Data Represintation - HW 3

1. On Circulant Matrices

a.) first well show what is the effect of applying J to a vector $\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = V \in \mathbb{R}^n$

$$\Rightarrow J \cdot V = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ 0 & 1 & & \vdots \\ & & \ddots & \vdots \\ 0 & \cdots & 0\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} V_n \\ V_1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix}$$ $\Rightarrow$ applying J to any column vector

simply cicly shifts of of its content 1 spot "downwards"

So $J^K = J^{K-1} \cdot J$ well be the same as cyclicly shifting all of J's coulumns K-1 slots down

In preticular, $J^n = J^{n-1} \cdot J$ well be ciclic n-1 shifts down which is nothing but one shift upwards, so we will get the Identity matrix (since J is I where each column is shifted down once)

$\Rightarrow J^n = I$

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1] cont.

b] let $\lambda \in \mathbb{R}$ be an Eigenvalue of J w.r.t. and $\vec{0} \neq V \in \mathbb{R}^n = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$ be a corresponding eigenvector

By definition

$$\Rightarrow \quad J \cdot V \overset{(a)}{=} \begin{bmatrix} V_n \\ V_1 \\ \vdots \\ V_{n-1} \end{bmatrix} = \lambda \cdot \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \Rightarrow \text{we get} \begin{cases} V_1 = \lambda V_2 \\ V_2 = \lambda V_3 \\ \vdots \\ V_n = \lambda V_1 \end{cases}$$

if we chose for $V_1 = 0$ we get $V = 0$ $\Rightarrow$ $V_i \neq 0$ $\forall_{1 \le i \le n}$

Therefore we can chose $V \in \mathbb{R}^n$ s.t $V_1 = 1$ (as $V_1 \neq 0$ and eigenvectors are up to scalar multiplication

$\Rightarrow$ we get $V_1 = \lambda$, $V_2 = \lambda^2$ ... $V_{n-1} = \lambda^{n-1}$, $\boxed{V_n = \lambda^n = 1}$

$\Rightarrow$ all $\lambda \in \mathbb{R}$ ~~that~~ s.t $\lambda^n = 1$ λ can be an Eigenvalue of J and there are exactly n such complex numbers: $e^{2\pi i k \cdot \frac{1}{n}}$ for $k = 0, 1, .., n-1$

~~and since it is easy to see that~~

$\Rightarrow$ J has exactly n eigenvalues $\Rightarrow$ $\lambda_k = e^{\frac{2\pi i k}{n}}$, $k = 0, 1, ..., n-1$.

(by definition, all λ that fullfils the requi equation is an eigenvalue)

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I] cont.

C.) as seen in (b.). we got that $\lambda_k = e^{+\frac{2\pi i k}{n}}$, $k=0,\dots,n-1$ are J's eigenvalues, but we've also got from the calculations, ~~the~~ the corresponding eigenvectors!

$$\underline{V_k} = \begin{bmatrix} \lambda_k^0 \\ \lambda_k^1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{n-1} \end{bmatrix} \Rightarrow \text{we get that } J \text{ is diagonalizable with } P = \begin{bmatrix} | & | & & | \\ \underline{V_1} & \underline{V_2} & \cdots & \underline{V_n} \\ | & | & & | \end{bmatrix}$$

Moreover we can see that P is made of the Fourier vector which as we know are orthogonal so after normalization we get that $U = \frac{1}{\sqrt{n}} P$ is a unitary Basis which diagonalize J! $U^* J U = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

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1] cont.

