Data Represintation - HW 3

1. On Circulant Matrices

a.) first well show what is the effect of applying J to a vector $\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = V \in \mathbb{R}^n$

$$\Rightarrow \quad J \cdot V = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ 0 & 1 & & \vdots \\ & & \ddots & \vdots \\ 0 & \cdots & 0\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} V_n \\ V_1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix}$$ $\Rightarrow$ applying J to any column vector

simply cicly shifts of of its content 1 spot "downwards"

So $J^K = J^{K-1} \cdot J$ well be the same as cyclicly shifting all of J's coulumns $K-1$ slots down

In preticular, $J^n = J^{n-1} \cdot J$ well be ciclic $n-1$ shifts down which is nothing but one shift upwards, so we will get the Identity matrix (since J is I where each column is shifted down once)

$\Rightarrow \quad J^n = I$

1] cont.

b] let $\lambda \in \mathbb{R}$ be an Eigenvalue of J ~~w.r.~~ and $\vec{v} \in \mathbb{R}^n = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be a corresponding eigenvector

By definition

$$\Rightarrow \quad J \cdot v \overset{(1)}{=} \begin{bmatrix} v_n \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} = \lambda \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow \text{ we get } \begin{cases} v_1 = \lambda v_2 \\ v_2 = \lambda v_3 \\ \vdots \\ v_n = \lambda v_1 \end{cases}$$

if we chose ~~fo~~ $v_1 = 0$ we get $v = 0 \Rightarrow v_i \neq 0 \ \forall_{1 \le i \le n}$

Therefore we can chose $v \in \mathbb{R}^n$ s.t $v_2 = 1$ (as $v_i \neq 0$ and eigenvectors are up to scalar multiplication

$\Rightarrow$ we get $v_1 = \lambda,\ v_2 = \lambda^2 \ldots\ v_{n-1} = \lambda^{n-1}$, $\boxed{v_n = \lambda^n = 1}$

$\Rightarrow$ all $\lambda \in \mathbb{R}$ ~~that~~ s.t $\lambda^n = 1$ λ can be an Eigenvalue of J

and there are exactly n such complex numbers: $e^{2\pi i k \cdot \frac{1}{n}}$ for $k = 0, 1, .., n-1$

~~and since it is easy to see that~~

$\Rightarrow$ J has exactly n eigenvalues $\Rightarrow \lambda_k = e^{\frac{2\pi i k}{n}}$, $k = 0, 1, \ldots, n-1$.

(by definition, all λ that fullfils the requi equation is an eigenvalue)

I] cont.

C.) as seen in (b.). we got that $\lambda_k = e^{+\frac{2\pi i k}{n}}$, $k=0,\dots,n-1$ are J's eigenvalues, <u>but</u> we've also got from the calculations, ~~the~~ the corresponding eigenvectors!

$$\underline{V_k} = \begin{bmatrix} \lambda_k^0 \\ \lambda_k^1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{n-1} \end{bmatrix} \Rightarrow \text{we get that } J \text{ is diagonalizable with } P = \begin{bmatrix} | & | & & | \\ \underline{V_1} & \underline{V_2} & \cdots & \underline{V_n} \\ | & | & & | \end{bmatrix}$$

<u>Moreover</u> we can see that P is made of the Fourier vector which as we know are orthogonal so after normalization we get that $U = \frac{1}{\sqrt{n}} P$ is a unitary Basis which diagonalize J! $U^* J U = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

1] cont.

d.) we can use what we learned about J^K in (a.) to find such a polinomial expression:

$$\Rightarrow P(A) = h_1 \cdot A + h_2 \cdot A^2 + \dots + h_{n-1} \cdot A^{n-1} + h_0 \cdot A^n$$

$$\Rightarrow \text{we get } P(J) = h_1 \cdot J + h_2 \cdot J^2 + \dots + h_{n-1} \cdot J^{n-1} + h_0 \cdot \underbrace{J^n}_{= I} = H.$$

e.) from Kily-Hamilton we get that $\forall_j\, j \le n$, $P(\lambda_j)$, s.t λ_j is an eigenvalue of J is an eigenvalue of $P(J) = H$, <u>with the same eigenvectors!</u>

$$\Rightarrow \lambda_{H,j} = P(\lambda_j) = P\left(e^{\frac{2\pi i j}{n}}\right) = h_0 + h_1 \cdot e^{-\frac{2\pi i j}{n}} + h_2 \cdot e^{-\frac{2\pi i j}{n} \cdot 2} + \dots + h_{n-1} \cdot e^{-\frac{2\pi i j}{n} \cdot (n-1)}$$

with $\underline{v}_{k,H} = \underline{v}_k = \begin{bmatrix} \lambda_k^0 \\ \lambda_k^1 \\ \vdots \\ \lambda_k^{n-1} \end{bmatrix}$

