,h^2}{h} \\
&= \lim_{h\to 0} [f(x)\,g'(x) + f'(x)\,g(x) + f'(x)\,g'(x)\,h] \\
&= f(x)\,g'(x) + f'(x)\,g(x)
\end{aligned}$$

Just as we already knew. Now we'll use the same technique to derive the chain rule. Suppose $h(x) = (f \circ g)(x) = f(g(x))$. The chain rule states that if both f and g are differentialble, then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Here goes.

[49]This concept will be discussed in class and is basically using an equation of a tangent line, in a local region, to approximate the curve. Actually for very small $h \neq 0$ we have

$$\frac{f(x+h) - f(x)}{h} \approx f'(x) \quad \Rightarrow \quad f(x+h) \approx f(x) + f'(x)\,h.$$

$$
\begin{aligned}
[f(g(x))]' &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(g(x) + g'(x)h) - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(g(x)) + f'(g(x))g'(x)h - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{f'(g(x))g'(x)h}{h} \\
&= f'(g(x))g'(x)
\end{aligned}
$$

Now we will learn how to use this rule. Consider the functions $f(x)$ and $g(x)$ given by the following graph (Figure 31, page 127).[50]

And define $h(x) = (f \circ g)(x) = f(g(x))$. Using the chain rule, we have

$$h'(x) = f'(g(x)) \cdot g'(x).$$

1. Remember that we defined $h(x) = (f \circ g)(x) = f(g(x))$, and using the chain rule, we have $h'(x) = f'(g(x)) \cdot g'(x)$. From the information given, find each of the following.
   (a) $h'(1)$.

   Solution: Using the chain rule we have.

   $$h'(1) = f'(g(1)) \cdot g'(1) = f'(3) \cdot 3 = \frac{4}{3} \cdot 3 = 4$$

   (b) $h'(0)$

[50] By inspection it is easily (I'm not being serious) determined that $g(x)$ is:

$$g(x) = \begin{cases} 3x & \text{if} \quad x \le 2 \\ \dfrac{30-3x}{4} & \text{if} \quad x > 2 \end{cases}$$

Also, by inspection it is easily (I'm not being serious) determined that $f(x)$ is:

$$f(x) = \begin{cases} \dfrac{x^2-2x+4}{3} & \text{if} \quad x \le 4 \\ \dfrac{8+x}{3} & \text{if} \quad x > 4 \end{cases}$$

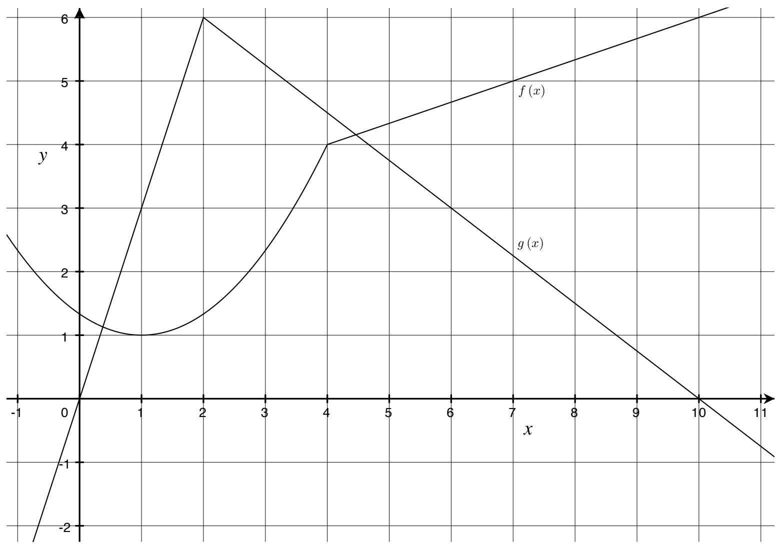

Figure 31: Both f and g are piecewise defined functions.

Solution: Using the chain rule we have.

$$h'(0) = f'(g(0)) \cdot g'(0) = f'(0) \cdot 3 = -\frac{2}{3} \cdot 3 = -2$$

(c) Does $h'(2)$ exist? If so, what does it equal?

Solution: Using the chain rule we see.

$$h'(2) = f'(g(2)) \cdot g'(2)$$

However, $g'(2)$ *does not exists*, so the derivative is undefined at 2.

2. Now define $k(x) = (g \circ f)(x) = g(f(x))$. Using the chain rule, we have

$$k'(x) = g'(f(x)) \cdot f'(x).$$

From the information given, find each of the following.

(a) $k'(1)$.

Solution: Using the chain rule we have.

$$k'(1) = g'(f(1)) \cdot f'(1) = g'(1) \cdot 0 = 3 \cdot 0 = 0$$

(b) $k'(0)$

Solution: Using the chain rule we have.

$$k'(0) = g'(f(0)) \cdot f'(0) = g'(4/3) \cdot (-2/3) = 3 \cdot (-2/3) = -2$$

(c) Does $k'(4)$ exist? If so, what does it equal?

Solution: Using the chain rule we see.

$$k'(4) = g'(f(4)) \cdot f'(4)$$

However, $f'(4)$ *does not exists*, so the derivative is undefined at 4.

19.2 Examples

Find the indicated derivatives of each of the following.

1. $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(x^2 - 2x + 3\right)^5\right] =$$

Solution: This will be discussed in class.

Final answer:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(x^2 - 2x + 3\right)^5\right] = 5\left(x^2 - 2x + 3\right)^4 \cdot \left(2x^2 - 2\right)$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{x^2+4}{x^3-1}\right)^5\right] =$$

Solution: This will be discussed in class.

Final answer:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{x^2+4}{x^3-1}\right)^5\right] = 5\left(\frac{x^2+4}{x^3-1}\right)^4 \cdot \frac{(x^3-1)\,2x - (x^2+4)\,3x^2}{(x^3-1)^2}$$

3. $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^2 x\right] =$$

Solution: This will be discussed in class.

Final answer:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^2 x\right] = 2\sin x \cdot \cos x$$

4. $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{\frac{1}{x+1}}\right] =$$

Solution: This will be discussed in class.

Final answer:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{\frac{1}{x+1}}\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)^{-1/2}\right] = -\frac{1}{2\,(x+1)\,\sqrt{x+1}}$$

5. $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^2 e^{x^2}\right] =$$

Solution: This will be discussed in class.

Final answer:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^2 e^{x^2}\right] = 4xe^{x^2}\sin e^{x^2}\cos e^{x^2} = 2xe^{x^2}\sin\left(2e^{x^2}\right)$$

Here's the Mathematica code (Figure 32, page 129) to do this problem. Learning to use a computer algebra system (CAS) such as Mathematica should be done well before you actually need it. Please take the time to play with a CAS system, even if it is just using a graphics calculator.

```
In[1]:=  D[(Sin[Exp[x^2]])^2, x]

Out[1]=  4 e^(x^2) x Cos[e^(x^2)] Sin[e^(x^2)]

In[2]:=  Simplify[%]

Out[2]=  2 e^(x^2) x Sin[2 e^(x^2)]
```

Figure 32: Mathematica code, program available in many student labs.

6. $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^3\left(x^2+2x-3\right)\right] =$$

Solution: This will be discussed in class.

Final answer:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^3\left(x^2+2x-3\right)\right] = -3\cos^2\left(x^2+2x-3\right)\cdot\sin\left(x^2+2x-3\right)\cdot(2x+2)$$

7. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{5x^2-2x+3}\right] =$

Solution: This will be discussed in class.

Final answer:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{5x^2-2x+3}\right] = \frac{10x-2}{2\sqrt{5x^2-2x+3}}$$

19.3 Assignment

You should read §3.7 and do the WebAssign assignment mth.121.03.07.

20 mth.121.03.08

20.1 Derivatives of Inverse Functions

Assuming that $f(x)$ is differentiable and one-to-one with inverse $f^{-1}(x) = g(x)$. If a belongs to the domain of $g(x)$, and $f'(g(a)) \neq 0$, then

$$g'(a) = \frac{1}{f'(g(a))}.$$

Here's a visual[51] (Figure 33, page 131) of what we are saying here.

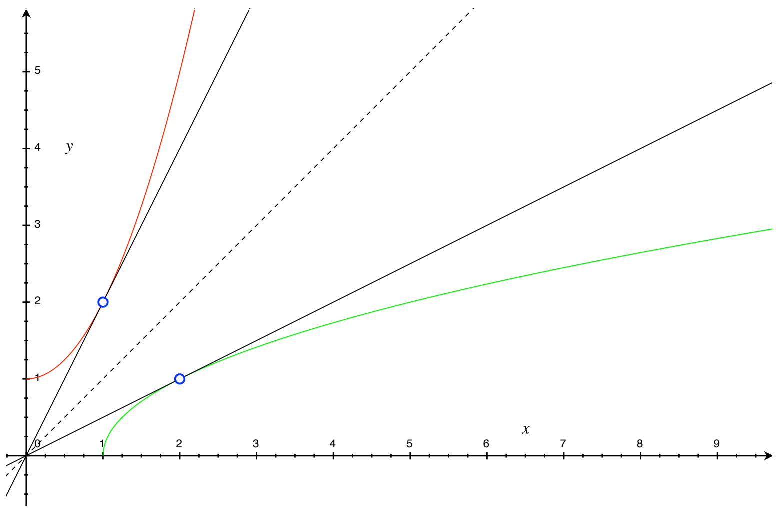

Figure 33: Partial graph of f, f^{-1}, $y = x$, and two tangent lines.

Example: Find the inverse of

$$f(x) = \frac{1}{1+x},$$

and then, using this inverse, compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f^{-1}(x)\right].$$

Now verify that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f^{-1}(x)\right] = \frac{1}{f'\left(f^{-1}(x)\right)}$$

[51]This will be labeled in class!

Solution: You should be able to show that

$$f^{-1}(x) = \frac{1-x}{x}.$$

And that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f^{-1}(x)\right] = -\frac{1}{x^2}.$$

And for the next step you'll need

$$f'(x) = -\frac{1}{(1+x)^2}.$$

And finally.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[f^{-1}(x)\right] &= \frac{1}{f'\left(f^{-1}(x)\right)} \\
-\frac{1}{x^2} &= \frac{1}{f'\left(\dfrac{1-x}{x}\right)} \\
&= \frac{1}{-\dfrac{1}{\left(1+\frac{1-x}{x}\right)^2}} \\
&= -\left(1+\frac{1-x}{x}\right)^2 \\
&= -\frac{1}{x^2}
\end{aligned}$$

20.1.1 Trigonometric Functions

This is mostly a review of what was covered in MTH-120. I'm not suggesting that you need to know *everything* that was covered in MTH-120, but you should be able to follow today's material without too much trouble if you learned the basics. Here's goes.

The point here is that the trigonometric functions are not invertible unless we restrict their domains. I believe you should know sine, cosine and tangent well enough to construct a reasonable domain restriction to make these three functions *one-to-one*. On the graph above you should be able to clearly identify all three functions and their inverses, the dashed-line $y = x$ is drawn on this graph as an aid.

Today we will derive the following derivatives.

1. Restrictions will be discussed later.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

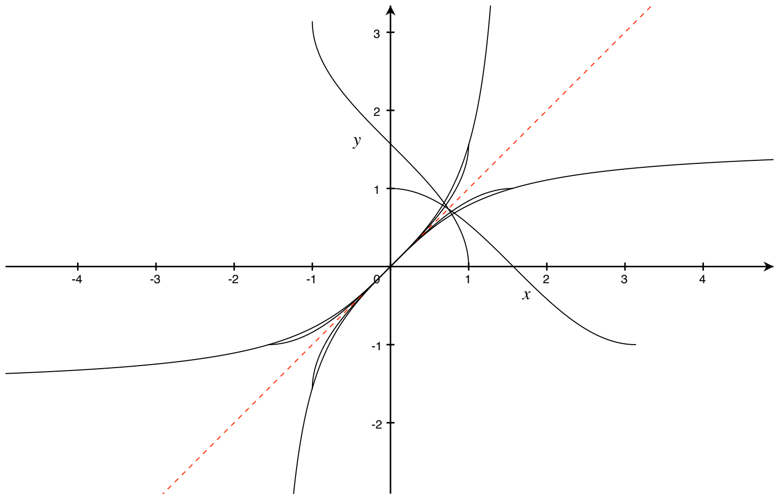

Figure 34: The three big trigonometric functions and their inverses, domain restrictions apply.

2. Restrictions will be discussed later.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

3. Restrictions will be discussed later.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$$

I will not cover[52] the derivatives of the other three trigonometric inverses, but here they are just in case you're interested:

1.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

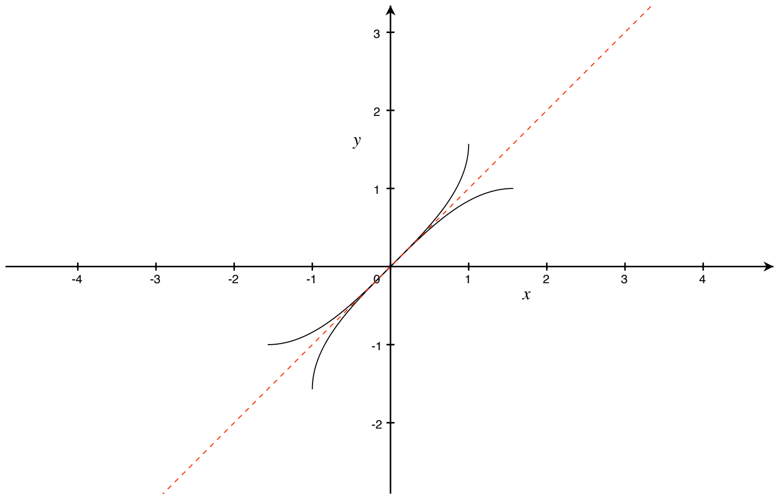

Figure 35: The sine and its inverse, arcsine.

20.2 Finding the Derivatives of Inverse Trigonometric Functions

Here's a graph (Figure 35, page 134) of the sine function and its inverse. Carefully look at the graph and answer the following questions.

1. Label directly on the graph the sine function.

   Solution: This will be *quickly* discussed in class.

2. What is the domain of this sine graph?

   Solution: This will be *quickly* discussed in class.

3. What is the range of this sine graph?

   Solution: This will be *quickly* discussed in class.

4. Label directly on the graph the arcsine function.

   Solution: This will be *quickly* discussed in class.

5. What is the domain of this arcsine graph?

[52]Most books also covers cosecant, secant and cotangent, however, I don't require them for this course. If needed, just look them up, or derive them on the fly.

Solution: This will be *quickly* discussed in class.

6. What is the range of this arcsine graph?

Solution: This will be *quickly* discussed in class.

7. To find the derivative of $y = \arcsin x$, we note that $\sin y = x$ and differentiate.

Solution: This will be discussed in class.

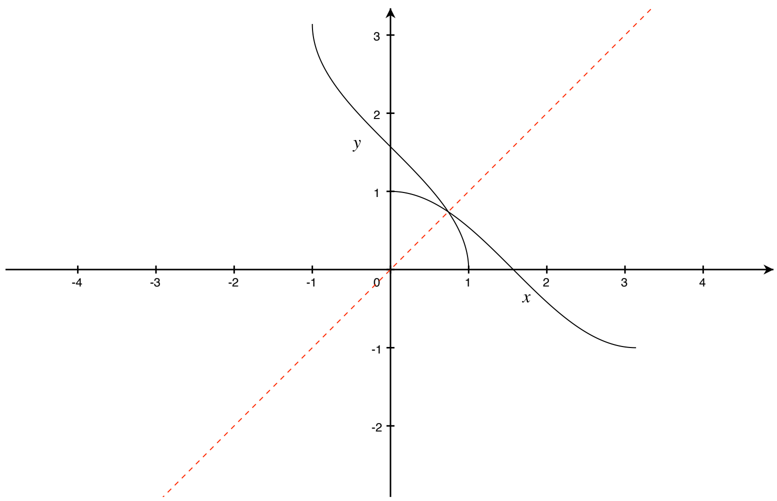

Figure 36: The cosine and its inverse, arccosine.

Here's a graph of the cosine (Figure 36, page 135) function and its inverse. Carefully look at the graph and answer the following questions.

1. Label directly on the graph the cosine function.

Solution: This will be discussed in class.

2. What is the domain of this cosine graph?

Solution: This will be *quickly* discussed in class.

3. What is the range of this cosine graph?

Solution: This will be *quickly* discussed in class.

4. Label directly on the graph the arccosine function.

Solution: This will be *quickly* discussed in class.

5. What is the domain of this arccosine graph?

Solution: This will be *quickly* discussed in class.

6. What is the range of this arccosine graph?

Solution: This will be *quickly* discussed in class.

7. To find the derivative of $y = \arccos x$, we note that $\cos y = x$ and differentiate. We'll do this in class.

Solution: This will be discussed in class.

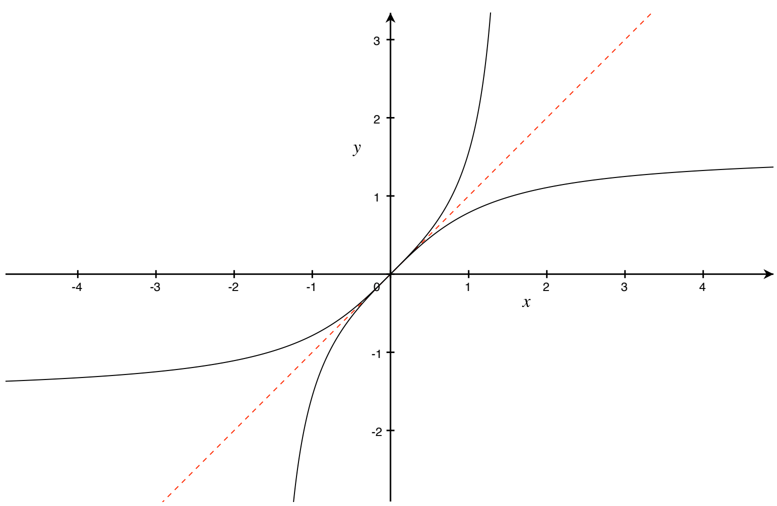

Figure 37: The tangent and its inverse, arctangent.

Here's a graph (Figure 37, page 136) of the tangent function and its inverse. Carefully look at the graph and answer the following questions.

1. Label directly on the graph the tangent function.

Solution: This will be *quickly* discussed in class.

2. What is the domain of this tangent graph?

Solution: This will be *quickly* discussed in class.

3. What is the range of this tangent graph?

Solution: This will be *quickly* discussed in class.

4. Label directly on the graph the arctangent function.

Solution: This will be *quickly* discussed in class.

5. What is the domain of this arctangent graph?

Solution: This will be *quickly* discussed in class.

6. What is the range of this arctangent graph?

Solution: This will be *quickly* discussed in class.

7. To find the derivative of $y = \arctan x$, we note that $\tan y = x$ and differentiate. We'll do this in class.

Solution: This will be discussed in class.

20.3 Examples

1. Find the derivative.

$$y = \sin^{-1} x^2$$

Solution: This will be discussed in class.

Final answer:

$$y' = \frac{2x}{\sqrt{1 - x^4}}$$

2. Find the derivative.

$$y = e^{\cos^{-1}\left(1-x^2\right)}$$

Solution: This will be discussed in class.

Final answer:

$$y' = \frac{2xe^{\cos^{-1}\left(1-x^2\right)}}{\sqrt{1-\left(1-x^2\right)^2}}$$

3. Find the derivative.

$$y = x\tan^{-1}x$$

Solution: This will be discussed in class.

Final answer:

$$y' = \tan^{-1}x + \frac{x}{1+x^2}$$

4. Find the derivative.

$$y = \sin^{-1}\frac{x}{1+x}$$

Solution: This will be discussed in class.

Final answer:

$$y' = \frac{1}{(1+x)^2\sqrt{1-\left(\frac{x}{1+x}\right)^2}}$$

5. Find the derivative.

$$y = \cos\left[\arcsin\left(x+1\right)\right]$$

Solution: This will be discussed in class.

Final answer:

$$y' = -\frac{1+x}{\sqrt{1-(1+x)^2}}$$

6. Given the following graph[53] (Figure 38, page 139),

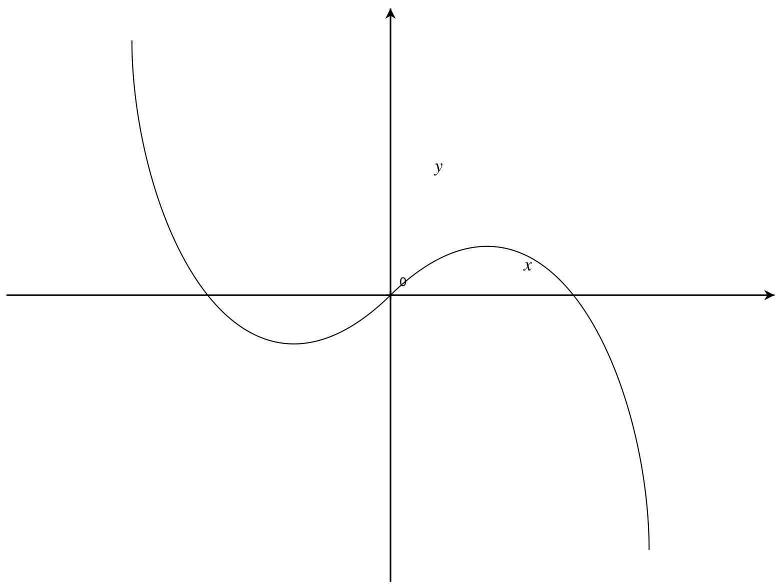

Figure 38: $f(x) = x \arcsin\left(1 - x^2\right)$

answer the following questions about $f(x)$:

(a) domain (interval notation);

Solution: First, you should know that the domain of the arcsine function is $[-1,\ 1]$, so we need to solve

$$\begin{aligned} -1 \leq 1 - x^2 &\leq 1 \\ -2 \leq -x^2 &\leq 0 \\ 0 \leq x^2 &\leq 2 \\ -\sqrt{2} \leq x &\leq \sqrt{2} \end{aligned}$$

So the domain is

$$\left[-\sqrt{2},\ \sqrt{2}\right] \approx [-1.4142,\ 1.4142]\,.$$

[53]Scale is not preserved.

(b) range (interval notation);

Solution: Using the result from part (a). Evaluating the left endpoint:

$$\begin{aligned} f\left(-\sqrt{2}\right) &= \left(-\sqrt{2}\right)\arcsin\left[1-\left(-\sqrt{2}\right)^2\right] \\ &= \left(-\sqrt{2}\right)\arcsin\left[1-2\right] \\ &= \left(-\sqrt{2}\right)\arcsin\left[-1\right] \\ &= \left(-\sqrt{2}\right)\left(-\frac{\pi}{2}\right) \\ &= \frac{\pi\sqrt{2}}{2} \end{aligned}$$

and evaluating the right endpoint:

$$\begin{aligned} f\left(\sqrt{2}\right) &= \left(\sqrt{2}\right)\arcsin\left[1-\left(\sqrt{2}\right)^2\right] \\ &= \left(\sqrt{2}\right)\arcsin\left[1-2\right] \\ &= \left(\sqrt{2}\right)\arcsin\left[-1\right] \\ &= \left(\sqrt{2}\right)\left(-\frac{\pi}{2}\right) \\ &= -\frac{\pi\sqrt{2}}{2} \end{aligned}$$

Finally the range is:

$$\left[-\frac{\pi\sqrt{2}}{2},\ \frac{\pi\sqrt{2}}{2}\right] \approx \left[-2.2214,\ 2.2214\right].$$

20.4 Assignment

You should read §3.8 and do the WebAssign assignment mth.121.03.08.

21 mth.121.03.09

21.1 Logarithmic Functions

Much of what we do in MTH-121 depends on material covered in MTH-119 and MTH-120. From MTH-119 you should recall that the general logarithmic function is of the form

$$\log_a x = y$$

where a is a positive constant, $a \neq 1$, and $x > 0$.

In MTH-119 you also learned how to graph and solve logarithmic equations. The basis for much of your work (either graphing or solving equations) was the following fact:

$$\log_a x = y \qquad \Leftrightarrow \qquad x = a^y.$$

So, I hope it is clear that the inverse of the logarithm (base a) is the exponential (base a), and *vise versa*. A simple graph illustrates this fact nicely.

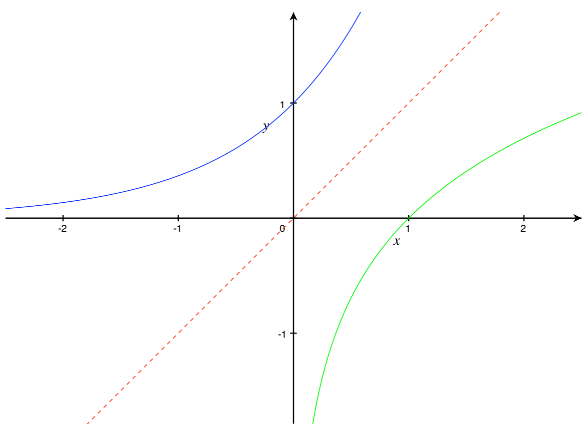

Figure 39: $y = e^x$ in blue; $y = x$ red dash; and $y = \ln x$ in green.

21.1.1 Derivatives of Natural Logarithmic Functions

We know from a prior worksheet that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x.$$

Let's use this fact to find the derivative of

$$y = \ln x.$$

Knowing small facts about e^x will allow us to find the derivative of $\ln x$. Here goes.

$$\begin{aligned} y &= \ln x \\ e^y &= x \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^y\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(x) \\ e^y\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{e^y}. \end{aligned}$$

This is not the usual way to write the derivative of $\ln x$, so we need to rewrite e^y in terms of x. Again, using what we already know about logarithms.

$$\begin{aligned} y &= \ln x \\ e^y &= x. \end{aligned}$$

Finally, we have:

$$y = \ln x \qquad \Rightarrow \qquad y' = \frac{1}{x}.$$

21.1.2 Other Bases?

Well you are bound to see other bases, for example, suppose you are asked to find the derivative of an exponential function whose base is not e? A general example, find the derivative of

$$y = a^x, \quad a > 0.$$

I personally don't believe it's worth memorizing, but you should be able to follow these steps.

$$\begin{aligned} y &= a^x \\ \ln y &= x \ln a \\ \frac{1}{y}y' &= \ln a \\ y' &= y \ln a \\ y' &= a^x \ln a \end{aligned}$$

So now we have another result.

$$y = a^x, \quad a > 0 \text{ and } a \neq 1 \qquad \Rightarrow \qquad y' = a^x \ln a$$

What about logarithms where the base is not e. I strongly suggest that you use the base change formula that you learned in MTH-119.[54]

$$y = \log_a x \qquad \Rightarrow \qquad y = \frac{\ln x}{\ln a}$$

[54] I'll review this in class.

So if you're asked to find the derivative of

$$y = \log_3 x,$$

I suggest that you rewrite this as

$$y = \frac{\ln x}{\ln 3},$$

and then differentiate.

$$y = \log_3 x \qquad \Rightarrow \qquad y' = \frac{1}{x \ln 3}$$

21.2 Examples

1. Differentiate.

$$y = \frac{1}{1 + \ln x}$$

Solution: This will be discussed in class. Final answer:

$$y' = -\frac{1}{x\,(1 + \ln x)^2}$$

2. Find the limit.

$$\lim_{x\to\infty} \left[\ln(2+x) - \ln(1+x)\right]$$

Solution: This will be discussed in class. Final answer:

$$0$$

3. Find the domain and range of $y = \ln(e^x - 2)$.

Solution: This will be discussed in class. Final answer: The domain is

$$(\ln 2,\ \infty)$$

and the range is

$$\mathbb{R}.$$

4. Find y', if $y = \ln\left(x^4 \sin^2 x\right)$.

Solution: This will be discussed in class.

Final answer:

$$y' = \frac{4}{x} + 2\cot x$$

5. Find the limit.

$$\lim_{x\to 3^+} \log_{10}\left(x^2 - 5x + 6\right)$$

Solution: This will be discussed in class.

Final answer:

$$-\infty$$

6. Given $y = \sqrt{x} e^{x^2} \left(x^2+1\right)^{10}$, answer each of the following questions.[55]

   (a) Take the natural log of both sides and expand the right side completely.

   Solution: This will be discussed in class.

   Final answer:

   $$\ln y = \frac{1}{2}\ln x + x^2 + 10\ln\left(x^2+1\right)$$

   (b) Differentiate both sides.

   Solution: This will be discussed in class.

   Final answer:

   $$\frac{1}{y}y' = \frac{1}{2x} + 2x + \frac{20x}{x^2+1}$$

   (c) Find y'.

   Solution: This will be discussed in class.

   Final answer:

   $$y' = \left(\sqrt{x} e^{x^2}\left(x^2+1\right)^{10}\right)\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$$

[55] An example of logarithmic differentiation.

7. Given

$$f(x) = \frac{1+e^x}{1-e^x},$$

answer each of the following questions.

(a) Find the domain of $f(x)$.

Solution: This will be discussed in class. Final answer:

$$\mathbb{R}, \ x \neq 0$$

(b) Find the range of $f(x)$.

Solution: This will be discussed in class.
Final answer:

$$(-\infty, \ -1) \cup (1, \ \infty)$$

(c) Find $f^{-1}(x)$ and indicate its domain and range.

Solution: This will be discussed in class.
Final answers: The inverse is:

$$f^{-1}(x) = \ln \frac{x-1}{x+1}.$$

The domain is:

$$(-\infty, \ -1) \cup (1, \ \infty)$$

The range is:

$$\mathbb{R}, \ x \neq 0$$

(d) Graph $f^{-1}(x)$ and $f(x)$ on the same axis.

Solution: This will be discussed in class. Final answer: strongly encourage everyone to use technology to construct graphs (Figure 40, page 146). For Mac OS X users I suggest Grapher (free); for Windows users I suggest WinPlot (free); and for Linux/UNIX users I suggest GNUPlot (free). A good web-based application is Wolfram Alpha.

8. Find y'.

$$x^y = y^x$$

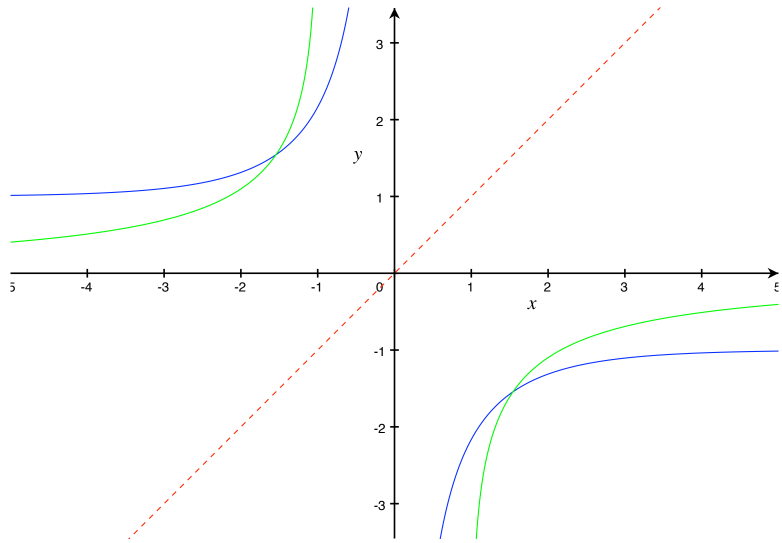

Figure 40: $f(x)$ in blue; $y = x$ red dash; and $f^{-1}(x)$ in green.

Solution: This will be discussed in class.

Final answer:

$$y' = \frac{\ln y - y/x}{\ln x - x/y} = \frac{y^2 - xy \ \ln y}{x^2 - xy \ \ln x}$$

21.3 Assignment

You should read §3.9 and do the WebAssign assignment math.121.03.09.

21.4 Hyperbolic Functions [Optional Material]

It's endless! No, but you'll see that many schools cover a lot of material in their Calculus classes, but I would prefer to cover fundamentals that will allow you to extend your knowledge without being overwhelmed with memorizing a lot of disconnected materials. So, please take a look at this *optional* material to see that knowing a little can in fact be used as a basis for knowing a lot. Just don't try to memorize what follows, but you should nonetheless be able to do this on your own. No, you won't be tested on this.

Well, I can't say, "once again!" That is, the hyperbolic functions were not covered in MTH-119 or MTH-120, so we must begin afresh. So let's start with hyperbolic sine, abbreviated sinh, and hyperbolic cosine, abbreviated cosh.[56] They are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Their graphs follow:

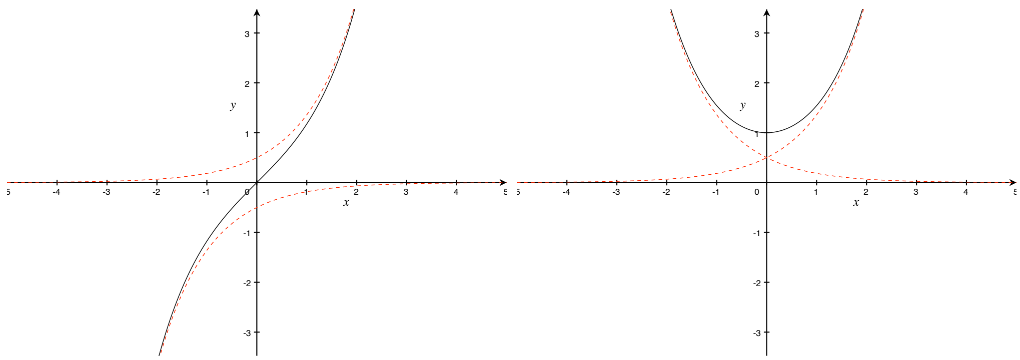

Figure 41: $y = \sinh x$ (left) and $y = \cosh x$.

The dashed red lines are not part of the graph, but I want to emphasize that these two functions are just simple combinations of e^x and e^{-x}. We'll discuss, in class, the dashed red lines on the above two graphs.

21.4.1 Some Questions? [Optional Material]

1. What is the domain of the hyperbolic sine?

   Solution: Answer: $\mathbb{R}$

2. What is the range of the hyperbolic sine?

[56] The graph of the hyperbolic cosine is called a *catenary*, the shape of a hanging cable.

Solution: Answer: $\mathbb{R}$

3. Is the hyperbolic sine *even*, *odd*, or *neither*?

Solution: Answer: *odd*

4. What is the domain of the hyperbolic cosine?

Solution: Answer: $\mathbb{R}$

5. What is the range of the hyperbolic cosine?

Solution: Answer: $[1, \infty)$

6. Is the hyperbolic cosine *even*, *odd*, or *neither*?

Solution: Answer: *even*

7. Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x.$$

Solution: Use the definitions!

8. Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x.$$

Solution: Use the definitions!

9. Show that
$$\cosh^2 t - \sinh^2 t = 1.$$

Solution: Use the definitions!