d.) we can use what we learned about J^K in (a.) to find such a polinomial expression:

$$\Rightarrow P(A) = h_1 \cdot A + h_2 \cdot A^2 + \ldots + h_{n-1} \cdot A^{n-1} + h_0 \cdot \underbrace{A^n}_{\overset{..}{I}}$$

$$\Rightarrow \text{we get } P(J) = h_1 \cdot J + h_2 \cdot J^2 + \ldots + h_{n-1} \cdot J^{n-1} + h_0 \cdot \underbrace{J^n}_{\overset{..}{I}} = H.$$

c.) from Kiley-Hamilton we get that $\forall_j\, j \le n$, $P(\lambda_j)$, s.t λ_j is an eigenvalue of J is an eigenvalue of $P(J) = H$, <u>with the same eigenvectors!</u>

$$\Rightarrow \lambda_{H,j} = P(\lambda_j) = P\left(e^{\frac{2\pi i j}{n}}\right) = h_0 + h_1 \cdot e^{-\frac{2\pi i j}{n}} + h_2 \cdot e^{-\frac{2\pi i j}{n} \cdot 2} + \ldots + h_{n-1} \cdot e^{-\frac{2\pi i j}{n} \cdot (n-1)}$$

with $\underline{v}_{k,H} = \underline{v}_k = \begin{bmatrix} \lambda_k^0 \\ \lambda_k^1 \\ \vdots \\ \lambda_k^{n-1} \end{bmatrix}$

$\Rightarrow$ H is diagonalizable by the same Unitary matrix $U = \frac{1}{\sqrt{n}} \begin{bmatrix} | & | & & | \\ \underline{v}_0 & \underline{v}_1 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}$

with eigenvalues as computed above

$$P(\lambda_j) = \sum_{k=0}^{n-1} h_k \cdot e^{+\frac{2\pi i k \cdot j}{n}}$$

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f.

$$H = F \Lambda F^*$$

$$H = \bar{H} = \bar{F}\,\overline{\Lambda}\,\overline{F^*} = \overline{F^{T}}\,\bar{\Lambda}\,\overline{\overline{F^{T}}} = F^*\bar{\Lambda}\,\bar{\bar{F}} = F^*\bar{\Lambda}F$$

H is real

F is symetric

F is symetric

1] cont.

g.) as weve shown, we know $WHW^* = \Delta$

, since W, W^* are symetric we get (and Δ is diagonal so also symetric)

$(WHW^*)^T = \Delta^T \Rightarrow W^* H^T W = \Delta$

now we'll multiply both sides by W^* (from the right) and since $WW^* = I$ we get:

$$W^* H^T = \Delta W^* = \overbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda & & & \\ \vdots & \lambda_1^2 & \lambda_2^2 & & \vdots \\ \vdots & \vdots & & \ddots & \\ 1 & \lambda_1^{n-1} & \cdots & & \lambda_n^{n-1} \end{bmatrix}}^{W^*} \begin{bmatrix} h_0 & \cdots & h_{n-1} \\ h_{n-1} & & \vdots \\ \vdots & & \vdots \\ h_1 & \cdots & h_0 \end{bmatrix} = \overbrace{\begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}}^{\Delta} \cdot \overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \cdots 1 \\ 1 & \cdot & & \\ 1 & & \ddots & \\ 1 & & & \\ 1 & & & \end{bmatrix}}^{W}$$

from rules of matrix multiplication we get that the multiplication of W^* by the first column of H^T is equal to the first colum in the resulting matrix, which is nothing but the Δ matrix multiplyed by the first colum of W which is only ones

$$\Rightarrow \quad W^* \cdot \begin{bmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{bmatrix} = \begin{bmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \\ & & & \lambda_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} \quad \blacksquare$$

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1] cont.

h.] let $H_1, H_2 \in \mathbb{R}^{n\times n}$, s.t H_1, H_2 are circulant.

I'll show $H_1 \cdot H_2 = H_2 \cdot H_1$.

as weve shown, all circulant matrices are unifidly diagonlizable by W = DFT

$$\Rightarrow H_1 \cdot H_2 = W\Delta_1 \underbrace{W^* \cdot W}_{I} \Delta_2 W^* = W \Delta_1 \Delta_2 W^*$$

Δ_1, Δ_2 are diagonal so they commute

$$= W \Delta_2 \Delta_1 W^* = W\Delta_2 \underbrace{W^* W}_{I} \Delta_1 W^* = H_2 \cdot H_1$$

$\Rightarrow$ All circulant Matrices commute.