$\Rightarrow$ H is diagonalizable by the same Unitary matrix $U = \frac{1}{\sqrt{n}} \begin{bmatrix} | & | & & | \\ \underline{v}_0 & \underline{v}_1 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}$

with eigenvalues as computed above

$$P(\lambda_j) = \sum_{k=0}^{n-1} h_k \cdot e^{+\frac{2\pi i k \cdot j}{n}}$$

1] cont.

g.) as weve shown, we know $WHW^* = \Delta$

, since W, W^* are symetric we get (and Δ is diagonal so also symetric)

$$(WHW^*)^T = \Delta^T \Rightarrow W^* H^T W = \Delta$$

now well multiply both sides by W^* (from the right) and since $WW^* = I$ we gets

$$W^* H^T = \Delta W^*$$

$$= \overbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda_1 & & & \vdots \\ \vdots & \lambda_1^2 & \lambda_2^2 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & \lambda_1^{n-1} & \cdots & \cdots & \lambda_n^{n-1} \end{bmatrix}}^{W^*} \begin{bmatrix} h_0 & \cdots & h_{n-1} \\ h_{n-1} & & \vdots \\ \vdots & \ddots & \vdots \\ h_1 & \cdots & h_0 \end{bmatrix} = \overbrace{\begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}}^{\Delta} \cdot \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \cdots 1 \\ 1 & \cdot & & \\ 1 & & & \\ 1 & & \ddots & \\ 1 & & & \\ 1 & & & \end{bmatrix}}^{W}$$

from rules of matrix multiplication we get that the multiplication of W^* by the first column of H^T is equal to the first coulm in the resulting matrix, which is nothing but the Δ matrix multiplyed by the first colum of W which is only ones

$$\Rightarrow \quad W^* \cdot \begin{bmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{bmatrix} = \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} \quad \blacksquare$$

1] cont.

h.] let $H_1, H_2 \in \mathbb{R}^{n\times n}$, s.t H_1, H_2 are circulent.

I'll show $H_1 \cdot H_2 = H_2 \cdot H_1$.

as weve shown, all circulent matrices are unifidly diagonlizable by $W = DFT$

$$\Rightarrow H_1 \cdot H_2 = W\Delta_1 \underbrace{W^* \cdot W}_{I} \Delta_2 W^* = W \Delta_1 \Delta_2 W^*$$

Δ_1, Δ_2 are diagonal so they commute

$$= W \Delta_2 \Delta_1 W^* = W\Delta_2 \underbrace{W^* W}_{I} \Delta_1 W^* = H_2 \cdot H_1$$

$\Rightarrow$ All circulant matrices commute.

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now well show that $H_1 \cdot H_2$ is circulant (or in other words, that circulant matrices are close to multiplication)

as seen in (d.) we can build in the exact same way a polinomial expression for each of the matrices P_1, P_2 s.t $P_1(J) = H_1$ and $P_2(J) = H_2$

$$\Rightarrow H_1 \cdot H_2 = P_1(J) \cdot P_2(J) = \sum_{k=0}^{n-1} h_{1,k} \cdot J^k \cdot \sum_{j=0}^{n-1} h_{2,j} \cdot J^j$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} h_{1,k} \cdot h_{2,j} \cdot J^{k+j} = (*)$$

$\Rightarrow$ as weve seen before for any $\alpha \in \mathbb{N}$, J^α is a circulant matrix and since circulant matrices are closed to addition:

$$\left(\begin{bmatrix} A_0 & A_{n-1} & & A_1 \\ A_1 & \ddots & & \\ & & \ddots & \\ & & A_1 & A_0 \end{bmatrix} + \begin{bmatrix} B_0 & & & B_1 \\ B_1 & \ddots & & \\ & & \ddots & \\ & & B_1 & B_0 \end{bmatrix} = \begin{bmatrix} A_0+B_0 & & & A_1+B_1 \\ A_1+B_1 & \ddots & & \\ & & \ddots & \\ & & A_1+B_1 & A_0+B_0 \end{bmatrix} \text{ is a circulant matrix} \right)$$

$\Rightarrow$ we get $(*)$ is circulant $\Rightarrow H_1 \cdot H_2$ is circulant.