10. If we define hyperbolic tangent as:
$$\tanh x = \frac{\sinh x}{\cosh x},$$
what is hyperbolic tangent in terms of e^x and e^{-x}?

Solution: Use the definitions!

By the way, many Calculus textbooks have interesting pictures[57] that relates the point,

$$P(\cosh t,\ \sinh t) = (x,\ y),$$

on a hyperbola to the origin $O(0,\ 0)$ using the identity derived in the problem set above, that is $\cosh^2 t - \sinh^2 t = 1$ or $x^2 - y^2 = 1$. Here's the graph of $x^2 - y^2 = 1$, which is a hyperbola.[58]

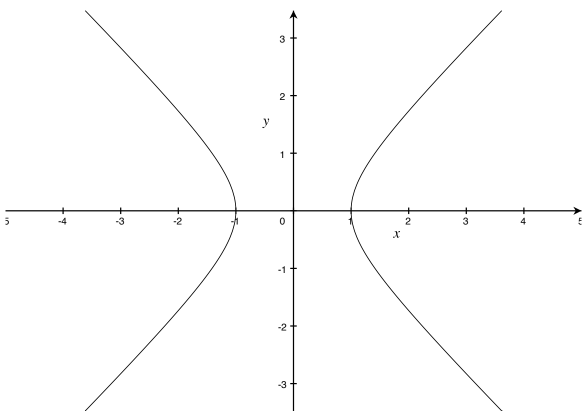

Figure 42: Graph of $x^2 - y^2 = 1$, which is a hyperbola.

Hence, the reason why they're called hyperbolic, just as the trigonometric functions are often called circular. That is, the point $P(\cos t,\ \sin t) = (x,\ y)$ on a circle is related to the origin $O(0,\ 0)$ using the identity $\cos^2 t + \sin^2 t = 1$ or $x^2 + y^2 = 1$.

Example: Pick the point $\left(\sqrt{2},\ 1\right)$ on the hyperbola, $x^2 - y^2 = 1$, and determine the exact value of t.[59]

[57] The parametric variable, t, when dealing with the trigonometric functions, represents the angle. Interestingly enough, the t is also related to the area of the sector of the unit circle; likewise, the same is true for the hyperbolic functions, but this time the area is not a sector, but the bounded region.

[58] The circular functions relate the point $P(\cos t,\ \sin t) = (x,\ y)$ on a unit circle, $x^2 + y^2 = 1$ to the origin $O(0,\ 0)$.

[59] Again, the point $P(\cosh t,\ \sinh t) = (x,\ y)$ is on $x^2 - y^2 = 1$.

Solution: Let's find t first.

$$\begin{aligned}
\sinh t &= 1 \\
\frac{e^t - e^{-t}}{2} &= 1 \\
e^t - e^{-t} &= 2 && \text{multiply both sides by } e^t \\
e^{2t} - 1 &= 2e^t && \text{solve for zero} \\
e^{2t} - 2e^t - 1 &= 0 && \text{let } u = e^t \text{ and note that } u > 0 \\
u^2 - 2u - 1 &= 0 && \text{solve for } u \text{ using the quadratic formula} \\
u &= 1 + \sqrt{2} && \text{now solve for } t, \text{ where } t = \ln u. \\
t &= \ln\left(1 + \sqrt{2}\right)
\end{aligned}$$

Now, using this value of t, verify $\cosh t = \sqrt{2}$.

$$\begin{aligned}
\cosh t &= \sqrt{2} \\
\frac{e^t + e^{-t}}{2} &= \sqrt{2} \\
\frac{e^{\ln\left(1+\sqrt{2}\right)} + e^{-\ln\left(1+\sqrt{2}\right)}}{2} &= \sqrt{2} \\
\frac{\left(1+\sqrt{2}\right) + \left(1+\sqrt{2}\right)^{-1}}{2} &= \sqrt{2} \\
\frac{\left(1+\sqrt{2}\right) + \left(\sqrt{2}-1\right)}{2} &= \sqrt{2} \\
\frac{2\sqrt{2}}{2} &= \sqrt{2} \\
\sqrt{2} &= \sqrt{2}
\end{aligned}$$

So it should be clear why the term hyperbolic is used, and the reason that we see *sine cosine* and *tangent* in these hyperbolic functions is mainly due to the fact that the identities they generate is reminiscent of the trigonometric identities.

21.4.2 Inverses [Optional Material]

From what we know about inverses, it is clear that the hyperbolic sine is invertible, but the hyperbolic cosine is not. However, just like the trigonometric functions, we are going to restrict the domain of the hyperbolic cosine to make it a one-to-one function. For sake of argument, let's restrict the domain of the hyperbolic cosine to $x \geq 0$.

Example: Try finding $\sinh^{-1} x$.

Solution:

$$
\begin{aligned}
\sinh x &= y \\
\sinh y &= x \\
\frac{e^y - e^{-y}}{2} &= x \\
e^y - e^{-y} &= 2x \qquad \text{multiply both sides by } e^y \\
e^{2y} - 1 &= 2xe^y \qquad \text{solve for zero} \\
e^{2y} - 2xe^y - 1 &= 0 \qquad \text{let } u = e^y \\
u^2 - 2xu - 1 &= 0 \\
u &= \frac{2x \pm \sqrt{4x^2+4}}{2} \qquad \text{quadratic formula} \\
e^y &= x + \sqrt{x^2+1} \qquad \text{note the } \pm \text{ is now } + \\
y &= \ln\left(x + \sqrt{x^2+1}\right)
\end{aligned}
$$

Side Note: Why only the $+$ is used from the expression $x \pm \sqrt{x^2+1}$, because we know that $e^y > 0$, hence $x \pm \sqrt{x^2+1} > 0$, and since $x < \sqrt{x^2+1}$ for all x we must exclude $x - \sqrt{x^2+1} < 0$.

21.4.3 What You May Need to Know? [Optional Material]

For this class, you do not need to know the hyperbolic functions, but some of you will enter fields that will use these functions. Just be aware that lots of named functions exists, and do look them up when needed. Have a good reference source, web-based or an actual textbook, whatever *floats your boat*!

Certainly, after reading this optional section, you should be able to recall the definitions for the hyperbolic sine and cosine, and then follow what was presented in this section so far.

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$
2. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$
3. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tanh\, x) = \mathrm{sech}^2\ x$
4. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{csch}\ x) = -\mathrm{csch}\ x\ \coth\ x$
5. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\ x) = -\mathrm{sech}\ x\ \tanh\ x$
6. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\coth\ x) = -\mathrm{csch}^2\ x$

7. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in \mathbb{R}$

8. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \in [1,\ \infty)$

9. $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad x \in (-1,\ 1)$

10. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{1+x^2}}$

11. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}}$

12. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$

13. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{csch}^{-1} x\right) = -\dfrac{1}{|x|\sqrt{1+x^2}}$

14. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{sech}^{-1} x\right) = -\dfrac{1}{x\sqrt{1-x^2}}$

15. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2}$

Believe it or not, the results above can be obtained by using the definitions of the hyperbolic sine and cosine alone. However, you also need to be familiar with some basic identities from trigonometry that relate sine and cosine to the other four trigonometric functions. The hyperbolic functions have similar fundamental identities.

21.4.4 Examples [Optional Material]

1. Prove the identity.

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Solution: Use the definitions!

2. Prove the identity.

$$\sinh 2x = 2\sinh x \cosh x$$

Solution: Use the definitions!

3. If $\sinh x = 3/4$, find the values of $\cosh x$ and $\tanh x$.

Solution: Use the definitions!

4. Find the derivative.

$$y = x \cosh x$$

Solution: Use the definitions!

5. Find the derivative.

$$y = \sinh x \cosh x$$

Solution: Use the definitions!

6. Find the derivative.

$$y = x^2 \sinh^{-1}(2x)$$

Solution: Use the definitions!

7. Evaluate.

$$\lim_{x \to \infty} \frac{\sinh x}{e^x}$$

Solution: Use the definitions!

22 mth.121.03.10

22.1 Implicit Differention

22.1.1 A Simple Visual Example

The figure below (Figure 43, page 154) is the graph of $x+y=\left(x^2+y^2\right)^2$ with three tangent lines drawn at the points: (0, 0) (0, 1) (1, 0). The graph is properly scaled, so slope is what it appears to be. There are no visual deceptions.

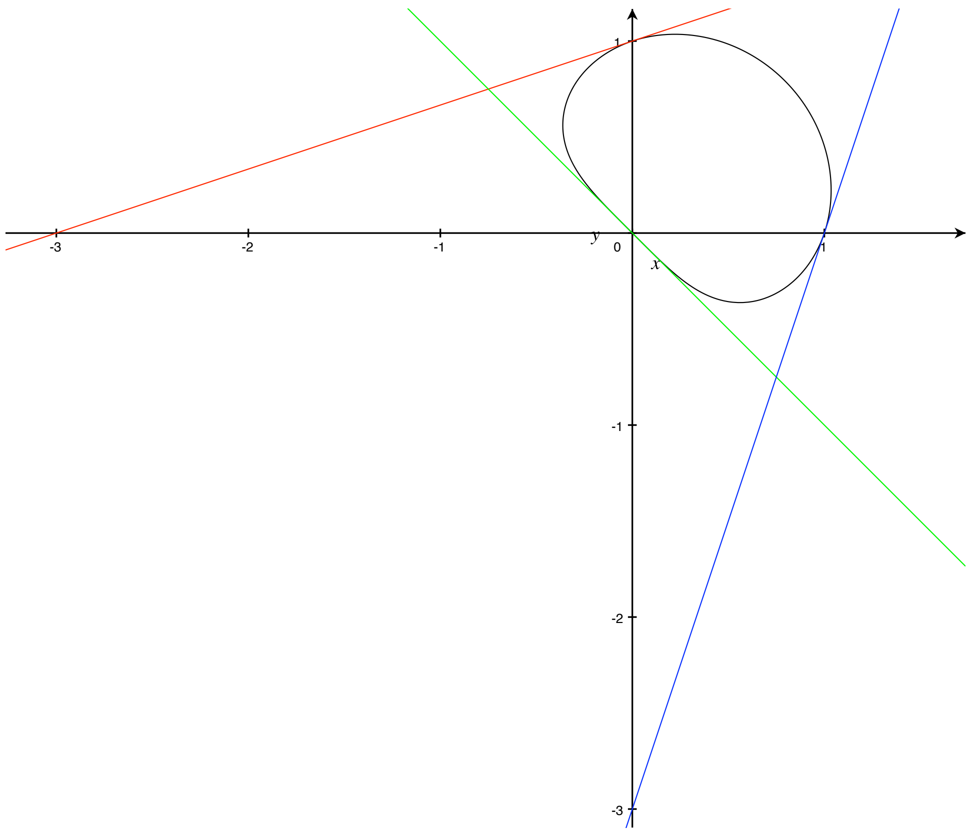

Figure 43: The graph of $x+y=\left(x^2+y^2\right)^2$, with three tangent lines to this graph.

1. From the graph, which is accurately constructed, find the equations of the three lines that are tangent to the curve $x+y=\left(x^2+y^2\right)^2$. The points of tangency are: (1, 0); (0, 0); and (0, 1). This requires no calculus! Indicate these equations directly on the lines themselves.

Solution: Work will be done in class.

1. Red Line (y-intercept at 1):
2. Blue Line (y-intercept at -3):

3. Green Line (y-intercept at 0):

2. After some introductory problems you should be able to differentiate $x + y = \left(x^2 + y^2\right)^2$ and verify that the slopes of these tangent lines and the derivative's values at these points agree. It is important that the visual provided is understood, and that the lines drawn appear to be tangents!

Solution: This will be discussed later.

22.2 Finding Derivatives Implicitly

In the past we were given a function to differentiate, but now we are being asked to differentiate a relationship between x and y that is not functional. Essentially we will need to differentiate both sides with respect to x, and go term by term as we always have.

Example: Find

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

for the equation of the unit circle centered at the origin.

Solution:

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + y^2\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[1] \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[y^2\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[1] \\ [2x]\frac{\mathrm{d}x}{\mathrm{d}x} + [2y]\frac{\mathrm{d}y}{\mathrm{d}x} &= [0]\frac{\mathrm{d}k}{\mathrm{d}x} \\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

Finally, solve for the $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

Example: Find the equation of the line tangent to the curve

$$x^2 - \sin(x \cdot y) + y^3 = 1$$

at the point $(1,\ 0)$. If possible, try to learn how to graph (Figure 44, page 156) this function using computer software.

Solution:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 - \sin(x\cdot y) + y^3\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[1] \\
2x - \cos(x\cdot y)\cdot(x\cdot y' + 1\cdot y) + 3y^2\cdot y' &= 0
\end{aligned}$$

We need to find y' when $x = 1$ and $y = 0$.

$$\begin{aligned}
2 - \cos(0)\cdot(y' + 0) + 0\cdot y' &= 0 \\
2 - y' &= 0 \\
2 &= y'
\end{aligned}$$

Using the point slope form we have

$$y - 0 = 2(x - 1)$$

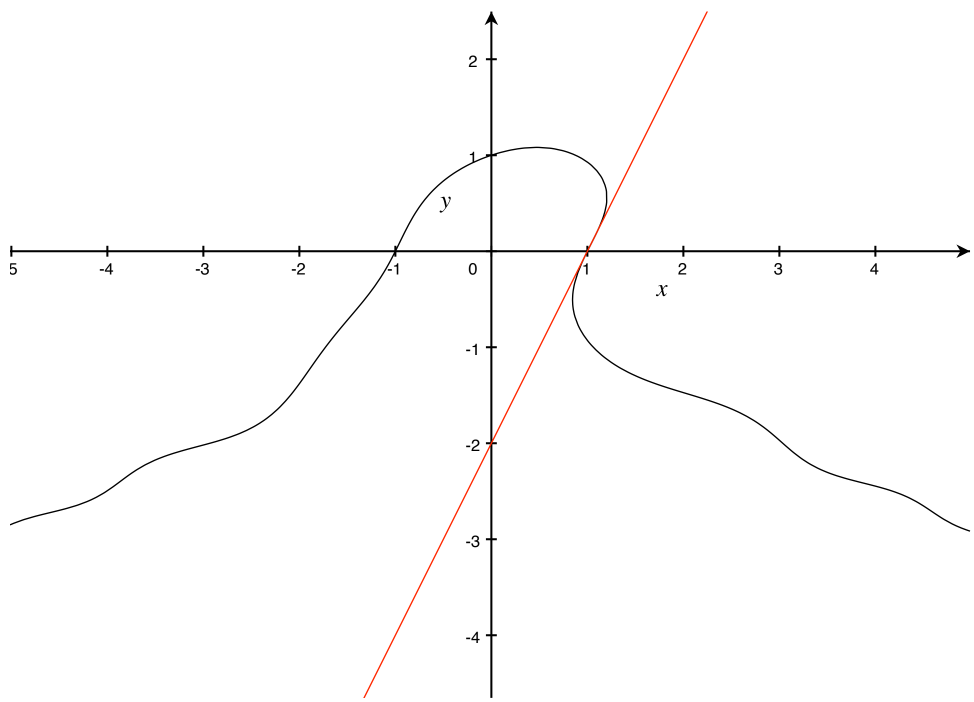

Figure 44: The graph of $x^2 - \sin(x\cdot y) + y^3 = 1$, and $y = 2x - 2$.

Differentiating usually is not difficult, but solving for y' can be.
Example: Using the last example, find y'.

Solution:

$$\begin{aligned}
2x - \cos(x \cdot y) \cdot (x \cdot y' + 1 \cdot y) + 3y^2 \cdot y' &= 0 \\
2x - x\cos(xy) \cdot y' - y\cos(xy) + 3y^2 \cdot y' &= 0 \\
2x - y\cos(xy) &= x\cos(xy) \cdot y' - 3y^2 \cdot y' \\
2x - y\cos(xy) &= \left[x\cos(xy) - 3y^2\right] \cdot y' \\
\frac{2x - y\cos(xy)}{x\cos(xy) - 3y^2} &= y'
\end{aligned}$$

22.3 Examples

1. Verify that the three tangents that you identified on page one are correct.

Solution: This will be discussed in class.

$$\begin{aligned}
x + y &= \left(x^2 + y^2\right)^2 \\
\frac{\mathrm{d}}{\mathrm{d}x}[x + y] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(x^2 + y^2\right)^2\right] \\
1 + y' &= 2\left(x^2 + y^2\right)(2x + 2y \cdot y')
\end{aligned}$$

Now, using the given points and the derivative.

1. Red Line: $x = 0$ and $y = 1$; and

$$\begin{aligned}
1 + y' &= 2\left(x^2 + y^2\right)(2x + 2y \cdot y') \\
1 + y' &= 2\left(0^2 + 1^2\right)(2 \cdot 0 + 2 \cdot 1 \cdot y') \\
1 + y' &= 4y' \\
y' &= \frac{1}{3}
\end{aligned}$$

As expected the equation of the line is:

$$y = \frac{1}{3}x + 1$$

2. Blue Line: $x = 1$ and $y = 0$; and

$$\begin{aligned}
1 + y' &= 2\left(x^2 + y^2\right)(2x + 2y \cdot y') \\
1 + y' &= 2\left(1^2 + 0^2\right)(2 \cdot 1 + 2 \cdot 0 \cdot y') \\
1 + y' &= 4 \\
y' &= 3
\end{aligned}$$

As expected the equation of the line is:

$$y = 3x - 3$$

3. Green Line: $x = 0$ and $y = 0$; and

$$\begin{aligned}1 + y' &= 2\left(x^2 + y^2\right)(2x + 2y \cdot y') \\ 1 + y' &= 2\left(0^2 + 0^2\right)(2 \cdot 0 + 2 \cdot 0 \cdot y') \\ 1 + y' &= 0 \\ y' &= -1\end{aligned}$$

As expected the equation of the line is:

$$y = -x$$

2. Consider $5x^2 + 3\sqrt{y} = x^3y^2$. Differentiate with respect to t where x and y vary.

Solution: This will be discussed in class.

$$\begin{aligned}5x^2 + 3\sqrt{y} &= x^3y^2 \\ 5x^2 + 3y^{1/2} &= x^3y^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[5x^2 + 3y^{1/2}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[x^3y^2\right] \\ 10x\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3}{2}y^{-1/2}\,\frac{\mathrm{d}y}{\mathrm{d}t} &= 3x^2y^2\,\frac{\mathrm{d}x}{\mathrm{d}t} + 2x^3y^2\,\frac{\mathrm{d}y}{\mathrm{d}t}\end{aligned}$$

3. If

$$[f(x)]^3 = [x + f(x)]^2 - 1$$

and $f(1) = 2$, find $f'(1)$

Solution: This will be discussed in class.

$$\begin{aligned}[f(x)]^3 &= [x + f(x)]^2 - 1 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left([f(x)]^3\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left([x + f(x)]^2 - 1\right) \\ 3[f(x)]^2 \cdot f'(x) &= 2[x + f(x)] \cdot [1 + f'(x)]\end{aligned}$$

Now let $x = 1$.

$$\begin{aligned}3[f(1)]^2 \cdot f'(1) &= 2[1 + f(1)] \cdot [1 + f'(1)] \\ 3[2]^2 \cdot f'(1) &= 2[1 + 2] \cdot [1 + f'(1)] \\ 12f'(1) &= 6 + 6f'(1) \\ f'(1) &= 1\end{aligned}$$

4. Find all points where the tangent line to $y^3 - xy = -6$ is either horizontal or vertical. A graph (Figure 45, page 159) of $y^3 - xy = -6$, and all vertical and horizontal tangents is provided.

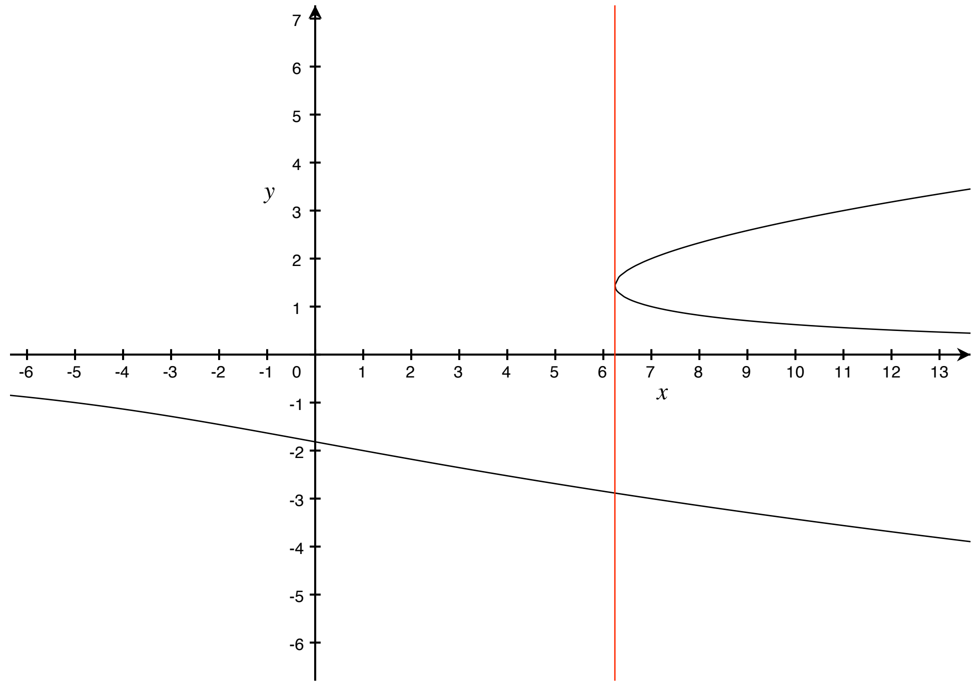

Figure 45: The graph of $y^3 - xy = -6$, and $x = 3\sqrt[3]{9}$.

Solution: This will be discussed in class. You should appreciate the usefulness of the provided graph in answering this question, and make sure you know how to graph technology to graph this relationship.

$$\begin{aligned}
y^3 - xy &= -6 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[y^3 - xy\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[-6\right] \\
3y^2y' - y - xy' &= 0 \\
y' &= \frac{y}{3y^2 - x}
\end{aligned}$$

A horizontal tangent is not possible (why?). The vertical tangent occurs when $3y^2 - x = 0$ or when $x = 3y^2$. Now, just replace x by $3y^2$ in the original equation to get the point.

$$\begin{aligned}
y^3 - xy &= -6 \\
y^3 - 3y^3 &= -6 \\
y^3 &= 3 \\
y &= \sqrt[3]{3} \\
x &= 3\sqrt[3]{9}
\end{aligned}$$

Finally, the only point is $\left(3\sqrt[3]{9},\ \sqrt[3]{3}\right)$.

22.4 Assignment

You should read §3.10 and do the WebAssign assignment mth.121.03.10.

23 mth.121.03.11

23.1 Word Problems

Word problems are the bane of mathematics. I once gave a colleague a word problem to check and she lamented that it was really tricky. She's a very bright woman, and has an excellent command of the English language. In fact I seek her advice on many occasions, and she is well-respected for her mathematical abilities as well as her intimate fluency with written language. So, why the silly preamble? Basically to tell you that even *we* in math get confused by words, so please don't despair, but please do try!

Words, however, are the *foundation* of mathematics. Without words, we would have never gotten to mathematical reason in the first place, nor we would have any need for numerical fluency. Words, after-all, is the main way we communicate with one another. Of course there are other ways in which we communicate, but it is words that make for the most lasting impressions.

Anaïs Nin[60] once said, "Truth is something which can't be told in a few *words*. Those who simplify the universe only reduce the expansion of its meaning."

23.1.1 Vexing Non-Calculus Examples [Optional Material]

Okay, here's some pretty tough non-calculus word problems that may interest you, but will not be covered in class.

1. An old car has to travel a two-mile route, uphill and downhill. Because it is old, the car can climb the first mile—the ascent—no faster than an average speed of fifteen miles per hour. How fast does the car have to travel the second mile—on the descent it can go faster, of course—in order to acheive an average speed of thirty miles per hour?[61]

2. Given:

$$\begin{cases} |x| + x + y = 10 \\ x + |y| - y = 12 \end{cases}$$

solve for $x + y$.[62]

[60] A famous French writer.

[61] This problem was sent to Albert Einstein by his friend Wertheimer. Einstein wrote this in reply to Wertheimer, "Your letter gave us a lot of amusement. The first intelligence test fooled both of us (Bucky and me). Only on working it out did I notice that no time is available for the downhill run! . . . [deleted text] . . . Such drolleries show us how stupid we are!"

[62] Students—and teachers—invariably struggle with this problem because it does not reflect the normal tone in which these two concepts (absolute value and systems of equations) are normally presented. It is here, however, where one's mathematical mettle is tested. Some students solve the problem rabidly, while others take many frustrating hours before seeing the simplicity. But perhaps the entire point here is not necessarily solving the problem, but getting students interested in trying. One student in particular worked on this one problem for five hours, and then Eureka! . . . he solved it. The look on his face reflected a joy in realizing his hours of frustration were not for naught.

23.1.2 Guidelines For Solving Related-Rate Word Problems

1. Read the problem, they're incredibly simple stories. Reading is perhaps the most fundamental activity in any academic environment and you need to understand what you read. You may also have to revisit word problems from prior courses.

2. Make a sketch of what's happening in the problem. Label your sketch and be sure to write down any know relationships. For example, if they're talking about the surface area of a cube, it would be a good idea to write down the algebraic relationship of a cube's surface area. Variables (things that change) should be clearly understood at this stage.

3. Write down any derivatives given and any derivatives wanted. You'll also need to find an algebraic/trigonometric relationship between the variables in those derivatives.

4. Differentiate this relationship with respect to time t.

5. Now just plug in the known quantities and solve for the unknown.

Example: Find the rate of change of the distance between the origin and a moving point on the graph of $y = x^2 + 1$ if $\mathrm{d}x/\mathrm{d}t = 2$ centimeters per second and the point is at $(1.1,\ 2.21)$.

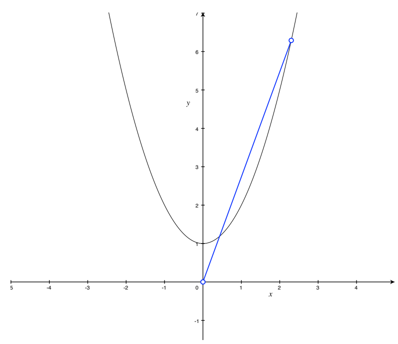

Figure 46: This is how I *see* the problem.

Solution: Details follow, but first think about the steps!

1. Okay, I read the problem. It looks simple enough.

2. Here's a picture (Figure 46, page 162). The blue line is the distance between the origin and some point on the parabola—the point is moving constantly so the point on the parabola is not static. The origin does not change, but the length of the blue line changes as the point moves along the parabola. The point on the parabola is $(x,\ x^2 + 1)$ and the distance between this point and the origin is:

$$d = \sqrt{(x-0)^2 + (x^2+1-0)^2} \qquad \Rightarrow \qquad d^2 = x^2 + \left(x^2+1\right)^2, \quad x \in \mathbb{R}, \quad d \geq 1.$$

3. Clearly they gave us $\mathrm{d}x/\mathrm{d}t = 2$, and they want $\mathrm{d}d/\mathrm{d}t$ when the moving point is at $(1.1,\ 2.21)$ on the parabola. Certainly, since x is getting bigger we would expect at this point that d is also getting bigger. The relationship between d and x from step 2 is:

$$d^2 = x^2 + \left(x^2+1\right)^2, \quad x \in \mathbb{R}, \quad d \geq 1.$$

4. Differentiate this relationship with respect to time t.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[d^2\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[x^2 + \left(x^2+1\right)^2\right] \\ 2d\,\frac{\mathrm{d}d}{\mathrm{d}t} &= \left[2x + 2\left(x^2+1\right)2x\right]\frac{\mathrm{d}x}{\mathrm{d}t} \end{aligned}$$

5. Now just plug in the knows quantities and solve for the unknown. We have $x = 1.1$, $d = \sqrt{(1.1)^2 + (1.1^2+1)^2} = \sqrt{6.0941}$, $\mathrm{d}x/\mathrm{d}t = 2$. Plugging in we get:

$$\begin{aligned} 2\sqrt{6.0941}\,\frac{\mathrm{d}d}{\mathrm{d}t} &= \left[2\cdot 1.1 + 2\left(1.1^2+1\right)2\cdot 1.1\right]2 \\ 2\sqrt{6.0941}\,\frac{\mathrm{d}d}{\mathrm{d}t} &= 23.848 \\ \frac{\mathrm{d}d}{\mathrm{d}t} &= \frac{23.848}{2\sqrt{6.0941}} \approx 4.83022 \end{aligned}$$

So the answer is 4.83 centimeters per second.

23.2 Examples

1. A particle moves along the curve $y = \sqrt{1+x^3}$. As it reaches the point $(2,\ 3)$, the y-coordinate is increasing at a rate of 4 cm/sec. How fast is the x-coordinate of the point changing at that instant?

Solution: This will also be discussed in class.

$$\begin{aligned} y &= \sqrt{1+x^3} \\ y &= \left(1+x^3\right)^{1/2} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{3x^2}{2\sqrt{1+x^3}}\frac{\mathrm{d}x}{\mathrm{d}t} \end{aligned}$$

Now plugging in.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{3x^2}{2\sqrt{1+x^3}}\frac{\mathrm{d}x}{\mathrm{d}t} \\
4 &= \frac{3\cdot 2^2}{2\sqrt{1+2^3}}\frac{\mathrm{d}x}{\mathrm{d}t} \\
4 &= \frac{12}{2\sqrt{9}}\,\frac{\mathrm{d}x}{\mathrm{d}t} \\
4 &= 2\,\frac{\mathrm{d}x}{\mathrm{d}t} \\
2 &= \frac{\mathrm{d}x}{\mathrm{d}t}
\end{aligned}$$

The x-coordinate of the point is changing at a rate of 2 cm/sec at that instant.

2. The minute hand of the clock is 8 centimeters long, and the hour hand is 5 centimeters long. How fast is the distance between the tips of the hands changing at 3 o'clock?

Solution: You'll be best served if you can visualize this. You should also realize that the hour hand is moving at a rate of

$$\frac{2\pi}{720\text{ min}},$$

and the minute hand is moving at a rate of

$$\frac{2\pi}{60\text{ min}}.$$

So the rate at which the angle is changing between these hands is given by

$$\frac{2\pi}{720\text{ min}} - \frac{2\pi}{60\text{ min}} = -\frac{11\pi}{360\text{ min}} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

If you see the distance between the hands forming a triangle you'll get this relationship where θ is the angle between the hands and x is the distance between the hands. I'm using the Law of Cosines here.

$$x^2 = 8^2 + 5^2 - 2\cdot 8\cdot 5\cos\theta$$

Differentiate this relationship with respect to t.

$$2x\,\frac{\mathrm{d}x}{\mathrm{d}t} = 80\sin\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

We want to know

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\theta=\pi/2,\ x=\sqrt{89}}$$

Okay, let's plug in. (I'm using units here.)

$$\begin{aligned}
2x\,\frac{\mathrm{d}x}{\mathrm{d}t} &= 80\ \text{cm}^2 \sin\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\\
2\left(\sqrt{89}\ \text{cm}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 80\ \text{cm}^2 \sin\left(\frac{\pi}{2}\right)\left(-\frac{11\pi}{360\ \text{min}}\right)\\
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{11\pi}{9\sqrt{89}}\ \text{centimeters per minute}\\
&= -0.40701\ \text{centimeters per minute}
\end{aligned}$$

3. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm?[63]

Solution: This will also be discussed in class. The surface area of a sphere is:

$$A = 4\pi r^2 \quad \Rightarrow \quad A = 4\pi\left(\frac{d}{2}\right)^2 = \pi d^2$$

Differentiate with respect to t.

$$\begin{aligned}
A &= \pi d^2\\
\frac{\mathrm{d}A}{\mathrm{d}t} &= 2\pi d\,\frac{\mathrm{d}d}{\mathrm{d}t}
\end{aligned}$$

Now plugging in.

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= 2\pi d\,\frac{\mathrm{d}d}{\mathrm{d}t}\\
-1 &= 2\pi 10\,\frac{\mathrm{d}d}{\mathrm{d}t}\\
-1 &= 20\pi\,\frac{\mathrm{d}d}{\mathrm{d}t}\\
-\frac{1}{20\pi} &= \frac{\mathrm{d}d}{\mathrm{d}t}
\end{aligned}$$

So the answer is

$$-\frac{1}{20\pi}\,\frac{\text{cm}}{\text{min}}$$

4. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing

[63] **Hint:** It's okay to look up a geometry formula. They're in your book!

at a rate of 3.44°/sec. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is 60°?

Solution: This will also be discussed in class. You should also convert all angles to radian.

$$\begin{aligned}
\frac{1}{2}\cdot 5\cdot 4\cdot \sin\left(\theta\right)\ \mathrm{m}^2 &= A\\
10\sin\left(\theta\right)\ \mathrm{m}^2 &= A\\
10\cos\left(\theta\right)\ \mathrm{m}^2\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}t}
\end{aligned}$$

Converting to radian.

$$60^\circ = \frac{\pi}{3}, \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{3.44^\circ}{\text{sec}}\cdot\frac{\pi}{180^\circ} = \frac{3.44\pi}{180\ \text{sec}}$$

Now plugging in.

$$\begin{aligned}
10\cos\left(\theta\right)\ \mathrm{m}^2\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}t}\\
10\cos\left(\frac{\pi}{3}\right)\ \mathrm{m}^2\cdot\frac{3.44\pi}{180\ \text{sec}} &= \frac{\mathrm{d}A}{\mathrm{d}t}\\
10\left(\frac{1}{2}\right)\ \mathrm{m}^2\cdot\frac{3.44\pi}{180\ \text{sec}} &= \frac{\mathrm{d}A}{\mathrm{d}t}\\
0.300196631343\ \frac{\mathrm{m}^2}{\text{sec}} &\approx \frac{\mathrm{d}A}{\mathrm{d}t}
\end{aligned}$$

23.3 Assignment

You should read §3.11 and do the WebAssign assignment mth.121.03.11.

24 mth.121.04.01

24.1 Linear Approximations and Differentials

24.1.1 Visual Approach

The graphs that we have seen so far—except for those with sharp[64] corners—can be locally linearized. For example, we have been using the tangent lines for a while now and you should appreciate that these lines are *good* fits to the curve in a local neighborhood. Here's the sine curve and lines tangent at the points where $x = 0$ and $x = \pi/3$ (Figure 47, page 167).

