HomeWork 2 - Data Rep.

Nitzan Leshem and Ron Benchetrit

1.

a. The k-term approximation of f in F using $Vec(\beta_{i_1}, \ldots \beta_{i_k})$ ortho

(a)

Linear combination of basis function

$$MSE(f,\hat{f}) = \int_I (f(x)-\hat{f}(x))^2 dx = \int_I \left(f(x) - \sum_{i=i_1}^{i_k} \hat{f}_i \beta_i(x)\right)^2 dx =$$

$$= \int_I f^2(x)dx - 2\sum_{i=i_1}^{i_k} \hat{f}_i \int_I f(x)\beta_i(x)dx + \sum_{i=i_1}^{i_k}\sum_{j=i_1}^{i_k} \hat{f}_i \hat{f}_j \int_I \underbrace{\beta_i(x)\beta_j(x)}_{\text{orthonormal}} dx =$$

$$= \int_I f^2(x)dx - 2\sum_{i=i_1}^{i_k} \hat{f}_i \int_I f(x)\beta_i(x)dx + \sum_{i=i_1}^{i_k} \hat{f}_i^2$$

$$\frac{\partial MSE}{\partial \hat{f}_i}(f,\hat{f}) = -2\int_I f(x)\beta_i(x)dx + 2\hat{f}_i$$

$\Downarrow$ when comparing to zero we get that

$$\hat{f}_i^* = \int_I f(x)\beta_i(x)dx = \langle \beta_i, f\rangle$$

The associated SE:

$$MSE(f,\hat{f}^*) = \int_I (f(x)-\hat{f}^*(x))^2 dx = \int_I \left(f(x) - \sum_{i=i_1}^{i_k} \hat{f}_i^* \beta_i(x)\right)^2 dx =$$

$$= \int_I f^2(x)dx - 2\sum_{i=i_1}^{i_k} \hat{f}_i^* \underbrace{\int_I f(x)\beta_i(x)dx}_{\hat{f}_i^*} + \sum_{i=i_1}^{i_k} (\hat{f}_i^*)^2 =$$

$$= \int_I f^2(x)dx - \sum_{i=i_1}^{i_k} (\hat{f}_i^*)^2$$

b.) as we've seen before the SE of the k-approximation is Equal to:

$$SE = \int_I f^2(x)dx - \underbrace{\sum_{i=i_1}^{i_k} (\hat{f}_i^*)^2}_{*} = SE(f(x), \hat{f}(x))$$

$\Rightarrow$ Therefore in order to minimize the SE we need to ~~choose~~ MAXimiz ~~the~~ ~~k largest~~ ⊛

and since $\hat{f}_i^*$ is independent of the other $\hat{f}_j$ $(j \neq i)$ we simply need to ~~pick~~ sort the values $|\langle f, b_i \rangle|$ and pick the k b_i's that correspond to the largest values.

The Squared Error will be the same as before where b_{i_1} correspond to the Largest value and b_{i_n} to the biggest

$$SE = \int_I f^2(x)dx - \sum_{i=i_0}^{i_k} (\hat{f}_i^*)^2$$

⊛ the representation is not unique since it is possible that there will be two b_i, b_j s.a $|\langle f, b_i \rangle| = |\langle f, b_j \rangle|$ and we could exchange one with the other and the SE will not change (we assume one of them is not in the original Representation if he is we can look at the same problem for a smaller k such as one of b_i / b_j will not be chosen)

6.

(a) as we've seen the k approximations using the two families are

$$\sum_{i=1}^{n} \langle f, \beta_i \rangle \beta_i \quad \text{and} \quad \sum_{i=1}^{n} \langle f, \tilde{\beta}_i \rangle \tilde{\beta}_i$$

moreover, Since F is of a finite dimension n and $\{\beta_i\}, \{\tilde{\beta}_i\}$ are both a set of n ortogonal function

=> They are both bases of F and span the same space (F..)

=> The projection of f on $\{\beta_i\}$ is the same as its projection to $\{\tilde{\beta}_i\}$, both are projections of f to F

b) we cant say much about the $k \neq n$ approximations on each family, it is possible that for some $k \in \{1, .., n-1\}$

the k-approx of f on one family is better then to the other and for a different K it is the other way around.

it is also possible that for $K < \tilde{K}$ $(k, \tilde{k} \in \{1, .., n-1\})$ the k-approx of f on one family will be better then the $\tilde{K}$-approx on the other one

—

~~we can say something about the~~

we can also say that when k goes from $1 \to n$ both fam K-approximations errors will converge to the same representation error (The n-approximation) however their Convergence rate might differ

2. Har Matrix & Walsh-Hadamard Matrix

a) We will show by definition That $H_4^H H = I$.

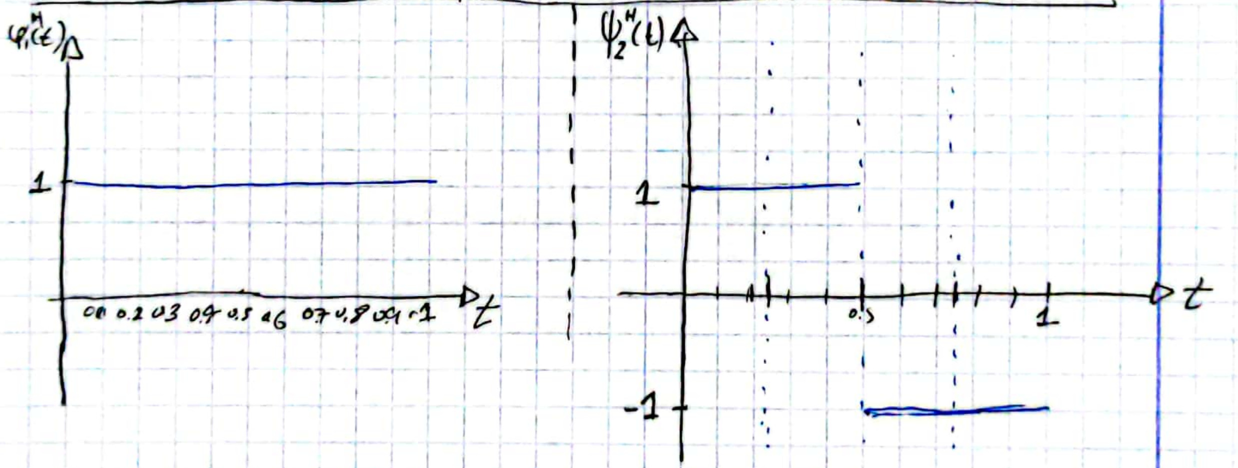
$$\left(\frac{1}{2}\right)\cdot\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\\sqrt{2}&-\sqrt{2}&0&0\\0&0&\sqrt{2}&-\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}1&1&\sqrt{2}&0\\1&1&-\sqrt{2}&0\\1&-1&0&\sqrt{2}\\1&-1&0&-\sqrt{2}\end{bmatrix}=\frac{1}{4}\cdot\begin{bmatrix}4&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&4\end{bmatrix}=I$$

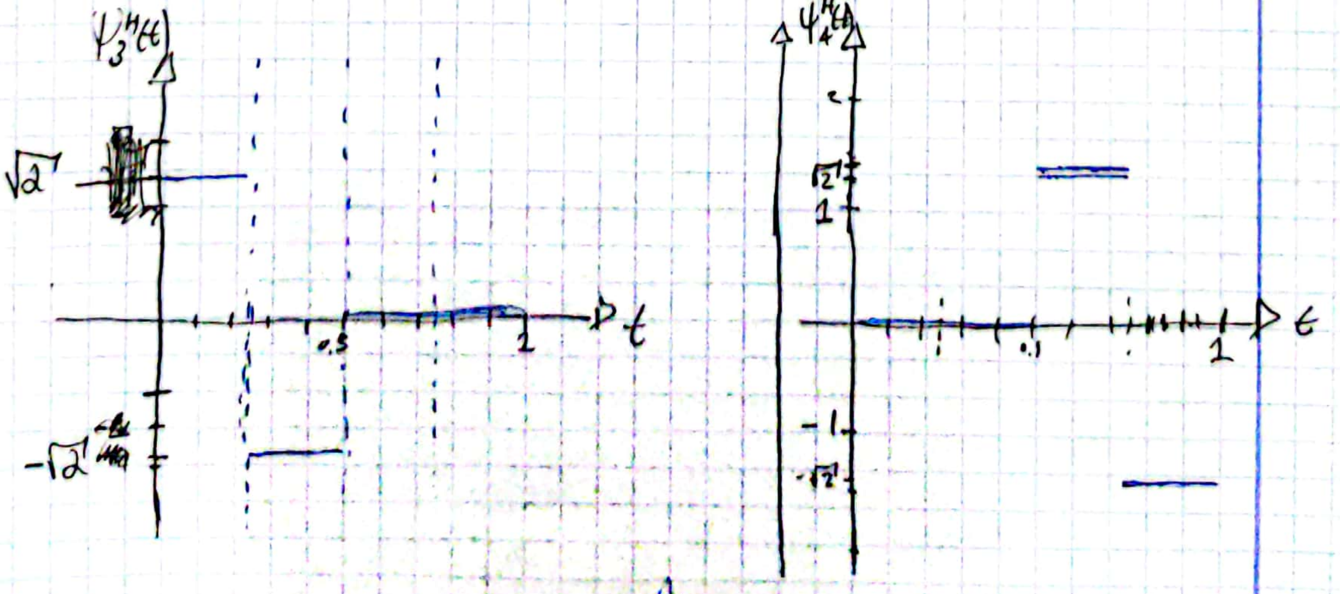
=> H_4 is Unitary.

b) in order to get the har functions we will multiply the har matrix (Hermit) by the standart basis.

$$\begin{bmatrix}\varphi_1^H(t)\\ \vdots \\ \varphi_4^H(t)\end{bmatrix} = H_4^H \cdot \sqrt{4}\cdot\begin{bmatrix}1\Delta_1\\1\Delta_2\\ \vdots \\1\Delta_3\end{bmatrix}$$, where the $\frac{1}{2}$ in H_4 is canceled by $\sqrt{4}$

=>





2] cont.

iii] $\phi(t) = \alpha + \beta\cdot\cos(2\pi t) + c\cdot\cos^2(\pi t)$

in order to find the best approximation of ϕ using $\{\varphi_i^H\}_{i=1}^{4}$ basis we need to project ϕ onto it.

as we've seen in the lecture & tutorial we will first find the coefficients $\varphi_i^* = \langle \phi(t), \varphi_i^H(t) \rangle$

$$\varphi_1^* = \langle \phi(t), \underbrace{1_{\Delta_1}\sqrt{4}}_{g(t)} \rangle = \int_0 \phi(t)\cdot\sqrt{4}\, g(t)\,dt = \sqrt{4}\cdot\int_0^{\Delta_1}\phi(t)\,dt \quad / \text{as } g(t)=0 \text{ when } t\notin\Delta_1$$

$$= \sqrt{4}\int_0^{0.25} \alpha + \beta\cdot\cos(2\pi t) + c\cdot\cos^2(\pi t)\,dt$$

$$= \sqrt{4}\Big(\tfrac{1}{4}\alpha + \tfrac{\beta}{2\pi}\cdot\sin(2\pi t) + \int_0^{0.25} c\cdot\cos^2(\pi t)\,dt$$

⊛ Short pause to calculate $\int\cos^2(\pi t)\,dt$.

$$\int\cos^2(\pi t)\,dt = \frac{1}{\pi}\sin(\pi t)\cdot\cos(\pi t) + \int\sin^2(\pi t)\,dt$$

$$= \frac{1}{2\pi}\sin(2\pi t) + \int 1-\cos^2(\pi t)\,dt$$

$$= \frac{1}{2\pi}\sin(2\pi t) + t - \int\cos^2(\pi t)\,dt$$

$$\Rightarrow \int\cos^2(\pi t)\,dt = \frac{1}{4\pi}\cdot\sin(2\pi t) + \frac{1}{2}t$$

In the next page we'll continue the sum.

2] Cont 2. we'll mark $\int_a^b$ by $F(a,b)$

- $\psi_1^* = \sqrt{4}\left(\frac{1}{4}\alpha + \frac{\beta}{2\pi}\sin(2\pi t) + c\cdot\left(\frac{1}{4\pi}\cdot\sin(2\pi t) + \frac{1}{2}t\right)\right)\Big|_{0=a}^{0.25=b}$

$$= \boxed{\frac{1}{2}\alpha + \frac{\beta}{\pi} + \frac{c}{2\pi} + \frac{c}{4}} = \psi_1^*$$

for the rest of the coefficiants we get the same integral & In different Intervals, so we'll simply skip to the, the repetetive computation.

$$\boxed{\psi_2^* = F\left(\tfrac{1}{4}, \tfrac{1}{2}\right) = \frac{1}{2}\alpha - \frac{\beta}{\pi} - \frac{c}{2\pi} + \frac{c}{4}}$$

$$\boxed{\psi_3^* = F\left(\tfrac{1}{2}, \tfrac{3}{4}\right) = \frac{1}{2}\alpha - \frac{\beta}{\pi} - \frac{c}{2\pi} + \frac{c}{4}}$$

$$\boxed{\psi_4^* = F\left(\tfrac{3}{4}, 1\right) = \frac{1}{2}\alpha + \frac{\beta}{\pi} + \frac{c}{2\pi} + \frac{c}{4}}$$

Since $\{\psi_i^4(t)\}$ is a Unitary basis we can use the MSE formula as seen in the tutorial.

$$MSE(\phi, \psi^*) = \int_0^1 \phi^2(t)\,dt - \sum_{i=1}^{4} |\psi_i^*|^2$$

$=$

more in the Next Page.

2 cont.

$$MSE = \underbrace{\int \left(\alpha + \beta\cos(2\pi t) + c\cdot\cos^2(\pi t)\right)^2 dt} - \sum_{i=1}^{4} (\psi_i^*)^2$$

$$\Rightarrow = \cancel{\int (\alpha+\beta\cos(2\pi t))^2 + 2(\alpha+\beta\cos(2\pi t))\cdot c\cdot\cos^2(\pi t) + c^2\cdot\cos^4(\pi t)}$$

$$= \int \left(\alpha + \beta\cos(2\pi t) + c\cdot\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\pi t)\right)\right)^2 dt$$

$$= \int \left(\alpha + (\beta - \tfrac{c}{2})\cdot\cos(2\pi t) + \tfrac{c}{2}\right)^2 dt$$

$$= \int (\alpha + \tfrac{c}{2})^2 + 2(\alpha + \tfrac{c}{2})(\beta - \tfrac{c}{2})\cos(2\pi t) + (\beta - \tfrac{c}{2})^2\cos^2(2\pi t)\, dt$$

$$= t(\alpha + \tfrac{c}{2})^2 + 2(\alpha + \tfrac{c}{2})(\beta - \tfrac{c}{2})\tfrac{1}{2\pi}\sin(2\pi t) + (\beta - \tfrac{c}{2})^2\cdot\left(\tfrac{1}{4\pi}\sin(2\pi t) + \tfrac{1}{2}t\right)\Big|_0^1$$

$$= (\alpha + \tfrac{c}{2})^2 + (\beta - \tfrac{c}{2})^2\cdot\tfrac{1}{2} = \underbrace{\alpha^2 + \alpha c + \tfrac{c^2}{4} + \tfrac{\beta^2}{2} - \tfrac{\beta\cdot c}{2} + \tfrac{c^2}{8}}$$

$$\Rightarrow \boxed{MSE = \alpha^2 + \alpha c + \frac{3c^2}{8} + \frac{\beta^2}{2} - \frac{\beta\cdot c}{2} - 2\left(\tfrac{1}{2}\alpha + \tfrac{\beta}{4} + \tfrac{c}{4} + \tfrac{c}{4}\right) - 2\left(\tfrac{1}{2}\alpha + \tfrac{\beta}{\pi} - \tfrac{c}{2\pi} + \tfrac{c}{4}\right)}$$

now that we have the standard coefficiants we can multiply them by H_4^H and get our desired $\{\psi_i^H\}$ coefs

$$\Rightarrow \begin{bmatrix} {}^H\psi_1^* \\ \vdots \\ (i) \\ \vdots \\ {}^H\psi_4^* \end{bmatrix} = H_4^H \cdot \begin{bmatrix} \psi_1^* \\ \vdots \\ \vdots \\ \psi_4^* \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2\alpha + c \\ 0 \\ \sqrt{2}\left(\frac{2b}{\pi} + \frac{c}{\pi}\right) \\ -\sqrt{2}\left(\frac{2b}{\pi} + \frac{c}{\pi}\right) \end{bmatrix} = \begin{bmatrix} \alpha + \frac{c}{2} \\ 0 \\ \sqrt{2}\left(\frac{b}{\pi} + \frac{c}{2\pi}\right) \\ -\sqrt{2}\left(\frac{b}{\pi} + \frac{c}{2\pi}\right) \end{bmatrix} = \begin{bmatrix} {}_1\psi_H^* \\ {}_2\psi_H^* \\ {}_3\psi_H^* \\ {}_4\psi_H^* \end{bmatrix}$$

6 (Exam 6 is duplicated :c)

2 cont.

iv) as we can see in the MSE function, we need to sort the coefficients by their absolute value (or squared) and choose them from the largest to the smallest.

Since $\alpha \geq \beta \geq 0$ and $c \geq 0$ we get:

$$|{}_2\psi_H^*| = |\alpha + \frac{c}{2}| \geq |\sqrt{2}(\frac{b}{\pi} + \frac{c}{2\pi})| = |{}_H\psi_3^*| = |{}_H\psi_4^*| \geq 0 = |{}_2\psi_H^*|$$

=> for $k = 1,2,3,4$ we need to chose the coefficient

by order of ${}_H\psi_1^* > {}_H\psi_3^* \geq {}_H\psi_4^* > {}_H\psi_2^*$ (i_1, i_2, i_3, i_4)

and once we choose our coefficiant our approximation would be $\tilde{f}_{H}(t) = \sum_{i=1}^{k} {}_H\psi_{i_i}^* \cdot \psi_{i_i}^H$

v.) now we get $\alpha = \frac{1}{\pi}$, $\beta = 1$, $c = \frac{3}{2}$

$${}_H\psi_1^* = \frac{1}{\pi} + \frac{3}{4}, \quad {}_H\psi_2^* = 0, \quad {}_H\psi_3^* = \frac{\sqrt{2}}{\pi} + \frac{3}{8\pi}, \quad {}_H\psi_4^* = -\left(\frac{\sqrt{2}}{\pi} + \frac{3}{8\pi}\right)$$

=> now we get that

$$|{}^*\psi_1^H| > |\psi_3^{H^*}| = |{}^*\psi_4^H| > |{}^*\psi_2^H|$$

and so the choices will be the same

(the slices $[0:k]$ of $[{}^*\psi_1^H, \underbrace{{}^*\psi_3^H, {}^*\psi_4^H}_{\text{can be exchanged}}, {}^*\psi_2^H]$

$$=> \tilde{\psi}_H(t) = \sum_{i=1}^{k} {}^H\psi_{i_i}^* \cdot \psi_{i_i}^H$$

7

2] Cont.

b] Walsh-Hadamard

i] I'll show $W_4^T W_4 = I$ and since $W_4 \in \mathbb{R}$ we know $W_4^H = W^T$

$$\Rightarrow \left(\frac{1}{2}\right)^2 \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{W_4^T} \cdot \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{W_4} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ W_4 is Unitary.

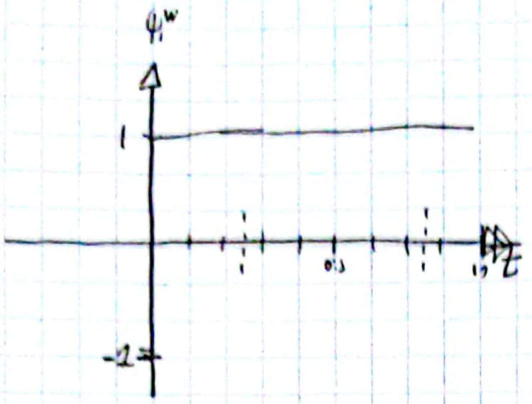
ii] as before, $\{\varphi_i^W(t)\}_i^4$ is recieved by applying W_4^H to the standard basis

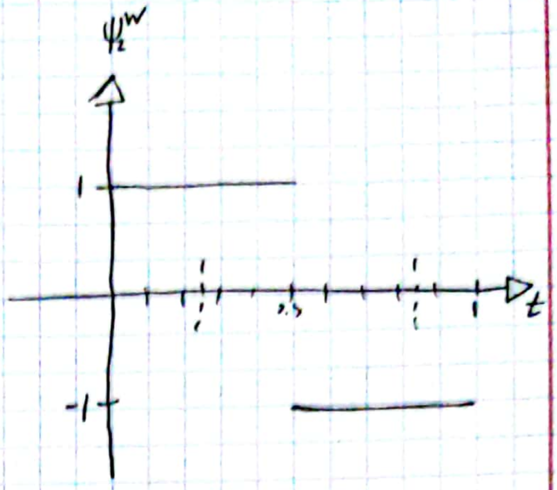
$$\begin{bmatrix} \varphi_1^W \\ \varphi_2^W \\ \vdots \\ \varphi_3^W \end{bmatrix} = W_4^H \cdot \begin{bmatrix} 1_{\Delta_1}\sqrt{4} \\ 1_{\Delta_2}\sqrt{4} \\ 1_{\Delta_3}\sqrt{4} \\ 1_{\Delta_4}\sqrt{4} \end{bmatrix} = \begin{bmatrix} 1_{\Delta_{1-4}} \\ 1_{\Delta_{1,2}} - 1_{\Delta_{3,4}} \\ 1_{\Delta_1} - 1_{\Delta_{2,3}} + 1_{\Delta_4} \\ 1_{\Delta_1} - 1_{\Delta_2} + 1_{\Delta_3} - 1_{\Delta_4} \end{bmatrix}$$

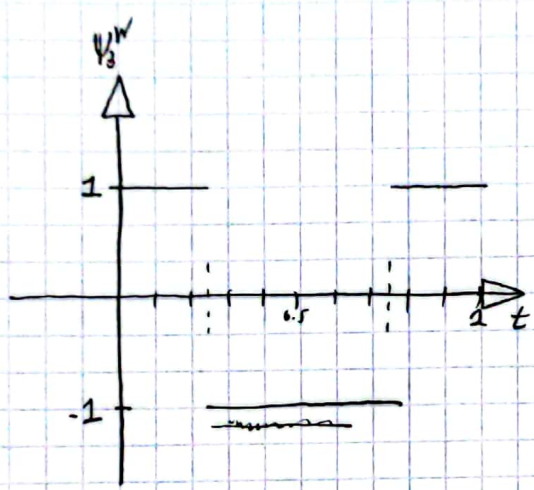
plots in the next page.

