2. Let's Randomise

$$Y = [m, \ldots m, M+L, m \ldots m]^T \qquad K \sim \text{Uni}(\{1,2,\ldots,N\})$$

$$E(m) = 0, \quad E(m^2) = c \qquad 0 < c < 1$$

$$L = \begin{cases} L_1, & \text{if } K \le \frac{N}{2} \\ L_2, & \text{otherwise} \end{cases}$$

$$E(L_1) = E(L_2) = 0$$

$$E(L_1^2) = Na, \quad E(L_2^2) = Nb \qquad 0 < a < 1, \quad 0 < b < 1$$

m, (L_1, L_2), K are independent

a. $$E(Y) = \sum_{K=1}^{N} P(K=k)\, E(Y \mid K=k) =$$

$$= \sum_{K=1}^{N} P(K=k)\, E([m, \ldots, m, \underset{\text{fixed } K}{m+L}, m, \ldots, m]^T) =$$

$$= \sum_{K=1}^{\frac{N}{2}} P(K=k)\, E([m, \ldots, m, m+L_1, m, \ldots m]^T) + \sum_{K=\frac{N}{2}+1}^{N} P(K=k)\, E([m \ldots m, m+L_2, m \ldots m]^T) =$$

$$= \sum_{K=1}^{\frac{N}{2}} P(K=k) \left([E(m) \ldots E(m), \underbrace{E(m+L_1)}_{E(m)+E(L_1)}, E(m), \ldots E(m)]^T\right) + \sum_{K=\frac{N}{2}+1}^{N} P(K=k)\, E\left([E(m), \ldots E(m), \underbrace{E(m+L_2)}_{E(m)+E(L_2)}, E(m) \ldots E(m)]^T\right) =$$

$$= \sum_{K=1}^{\frac{N}{2}} \frac{1}{N} (0, \ldots 0) + \sum_{K=\frac{N}{2}+1}^{N} \frac{1}{N} (0, \ldots 0) = \bar{0}$$

6. we will compute the i,j element of R_ℓ

$$R_\ell(i,j) = E(\ell_i \ell_j^*)$$

1. $i \neq j$,

$$E(\ell_i \cdot \ell_j^*) = \cancel{E(\ell_i \ell_j^*)}\ P(K=i)\,E(\ell_i \ell_j^* \mid K=i) + P(K=j)\,E(\ell_i \ell_j^* \mid K=j) +$$

$$+\ P(K \neq i,j)\,E(\ell_i \ell_j^* \mid K \neq i,j) =$$

$$= \frac{1}{N} E\big((m+L)m\big) + \frac{1}{N} E\big(m(m+L)\big) + \frac{N-2}{N} E(m^2) =$$

$$= \frac{2}{N}\Big(E(m^2) + \underbrace{\cancel{E(mL)}}_{\substack{\text{independent variables} \\ E(m)E(L)}}\Big) + \frac{N-2}{N} E(m^2) = E(m^2) = c$$

2. $i = j$, $i,j \leq \frac{N}{2}$

$$E(\ell_i \ell_j^*) = P(K=i,j)\,E(\ell_i \ell_j^* \mid K=i,j) + P(K \neq i,j)\,E(\ell_i \ell_j \mid K \neq i,j) =$$

$$= \frac{1}{N} E\big((m+L_1)^2\big) + \frac{N-1}{N} E(m^2) = \frac{1}{N}\Big[E(m^2) + 2\underbrace{\cancel{E(mL_1)}}_{E(m)E(L_1)} + E(L_1^2)\Big] + \frac{N-1}{N} E(m^2) =$$

$$= \frac{1}{N} E(m^2) + \frac{1}{N} E(L_1^2) + \frac{N-1}{N} E(m^2) = E(m^2) + \frac{1}{N} E(L_1) = c + a$$

3. $i = j$, $i,j > \frac{N}{2}$

as above, we will get that $E(\ell_i \ell_j^*) = c + b$

Overall, The matrix structure of R_ℓ is:

$$R_\ell = \begin{pmatrix} c+a & c & \cdots c & c & \cdots & c \\ c & c+a & & \vdots & & \vdots \\ \vdots & & \ddots & \vdots & & \vdots \\ \vdots & & c+a & \vdots & & c \\ \vdots & & \vdots & c+b & & c \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ c & \cdots & c & c & \cdots & c+b \end{pmatrix}$$

c. The most general condition is $a=b$.

If $a=b$ then R_ℓ is circulant and can be diagonalized by the DFT^* matrix.

$$R_\ell = DFT^* \Lambda \, DFT$$

Also, ~~in matter~~ in order to have the PCA matrix identical to the DFT^* matrix we need to satisfy $\lambda_\ell > \lambda_{\ell+1}$ for $\ell \geq 0$.

Since $R_\ell = (c+a)\cdot J^0 + c\cdot J^1 + c\cdot J^2 + \dots c\cdot J^{N-1}$

we have that

$$\lambda_\ell = (c+a)\cdot w^{-0\ell} + c\cdot w^{-\ell} + c\cdot w^{-2\ell} + \dots + c\cdot w^{-(n-1)\ell} =$$

$$= a + c\cdot \sum_{i=0}^{n-1} w^{-i\ell} = \begin{cases} a + c\cdot N & \ell = 0 \\ a & \ell > 0 \end{cases}$$

for $\ell \neq 0$ $\lambda_\ell = \lambda_{\ell+1}$. for $\ell = 0$

$\lambda_0 > \lambda_1$ iff $a + c\cdot N > a$ iff $c\cdot N > 0$ iff $a>0$, $N>0$

and the last condition holds because $0<a<1$, $N>0$