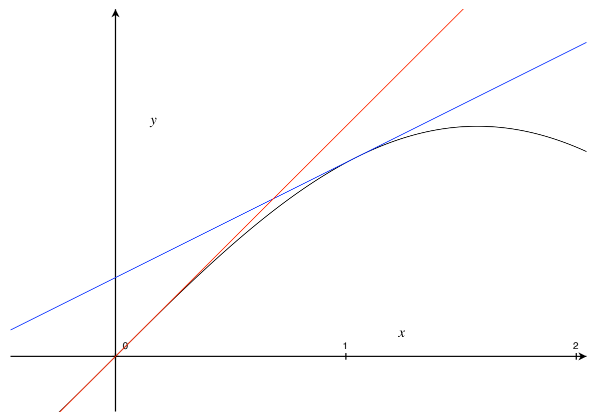

Figure 47: The red and green tangent lines at $x = 0$ and $x = \pi/3$.

Which tangent line *fits* the graph better?[65] Zooming in, centered at the points of tangency, and using an interval of 0.5 it becomes clearer which is better. Accurate scale is preserved in all graphs (Figure 49, page 167).

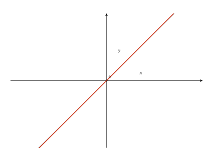

Figure 48: Centered at $x = 0$.

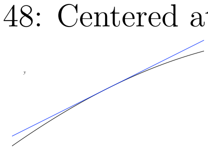

Figure 49: Centered at $x = \pi/3$.

Clearly the tangent line at $x = 0$ is a much better fit.

[64] Absolute values and some piece-wise defined functions.
[65] By this, I mean, which line fits over a wider region.

24.1.2 Linearization

We will be using the definition[66] of the derivative to get this linear relationship

$$f(x) \approx f(a) + f'(a)(x-a).$$

The closer x gets to a, the better these approximations become. In fact if we take the limit of each side, as $x \to a$, we get an equality.

As an example, we will use a linearization of $f(x) = y = \sqrt{x}$ to approximate the value of $\sqrt{15.57}$. We basically need to find an *easy* x value near 15.57 that's on $f(x) = y = \sqrt{x}$, and I think you'll agree that 16 is close and easy! So our $a = 16$ and our $x = 15.57$,

$$\begin{aligned}
f(x) &\approx f(a) + f'(a)(x-a) \\
f(15.57) &\approx f(16) + f'(16)(15.57-16) \\
f(15.57) &\approx \sqrt{16} + \frac{1}{2\sqrt{16}}(-0.43) \\
f(15.57) &\approx 4 + \frac{1}{8}(-0.43) \\
f(15.57) &\approx 3.94625 \qquad \text{I cheated and used a calculator}
\end{aligned}$$

I'm not saying that this is entirely useful, especially nowadays where a calculator would be used[67] to find the approximate square root of 15.57. However, this is just one more example of using the derivative to see that there's little difference between the graph of f and a line tangent to f at a point, as long as we stay within a small region of a and f is differentiable at a.[68]

24.1.3 Differentials

The basic idea behind what we just did with local linearization is often used to introduce a new notation called differentials. Suppose y is a function of x, where f is differentiable, then the differential $\mathrm{d}x$ is treated as an independent variable, where we can write

$$\mathrm{d}y = f'(x)\,\mathrm{d}x.$$

It appears that if $\mathrm{d}x$ is taken as the independent variable, then $\mathrm{d}y$ would be the dependent variable. For example if you are asked to find the differential of

$$y = xe^x,$$

[66] Once again, this concept will be discussed in class and we are basically just using an equation of a tangent line, in a local region, to approximate the curve. Actually for very small differences between x and a ($x \neq a$) we have

$$\frac{f(x) - f(a)}{x-a} \approx f'(a) \quad \Rightarrow \quad f(x) \approx f(a) + f'(a)(x-a).$$

[67] $\sqrt{15.57} \approx 3.94588393139$

[68] Remember the limit, as defined for the derivative, must exist for f to be differentiable at a.

you should write

$$\begin{aligned} \mathrm{d}y &= f'(x)\,\mathrm{d}x \\ \mathrm{d}y &= (xe^x + e^x)\,\mathrm{d}x. \end{aligned}$$

This is an important area, and I just want to review the notation one more time. More often than not, notation gets in the way of reason. The suggestion here is that we need to have a clear idea of what our notation means. So, let's proceed.

$$\Delta x = x_2 - x_1 \qquad x_2 \neq x_1 \tag{1}$$

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \tag{2}$$

$$\Delta f = f(a + \Delta x) - f(a) \tag{3}$$

And now, for a small change in x, we have

$$\Delta f \approx f'(a) \cdot \Delta x,$$

which is an estimate of how much f changes. In this section we're going to use *differentials* to express this relationship. Where we have

$$\mathrm{d}y = f'(a)\ \mathrm{d}x,$$

and

$$\Delta y = f(a + \mathrm{d}x) - f(a) = f(a + \Delta x) - f(a).$$

And once again, for a small change in x, we have a *linear approximation* that is given by

$$\Delta y \approx \mathrm{d}y, \quad \text{or} \quad \Delta f \approx f'(a) \cdot \Delta x.$$

The difference between Δf and $f'(a) \cdot \Delta x$ is called the error. For example, suppose we have a function $f(x) = \sqrt[3]{x}$, and we're interested in the difference between $\sqrt[3]{8.1}$ and $\sqrt[3]{8}$.

$$\Delta y = f(8 + \mathrm{d}x) - f(8) \tag{4}$$

$$= \sqrt[3]{8 + 0.1} - \sqrt[3]{8} \tag{5}$$

$$= \sqrt[3]{8.1} - 2 \tag{6}$$

$$\mathrm{d}y = f'(8)\ \mathrm{d}x \tag{7}$$

$$= \frac{1}{12} \cdot 0.1 \approx 0.00833333333333333 \tag{8}$$

$$\Delta y \approx \mathrm{d}y \tag{9}$$

We could use this information to approximate the $\sqrt[3]{8.1}$ to be 2.0083. You may not bother with this though, especially considering that you can use software to get many places of accuracy.[69]

[69] For example, $\sqrt[3]{8.1} = 2.0082988502465085656479779739197923799102530615859$ to 49 decimal places.

24.2 Examples

1. Find the linearization $L(x)$ at $x = 9$ for the function

$$f(x) = \sqrt{x}.$$

And use this linearization to estimate $\sqrt{9.1}$.

Solution:

$$\begin{aligned} f(x) &= \sqrt{x} \\ f(x) &= \frac{1}{2\sqrt{x}} \\ y = L(x) &= f'(9)(x-9) + f(9) \\ y = L(x) &= \frac{1}{6}(x-9) + 3 \\ L(9.1) &= \frac{1}{6}(9.1-9) + 3 \approx 3.0167 \end{aligned}$$

2. Use the linear approximation to estimate $\Delta f = f(8.02) - f(8)$ for $f(x) = x^4$.

Solution:

$$\begin{aligned} f(x) &= x^4 \\ f'(x) &= 4x^3 \\ \Delta x &= 8.02 - 8 = 0.02 \\ \Delta f &\approx f'(8)\,\Delta x = 4 \cdot 8^3 \cdot 0.02 = 40.96 \end{aligned}$$

3. Use linear approximation to estimate Δf for

$$f(x) = \frac{8}{2 + x^2}$$

when $a = 2$ and $\Delta x = -0.5$. Also compute the error in the linear approximation, and percent error in the linear approximation. (Round all answers to three decimal places.)

Solution:

$$\begin{aligned}
f(x) &= \frac{8}{2+x^2} \\
f'(x) &= -\frac{16x}{\left(2+x^2\right)^2} \\
\Delta x &= -0.5 \\
\Delta f &\approx f'(2)\,\Delta x = -\frac{32}{36}\cdot(-0.5) = 0.444
\end{aligned}$$

The actual change is:

$$\Delta f = f(a+\Delta x) - f(a) = f(1.5) - f(2) = \frac{28}{51} = 0.549$$

The error in the linear approximation is

$$|0.549 - 0.444| = 0.105$$

In percentage terms, the error is:

$$\left|\frac{0.105}{0.549}\right| \cdot 100\% \approx 19.125\%$$

24.3 Assignment

You should read §4.1 and do the WebAssign assignment mth.121.04.01.

25 mth.121.04.02

25.1 Important Features of f

Ah, here we are, once again, back in MTH-119 trying to graph a polynomial function.

$$f(x) = x^3 - 9x^2 - 4x + 36$$

As I hope you recall from your MTH-119 days, you need to factor[70] this polynomial, determine the x-intercepts, y-intercept, and do some very simple sign-analysis. After doing all this you were expected to make a pre-calculus sketch. You should also recall that your MTH-119 teacher most likely said that this was a rather crude method, and that you'd have to wait for calculus for better detail.

Let's start the MTH-119 process.

$$\begin{aligned} f(x) &= x^3 - 9x^2 - 4x + 36 \\ &= x^2(x-9) - 4(x-9) \\ &= (x-9)\left(x^2-4\right) \\ &= (x-9)(x-2)(x+2) \end{aligned}$$

Clearly the x-intercepts are: $(-2,\ 0)$, $(2,\ 0)$, and $(9,\ 0)$. And the y-intercept is: $(0,\ 36)$. Plotting thes points and then doing the sign-analysis results in the following graph, albeit not as refined as my computer's version.

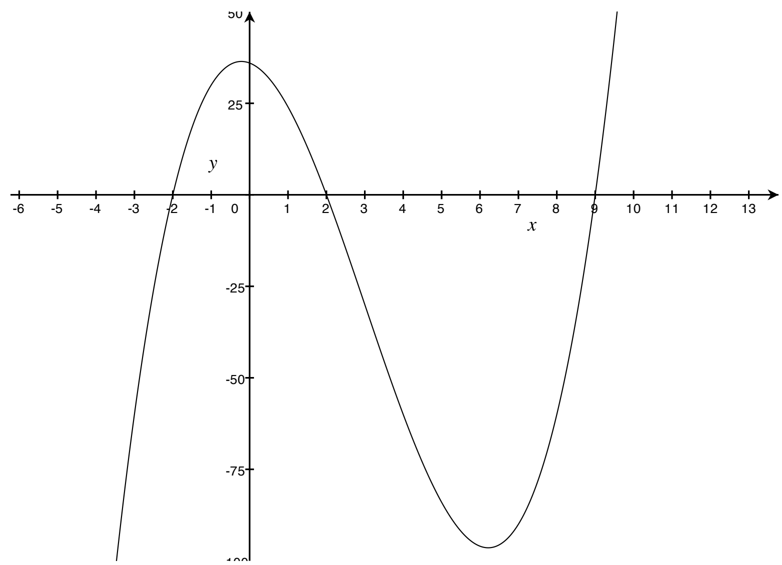

Figure 50: $f(x) = x^3 - 9x^2 - 4x + 36$

Now we will start the process of using the derivatives (calculus) to refine our understanding of f. Here are what[71] the derivatives tell us about the shape of a curve?

[70] Rational Root Theorem, long division, *etc.*.
[71] We'll appeal to your intuition here.

1. If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.
3. If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval.
4. If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.

So let's take a look at $f(x)$, $f'(x)$, and $f''(x)$ on the same graph (Figure 51, page 173).

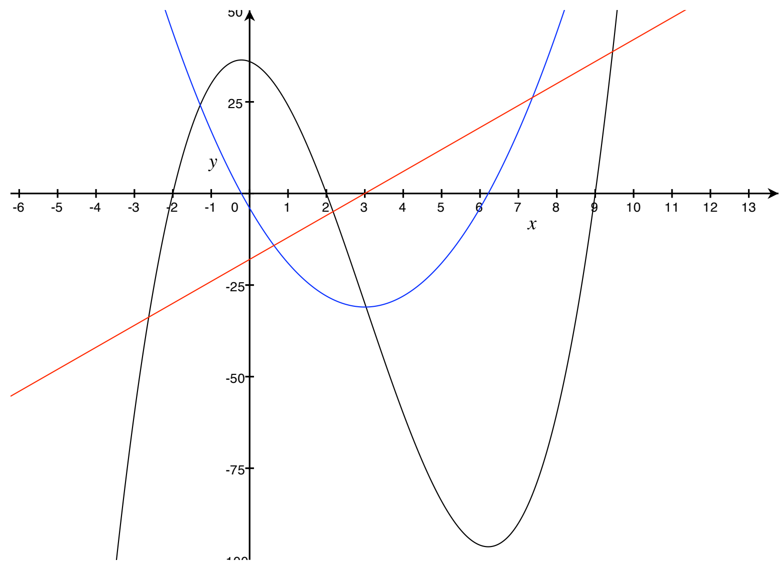

Figure 51: $f(x)$, $f'(x)$ in blue, and $f''(x)$ in red.

Time to think!

1. Find $f'(x)$ and determine on what intervals it is positive/negative. It should agree with the graph.

Solution:

$$\begin{aligned} f(x) &= x^3 - 9x^2 - 4x + 36 \\ f'(x) &= 3x^2 - 18x - 4 \end{aligned}$$

As I hope you recall from MTH-119, the necessary step is to find where $f'(x) = 0$ and then to use simple sign-analysis to determine when $f'(x) < 0$ and when $f'(x) > 0$.

$$3x^2 - 18x^2 - 4 = 0 \qquad \Rightarrow \qquad x = \frac{18 \pm \sqrt{18^2 + 4 \cdot 3 \cdot 4}}{6} = \frac{9 \pm \sqrt{93}}{3}$$

You might want to make *sense* out of this irrational number by approximating (-0.215 and 6.215) it with a rational number. But I am interested in exact answers! So here's the interval where $f'(x) < 0$:

$$\left(\frac{9-\sqrt{93}}{3},\ \frac{9+\sqrt{93}}{3}\right),$$

and here's the intervals where $f'(x) > 0$:

$$\left(-\infty,\ \frac{9-\sqrt{93}}{3}\right) \cup \left(\frac{9+\sqrt{93}}{3},\ \infty\right).$$

2. Find $f''(x)$ and determine on what intervals it is positive/negative. It should agree with the graph.

Solution:

$$\begin{aligned} f'(x) &= 3x^2 - 18x - 4 \\ f''(x) &= 6x - 18 \end{aligned}$$

It is getting easier, I hope. The interval where $f''(x) > 0$:

$$(3,\ \infty),$$

and he interval where $f''(x) < 0$:

$$(-\infty,\ 3).$$

3. Find the local extrema on $f(x)$. What happened to f' at this point?

Solution: The local extrema occurs on f where the first derivative is zero. The local maximum is:

$$\left(\frac{9-\sqrt{93}}{3},\ f\left(\frac{9-\sqrt{93}}{3}\right)\right),$$

and the local minimum is

$$\left(\frac{9+\sqrt{93}}{3},\ f\left(\frac{9+\sqrt{93}}{3}\right)\right).$$

4. Find the point on $f(x)$ where the concavity changes. What happened to f'' at this point?

Solution: This point is called an inflection point.

$$(3,\ f(3)).$$

Okay, that was essentially just an introduction to using derivatives to graph. If you could follow that example you're ready to move forward!

25.2 Definitions and Theorems

1. **Definition:** A function f has a *absolute maximum* (or *global maximum*) at c if $f(c) \geq f(x)$ for all x in f's domain. The number $f(c)$ is called the *maximum value* of f on f's domain. Similarly, f has a *absolute minimum* (or *global minimum*) at c if $f(c) \leq f(x)$ for all x in f's domain. The number $f(c)$ is called the *minimum value* of f on f's domain. The minimum and maximum values of f are called *extreme values* of f.

2. **Definition:** A function f has a *local maximum* (or *relative maximum*) at c if $f(c) \geq f(x)$ when x is near c.[72] The number $f(c)$ is called the *local maximum value* of f. Similarly, f has a *local minimum* (or *relative minimum*) at c if $f(c) \leq f(x)$ when x is near c.[73] The number $f(c)$ is called the *local minimum value* of f. The local minimum and local maximum values of f are called *local extreme values* of f.

3. **Extreme Value Theorem:** If f is continuous on a closed interval $[a,\ b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,\ b]$.

4. **Fermat's Theorem:** If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$

5. **Definition:** A *critical number* of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

6. **Theorem:** If f has a local maximum or minimum at c, then c is a critical number of f.

7. **Rolle's Theorem:** Let f be a function that satisfies the following three hypothesis:
   (a) f is continuous on the closed interval $[a,\ b]$.
   (b) f is differentiable on the open interval $(a,\ b)$.
   (c) $f(a) = f(b)$.

[72] The c must be in some open interval containing c.
[73] Again, the c must be in some open interval containing c.

Then there is a number c in $(a,\ b)$ such that $f'(c) = 0$.

8. **Mean Value Theorem:** Let f be a function that satisfies the following two hypothesis:

   (a) f is continuous on the closed interval $[a,\ b]$.

   (b) f is differentiable on the open interval $(a,\ b)$.

   Then there is a number c in $(a,\ b)$ such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

9. **Closed Interval Method:** To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,\ b]$:

   (a) Find the values of f at the critical numbers of f in the open interval $(a,\ b)$.

   (b) Find the values of f at the endpoints of the interval.

   (c) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

25.3 Examples

1. Find the critical points of $f(x) = x^3 - 9x^2 + 24x - 17$.

   Solution: We'll discuss this in class.

   Final answer: $c = 2$ and $c = 4$.

2. Find all critical points of the function.

   $$f(x) = x^{-2} - x^{-3}$$

   Solution: We'll discuss this in class. I suggest that you also graph this function and look at it!

   Final answer: $c = 3/2$.

3. Find all critical points of the function.

   $$f(x) = \frac{1}{x-8} - \frac{1}{x}$$

Solution: We'll discuss this in class. I suggest that you also graph this function and look at it!

Final answer: $c = 4$.

4. Find the critical points of $f(x) = |x^2 - 1|$.

Solution: We'll discuss this in class. I suggest that you also graph this function and look at it!

Final answer: $c = -1$, $c = 0$ and $c = 1$.

5. Use the Closed Interval Method to find the absolute maximum and absolute minimum values of $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1,\ 4]$.

Solution: Although not necessary, a graph (Figure 52, page 177) is provided.

We know that f is a polynomial and is continuous, and we're given a closed interval.

- $f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1)$, the critical numbers are $x = 1$ and $x = 3$ and they are in the open interval $(-1,\ 4)$. $f(1) = 6$ and $f(3) = 2$.
- $f(-1) = -14$ and $f(4) = 6$.
- The absolute maximum value of f is 6; the absolute minimum value of f is -14.

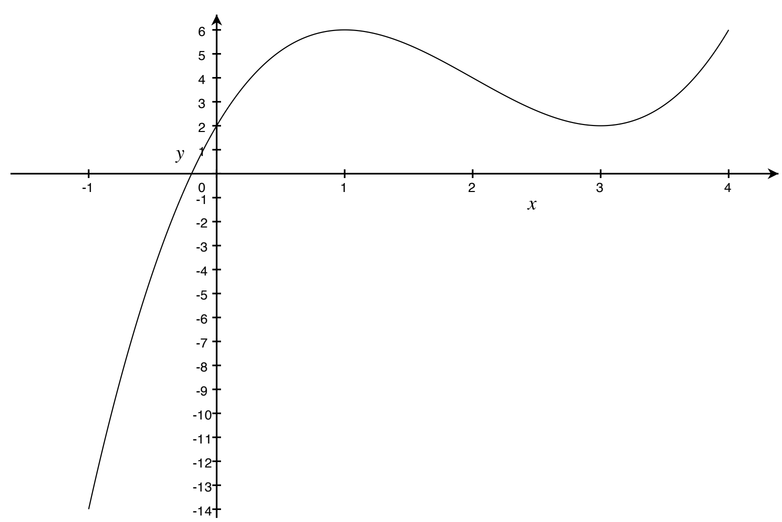

Figure 52: Complete graph of $f(x) = x^3 - 6x^2 + 9x + 2$ on $[-1,\ 4]$.

6. Verify that $f(x) = \dfrac{x}{x+2}$, on the interval $[1,\ 4]$ satisfies all two conditions of the Mean Value Theorem and then find the value for c.

Solution: First of all, $f(x)$ is continuous on its domain which includes the interval $[1,\ 4]$. (It's a rational function whose domain is all real numbers except -2 and is differentiable everywhere in its domain.) Its derivative is

$$f'(x) = \frac{2}{(x+2)^2}.$$

Then there is a number c in $(a,\ b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Making the substitutions and solving for c, we have.

$$\begin{aligned}
\frac{2}{(c+2)^2} &= \frac{f(4) - f(1)}{4-1} \\
\frac{2}{(c+2)^2} &= \frac{1}{9} \\
18 &= (c+2)^2 \\
\pm 3\sqrt{2} &= c+2 \\
\pm 3\sqrt{2} - 2 &= c
\end{aligned}$$

However we only have one value for c that's in the interval, that is $c = 3\sqrt{2} - 2$. Here's a graph (Figure 53, page 178).

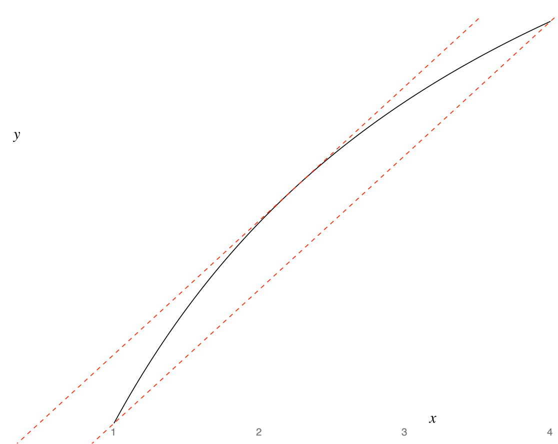

Figure 53: The graph of $f(x)$ and the tangent line at $x = 3\sqrt{2} - 2$.

It should be further noted that the line drawn between the endpoints is parallel to the tangent line. You should be clear about which one of these two lines is tangent to f.

7. Find the minimum and maximum values of the function $f(x) = \sin x + \cos^2 x$ on the interval $[0,\ 2\pi]$.

Solution: We'll discuss this in class.

The function f takes on a maximum at: $x = \pi/6$ and $x = 5\pi/6$. The function f takes on a minimum at: $x = 3\pi/2$. Graphing will help!

8. Verify that $f(x) = x^3 - 3x^2 + 2x + 5$, on the interval $[0,\ 2]$ satisfies all three conditions of Rolle's Theorem and then find the value for c.

Solution: First of all, $f(x) = x^3 - 3x^2 + 2x + 5$ is continuous everywhere,[74] so it is continuous on the interval $[0,\ 2]$. Polynomials are differentiable everywhere, and its derivative is $f'(x) = 3x^2 - 6x^2 + 2$. Evaluating f at its endpoints we have $f(0) = f(2) = 5$. To find the $c \in (0,\ 2)$ we need to solve

$$\begin{aligned} f'(x) &= 0 \\ 0 &= 3x^2 - 6x^2 + 2 \\ x &= \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{3 \pm \sqrt{3}}{3} \end{aligned}$$

Both answers are on the interval, so we found two values for c. Here's a graph (Figure 54, page 179).

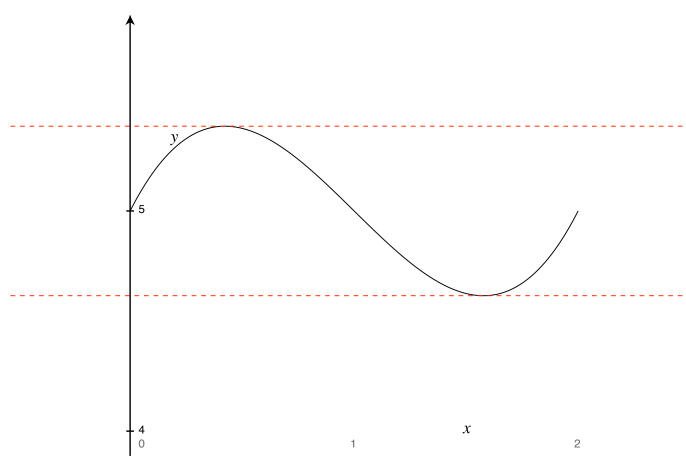

Figure 54: The graph of $f(x)$ and the two tangent lines at $x = \dfrac{3 \pm \sqrt{3}}{3}$.

25.4 Assignment

You should read §4.2 and do the WebAssign assignment mth.121.04.02.

26 mth.121.04.03

26.1 Let's Continue ...

Given the following graph (Figure 55, page 180).

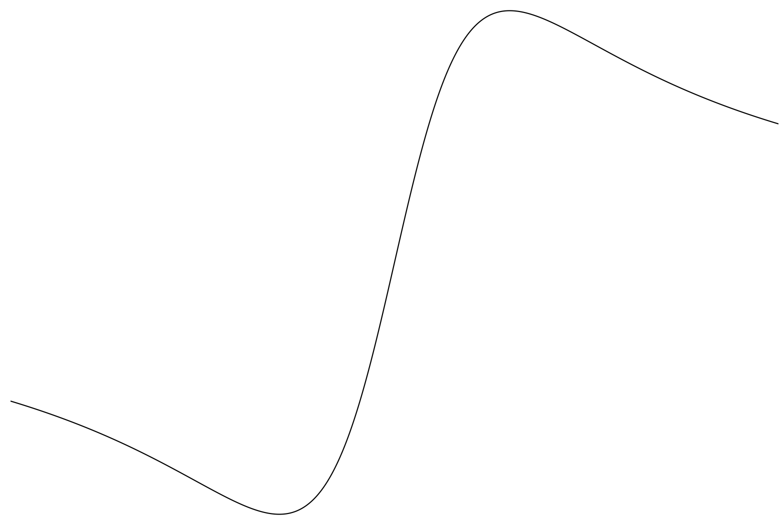

Figure 55: Partial graph of $f(x) = \dfrac{x}{x^2+9}$.

Answer the following questions.

1. What is the domain?

   Solution: All real numbers, that is $\mathbb{R}$.

2. Are there any x-intercepts?

   Solution: Yes, $f(x) = 0$ when $x = 0$, so the x-intercept is $(0,\ 0)$.

3. Are there any y-intercepts?

   Solution: Yes, $f(0) = 0$, so the y-intercept is $(0,\ 0)$.

4. Is there any symmetry?

   Solution: Yes, since $f(x) = -f(-x)$ the symmetry is *odd*.

5. Find and simplify $f'(x)$.

Solution:

$$
\begin{aligned}
f'(x) &= \frac{(x^2+9) - x(2x)}{(x^2+9)^2} \\
&= \frac{9 - x^2}{(x^2+9)^2} \\
&= \frac{(3-x)(3+x)}{(x^2+9)^2}
\end{aligned}
$$

6. Find the point(s) on f that are local extrema.

Solution: Simple sign analysis along with looking at the graph is all one needs to do. The global maximum is $(3,\ 1/6)$, and the global minimum is $(-3,\ -1/6)$.

7. Find the interval(s) where f is increasing.

Solution: $(-3,\ 3)$

8. Find the interval(s) where f is decreasing.

Solution: $(-\infty,\ -3) \cup (3,\ \infty)$

9. Find and simplify $f''(x)$.

Solution: It's getting tougher, but please make sure that your first derivative is correct or you're wasting your time.

$$
\begin{aligned}
f''(x) &= \frac{(x^2+9)^2(-2x) - 2(x^2+9)(2x)(9-x^2)}{(x^2+9)^4} \\
&= \frac{2x(x^2+9)\left[(x^2+9)(-1) - 2(9-x^2)\right]}{(x^2+9)^4} \\
&= \frac{2x(x^2-27)}{(x^2+9)^3}
\end{aligned}
$$

10. Find the inflection point(s) on f.

Solution: Simple sign analysis along with looking at the graph is all one needs to do. The points of inflection are: $\left(-3\sqrt{3},\ -\sqrt{3}/12\right)$, $(0,\ 0)$, and $\left(3\sqrt{3},\ \sqrt{3}/12\right)$.

11. Find the interval(s) where f is concave up.

Solution: $\left(-3\sqrt{3},\ 0\right) \cup \left(3\sqrt{3},\ \infty\right)$

12. Find the interval(s) where f is concave down.

Solution: $\left(-\infty,\ -3\sqrt{3}\right) \cup \left(0,\ 3\sqrt{3}\right)$

26.2 Let's Summarize

What Derivatives Tell Us About a Function and its Graph

- If $f' > 0$ on an interval, then f is *increasing* on that interval.
- If $f' < 0$ on an interval, then f is *decreasing* on that interval.
- If $f'' > 0$ on an interval, then f is *concave up* on that interval.
- If $f'' < 0$ on an interval, then f is *concave down* on that interval.

Local Maxima and Minima

Suppose a is a point in the domain of f

- f has a *local minimum* at a if $f(a)$ is less than or equal to values of f for points near a. Near a means that there needs to be points to the left and right of a.
- f has a *local maximum* at a if $f(a)$ is greater than or equal to values of f for points near a. Again, near a means that there needs to be points to the left and right of a

Finding Local Extrema

For any function f, a point a in the domain of f where $f'(a) = 0$ or $f'(a)$ is undefined is called a *critical point* of the function. In addition, the point $(a,\ f(a))$ on the graph of f is also called a *critical point*.

The first-derivative test for local maxima and minima. Suppose a is a *critical point* of a continuous function f.

- If f' changes from negative to positive at a, then f has a local minimum at a.
- If f' changes from positive to negative at a, then f has a local maximum at a.

The second-derivative test for local maxima and minima. Suppose a is a *critical point* of a continuous function f.

- If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a.
- If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at a.
- If $f'(a) = 0$ and $f''(a) = 0$, then the test tells us nothing.

Concavity and Points of Inflection

A point at which the graph of f changes concavity is called an *inflection point*.

26.3 Examples

1. Use the first derivative to find the minimum and maximum of f, indicate if these are absolute/global or relative/local. Also indicate the intervals on which f is increasing or decreasing.

$$f(x) = x^2 + (10 - x)^2$$

Solution: We'll discuss this in class.

Final answer: The critical number is $c = 5$. We have a global minimum at $x = 5$. f is increasing on $(5,\ \infty)$, and decreasing on $(-\infty,\ 5)$.

2. Use the first derivative to find the minimum and maximum of f, indicate if these are absolute/global or relative/local. Also indicate the intervals on which f is increasing or decreasing.

$$f(x) = x^{5/2} - x^2, \quad x > 0$$

Solution: We'll discuss this in class.

Final answer: The critical number is $c = 16/25$. We have a global minimum at $x = 16/25$. f is increasing on $(16/25,\ \infty)$, and decreasing on $(0,\ 16/25)$.

3. Use the first derivative to find the minimum and maximum of f, indicate if these are absolute/global or relative/local. Also indicate the intervals on which f is increasing or decreasing.

$$f(x) = \frac{x^3}{x^2 - 3}$$

Solution: We'll discuss this in class.

Final answer: The critical numbers are $c = 0$, and $c = \pm 3$. We have a local maximum at $x = -3$. We have a local minimum at $x = 3$. f is increasing on $(-\infty,\ -3) \cup (3,\ \infty)$, and decreasing on $\left(-3,\ -\sqrt{3}\right) \cup \left(-\sqrt{3},\ 0\right) \cup \left(0,\ \sqrt{3}\right) \cup \left(\sqrt{3},\ 3\right)$.

4. Use the first derivative to find the minimum and maximum of f, indicate if these are absolute/global or relative/local. Also indicate the intervals on which f is increasing or decreasing.

$$f(x) = \sin x + \sqrt{3}\cos x$$

Solution: We'll discuss this in class.

Final answer: The critical numbers are $c = \pi/6 + \pi k,\ k \in \mathbb{Z}$. We have a global maximums at $x = \pi/6 + 2\pi k,\ k \in \mathbb{Z}$. We have a global minimum at $x = 7\pi/6 + 2\pi k$, $k \in \mathbb{Z}$. f is increasing on $(7\pi/6 + 2\pi k,\ 13\pi/6 + 2\pi k),\ k \in \mathbb{Z}.$, and decreasing on $(\pi/6 + 2\pi k,\ 7\pi/6 + 2\pi k),\ k \in \mathbb{Z}..$

5. Use the first derivative to find the minimum and maximum of f, indicate if these are absolute/global or relative/local. Also indicate the intervals on which f is increasing or decreasing.

$$f(x) = \frac{\ln x}{x}, \quad x > 0$$

Solution: We'll discuss this in class.

Final answer: The critical number is $c = e$. We have a global minimum at $x = e$. f is increasing on $(0,\ e)$, and decreasing on $(e,\ \infty)$.

6. Find a point c satisfying the conclusion of the MVT for the given function and interval.

$$y = \sqrt{x-1}, \quad [5,\ 26]$$

Solution: We'll discuss this in class.

Final answer: $c = 53/4$, and the point is $(53/4,\ 7/2)$

7. Show using the Intermediate Value Theorem and Rolle's Theorem that

$$f(x) = x^7 + 4x^5 + 3x + 5$$

has exactly one real root and that root is between -1 and 0.

Solution: Since $f(x)$ is continuous everywhere and $f(-1) = -3 < 0$ and $f(0) = 5 > 0$ we know that by the Intermediate Value Theorem that a root exists between -1 and 0, that is, there exists at least one $c \in (-1,\ 0)$ such that $f(c) = 0$.

Now, the derivative of $f(x)$ is $f'(x) = 7x^6 + 20x^4 + 3$ is always positive. So now assume that two roots exists, $f(a) = f(b) = 0$, where $a < b$, since the polynomial is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$, Rolle's Theorem implies that there is a number $r \in (a,\ b)$ such that $f'(r) = 0$, but that's not possible because $f'(x) = 7x^6 + 20x^4 + 3$ is always positive. This contradiction shows that $f(x)$ cannot have two real roots. Since we've already shown the existence of a root, the conclusion is that there's only one real root. Here's the graph (Figure 56, page 185).

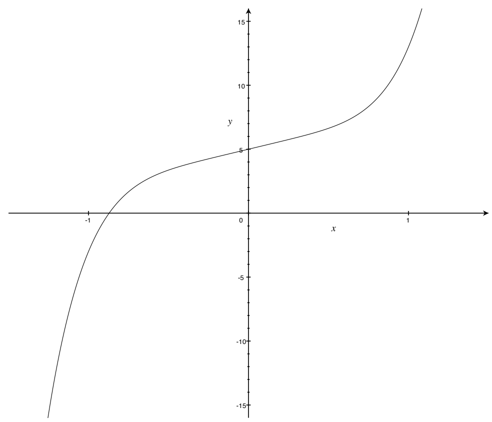

Figure 56: Partial graph of $f(x) = x^7 + 4x^5 + 3x + 5$.

26.4 Assignment

You should read §4.3 and do the WebAssign assignment mth.121.04.03.

27 mth.121.04.04

27.1 Let's Continue ...

God is in the details! Or, how the *devil* is revealed in the calculus! Here's a graph (Figure 57, page 186) that has many details that are hard to appreciate. The graph of this function,

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5},$$

is properly scaled, but with no references to tag it down.