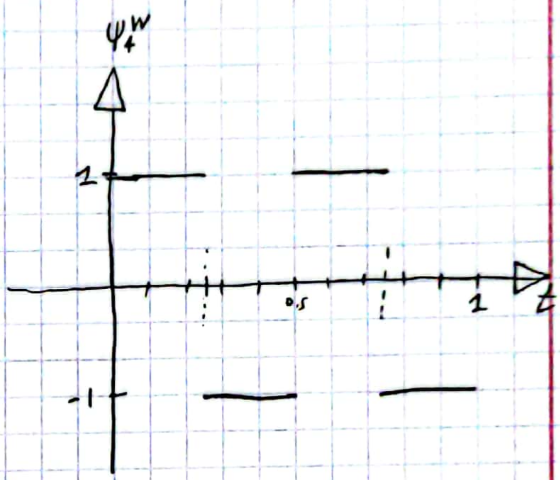
2' cont.

5. cont









---

iii as before, we'll ~~first find~~ use the standard coefficients of $\phi(t)$ by multiplying them by W_4^H and get our ψ_i^w coefficients:

$$\begin{bmatrix} \psi_1^{w*} \\ \vdots \\ \psi_4^{w*} \end{bmatrix} = W_4^H \begin{bmatrix} \psi_1^* \\ \vdots \\ \psi_4^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\alpha + c \\ 0 \\ \frac{4\beta}{\pi} + \frac{2c}{\pi} \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha + \frac{c}{2} \\ 0 \\ \frac{2}{\pi}\beta + \frac{c}{\pi} \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_1^{w*} \\ \psi_2^{w*} \\ \psi_3^{w*} \\ \psi_4^{w*} \end{bmatrix}$$

the best approximation of $\phi(t)$ will be to project it onto $\{\psi_i^w\}_{i=1}^4$ but since it spans the same space as the standart basis and ~~the~~ ~~$\{\psi_i\}_{i=1}^4$~~ from the 1st ~~exer~~ exercise we know that ~~MSE will~~ aswell as ~~while~~ the optimal projection will be the same, the MSE

$$\Rightarrow MSE = \alpha^2 + \alpha c + \frac{3c^2}{2} + \frac{\beta^2}{3} - \frac{\beta c}{2} - 2\left(\frac{1}{2}\alpha + \frac{\beta}{4} + \frac{c}{\pi} + \frac{c}{4}\right) + 2\left(\frac{1}{2}\alpha - \frac{\beta}{\pi} - \frac{c}{2\pi} + \frac{c}{4}\right)$$

2] cont.

b. cont.

iv) assuming $\alpha \geq \beta > 0$, $c \geq 0$ we get $\alpha + \frac{c}{2} > \frac{2}{\pi}\beta + \frac{c}{2}$

$\Rightarrow \quad |\dot{\psi}_1^w| > |\dot{\psi}_3^w| > |\dot{\psi}_2^w| = |\dot{\psi}_4^w|$

and so as before well choose the first K coefficient in that order, where the optimal K approx for $k=3,4$ will be the same as $k=2$ since we have 2 coefficient that are equal to zero

$\Rightarrow \quad K=1: \phi_1(t) = s \cdot {}^*\psi_1^w \cdot \psi_1^w$

$K=2: \phi_2(t) = {}^*\psi_1^w \cdot \psi_1^w + {}^*\psi_3^w \cdot \psi_3^w$

$K=3,4: \phi_3(t) = \phi_4(t) = \phi_2(t) = f$

v) for $\alpha = \frac{1}{\pi}$, $b=1$, $c = \frac{3}{2}$

we get

${}^*\psi_1^w = \frac{1}{\pi} + \frac{3}{4}$, ${}^*\psi_3^w = \frac{2}{\pi} + \frac{3}{2\pi}$

$\Rightarrow |{}^*\psi_3^w| > |{}^*\psi_1^w|$ (by checked with calculator)

$\Rightarrow \quad K_1: \phi_1(t) = {}^*\psi_3^w \cdot \psi_3^w$

$K_2: \phi_2(t) = {}^*\psi_3^w \cdot \psi_3^w + {}^*\psi_1^w \cdot \psi_1^w$

$K_{3,4} = K_2 = \phi_2(t) =$ optimal approximation of $f(t)$ in the standart basis

3) On Hadamard Matrices

a.) ~~First~~ I'll show that H_N is symatric real & Unitary and that it can be written as $H_N = \lambda_N A$ where $\lambda_N \in \mathbb{R}$ and A is a matrix with only ± 1 entries

Proof by Induction on $n \in N$

for $n=1$:

$$H_2 \cdot H_2 = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{2}{2} I \qquad H_2 = \overbrace{\frac{1}{2}}^{\lambda_2} \cdot \overbrace{\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}}^{A_2}$$

=> ~~as can be seen $\frac{1}{2}I$ is real symetric and unitary~~

~~and can be written as ... where $\lambda_2 = \frac{1}{2}$~~

as can be seen H_2 is real symatric and unitary

lets assume the claim is true for $n \in \mathbb{N}$ and can be written as $H_2 = \lambda_2 \cdot A_2$

prove it for $n+1$:

$$H_{2^{n+1}} = H_2 \otimes H_{2^n} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n}\end{bmatrix}$$

• Symatric:

$$H_{2^{n+1}} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n}\end{bmatrix} \underset{\substack{\text{Since } H_{2^n} \text{ is} \\ \text{symatric}}}{=} \begin{bmatrix} H_{2^n}^T & H_{2^n}^T \\ H_{2^n}^T & -H_{2^n}^T\end{bmatrix} = H_{2^{n+1}}^T$$

once will show it is also real the proof for symetric will be done as $H_{2^{n+1}}^H$ will be equal to $H_{2^{n+1}}^T$

real: Since H_{2^n} is real then ofcourse $H_{2^{n+1}}$ is real

Continue at the next page

3] cont.

Unitary:

$$H_{2^{n+1}}^H \cdot H_{2^{n+1}} = H_{2^{n+1}}^T \cdot H_{2^{n+1}} = \frac{1}{2}\begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix} \cdot \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} H_{2^n}H_{2^n} + H_{2^n}H_{2^n} & H_{2^n}H_{2^n} - H_{2^n}H_{2^n} \\ -H_{2^n}H_{2^n} + H_{2^n}H_{2^n} & H_{2^n}H_{2^n} + H_{2^n}H_{2^n} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$$

this beg. from the daim of the induction, $H_{2^n} \cdot H_{2^n}^H = I$.

$\Rightarrow$ $H_{2^{n+1}}$ is Unitary.

Finally: from the claim of Induction H_{2^n} can be written as $\lambda_{2^n} A$ where $\lambda_{2^n} \in R$ and A is a matrix with ± 1 entries

$$\Rightarrow H_{2^{n+1}} = \frac{1}{2}\begin{bmatrix} \lambda_{2^n}A & \lambda_{2^n}A \\ \lambda_{2^n}A & -\lambda_{2^n}A \end{bmatrix} = \underbrace{\frac{1}{2}\lambda_{2^n}}\underbrace{\begin{bmatrix} A & A \\ A & -A \end{bmatrix}}_{A^*}$$

where $\frac{1}{2}\lambda_{2^n} \in \mathbb{R}$, and A^* is a matrix with the same enteris as A and $-A$ which are ± 1.

and with this the proof is done,

Since the claim holds for $n=1$, and then it is holds for every $n \in \mathbb{N}$.

3] cont.

b] i] we will split to different cases.

1: s_1 ends in the same sign that s_2 begins with

in that case $S(s_1 s_2) = S(s_1) + S(s_2)$

2: s_1 ends with a different sign then s_2 begins with

$\therefore$ $S(s_1 s_2) = S(s_1) + S(s_2) + 1$

since we've added another sign change in the place of the concatination.

ii] ~~as~~ we will again prove in induction that $\{S(r_1), \dots, S(r_N)\} = \{0, \dots, N-1\}$

$n=1$: $H_{2^1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ as shown before.

as can be seen $S(r_0) = 0$, $S(r_1) = 1$ and so $\{S(r_0), S(r_1)\} = \{0, 1\}$ and the claim holds.

well assume it holds for n and prove for $n+1$:

$H_{2^{n+1}} = \frac{1}{2} \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$, as we can see, in the upper half of the matrix

we concatinate H_{2^n} with itself, and in the bottom half with itself $\cdot -1$ ~~where~~

~~since the claim holds we know that~~

{ Continues in the next page.

3) cont.

let $n^* \in_{\mathbb{Z}} \{0,1,\dots,N-1\}$, I'll show $n^* \in \{S(r_1),\dots,S(r_N)\}$

~~we'll split to 2 cases:~~

~~1) n^* is even:~~ $\Rightarrow$ n^* ~~can be written~~

$\Rightarrow$ as we know $\lfloor \frac{n^*}{2} \rfloor \in \{0,1,\dots,2^n-1\}$

and so there is a $j \in \{0,1,\dots,2^n-1\}$ s.t

$S(\bar{r}_j) = \lfloor \frac{n^*}{2} \rfloor$ where $\bar{r}_j$ is a row of H_{2^n}

~~we also know that $\frac{n^*}{2}$ is even and so it starts and ends with the same sign~~

~~$S(\bar{r}_j \bar{r}_j) = S(\bar{r}_j) + S(\bar{r}_j) =$~~

let's if we look at r_j and r_{2j} of $H_{2^{n+1}}$

$r_j = \underbrace{\bar{r}_j}_{s_1}\underbrace{\bar{r}_j}_{s_2}$, $\quad r_{2j} = (\underbrace{\bar{r}_j}_{s_1}, \underbrace{-(\bar{r}_j)}_{s_2})$

$\Rightarrow$ in one of the cases we will be in the case where

s_1 ends in the sign that s_2 begins with

and the other row will be in the case where

s_1 ends in the same sign that s_2 begins with

$\Rightarrow$ either $S(r_j) = \lfloor \frac{n^*}{2} \rfloor + \lfloor \frac{n^*}{2} \rfloor + 1$ and $S(r_{2j}) = \lfloor \frac{n^*}{2} \rfloor + \lfloor \frac{n^*}{2} \rfloor$

or the other way around

~~and so we know that both n^* and~~

and so if n^* is even then $\lfloor \frac{n^*}{2} \rfloor + \lfloor \frac{n^*}{2} \rfloor = n^*$

and if it is odd then $\lfloor \frac{n^*}{2} \rfloor + \lfloor \frac{n^*}{2} \rfloor + 1 = n^*$

either way $n^* \in \{S(r_1),\dots,S(r_N)\}$

$\Rightarrow \{0,1,\dots,N-1\} \subseteq \{S(r_1),\dots,S(r_N)\}$

we also know that $|\{0,\dots,N-1\}| = |\{S(r_1),\dots,S(r_N)\}| = N$

$\Rightarrow \{S(r_1),\dots,S(r_N)\} = \{0,\dots,N-1\}$.

$\Rightarrow$ the claim holds for all $n \in \mathbb{N}$. $\blacksquare$

$\boxed{14}$

4 all Haar Matrices

a) no, lets look at $H_{2,1}$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} = H_2^T \quad \Rightarrow \quad H_2 \text{ is not symetric.}$$

b) no, lets look at $H_4^H \cdot H_4 = H_4^T \cdot H_4$

$$\Rightarrow H_4^H \cdot H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow$ we can see that $H_4^H \cdot H_4 \neq I$

$\Rightarrow$ Since H_4 is real it is neither orthogonal nor Unitary

c) as shown before H_4 is not Unitary.

well continue at the next page.

4] cont.

d] we will recursivly define $\tilde{H}$ in the following way

$$\tilde{H}_2 = \frac{1}{\sqrt{2}} H_2$$

$$\tilde{H}_{2(N+1)} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} \tilde{H}_{2N} \otimes [1,1] \\ I_{2N} \otimes [1,1] \end{bmatrix}$$

e] well prove $(A \otimes B)^T = A^T \otimes B^T$ for all A, B MATRIXES.

$$(A \otimes B)^T = \begin{bmatrix} a_{11} \cdot B, & a_{1,2} \cdot B & \cdots & a_{1n} \cdot B \\ a_{21} B, & a_{22} B & \cdots & a_{2n} \cdot B \\ \vdots & & & \vdots \\ a_{n1} \cdot B & a_{n,2} \cdot B & \cdots & a_{nn} \cdot B \end{bmatrix}^T =$$

$$= \begin{bmatrix} a_{11} B^T & a_{2,1} B^T & \cdots & a_{n,1} B^T \\ a_{1,2} B^T & a_{22} B^T & & a_{n,2} B^T \\ \vdots & & & \\ a_{1,n} B^T & a_{2,n} B^T & \cdots & a_{nn} B^T \end{bmatrix} = A^T \otimes B^T.$$

f] well define: $\hat{H}_2 = \tilde{H}_2^T$

and we want $\underline{\hat{H}_{2(N+1)}} = \tilde{H}^T_{2(N+1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{H}_{2N} \otimes [1,1] \\ I_{2N} \otimes [1,-1] \end{pmatrix}^T$

$$= \frac{1}{\sqrt{2}} \cdot \left[(\tilde{H}_{2N} \otimes [1,1])^T \; (I_{2N} \otimes [1,-1])^T \right] = \frac{1}{\sqrt{N}} \left[\hat{H}_{2N} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \; I_{2N} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]$$

1. Numerical and Practical Bit allocation for Two-Dimensional Signals

$$\phi(x,y) = Acos(2\pi w_x x)\sin(2\pi w_y y)$$
$$A = 2500, w_x = 2, w_y = 7$$

a. From differentiating and integrating we get that:

i. $\frac{d\phi}{dx} = -2\pi A w_x \sin(2\pi w_x x)\sin(2\pi w_y y)$

ii. $\frac{d\phi}{dy} = 2\pi A w_y \cos(2\pi w_x x)\cos(2\pi w_y y)$

iii. $\int_0^1\int_0^1 \left(\frac{d\phi}{dx}\right)^2 dxdy = \int_0^1\int_0^1 \left(-2\pi A w_x \sin(2\pi w_x x)\sin(2\pi w_y y)\right)^2 dxdy =$

$$= 4\pi^2 A^2 w_x^2 \int_0^1 \sin(2\pi w_x x)^2 dx \int_0^1 \sin(2\pi w_y y)^2 dy =$$

$$=_{\sin^2 x = 0.5 - cos2x} 4\pi^2 A^2 w_x^2 \int_0^1 0.5 - 0.5\cos(4\pi w_x x)\, dx \int_0^1 0.5 - 0.5\cos(4\pi w_x x) dy =$$

$$= 4\pi^2 A^2 w_x^2 \left(0.5 - \frac{\sin 4\pi w_x}{8\pi w_x}\right)\left(0.5 - \frac{\sin 4\pi w_x}{8\pi w_x}\right) =$$

$$=_{plugging\ A, w_x, w_y} 4\pi^2 * A^2 w_x^2 * (0.5) * (0.5) = (\pi * A * w_x)^2$$

$$= 246{,}740{,}110{,}0.2723$$

iv. $\int_0^1\int_0^1 \left(\frac{d\phi}{dy}\right)^2 dxdy = \int_0^1\int_0^1 \left(2\pi A w_y \cos(2\pi w_x x)\cos(2\pi w_y y)\right)^2 dxdy =$

$$= 4\pi^2 A^2 w_y^2 \int_0^1 \cos(2\pi w_x x)^2 dx \int_0^1 \cos(2\pi w_y y)^2 dy =$$

$$= 4\pi^2 A^2 w_y^2 \int_0^1 0.5 + 0.5\cos 4\pi w_x x\, dx \int_0^1 0.5 + 0.5\cos 4\pi w_y y\, dy =$$

$$= 4\pi^2 A^2 w_y^2 \left(0.5 + \frac{\sin 4\pi w_x}{8\pi w_x}\right)\left(0.5 + \frac{\sin 4\pi w_y}{8\pi w_y}\right) =$$

$$=_{plugging\ A, w_x, w_y} 4\pi^2 * A^2 w_y^2 * (0.5) * (0.5) = (\pi * A * w_y)^2$$

$$= 3{,}022{,}566{,}347.8336$$

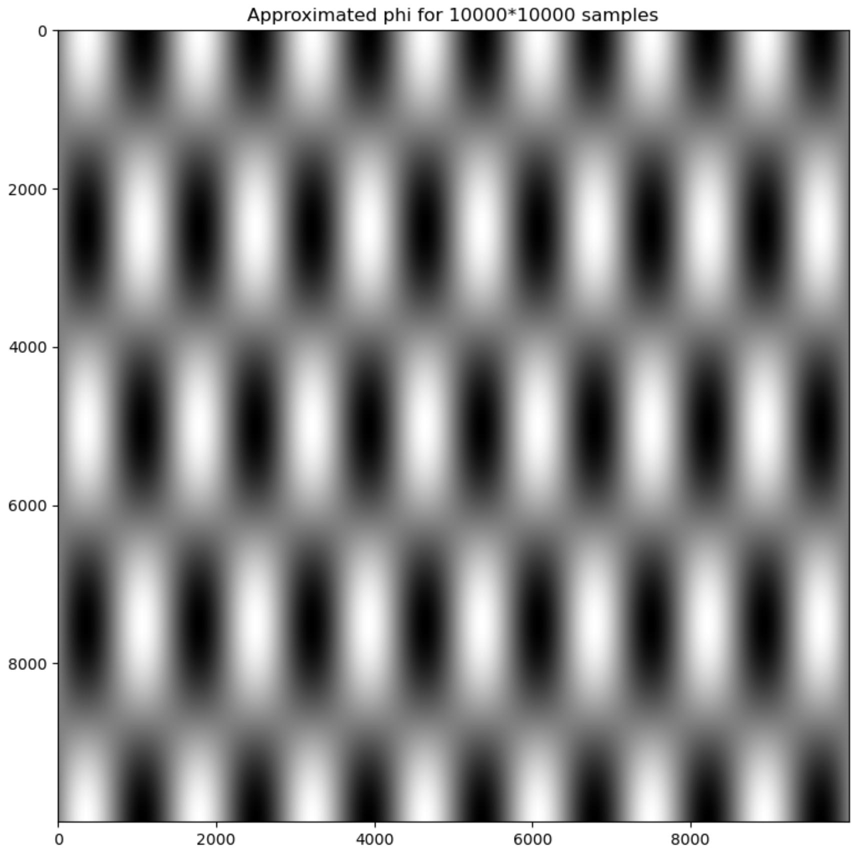
Because $-1 \le \sin x \le 1\ and\ -1 \le \cos x \le 1\ for\ any\ x\ and\ because\ a\ maximum\ and\ minimum\ points\ are\ found\ within\ function\ range$:

v. $\phi_{high} = A = 2500$

vi. $\phi_{low} = -A = -2500$

vii. Value-range = $\phi_{high} - \phi_{low} = 5000$

b. Below is the approximated signal.



c. From numerically computing the vertical derivative energy, horizontal derivative energy and the value range:

```
approximated vertical derivative energy: 3022561475.352848
approximated horizontal derivative energy: 246740077.55753663
approximated value range: 5000.0
```

The results computed analytically are 3,022,566,347.8336, 246,740,110,0.2723 and 5000 in that order.

The numerical computed results deviate slightly. A higher resolution would generate a smaller deviation.

e. The obtained results are displayed in the table below:

Bit-budget	5000	50000
$N_x, N_y, b \in R$	20.73, 72.54, 3.32	53.53, 187.32, 4.9855
$N_x, N_y, b \in N$	21, 79, 3	54, 185, 5

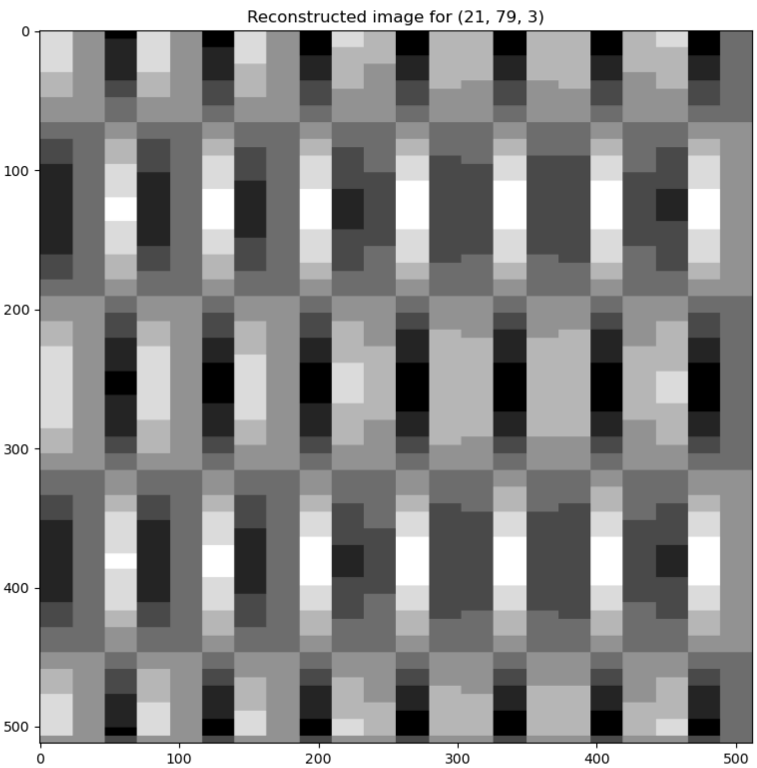
f. Implemented in code.