---

now well show that $H_1 \cdot H_2$ is circulant (and in other words, that circulant Matrices are close to multiplication)

as seen in (d.) we can build in the exact same way a polinomial expression for each of the Matrices P_1, P_2 s.t $P_1(J) = H_1$ and $P_2(J) = H_2$

$$\Rightarrow H_1 \cdot H_2 = P_1(J) \cdot P_2(J) = \sum_{k=0}^{n-1} h_{1,k} \cdot J^k \cdot \sum_{j=0}^{n-1} h_{2,j} \cdot J^j$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} h_{1,k} \cdot h_{2,j} \cdot J^{k+j} = (*)$$

$\Rightarrow$ as weve seen before for any $\alpha \in \mathbb{N}$, J^α is a circulant matrix and since circulant Matrices are closed to addition:

$$\left(\begin{bmatrix} A_0 & A_{n-1} & & \\ A_1 & \ddots & \ddots & \\ & & A_1 & A_0 \end{bmatrix} + \begin{bmatrix} B_0 & & & B_1 \\ B_1 & \ddots & & \\ & & B_1 & B_0 \end{bmatrix} = \begin{bmatrix} A_0+B_0 & & A_1+B_1 \\ A_1+B_1 & A_1+B_1 & \\ & A_1+B_1 & A_0+B_0 \end{bmatrix} \text{ is a circulant Matrix} \right)$$

$\Rightarrow$ we get $(*)$ is circulant $\Rightarrow H_1 \cdot H_2$ is circulant.

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i. DFT^k

$k=2$:

$$DFT^2 = \frac{1}{\sqrt{m}}\begin{pmatrix} w^{00} & w^{10} & \cdots & w^{(m-1)0} \\ w^{01} & w^{11} & & w^{(m-1)1} \\ \vdots & & & \vdots \\ w^{0(m-1)} & w^{1(m-1)} & & w^{(m-1)^2} \end{pmatrix}^* \frac{1}{\sqrt{m}}\begin{pmatrix} w^{00} & w^{10} & \cdots & w^{(m-1)0} \\ \vdots & & & \vdots \\ w^{0(m-1)} & & \cdots & w^{(m-1)^2} \end{pmatrix} =$$

$$\Rightarrow DFT^2_{l,j} = \frac{1}{m}\left(w^{0l^*} w^{j\cdot 0^*} + w^{1l^*} w^{j\cdot 1^*} + \dots + w^{ml^*} w^{j\cdot(m)^*}\right) =$$

$$= \frac{1}{m}\sum_{k=0}^{m-1} (w^{kl})^* (w^{jk})^* = \frac{1}{m}\sum_{k=0}^{m-1} e^{-\frac{i2\pi kl}{m}} e^{-\frac{i2\pi kj}{m}} =$$

$$= \frac{1}{m}\sum_{k=0}^{m-1} e^{-\frac{i2\pi k}{m}(l+j)} = \begin{cases} 1, & l+j = 0 \mod m \\ 0, & \text{else} \end{cases}$$

Therefore, the result is:

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & \cdots & m-2 & m-1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ m-2 \\ m-1 \end{matrix} & \begin{pmatrix} 1 & 0 & & & 0 \\ 0 & 0 & & 0 & 1 \\ & & 0 & 1 & 0 \\ & 0 & 1 & 0 & \\ 0 & 1 & & & 0 \end{pmatrix} \end{array} = Ac$$

$k=4$:

$$DFT^4 = DFT^2\, DFT^2 = Ac^2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & 1 \\ & & \iddots & \\ 0 & 1 & & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & 1 \\ \vdots & & \iddots & \\ 0 & 1 & \cdots & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & & 1 \end{pmatrix} = I$$