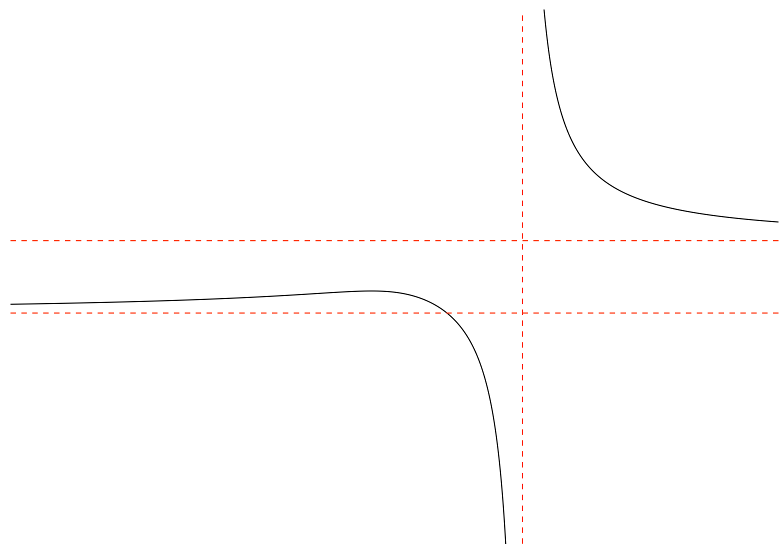

Figure 57: A lone graph of $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$.

1. Are there any x-intercepts?

Solution: Well, $2x^2 + 1$ is never zero if we're restricted to using real numbers, so $f(x)$ can not be zero. That is, no x-intercepts.

2. Are there any y-intercepts?

Solution: Just set $x = 0$, and you'll get $(0,\ -1/5)$.

3. A vertical asymptote is indicated, what is its equation?

Solution: Easy, at least I think so. Just set $3x-5$ to zero and solve for x. Of course you'll get $x=5/3$. Remember that division by zero is a *mortal sin* and any sensible graph will avoid it like the plague. You should also look at the following two limits.

$$\begin{aligned}\lim_{x\to 5/3^+}\frac{\sqrt{2x^2+1}}{3x-5} &= \infty\\ \lim_{x\to 5/3^-}\frac{\sqrt{2x^2+1}}{3x-5} &= -\infty\end{aligned}$$

4. Two horizontal asymptotes are indicated, what are their equations?

Solution: Clearly they are different. We'll need to take limits as $x\to\pm\infty$.

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5} &= \frac{\sqrt{2}}{3}\\ \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5} &= -\frac{\sqrt{2}}{3}\end{aligned}$$

So the two vertical asymptotes are:

$$y=-\frac{\sqrt{2}}{3}\qquad\text{and}\qquad y=\frac{\sqrt{2}}{3}.$$

5. Find the point(s) on f that are local extrema. But first verify that

$$f'(x)=-\frac{10x+3}{(3x-5)^2\sqrt{2x^2+1}}.$$

Solution: You need to spend *time* verifying this derivative. It's tough, but your algebra skills will improve by doing these verifications. However, you should also consider using technology to do these derivatives—and then do the hand work. The only candidate for extrema is $x=-3/10$. The derivative on $(-\infty,\ -3/10)$ is always positive, and the derivative on $(-3/10,\ 5/3)$ is always negative. Certainly the graph, although not tagged down to an axis is indicating this, so we clearly have a local maximum at

$$\left(-\frac{3}{10},-\sqrt{\frac{2}{59}}\right)$$

6. Find the interval(s) where f is increasing.

Solution: The derivative on $(-\infty,\ -3/10)$ is always positive, that is where f is increasing.

7. Find the interval(s) where f is decreasing.

Solution: The derivative on $(-3/10,\ 5/3) \cup (5/3,\ \infty)$ is always negative, that is where f is decreasing.

8. Find the inflection point(s) on f. But first verify that

$$f''(x) = \frac{2\left(60x^3 + 27x^2 + 34\right)}{(3x-5)^3\left(2x^2+1\right)^{3/2}}.$$

Solution: You need to spend *time* verifying this derivative, and it's much more difficult than getting the first derivative. Again, you should also consider using technology to do these derivatives—and then do the hand work. The only candidate for extrema is the solution(s) of $60x^3 + 27x^2 + 34 = 0$—not easy to do and I suggest that you use your calculator to solve. You should be able to verify that the only real solution is approximately $x \approx -1.00786$. You can also analyze the second derivative to verify that it changes sign at this value, so it is a POI and its approximate value is:

$$(-1.00786,\ f(-1.00786)).$$

27.2 Examples

1. Find the points of inflection of the function.

$$f(x) = \frac{x^4}{12} - \frac{x^2}{2} + \frac{3x}{5} - \frac{9}{7}$$

(a) Point(s)-of-inflection.

Solution: We'll discuss this in class.

Final answer: There's two and they occur at $x = \pm 1$.

(b) Determine the intervals on which the function is concave up or concave down. (Enter your answers using interval notation.)

(a) Concave up.

Solution: We'll discuss this in class
Final answer: $(-\infty,\ -1) \cup (1,\ \infty)$

(b) Concave down.

Solution: We'll discuss this in class
Final answer: $(-1,\ 1)$

2. Find the points of inflection of the function.

$$f(\theta) = 7\theta + 7\cos^2\theta, \quad [0,\ 2\pi]$$

Solution: We'll discuss this in class.

A graph (Figure 58, page 189) is provided and I suggest that you do this on your own. Final answer: There's four and they occur at $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$, and $\theta = 7\pi/4$. I've drawn red dashed lines where the concavity is changing.

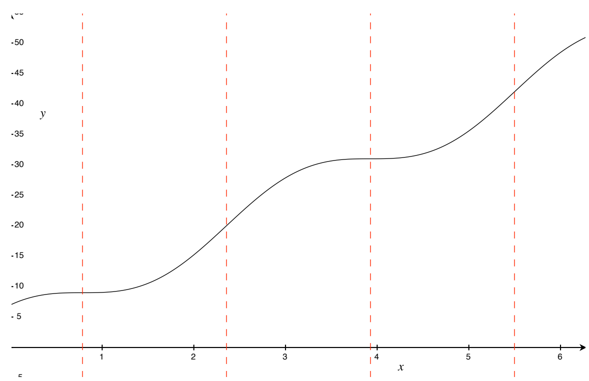

Figure 58: $f(\theta) = 7\theta + 7\cos^2\theta, \quad [0,\ 2\pi]$

3. Find the points of inflection of the function.

$$f(\theta) = \frac{x}{1+x^2}$$

Solution: We'll discuss this in class.

Final answer: You should be able to easily find

$$f''(\theta) = \frac{2x\left(x^2-3\right)}{\left(1+x^2\right)^3},$$

and the verify that the POIs occur at $x = \pm\sqrt{3}$ and $x = 0$. Here's a graph (Figure 59, page 190) of f.

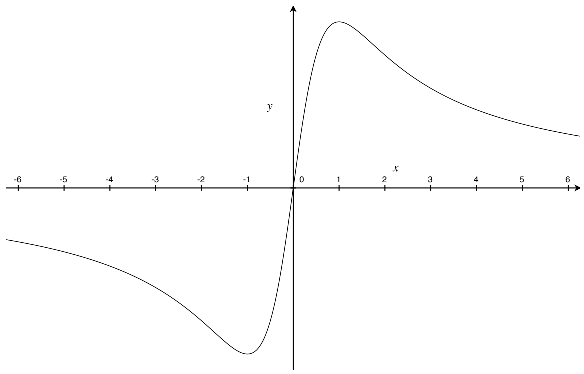

Figure 59: $f(x) = \dfrac{x}{1+x^2}$

4. Find the points of inflection of the function.

$$f(\theta) = x^2 \ln x, \quad x > 0$$

Solution: We'll discuss this in class.

Final answer: You should be able to easily find

$$f''(\theta) = 2\ln x + 3,$$

and the verify that the POI occur at $x = e^{-3/2}$. Here's a graph (Figure 60, page 190) of f.

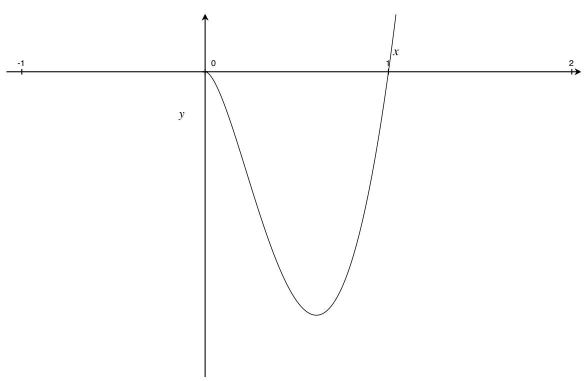

Figure 60: $f(\theta) = x^2 \ln x, \quad x > 0$

5. The following graph (Figure 61, page 191) is given as a guide.

   Answer the following questions about f.

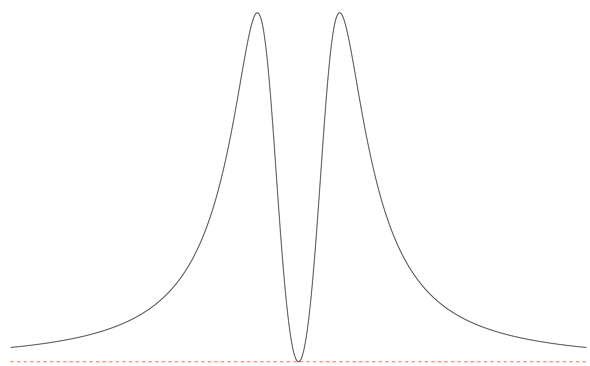

Figure 61: Partial graph of $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$.

(a) Are there any x-intercepts?

Solution: The function can not equal zero when x is restricted to the real numbers. So we can safely say there are none.

(b) Are there any y-intercepts?

Solution: Just set $x = 0$ and you'll get $(0,\ 2)$.

(c) A horizontal asymptote is indicated, what is its equation?

Solution: Clearly one is indicated by the red dashed line. We'll need to take limits as $x \to \pm\infty$.

$$\begin{aligned} \lim_{x\to\infty} f(x) = 2 + \frac{x^2}{x^4+1} &= 2 \\ \lim_{x\to-\infty} f(x) = 2 + \frac{x^2}{x^4+1} &= 2 \end{aligned}$$

So we have $y = 2$.

(d) Find the point(s) on f that are local extrema. But first verify that

$$f'(x) = -\frac{2x\left(x^4-1\right)}{\left(x^4+1\right)^2}.$$

Solution: Yes, you should be able to verify this. If you can't, you need to review algebra. Now, to find extrema, we'll first need to find the critical points. Since $x^4 + 1$ is never zero, I just need to solve $2x\left(x^4 - 1\right) = 0$, which easily yields $\{-1, 0, 1\}$. Simple sign analysis of the first derivative, tells us that the function is increasing on $(-\infty,\ -1) \cup (0,\ 1)$ and is decreasing on $(-1,\ 0) \cup (1,\ \infty)$.

The minimum point is global and is $(0,\ 2)$.
The maximum points (there are two) are global and are $(-1,\ 2.5)$ and $(1,\ 2.5)$.

(e) Find the interval(s) where f is increasing.

Solution: From above, simple sign analysis of the first derivative, tells us that the function is increasing on $(-\infty,\ -1) \cup (0,\ 1)$.

(f) Find the interval(s) where f is decreasing.

Solution: From above, simple sign analysis of the first derivative, tells us that the function is decreasing on $(-1,\ 0) \cup (1,\ \infty)$.

(g) Find the inflection point(s) on f. But first verify that

$$f''(x) = \frac{2\left(3x^8 - 12x^4 + 1\right)}{\left(x^4+1\right)^3}.$$

Solution: Yes, you should be able to verify this. If you can't, you need to review algebra ... *AGAIN*! Now, to find points of inflection, we'll first need to find where the second derivative is zero or undefined. Since $x^4 + 1$ is never zero, I just need to solve $3x^8 - 12x^4 + 1 = 0$, which is a eighth-degree polynomial. I suggest that making a substitution, $u = x^4$, is helpful and is exactly what was presented in MTH119.

$$\begin{aligned} u &= x^4 \\ 3x^8 - 12x^4 + 1 &= 0 \\ 3u^2 - 12u + 1 &= 0 \\ u &= \frac{6 \pm \sqrt{33}}{3} \\ x &= \pm\sqrt[4]{\frac{6 \pm \sqrt{33}}{3}} \end{aligned}$$

There's actually four inflection points and they are of the form:

$$\left(\pm\sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}, f\left(\pm\sqrt[4]{\frac{6 \pm \sqrt{33}}{3}} \right) \right)$$

(h) Find the interval(s) where f is concave up.

Solution: Okay, things are getting *nasty*. But there's point to be made, and that's exactly what I am trying to get across here. Now is probably the time to be using technology to help. So please consider learning whatever technology you have access to. For example, calculators are quite capable of solving equations, finding derivatives, and creating graphs. Furthermore, ECC provides access to Mathematica in the computer labs on the third floor.

One of the greatest movies ever made (*2001: A Space Odyssey*), in my humble opinion, had a religious sub-theme that actually made the point that our evolution as a species depends on the intelligent use of tools. Essentially those that can't adapt to a changing tool-set are doomed ... so how does one learn a tool? Some watch in awe, but just take a look at the beginning of *2001: A Space Odyssey* to see a prime example that watching others use tools to their advantage is a very bad strategy. If anyone is interested, I have a DVD copy of *2001: A Space Odyssey* for anyone who wants to watch. As Clarke wrote in 1972: "Quite early in the game I went around saying, not very loudly, 'MGM doesn't know this yet, but they're paying for the first $10,000,000 religious movie.' "

Okay, finally the answer is:

$$\left(-\infty,\ -\sqrt[4]{\frac{6+\sqrt{33}}{3}}\right) \cup \left(-\sqrt[4]{\frac{6-\sqrt{33}}{3}},\ \sqrt[4]{\frac{6-\sqrt{33}}{3}}\right) \cup \left(\sqrt[4]{\frac{6+\sqrt{33}}{3}},\ \infty\right).$$

(i) Find the interval(s) where f is concave down.

Solution: Enough talk already! Here's the answer:

$$\left(-\sqrt[4]{\frac{6+\sqrt{33}}{3}},\ -\sqrt[4]{\frac{6-\sqrt{33}}{3}}\right) \cup \left(\sqrt[4]{\frac{6-\sqrt{33}}{3}},\ \sqrt[4]{\frac{6+\sqrt{33}}{3}}\right).$$

27.3 Assignment

You should read §4.4 and do the WebAssign assignment mth.121.04.04.

28 mth.121.04.05

28.1 l'Hôpital's Rule

Guillaume de l'Hôpital, a French mathematician, published (1696) the first successful differential calculus book, *Analyse des Infiniment Petits*, which I believe literally translates to: *Analysis of the Infinitely Small.* However, he is better known for a rule that bears his name, but the rule was actually the work of a Swiss mathematician, named Johann Bernoulli. Let's take a look ...

28.2 An Example Limit

Suppose you are asked to compute the following limit.

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

Substituting in $x = 0$ gives $0/0$, which is undefined and cancelations are not algebraically obvious as there were before. Let's instead look at the graph (Figure 62, page 194) and try to make a guess. Clearly we know that the function is not defined at zero, but to the left and

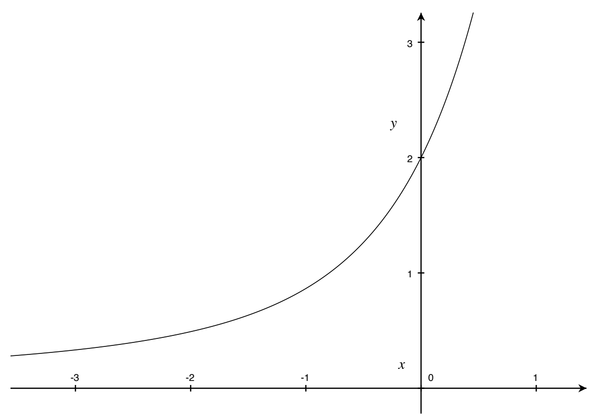

Figure 62: $f(x) = \dfrac{e^{2x} - 1}{x}$

right of zero it appears (see graph) that the function is going towards 2 as x gets *close* to 0. You may recall a similar trigonometric (cancelations are not algebraically obvious) limit that that you've seen before.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

It was a rather long and lengthy geometric argument that showed that limit exists and is 1. Geometric arguments are nice, and l'Hôpital's rule may best be understood by looking at the geometry, but in a rather different way.

In a geometric sense we are going to return to using derivatives to create local lines that fit the numerator and denominator at the point of interest. For example, the numerator $e^{2x} - 1$ looks very much like

$$y = 2x$$

when $x = 0$, and of course the denominator x, always looks like a line. So when we're near zero, the numerator and denominator ratio becomes approximately the same as

$$\frac{e^{2x} - 1}{x} \approx \frac{2x}{x} = \frac{2}{1}.$$

You should note that if we let

$$f(x) = e^{2x} - 1, \quad f'(x) = 2e^{2x}$$

and

$$g(x) = x, \quad g'(x) = 1$$

we have

$$f'(0) = 2 \qquad \text{and} \qquad g'(0) = 1.$$

For this example it should be clear that:

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2$$

Here's a visual (Figure 63, page 195) that will hopefully clarify the relationship between $y = 2x$ and $y = e^{2x} - 1$. It should be clear that these two graphs are nearly identical near the origin.

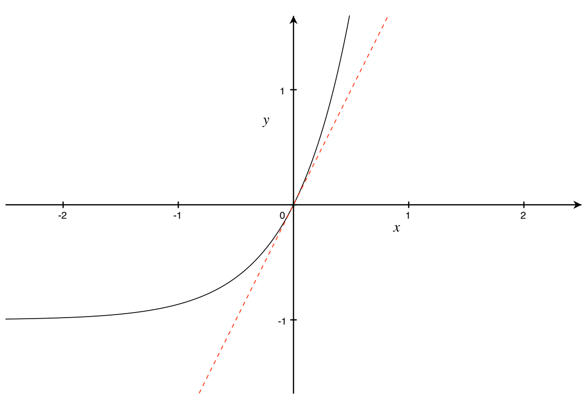

Figure 63: $y = e^{2x} - 1$ and $y = 2x$.

l'Hôpital's rule: If f and g are differentiable, $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Using the definition of derivative, here's a justification of l'Hôpital's rule.

$$\begin{aligned}
\frac{f'(a)}{g'(a)} &= \frac{\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}}{\lim\limits_{h\to 0}\frac{g(a+h)-g(a)}{h}} \\
&= \frac{\lim\limits_{h\to 0} f(a+h)-f(a)}{\lim\limits_{h\to 0} g(a+h)-g(a)} \\
&= \frac{\lim\limits_{h\to 0} f(a+h)}{\lim\limits_{h\to 0} g(a+h)} \\
&= \lim_{x\to a}\frac{f(x)}{g(x)}
\end{aligned}$$

A more general form of l'Hôpital's rule is: If f and g are differentiable, $f(a) = g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the right hand limit exists.

l'Hôpital's rule also applies when f and g limits at a are $\pm\infty$. Here the rule is stated as: provided f and g are differentiable:

- When $x \to a$, both f and g go towards $\pm\infty$, or
- When $a = \pm\infty$ and as $x \to a$, both f and g go towards $\pm\infty$.

  $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

  where $a = \pm\infty$, provided the right hand limit exists.

These forms are often referred to as indeterminate forms. Here's the list:

- $\boxed{\text{Indeterminate form of type } 0/0.}$
- $\boxed{\text{Indeterminate form of type } \infty/\infty.}$
- Indeterminate form of type $0 \cdot \infty$.
- Indeterminate form of type $\infty - \infty$.
- Indeterminate form of type 0^0.
- Indeterminate form of type ∞^0.
- Indeterminate form of type 1^∞.

Here's some general notes that most calculus textbooks will state:

Note 1 l'Hôpital's rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied.

Note 2 l'Hôpital's rule is also valid for one-sided limits and for limits at $\pm$ infinity.

28.3 Examples

1. For students planning to major in mathematics, please carefully look over the proof of l'Hôpital's rule in any calculus textbook.

   Solution: Again, you need to *read* about mathematics if you ever plan to move forward. Reading and writing are perhaps the strongest glue of any civilization—and many civil societies have collapsed due to their inability to use language.

2. Show using l'Hôpital's rule that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

   Solution: It's pretty typical to use the following notation when using l'Hôpital's rule.

   $$\lim_{x\to 0} \frac{\sin x}{x} \stackrel{H}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

3. Show using l'Hôpital's rule that $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1} = 1$.

   Solution: We'll do the work in class.

4. Show using l'Hôpital's rule that $\lim\limits_{x\to\infty} xe^{-x} = 0$.

   Solution: First you'll need to rewrite xe^{-x} as

   $$\frac{x}{e^x}$$

   We'll do the remaining work in class.

5. Show using l'Hôpital's rule that $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \infty$.

   Solution: We'll do the work in class.

6. Show using l'Hôpital's rule that $\lim\limits_{x\to\infty} \dfrac{5x + e^{-x}}{7x} = \dfrac{5}{7}$.

   Solution: We'll do the work in class.

7. Don't use l'Hôpital's rule but show that $\lim\limits_{x\to\infty} \dfrac{x^2 + \sin x}{x^2} = 1$. Now try using l'Hôpital's rule to see what happens.

Solution: We'll do the work in class. You should make special note that the limit is reasonable and initially looks *easy* to do using l'Hôpital's rule, but l'Hôpital's rule fails.

8. Show using l'Hôpital's rule that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Solution: You'll need to rewrite first!

$$\begin{aligned} y &= \left(1+\frac{1}{x}\right)^x \\ \ln y &= x \cdot \ln\left(1+\frac{1}{x}\right) \\ y &= e^{\frac{\ln\left(1+x^{-1}\right)}{x^{-1}}} \end{aligned}$$

We'll do the remaining work in class.

9. Show using l'Hôpital's rule that $\lim_{x\to 0}\frac{1-\cosh(3x)}{x} = 0$.[75]

Solution: We'll do the work in class.

10. I think these are two difficult problems, and I think everyone should give them a try.

 (a) $\lim_{x\to 0^+}\left[\sin\left(\frac{\pi}{2}+x\right)\right]^{\frac{1}{x}} = 1$

 Solution: Rewrite!

$$\begin{aligned} y &= \left[\sin\left(\frac{\pi}{2}+x\right)\right]^{\frac{1}{x}} \\ \ln y &= \frac{\ln\left[\sin\left(\frac{\pi}{2}+x\right)\right]}{x} \\ y &= e^{\frac{\ln[\sin(\pi/2+x)]}{x}} \end{aligned}$$

 We'll do the remaining work in class.

 (b) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \frac{1}{2}$

[75] Here's the definition of this hyperbolic function.

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

These functions may have been covered in a prior mathematics course and are also included in this guide as optional material. You *do not need* to memorize the definition of any hyperbolic function.

Solution: Rewrite!

$$\begin{aligned} y &= \frac{1}{\ln x} - \frac{1}{x-1} \\ &= \frac{x - 1 - \ln x}{(x-1)\ln x} \end{aligned}$$

We'll do the remaining work in class.

11. Marquis l'Hôpital's used this example in his textbook.

$$\lim_{x \to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}, \quad a > 0$$

Evaluate this limit.

Solution: You should look over this solution to see how simple (if you can manipulate and do the algebra) this question is. More importantly, you should note that calculus problems *haven't changed much* since l'Hôpital's times.

$$\begin{aligned} \lim_{x \to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} &\overset{H}{=} \lim_{x \to a} \frac{\frac{1}{2} \cdot \left(2a^3x - x^4\right)^{-1/2} \cdot \left(2a^3 - 4x^3\right) - \frac{a}{3} \cdot \left(a^2x\right)^{-2/3} \cdot a^2}{-\frac{1}{4} \cdot \left(ax^3\right)^{-3/4} \cdot 3ax^2} \\ &= \frac{16}{9}a \end{aligned}$$

12. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

Solution: This is a lot easier than it looks!

$$\begin{aligned} \lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x} &\overset{H}{=} \lim_{x \to 0} \frac{3f'(2 + 3x) + 5f'(2 + 5x)}{1} \\ &= 3f'(2) + 5f'(2) \\ &= 3 \cdot 7 + 5 \cdot 7 \\ &= 56 \end{aligned}$$

13. Given that $f(x) = x^{-x}$ where $x > 0$, and its graph (Figure 64, page 200).
    (a) Use l'Hôpital's rule to explain the behavior as $x \to 0^+$.

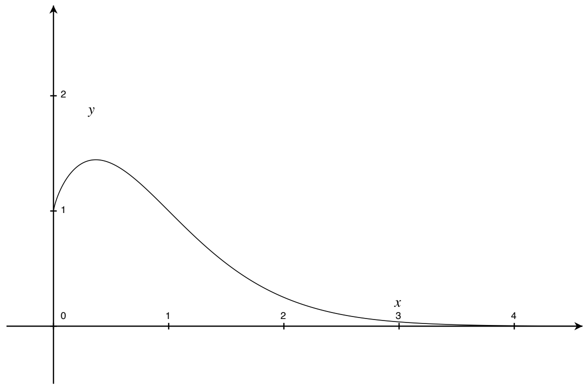

Figure 64: $f(x)$

Solution: You'll need to rewrite!

$$\begin{aligned} y &= x^{-x} \\ \ln y &= -x \ln x \\ \ln y &= -\frac{\ln x}{x^{-1}} \\ y &= e^{-\frac{\ln x}{x^{-1}}} \end{aligned}$$

We'll do the remaining work in class.

(b) Use calculus to find the maximum value of f.

Solution: We'll do the work in class. But please be clear that a graph (Figure 64, page 200) is very *helpful* and you should consider graphing this on your own even though a graph is provided. The answer is $f(e^{-1})$.

(c) What is f's range?

Solution: We'll do the work in class. Again, the graph (Figure 64, page 200) is worth looking at! The answer is $(0, \sqrt[e]{e}\,]$

28.4 Assignment

You should read §4.5 and do the WebAssign assignment mth.121.04.05.

29 mth.121.04.06

29.1 Curve Sketching

Technology is important, and I strongly suggest that you *intermingle* paper-and-pencil analysis with use of technology. Here, in this section, I will be pointing out *when* and *when not* to use technology. You are required to use paper-and-pencil to answer exam questions, but you'll still need to learn how to use a computer if you plan to do real world problems.

29.2 Examples

1. Answer (a) through (d) below.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5.$$

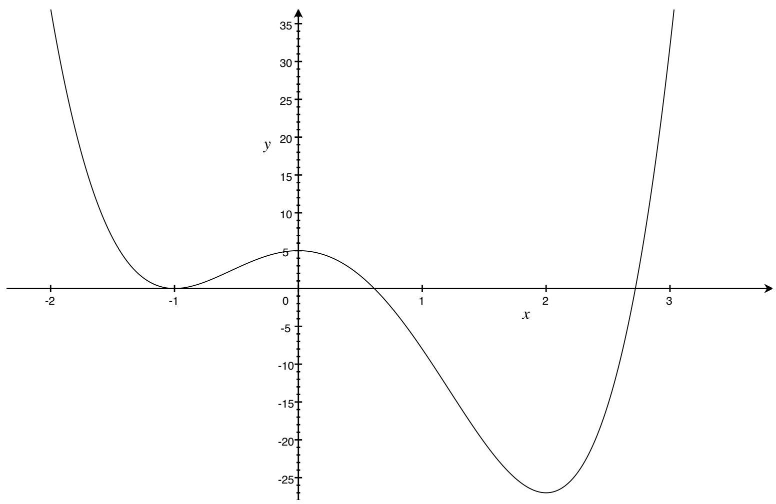

Figure 65: Partial graph of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution: Before proceeding with any *curve sketching* problem you should get the first and second derivative. I would also like to suggest that you use your calculators for more difficult problems. Even if you have to *show work*, it sure is nice to be able to use technology first and then proceed with paper-and-pencil analysis. A graph (Figure 65, page 201) is provided as a guide, and you should be able to do this on your own.

You'll need the first derivative

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2),$$

and where it changes sign, clearly at $x = -1$, $x = 0$ and $x = 2$.

And you'll also need the second derivative

$$f''(x) = 36x^2 - 24x - 24 = 12\left(3x^2 - 2x - 2\right),$$

and where it changes sign, clearly at $x = \dfrac{1 \pm \sqrt{7}}{3}$.

(a) Find the interval(s) where f is increasing.

Solution: Using the first derivative and sign analysis, $(-1,\ 0)$, $(2,\ \infty)$.

(b) Find the interval(s) where f is decreasing.

Solution: Again, using the first derivative and sign analysis $(-\infty,\ -1)$, $(0,\ 2)$.

(c) Find the interval(s) where f is concave-up.

Solution: Using the second derivative and sign analysis,
$\left(-\infty,\ \dfrac{1-\sqrt{7}}{3}\right)$, $\left(\dfrac{1+\sqrt{7}}{3},\ \infty\right)$.

(d) Find the interval(s) where f is concave-down.

Solution: Again, using the second derivative and sign analysis,
$\left(\dfrac{1-\sqrt{7}}{3},\ \dfrac{1+\sqrt{7}}{3}\right)$.

2. Given that

$$\begin{aligned} f(x) &= \sqrt{\frac{x^2+1}{x+1}} \\ f'(x) &= \frac{x^2+2x-1}{2(x+1)\sqrt{(x^2+1)(x+1)}} \end{aligned}$$

and the following partial graph of $f(x)$.

You need to *make sure* that you are capable of using your calculator/computer to create such a graph (Figure 66, page 203) *on your own*! Do not depend on a graph being provided, and you should see the calculator/computer as an aid to *learning* how calculus relates to the graph. Yes, it takes practice.

Answer each of the following questions.

(a) The domain of $f(x)$.

Solution: Here, although obvious, we must have

$$\frac{x^2+1}{x+1} \geq 0 \qquad \Rightarrow \qquad x \in (-1,\ \infty),$$

and this is certainly indicated by the partial graph of $f(x)$.

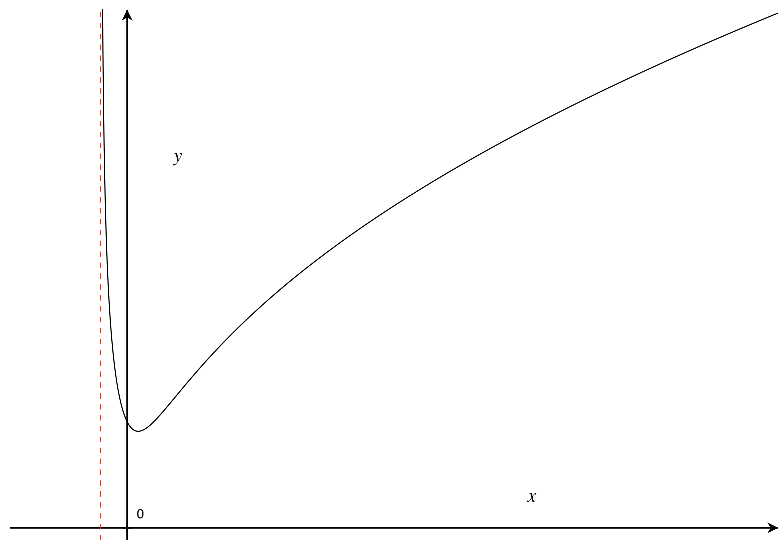

Figure 66: Partial graph of $f(x)$.

(b) The y-intercept.

Solution: Just set $x = 0$ and you'll get $(0,\ 1)$.

(c) The equation of the vertical asymptote.

Solution: As $x \to -1^+$ we have $f(x) \to \infty$ so the vertical asymptote is $x = -1$.

(d) Evaluate $\lim_{x\to\infty} f(x)$.

Solution:

$$\lim_{x\to\infty} f(x) = \infty$$

(e) The global minimum point on $f(x)$.

Solution: Here you will need to find the critical numbers in the domain of f. Looking at the derivative, the only possibility is where the numerator is zero.

$$x^2 + 2x - 1 = 0 \qquad \Rightarrow \qquad x = -1 \pm \sqrt{2}$$

Only $-1+\sqrt{2}$ is in the domain. It should be further noted that f is decreasing on $\left(-1,\ -1+\sqrt{2}\right)$ and is increasing on $\left(-1+\sqrt{2},\ \infty\right)$, therefore we have a minimum at

$$\left(\sqrt{2}-1,\ f\left(\sqrt{2}-1\right)\right),$$

or if you, or your calculator, does the work,

$$\left(\sqrt{2}-1,\ \sqrt{2\sqrt{2}-2}\right).$$

(f) The range of $f(x)$.

Solution: Using the work from above, we have

$$[f(\sqrt{2}-1),\ \infty),$$

or if you, or your calculator, does the work,

$$\left(\sqrt{2\sqrt{2}-2},\ \infty\right).$$

3. Given that

$$\begin{aligned} f(x) &= \frac{x^3-x^2-1}{x^2+1} \\ f'(x) &= \frac{x^2(x^2+3)}{(x^2+1)^2} \\ f''(x) &= \frac{2x(3-x^2)}{(x^2+1)^3} \end{aligned}$$

and the following partial graph (Figure 67, page 204) of $f(x)$.

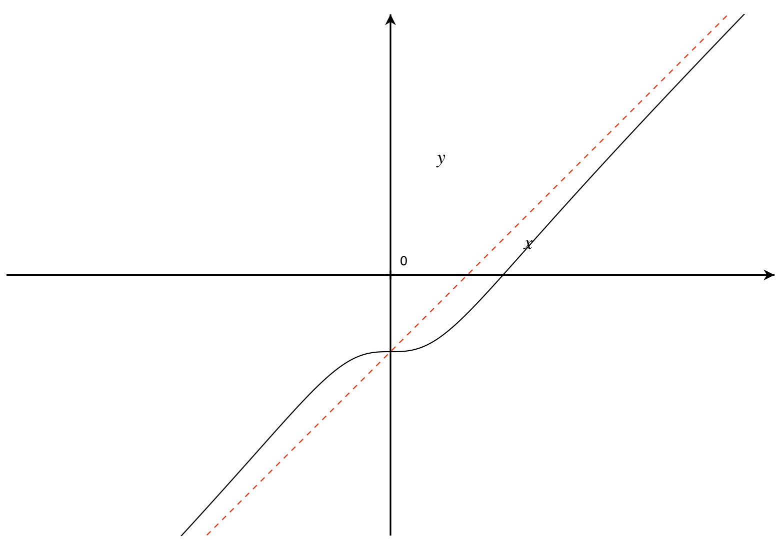

Figure 67: Partial graph of $f(x)$.

Answer each of the following questions.

(a) The domain of $f(x)$.

Solution: No troubles here, so the domain is $\mathbb{R}$.

(b) The y-intercept.

Solution: Just set $x = 0$ and you'll get $(0,\ -1)$.