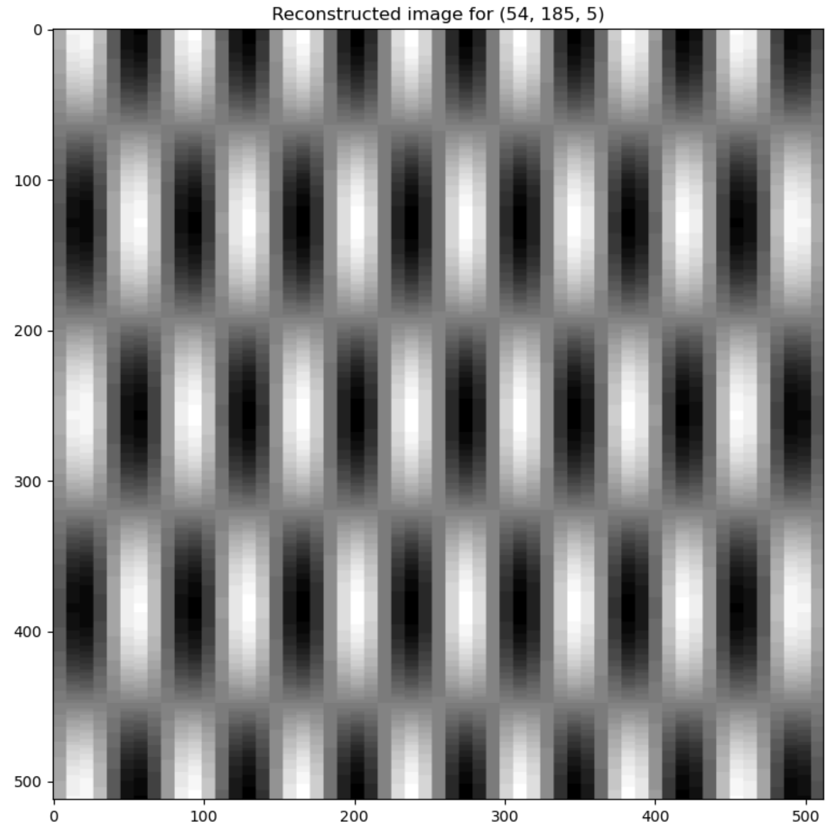
g. The results obtained by the search procedure are displayed in the table below:

Bit-budget	5000	50000
$N_x, N_y, b \in R$	20.69, 72.58, 3.32	53.49, 187.42, 4.9866
$N_x, N_y, b \in N$	21, 79, 3	54, 185, 5

The "real" results differ slightly. But the rounded results are identical. This is because we round the results to the best feasible solution.

The reconstructed images obtained in the experiment are:





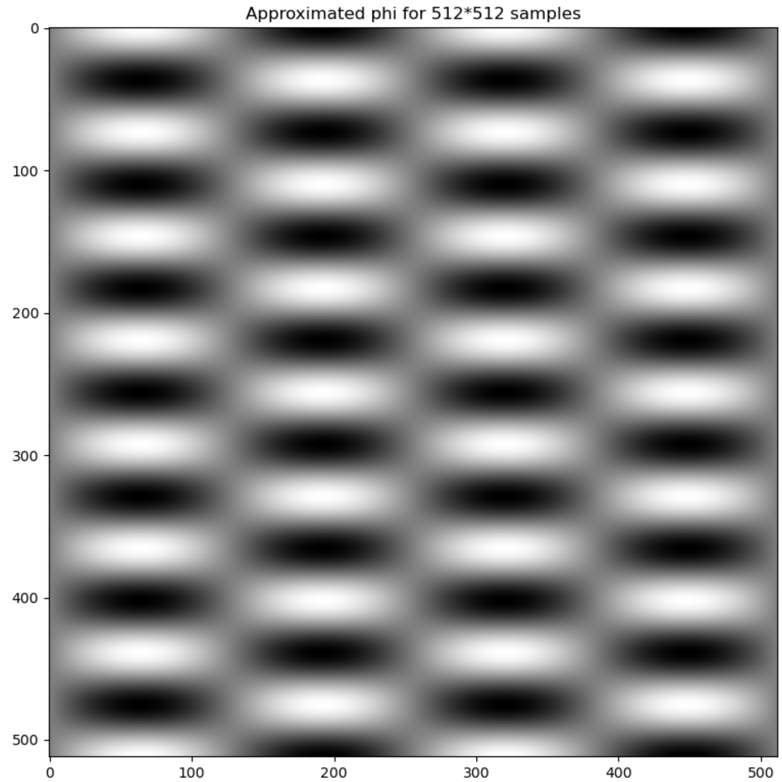
The images for both experiments are the same because the N_x, N_y and b parameters are the same.

h. $A = 2500, w_x = 7, w_y = 2$

i. From the same analysis we get that:

1. $\frac{d\phi}{dx} = -2\pi A w_x \sin(2\pi w_x x)\sin(2\pi w_y y)$
2. $\frac{d\phi}{dy} = 2\pi A w_y \cos(2\pi w_x x)\cos(2\pi w_y y)$
3. $\int_0^1 \int_0^1 \left(\frac{d\phi}{dx}\right)^2 dxdy = (\pi * A * w_x)^2 = 3{,}022{,}566{,}347.8336$
4. $\int_0^1 \int_0^1 \left(\frac{d\phi}{dy}\right)^2 dxdy = \left(\pi * A * w_y\right)^2 = 246{,}740{,}110{,}0.2723$
5. $value - range = 2 * A = 5000$

ii. The approximated image:



iii. The numerical calculated results are:

1. Value-range = 5000
2. horizontal derivative energy: 3020708099.8682528
3. vertical derivative energy: 246727724.06950286

By no surprise, the results are the same only that the horizontal and vertical derivatives energy are swapped (actually the results differ slightly because we used a lower resolution when sampling ϕ)

iv. The obtained results are displayed in the table below:

Bit-budget	5000	50000
$N_x, N_y, b \in R$	72.54, 20.73, 3.32	187.32, 53.53, 4.9855
$N_x, N_y, b \in N$	79, 21, 3	185, 54, 5

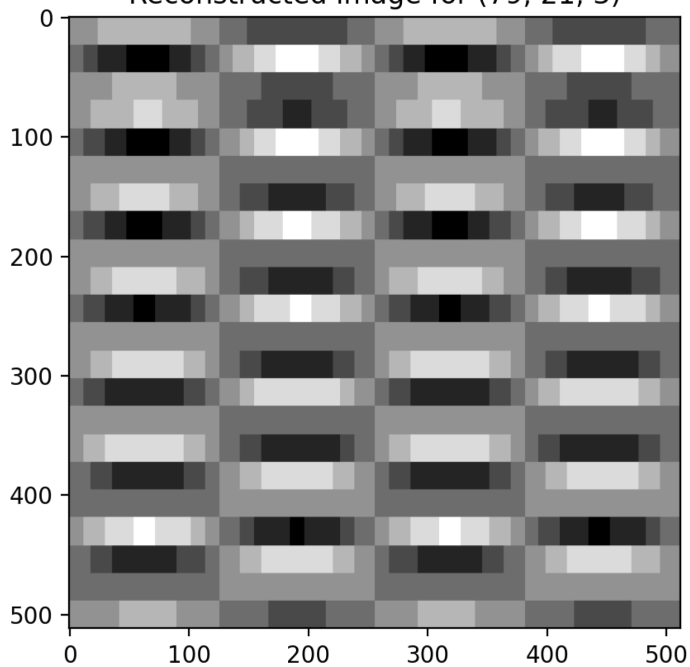
v. The results obtained by the search procedure are displayed in the table below:

Bit-budget	5000	50000
$N_x, N_y, b \in R$	72.58, 20.69, 3.32	187.42, 53.49, 4.9866
$N_x, N_y, b \in N$	79, 21, 3	185, 54, 5

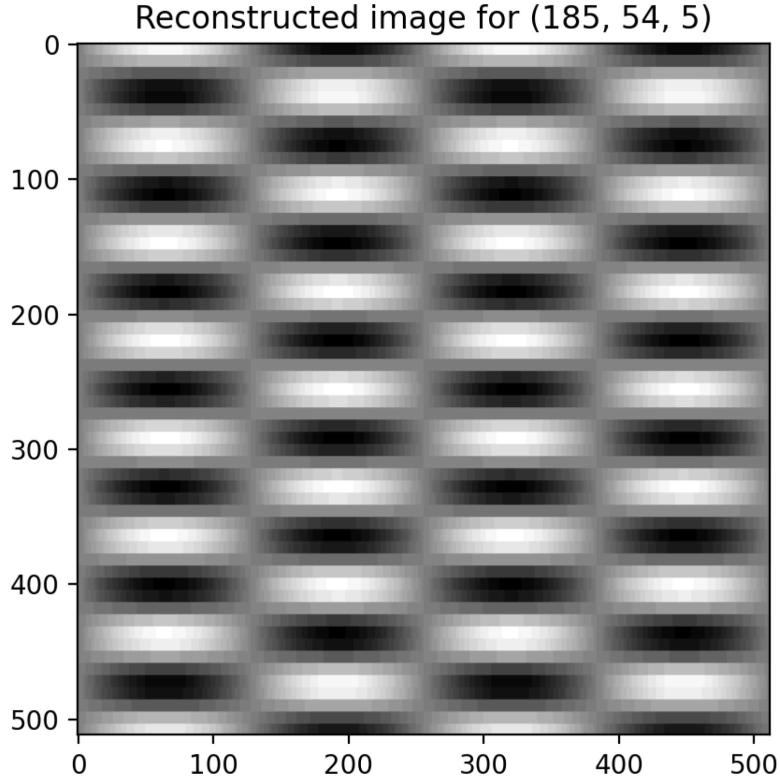
As expected, these are the same results where N_x, N_y are swapped.

The reconstructed images obtained in the experiment are:

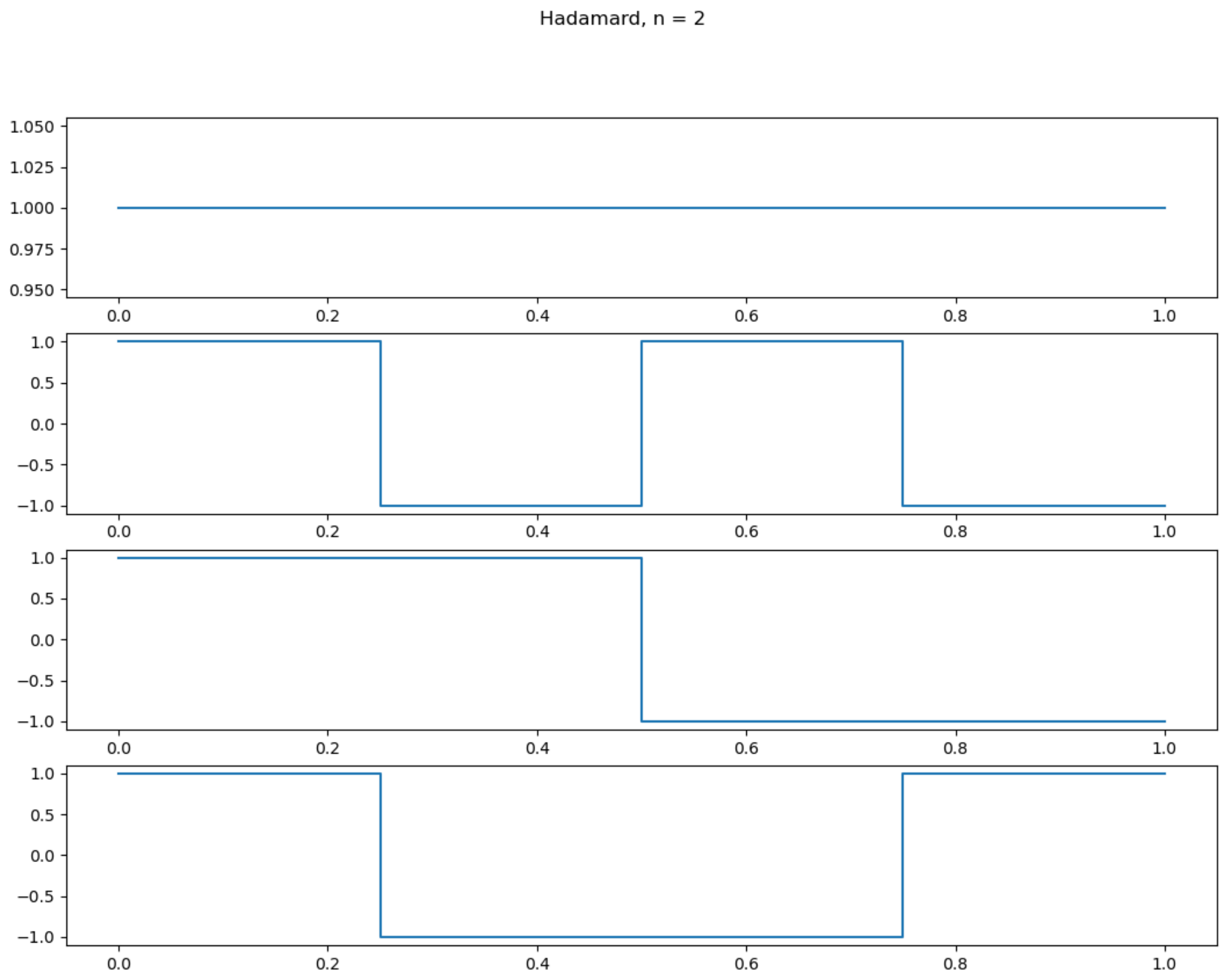
Reconstructed image for (79, 21, 3)



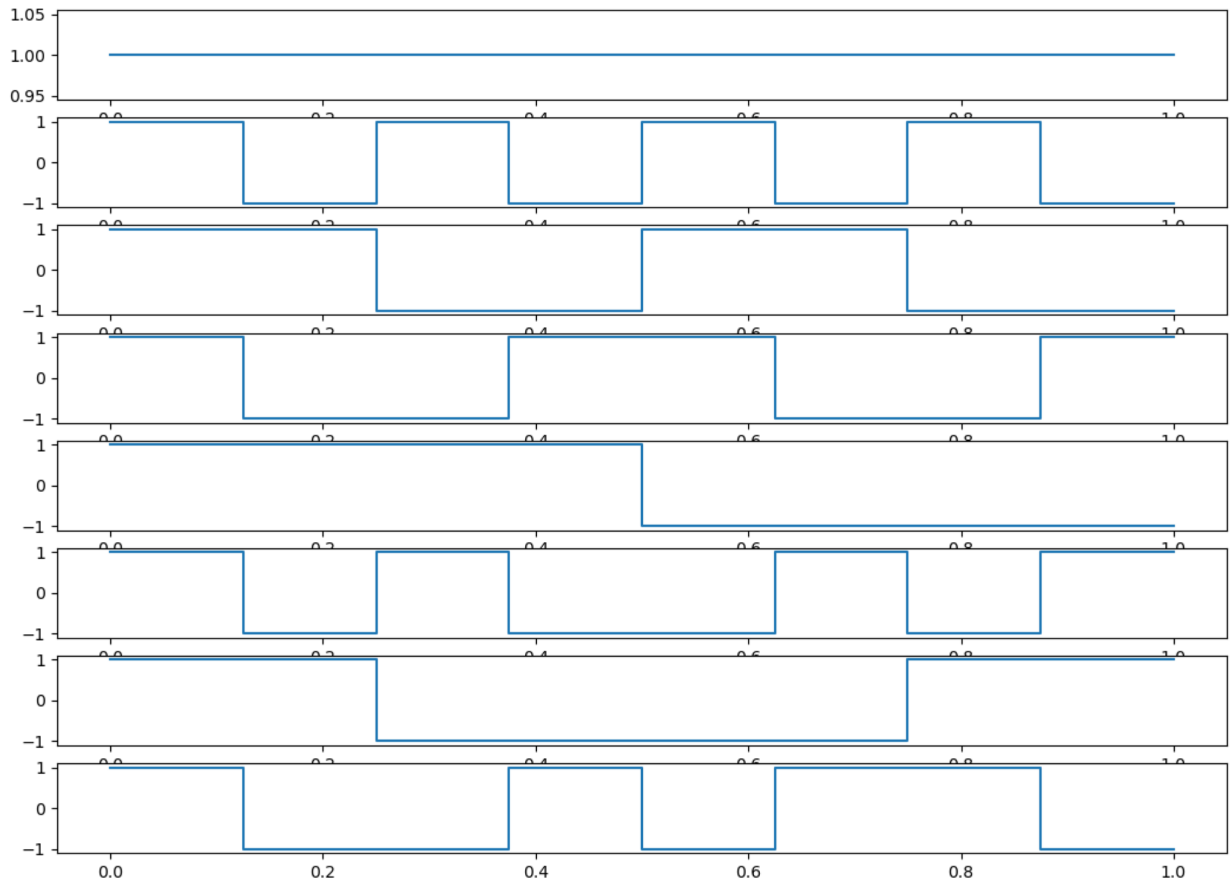
Reconstructed image for (185, 54, 5)



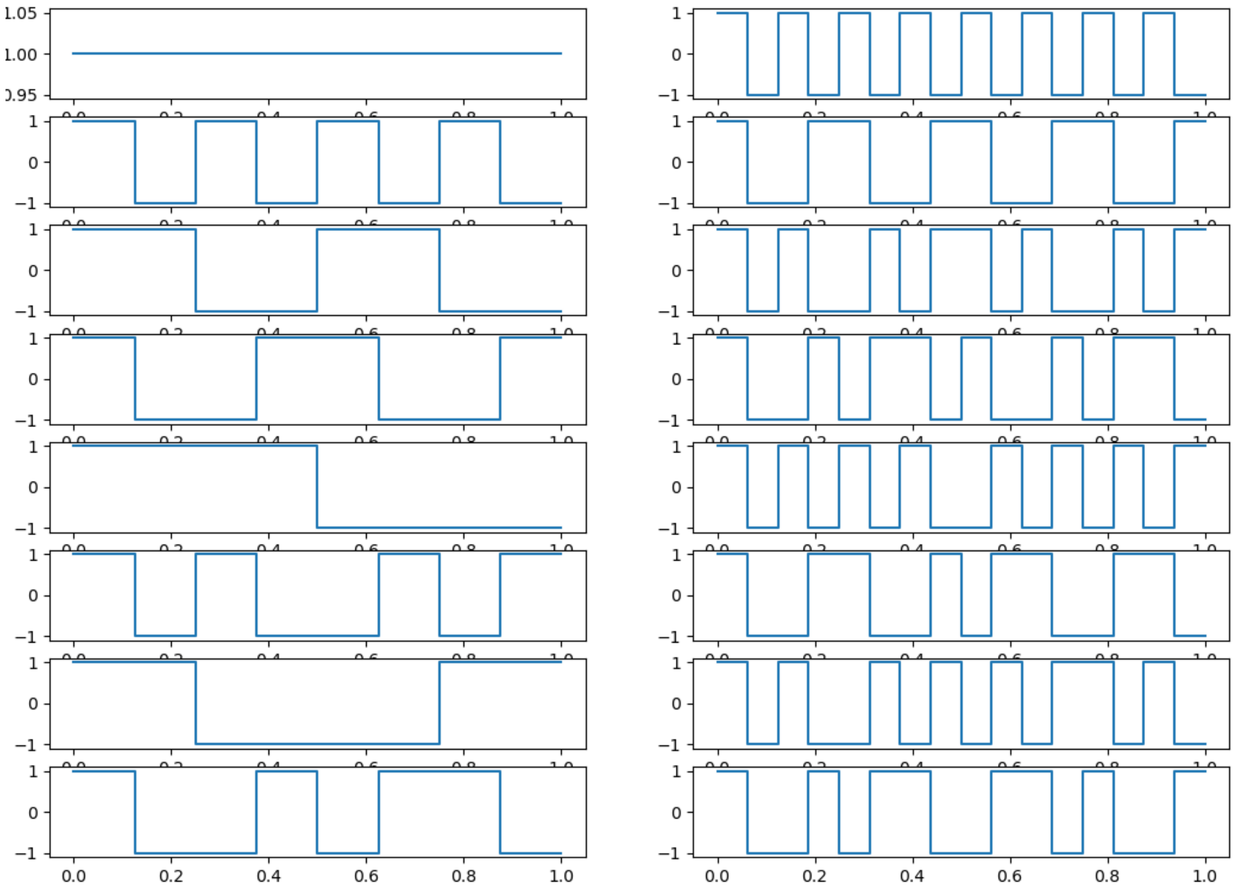
2. **Hadamard, Hadamard-Walsh, and Haar matrices**
    a. Implemented in code.
    b. Below are the results:



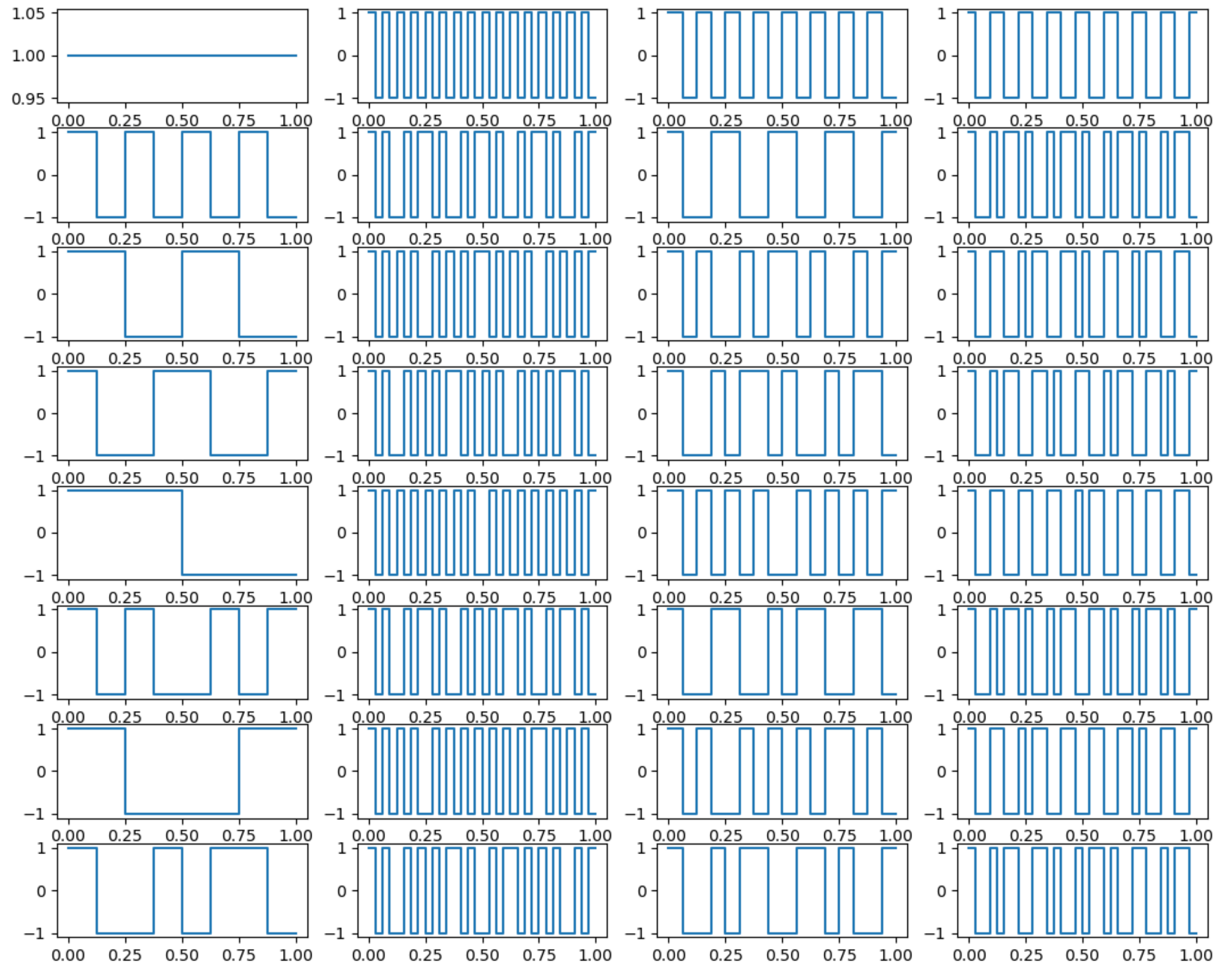
Hadamard, n = 3

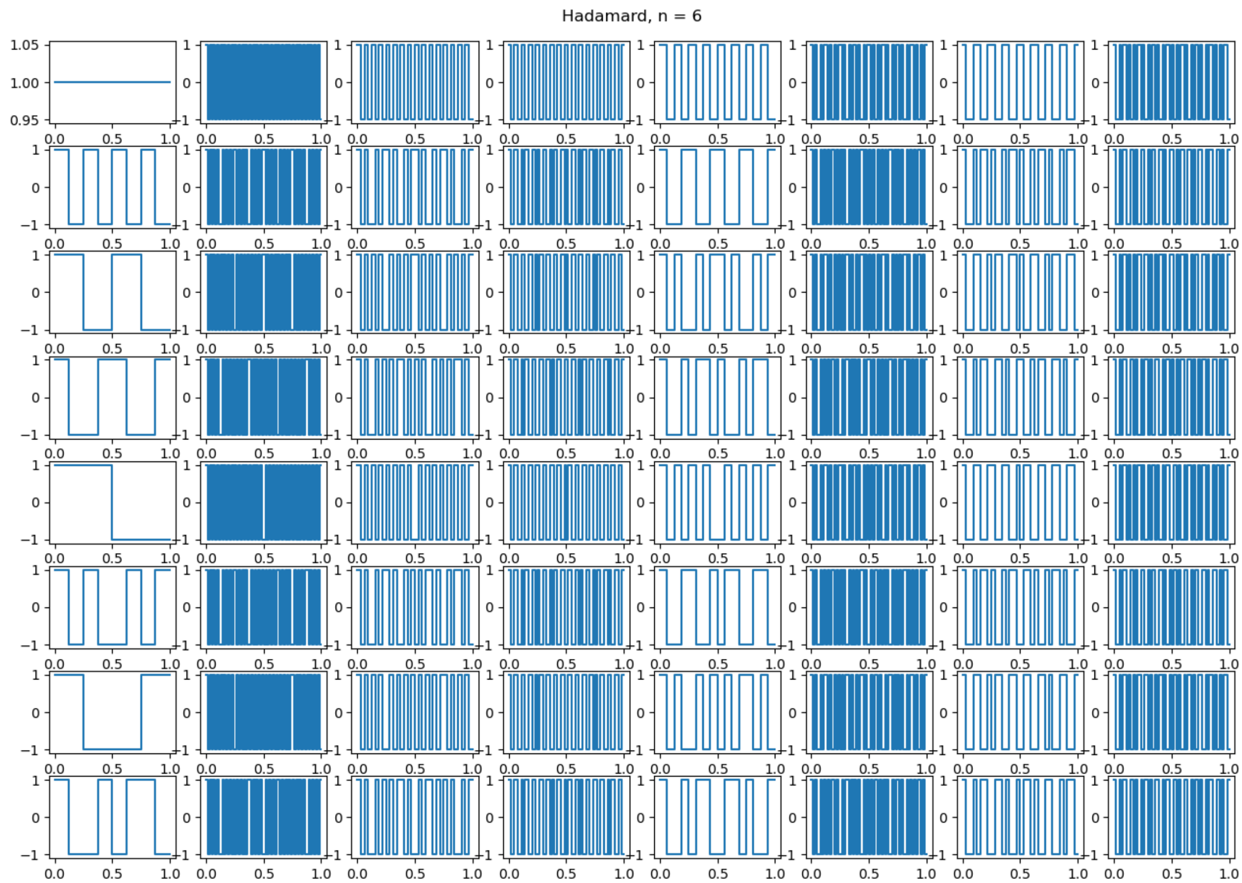


Hadamard, n = 4



Hadamard, n = 5

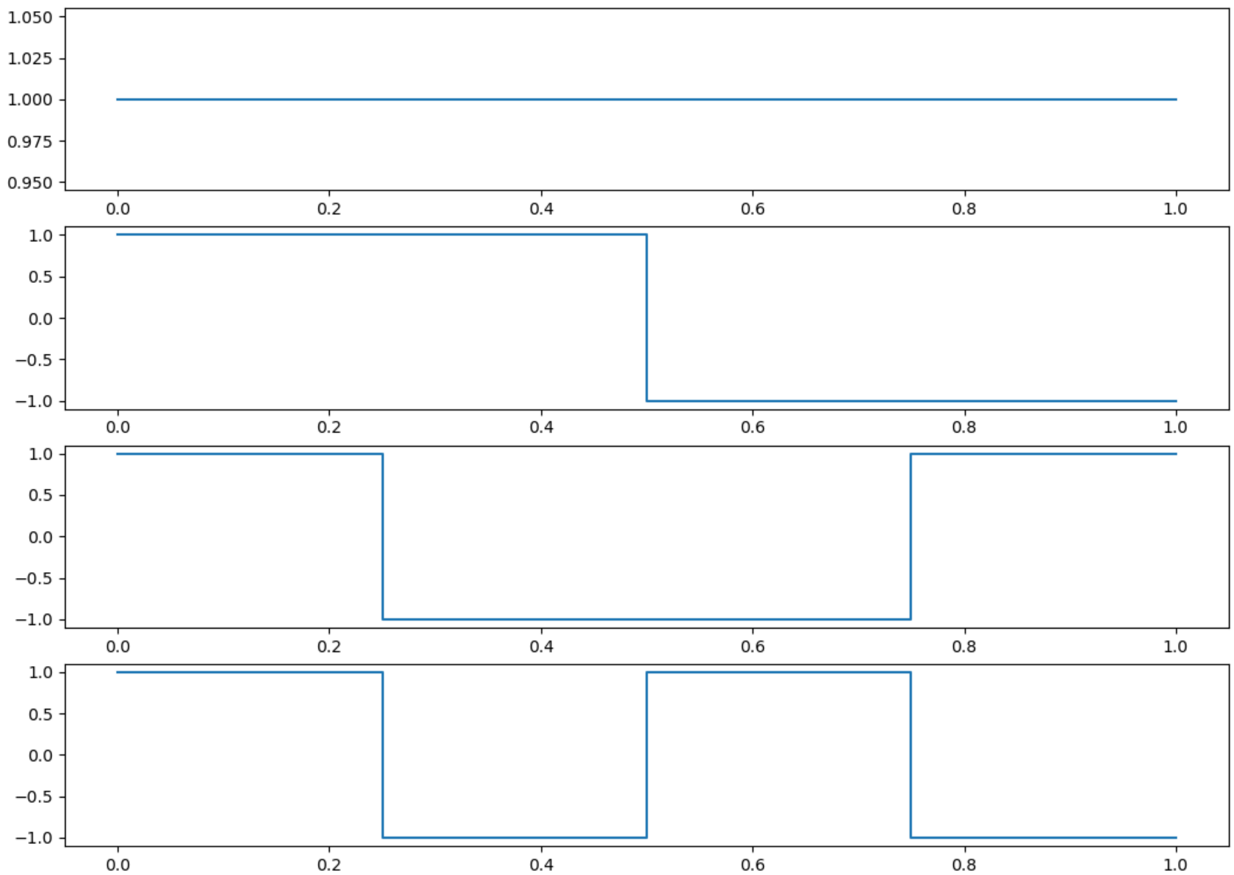


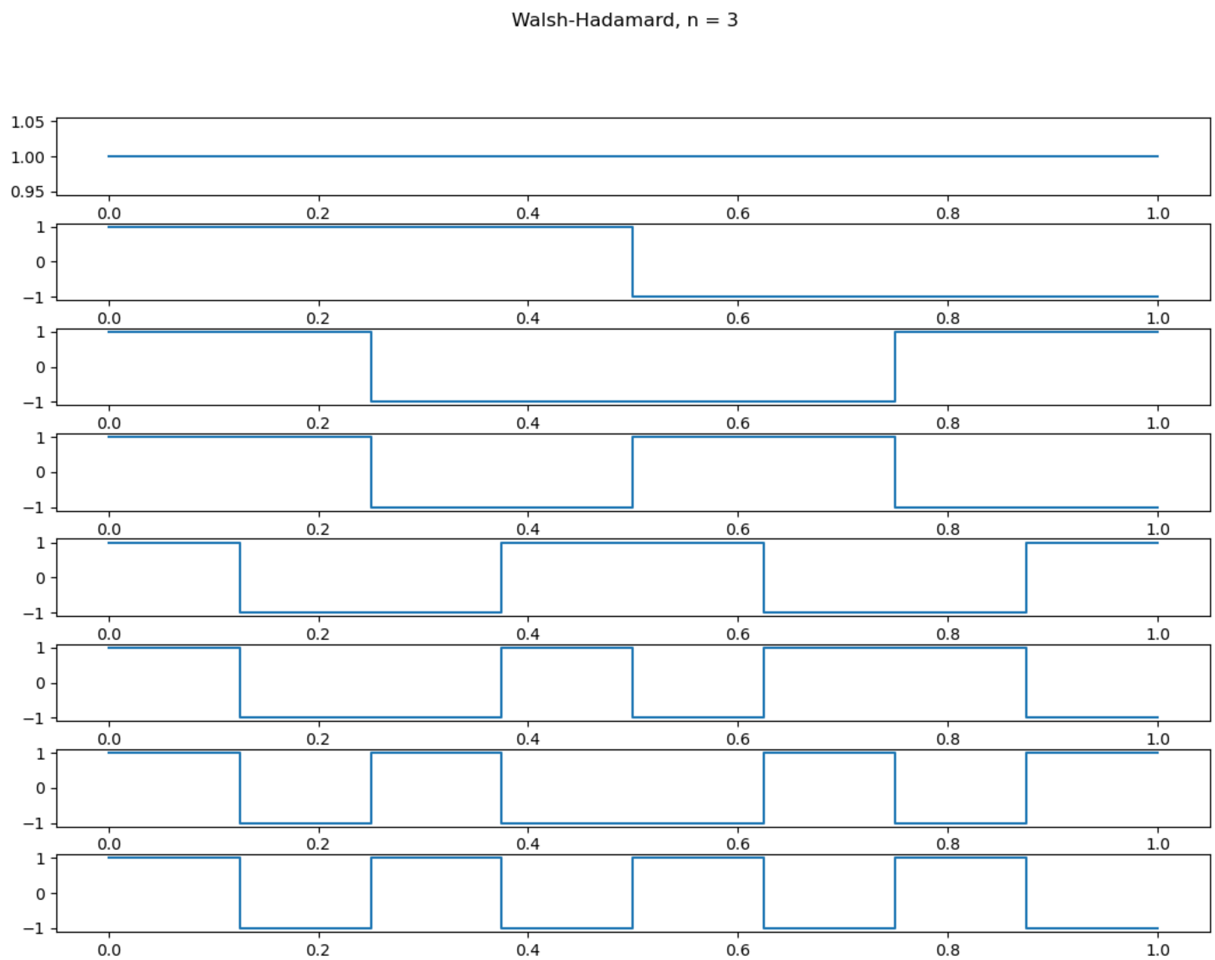


c. Implemented in code.

d. Below are the results:

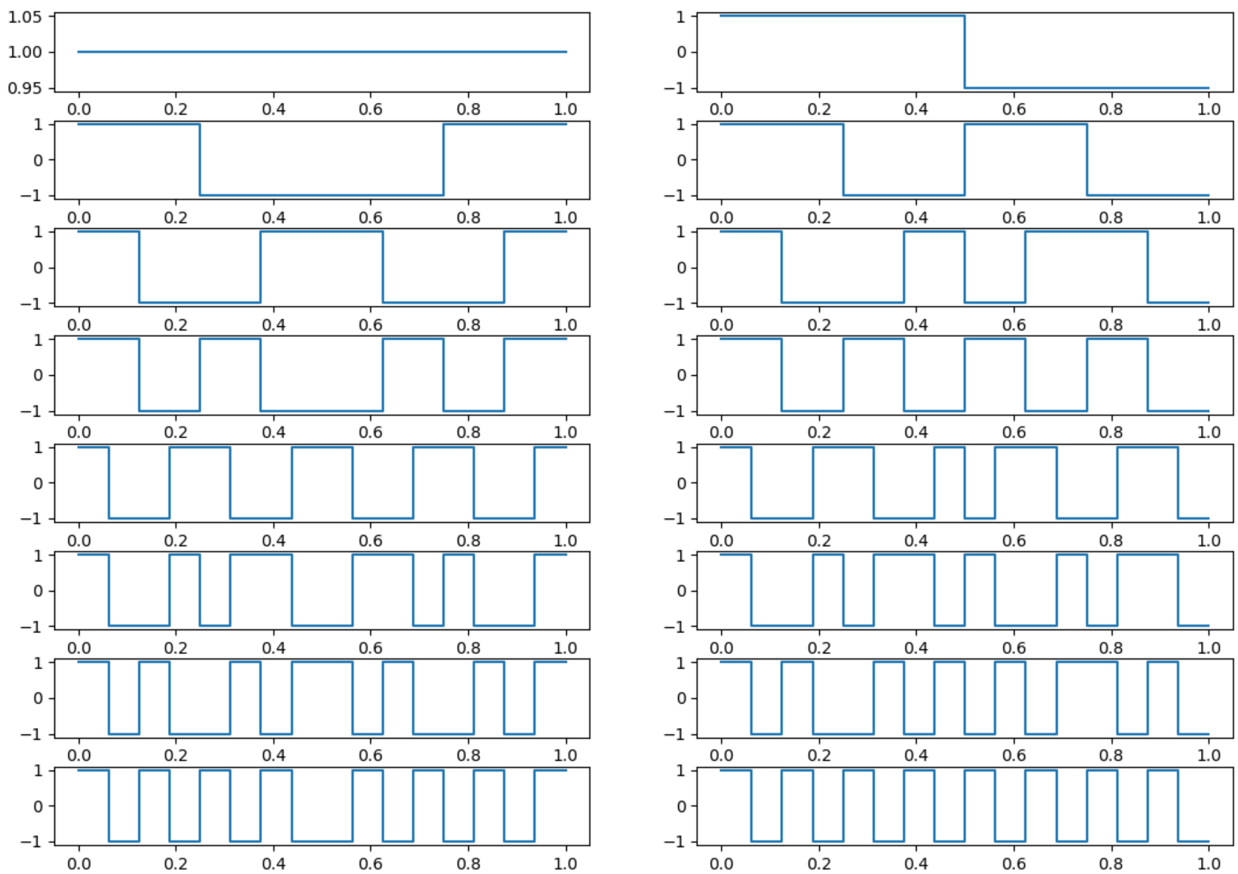
Walsh-Hadamard, n = 2



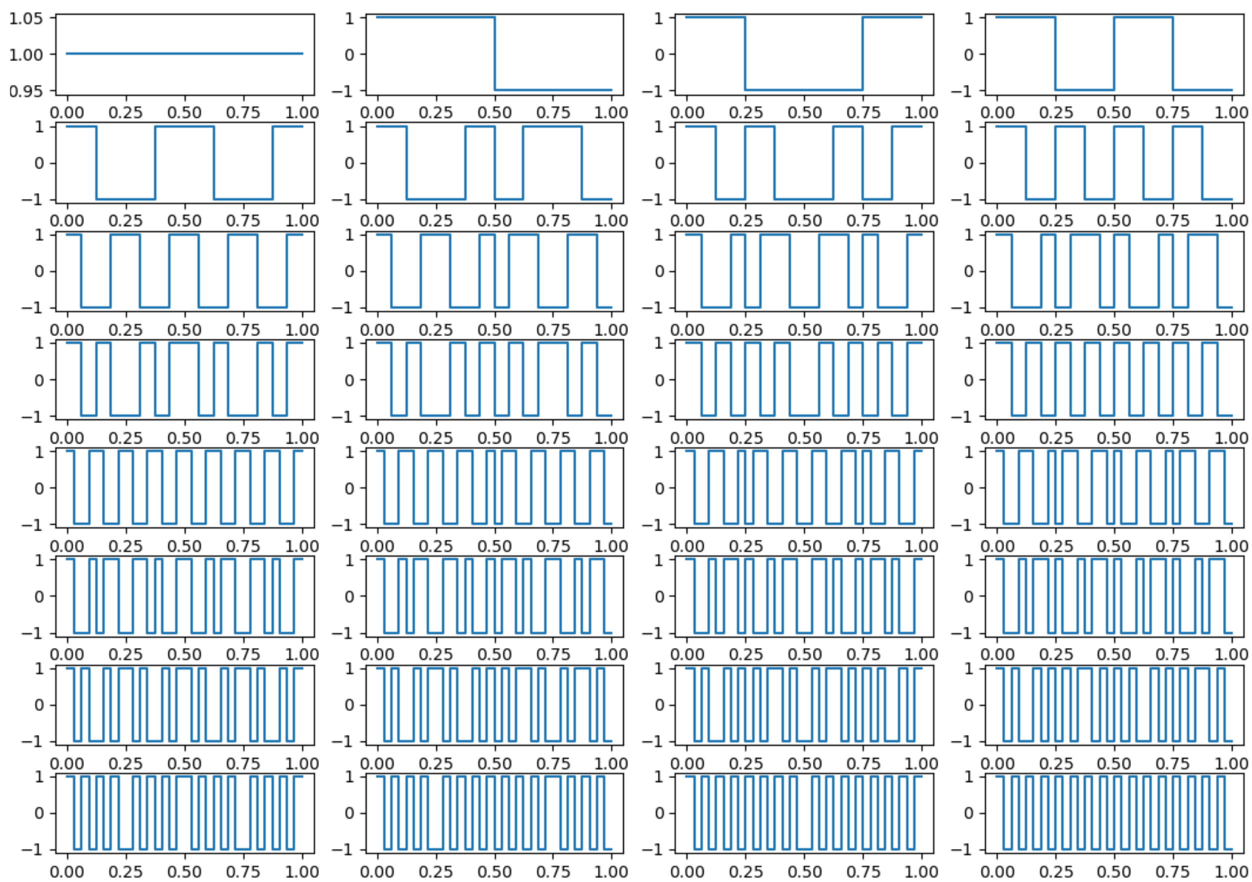


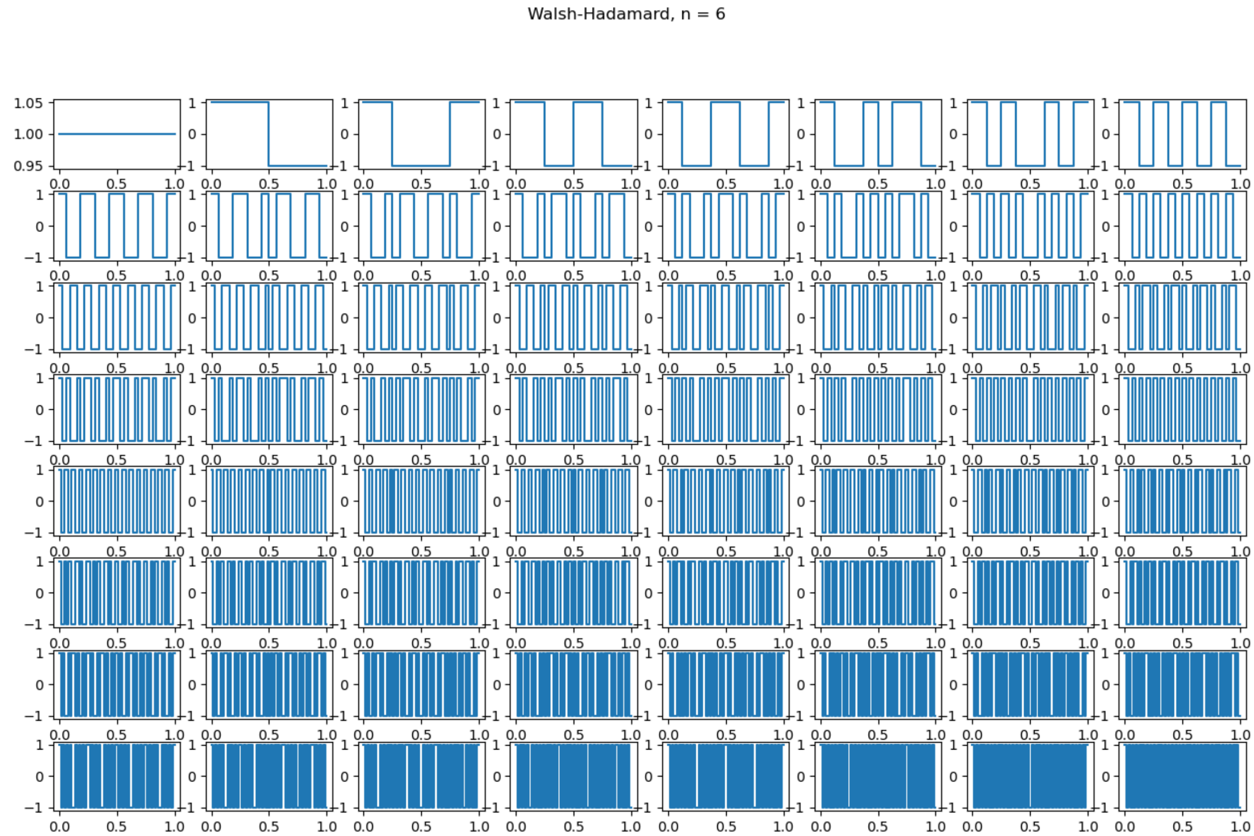
Walsh-Hadamard, n = 3
1.05
1.00
0.95
1
0
−1
0.0
0.2
0.4
0.6
0.8
1.0