$k=3$:

$$DFT^3 = DFT^4\, DFT^* = DFT^*$$

In conclusion:

$$DFT^k = \begin{cases} DFT & k = 1 \mod 4 \\ Ac & k = 2 \mod 4 \\ DFT^* & k = 3 \mod 4 \\ I & k = 0 \mod 4 \end{cases}$$

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1] cont.

j) first, we can write the convolution of $x \circledast y$ in matrix form by building a circulant matrix X s.t its first row (which fully defines it) is x and multiplying y by its transpose

$\Rightarrow \quad x \circledast y = X^T y$

$\Rightarrow \quad z = x \circledast y = X^T y \quad \Rightarrow$ ~~DFT(z)=~~

$$\Rightarrow \overset{W}{\text{DFT}} \cdot z = \boxed{Wz =} W X^T y = W X^T \underbrace{W^* W}_{I} y \overset{①}{=} \underbrace{W^* X W} \cdot W y \overset{②}{=} \boxed{W x \odot W y \cdot \sqrt{n}}$$

~~① $W X^T W^*$ ...~~

① we'll show $W X^T W^* = W^* X W$:

X^T is circulant $\Rightarrow$ W diagonlize her

$\Rightarrow \quad W^* X W$ is diagonal $\Rightarrow$ ~~$(W^* X W)^T = W^* X W$~~ $\Rightarrow$ symetric

$\Rightarrow \quad W^* X W = (W^* X W)^T = W^T X^T W^{*T} = W X^T W^* \quad$ (W, W^* are symetric).

② we'll show ~~...~~ $\sqrt{n}\, \underline{W^* X W} \cdot W y = \sqrt{n}\, W x \odot W y$

* as we've seen $W^* X W$ diagonlizes X, therefore it is a diagonal matrix where it's values on the diag is it's eigenvalues, and since X is built using the x we know from (g.) that $\sqrt{n}\, W x = \begin{bmatrix} \lambda_{0,x} \\ \vdots \\ \lambda_{n-1,x} \end{bmatrix}$ of X

$\Rightarrow W x = \frac{1}{\sqrt{n}} \begin{bmatrix} \lambda_{0,x} \\ \vdots \\ \lambda_{n-1,x} \end{bmatrix} \Rightarrow \lambda_{i,x} = \sqrt{n}\, \lambda_{i,x}$

$$\Rightarrow \sqrt{n}\, W^* X W \cdot W y = \sqrt{n} \begin{bmatrix} \lambda_{0,x} & & \\ & \lambda_{1,x} & \\ & & \ddots \\ & & & \lambda_{n-1,x} \end{bmatrix} \cdot \begin{bmatrix} \lambda_{0,y} \\ \lambda_{1,y} \\ \vdots \\ \lambda_{n-1,y} \end{bmatrix} = \begin{bmatrix} \lambda_{0,x} \\ \vdots \\ \lambda_{n-1,x} \end{bmatrix} \odot \begin{bmatrix} \lambda_{0,y} \\ \lambda_{1,y} \\ \vdots \\ \lambda_{n-1,y} \end{bmatrix}$$

$= W x \odot W y \quad \blacksquare$

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2 Fourier Transform

a.) we are given $f(t) * g(t) = h(t)$

we will show $f(t-1) * g(t+1) = h(t)$:

First we'll define two new functions: $\tilde{f}(t) = f(t-1), \tilde{g}(t) = g(t+1)$

$$\Rightarrow \boxed{f(t-1) * g(t+1)} = \tilde{f}(t) * \tilde{g}(t) = \int_{-\infty}^{\infty} \tilde{f}(\xi) \cdot \tilde{g}(t-\xi) \cdot d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi-1) \cdot g(t-(\xi-1)) \cdot d\xi = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) \cdot d\tau$$

$\left\{ \begin{array}{l} \text{var change:} \\ \tau = \xi - 1, \ \frac{d\tau}{d\xi} = 1 \Rightarrow d\tau = d\xi \end{array} \right\}$

$$\overset{\text{conv definition}}{=} f(t) * g(t) = \boxed{h(t)}$$

Another maybe easier way of showing this is by using the fact that τ_ξ is LSI as shown in class

and by using: $\tau_{-1}(\tau_{+1}(\varphi(t))) = \tau_{-1}(\varphi(t+1)) = \varphi(t)$