(c) The equation of the slant asymptote.

Solution: You'll need to do the *dreaded long division*, and then you'll find that the slant asymptote is $y = x - 1$.

(d) Find all inflection points on $f(x)$. It's hard to see them on the graph, but they're there.

Solution: The second derivative

$$f''(x) = \frac{2x\left(3 - x^2\right)}{\left(x^2+1\right)^3},$$

is changing signs at $x = 0$, $x = \pm\sqrt{3}$. So the points of inflection are

$$\left(-\sqrt{3},\ f\left(-\sqrt{3}\right)\right), \left(0,\ f(0)\right), \left(\sqrt{3},\ f\left(\sqrt{3}\right)\right);$$

or if you prefer

$$\left(-\sqrt{3},\ -\frac{3\sqrt{3}+4}{4}\right), \left(0,\ -1\right), \left(\sqrt{3},\ \frac{3\sqrt{3}-4}{4}\right).$$

4. You should be able to look at the following graph (Figure 68, page 206) and discuss where the function is increasing, and decreasing; furthermore you should be able to determine where the derivative is positive, negative, undefined, and zero; better still, you should be able to determine where the function is concave up and concave down.

Solution: We'll briefly discuss this in class.

5. Given[76]

$$f(x) = \sqrt[3]{x}e^{-x^2},$$

and a partial graph (Figure 69, page 206) of f. You should make special note of the points indicated on the graph (Figure 69, page 206) and be aware that most calculators can easily identify these points: *min*, *max*, and *POI*.

[76]This question probably requires—at least for most students—the use of technology to answer. I (Ron Bannon) will probably be posting a video of how one would do this problem using Mathematica, but you should also check to see if your calculator can do this. Be sure to ask, if I fail to post a video and you'd like to see how it's done using technology available to students at Essex County College.

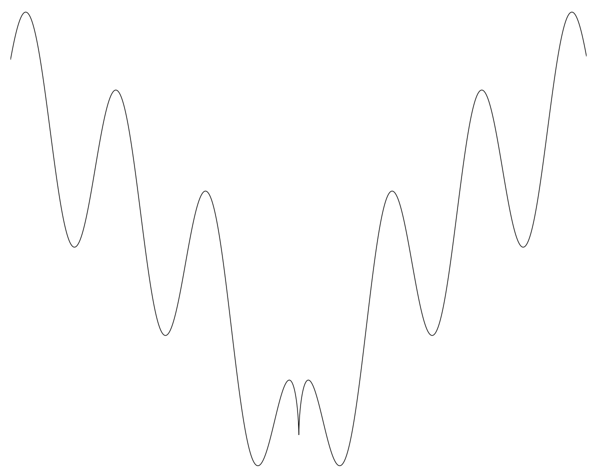

Figure 68: Partial graph of $f(x) = \sqrt{|x|} + \cos x$.

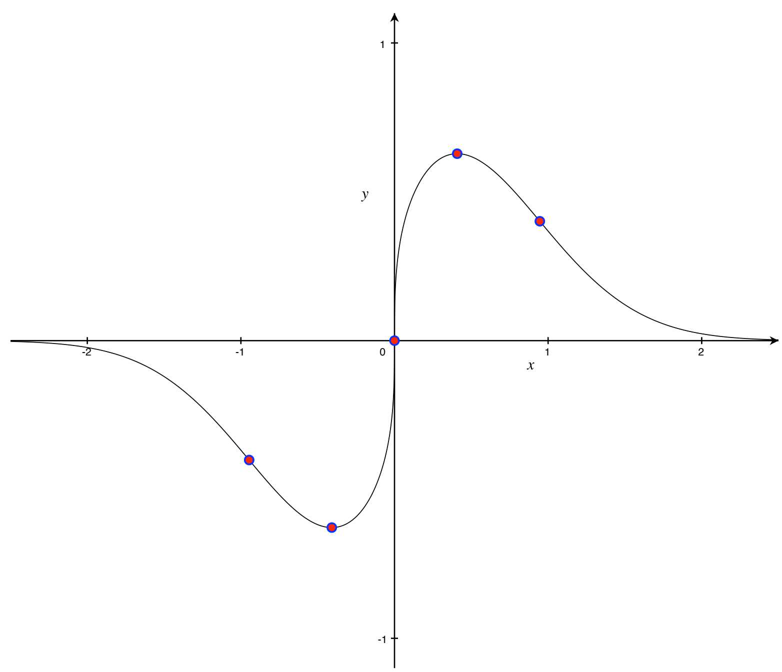

Figure 69: Partial graph of f with some important points indicated.

Answer each of the following questions. I don't need to see your work because you should be using a computer to do this work, but do not use *decimals*—exact answers only!

(a) Find and simplify f'.

Solution:

$$f'(x) = \frac{1 - 6x^2}{3e^{x^2}\sqrt[3]{x^2}}$$

(b) Find and simplify f''.

Solution:

$$f''(x) = \frac{2\left(18x^4 - 15x^2 - 1\right)}{9xe^{x^2}\sqrt[3]{x^2}}$$

(c) Find the maximal point on f.

Solution: Certainly the graph helps and it should be clear that a maximum (both global and local) occurs at:

$$\left(\frac{1}{\sqrt{6}},\ f\left(\frac{1}{\sqrt{6}}\right)\right) \quad \text{or} \quad \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt[6]{6e}}\right)$$

Approximating this point may prove helpful.

$$(0.41,\ 0.63)$$

(d) Find the minimal point on f.

Solution: Certainly the graph helps and it should be clear that a minimum (both global and local) occurs at:

$$\left(-\frac{1}{\sqrt{6}},\ f\left(-\frac{1}{\sqrt{6}}\right)\right) \quad \text{or} \quad \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt[6]{6e}}\right)$$

Approximating this point may prove helpful.

$$(-0.41,\ -0.63)$$

(e) Find the points-of-inflection on f.

Solution: Certainly the graph helps and it should be clear that a there are three points-of-inflection and they occur at the following x values:

$$-\sqrt{\frac{5+\sqrt{3}}{12}},\ 0, \sqrt{\frac{5+\sqrt{3}}{12}}$$

The points are:

$$\left(-\sqrt{\frac{5+\sqrt{3}}{12}},\ f\left(-\sqrt{\frac{5+\sqrt{3}}{12}}\right)\right),\ (0,\ 0),\ \left(\sqrt{\frac{5+\sqrt{3}}{12}},\ f\left(\sqrt{\frac{5+\sqrt{3}}{12}}\right)\right)$$

Approximating these points may prove helpful.

$$(-0.95,\ -0.40),\ (0.00,\ 0.00),\ (0.95,\ 0.40)$$

29.3 Assignment

You should read §4.6 and do the WebAssign assignment mth.121.04.06.

30 mth.121.04.07

30.1 Optimization and Modeling

Yes, we are once again back doing word problems. Using calculus to solve problems that could not have been done previously, and this time we're trying to maximize or minimize something. Usually, I'd like to think anyway, that we're trying to maximize usage, or to minimize loss.

- Well, I can't say it enough, read! Yes, try to understand the story. Read it several times if you have to.
- Since we're dealing with a variable, try to identify what is being said in the word problem with respect to this unknown. That is, they're usually trying to maximize or minimize something and it should be related to some unknown.
- Draw a picture and try to relate the variable to the picture itself.
- Now try to find a function of the variable, the function is what you're trying to optimize. The domain of the function should be clear.
- Differentiate and try to optimize using the derivative to clearly answer the question.

Example: Find the point on the parabola $y = x^2$ which is closest to the point $(3,\ 0)$.

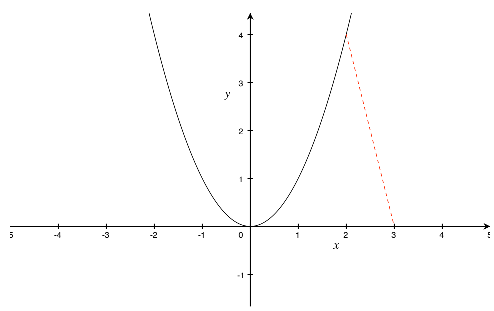

Figure 70: $y = x^2$ and a line segment to be minimized.

Solution: The work for this problem should *proceed in steps* until you become more familiar with the process. Giving the problem some thought before proceeding will usually save you time in the long run.

- Okay, I read it and I understand that I need to find a point on the parabola $y = x^2$ which is closest to the point $(3,\ 0)$. I really can't find a simpler way to say this.
- We're trying to minimize the distance between the point on the parabola, $(x,\ x^2)$, and the point $(3,\ 0)$. By the way, if I know x I know y, after-all it's a point on the parabola.

- Okay, here's a picture (Figure 70, page 209). Again, make sure you can draw your own pictures/diagrams—you need to be able to visualize what is being asked.

- We're trying to minimize the distance between the point on the parabola and the point (3, 0), there's a known relationship here, and it's the basic distance between points formula.

$$\begin{aligned} d &= \sqrt{(x-3)^2 + (x^2 - 0)^2} \\ &= \sqrt{x^4 + x^2 - 6x + 9} = f(x) \end{aligned}$$

  The domain of this function is $\mathbb{R}$.

- Now differentiate.

$$\begin{aligned} f(x) &= \sqrt{x^4 + x^2 - 6x + 9} \\ f'(x) &= \frac{4x^3 + 2x - 6}{2\sqrt{x^4 + x^2 - 6x + 9}} \end{aligned}$$

  This derivative is not easy to analyze, but I do know the bottom is never zero, just take a look at $(x-3)^2 + (x^2-0)^2$ to see why. However I need to solve the numerator for zero, and cubics are generally tough—you might want to review the Rational Root Theorem, but what if it's irrational? Using a calculator to graph the numerator may be helpful and I do suggest that you do it—actually doing this will tell you that there's only one real root[77] and it looks rational. I hope you will at least verify that $x = 1$ makes the numerator zero. So we have after analyzing the derivative on a number line: $f(x)$ decreasing on $(-\infty,\ 1)$; and $f(x)$ increasing on $(1,\ \infty)$. Clearly there's a minimum at $x = 1$. The point then is $(1,\ 1)$. Here's another snapshot (Figure 71, page 210), but this time I also have the line tangent to $y = x^2$ at $(1,\ 1)$ indicated (it's green).

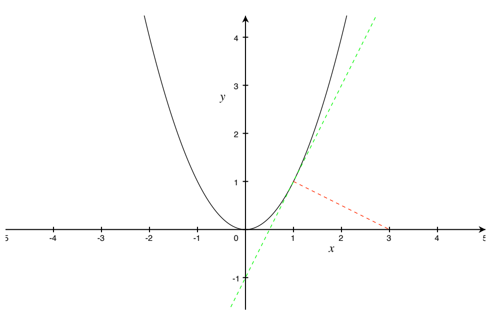

Figure 71: $y = x^2$ and the shortest segment.

30.2 Examples

1. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

Solution: Let x be a positive number, that is $x > 0$. We are trying to minimize (Figure 72, page 211)

$$f(x) = x + \frac{1}{x} \quad x > 0.$$

Taking the derivative.

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Since $x > 0$ we only have one critical number $x = 1$. You should now verify that $f'(x) < 0$ on $(0,\ 1)$; and $f'(x) > 0$ on $(1,\ \infty)$. So the answer is 1.

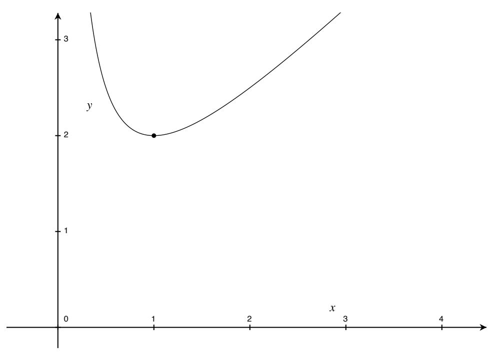

Figure 72: Partial graph of $f(x)$, and indicated minimum.

2. At which points on the curve $y = 1 + 4x^3 - 3x^5$ does the tangent line have the largest slope?

Solution: Okay, this one may be a bit tricky. First you'll need to find the function to maximize. Since it is the slope of the tangent we need to first take the derivative of f.

$$f'(x) = g(x) = 12x^2 - 15x^4$$

To maximize g we need to take its derivative.

$$g'(x) = 24x - 60x^3 = 12x\left(2 - 5x^2\right)$$

Here we need to find the critical numbers and do the analysis. The critical numbers are:

$$-\sqrt{\frac{2}{5}},\ 0,\ \sqrt{\frac{2}{5}}$$

The analysis shows that both

$$-\sqrt{\frac{2}{5}},\ \sqrt{\frac{2}{5}}$$

are maximums. Plugging in we find that

$$g\left(-\sqrt{\frac{2}{5}}\right) = g\left(\sqrt{\frac{2}{5}}\right)$$

So the points are:

$$\left(-\sqrt{\frac{2}{5}},\ f\left(-\sqrt{\frac{2}{5}}\right)\right) \qquad \text{and} \qquad \left(\sqrt{\frac{2}{5}},\ f\left(\sqrt{\frac{2}{5}}\right)\right)$$

A graph (Figure 73, page 212) with indicated points should help in understanding this answer. You should also note that the curve is actually steeper elsewhere, but the slopes there are negative.

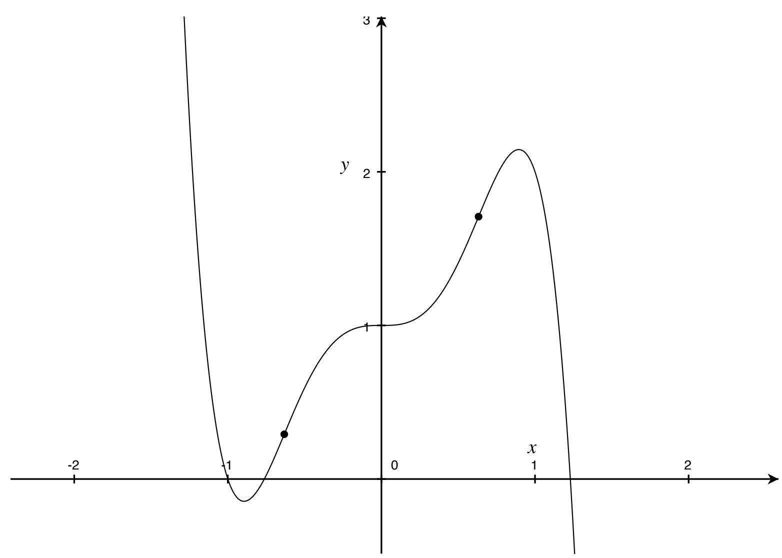

Figure 73: Partial graph of $f(x)$, and indicated points.

3. Find an equation of a line through the point $(3,\ 5)$ that cuts off the least area from the first quadrant.

Solution: There's some value $x, y > 0$, where we have the following relationship determined by slope. (The points along the x and y axis are: $(0,\ y)$ and $(x,\ 0)$.)

$$\begin{aligned}
\frac{y-5}{0-3} &= \frac{0-5}{x-3} \\
y-5 &= \frac{15}{x-3} \\
y &= \frac{15}{x-3}+5
\end{aligned}$$

The area in the first quadrant formed by a line through $(3,\ 5)$ is:

$$A=\frac{1}{2}xy.$$

Rewriting this formula to get area as a function of one variable.

$$\begin{aligned}
A &= \frac{1}{2}xy \\
A\left(x\right) &= \frac{1}{2}x\left(\frac{15}{x-3}+5\right) \\
A\left(x\right) &= \frac{x}{2}\left(\frac{15+5x-15}{x-3}\right) \\
A\left(x\right) &= \frac{x}{2}\left(\frac{5x}{x-3}\right) \\
A\left(x\right) &= \frac{5x^2}{2x-6}
\end{aligned}$$

Differentiating.

$$\begin{aligned}
A\left(x\right) &= \frac{5x^2}{2x-6} \\
A'\left(x\right) &= \frac{\left(2x-6\right)10x-\left(2\right)5x^2}{\left(2x-6\right)^2} \\
A'\left(x\right) &= \frac{20x^2-60x-10x^2}{\left(2x-6\right)^2} \\
A'\left(x\right) &= \frac{10x\left(x-6\right)}{\left(2x-6\right)^2}
\end{aligned}$$

Here we need to find the critical numbers and do the analysis. The critical numbers are:

$$0,\ 3,\ 6$$

The analysis shows that $x = 6$ is the value which minimizes A. So we just need to find the corresponding y and slope.

$$y - 10 = -\frac{5}{3}(x - 6)$$

30.3 Assignment

You should read §4.7 and do the WebAssign assignment mth.121.04.07.

31 mth.121.04.08

31.1 Newton's Method

Yes, you're going to have to use software to do Newton's Method. However, Having access to software may negate the whole need of using Newton's Method in the first place. For example, if I were asked to solve

$$\ln(x+4) = x,$$

for x, I would probably try to do it by hand first.[78] In many of your prior mathematics courses you were asked to solve problems that have nice simple answers. Not all problems have nice simple answers, and you should never assume you can solve a problem using simple methods. Anyway, it's tough to solve this problem by hand, and if you're inexperienced with software you might get the following nonsense (Figure 74, page 215).

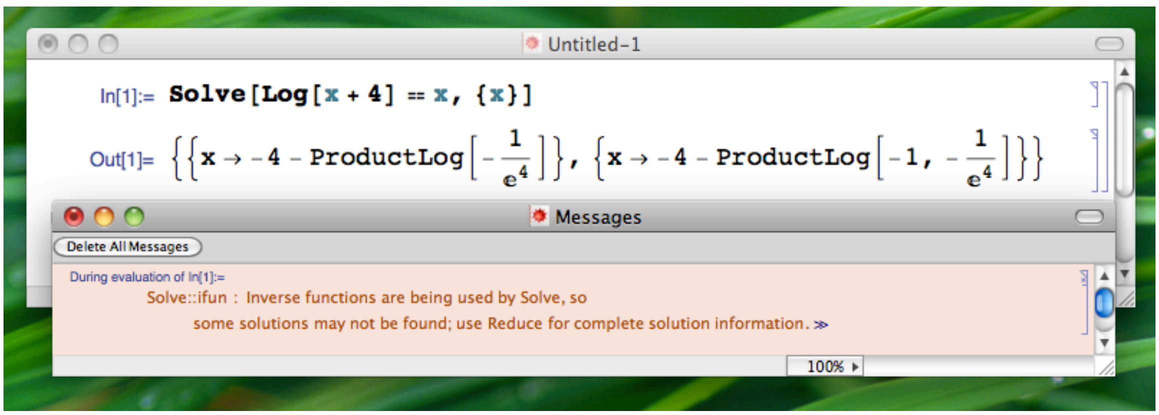

Figure 74: Mathematica Code and Output

It's doubtful that anyone is going to understand this output—it even has a message of warning. Here, I want to suggest using your head first, by graphing (using software again) the following two equations and see if they intersect.

$$y = \ln(x+4) \quad \text{and} \quad y = x$$

After inspecting the graph (Figure 75, page 216) you should clearly see two points of intersection and you should be able to approximate these solutions to at least two decimal place. My guess is $x = -3.98$ and $x = 1.75$. But let's agree that I want better approximations than that. There are many methods to get better solutions, but here we'll only concentrate on Newton's Method because it relates nicely to calculus. Your teacher will discuss the derivation in class, but you can also find an excellent explanation in just about any calculus textbook available.

[78] Some students may prefer to start with a simpler example first. For example, if you're asked to solve $x^2 = 5$ using Newton's method. Towards the end of this sheet I will present this easy example.

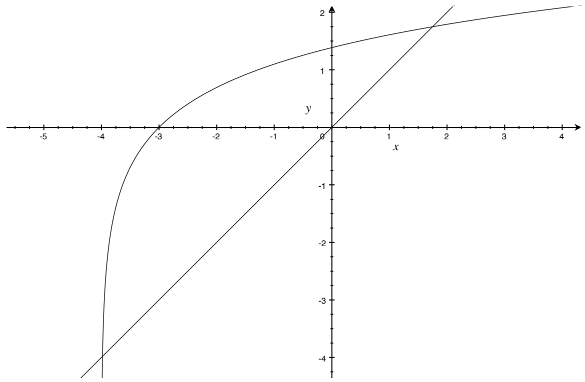

Figure 75: Partial graph of $y = \ln(x + 4)$ and $y = x$.

Newton's Method: To find a sequence of numerical approximations that *may* approach the root[79] of $f(x)$, you need to begin with an initial guess, typically referred to as x_0; and then, using this initial guess, construct the sequence

$$x_0,\ x_1,\ x_2,\ x_3,\ \ldots$$

using the formula[80]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For our example, the first step is to solve the equation for zero and call this our function.

$$\begin{aligned} \ln(x+4) &= x \\ \ln(x+4) - x &= 0 \\ f(x) &= \ln(x+4) - x \end{aligned}$$

Then its derivative.

$$f'(x) = \frac{1}{x+4} - 1 = -\frac{x+3}{x+4},$$

Then try to simplify

$$x - \frac{f(x)}{f'(x)} = x + [\ln(x+4) - x] \cdot \frac{x+4}{x+3}$$

Now graph f (Figure 76, page 217) and try to find a root, clearly we have two. And now, for the nasty part, the actual computations using an initial guess of $x_0 = -3.98$ obtained by

[79]The root is simply where our function takes on a zero.
[80]This will be explained in class.

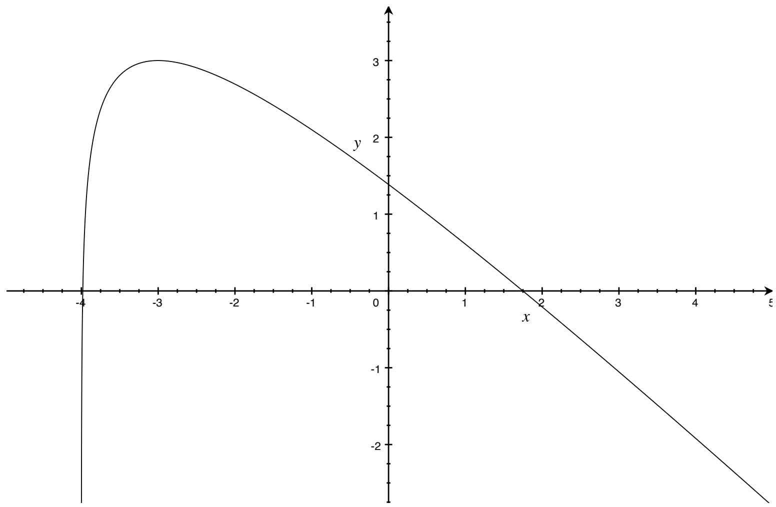

Figure 76: Partial graph of $y = \ln(x+4) - x$.

looking at the graph.[81]

$$\begin{aligned} x_0 &= -3.98 \\ x_1 &= -3.98138728560351 \\ x_2 &= -3.98133943364673 \\ x_3 &= -3.98133937091142 \\ x_4 &= -3.98133937091132 \\ x_5 &= -3.98133937091132 \qquad \text{There's no change!} \end{aligned}$$

I'm not saying this is the *solution*, but it's accurate to fourteen decimal places.[82] Not bad. You could continue with the iterations, but the spreadsheet that I am using is unable to go beyond fourteen decimal places. Your calculator may have a similar precision constraint.

31.2 A Very Simple Example

If you're reading the footnotes—I mention that some student's prefer a simpler example to start with. Here's the solution to that simple problem. For this example, the first step is to solve our equation for zero and then assign this to $f(x)$.

$$\begin{aligned} x^2 &= 5 \\ 0 &= x^2 - 5 \\ f(x) &= x^2 - 5 \end{aligned}$$

[81] I suggest you *learn* how to use your calculator to do this, or as I am doing here, a spreadsheet program. A "how-to" (video) is available for a simple example for students needing some help.

[82] Using better software, I was able to get this approximation:

$$x \approx -3.98133937091131665754166472115047349241490025745400505202095208720722004606106026556797 9$$

Then its derivative.

$$f'(x) = 2x,$$

Then try to simplify

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 5}{2x} = \frac{x^2 + 5}{2x}$$

And now, for the nasty part, the actual computations, but first you'll need to make a guess.[83]

$$\begin{aligned}
x_0 &= 2.24 \\
x_1 &= 2.23607142857143 \\
x_2 &= 2.23606797750245 \\
x_3 &= 2.23606797749979 \\
x_4 &= 2.23606797749979 \qquad \text{There's no change!}
\end{aligned}$$

This should come as no surprise. Just check by finding $\sqrt{5}$ on your calculator.

31.3 Visual Demonstartion

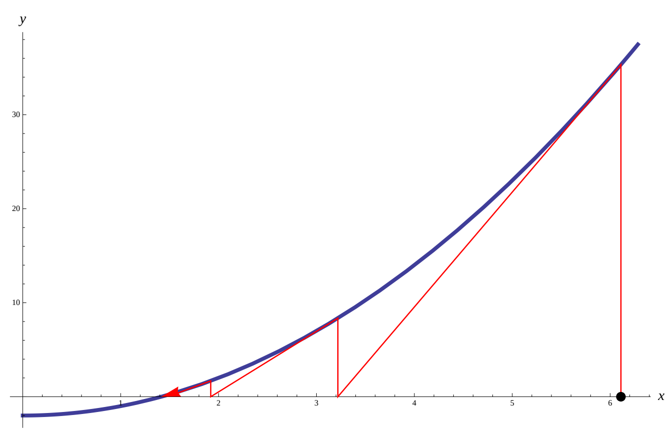

Figure 77: Mathematica Demo

I strongly suggest that you visit http://demonstrations.wolfram.com/ and search for a demo (Figure 77, page 218) on Newton's Method. You'll be shocked by the quality of demos available and best of all, they're *free*!

[83]Since we know the solution is $x = \pm\sqrt{5}$, a good guess is ± 2.24.

31.4 Examples

1. Redo the first example[84] with $x_0 = 1.75$.

Solution: Work will be presented in class.

$$\begin{aligned} x_0 &= 1.75 \\ x_1 &= 1.74903140319016 \\ x_2 &= 1.74903138601270 \\ x_3 &= 1.74903138601270 \qquad \text{There's no change!} \end{aligned}$$

2. Find the largest possible root of

$$f(x) = x^5 - 20x + 10.$$

Solution: Work will be presented in class.

You should be able to get at least ten digits beyond the decimal point.

$$x = 1.965067827983419123868142857118508095062606750326506608216845958950 42$$

Here's the graph (Figure 78, page 219).

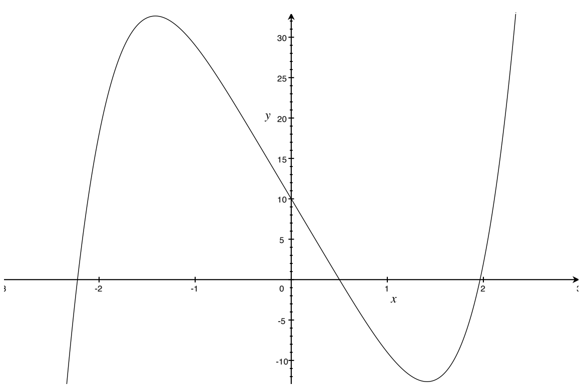

Figure 78: Partial graph of $y = x^5 - 20x + 10$.

31.5 Assignment

You should read §4.8 and do the WebAssign assignment mth.121.04.08.

[84] $f(x) = \ln(x+4) - x$

32 mth.121.04.09

32.1 Antiderivatives

Working backwards from the derivative can be as maddening as backing up a tractor trailer at 60 mph. In fact, as you'll learn in MTH-122, many *tricky* techniques exists just to allow us to do so. And, as I hope you know, backing up while driving (or undoing differentiation) requires practice and patience. Yes, you'll need to go slow at first, and once done you'll need to retrace your steps forward to see if all is going as planned! Let's proceed.

Definition: A function F is called the *antiderivative* of f on the interval I if

$$F'(x) = f(x)$$

for all x in I.

I am going to try to avoid[85] the "big F" notation and stick with the more familiar f and f' only. So using the above definition again. we would now have:

Definition: A function f is called the *antiderivative* of f' on the interval I if

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] = f'(x)$$

for all x in I.

Here's an example. Suppose $f'(x) = 3x^2 + 2x - 3$, find the antiderivative $f(x)$. Although this might be a bit frightening at first, you should realize that we're just looking for an f, that when differentiated with look like f'. Going term-by-term we get

$$f(x) = x^3 + x^2 - 3x + C.$$

Yes, you should note the C, and you should also check to verify that differentiating f will in fact give you f'. Let's see.

$$\begin{aligned} f(x) &= x^3 + x^2 - 3x + C \\ f'(x) &= 3x^2 + 2x - 3 \end{aligned}$$

Yes, all is well! Now, here's the reason for the C.

Theorem: If f is the antiderivative of f' on the interval I, then the most general antiderivative of f' on I is

$$f(x) + C,$$

where C is an arbitrary constant. The C, as you should know, will always differentiate to zero!

A short list of antiderivatives follow.

[85] Not always!

Table 1: Table of General Antiderivatives

Function	Antiderivative
$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1} + C$
x^{-1}	$\ln\|x\| + C$
e^x	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\dfrac{1}{1+x^2}$	$\arctan x + C$

32.2 Checking the Antiderivatives

I will discuss each of these antiderivatives in class. I'd like to believe that you are already familiar with most of these (you may have to review the prior notes though). However, I don't care how trivial the problem is, you must always check yourself.

To check each of these antiderivatives we just need to differentiate and see what it gives back. Again, we'll discuss these in class.

$$x^n = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^{n+1}}{n+1} + C\right],\ n \neq -1 \tag{10}$$

$$x^{-1} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln|x| + C\right] \tag{11}$$

$$e^x = \frac{\mathrm{d}}{\mathrm{d}x}\left[e^x + C\right] \tag{12}$$

$$\cos x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x + C\right] \tag{13}$$

$$\sin x = \frac{\mathrm{d}}{\mathrm{d}x}\left[-\cos x + C\right] \tag{14}$$

$$\sec^2 x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan x + C\right] \tag{15}$$

$$\sec x \tan x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sec x + C\right] \tag{16}$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\arcsin x + C\right] \tag{17}$$

$$\frac{1}{1+x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\arctan x + C\right] \tag{18}$$

Certainly we will differentiate to check out antiderivatives, but I will say that (11) is a bit tricky. To see why we need to compute the derivative of $y = \ln|x|$. Recall that using definition of the absolute value function will result in following breakdown:

$$y = \ln|x| = \begin{cases} \ln x & \text{if} \quad x > 0 \\ \ln(-x) & \text{if} \quad x < 0 \end{cases}.$$

You should notice that zero is excluded. Now let's differentiate.

$$y' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln|x|\right) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x > 0 \\ \dfrac{1}{x} & \text{if} \quad x < 0 \end{cases}.$$

32.3 Examples

1. Find f, where $f'(x) = \cos x - 5e^x$.

Solution: We'll discuss this in class.

$$\begin{aligned} f'(x) &= \cos x - 5e^x \\ f(x) &= \sin x - 5e^x + C \end{aligned}$$

2. Find f, where $f'(x) = 6/x^7$.

Solution: We'll discuss this in class.

$$\begin{aligned} f'(x) &= 6/x^7 \\ f'(x) &= 6x^{-7} \\ f'(x) &= -x^{-6} + C \end{aligned}$$

3. Find f, where $f'(x) = \sqrt[5]{x^3} - \sin x$.

Solution: We'll discuss this in class.

$$
\begin{aligned}
f'(x) &= \sqrt[5]{x^3} - \sin x \\
f'(x) &= x^{3/5} - \sin x \\
f(x) &= \frac{5x^{8/5}}{8} + \cos x + C
\end{aligned}
$$

4. Find f, where $f'(x) = (2x-5)^2$.

Solution: We'll discuss this in class.

$$
\begin{aligned}
f'(x) &= (2x-5)^2 \\
f'(x) &= 4x^2 - 20x + 25 \\
f(x) &= \frac{4}{3}x^3 - 10x^2 + 25x + C
\end{aligned}
$$

5. Find f, where $f''(x) = 6x + 12x^2$.

Solution: We'll discuss this in class.

$$
\begin{aligned}
f''(x) &= 6x + 12x^2 \\
f'(x) &= 3x^2 + 4x^3 + C_1 \\
f(x) &= x^3 + x^4 + C_1 x + C_2
\end{aligned}
$$

6. Find f, where $f'''(x) = 5x + \cos x$.

Solution: We'll discuss this in class.

$$
\begin{aligned}
f'''(x) &= 5x + \cos x \\
f''(x) &= \frac{5}{2}x^2 + \sin x + C_1 \\
f'(x) &= \frac{5}{6}x^3 - \cos x + C_1 x + C_2 \\
f(x) &= \frac{5}{24}x^4 - \sin x + \frac{C_1}{2}x^2 + C_2 x + C_3
\end{aligned}
$$

7. Find f, where $f'(x) = 1 - 6x$ and $f(0) = 7$.

Solution: We'll discuss this in class.

$$
\begin{aligned}
f'(x) &= 1 - 6x \\
f(x) &= x - 3x^2 + C \\
f(0) &= C = 7 \\
f(x) &= x - 3x^2 + 7
\end{aligned}
$$

8. Find f, where $f''(x) = 2e^x + 3\sin x$, $f(0) = 0$, and $f(\pi) = 0$.

Solution: We'll discuss this in class.

$$
\begin{aligned}
f''(x) &= 2e^x + 3\sin x \\
f'(x) &= 2e^x - 3\cos x + C_1 \\
f(x) &= 2e^x - 3\sin x + C_1 x + C_2 \\
f(0) &= 2 + C_2 = 0, \quad C_2 = -2 \\
f(\pi) &= 2e^\pi + \pi C_1 - 2 = 0, \quad C_1 = \frac{2 - 2e^\pi}{\pi} \\
f(x) &= 2e^x - 3\sin x + \left(\frac{2 - 2e^\pi}{\pi}\right) x - 2
\end{aligned}
$$

9. **Optional Question:** Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial *downward*

velocity of 10 m/s and its *downward* acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if} \quad 0 \le t \le 10 \\ 0 & \text{if} \quad t > 10 \end{cases}.$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?[86]

Solution: We'll discuss this in class.

Note: I think this last problem is difficult and we'll discuss it in class. Here's my solution.