Walsh-Hadamard, n = 4



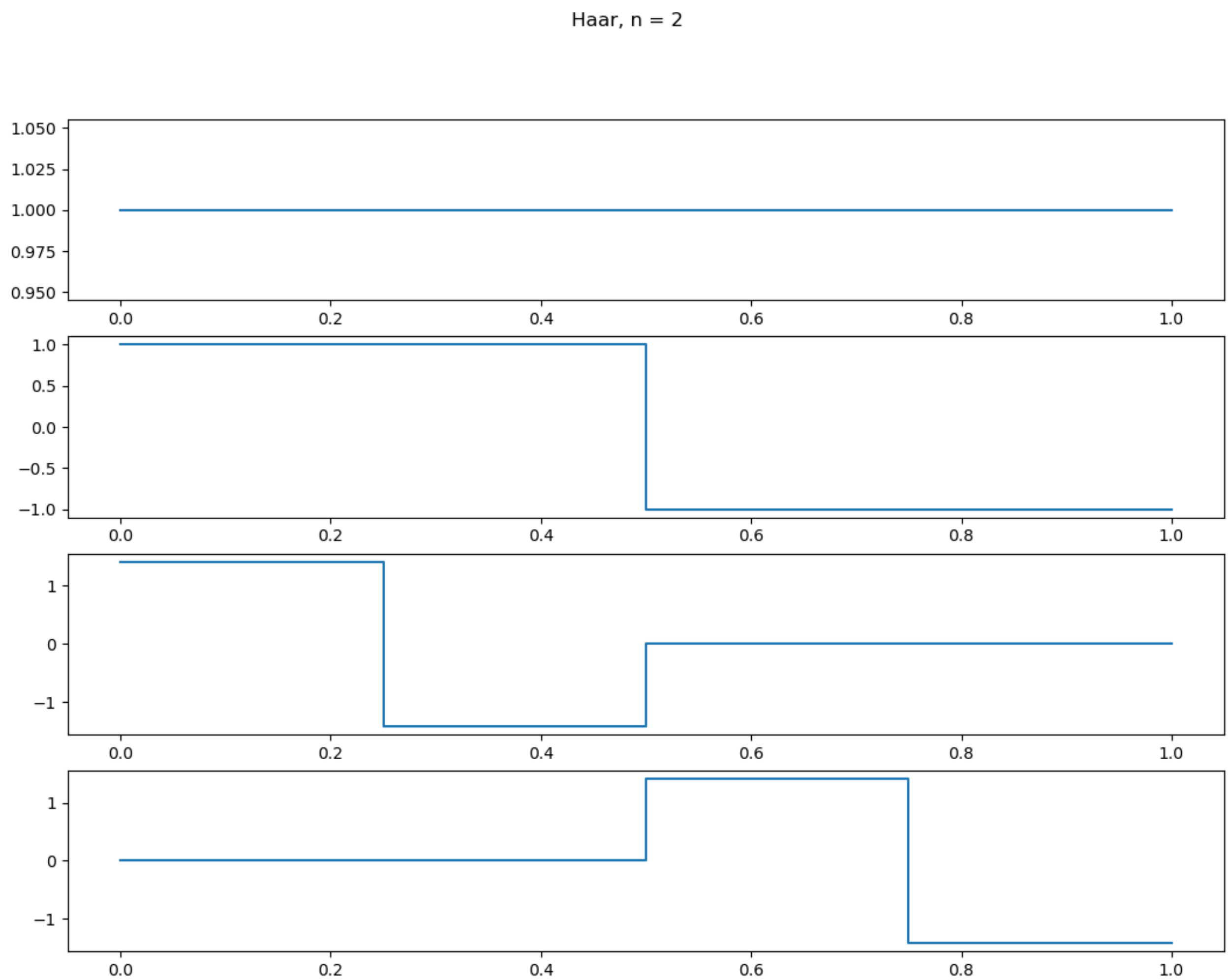
Walsh-Hadamard, n = 5



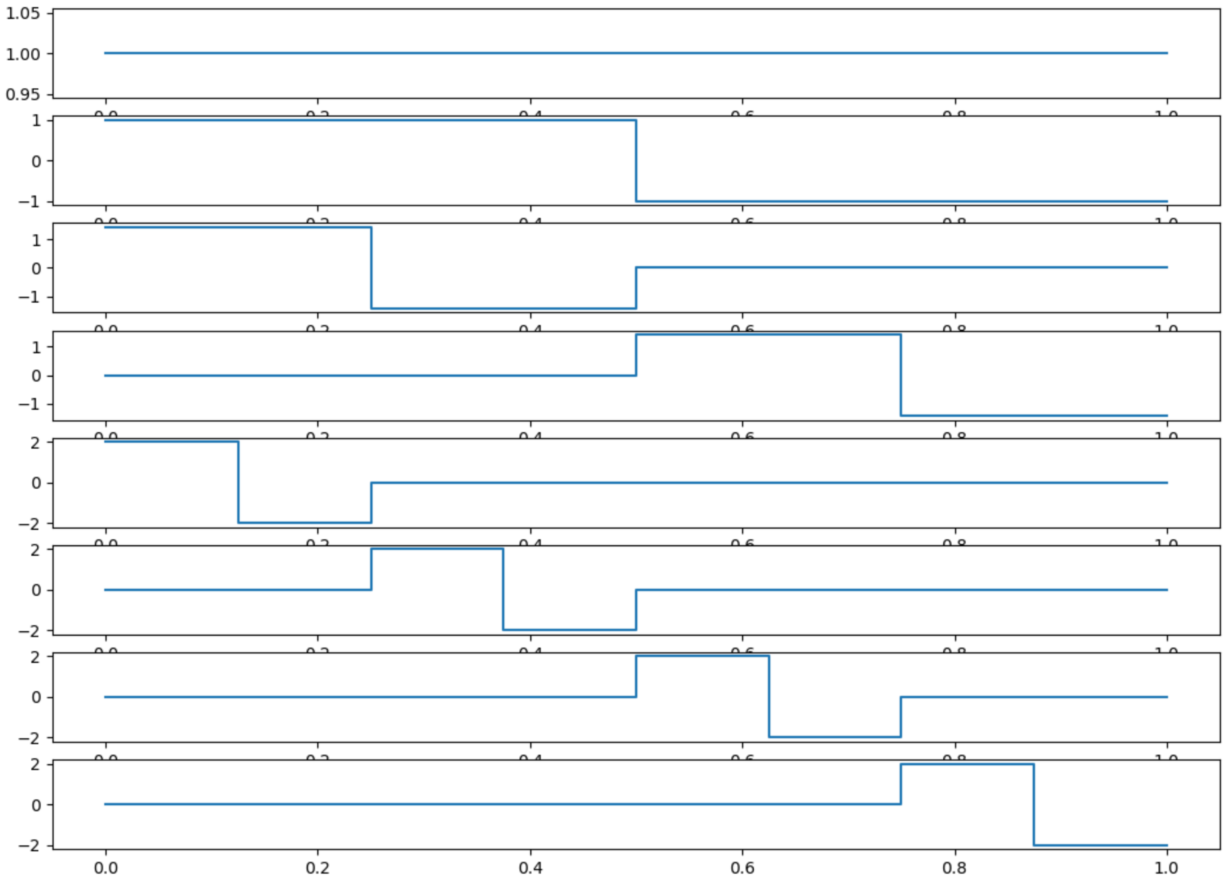


e. Implemented in code.

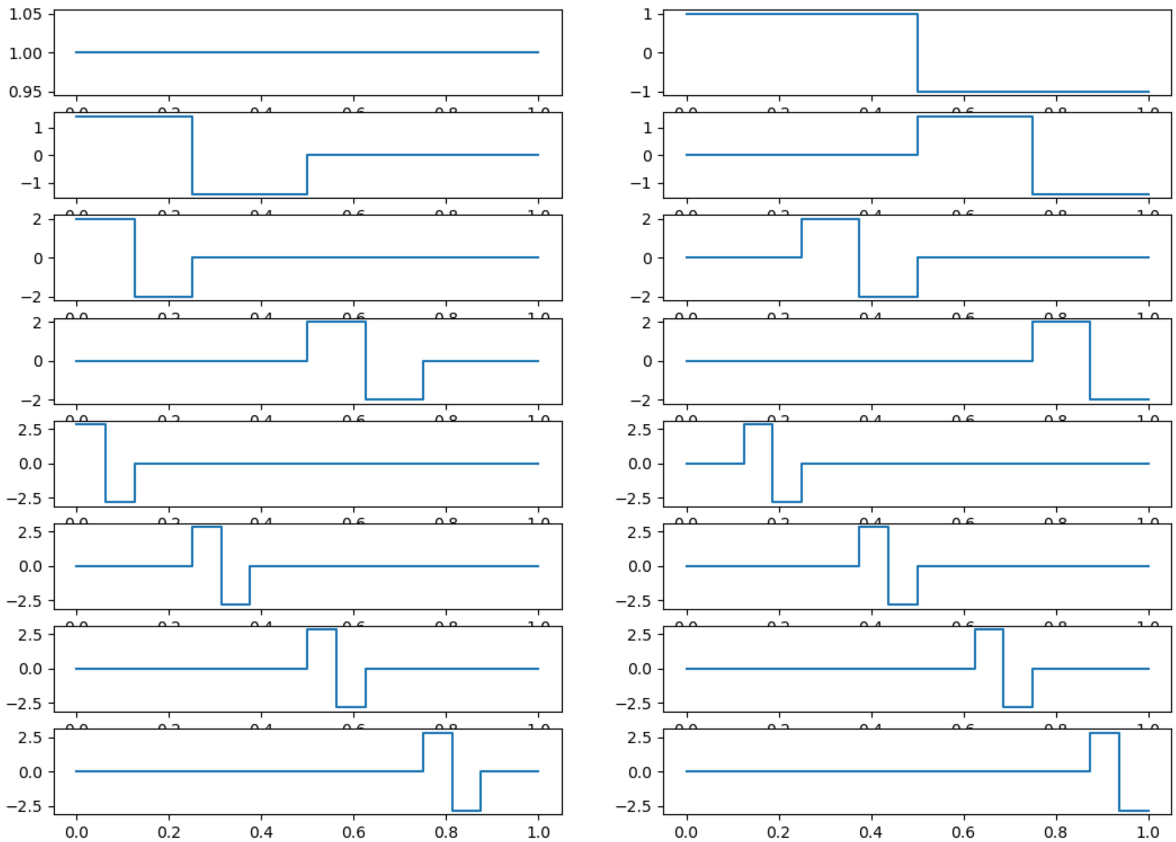
f. The results are presented below:



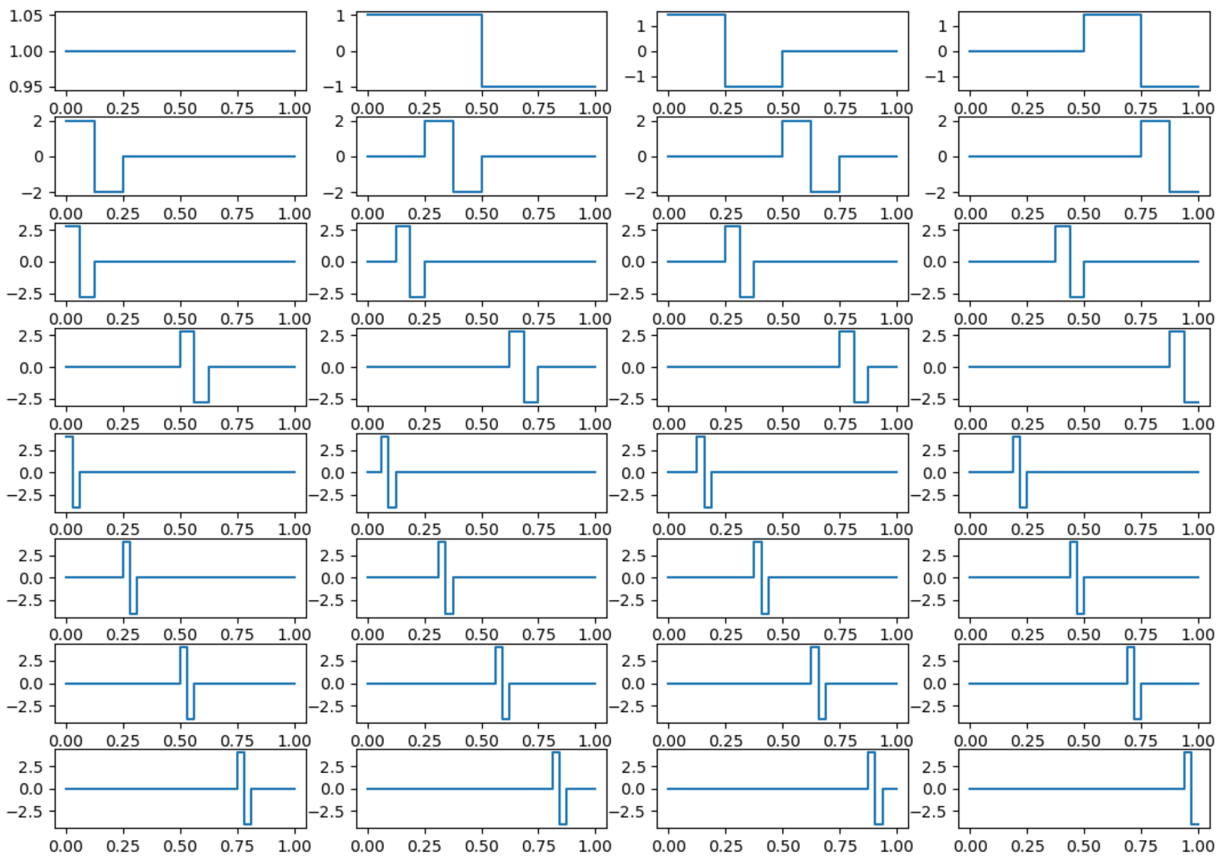
Haar, n = 3



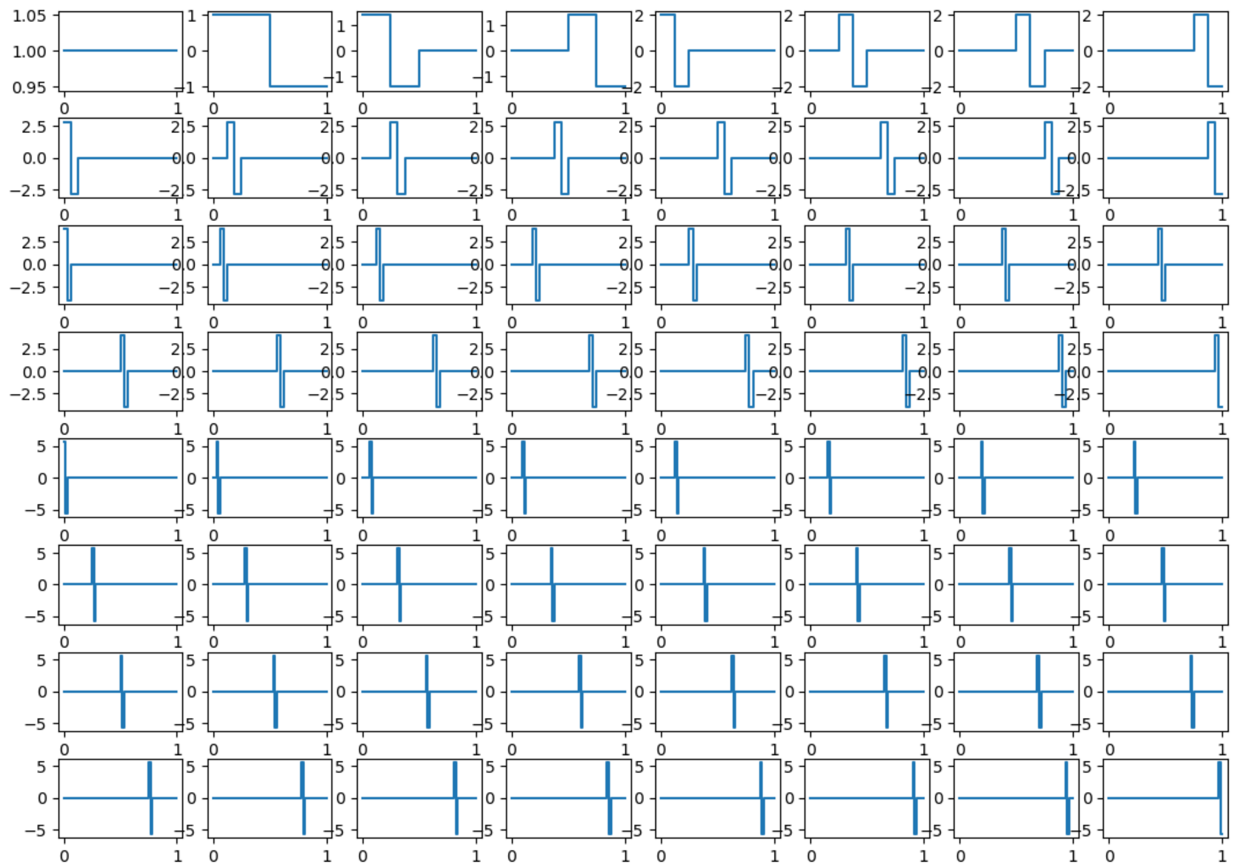
Haar, n = 4



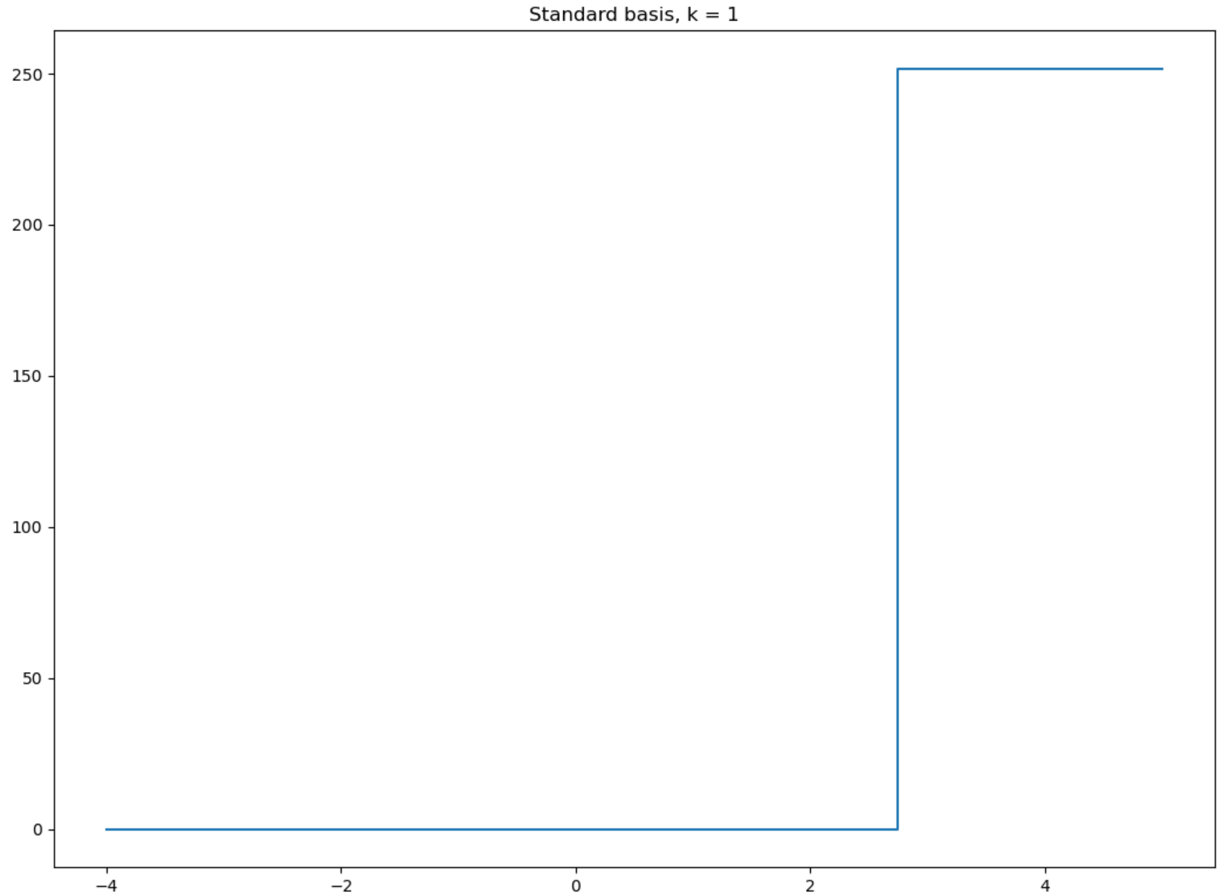
Haar, n = 5

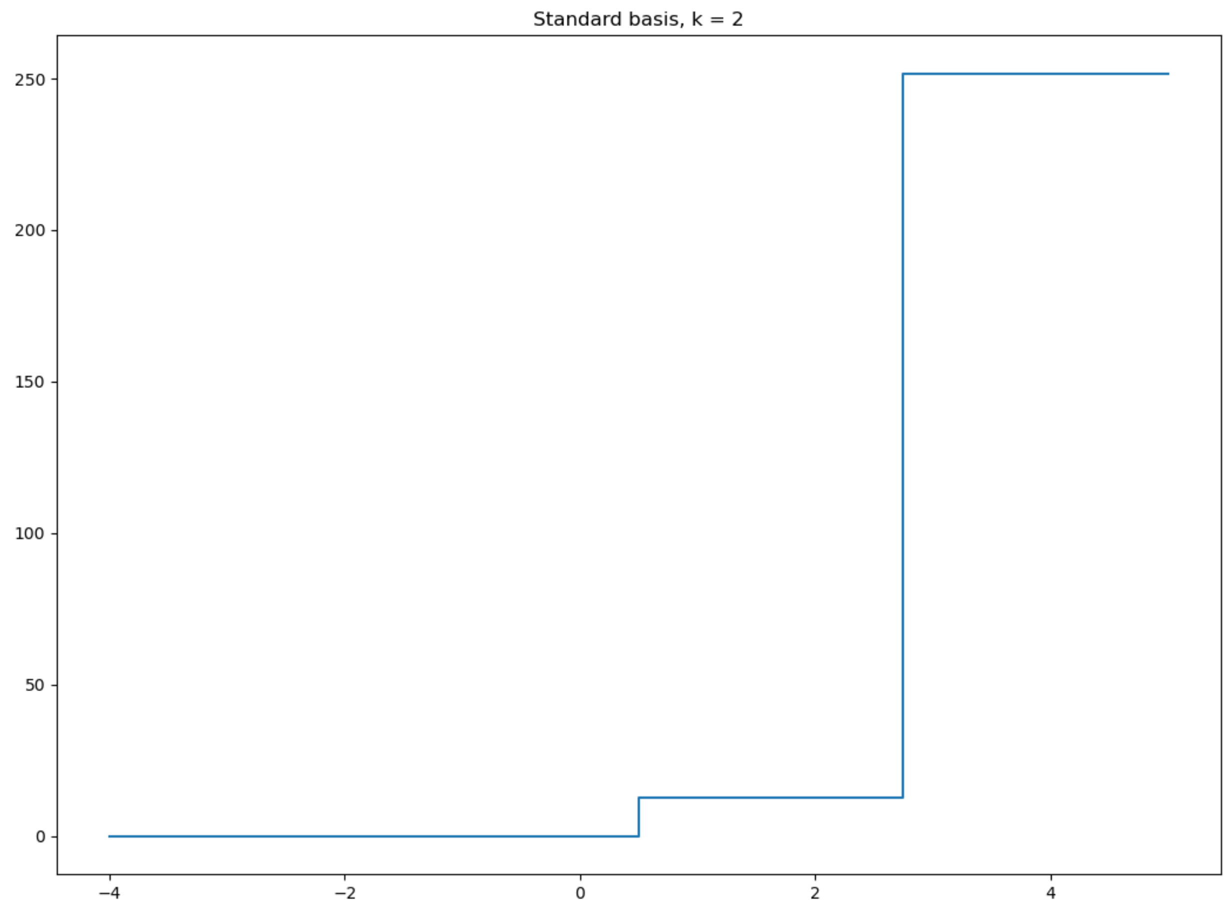


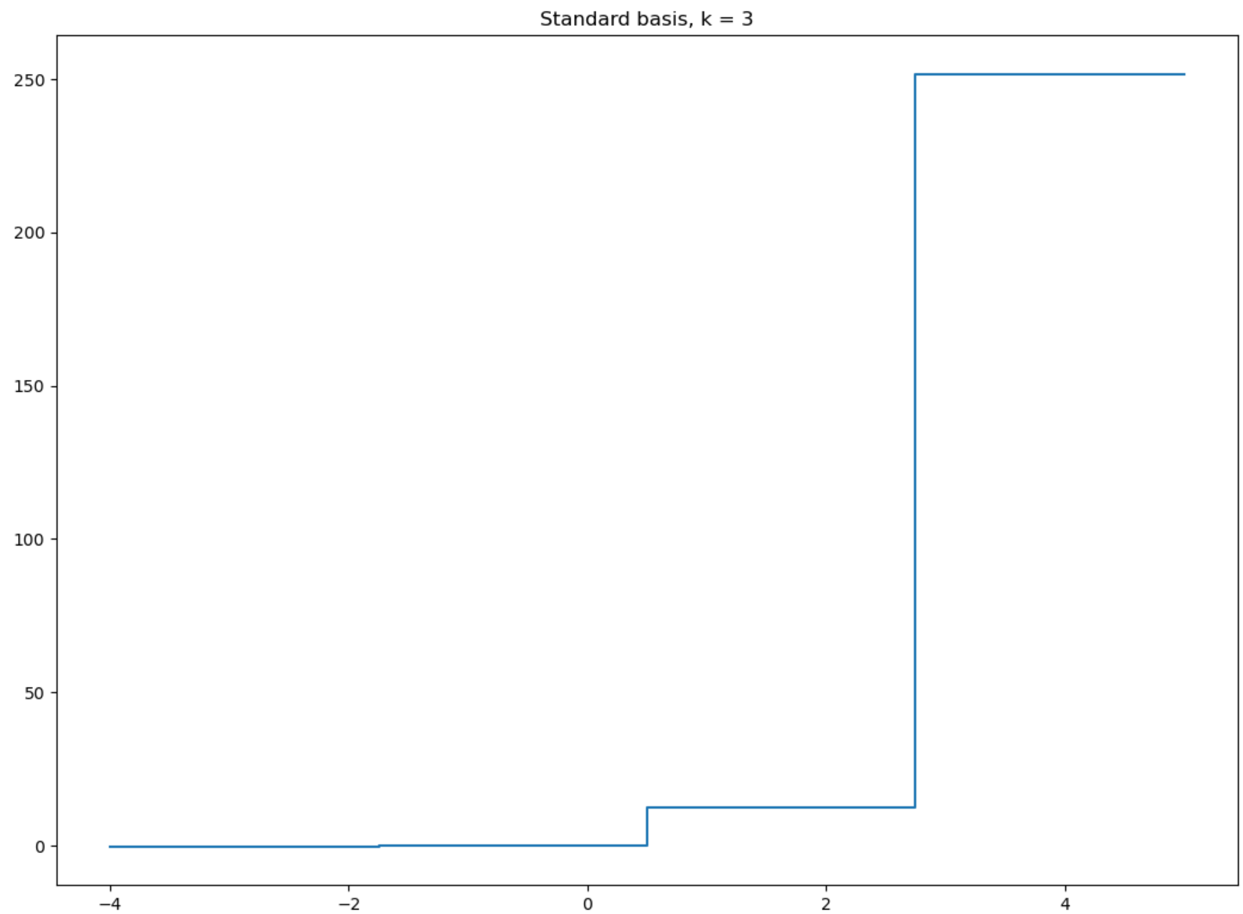
Haar, n = 6

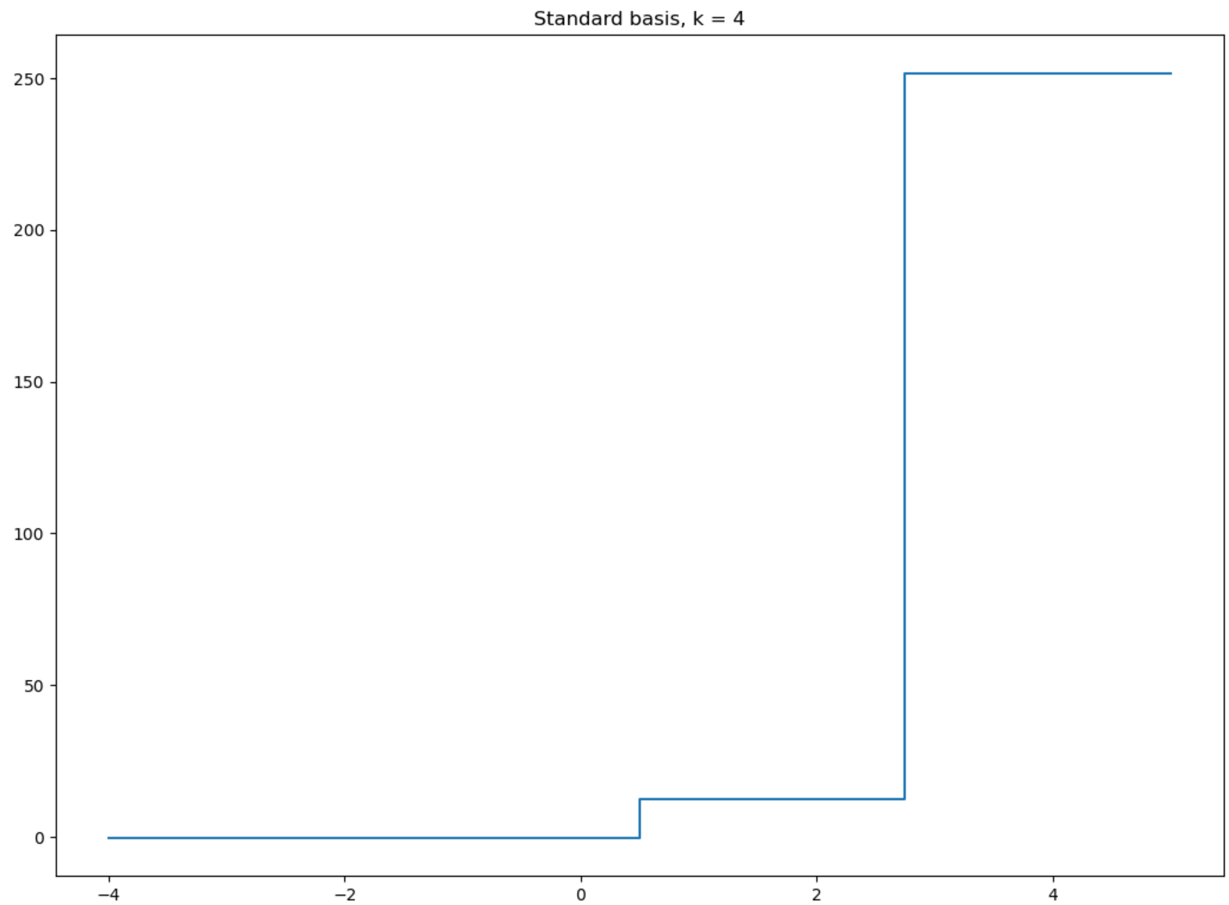


g. Below are the approximations and their corresponding MSE.

