3. Let's Randomise Again!

$$\varphi = [n, \dots, n, n+L, n, \dots n, n+L, n, \dots, n]^T$$

$k \qquad\qquad k+\frac{N}{2}$

$k \sim Uni(\{1, \dots, \frac{N}{2}\})$

$E(n) = 0$

$E(n^2) = c$

$E(L) = 0$

$E(L^2) = \frac{N}{2}(1-c), \quad 0<c<1$

a. we will compute the i,j element of R_φ

$R_\varphi(i,j) = E(\varphi_i \varphi_j^*)$

1. $i = j \mod \frac{N}{2}$:

$$E(\varphi_i\varphi_j^*) = P(k \overset{\mod \frac{N}{2}}{=} i) E(\varphi_i\varphi_j^* \mid k \overset{\mod \frac{N}{2}}{=} i) + P(k \overset{\mod \frac{N}{2}}{\neq} i) E(\varphi_i \varphi_j^* \mid k \overset{\mod \frac{N}{2}}{\neq} i) =$$

$$= \frac{1}{\frac{N}{2}} E((n+L)^2) + \frac{\frac{N}{2}-1}{\frac{N}{2}} E(n^2) =$$

$$= \frac{2}{N}[E(n^2) \overset{+2E(nL) = \cancel{E(n)E(L)}}{+} E(L^2)] + \frac{N-2}{N} E(n^2) = E(n^2) + \frac{2}{N} E(L^2) =$$

$$= c + \frac{\cancel{2}}{\cancel{N}} \frac{\cancel{N}}{\cancel{2}}(1-c) = 1$$

2. $i \neq j \mod \frac{N}{2}$

$$E(\varphi_i\varphi_j^*) = P(k = i \mod \tfrac{N}{2}) E(\varphi_i \varphi_j^* \mid k = i \mod \tfrac{N}{2}) +$$

$$+ P(k = j \mod \tfrac{N}{2}) E(\varphi_i \varphi_j^* \mid k = j \mod \tfrac{N}{2}) + P(k \neq i, j \mod \tfrac{N}{2}) E(\varphi_i\varphi_j^* \mid k \neq i,j \mod \tfrac{N}{2}) =$$

$$= \frac{1}{\frac{N}{2}} E(\underbrace{n(n+L)}_{E(n^2)+E(nL) = E(n^2)+E(n)E(L)}) + \frac{1}{\frac{N}{2}} E(n(n+L)) + \frac{\frac{N}{2}-2}{\frac{N}{2}} E(n^2) =$$

$$= \frac{4}{N} E(n^2) + \frac{N-4}{N} E(n^2) = E(n^2) = c$$

overall, the structure of R_ℓ is:

$$R_\ell = \begin{pmatrix} 1 & c & \cdots & c & 1 & c & \cdots & c \\ c & 1 & c & \cdots & & 1 & \ddots & \vdots \\ \vdots & c & \ddots & & & & \ddots & c \\ c & & & \ddots & & & & 1 \\ 1 & c & & & \ddots & & & \vdots \\ c & 1 & \ddots & & & \ddots & & \vdots \\ \vdots & & 1 & \ddots & & & \ddots & c \\ c & \cdots & c & 1 & c & \cdots & c & 1 \end{pmatrix}$$

We can see that R_ℓ is indeed circulant.

b. • The first way to compute λ is to use the fact that R_ℓ is circulant and therefore can be written as a polynom of J.

$$R_\ell = J^0 + c \cdot J^1 + \dots + c J^{\frac{n}{2}-1} + 1 \cdot J^{\frac{n}{2}} + c \cdot J^{\frac{n}{2}+1} + \dots + c \cdot J^{n-1}$$

$$\lambda_\ell = (w^0)^\ell + c \cdot w^{-\ell} + \dots c \cdot w^{-(\frac{n}{2}-1)\ell} + w^{-\frac{n}{2} \cdot \ell} + c \cdot w^{-(\frac{n}{2}+1)\ell} + \dots + c \cdot w^{-(n-1)\ell} =$$

$$= 2 + c \cdot Re(w^{-\ell}) + c \cdot Re(w^{-2\ell}) + \dots + c \cdot Re(w^{-2(\frac{n}{2}-1) \cdot \ell}) =$$

$$= 2 + 2c\left[\cos\left(\frac{2\pi\ell}{n}\right) + \dots + \cos\left(\frac{2\pi}{n}\ell \cdot \left(\frac{n}{2}-1\right)\right)\right]$$

• The second way is using the next theorem

$$\begin{pmatrix} \lambda_0 \\ \vdots \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n}\, DFT^* \begin{pmatrix} 1 \\ c \\ \vdots \\ c \\ \vdots \\ c \end{pmatrix} = \frac{\sqrt{n}}{\sqrt{n}} \begin{pmatrix} w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & & w^{n-1} \\ \vdots & & & \\ w^0 & w^\ell & \cdots & w^{(n-1)\ell} \\ \vdots & & & \\ w^0 & & & w^{(n-1)^2} \end{pmatrix} \begin{pmatrix} 1 \\ c \\ \vdots \\ c \\ \vdots \\ c \end{pmatrix} =$$

$$\lambda_\ell = w^0 + c \cdot w^{-\ell} + \dots c \cdot w^{-(\frac{n}{2}-1)\ell} + c \cdot w^{-\frac{n}{2} \cdot \ell} + c \cdot w^{(\frac{n}{2}+1)\ell} + \dots + c^{-(n-1)\ell}$$