It *clearly* states that the initial velocity is *downward*, so

$$v(0) = -10.$$

Furthermore, the acceleration function given is also *downward*,[87] so

$$a(t) = \begin{cases} 0.9t - 9 & \text{if} \quad 0 \le t \le 10 \\ 0 & \text{if} \quad t > 10 \end{cases}.$$

Now we need to work backwards to find both the velocity function, $v(t)$, and the position function, $s(t)$. Working from acceleration to velocity first.

$$\begin{aligned} a(t) = v'(t) &= 0.9t - 9 \\ v(t) &= 0.45t^2 - 9t + v_0 \\ v(0) &= -10 = v_0 \\ v(t) &= 0.45t^2 - 9t - 10 \end{aligned}$$

Then from velocity to position next.

$$\begin{aligned} v(t) = s'(t) &= 0.45t^2 - 9t - 10 \\ s(t) &= 0.15t^3 - 4.5t^2 - 10t + s_0 \\ s(0) &= s_0 = 500 \\ s(t) &= 0.15t^3 - 4.5t^2 - 10t + 500 \end{aligned}$$

When $t = 10$ the raindrop reaches its terminal velocity and it is

$$s(10) = 150 - 450 - 100 + 500 = 100 \text{ m}$$

above the ground and its terminal velocity is

$$v(10) = 45 - 90 - 10 = -55 \text{ m/sec}.$$

Now, finally, at this constant rate it will take an additional 1.8 seconds to reach the ground. So the total time for the raindrop to fall 500 meters using this model is $\boxed{11.8 \text{ seconds}}$.

[86] This problem has been known to pop a few brains open, but it's also the type of problem that may finally pop your mind closed to mathematics.

Visual Note: Here's the position graph(Figure 79, page 226).

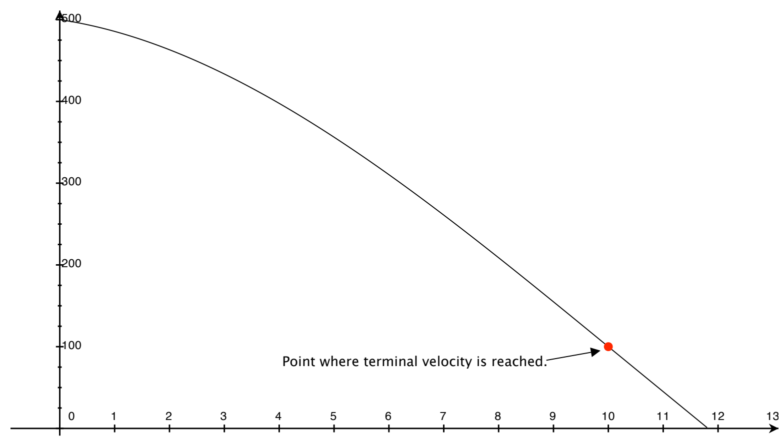

Figure 79: The vertical axis is position (meters) and the horizontal is time (seconds).

The curve to the left of the red point is $s(t) = 0.15t^3 - 4.5t^2 - 10t + 500$ and to the right is $s(t) = -55(x - 10) + 100$.

Here's the velocity graph (Figure 80, page 226).

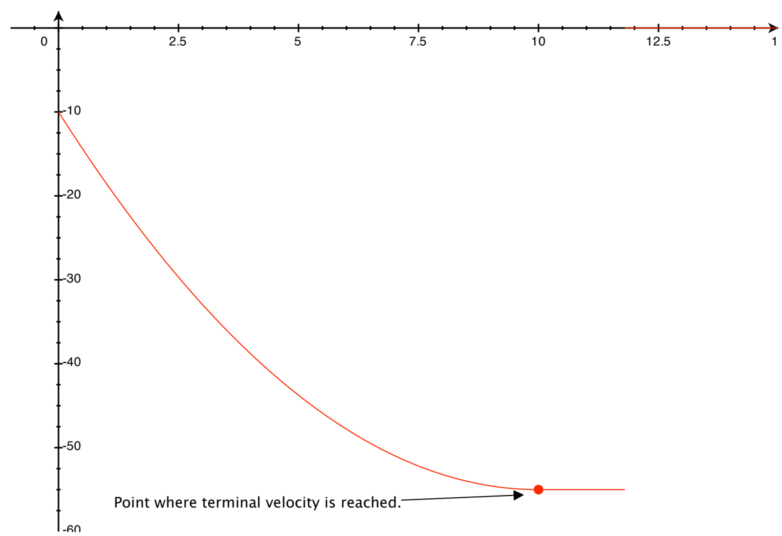

Figure 80: The vertical axis is velocity (m/s) and the horizontal is time (seconds).

The curve to the left of the red point is $v(t) = 0.45t^2 - 9t - 10$ and to the right is $v(t) = -55$ until it hits the ground at 11.8 seconds, where the velocity becomes zero.

32.4 Assignment

You should read §4.9 and do the WebAssign assignment mth.121.04.09.

33 mth.121.05.01

33.1 Introduction to Areas

Here at ECC we spend some time on sums in MTH 120, here's what you need to know from MTH 120 for chapter of the textbook. First off, when i is used here it represents an integer counter and not the imaginary unit, and k is any real constant.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n \tag{19}$$

$$\sum_{i=1}^{n} ka_i = ka_1 + ka_2 + ka_3 + \cdots + ka_n = k \sum_{i=1}^{n} a_i \tag{20}$$

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \tag{21}$$

$$\sum_{i=1}^{n} k = k + k + k + \cdots + k = nk \tag{22}$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{23}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{24}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{25}$$

You may recall that you proved some of these formulas by the Principle of Mathematical Induction. Here, however, you will be expected to use them to find sums and this may require a that you review what you've learned, or were supposed to have learned in pre-calculus.

The reason we are going to need sum formulas is that we are going to divide areas into small, well-defined pieces, and then add them together. The book does this in a variety of ways, and you should be aware that while you are doing the homework problems that you may have to look carefully at the book (notes too) to understand what is being asked. I will certainly discuss this in class, and I will mainly concentrate on doing each problem using similar techniques.

Example: Try computing the following finite sum.[88]

$$\sum_{j=0}^{50} j\,(j-1)$$

Solution:

$$\begin{aligned}
\sum_{j=0}^{50} j\,(j-1) &= \sum_{j=0}^{50} j^2 - \sum_{j=0}^{50} j \\
&= \sum_{j=1}^{50} j^2 - \sum_{j=1}^{50} j \\
&= \frac{50\cdot 51\cdot 101}{6} - \frac{50\cdot 51}{2} \\
&= 42925 - 1275 \\
&= 41650
\end{aligned}$$

Example: Evaluate the limit.[89]

$$\lim_{N\to\infty} \sum_{j=1}^{N} \frac{j^3}{N^4}$$

Solution:

$$\begin{aligned}
\lim_{N\to\infty} \sum_{j=1}^{N} \frac{j^3}{N^4} &= \lim_{N\to\infty} \frac{1}{N^4} \sum_{j=1}^{N} j^3 \\
&= \lim_{N\to\infty} \frac{1}{N^4} \left[\frac{N\,(N+1)}{2}\right]^2 \\
&= \lim_{N\to\infty} \frac{N^2+2N+1}{4N^2} \\
&= \frac{1}{4}
\end{aligned}$$

33.2 Area

We'll be looking at computing areas of bounded regions, some very familiar, and others very unfamiliar. You should have some basic skills in computing area of irregularly/regularly

[88] 41650
[89] 1/4

shaped regions, and be able to deal with a variety of units. For example, if we have a rectangle with a base of 6 cm/hr, and a height of 15 minutes, you should be able to computes it area.[90] Your ability to visualize a rectangle is essential, but you should also know the area of a triangle[91] and circle[92].

33.3 The Definite Integral

Suppose f is continuous for $a \le t \le b$. The **definite integral** of f from a to b, written

$$\int_a^b f(t)\ \mathrm{d}t,$$

is the limit of the left-hand or right-hand sums with n subdivisions as n gets arbitrarily large. In other words, for left-hand[93] sums, we have

$$\int_a^b f(t)\ \mathrm{d}t = \lim_{n\to\infty}\left(\sum_{i=0}^{n-1} f(t_i)\,\Delta t\right),$$

and for right-hand[94] sums, we have

$$\int_a^b f(t)\ \mathrm{d}t = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(t_i)\,\Delta t\right).$$

Each of these sums is called a **Riemann sum**, f is called **integrand**, and a and b are called the **limits of integration**. These strange looking stuff will be discussed in class as it relates to area. For now, I want to suggest that you concentrate on computing areas, and noting how best to approximate areas using rectangles.

You're expected to be able to compute both finite and infinite sums. This takes some effort to understand, but I will try my best to extend the finite to the infinite!

33.4 Examples

1. Find the area bounded by $y = 2x + 1$, $x = 1$, $x = 3$, and $y = 0$. This problem can be written as:

   $$\int_1^3 2x + 1\ \mathrm{d}x.$$

   And its area is represented by the shaded region in the graph (Figure 81, page 230).

[90] $3/2$ cm.
[91] $A = \dfrac{bh}{2}$
[92] $A = \pi r^2$
[93] Usually denoted L.
[94] Usually denoted R.

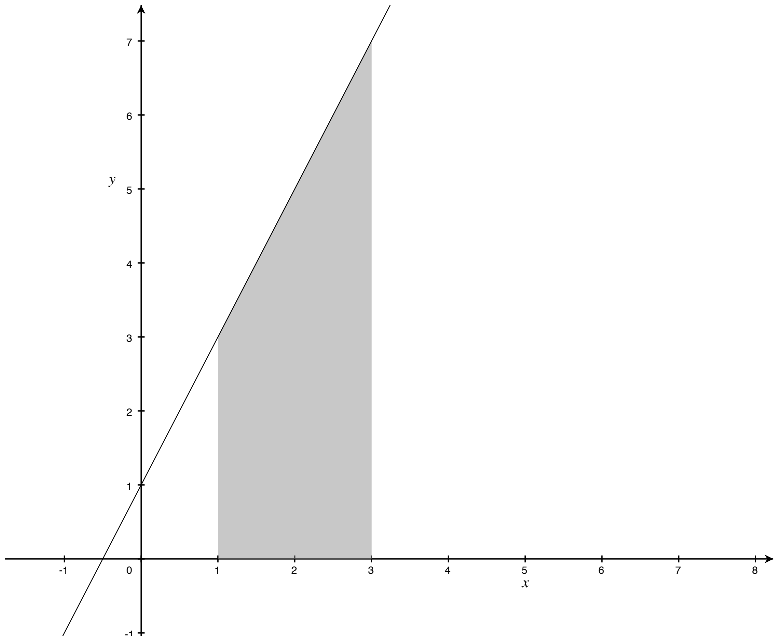

Figure 81: Shaded area of interest.

Solution: Basic geometry where we split the shaded region into two parts, a rectangle $(2 \times 3 = 6)$ and a triangle $(1/2 \times 2 \times 4 = 4)$. Now combine the two areas to get $\boxed{10 \text{ square units}}$.

2. Find the area bounded by $y = \sqrt{1 - x^2}$, $x = 0$, $x = 1$, and $y = 0$. This problem can be written as:

$$\int_0^1 \sqrt{1 - x^2}\, \mathrm{d}x.$$

And its area is represented by the shaded region in the graph (Figure 82, page 231).

Solution: Again, basic geometry because this is just one quarter of a circle of radius 1, that is $\boxed{\pi/4 \text{ square units}}$.

3. For time, t, in hours, $0 \leq t \leq 1$, a bug is crawling at a velocity, v, in meters/hour given by

$$v(t) = \frac{1}{1+t}.$$

(a) Estimate the area represented by the shaded region (Figure 83, page 231). What is the unit of this area?

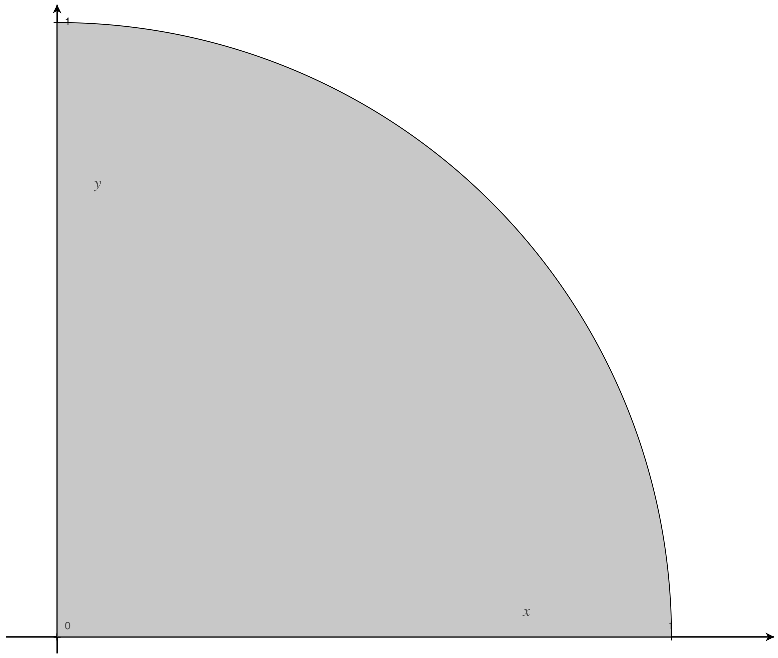

Figure 82: Shaded area of interest.

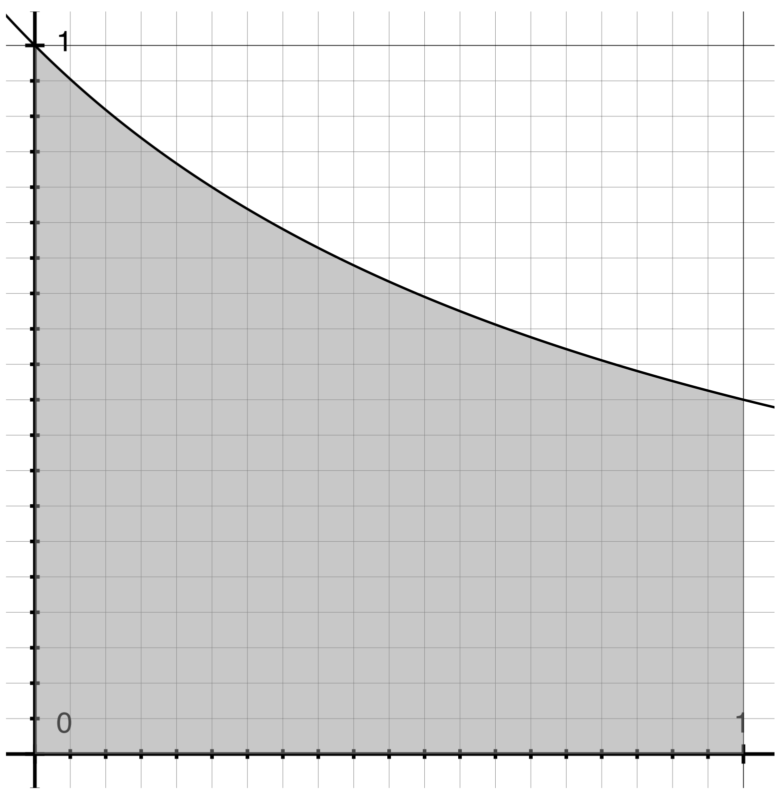

Figure 83: Vertical axis, $v(t)$, in mph; and the horizontal axis, t, is in hours.

Solution: We have a 20×20 grid, which represents 400 square units, since the horizontal axis represents time in hours and the vertical axis represents meters per hour, the product represents meters. My estimate[95] of the number of shaded squares is 277.25, so I get

$$\frac{277.25}{400} = 0.693125 \text{ meter.}$$

You should note that this area represents distance, that is, the approximate distance that the bug travels during this one hour time period.

(b) Use $\Delta t = 0.5$ to estimate the distance traveled during this hour. Do both and upper and lower estimate.[96] Then average the two.

Solution: The upper esitmate first, which is

$$0.5\left[v\left(0\right) + v\left(0.5\right)\right].$$

You can do this by hand, or use a calculator. Here's the shorthand version.

$$\sum_{i=0}^{1} \frac{1}{2} v\left(0 + i \cdot \frac{1}{2}\right) = \frac{5}{6} \approx 0.833.$$

So the upper estimate is $5/6$.
The lower estimate is

$$0.5\left[v\left(0.5\right) + v\left(1\right)\right].$$

You can do this by hand, or use a calculator. Here's the shorthand version.

$$\sum_{i=1}^{2} \frac{1}{2} v\left(0 + i \cdot \frac{1}{2}\right) = \frac{7}{12} \approx 0.583.$$

So the lower estimate is $7/12$.
The average is

$$\frac{1}{2}\left(\frac{5}{6} + \frac{7}{12}\right) = \frac{17}{24} \approx 0.708.$$

(c) Use $\Delta t = 0.1$ to estimate the distance traveled during this hour. Do both and upper and lower estimate. Then average the two.

[96] The book uses both right and left endpoints, and midpoints, using the letters R, L and M respectively. You will see, on occasion, L_n, R_n and M_n where the n indicates the number of division needed.

Solution: The upper esitmate first, which is

$$0.1\left[v\left(0\right)+v\left(0.1\right)+\cdots+v\left(0.8\right)+v\left(0.9\right)\right].$$

You can do this by hand, or use a calculator. Here's the shorthand version.

$$\sum_{i=0}^{9} 0.1v\left(0+i\cdot 0.1\right)=\frac{33464927}{46558512}\approx 0.719.$$

So the upper estimate is $\frac{33464927}{46558512}\approx 0.719$.
The lower estimate is

$$0.1\left[v\left(0.1\right)+v\left(0.2\right)+\cdots+v\left(0.9\right)+v\left(1\right)\right].$$

You can do this by hand, or use a calculator. Here's the shorthand version.

$$\sum_{i=1}^{10} 0.1v\left(0+i\cdot 0.1\right)=\frac{155685007}{232792560}\approx 0.669.$$

So the upper estimate is $\frac{155685007}{232792560}\approx 0.669$.
The average is

$$\frac{1}{2}\left(\frac{155685007}{232792560}+\frac{33464927}{46558512}\right)=\frac{161504821}{232792560}\approx 0.694.$$

(d) Use $\Delta t = 0.05$ to estimate the distance traveled during this hour. Do both and upper and lower estimate. Then average the two.

Solution: The upper esitmate first, which is

$$0.05\left[v\left(0\right)+v\left(0.05\right)+\cdots+v\left(0.9\right)+v\left(0.95\right)\right].$$

You can do this by hand, or use a calculator. Here's the shorthand version.

$$\sum_{i=0}^{19} 0.05v\left(0+i\cdot 0.05\right)=\frac{3771059091081773}{5342931457063200}\approx 0.706.$$

So the upper estimate is $\frac{3771059091081773}{5342931457063200}\approx 0.706$.
The lower estimate is

$$0.05\left[v\left(0.05\right)+v\left(0.1\right)+\cdots+v\left(0.95\right)+v\left(1\right)\right].$$

You can do this by hand, or use a calculator. Here's the shorthand version.

$$\sum_{i=1}^{20} 0.05v\left(0+i\cdot 0.05\right)=\frac{3637485804655193}{5342931457063200}\approx 0.681.$$

So the upper estimate is $\frac{3637485804655193}{5342931457063200} \approx 0.681$.
The average is

$$\frac{1}{2}\left(\frac{3637485804655193}{5342931457063200} + \frac{3771059091081773}{5342931457063200}\right) = \frac{3704272447868483}{5342931457063200} \approx 0.693.$$

As an aside, if you were to continue with smaller-and-smaller partitions, that is, as the number of partitions tend towards infinity, this numerical sequence would go towards

$$\ln 2 = 0.693147180559945309417232121458176568075500134360255254 1\ldots.$$

That's a lot of work though. Since you have experience with Mathematica, you might try the following (Figure 84, page 235) code. This can also be done on a TI-89 or similar calculator.

4. Consider $f(x) = 4x + 1$ on $[0,\ 3]$. Compute L_6, R_6, M_6, and using geometry, the exact area.

Solution:

$$\begin{aligned} L_6 &= \frac{1}{2}\left[f(0) + f(0.5) + f(1.0) + f(1.5) + f(2.0) + f(2.5)\right] = 18 \\ R_6 &= \frac{1}{2}\left[f(0.5) + f(1.0) + f(1.5) + f(2.0) + f(2.5) + f(3.0)\right] = 24 \\ M_6 &= \frac{1}{2}\left[f(0.25) + f(0.75) + f(1.25) + f(1.75) + f(2.25) + f(2.75)\right] = 21 \\ A &= 1 \cdot 3 + \frac{1}{2} \cdot 3 \cdot 12 = 21 \end{aligned}$$

5. Find the area bounded by $y = x^2 - 2x + 2$, $x = -1$, $x = 3$, and $y = 0$. This problem can be written as:

$$\int_{-1}^{3} x^2 - 2x + 2 \, \mathrm{d}x.$$

And its area is represented by the shaded region in the graph (Figure 85, page 236). This is a much more difficult problem and we will work towards finding an exact answer.

(a) Approximate the area using triangles and rectangles (or trapezoids).

Solution: Answers may vary, but this is what I see. I am divided the shaded region into 4 distinct regions of equal width.

$$\left[1 \times 2 + \frac{1 \times 5}{2}\right] + \left[1 \times 1 + \frac{1 \times 1}{2}\right] + \left[1 \times 1 + \frac{1 \times 1}{2}\right] + \left[1 \times 2 + \frac{1 \times 5}{2}\right] = 9$$

```
In[31]:= v[t_] := 1 / (1 + t)

In[32]:= Sum[1 / 20 * v[0 + 1 / 20 * i], {i, 0, 19}]
```

Out[32]= $\dfrac{3771059091081773}{5342931457063200}$

```
In[33]:= Sum[1 / 20. * v[0 + 1 / 20 * i], {i, 0, 19}]

Out[33]= 0.705803

In[34]:= Sum[1 / 20 * v[0 + 1 / 20 * i], {i, 1, 20}]
```

Out[34]= $\dfrac{3637485804655193}{5342931457063200}$

```
In[35]:= Sum[1 / 20. * v[0 + 1 / 20 * i], {i, 1, 20}]

Out[35]= 0.680803
```

In[36]:= $1 / 2 * \left(\dfrac{3637485804655193}{5342931457063200} + \dfrac{3771059091081773}{5342931457063200}\right)$

Out[36]= $\dfrac{3704272447868483}{5342931457063200}$

In[37]:= $1 / 2. * \left(\dfrac{3637485804655193}{5342931457063200} + \dfrac{3771059091081773}{5342931457063200}\right)$

```
Out[37]= 0.693303

In[40]:= Sum[1 / 1000000. * v[0 + 1 / 1000000 * i], {i, 1, 1000000}]

Out[40]= 0.693147

In[41]:= Log[2.0]

Out[41]= 0.693147
```

Figure 84: Simple Mathematica code that you can all try.

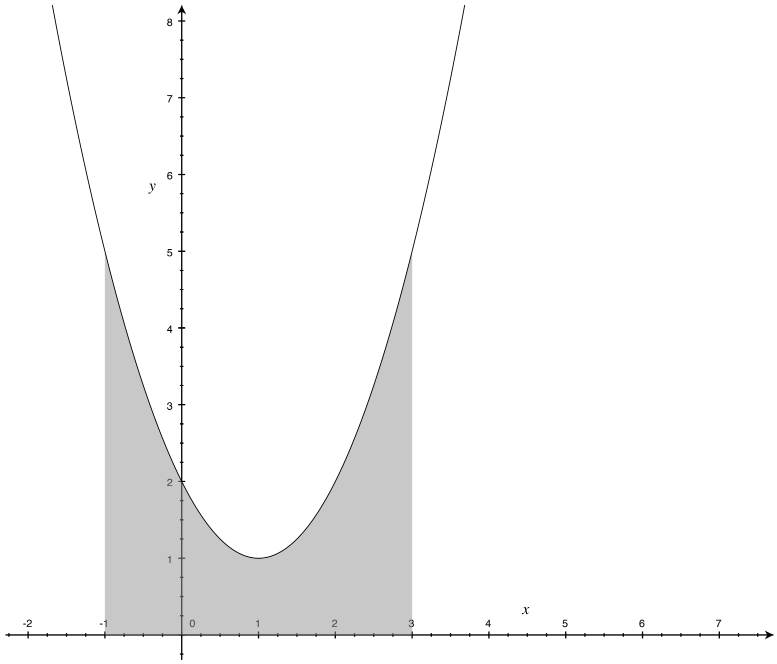

Figure 85: Shaded area of interest.

This is a fairly good estimate. We will actually show that the exact area is $9\frac{1}{3}$.

(b) Evaluate a Riemann sum for $f(x) = x^2 - 2x + 2$ taking sample points to be right endpoints[97] and $a = -1$ and $b = 3$, and $n = 16$.

Solution: Using right endpoints we have

$$\frac{1}{4}\left[f(-0.75) + f(-0.50) + \cdots + f(2.75) + f(3)\right].$$

You can do this by hand, or use a calculator. Here's the shorthand version.

$$\sum_{i=1}^{16} 0.25 f(-1 + i \cdot 0.25) = \frac{75}{8} \approx 9.375.$$

So the right endpoint estimate is $\frac{75}{8} \approx 9.375$. This is even better.

(c) Evaluate

$$\int_{-1}^{3} x^2 - 2x + 2 \, \mathrm{d}x.$$

by taking the limit (right end-point Riemann sum as $n \to \infty$).

[97] You should also be able to do left endpoints too.

Solution: First the general sum for n sub-intervals.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{4}{n} f\left(-1 + i \cdot \frac{4}{n}\right) &= \sum_{i=1}^{n} \frac{64i^2}{n^3} - \frac{64i}{n^2} + \frac{20}{n} \\
&= \frac{32\left(2n^2 + 3n + 1\right)}{3n^2} - \frac{32\left(n+1\right)}{n} + 20 \\
&= \frac{64}{3} + \frac{32}{n} + \frac{32}{3n^2} - 32 - \frac{32}{n} + 20 \\
&= \frac{28}{3} + \frac{32}{3n^2}
\end{aligned}$$

Now let $n \to \infty$,

$$\lim_{n\to\infty} \frac{28}{3} + \frac{32}{3n^2} = \frac{28}{3} = 9\frac{1}{3}$$

33.5 Assignment

You should read §5.1 and do the WebAssign assignment mth.121.05.01.

34 mth.121.05.02

34.1 Visual Approach to Integration

34.1.1 Computing Signed Areas

What follows is an integration and the signed areas it represent.

Example: Graph $f(x) = 2x$ and shade the region between the x-axis and $f(x)$ from $x = -3$ to $x = 2$. Area below the x-axis is negative; area above the x-axis is positive. Show that the total signed area is -5. Here's what we're doing:

$$\int_{-3}^{2} 2x \, \mathrm{d}x = -5.$$

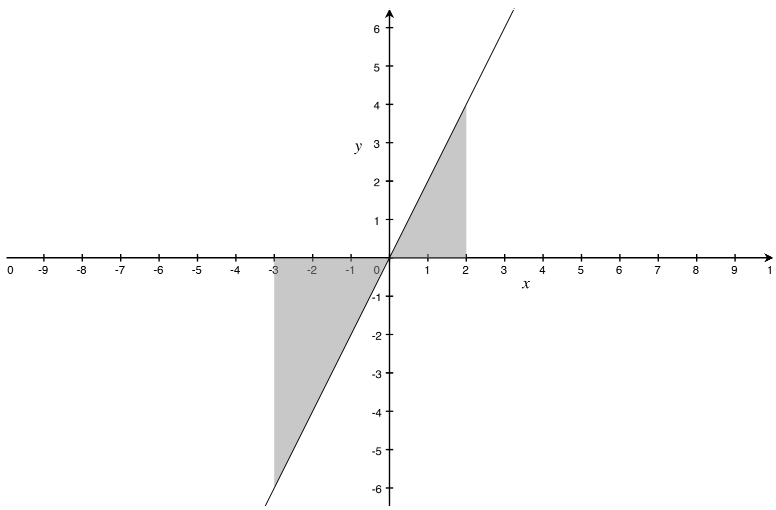

Figure 86: Region of interest.

Solution: Here's the graph (Figure 86, page 238) and you should be able to do such a graph on your own. You should be clear that two triangles exist and their areas are easy to compute. The one to the right of zero is above the x-axis and its area is four square units; the one to the left of zero is below the x-axis and its area is nine square units. Be aware that we're dealing with *signed* areas, and what is above the x-axis is positive and below the x-axis is negative.

$$\int_{-3}^{2} 2x \, \mathrm{d}x = 4 + (-9) = -5$$

34.1.2 Riemann Sums

A Riemann sum, denoted $R(f,\ P,\ C)$, for the interval $[a,\ b]$ is defined by choosing a partition

$$P : a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

and sample points $C = \{c_i\}$, where $c_i \in [x_{i-1},\ x_i]$. If we let $\Delta x_i = x_i - x_{i-1}$, then

$$R(f,\ P,\ C) = \sum_{i=1}^{N} f(c_i)\ \Delta x_i.$$

The maximum of the widths Δx_i is called the norm of the partition, denoted $\|P\|$. The definite integral is the limit of the Riemann sum, if it exists, and is denoted by:

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{\|P\|\to 0} R(f,\ P,\ C)$$

Example: Calculate the Riemann sum for

$$f(x) = x, \quad P = \{1,\ 1.2,\ 1.5,\ 2\}, \quad C = \{1.1,\ 1.4,\ 1.9\}.$$

Sketch the graph, and geometrically compute the limit as $\|P\| \to 0$.

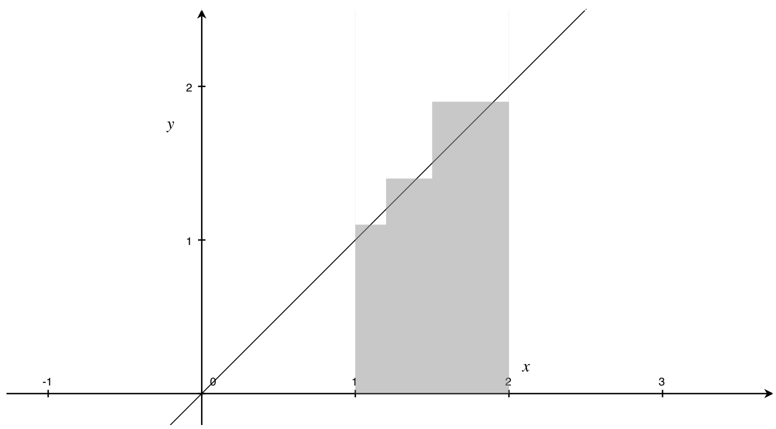

Figure 87: Region of interest.

Solution: This notation is actually pretty cryptic, but once you see how it relates to the graph (Figure 87, page 239) you should be able to compute these without much trouble.

$$(1.2 - 1) \cdot f(1.1) + (1.5 - 1.2) \cdot f(1.4) + (2 - 1.5) \cdot f(1.9) = 1.59$$

The graph (Figure 87, page 239) should help.

$$\lim_{\|P\|\to 0} R(x,\ P,\ C) = \int_1^2 x\ \mathrm{d}x = \frac{3}{2}$$

34.2 Examples

1. Graph $f(x) = \sqrt{25 - x^2}$ and shade the region between the x-axis and $f(x)$ from $x = 0$ to $x = 5$. Area below the x-axis is negative; area above the x-axis is positive. Show that the total signed area is $25\pi/4$. We just showed that

$$\int_0^5 \sqrt{25 - x^2}\, dx = \frac{25\pi}{4}.$$

Solution: This will be discussed in class.

2. Graph $f(x) = |x|$ and shade the region between the x-axis and $f(x)$ from $x = -2$ to $x = 3$. Area below the x-axis is negative; area above the x-axis is positive. Show that the total signed area is $13/2$. We just showed that

$$\int_{-2}^{3} |x|\, dx = \frac{13}{2}.$$

Solution: This will be discussed in class.

3. Calculate the Riemann sum for

$$f(x) = x^2 + x, \quad P = \{2,\ 3,\ 4.5,\ 5\}, \quad C = \{2,\ 3.5,\ 5\}.$$

Solution: This will be discussed in class. Make sure you can do this on your own!

$$(3-2) \cdot f(2) + (4.5-3) \cdot f(3.5) + (5-4.5) \cdot f(5) \quad = \quad 23.625$$

Here's the graph (Figure 88, page 241).

4. Graph $f(x) = \sin x$ and shade the region between the x-axis and $f(x)$ from $x = -\pi/2$ to $x = 3\pi/2$. Area below the x-axis is negative; area above the x-axis is positive. Show that the total signed area is 0. We just showed that

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x\, dx = 0.$$

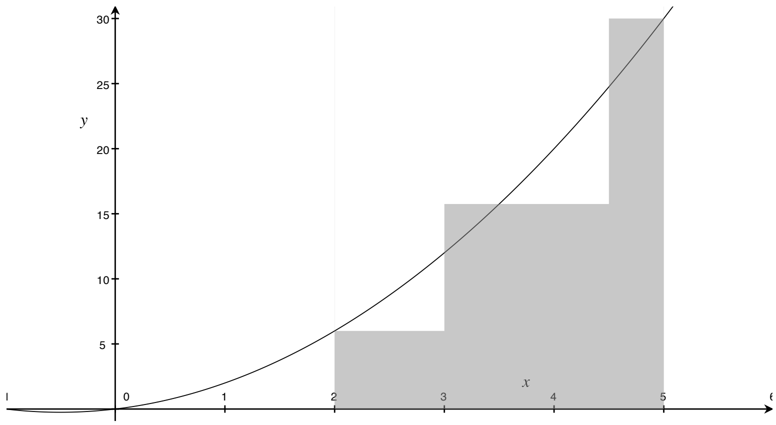

Figure 88: Region of interest.

Solution: Although I am providing a graph (Figure 89, page 241) you should be able to graph this quickly on your own. Furthermore, you should be able to explain why the integration evaluates to zero.

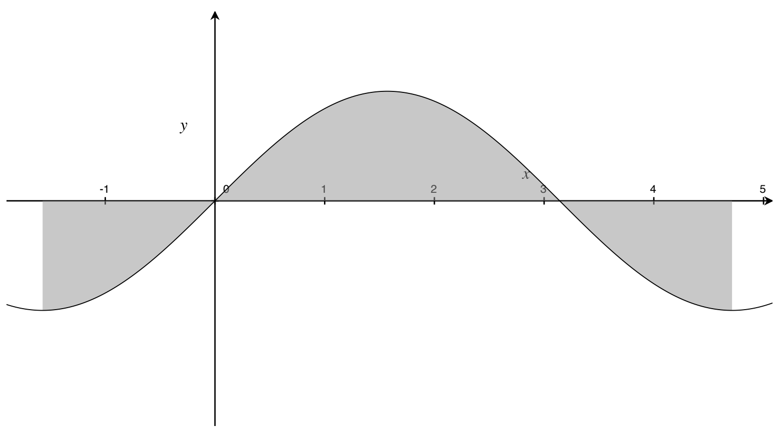

Figure 89: Region of interest.

34.3 Assignment

You should read §5.2 and do the WebAssign assignment mth.122.05.02.