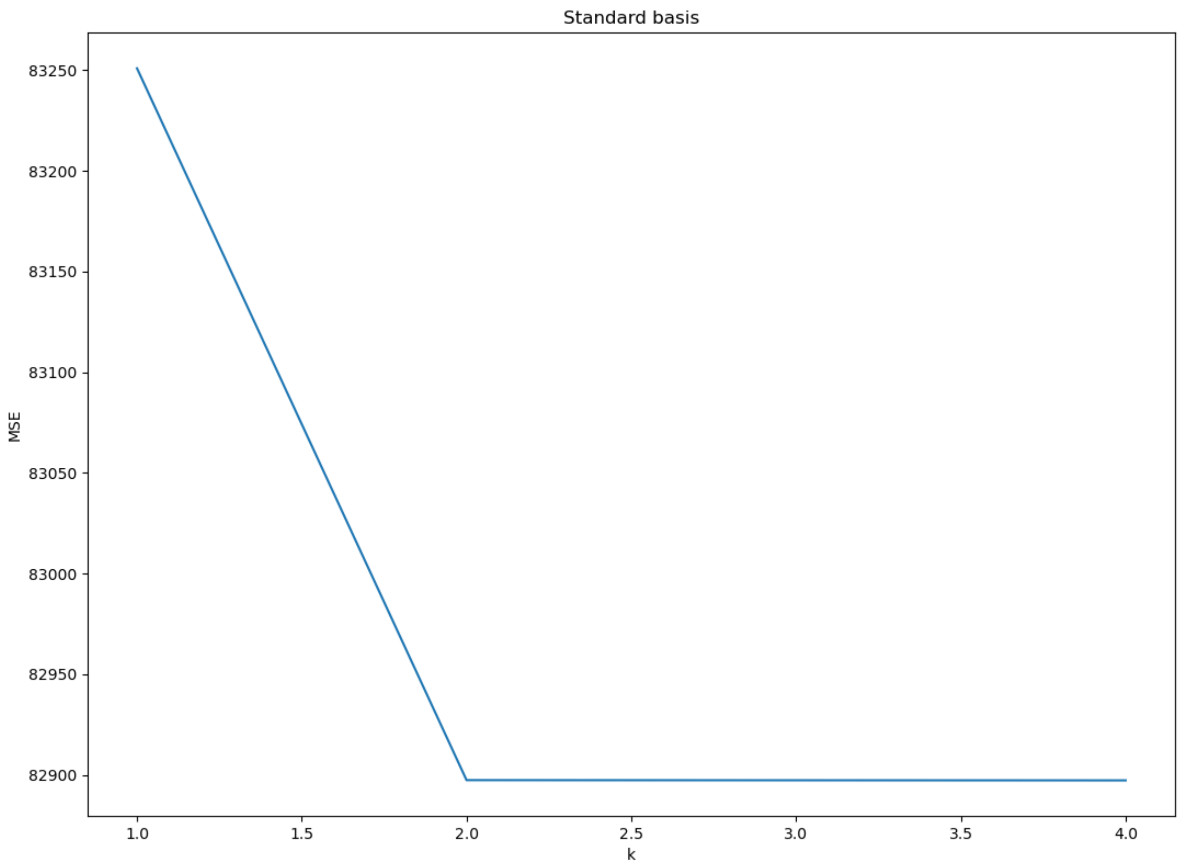


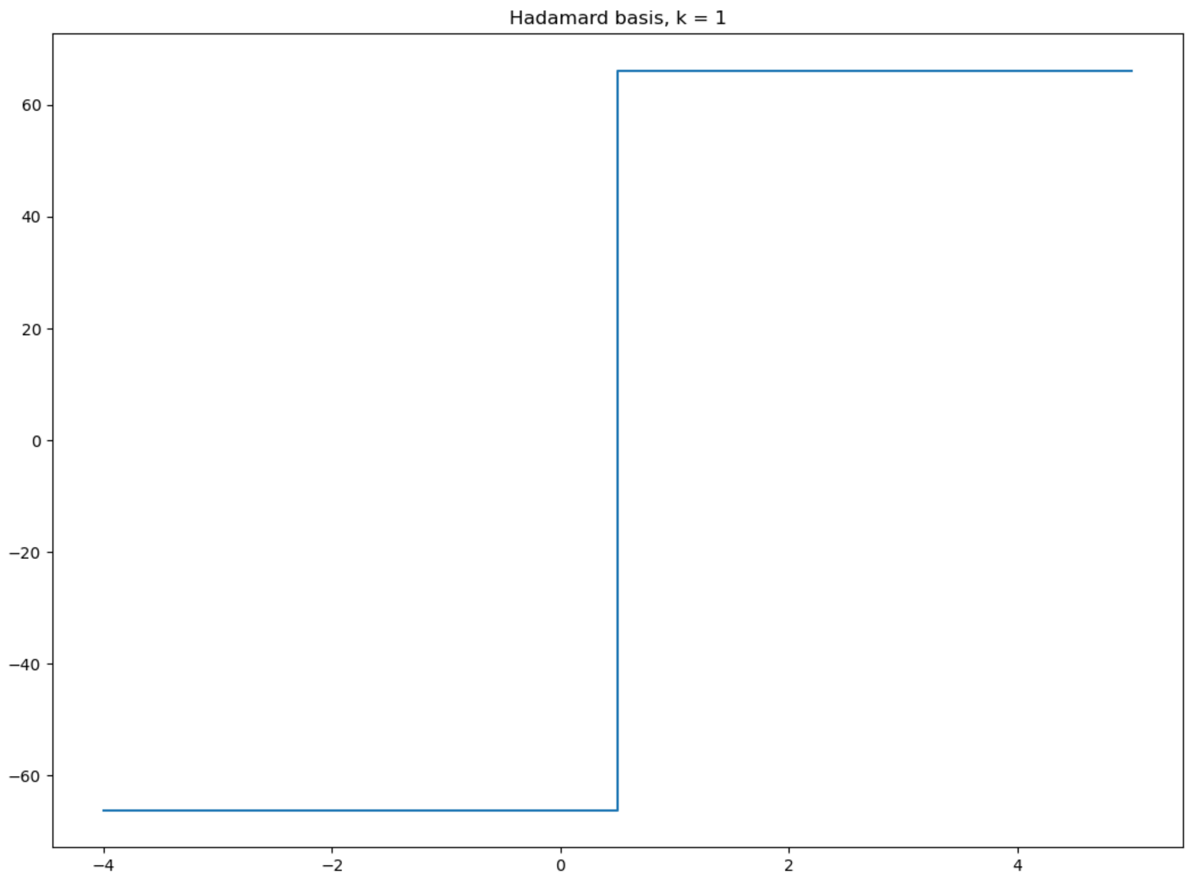

Standard basis, k = 2
250
200
150
100
50
0
−4
−2
0
2
4

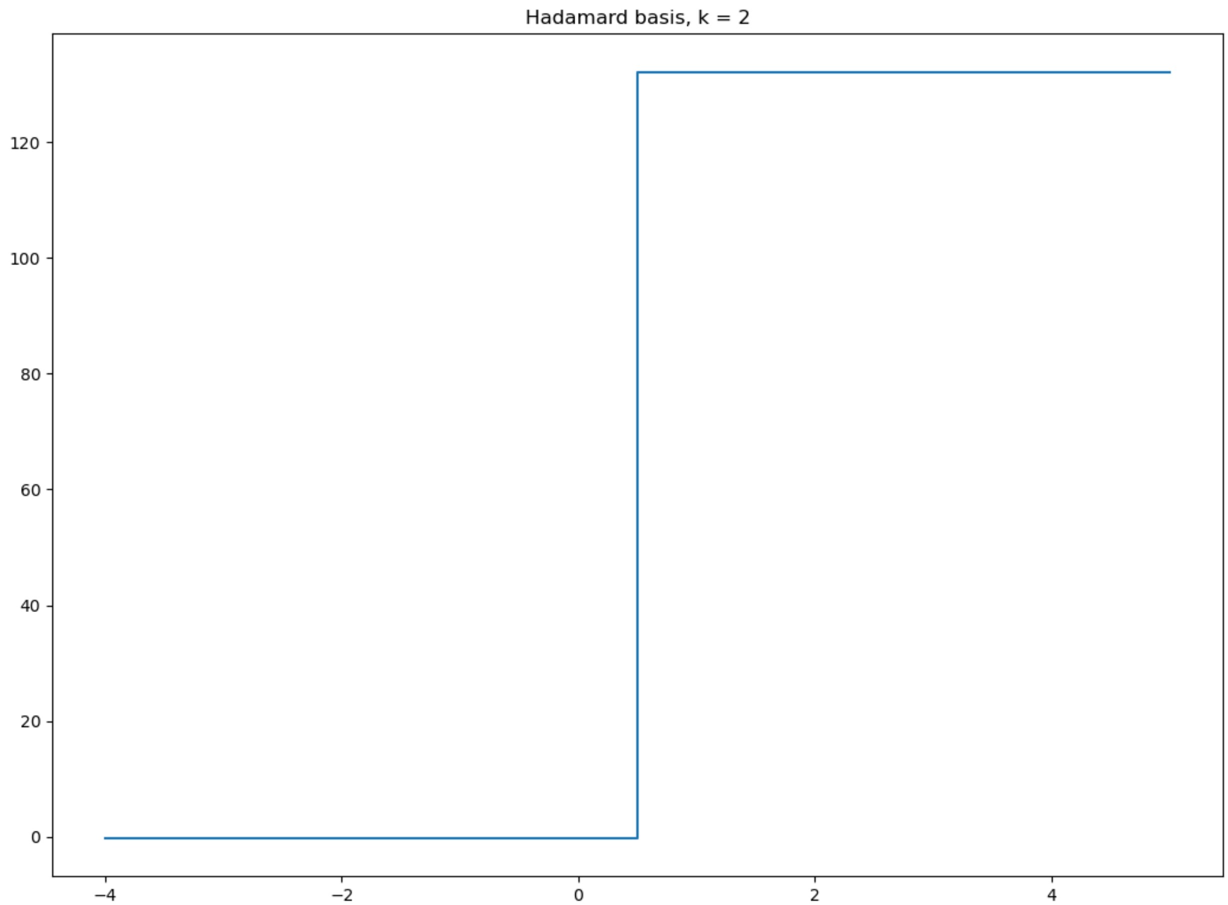

Standard basis, k = 3
250
200
150
100
50
0
−4
−2
0
2
4

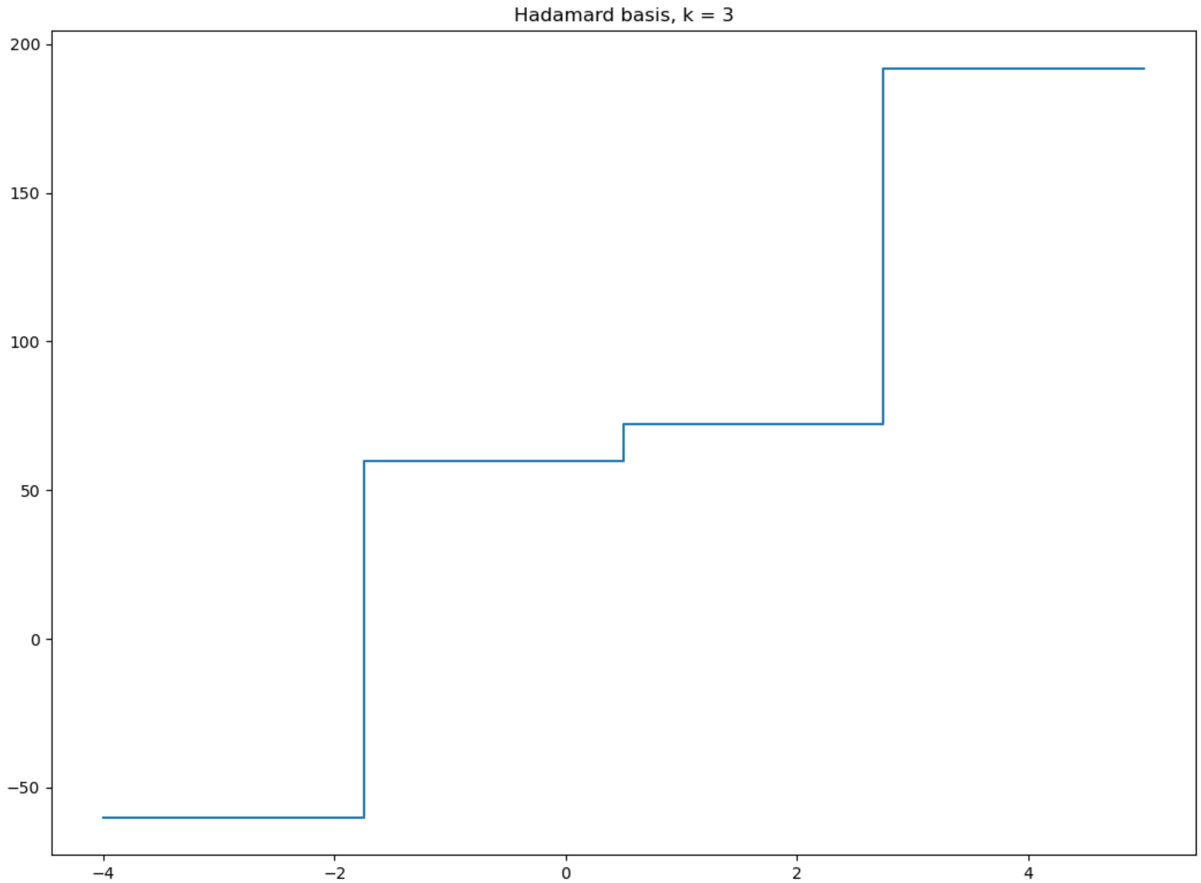


Standard basis, k = 4
250
200
150
100
50
0
−4
−2
0
2
4

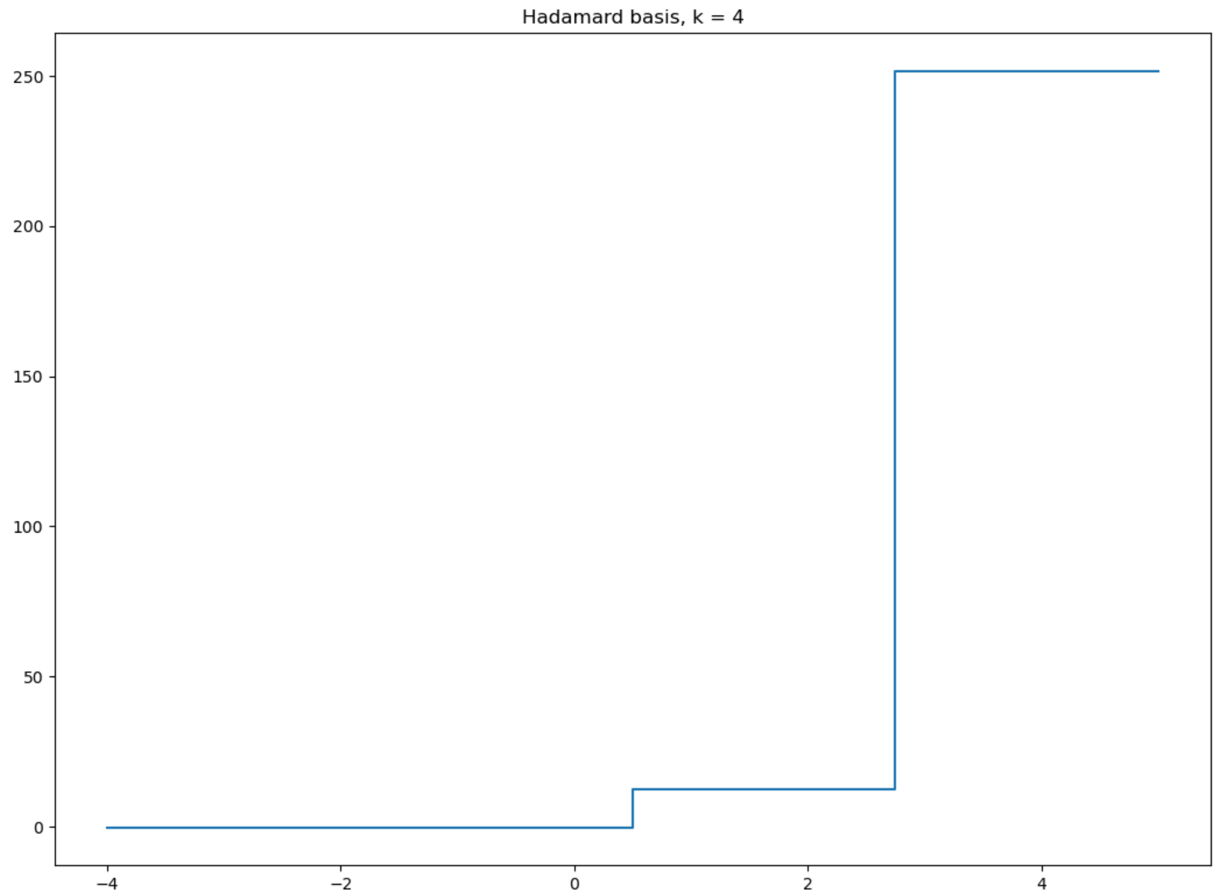

Standard basis
MSE
83250
83200
83150
83100
83050
83000
82950
82900
1.0
1.5
2.0
2.5
3.0
3.5
4.0
k

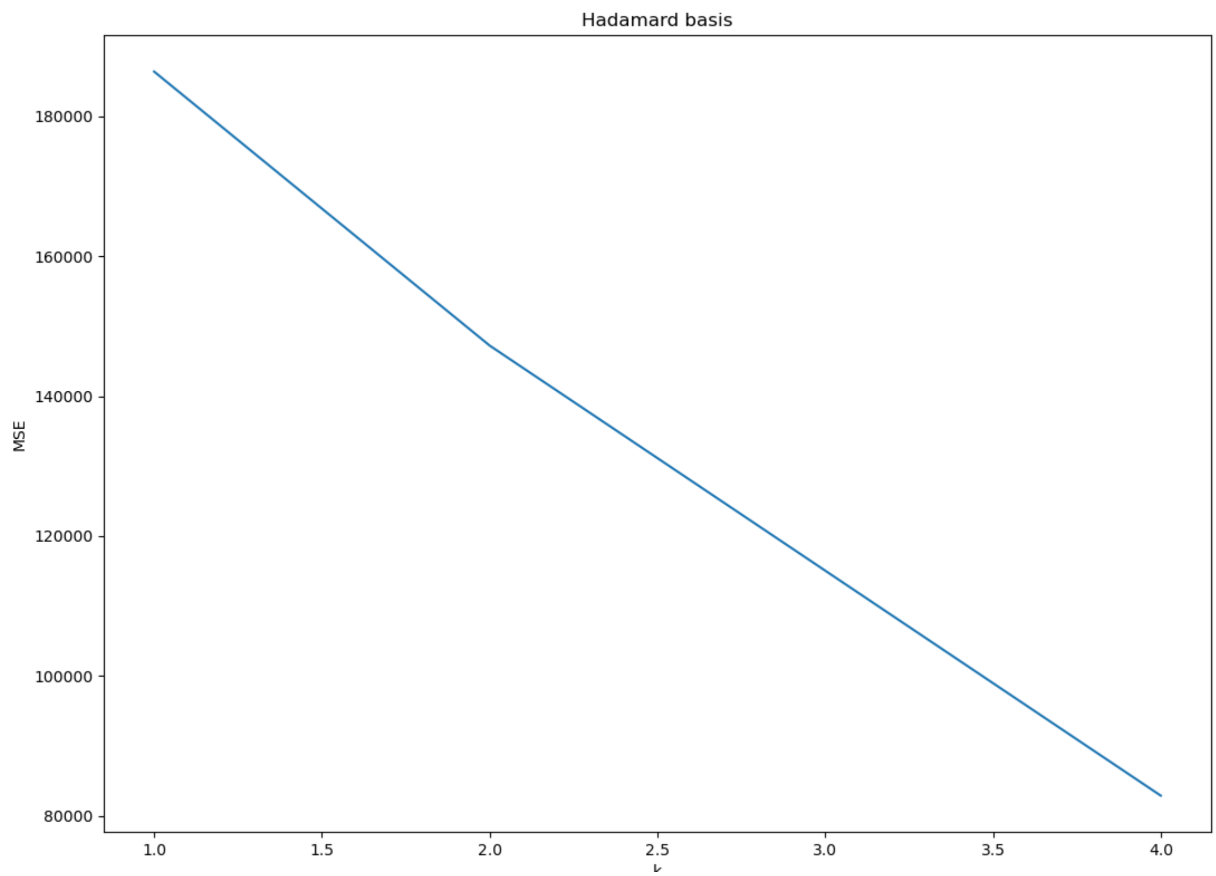

Hadamard basis, k = 1
60
40
20
0
−20
−40
−60
−4
−2
0
2
4


Hadamard basis, k = 2
120
100
80
60
40
20
0
−4
−2
0
2
4

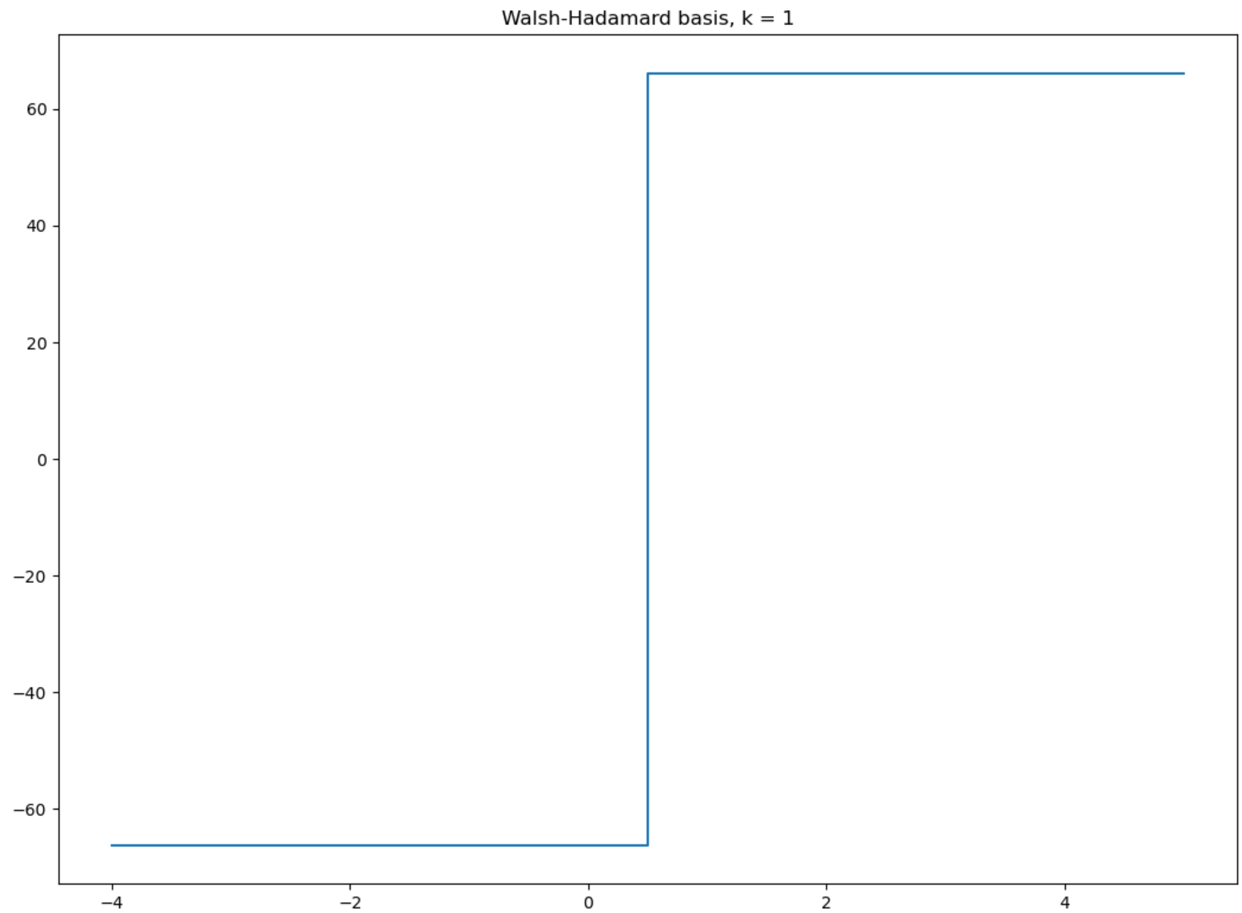


Hadamard basis, k = 3
200
150
100
50
0
−50
−4
−2
0
2
4

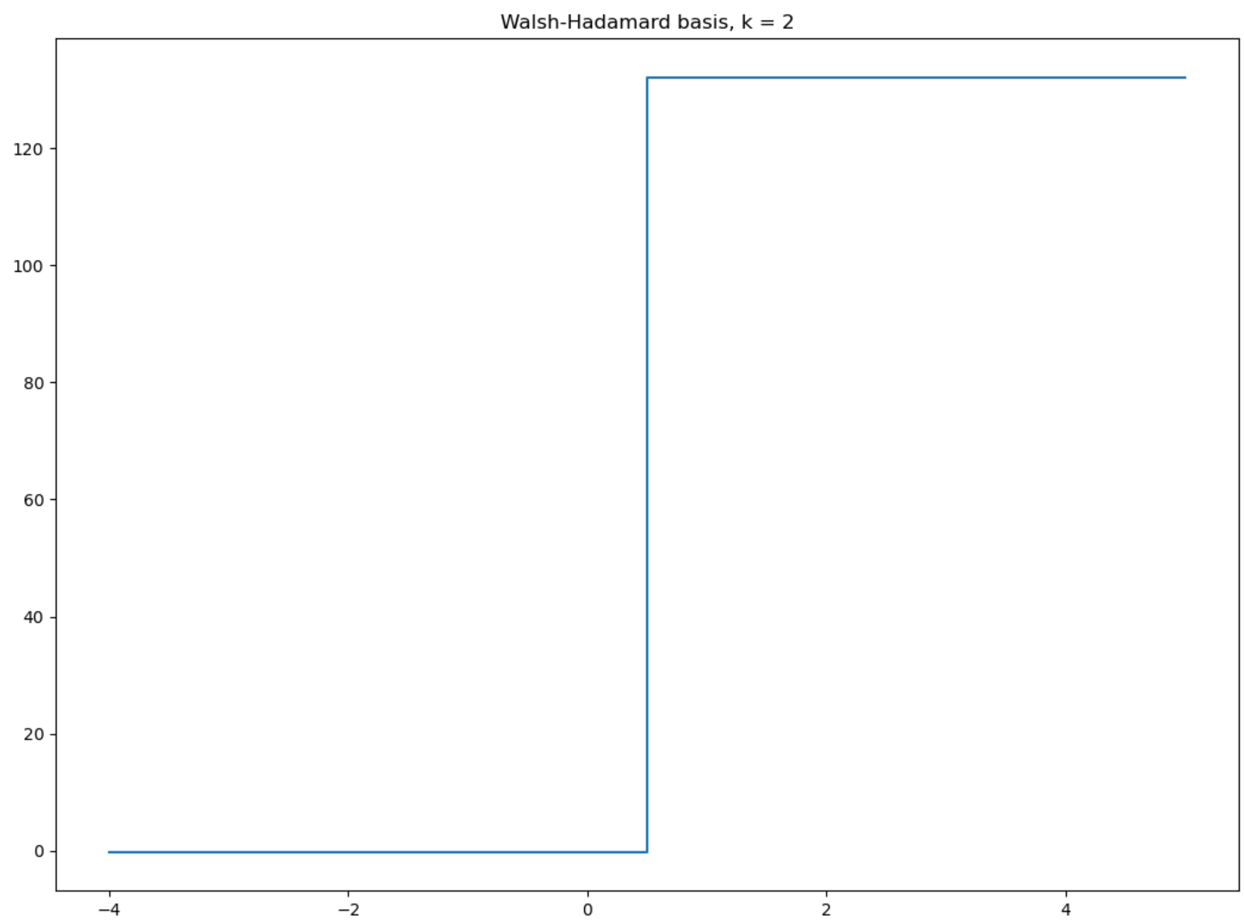

Hadamard basis, k = 4
250
200
150
100
50
0
−4
−2
0
2
4



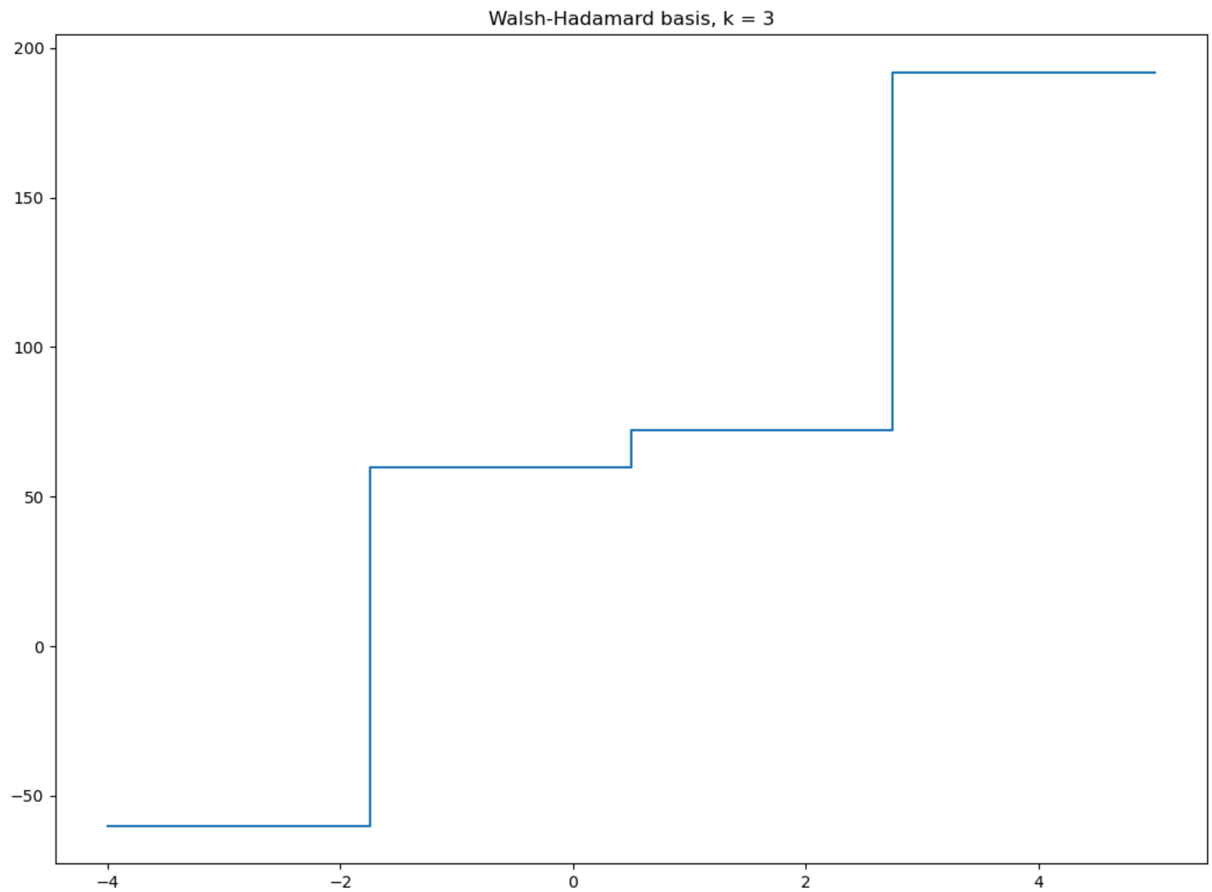
Hadamard basis
180000
160000
140000
120000
100000
80000
MSE
1.0
1.5
2.0
2.5
3.0
3.5
4.0
ν