$\Rightarrow$ we'll define $h_{\tau_{-1}}(t), h_{\tau_{+1}}(t)$ to be the Impulse responses of τ_{-1}, τ_{+1}

$$\Rightarrow \boxed{f(t) * g(t)} = \tau_{-1}(\tau_{+1}(f(t) * g(t))) = h_{\tau_{-1}}(t) * (h_{\tau_{+1}}(t) * (f(t) * g(t)))$$

$$\overset{\text{(LSI commutativity as seen in Lecture 7)}}{=} h_{\tau_{-1}}(t) * (f(t) * (h_{\tau_{+1}}(t) * g(t)))$$

$$= (h_{\tau_{-1}}(t) * (f(t))) * (h_{\tau_{+1}}(t) * g(t)) = \tau_{-1}(f(t)) * \tau_{+1}(g(t))$$

$$= \boxed{f(t-1) * g(t+1)}$$

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2] cont.

b] we need to show: $\int_{-\infty}^{\infty} f(t)\cdot g(t)\,dt = \int_{-\infty}^{\infty} F(u)\cdot G(u)\,du$

where $F(u), G(u)$ are the Fourier transforms of $f(t), g(t)$.

first well define $h(\xi) = f(\xi) * g(\xi) = \int_{-\infty}^{\infty} f(t)\cdot g(\xi - t)\,dt$

well notice $\boxed{h(0) = \int_{-\infty}^{\infty} f(t)\cdot g(-t)\,dt}$ 1

Moreover, as we learnd in the Lectures, we can fully represent any function by projecting it to the Fourier Family

$\Rightarrow h(t) = \int_{-\infty}^{\infty} H(u)\cdot e^{2\pi i u t}\,du$ when $H(u)$ is the Fourier transform of $h(t)$.

$\Rightarrow \boxed{h(0) = \int_{-\infty}^{\infty} H(u)\,du}$ 2

now, lets find out who is $H(u)$.

first as we said before we can look at $f(t), g(t)$ in their Fourier representations:

$f(t) = \int_{-\infty}^{\infty} F(u)\cdot e^{2\pi i u t}\,du, \quad g(t) = \int_{-\infty}^{\infty} G(u)\cdot e^{-2\pi i u t}\,du$

$$\cancel{H(u) = \int_{-\infty}^{\infty} h(t)\cdot e^{-2\pi i u t}\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\cdot g(t-\xi)\,d\xi\, e^{-2\pi i u t}\,d\xi\,dt}$$

$$\cancel{= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(u)\cdot e^{2\pi i u \xi}\,du\cdot\int_{-\infty}^{\infty} G(u)\cdot e^{2\pi i u(t-\xi)}\,du\right) e^{-2\pi i u t}\,d\xi\,dt}$$

More in the next page

2] b. cont.

recap. from (2) we know $h(0) = \int_{-\infty}^{\infty} H(u)\,du$

and from (1) we know $h(0) = \int_{-\infty}^{\infty} f(t)\cdot g(-t)\,dt$

$\Rightarrow$ we need only to show $\int_{-\infty}^{\infty} H(u)\,du = \int_{-\infty}^{\infty} F(u)\,G(u)\,du$ (3)

let us look at $F(u)\cdot G(u)$

$$\boxed{F(u)\cdot G(u)} = \int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi i u x}\,dx \cdot \int_{-\infty}^{\infty} g(y)\cdot e^{-2\pi i u y}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\,g(y)\cdot e^{-2\pi i u(x+y)}\,dy\,dx = \left|\ \begin{array}{l}\text{change variable from } y \text{ to } \xi - x\\ y = \xi - x,\quad \frac{dy}{d\xi} = +1 \Rightarrow dy = +d\xi\end{array}\right.$$

$$(*) = \int_{-\infty}^{\infty} e^{-2\pi i u \xi} \cdot \underbrace{\int_{-\infty}^{\infty} f(x)\cdot g(\xi - x)\,dx}\,d\xi = \int_{-\infty}^{\infty} e^{-2\pi i u \xi}\cdot h(\xi)\,d\xi$$

$$= \boxed{H(u)}$$

(*) we also changed the order of integration from $d\xi\,dx$ to $dx\,d\xi$ (Fubini)