35 mth.121.05.03

35.1 The Fundamental Theorems of Calculus (FTC)

35.1.1 FTC Part I

If f is continuous on the interval $[a,\ b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)\ \mathrm{d}t = F(b) - F(a) = F(t)\big|_a^b.$$

35.1.2 FTC Part II

This is often stated in another way, because if

$$\int_a^x f(t)\ \mathrm{d}t = F(x) - F(a),$$

then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\ \mathrm{d}t\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[F(x) - F(a)\right] = f(x).$$

This, of course, allows us to differentiate integrals. It should be noted that $x \in [a,\ b]$.

35.1.3 Making Sense Out of the FTC, Part I

First let

$$g(b) = \int_a^b f(t)\ \mathrm{d}t$$

If we take one of these partitions, say from x to $x + h$.[98] with the left endpoint evaluation method, we get its area to be

$$g(x+h) - g(x) \approx hf(x),$$

or for $h \neq 0$ we have

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

Taking limits as $h \to 0$ we get

$$g'(x) = f(x).$$

Here's a proof that if f is continuous on $[a,\ b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\ \mathrm{d}t \qquad a \leq x \leq b$$

[98] $h > 0$ and $a < x + h < b$ I'll draw a sketch in class

is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$, and $g'(x) = f(x)$.

Partial Proof: If x and $x+h$ are in $(a,\ b)$, then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\ \mathrm{d}t - \int_a^x f(t)\ \mathrm{d}t \\ &= \left(\int_a^x f(t)\ \mathrm{d}t + \int_x^{x+h} f(t)\ \mathrm{d}t\right) - \int_a^x f(t)\ \mathrm{d}t \\ &= \int_x^{x+h} f(t)\ \mathrm{d}t \end{aligned}$$

and so, for $h \neq 0$, we have

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\ \mathrm{d}t$$

For $h > 0$, we know[99] that f will take on a minimum $(m = f(u))$ and maximum $(M = f(v))$ value on $[x,\ x+h]$ where $u, v \in [x,\ x+h]$, that is

$$\begin{array}{rcl} mh \leq & \int_x^{x+h} f(t)\ \mathrm{d}t & \leq Mh \\ f(u)\,h \leq & \int_x^{x+h} f(t)\ \mathrm{d}t & \leq f(v)\,h \\ f(u) \leq & \frac{1}{h}\int_x^{x+h} f(t)\ \mathrm{d}t & \leq f(v) \\ f(u) \leq & \dfrac{g(x+h) - g(x)}{h} & \leq f(v) \end{array}$$

Now let $h \to 0$ which means that both $u, v \to x$, therefor

$$\begin{array}{rcl} \lim\limits_{u\to x} f(u) \leq & \lim\limits_{h\to 0} \dfrac{g(x+h) - g(x)}{h} & \leq \lim\limits_{v\to x} f(v) \\ f(x) \leq & g'(x) & \leq f(x) \end{array}$$

That is, we have $f(x) = g'(x)$.

You may also want to read the textbook to see other ways to make sense of the FTC, Part I. However, for now I just want use the FTC, Part I to evaluate integrals. As you should know by now, evaluating integrals without the FTC, Part I was indeed tedious. However, don't think the FTC, Part I is easy, because it depends on finding an anti-derivative—and that, in general, is quite difficult!

35.2 Examples

1. Evaluate the integral using the FTC, Part I.

$$\int_1^3 2\ \mathrm{d}x$$

[99] Extreme Value Theorem

Solution: We'll do the *work* in class.

$$\int_1^3 2 \, \mathrm{d}x = 4$$

2. Evaluate the integral using the FTC, Part I.

$$\int_{-1}^{2} (x-1)^3 \, \mathrm{d}x$$

Solution: We'll do the *work* in class.

$$\int_{-1}^{2} (x-1)^3 \, \mathrm{d}x = -\frac{15}{4}$$

3. Evaluate the integral using the FTC, Part I.

$$\int_{8/27}^{1} \frac{10x^{4/3} - 8x^{1/3}}{x^2} \, \mathrm{d}x$$

Solution: We'll do the *work* in class.

$$\int_{8/27}^{1} \frac{10x^{4/3} - 8x^{1/3}}{x^2} \, \mathrm{d}x = -5$$

4. Evaluate the integral using the FTC, Part I.

$$\int_{-2}^{-1} \left(x - \frac{1}{x}\right)^2 \, \mathrm{d}x$$

Solution: We'll do the *work* in class.

$$\int_{-2}^{-1} \left(x - \frac{1}{x}\right)^2 \, \mathrm{d}x = \frac{5}{6}$$

5. Evaluate the integral using the FTC, Part I.

$$\int_{-2}^{1} \left|x^2 - 1\right| \, \mathrm{d}x$$

Solution: We'll do the *work* in class.

$$\int_{-2}^{1} \left|x^2 - 1\right| \, \mathrm{d}x = \frac{8}{3}$$

6. Find a formula for the area function of $f(x) = 2x + 4$ with lower limit $a = 0$.

Solution: We'll do the *work* in class.

$$A(x) = \int_0^x 2t + 4 \, \mathrm{d}t = x^2 + 4x$$

You should do this visually, and make special note that calculus is not needed for this problem.

7. Verify both antiderivatives are correct.

$$\begin{aligned} \int \frac{1}{(1-x)^2} \, \mathrm{d}x &= \frac{1}{1-x} + C_1 \\ \int \frac{1}{(1-x)^2} \, \mathrm{d}x &= \frac{x}{1-x} + C_2 \end{aligned}$$

Solution: We'll do the *work* in class. Here's something to think about. First note that

$$\frac{1}{1-x} - \frac{x}{1-x} = 1, \quad x \neq 1.$$

That is the antiderivatives differ by a constant! Here's why: Suppose $F_1(x)$ and $F_2(x)$ are two different antiderivatives of $f(x)$ on some interval $[a,\ b]$.

$$\begin{aligned} \frac{\mathrm{d}F_1(x)}{\mathrm{d}x} = \frac{\mathrm{d}F_2(x)}{\mathrm{d}x} &= f(x), \quad x \in [a,\ b] \\ \frac{\mathrm{d}\left[F_1(x) - F_2(x)\right]}{\mathrm{d}x} &= 0 \quad x \in [a,\ b] \end{aligned}$$

This implies $F_1(x) - F_2(x)$ is a constant on the interval $[a,\ b]$, which can be written as

$$F_1(x) = F_2(x) + C, \quad x \in [a,\ b]$$

Thus any two antiderivative of the same function on any interval, can differ only by a constant. The antiderivative is therefore not unique, but is *unique up to a constant.* You should now verify that

$$\frac{1}{1-x} = \frac{x}{1-x} + 1,$$

is true for all $x \neq 1$.

8. Find a formula for

$$\int_{3x}^{9x+2} e^{-t}\ \mathrm{d}t.$$

Solution: We'll do the *work* in class.

$$\int_{3x}^{9x+2} e^{-t}\ \mathrm{d}t = e^{-3x} - e^{-9x-2}$$

9. Try to follow these steps and write an English phase next to each step to explain what's going on.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\ \mathrm{d}t\right) &= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\ \mathrm{d}t - \int_a^x f(t)\ \mathrm{d}t}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\ \mathrm{d}t \\
&= \lim_{h\to 0} \frac{1}{h} f(x^*)\,\Delta x, \qquad x \leq x^* \leq x+h \\
&= \lim_{h\to 0} \frac{hf(x^*)}{h}, \qquad \Delta x = (x+h) - (x) \\
&= f(x)
\end{aligned}$$

Solution: We'll discuss this in class.

10. The antiderivatives of $f(x)$ is given by

$$\int_a^x f(t)\ \mathrm{d}t,$$

and we now know that all antiderivatives differ by a constant. Let $G(x)$ be one of these antiderivatives. Try to follow these steps and write an English phase next to each step to explain what's going on.

$$\begin{aligned}
\int_a^x f(t)\ \mathrm{d}t - G(x) &= C \\
\int_a^a f(t)\ \mathrm{d}t - G(a) &= C \\
-G(a) &= C \\
\int_a^b f(t)\ \mathrm{d}t - G(b) &= -G(a) \\
\int_a^b f(t)\ \mathrm{d}t &= G(b) - G(a)
\end{aligned}$$

35.3 Assignment

You should read §5.3 and do the WebAssign assignment mth.121.05.03.

36 mth.121.05.04

36.1 FTC II

We covered both parts of the FTC in the last section. To review

FTC Part I: If f is continuous on the interval $[a,\ b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)\ \mathrm{d}t = F(b) - F(a) = F(t)|_a^b.$$

FTC Part II: This is often stated in another way, because if

$$\int_a^x f(t)\ \mathrm{d}t = F(x) - F(a),$$

then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\ \mathrm{d}t\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[F(x) - F(a)\right] = f(x).$$

This, of course, allows us to differentiate integrals. It should be noted that $x \in [a,\ b]$.

36.2 Examples

1. Let

$$G(x) = \int_1^x t^2 - 2\ \mathrm{d}t.$$

Calculate $G(1)$, $G'(1)$, and $G'(2)$. Then find a formula for $G(x)$

Solution: We'll do the *work* in class.

$$\begin{aligned} G(1) &= 0 \\ G'(1) &= -1 \\ G'(2) &= 2 \\ G(x) &= \frac{x^3}{3} - 2x + \frac{5}{3} \end{aligned}$$

2. Calculate the derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{x^2}^{x^4} \sqrt{t}\ \mathrm{d}t\right]$$

Solution: We'll do the work in class.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{x^2}^{x^4} \sqrt{t}\ \mathrm{d}t\right] = \sqrt{x^4}\cdot 4x^3 - \sqrt{x^2}\cdot 2x = 4x^5 - 2x\,|x|$$

3. Calculate the derivative.

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[\int_{-u}^{3u} \sqrt{x^2+1}\ \mathrm{d}x\right]$$

Solution: We'll discuss this in class.

You're not going to be able to do this by first integrating and then differentiating. You will need to understand the FTC II and the chain rule! Here I am letting $f(x) = \sqrt{x^2+1}$ and $F(x)$ is its antiderivative, that is $F'(x) = f(x)$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\left[\int_{-u}^{3u} \sqrt{x^2+1}\ \mathrm{d}x\right] &= \frac{\mathrm{d}}{\mathrm{d}u}\left[F(3u) - F(-u)\right] \\
&= F'(3u)\cdot\frac{\mathrm{d}}{\mathrm{d}u}[3u] - F'(-u)\cdot\frac{\mathrm{d}}{\mathrm{d}u}[-u] \\
&= f(3u)\cdot 3 - F'(-u)\cdot(-1) \\
&= 3\sqrt{9u^2+1} + \sqrt{u^2+1}
\end{aligned}$$

36.3 Assignment

You should read §5.4 and do the WebAssign assignment mth.121.05.04.

37 mth.121.05.05

37.1 Net Change as the Integral of Rate

Theorem: The net change in $f(t)$ over an interval $[t_1,\ t_2]$ is given by

$$f(t_2) - f(t_1).$$

This can also be given as an integral

$$\int_{t_1}^{t_2} f'(t)\ \mathrm{d}t = f(t_2) - f(t_1).$$

Example: A population of insects increases at a rate $190 + 6t + 0.9t^2$ insects per day. Find the insect population after 3 days, assuming that there are 50 insects at $t = 0$. (Round your answer to the nearest insect.)

Solution: We're given

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P'(t) = 190 + 6t + 0.9t^2$$

Now integrate.

$$\int 190 + 6t + 0.9t^2\ \mathrm{d}t = 190t + 3t^2 + 0.3t^3 + C = P(t)$$

We're told that at $t = 0$ that the P is 50, so that means $C = 50$, and $P(3) = 655.1$. Using the notation above, we have

$$\begin{aligned} \int_0^3 190 + 6t + 0.9t^2\ \mathrm{d}t\ \mathrm{d}t &= P(3) - P(0) \\ 605.1 &= P(3) - 50 \\ 655.1 &= P(3) \end{aligned}$$

37.2 Examples

1. The marginal cost[100] of producing x widgets (in units of 1000) is

$$C'(x) = 300x^2 - 4000x + 40000 \quad \text{dollars per 1000 widgets.}$$

   (a) Determine the cost $(C(x))$ function if $C(0) = 10000$.

[100] If $C(x)$ is the cost function, then $C'(x)$ is called the marginal cost.

Solution:

$$\begin{aligned}
\int 300x^2 - 4000x + 40000 \, \mathrm{d}x &= 100x^3 - 2000x^2 + 40000x + D \\
C(x) &= 100x^3 - 2000x^2 + 40000x + D \\
C(0) &= D = 10000 \\
C(x) &= 100x^3 - 2000x^2 + 40000x + 10000
\end{aligned}$$

(b) Determine the cost of increasing production from 1000 to 3000 widgets. Do this as an integration and as a net change.

Solution:

$$\begin{aligned}
\int_1^3 300x^2 - 4000x + 40000 \, \mathrm{d}x &= 66600 \\
C(3) - C(1) &= 66600
\end{aligned}$$

(c) Determine the total cost of producing 7000 widgets

Solution:

$$\begin{aligned}
C(7) - C(0) &= \int_0^7 300x^2 - 4000x + 40000 \, \mathrm{d}x \\
C(7) - 10000 &= 216300 \\
C(7) &= 226300
\end{aligned}$$

2. A particle has velocity $v(t) = t^3 - 10t^2 + 24t$ meters per second. Compute:

(a) Displacement[101] over $[0,\ 6]$.

Solution: A little though is probably more useful than the formula!

$$\begin{aligned}
s(t) &= \int t^3 - 10t^2 + 24t \, \mathrm{d}t = \frac{t^4}{4} - \frac{10t^3}{3} + 12t^2 + C \\
s(6) - s(0) &= 36
\end{aligned}$$

You could also do this using the formula.

$$\int_0^6 t^3 - 10t^2 + 24t \, \mathrm{d}t = 36$$

The displacement is 36 meters.

[101] Displacement during $[t_1,\ t_2]$ is given by

$$\int_{t_1}^{t_2} v(t) \, \mathrm{d}t$$

(b) Total distance[102] traveled over $[0,\ 6]$.

Solution: Distance traveled takes a little more thought! The velocity is

$$v(t) = t^3 - 10t^2 + 24t = t(t-6)(t-4),$$

and using simple sign-analysis we get the object moving right (positive velocity) between $[0,\ 4]$; and we get the object moving left (negative velocity) between $[4,\ 6]$.

$$\begin{aligned} s(4) - s(0) &= \frac{128}{3}, \quad \text{to right, so it's positive} \\ s(6) - s(4) &= -\frac{20}{3}, \quad \text{to left, so it's negative.} \end{aligned}$$

The total distance is

$$\frac{148}{3} \text{ meters}$$

You could also do this using the formula.

$$\int_0^6 \left| t^3 - 10t^2 + 24t \right| \, \mathrm{d}t = \frac{148}{3}$$

37.3 Assignment

You should read §5.5 and do the WebAssign assignment mth.121.05.05.

[102]Total distance during $[t_1,\ t_2]$ is given by

$$\int_{t_1}^{t_2} |v(t)| \, \mathrm{d}t$$

38 mth.121.05.06

38.1 Integration

38.1.1 Recognition

Here's a partial list of integrals that I think everyone should be familiar with, although I don't think there's universal agreement on this list, it's a good basis for moving forward. I'm placing a boxed star $\boxed{\star}$ next to those that you should *absolutely* know, where the exponent indicates importance![103]

1. $\displaystyle\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1 \qquad \boxed{\star^{100}}$
2. $\displaystyle\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + C \qquad \boxed{\star^{50}}$
3. $\displaystyle\int e^x \, \mathrm{d}x = e^x + C \qquad \boxed{\star^{\infty}}$
4. $\displaystyle\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + C$
5. $\displaystyle\int \sin x \, \mathrm{d}x = -\cos x + C \qquad \boxed{\star^{25}}$
6. $\displaystyle\int \cos x \, \mathrm{d}x = \sin x + C \qquad \boxed{\star^{25}}$
7. $\displaystyle\int \sec^2 x \, \mathrm{d}x = \tan x + C$
8. $\displaystyle\int \csc^2 x \, \mathrm{d}x = -\cot x + C$
9. $\displaystyle\int \sec x \, \tan x \, \mathrm{d}x = \sec x + C$
10. $\displaystyle\int \csc x \, \cot x \, \mathrm{d}x = -\csc x + C$
11. $\displaystyle\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C$
12. $\displaystyle\int \csc x \, \mathrm{d}x = \ln|\csc x - \cot x| + C$
13. $\displaystyle\int \tan x \, \mathrm{d}x = \ln|\sec x| + C$

[103] At least while taking courses related to calculus.

14. $\displaystyle\int \cot x \, \mathrm{d}x = \ln|\sin x| + C$

15. $\displaystyle\int \sinh x \, \mathrm{d}x = \cosh x + C$

16. $\displaystyle\int \cosh x \, \mathrm{d}x = \sinh x + C$

17. $\displaystyle\int \frac{1}{1+x^2} \, \mathrm{d}x = \arctan x + C$ $\boxed{\star^3}$

18. $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arcsin x + C$ $\boxed{\star^2}$

38.1.2 u-Substitution in Action!

Recognition is the single most important method of integration, but when recognition fails we will need to proceed with a method of integration called u-substitution. Next semester you will learn many more techniques of integration (parts, partial fractions, trigonometric substitution, *etc.*). Always keep in mind that recognition and then u-substitution should be at the top of your list when doing an integration.

We have the general form to recognize.

$$\int f(g(x))\, g'(x) \, \mathrm{d}x$$

Now let F be the antiderivative of f, that is $F' = f$. By the chain rule we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(g(x))\right] = F'(g(x))\, g'(x).$$

Therefore we have:

$$\int f(g(x))\, g'(x) \, \mathrm{d}x = F(g(x)) + C.$$

Now let

$$u = g(x) \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = g'(x),$$

it follows that

$$\int f(g(x))\, g'(x) \, \mathrm{d}x = \int f(u) \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x = F(u) + C,$$

but we also know that

$$\int f(u) \, \mathrm{d}u = F(u) + C.$$

Thus we finally have

$$\int f(u) \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x = \int f(u) \, \mathrm{d}u = F(u) + C.$$

Okay, you're probably totally confused by now, so let's do some examples.

38.2 Examples

1. Find

$$\int 3x^2 \cos x^3 \, \mathrm{d}x,$$

by using $u = x^3$.

Solution: We'll discuss this in class.

Taking the suggestion we have:

$$u = x^3 \quad \Rightarrow \quad \mathrm{d}u = 3x^2 \, \mathrm{d}x.$$

Now make the substitution.

$$\begin{aligned} \int 3x^2 \cos x^3 \, \mathrm{d}x &= \int^* \cos u \, \mathrm{d}u \\ &= \sin u + C \\ &= \sin x^3 + C \end{aligned}$$

2. Find

$$\int x^3 \sqrt{x^4 + 5} \, \mathrm{d}x,$$

by using $u = x^4 + 5$.

Solution: We'll discuss this in class.

Taking the suggestion we have:

$$u = x^4 + 5 \quad \Rightarrow \quad \mathrm{d}u = 4x^3 \, \mathrm{d}x \quad \Rightarrow \quad \frac{1}{4}\mathrm{d}u = x^3 \, \mathrm{d}x.$$

Now make the substitution.

$$\begin{aligned} \int x^3 \sqrt{x^4 + 5} \, \mathrm{d}x &= \frac{1}{4} \int^* u^{1/2} \, \mathrm{d}u \\ &= \frac{u^{3/2}}{6} + C \\ &= \frac{(x^4 + 5)\sqrt{x^4 + 5}}{6} + C \end{aligned}$$

3. Find

$$\int \sqrt{1 + \sqrt{x}} \, \mathrm{d}x,$$

by using $u = 1 + \sqrt{x}$.

Solution: We'll discuss this in class.

This suggestion is a bit tricky.

$$\begin{aligned} u &= 1 + \sqrt{x} \\ u - 1 &= \sqrt{x} \\ (u-1)^2 &= x \\ 2(u-1)\ \mathrm{d}u &= \mathrm{d}x \end{aligned}$$

Now make the substitution.

$$\begin{aligned} \int \sqrt{1+\sqrt{x}}\ \mathrm{d}x &= 2\int^{*} \sqrt{u}\,(u-1)\ \mathrm{d}u \\ &= 2\int^{*} u^{3/2} - u^{1/2}\ \mathrm{d}u \\ &= \frac{4u^{5/2}}{5} - \frac{4u^{3/2}}{3} + C \\ &= \frac{4\,(1+\sqrt{x})\,(3\sqrt{x}-2)\,\sqrt{1+\sqrt{x}}}{15} + C \end{aligned}$$

4. Evaluate

$$\int_{-1}^{2} \sqrt{x+2}\ \mathrm{d}x,$$

by using $u = x + 2$.

Solution: We'll discuss this in class.

Taking the suggestion we have:

$$u = x + 2 \quad \Rightarrow \quad \mathrm{d}u = \mathrm{d}x,$$

and when

$$x = -1 \quad \Rightarrow \quad u = 1 \qquad \text{and} \qquad x = 2 \quad \Rightarrow \quad u = 4.$$

Using this information, our integral becomes:

$$
\begin{aligned}
\int_{-1}^{2} \sqrt{x+2}\, \mathrm{d}x &= \int_{1}^{4} \sqrt{u}\, \mathrm{d}u \\
&= \int_{1}^{4} u^{1/2}\, \mathrm{d}u \\
&= \left.\frac{2u\sqrt{u}}{3}\right|_{1}^{4} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}
\end{aligned}
$$

The graphs (Figure 90, page 257) are a visual of the original integral (left) and the equivalent transformed integral (right).

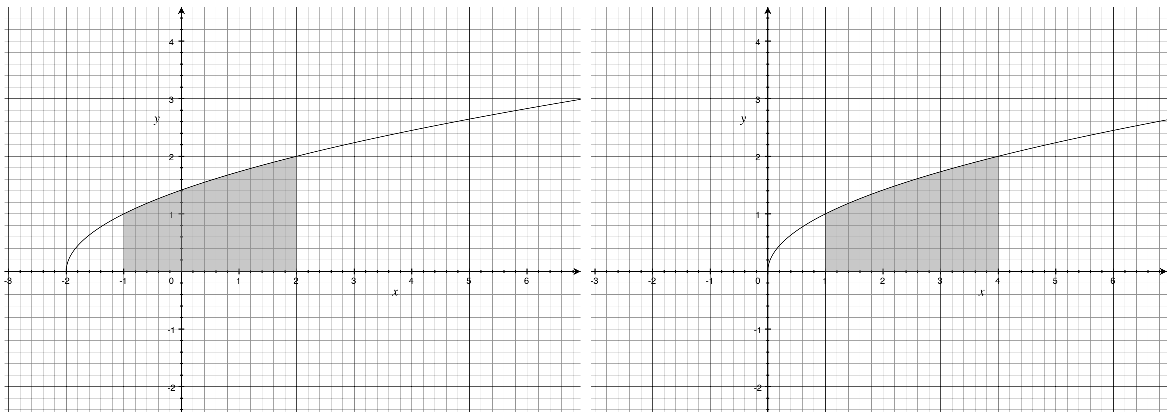

Figure 90: Side-by-side comparison of the transformation.

5. Evaluate

$$\int_{0}^{\pi} \cos\left(x+\pi\right)\, \mathrm{d}x,$$

by using $u = x + \pi$.

Solution: We'll discuss this in class.

Taking the suggestion we have:

$$u = x + \pi \quad \Rightarrow \quad \mathrm{d}u = \mathrm{d}x,$$

and when

$$x = 0 \quad \Rightarrow \quad u = \pi \qquad \text{and} \qquad x = \pi \quad \Rightarrow \quad u = 2\pi.$$

Using this information, our integral becomes:

$$\begin{aligned}\int_0^{\pi} \cos(x+\pi)\ \mathrm{d}x &= \int_{\pi}^{2\pi} \cos(u)\ \mathrm{d}u \\ &= \sin u\big|_{\pi}^{2\pi} \\ &= 0 - 0 = 0\end{aligned}$$

The graphs (Figure 91, page 258) are a visual of the original integral (left) and the equivalent transformed integral (right).

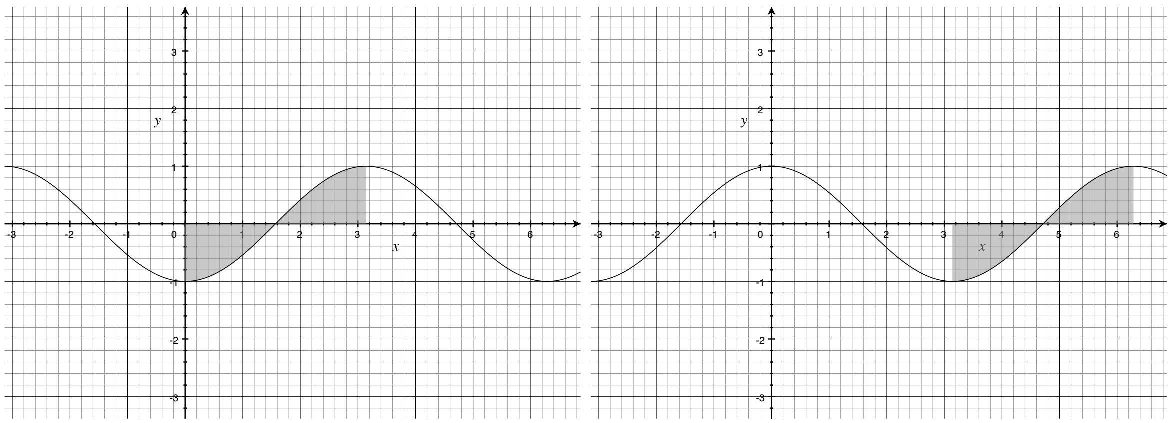

Figure 91: Side-by-side comparison of the transformation.

6. Evaluate

$$\int_0^2 \frac{x}{(1+x^2)^2}\ \mathrm{d}x,$$

by using $u = 1 + x^2$.

Solution: We'll discuss this in class.

Taking the suggestion we have:

$$u = 1 + x^2 \quad \Rightarrow \quad \mathrm{d}u = 2x\mathrm{d}x,$$

and when

$$x = 0 \quad \Rightarrow \quad u = 1 \qquad \text{and} \qquad x = 2 \quad \Rightarrow \quad u = 5.$$

Using this information, our integral becomes:

$$\begin{aligned}\int_0^2 \frac{x}{(1+x^2)^2}\,\mathrm{d}x &= \frac{1}{2}\int_1^5 u^{-2}\,\mathrm{d}u\\ &= -\frac{1}{2u}\bigg|_1^5\\ &= -\frac{1}{10}+\frac{1}{2}=\frac{2}{5}\end{aligned}$$

The graphs (Figure 92, page 259) are a visual of the original integral (left) and the equivalent transformed integral (right).

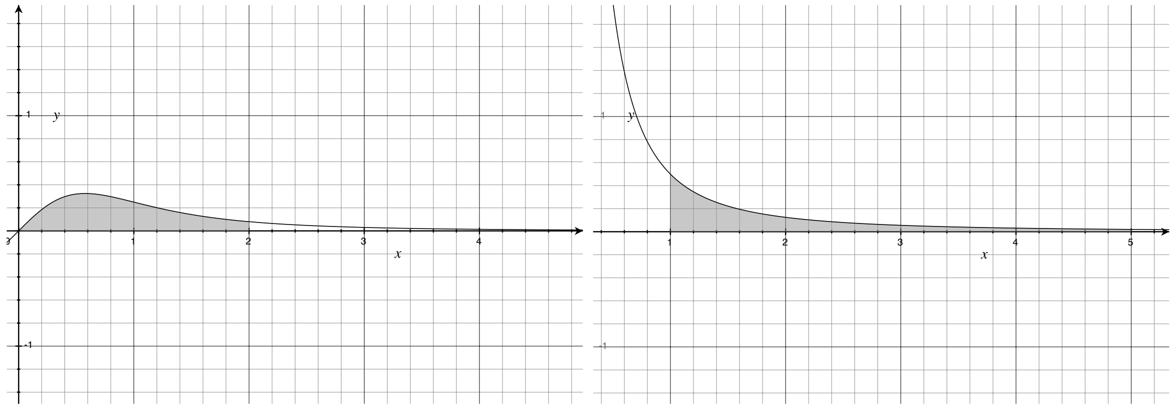

Figure 92: Side-by-side comparison of the transformation.

7. Evaluate

$$\int_3^7 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x,$$

by using $u = x + 1$.

Solution: We'll discuss this in class.

Taking the suggestion we have:

$$u = x + 1 \quad\Rightarrow\quad \mathrm{d}u = \mathrm{d}x,$$

and when

$$x = 3 \quad\Rightarrow\quad u = 4 \qquad\text{and}\qquad x = 7 \quad\Rightarrow\quad u = 8.$$

Using this information, our integral becomes:

$$\begin{aligned}
\int_3^7 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x &= \int_4^8 \frac{u-1}{\sqrt{u}}\,\mathrm{d}u \\
&= \int_4^8 u^{1/2} - u^{-1/2}\,\mathrm{d}u \\
&= \frac{2u\sqrt{u}}{3} - 2\sqrt{u}\Bigg|_4^8 = \frac{20\sqrt{2}}{3} - \frac{4}{3} = \frac{20\sqrt{2}-4}{3}
\end{aligned}$$

The graphs (Figure 93, page 260) are a visual of the original integral (left) and the equivalent transformed integral (right).

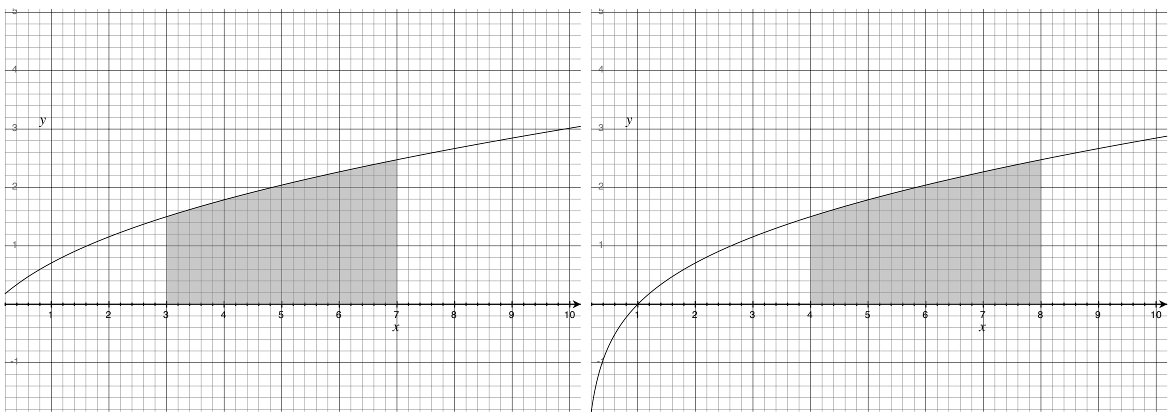

Figure 93: Side-by-side comparison of the transformation.

8. Evaluate

$$\int_0^{\frac{\pi}{4}} \frac{\tan^3\theta}{\cos^2\theta}\,\mathrm{d}\theta,$$

by using $u = \tan\theta$.

Solution: We'll discuss this in class.

Taking the suggestion we have:

$$u = \tan\theta \quad \Rightarrow \quad \mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta,$$

and when

$$\theta = 0 \quad \Rightarrow \quad u = 0 \qquad \text{and} \qquad \theta = \frac{\pi}{4} \quad \Rightarrow \quad u = 1.$$

Using this information, our integral becomes:

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\tan^3\theta}{\cos^2\theta}\,\mathrm{d}\theta &= \int_0^{\frac{\pi}{4}} \tan^3\theta\sec^2\theta\,\mathrm{d}\theta \\ &= \int_0^1 u^3\,\mathrm{d}u \\ &= \left.\frac{u^4}{4}\right|_0^1 \\ &= \frac{1}{4} - 0 = \frac{1}{4}\end{aligned}$$

The graphs (Figure 94, page 261) are a visual of the original integral (left) and the equivalent transformed integral (right).

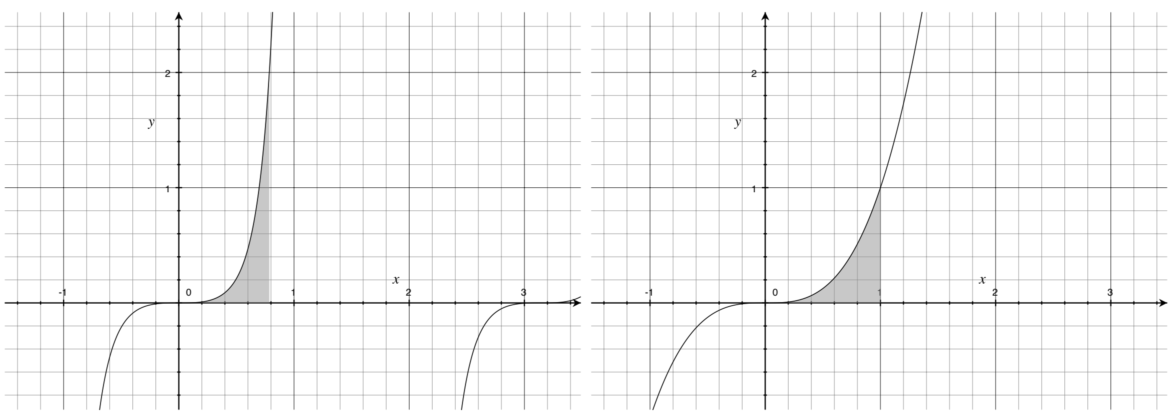

Figure 94: Side-by-side comparison of the transformation.

38.2.1 Mathematica Code

Here's a screen capture of the Mathematica code (Figure 95, page 262). Yes, everyone should learn how to integrate using software, especially on a calculator.

