Homework 4

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1. Inverting the Second Derivative operator

$$\varphi_{data} = H\varphi \qquad \varphi, \varphi_{data} \in \mathbb{R}^m, \; H \in \mathbb{R}^{m \times m}$$

$$\varphi_{data,j} = -\frac{1}{12}\varphi_{j-2 \bmod m} + \frac{4}{3}\varphi_{j-1 \bmod m} - \frac{5}{2}\varphi_{j \bmod m} + \frac{4}{3}\varphi_{j+1 \bmod m} - \frac{1}{12}\varphi_{j+2 \bmod m}$$

a. We can denote $\varphi_{data} = h \circledast \varphi$ — cyclic convolution

where $h = \left[-\frac{5}{2}, \frac{4}{3}, -\frac{1}{12}, 0, \dots, 0, -\frac{1}{12}, \frac{4}{3}\right]^T$

Therfore,

$$H = \xcancel{\begin{pmatrix} -\frac{5}{2} & \frac{4}{3} & \cdots & \frac{4}{3} & \frac{4}{3} \\ \frac{4}{3} & -\frac{1}{12} & & & \\ -\frac{1}{12} & 0 & & & \\ 0 & \vdots & & & \\ \vdots & 0 & & & \\ 0 & -\frac{1}{12} & & & \\ -\frac{1}{12} & \frac{4}{3} & & & \\ \frac{4}{3} & -\frac{5}{2} & \cdots & & -\frac{5}{2} \end{pmatrix}}$$

$$H = \begin{pmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & -\frac{1}{12} & \frac{4}{3} \\ \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & & & & -\frac{1}{12} \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \ddots & & & 0 \\ 0 & -\frac{1}{12} & \frac{4}{3} & \ddots & \ddots & & \vdots \\ \vdots & 0 & & \ddots & \ddots & & 0 \\ 0 & \vdots & & & \ddots & & -\frac{1}{12} \\ -\frac{1}{12} & 0 & & & & & \frac{4}{3} \\ \frac{4}{3} & -\frac{1}{12} & \cdots & & & & -\frac{5}{2} \end{pmatrix}$$

6. we will remember that any circulant matrix can be written as a polynom of J where

$$J = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}$$

, Therfore we can write $H = p(J)$

$$H = -\frac{5}{2}\underbrace{J^0}_{I} + \frac{4}{3}J^1 - \frac{1}{12}J^2 - \frac{1}{12}J^{m-2} + \frac{4}{3}J^{m-1}$$

we will also remember that the eigenvalues of J are $\alpha_\ell = \bar{w}^\ell = e^{-\frac{i2\pi\ell}{m}}$.

From caley-Hamilton we get that the eigenvalues of H are a polynom of α. $\lambda = p(\alpha)$. Therfore,

$$\lambda_\ell = p(\alpha_\ell) = -\frac{5}{2}\alpha_\ell^0 + \frac{4}{3}\alpha_\ell - \frac{1}{12}\alpha_\ell^2 - \frac{1}{12}\alpha_\ell^{m-2} + \frac{4}{3}\alpha_\ell^{m-1} =$$

$$= -\frac{5}{2} + \frac{4}{3}w^{-\ell} - \frac{1}{12}w^{-2\ell} - \frac{1}{12}w^{-(m-2)\ell} + \frac{4}{3}w^{-(m-1)\ell} =$$

$$= -\frac{5}{2} + \frac{8}{3}Re(w^{-\ell}) - \frac{1}{6}Re(w^{-2\ell}) = -\frac{5}{2} + \frac{8}{3}\cos\left(\frac{2\pi\ell}{m}\right) - \frac{1}{6}\cos\left(\frac{4\pi\ell}{m}\right)$$

$\lambda_0 = 0$

$\lambda_1 = -\frac{5}{2} + \frac{8}{3}\cos\left(\frac{2\pi}{m}\right) - \frac{1}{6}\cos\left(\frac{4\pi}{m}\right)$

λ_2

$\vdots$

λ_{m-1}

The eigenvectores remain the same as J. which is The Fourier. Therfore

$$H = DFT^* \Lambda \, DFT$$

$$\Lambda = \begin{pmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{m-1} \end{pmatrix}$$

We want to design M to minimise the reconstruction error

$$M = V \Sigma V^* = DFT^* \Sigma DFT$$

$$\partial_\ell = \begin{cases} 0, & \lambda_\ell = 0 \\ \frac{1}{\lambda_\ell} & \lambda_\ell \neq 0 \end{cases}$$

Since only $\lambda_0 = 0$ for $1 \le \ell \le n-1$ we will get that

$$\Sigma = \begin{pmatrix} 0 & & & \\ & \frac{1}{\lambda_1} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_{n-1}} \end{pmatrix}$$

C. No, we can see right away that since $\lambda_0 = \partial_0 = 0$

~~not any signal~~ and $rank(H) = rank(M) < m$ not

any signal φ can be perfectly recoverd.

The ~~easiest~~ simplest example of such signals would be $Ker(H)$.

we will look for a set of input signals φ s.t $\hat{\varphi} = \varphi$.

meaning $\hat{\varphi} = M\varphi_{down} = MH\varphi = \underset{\hat{F}\Sigma F^*}{F^*}\Sigma F \cancel{F^*} \Lambda F\varphi = F^*\Sigma\Lambda F\varphi$

$$F\hat{\varphi} = \Sigma\Lambda F\varphi$$

$$\hat{\varphi}^F = \Sigma\Lambda\varphi^F$$

$$\hat{\varphi}^F = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{m-1} \end{pmatrix}\varphi^F$$

$$\varphi^F = \begin{pmatrix} 0 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}\varphi^F = \begin{pmatrix} 0\cdot\varphi^F_0 \\ \varphi^F_1 \\ \vdots \\ \varphi^F_{m-1} \end{pmatrix}$$

$\varphi^F = \varphi^F_0 \beta^F_0 + \dots \varphi^F_{m-1}\beta^F_{m-1}$

and this is true iff $\varphi^F_0 = 0$

iff ~~$\varphi^F \in \mathbb{R}^m \setminus ker(H)$ iff~~

$\varphi^F \in F^m \setminus ker(H)$ iff $\varphi^F \in span\{\beta^F_1, \dots \beta^F_{m-1}\}$

iff The first Fourier coefficient is zero