$\Rightarrow$ since we've shown $H(u) = F(u)\cdot G(u)$ we get (3) instantly

$$\Rightarrow \underline{\int_{-\infty}^{\infty} F(u)\cdot G(u)\,du} = \int_{-\infty}^{\infty} H(u)\,du = h(0) = \underline{\int_{-\infty}^{\infty} f(t)\,g(-t)\,dt} \quad \blacksquare$$

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3. Discrete Fourier Transform

$$\phi = \left[1, \tfrac{1}{2}, 0, \dots, 0, \tfrac{1}{2}\right]^T \qquad n = 2N$$

a. $$\frac{1}{\sqrt{n}} W_n^* \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} w^{0\cdot 0^*} & w^{1\cdot 0^*} & \dots & w^{(n-1)0^*} \\ \vdots & \vdots & & \vdots \\ w^{0 l^*} & w^{1 l^*} & \dots & w^{(n-1) l^*} \\ \vdots & \vdots & & \vdots \\ w^{0\cdot n-1^*} & w^{1\cdot n-1^*} & \dots & w^{(n-1)^{2*}} \end{pmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix} =$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} w^{0\cdot 0^*} + \frac{1}{2} w^{1\cdot 0^*} + \frac{1}{2} w^{(n-1)\cdot 0^*} \\ \vdots \\ w^{0 l^*} + \frac{1}{2} w^{1 l^*} + \frac{1}{2} w^{(n-1) l^*} \\ \vdots \\ w^{0\cdot(n-1)^*} + \frac{1}{2} w^{1\cdot(n-1)^*} + \frac{1}{2} w^{(n-1)^{2*}} \end{bmatrix} \overset{*}{=} \qquad * \left(w^{(n-1)^*} = e^{i\frac{2\pi}{n}(n-1)} = e^{-\frac{i 2\pi}{n}} = w \right)$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} 2 \\ \vdots \\ 1 + \frac{1}{2} w^{l} + \frac{1}{2} w^{l^{**}} \\ \vdots \\ 1 + \frac{1}{2} w^{(n-1)^*} + \frac{1}{2} w^{(n-1)^*} \end{bmatrix} \quad * \frac{z+z^*}{2} = \mathrm{Re}(z) \quad = \frac{1}{\sqrt{n}} \begin{bmatrix} 2 \\ \vdots \\ 1 + \mathrm{Re}(w^{l}) \\ \vdots \\ 1 + \mathrm{Re}(w^{(n-1)}) \end{bmatrix} \quad e^{\left(\frac{i 2\pi}{n}\right) l} = \cos\left(\frac{2\pi l}{n}\right) + i \sin\left(\frac{2\pi l}{n}\right)$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} 2 \\ \vdots \\ 1 + \cos\left(\frac{2\pi l}{n}\right) \\ \vdots \\ 1 + \cos\left(\frac{2\pi(n-1)}{n}\right) \end{bmatrix} = \frac{1}{\sqrt{2N}} \cancel{\begin{bmatrix} 2 \\ 1 + \cos\left(\frac{\pi}{N}\right) \\ 1 + \cos\left(\frac{2\pi}{N}\right) \\ \vdots \\ 1 + \cos\left(\frac{(N-2)\pi}{N}\right) \\ 1 + \cos\left(\frac{(N-1)\pi}{N}\right) \end{bmatrix}}$$

$$= \frac{1}{\sqrt{2N}} \begin{bmatrix} 2 \\ 1 + \cos\left(\frac{\pi}{N}\right) \\ \vdots \\ 1 + \cos\left(\frac{N-1}{N}\pi\right) \\ 2 \\ 1 + \cos\left(\frac{\pi}{N}\right) \\ \vdots \\ 1 + \cos\left(\frac{N-1}{N}\pi\right) \end{bmatrix} //$$