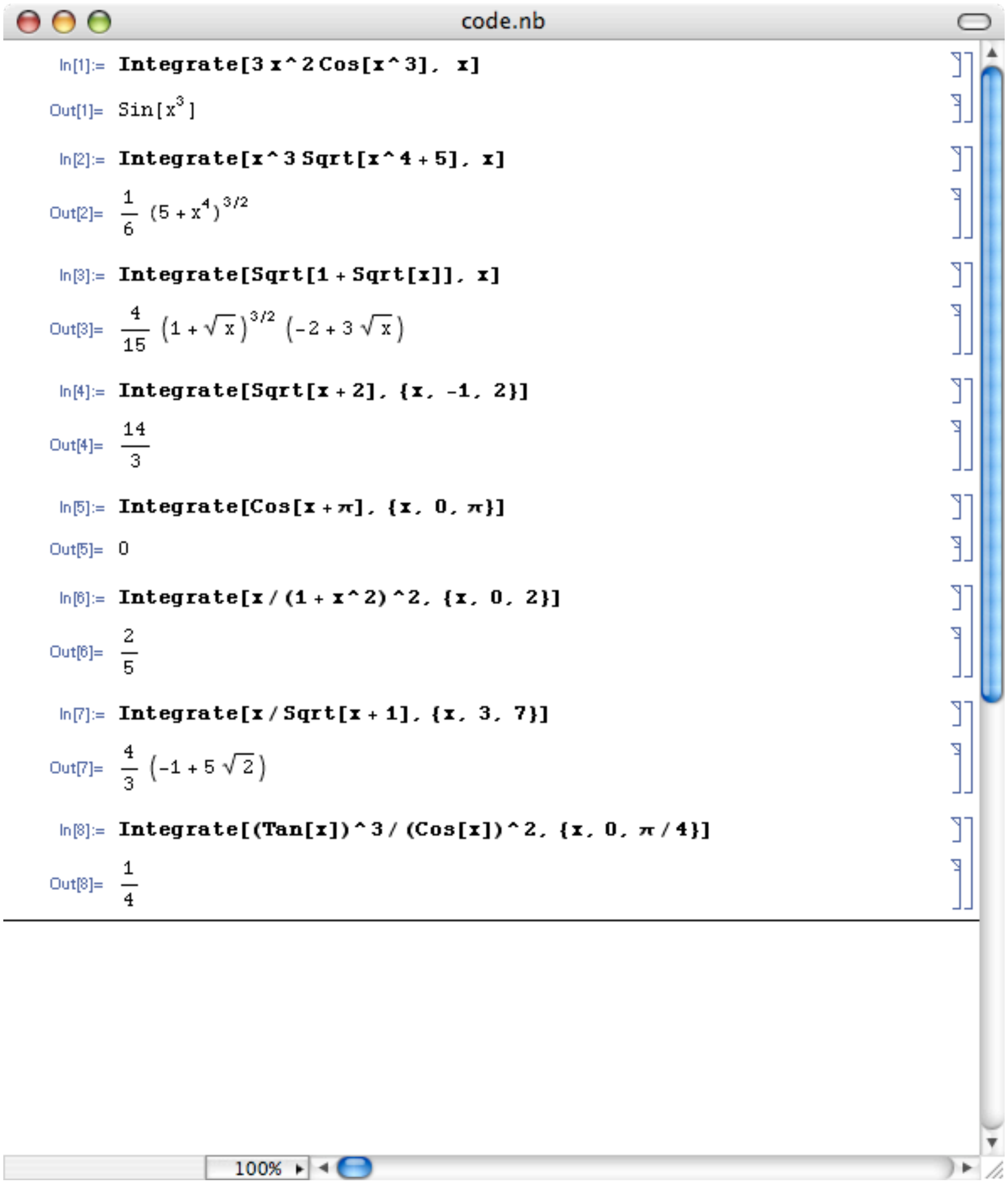

Figure 95: Mathematica Code

38.3 Assignment

You should read §5.6 and do the WebAssign assignment mth.122.05.06.

39 mth.121.06.01

39.1 Applications of Integration I

39.1.1 Area Between Curves

This material should be considered a natural extension of what you have learned so far about integration.

1. Compute the area between $f(x) = -\sin x$ and $g(x) = -2\sin x$ from 0 to π. You should first estimate this area, the graph (Figure 96, page 264) is drawn with proper aspect ratio.

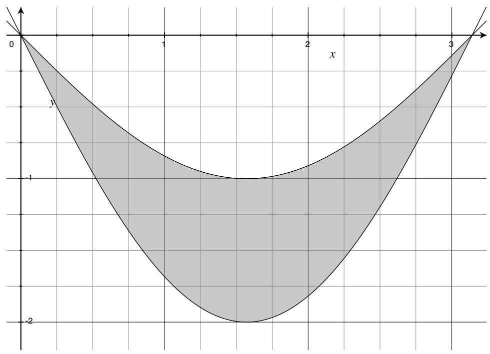

Figure 96: $f(x) = -\sin x$ and $g(x) = -2\sin x$ from 0 to π

Solution: We'll discuss this in class.

Setting up the integral is the first step, and easily relates to what you learned about integration so far. Basically you will need to visual rectangles and then sum them together using integration.

$$\int_0^{\pi} [-\sin x - (-2\sin x)] \, \mathrm{d}x = \int_0^{\pi} \sin x \, \mathrm{d}x = 2$$

2. Compute the area between $f(x) = x^2$ and $g(x) = x^3$ from -1 to 2. The graph (Figure 97, page 265) is *not* drawn with proper aspect ratio.

Solution: We'll discuss this in class.

Again, just like before we need to set-up the integration and this will be throughly discussed in class!

$$\int_{-1}^{2} \left|x^2 - x^3\right| \, \mathrm{d}x = \int_{-1}^{1} x^2 - x^3 \, \mathrm{d}x + \int_{1}^{2} x^3 - x^2 \, \mathrm{d}x = \frac{25}{12}$$

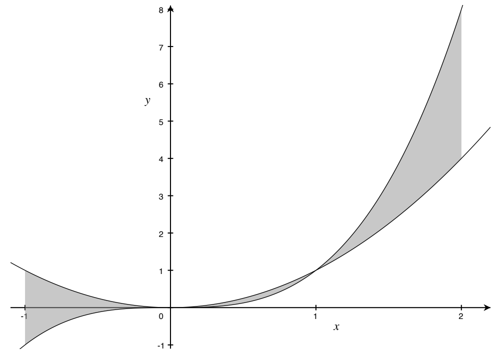

Figure 97: $f(x) = x^2$ and $g(x) = x^3$ from -1 to 2

3. Compute the area between $y = \sqrt[5]{x+1}$ and $x = 3y - y^2$. The graph (Figure 98, page 265) is *not* drawn with proper aspect ratio. The approximate points of intersection are $(-1.00257,\ -0.30349)$ and $(2.19305,\ 1.26137)$ and they are not easy to find—however, *everyone* should be able to use technology to solve problems like this.

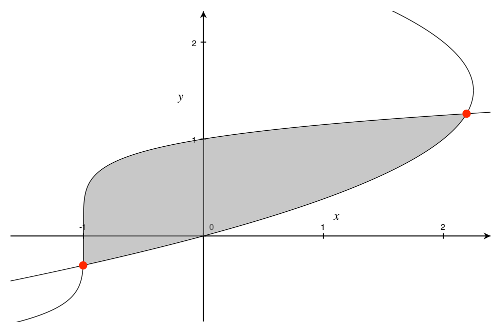

Figure 98: Points of intersection indicated red.

Solution: We'll discuss this in class.

Again, just like before we need to set-up the integration and this will be throughly discussed in class! (Yes, I am using a calculator to do the arithmetic.)

$$\int_{-0.30349}^{1.26137} 3y - y^2 - y^5 + 1 \text{ d}y \approx 2.46385$$

39.2 Examples

1. Consider the following.

$$f(x) = 8x - 10$$
$$g(x) = x^2 - 4x + 10$$

(a) Find the points of intersection of the graphs.

Solution: We'll discuss this in class.

The points of intersection occur at $x = 2$ and $x = 10$.

(b) Compute the area of the region below the graph of f and above the graph of g.

Solution: We'll discuss this in class.

A graph (Figure 99, page 266) is provided, but you should be able to do this on your own!

$$\int_2^{10} (8x - 10) - \left(x^2 - 4x + 10\right) \, \mathrm{d}x = \frac{256}{3}$$

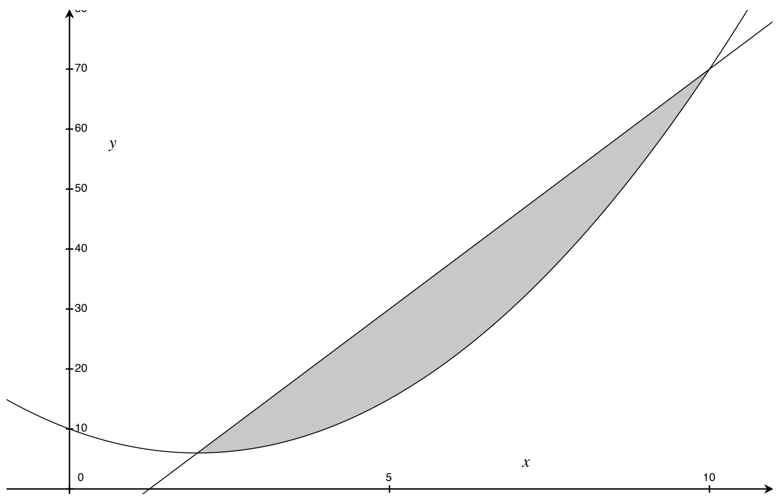

Figure 99: Area of interest

2. Find the area of the region enclosed by the graphs $f(x) = 20 + x - x^2$ and $g(x) = x^2 - 5x$.

Solution: You should be able to graph these two curves quickly and algebraically determine the points of intersection—this must be done, and once done the integration should be easy to write down. Here's the integration.

$$\int_{-2}^{5} \left(20 + x - x^2\right) - \left(x^2 - 5x\right) \, \mathrm{d}x = \frac{343}{3}$$

You should be able to do the integration, but for now I am more interested in setting it up. That, for many, is the tough part.

3. Find the area of the region enclosed by the graphs of $x = 9 - y^2$ and $x = 7$.

Solution: Again, you need to do the graph and then set-up the integration.

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left(9 - y^2\right) - (7)\ \mathrm{d}y = \frac{8\sqrt{2}}{3}$$

39.3 Assignment

You should read §6.1 and do the WebAssign assignment mth.121.06.01.

40 mth.121.06.02

40.1 Applications of Integration II

40.1.1 Volumes Part I

Again, your ability to visualize these problems will allow you to proceed quickly. The extension here is going from rectangles (two-dimensional) to boxes (three-dimensional). The process will be discussed in class.

Example: Find the volume of the solid whose cross-sections perpendicular to the x-axis are isosceles right triangles (Figure 100, page 268).

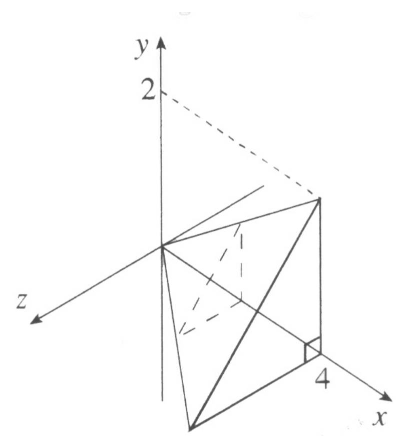

Figure 100: Isosceles right triangles

Solution: We'll discuss this in class.

$$\int_0^4 \frac{x^2}{8}\, \mathrm{d}x = \frac{8}{3}$$

40.1.2 Average Value of a Data Set

You should all know how to take a simple average of a finite data set where each value in the data set is of equal importance. For example, if the data set is composed of n equally important data points, the average is:

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}.$$

However, this finite example may extend to the infinite. For example, the infinite data set

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \cdots\ \frac{1}{2^n},\ \cdots$$

also has an average. Using the formula above we have,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2^i} = \frac{1}{n}\left(1-2^{-n}\right).$$

Now taking a limit as $n \to \infty$ we get

$$\lim_{n\to\infty}\frac{1}{n}\left(1-2^{-n}\right) = 0.$$

Wow, an average that's not even in the data set.

40.1.3 Average Value of a Function

The average value of a function on an interval $[a,\ b]$ is given by

$$f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\ \mathrm{d}x.$$

This formula will be discussed in class.

Example: Given the graph of $f(x) = x^2\sqrt{10-x^3},\ [0,\ 2]$, find its average value.

Solution: This will be discussed in class.[104]

$$f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\ \mathrm{d}x = \frac{1}{2}\int_0^2 f(x)\ \mathrm{d}x = \frac{10\sqrt{10}-2\sqrt{2}}{9} \approx 3.19937$$

The graph (Figure 101, 270) should be used to *make sense* of what average value is for these problems.

Using f, you should notice that there's a $c \in [0,\ 2]$, such that

$$f(c) = \frac{10\sqrt{10}-2\sqrt{2}}{9}.$$

Approximate this c. Well, it certainly appears to be slightly greater than 1, but I decided to use my calculator and found that the c is approximately 1.0359231772. Yes, calculators are very useful.

The Mean Value Theorem for Integrals If f is continuous on $[a,\ b]$, then there exists a number c in $[a,\ b]$ such that

$$\int_a^b f(x)\ \mathrm{d}x = f(c)\cdot(b-a)$$

Here's a visual (Figure 102, page 270) of theorem using the prior example.

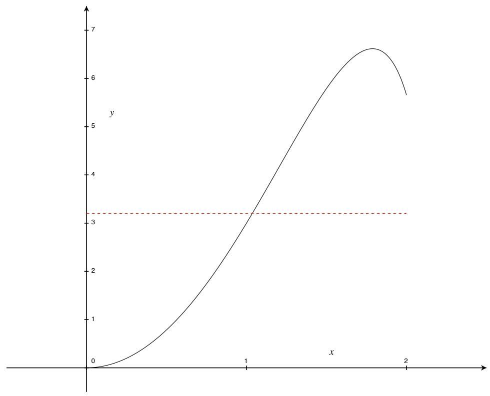

Figure 101: $f(x) = x^2\sqrt{10 - x^3}$, $[0,\ 2]$

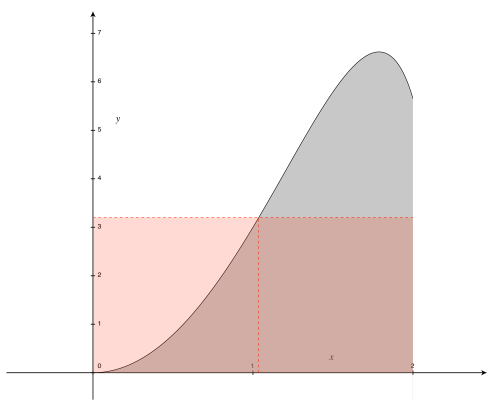

Figure 102: The Mean Value Theorem for Integrals Visualized!

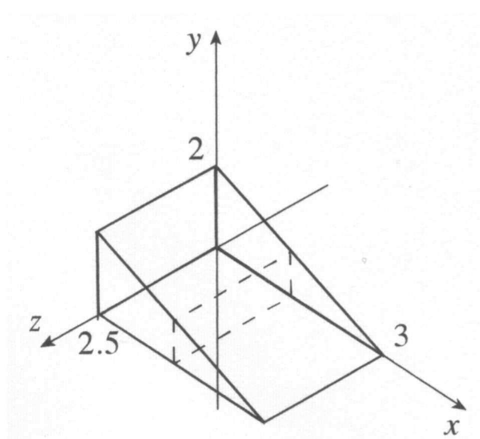

Figure 103: Rectangles

40.2 Examples

1. Find the volume of the solid whose cross-sections perpendicular to the x-axis are rectangles (Figure 103, page 270).

Solution: We'll discuss this in class.

$$\int_0^3 5 - \frac{5x}{3} \, dx = \frac{15}{2}$$

2. Find the volume of the solid whose cross-sections perpendicular to the x-axis are half-circles (Figure 104, page 271).

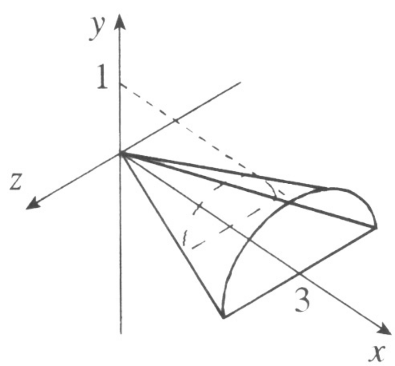

Figure 104: Half-circles

Solution: We'll discuss this in class.

$$\int_0^3 \frac{\pi x^2}{18} \, dx = \frac{\pi}{2}$$

3. Assuming that the given graph (Figure 105, page 272) is composed of circular regions of equal radii, find the area under this curve and then the value(s) for c.[105]

[105] The area is:

$$\left(1 + \frac{\pi}{4}\right) + \left(1 - \frac{\pi}{4}\right) + \left(1 - \frac{\pi}{4}\right) + \left(1 + \frac{\pi}{4}\right) = 4.$$

The value for c is ± 1.

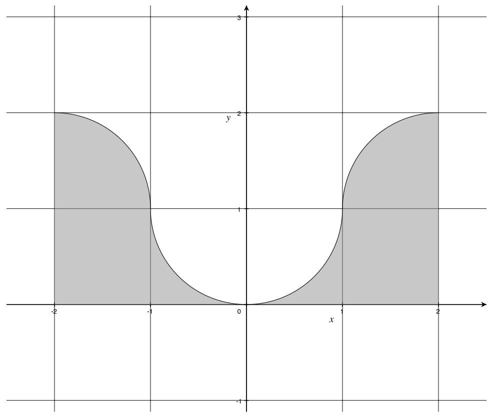

Figure 105: A visual example.

4. Calculate the average over the given interval. (Round your answer to three decimal places.)

$$f(x) = e^{-3x}, \quad [-1,\ 9]$$

Solution: We'll discuss this in class.

$$\frac{1}{9-(-1)} \int_{-1}^{9} e^{-3x}\, \mathrm{d}x = \frac{e^{30}-1}{30e^{27}}$$

40.3 Assignment

You should read §6.2 and do the WebAssign assignment mth.121.06.02.

41 mth.121.06.03

41.1 Volumes of Revolutions

41.1.1 Method of Disks/Washers

If $f(x)$ is continuous and $f(x) \geq 0$ on $[a,\ b]$, then the solid obtained by rotating the region under the graph about the x-axis has volume

$$V = \int_a^b \pi \left[f(x) \right]^2 \, \mathrm{d}x.$$

This formula will be discussed in class. And I suggest that you make a good effort in attempting to understand the process (we'll do examples) and not just memorize.

Example: Find the volume of the solid obtained (Figure 106, page 273) by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

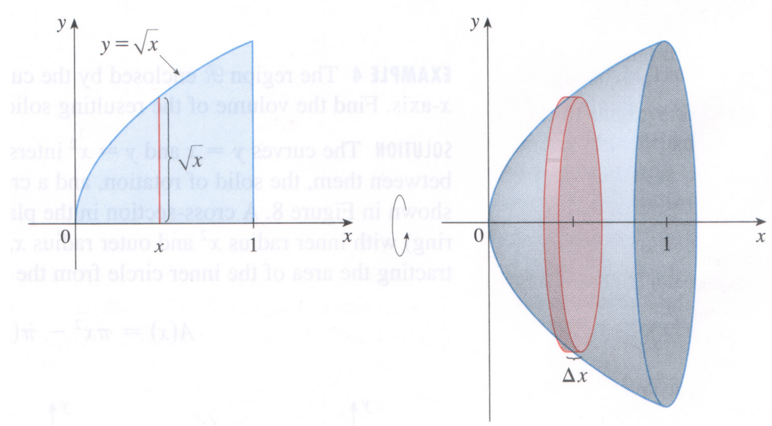

Figure 106: Indicated volume

Solution: This will be discussed in class.

Final answer:

$$\int_0^1 \pi x \, \mathrm{d}x = \frac{\pi}{2}$$

41.2 Examples

1. Find the volume of the solid obtained (Figure 107, page 274) by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

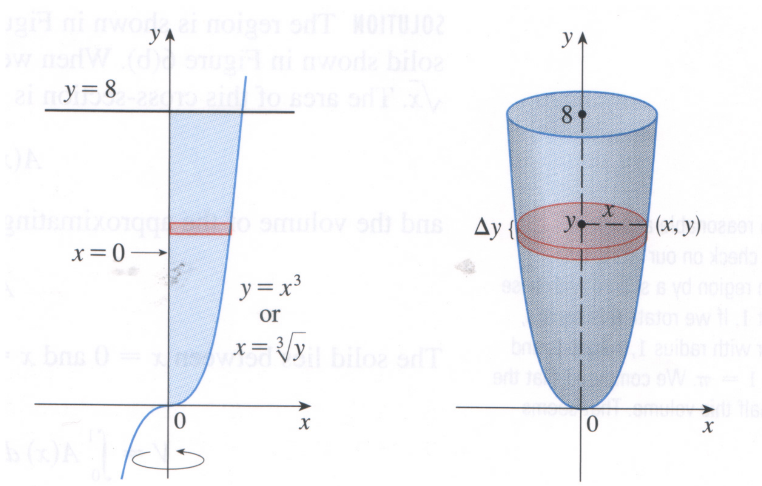

Figure 107: Indicated Volume

Solution: This will be discussed in class.

Final answer:

$$\int_0^8 \pi y^{2/3}\ \mathrm{d}y = \frac{96\pi}{5}$$

2. The region $\Re$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid (Figure 108, page 274).

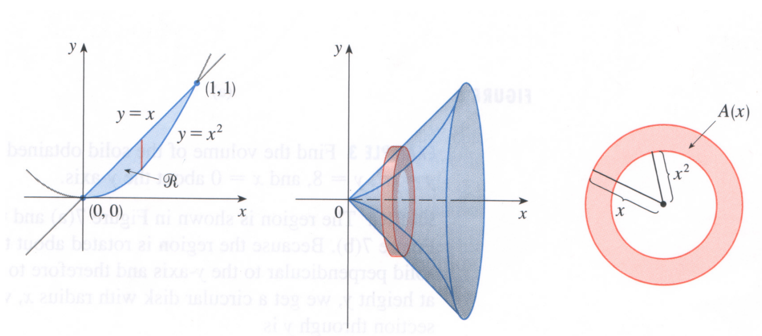

Figure 108: Indicated volume

Solution: This will be discussed in class.

Final answer:

$$\int_0^1 \pi\left(x^2 - x^4\right)\ \mathrm{d}x = \frac{2\pi}{15}$$

41.3 Assignment

You should read §6.3 and do the WebAssign assignment mth.121.06.03.

42 mth.121.06.04

42.1 Volumes of Revolutions

42.1.1 The Method of Cylindrical Shells

You should now be familiar with the disk (washer) method to find the volume of a rotated region. In fact the the disk (washer) method is the only method taught at some schools (particularly high schools), but I find that this method can in fact be quite difficult when dealing with some problems. However, it is your option to do use whatever method you like. Here, however, I will outline the method of cylindrical shells.

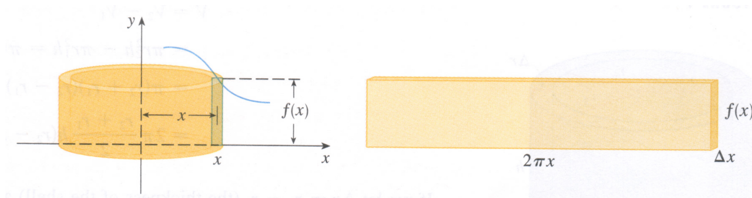

Figure 109: Indicated volume

As you can see from this visual (Figure 109, page 276), you need to see that the cylindrical shell is cut and then spread out to form a rectangular box. That is, this particular shell has the volume

$$\text{volume} \; = \text{length} \; \times \text{height} \; \times \text{width},$$

which symbolically is

$$V = 2\pi x \times f(x) \times \Delta x.$$

If you decide to use this method you will need to visualize the cylindrical shells and construct an integral that represents the volume.

42.2 Examples

1. Find the volume of the solid (Figure 110, page 277) obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

   Solution: We'll discuss this in class.

   Cylindrical shells:

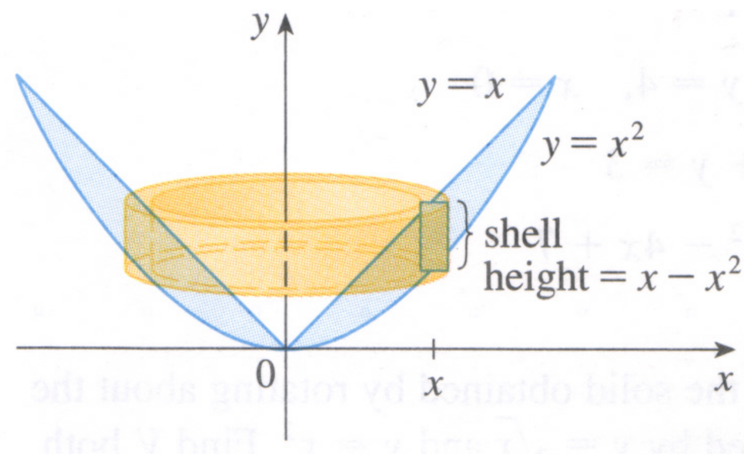

Figure 110: Indicated volume

$$
\begin{aligned}
\int_0^1 2\pi x \cdot \left(x - x^2\right)\ \mathrm{d}x &= 2\pi \int_0^1 x^2 - x^3\ \mathrm{d}x \\
&= 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\
&= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) \\
&= \frac{\pi}{6}
\end{aligned}
$$

Disc (washer): .

$$
\begin{aligned}
\int_0^1 \pi \left[\sqrt{y}\right]^2\ \mathrm{d}y - \int_0^1 \pi \left[y\right]^2\ \mathrm{d}y &= \pi \int_0^1 y\ \mathrm{d}y - \pi \int_0^1 y^2\ \mathrm{d}y \\
&= \pi \int_0^1 y - y^2\ \mathrm{d}y \\
&= \pi \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 \\
&= \pi \left(\frac{1}{2} - \frac{1}{3}\right) \\
&= \frac{\pi}{6}
\end{aligned}
$$

42.3 Assignment

You should read §6.4 and do the WebAssign assignment mth.121.06.04.

43 How to Present [Optional Material]

43.1 The Art of $\LaTeX\,2_\varepsilon$

Yes, most mathematics teachers fear looking at students typewritten work as if it were the plague. I should make copies of all the typewritten papers I receive and publish a book. Just take a look at the following anonymous MTH-121 student example (Figure 111, page 278). It's just plain bizarre! Again, I can't say it enough, "learn how to present information

Y= sin(x+y)

Y^1= cos(x+y)(1+y)

Y^1= cos(x+y) + Y^1(cos(x+y))

Y^1-Y^1 cos(x+y) = cos(x+y)

Y^1(1-cos(x+y)) = cos(x+y)

Y^1= cos(x+y)/1-cos(x+y)

Given, Y^1 = 0

Then,

Y^1 = cos(x+y)/1-cos(x+y) = 0

Therefore,

Cos(x+y) = 0

(x+y) = cos^{-1}0

Figure 111: A MTH-121 student's work.

using a computer." Even if it's just using Microsoft's Word and Design Science's MathType, you all need to learn how to typeset mathematics, including graphs. If, for whatever reason, you believe that you need to be explicitly told what to do and how to do it, you are not only wrong, but you're being lead down a *mediocre* path. Learn to do what works for you. More importantly, you're being judge by not only what you present, but how it *looks*. First impressions are all too important.

43.2 An Example

Suppose you're asked to find the volume when a region in the first quadrant, bounded by

$$y = \sin x \qquad \text{and} \qquad y = x^3 - 1,$$

is revolved about the line $x = 5$. I have consistently tried to emphasize the importance of using technology to create a *good* graph and determining the points of intersection. Here's a fairly good picture (Figure 112, page 279), although not properly scaled. This visual is

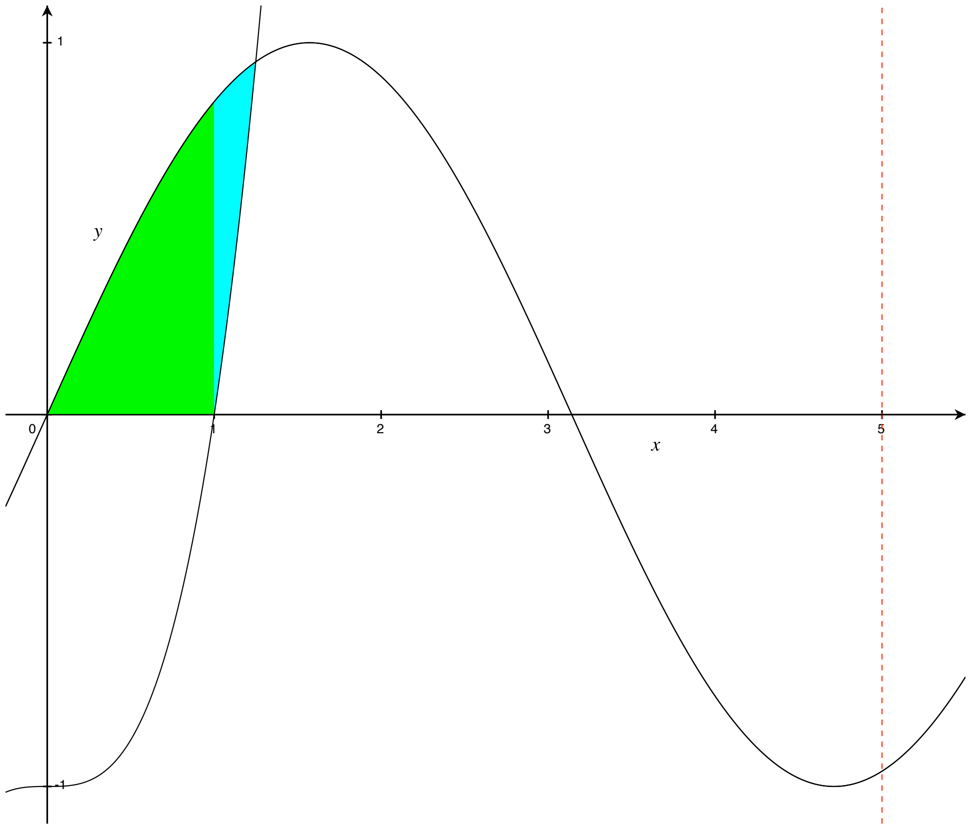

Figure 112: The area of interest.

actually better than one can get with a handheld calculator—mainly because I can use color to emphasize the nature of this region. Yes, it splits!

Setting up an integral is important, but without a visual we're bound to make mistakes. Please, I can not emphasize this enough, and I am somewhat dismayed by how long it takes people to get their arse's moving—drawing a damn picture with your calculator is not difficult and should be done quickly.

Method of Discs I will discuss this set-up in class. If you don't have a calculator capable of doing this, then use Mathematica (Figure 113, page 280)!

$$\int_0^{0.9487} \pi\left(5 - \arcsin y\right)^2 \; \mathrm{d}y - \int_0^{0.9487} \pi\left(5 - \sqrt[3]{y+1}\right)^2 \; \mathrm{d}y \approx 15.3681$$

Method of Shells I will discuss this set-up in class. If you don't have a calculator capable of doing this, then use Mathematica (Figure 113, page 280)!

$$\int_0^{1} 2\pi\left(5 - x\right)\sin x \; \mathrm{d}x + \int_1^{1.2491} 2\pi\left(5 - x\right)\left(\sin x - x^3 + 1\right) \; \mathrm{d}x \approx 15.3681$$

Figure 113: Mathematica Code

Here's the Mathematica code (Figure 113, page 280). Yes, if you've read this far I am hopeful that you've learned something. And yes, this should prove helpful to those diligent enough to take stock in my warnings!

43.3 Another Example

I think one of your homework problems asked for a proof of this:

$$\int_0^{\frac{\pi}{2}} x \sin x \, \mathrm{d}x \leq \frac{\pi^2}{8}.$$

You're not being asked to do the integral, just come up with a convincing argument as to why this is true. Yes, I think you should turn the lights on by using technology. Here's a graph (Figure 114, page 280):

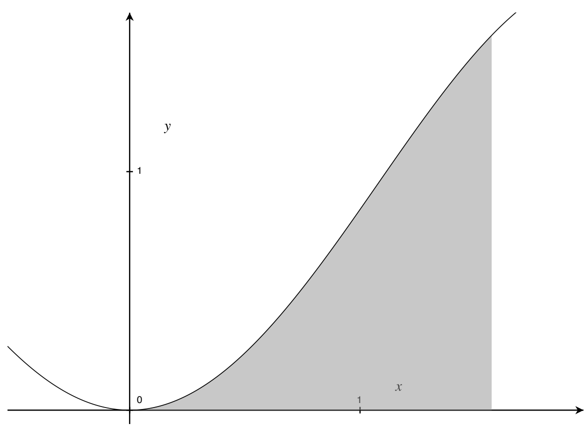

Figure 114: The area of interest.

This visual can offer great insight into what is being asked. Although I did not mention some very important properties of integrals, I want to suggest that this proof can actually be done by noting that

$$0 \leq \sin x \leq 1$$

for all $x \in \left[0,\ \frac{\pi}{2}\right]$. From this inequality, it follows that

$$0 \leq x \sin x \leq x$$

for all $x \in \left[0,\ \frac{\pi}{2}\right]$ as well. Here's a graph (Figure 115, page 281) illustrating that $0 \le x \sin x \le x$ for all $x \in \left[0,\ \frac{\pi}{2}\right]$. So, I think from what we know about Riemann Sums, it is

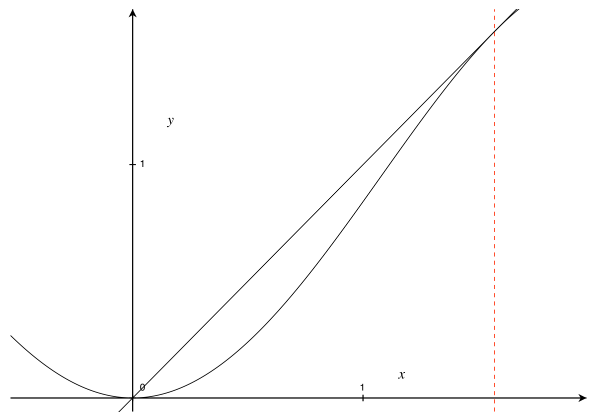

Figure 115: The area of interest.

clear that

$$\int_0^{\frac{\pi}{2}} x \sin x \, \mathrm{d}x \le \int_0^{\frac{\pi}{2}} x \, \mathrm{d}x = \frac{\pi^2}{8}.$$

Actually, if you take MTH122 you will be able to evaluate

$$\int_0^{\frac{\pi}{2}} x \sin x \, \mathrm{d}x = 1 \le \frac{\pi^2}{8}.$$

43.4 Got LaTeX?

If you're interested in learning how to typeset mathematics, I suggest you visit my m11 website and start the process. The URL is http://m11.mathography.org and you'll even find a *free* book on introductory LaTeX there. If you want to study mathematics with the *big-people*, you're just going to have to learn LaTeX! As for graphing, that's more involved and depends on your operating system. Mathematica[106] (Windows, UNIX, LINUX, MAC OS X) has loads of ways to generate cool graphs, however most of my work is done with Grapher (Mac OS X only). Grapher is easier to use than Mathematica, but it lacks many of Mathematica's options. Anyway, look around for what best works for you.

Oh, one more idea. Now that you're starting to see math more broadly, you might want to visit http://m12.mathography.org to learn more about using mathematics to solve tricky problems. Your education is in your hands, don't squander the opportunity!

[106] Visit http://m10.mathography.org for more information about learning Mathematica at Essex County College.