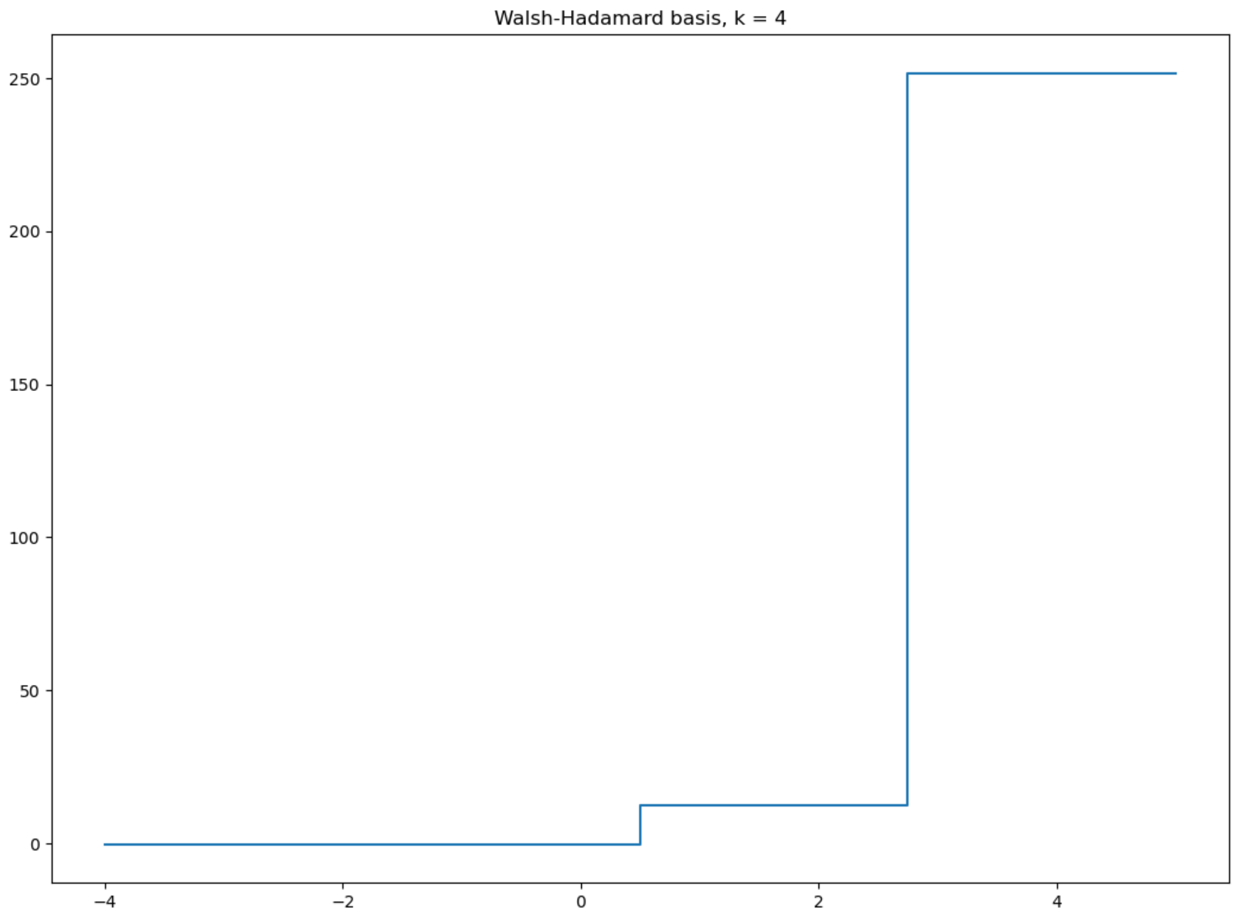


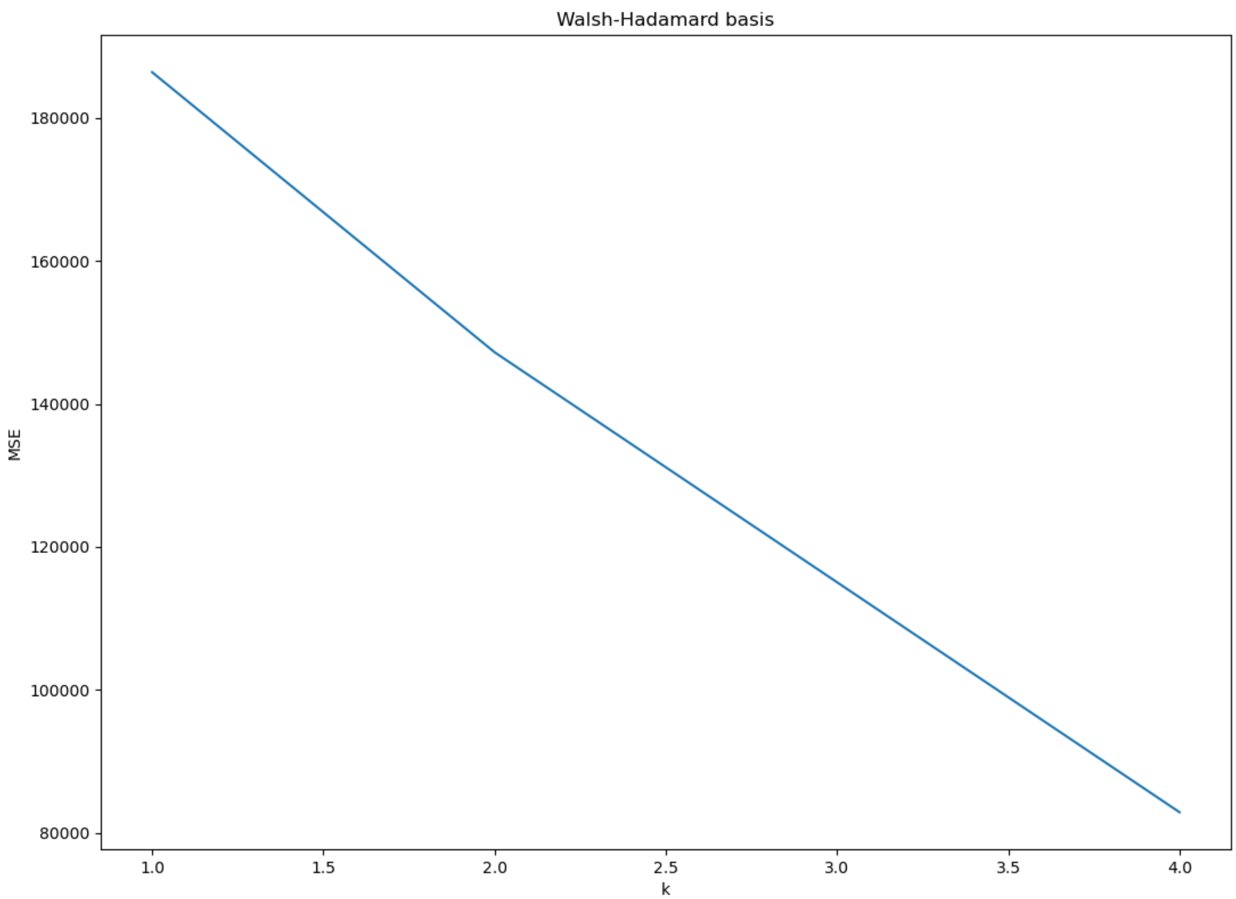
Walsh-Hadamard basis, k = 1
60
40
20
0
−20
−40
−60
−4
−2
0
2
4

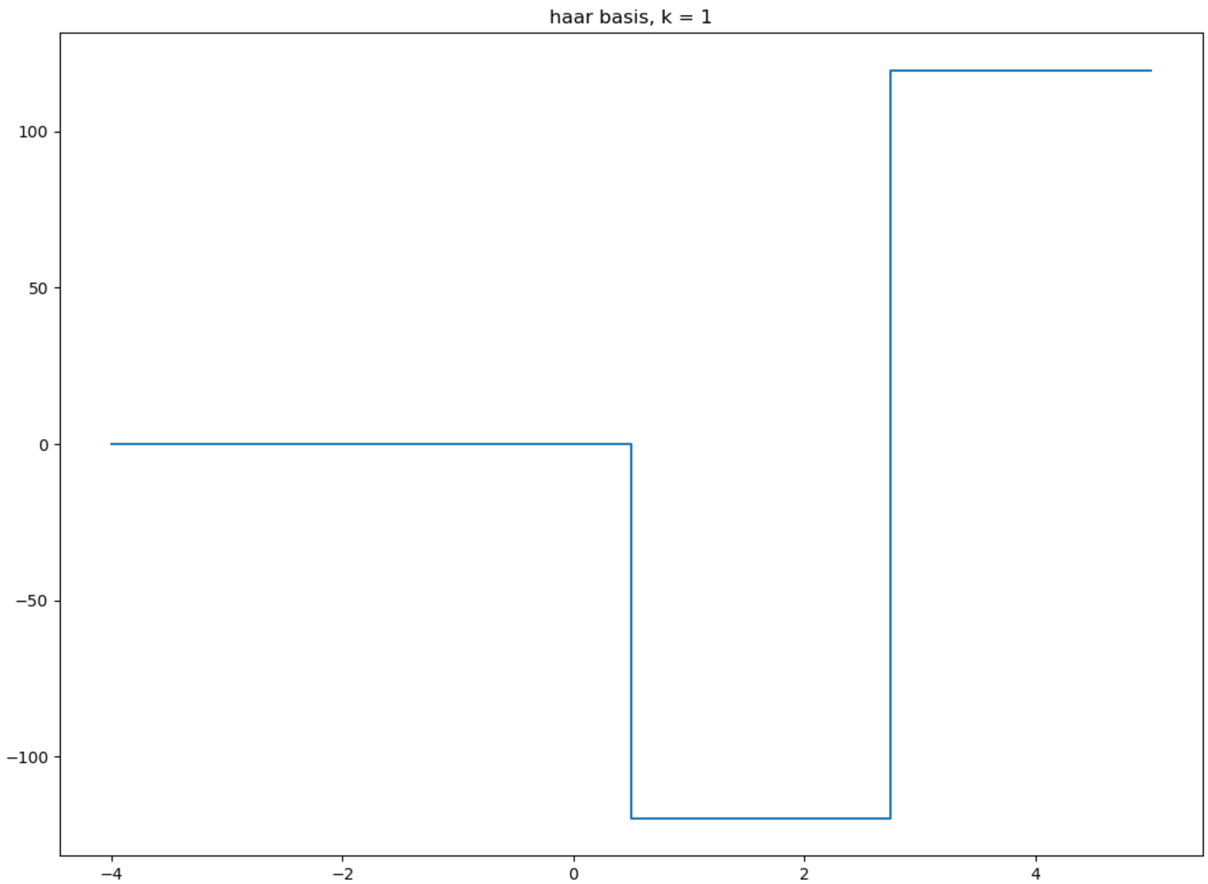


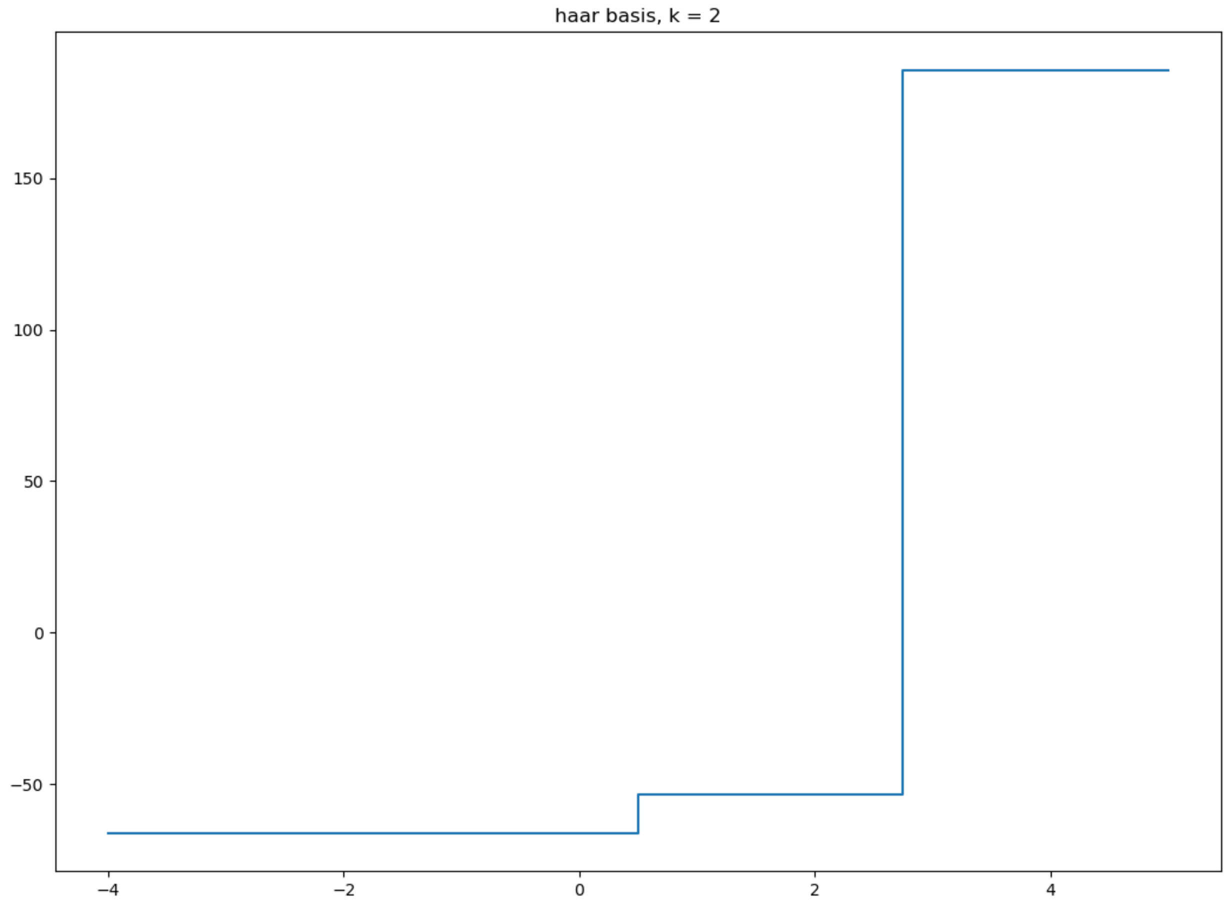
Walsh-Hadamard basis, k = 2
120
100
80
60
40
20
0
−4
−2
0
2
4

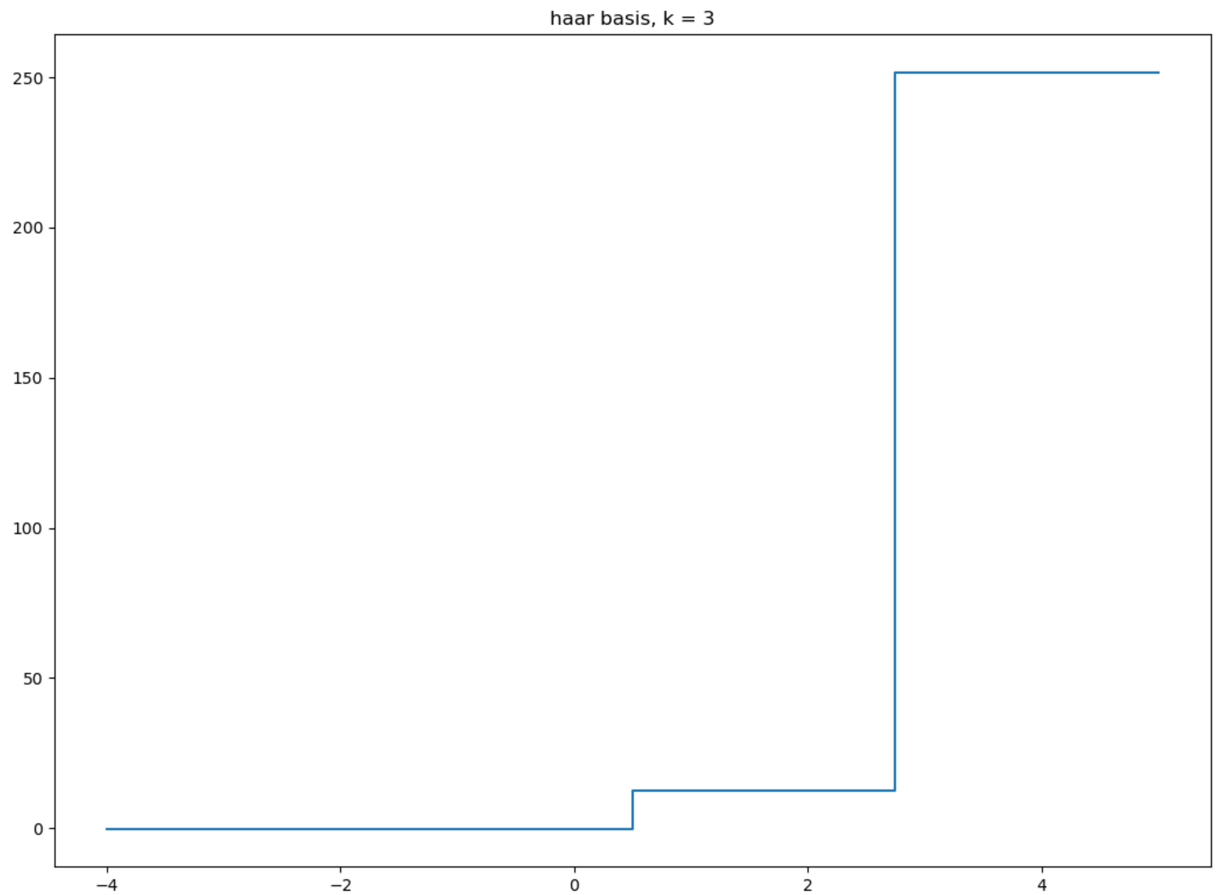

Walsh-Hadamard basis, k = 3
200
150
100
50
0
−50
−4
−2
0
2
4

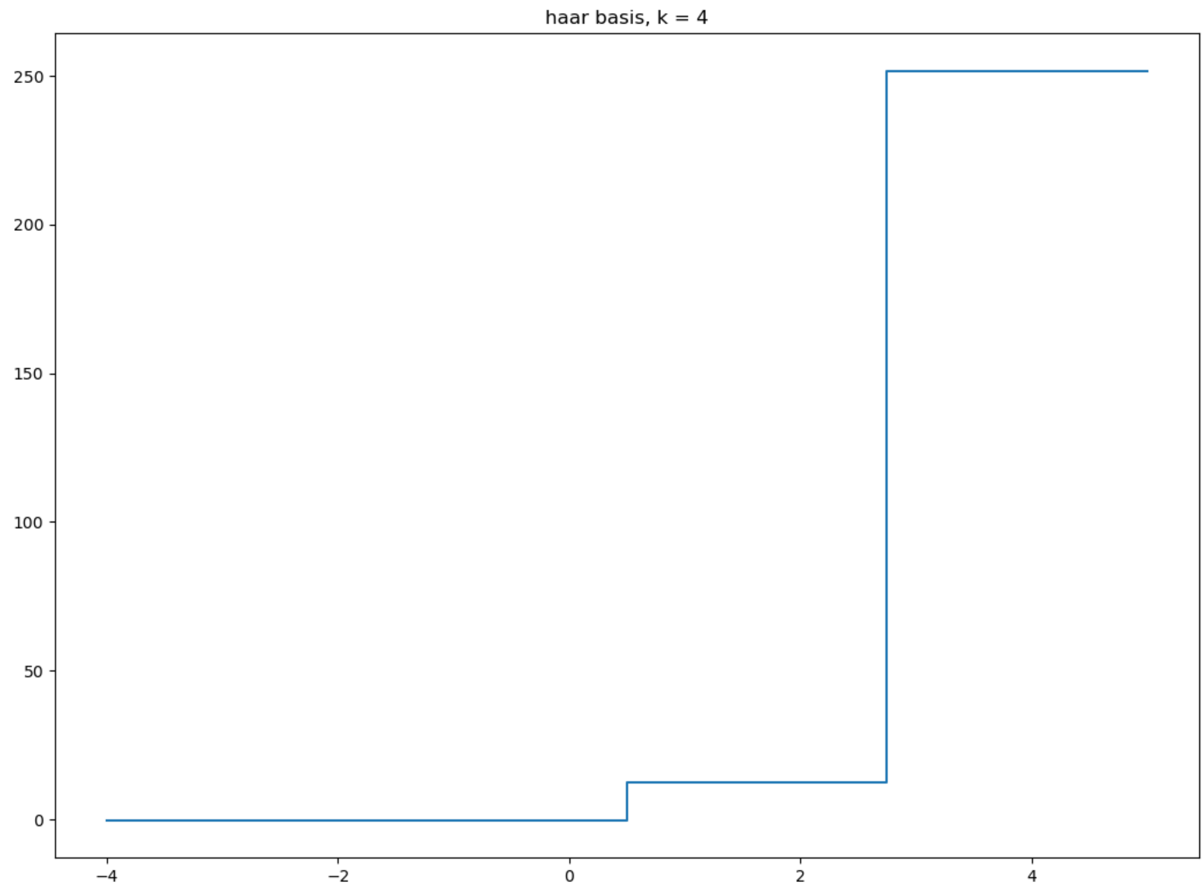

Walsh-Hadamard basis, k = 4
250
200
150
100
50
0
−4
−2
0
2
4

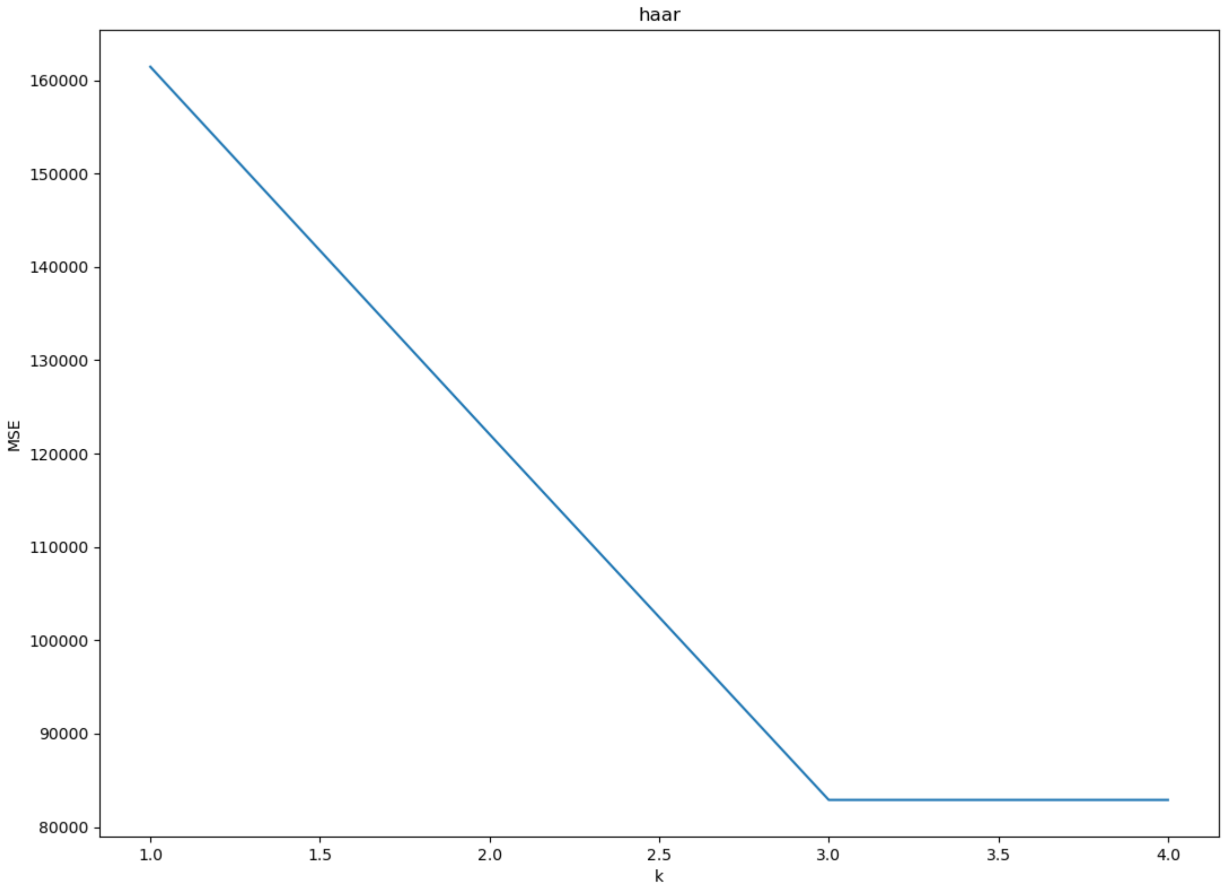

Walsh-Hadamard basis
MSE
180000
160000
140000
120000
100000
80000
1.0
1.5
2.0
2.5
3.0
3.5
4.0
k


haar basis, k = 1
100
50
0
−50
−100
−4
−2
0
2
4


haar basis, k = 2
150
100
50
0
−50
−4
−2
0
2
4


haar basis, k = 3
250
200
150
100
50
0
−4
−2
0
2
4


haar basis, k = 4
250
200
150
100
50
0
−4
−2
0
2
4


haar
MSE
k
160000
150000
140000
130000
120000
110000
100000
90000
80000
1.0
1.5
2.0
2.5
3.0
3.5
4.0