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b. $\gamma = [\psi_0, 0, \psi_1, 0, \psi_2, 0, \dots, \psi_{N-1}, 0]^T \in \mathbb{R}^{2N}$

- let $H \in \{0,1\}^{2N \times N}$ s.t. $\gamma = H\psi$

for example, for $N=2$ $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note the H is a linear operator.

$\psi^F = W_N^* \psi$, $\psi = W_N \psi^F$

$$\gamma^F = W_{2N}^* \gamma = W_{2N}^* H \psi = W_{2N}^* H W_N \psi^F =$$

$$= \frac{1}{\sqrt{2N}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & w^{1*} & & w^{(2N-1)*} \\ \vdots & & & \\ 1 & w^{(2N-1)} & & w^{*(2N-1)^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}_{2N \times N} \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & w^{(N-1)^2} \end{pmatrix}_{N \times N} \psi^F =$$

$$= \frac{1}{\sqrt{2N}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & w^{1*} & \dots & w^{(2N-1)*} \\ \vdots & \vdots & & \\ 1 & w^{(2N-1)} & \dots & w^{(2N-1)^2*} \end{pmatrix}_{2N \times 2N} \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & & 0 \\ 1 & w & & w^{(N-1)} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 1 & w^{(N-1)} & & w^{(N-1)^2} \\ 0 & 0 & & 0 \end{pmatrix}_{2N \times N} \psi^F = \; *$$

$$= \frac{1}{\sqrt{2} \cdot N} \begin{pmatrix} N & & & \\ & N & & \\ & & \ddots & \\ & & & N \\ N & & & \\ & N & & \\ & & \ddots & \\ & & & N \end{pmatrix} \psi^F = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^F \\ \psi^F \end{pmatrix}$$

$*$ The i,j entry in the multiplication is:

$$w^{0 \cdot i *} \cdot w^{j \cdot 0 *} + w^{2i *} \cdot w^{j \cdot 1 *} + \dots + w^{(2N-2)i *} \cdot w^{j \cdot N *} =$$

$$= \sum_{k=0}^{N-1} (w^{2k \cdot i})^* \cdot (w^{j \cdot k}) = \sum_{k=0}^{N-1} e^{-i\frac{2\pi}{2N} \cdot (2ki)} \cdot e^{i\frac{2\pi}{N} jk} =$$

$$= \sum_{k=0}^{N-1} e^{-i\frac{2\pi}{N}(k(i-j))} = \begin{cases} N, & i = j \pmod N \\ 0, & i \neq j \pmod N \end{cases}$$

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c. $$h = \gamma * \phi \overset{\text{commutability}}{=} \phi * \gamma = \begin{pmatrix} 1 & \frac{1}{2} & 0 & & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & & \\ 0 & \frac{1}{2} & 1 & & 1 \\ \vdots & 0 & \frac{1}{2} & & \vdots \\ \vdots & \vdots & \vdots & & 0 \\ 0 & & & & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ 0 \\ \psi_1 \\ 0 \\ \vdots \\ \psi_{N-1} \\ 0 \end{pmatrix} =$$

$$= \left[\psi_0 ,\ \frac{\psi_0 + \psi_1}{2} ,\ \psi_1 ,\ \dots ,\ \psi_{N-1} ,\ \frac{\psi_{N-1} + \psi_0}{2} \right]^T$$