1] cont.

j) first, we can write the convolution of $x \circledast y$ in matrix form by building a circulant matrix X s.t its first row (which fully defines it) is x and multiplying y by its transpose

$$\Rightarrow \quad x \circledast y = X^T y$$

$$\Rightarrow \quad z = x \circledast y = X^T y \quad \Rightarrow \text{~~DFT(z)=~~}$$

$$\Rightarrow \quad \overset{W}{\overbrace{DFT}} \cdot z = \boxed{Wz =} W X^T y = W X^T \underbrace{W^* W}_{I} y \overset{①}{=} \underbrace{W^* X W}\cdot W y \overset{②}{=} \boxed{\underbrace{W x \odot W y}\cdot \sqrt{n}}$$

~~① $W X^T W^*$ ...~~

① well show $W X^T W^* = W^* X W$:

X^T is circulant $\Rightarrow$ W diagonlize her

$\Rightarrow$ $W^* X W$ is diagonal $\Rightarrow$ $\cancel{(W^* X W)^T = W^* X W}$ $\Rightarrow$ symetric

$\Rightarrow$ $W^* X W = (W^* X W)^T = W^T X^T W^{*T} = W X^T W^*$ (W, W^* are symetric).

② well show ~~...~~ $\sqrt{n}\,\underline{W^* X W}\cdot Wy = \sqrt{n}\, Wx \odot Wy$

* as weve seen $W^* X W$ diagonlizes X, therefore it is a diagonal matrix where it's values on the diag is it's eigenvalues, and since X is built using the x we know from (g.) that $\sqrt{n}\, Wx = \begin{bmatrix} \lambda_{0,x} \\ \vdots \\ \lambda_{n-1,x} \end{bmatrix}$ of X

$$\Rightarrow Wx = \frac{1}{\sqrt{n}} \begin{bmatrix} \lambda_{0,x} \\ \vdots \\ \lambda_{n-1,x} \end{bmatrix} \Rightarrow \tilde{\lambda}_{i,x} = \frac{1}{\sqrt{n}} \lambda_{i,x}$$

$$\Rightarrow \sqrt{n}\, W^* X W \cdot Wy = \sqrt{n} \begin{bmatrix} \tilde{\lambda}_{0,x} & & & \\ & \tilde{\lambda}_{1,x} & & \\ & & \ddots & \\ & & & \tilde{\lambda}_{n-1,x} \end{bmatrix} \cdot \begin{bmatrix} \tilde{\lambda}_{0,y} \\ \tilde{\lambda}_{1,y} \\ \vdots \\ \tilde{\lambda}_{n-1,y} \end{bmatrix} = \begin{bmatrix} \lambda_{0,x} \\ \vdots \\ \lambda_{n-1,x} \end{bmatrix} \odot \begin{bmatrix} \tilde{\lambda}_{0,y} \\ \tilde{\lambda}_{1,y} \\ \vdots \\ \tilde{\lambda}_{n-1,y} \end{bmatrix}$$

$$= Wx \odot Wy \quad \boxtimes$$

2 Fourier Transform

a.) we are given $f(t)*g(t)=h(t)$

we will show $f(t-1)*g(t+1)=h(t)$:

First we'll define two new functions: $\tilde{f}(t)=f(t-1), \tilde{g}(t)=g(t+1)$

$$\Rightarrow \boxed{f(t-1)*g(t+1)}=\tilde{f}(t)*\tilde{g}(t)=\int_{-\infty}^{\infty}\tilde{f}(\xi)\cdot\tilde{g}(t-\xi)\cdot d\xi$$

$$=\int_{-\infty}^{\infty} f(\xi-1)\cdot g(t-(\xi-1))\cdot d\xi = \int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\cdot d\tau$$

$$\left\{\begin{array}{l}\text{var change:}\\ \tau=\xi-1,\ \frac{d\tau}{d\xi}=1\Rightarrow d\tau=d\xi\end{array}\right\}$$

conv definition

$$= f(t)*g(t)=\boxed{h(t)}$$

Another maybe easier way of showing this is by using the fact that τ_ξ is LSI as shown in class

and by using: $\tau_{-1}(\tau_{+1}(\varphi(t)) = \tau_{-1}(\varphi(t+1)) = \varphi(t)$