d. from Q.1.i we get that

$$(DFT)h = (DFT)\gamma \odot (DFT)\phi =$$

$$\cancel{= \begin{pmatrix} \psi^F \\ \vdots \\ \psi^F \end{pmatrix} \begin{pmatrix} 2 \\ 1+\cos\left(\frac{\pi}{N}\right) \\ 1+\cos\left(\frac{2\pi}{N}\right) \\ \vdots \\ 1+\cos\left(\frac{(N-1)\pi}{N}\right) \end{pmatrix} = \begin{pmatrix} 2\psi_1^F \\ 1+\cos\left(\frac{\pi}{N}\right)\psi_2^F \\ \vdots \end{pmatrix}}$$

$$= \begin{pmatrix} \vdots \\ \psi^F \\ \vdots \\ \vdots \\ \psi^F \\ \vdots \end{pmatrix} \begin{pmatrix} 2 \\ \vdots \\ 1+\cos\left(\frac{N-1}{N}\pi\right) \\ 2 \\ \vdots \\ 1+\cos\left(\frac{N-1}{N}\pi\right) \end{pmatrix} = \begin{pmatrix} 2\psi_1^F \\ \left(1+\cos\left(\frac{\pi}{N}\right)\right)\psi_2^F \\ \vdots \\ \left(1+\cos\left(\frac{N-1}{N}\pi\right)\right)\psi_N^F \\ 2\psi_1^F \\ 1+\cos\left(\frac{\pi}{N}\right)\psi_2^F \\ \vdots \\ 1+\cos\left(\frac{N-1}{N}\pi\right)\psi_N^F \end{pmatrix} //$$

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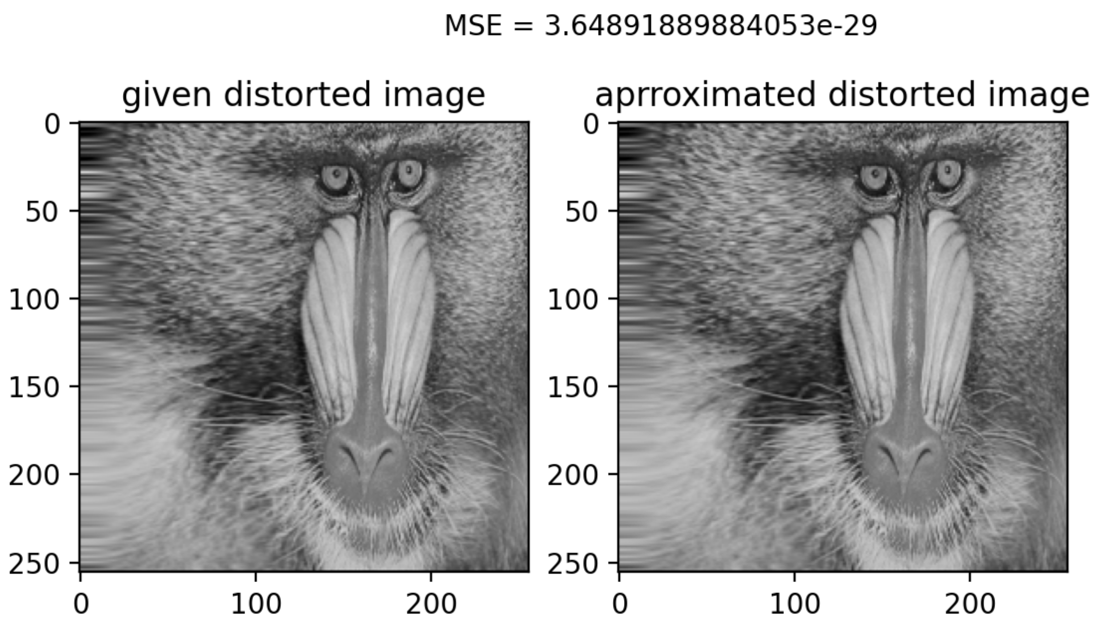
1. **Image example**
    a. Implemented in code.
    b. Implemented in code.
    c. Below is a comparison of the images and the computed MSE:



2. Audio example

Below is a comparison between the signals and the corresponding MSE.