$\Rightarrow$ we'll define $h_{\tau_{-1}}(t), h_{\tau_{+1}}(t)$ to be the Impulse responses of τ_{-1}, τ_{+1}

$$\Rightarrow \boxed{f(t)*g(t)} = \tau_{-1}(\tau_{+1}(f(t)*g(t)) = h_{\tau_{-1}}(t)*(h_{\tau_{+1}}(t)*(f(t)*g(t)))$$

$\Leftarrow$ (LSI commutativity as seen in Lecture 7)

$$= h_{\tau_{-1}}(t)*(f(t)*(h_{\tau_{+1}}(t)*g(t)))$$

$$=(h_{\tau_{-1}}(t)*(f(t))*(h_{\tau_{+1}}(t)*g(t)) = \tau_{-1}(f(t))*\tau_{+1}(g(t))$$

$$=\boxed{f(t-1)*g(t+1)}$$

2] cont.

b] we need to show: $\int_{-\infty}^{\infty} f(t)\cdot g(t)\,dt = \int_{-\infty}^{\infty} F(u)\cdot G(u)\,du$

where $F(u), G(u)$ are the Fourier transforms of $f(t), g(t)$.

first well define $h(\xi) = f(\xi) * g(\xi) = \int_{-\infty}^{\infty} f(t)\cdot g(\xi - t)\,dt$

well notice $\boxed{h(0) = \int_{-\infty}^{\infty} f(t)\cdot g(-t)\,dt}$ 1

Moreover, as we learned in the Lectures, we can fully represent any function by projecting it to the Fourier Family

$\Rightarrow h(t) = \int_{-\infty}^{\infty} H(u)\cdot e^{2\pi i u t}\,du$ when $H(u)$ is the Fourier transform of $h(t)$.

$\Rightarrow \boxed{h(0) = \int_{-\infty}^{\infty} H(u)\,du}$ 2

now, lets find out who is $H(u)$.

first as we said before we can look at $f(t), g(t)$ in their Fourier representations:

$$f(t) = \int_{-\infty}^{\infty} F(u)\cdot e^{2\pi i u t}\,du, \quad g(t) = \int_{-\infty}^{\infty} G(u)\cdot e^{-2\pi i u t}\,du$$

$$\cancel{H(u) = \int_{-\infty}^{\infty} h(t)\cdot e^{-2\pi i u t}\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\cdot g(t-\xi)\,d\xi\, e^{-2\pi i u t}\,d\xi\,dt}$$

$$\cancel{= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(u)\cdot e^{2\pi i u \xi}\,du\right)\cdot\left(\int_{-\infty}^{\infty} G(u)\cdot e^{2\pi i u(t-\xi)}\,du\right)\cdot e^{-2\pi i u t}\,d\xi\,dt}$$

More in the next page

2] b. cont.

recap. from (2) we know $h(0) = \int_{-\infty}^{\infty} H(u)\,du$

and from (1) we know $h(0) = \int_{-\infty}^{\infty} f(t)\cdot g(-t)\,dt$

$\Rightarrow$ we need only to show $\int_{-\infty}^{\infty} H(u)\,du = \int_{-\infty}^{\infty} F(u)\,G(u)\,du$ (3)

let us look at $F(u)\cdot G(u)$

$$\boxed{F(u)\cdot G(u)} = \int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi iux}\,dx \cdot \int_{-\infty}^{\infty} g(y)\cdot e^{-2\pi iuy}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\,g(y)\cdot e^{-2\pi iu(x+y)}\,dy\,dx = \left|\begin{array}{l}\text{change variable from } y \text{ to } \xi\text{:}\\ y = \xi - x,\ \frac{dy}{d\xi} = +1 \Rightarrow dy = +d\xi\end{array}\right.$$

$$\circledast = \int_{-\infty}^{\infty} e^{-2\pi iu\xi}\cdot \underbrace{\int_{-\infty}^{\infty} f(x)\cdot g(\xi - x)\,dx}\,d\xi = \int_{-\infty}^{\infty} e^{-2\pi iu\xi}\cdot h(\xi)\,d\xi$$

$$= \boxed{H(u)}$$

$\circledast$ we also changed the order of integration from $d\xi\,dx$ to $dx\,d\xi$ (Fubini)

$\Rightarrow$ since we've shown $H(u) = F(u)\cdot G(u)$ we get (3) instantly

$$\Rightarrow \underline{\int_{-\infty}^{\infty} F(u)\cdot G(u)\,du} = \int_{-\infty}^{\infty} H(u)\,du = h(0) = \underline{\int_{-\infty}^{\infty} f(t)\,g(-t)\,dt} \quad \blacksquare$$