3. Discrete Fourier Transform

$$\phi = \left[1, \tfrac{1}{2}, 0, \dots, 0, \tfrac{1}{2}\right]^T \qquad m = 2N$$

a. $$\frac{1}{\sqrt{m}} W_m^* \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{pmatrix} w^{0\cdot 0^*} & w^{1\cdot 0^*} & \dots & w^{(m-1)0^*} \\ \vdots & \vdots & & \vdots \\ w^{0 l^*} & w^{1 l^*} & \dots & w^{(m-1) l^*} \\ \vdots & \vdots & & \vdots \\ w^{0\cdot m-1^*} & w^{1\cdot m-1^*} & \dots & w^{(m-1)^{2^*}} \end{pmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix} =$$

$$= \frac{1}{\sqrt{m}} \begin{bmatrix} w^{0\cdot 0^*} + \frac{1}{2} w^{1\cdot 0^*} + \frac{1}{2} w^{(m-1)\cdot 0^*} \\ \vdots \\ w^{0 l^*} + \frac{1}{2} w^{1 l^*} + \frac{1}{2} w^{(m-1) l^*} \\ \vdots \\ w^{0(m-1)^*} + \frac{1}{2} w^{1\cdot(m-1)^*} + \frac{1}{2} w^{(m-1)^{2^*}} \end{bmatrix} = \quad * \left(w^{(m-1)^*} = e^{i\frac{2\pi}{m}(m-1)} = e^{-\frac{i2\pi}{m}} = w \right)$$

$$= \frac{1}{\sqrt{m}} \begin{bmatrix} 2 \\ \vdots \\ 1 + \frac{1}{2} w^{l^*} + \frac{1}{2} w^{l^{**}} \\ \vdots \\ 1 + \frac{1}{2} w^{(m-1)^*} + \frac{1}{2} w^{(m-1)^{**}} \end{bmatrix} \; * \; \frac{z+z^*}{2} = \mathrm{Re}(z) \; = \frac{1}{\sqrt{m}} \begin{bmatrix} 2 \\ \vdots \\ 1 + \mathrm{Re}(w^l) \\ \vdots \\ 1 + \mathrm{Re}(w^{(m-1)}) \end{bmatrix} \quad e^{\left(\frac{i2\pi}{m}\right) l} = \cos\left(\frac{2\pi l}{m}\right) + i\sin\left(\frac{2\pi l}{m}\right)$$

$$= \frac{1}{\sqrt{m}} \begin{bmatrix} 2 \\ \vdots \\ 1 + \cos\left(\frac{2\pi l}{m}\right) \\ \vdots \\ 1 + \cos\left(\frac{2\pi(m-1)}{m}\right) \end{bmatrix} = \frac{1}{\sqrt{2N}} \cancel{\begin{bmatrix} 2 \\ 1 + \cos\left(\frac{\pi}{N}\right) \\ 1 + \cos\left(\frac{2\pi}{N}\right) \\ \vdots \\ 1 + \cos\left(\frac{(N-2)\pi}{N}\right) \\ 1 + \cos\left(\frac{(N-1)\pi}{N}\right) \end{bmatrix}}$$

$$= \frac{1}{\sqrt{2N}} \begin{bmatrix} 2 \\ 1 + \cos\left(\frac{\pi}{N}\right) \\ \vdots \\ 1 + \cos\left(\frac{N-1}{N}\pi\right) \\ 2 \\ 1 + \cos\left(\frac{\pi}{N}\right) \\ \vdots \\ 1 + \cos\left(\frac{N-1}{N}\pi\right) \end{bmatrix} //$$

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b. $\gamma = [\psi_0, 0, \psi_1, 0, \psi_2, 0, \dots, \psi_{N-1}, 0]^T \in \mathbb{R}^{2N}$

- let $H \in \{0,1\}^{2N \times N}$ s.t $\gamma = H\psi$

for example, for $N=2$ $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note the H is a linear operator.

$\psi^F = W_N^* \psi$, $\psi = W_N \psi^F$

$$\gamma^F = W_{2N}^* \gamma = W_{2N}^* H \psi = W_{2N}^* H W_N \psi^F =$$

$$= \frac{1}{\sqrt{2N}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & w^{1*} & & w^{(2N-1)*} \\ \vdots & & & \\ 1 & w^{(2N-1)} & & w^{*(N-1)^2} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}}_{2N \times N} \frac{1}{\sqrt{N}} \underbrace{\begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & w^{(N-1)^*} \end{pmatrix}}_{N \times N} \psi^F =$$

$$= \frac{1}{\sqrt{2N}} \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & w^{1*} & \dots & w^{(2N-1)*} \\ \vdots & \vdots & & \\ 1 & w^{(2N-1)} & \dots & w^{(N-1)^2*} \end{pmatrix}}_{2N \times 2N} \frac{1}{\sqrt{N}} \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & & 0 \\ 1 & \cancel{w} & & w^{(N-1)*} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 1 & w^{(N-1)*} & & w^{(N-1)*} \\ 0 & & & 0 \end{pmatrix}}_{2N \times N} \psi^F = \;\;*$$

$$= \frac{1}{\sqrt{2 \cdot N}} \cdot \begin{pmatrix} N & & & \\ & N & & \\ & & \ddots & \\ & & & N \\ N & & & \\ & N & & \\ & & \ddots & \\ & & & N \end{pmatrix} \psi^F = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^F \\ \psi^F \end{pmatrix}$$

$*$ The i,j entry in the multiplication is:

$$w^{0 \cdot i *} \cdot w^{j \cdot 0 *} + w^{2 i *} \cdot w^{j \cdot 1 *} + \dots + w^{(2N-2) i *} \cdot w^{j \cdot N *} =$$

$$= \sum_{k=0}^{N-1} (w^{2k \cdot i})^* \cdot (w^{j \cdot k}) = \sum_{k=0}^{N-1} e^{-i \frac{2\pi}{2N} \cdot (2ki)} \cdot e^{\frac{2\pi}{N} j \cdot k} =$$

$$= \sum_{k=0}^{N-1} e^{-\frac{i 2\pi}{N} (k(i-j))} = \begin{cases} N, & i = j \pmod N \\ 0, & i \neq j \pmod N \end{cases}$$

c. $h = \gamma * \phi \overset{\text{commutability}}{=} \phi * \gamma = \begin{pmatrix} 1 & \frac{1}{2} & 0 & & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & & \vdots \\ 0 & \frac{1}{2} & 1 & & \vdots \\ \vdots & 0 & \frac{1}{2} & & 0 \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{2} \\ 0 & & & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 & & \end{pmatrix} \begin{pmatrix} \psi_0 \\ 0 \\ \psi_1 \\ 0 \\ \vdots \\ \psi_{N-1} \\ 0 \end{pmatrix} =$

$= \left[\psi_0 ,\ \frac{\psi_0 + \psi_1}{2} ,\ \psi_1 ,\ \dots ,\ \psi_{N-1} ,\ \frac{\psi_{N-1} + \psi_0}{2} \right]^T$

d. from Q.1.i we get that

$(DFT)h = (DFT)\gamma \odot (DFT)\phi =$

$= \begin{pmatrix} \vdots \\ \psi^F \\ \vdots \\ \psi^F \end{pmatrix} \odot \begin{pmatrix} 2 \\ 1+\cos\left(\frac{\pi}{N}\right) \\ 1+\cos\left(\frac{2\pi}{N}\right) \\ \vdots \\ 1+\cos\left(\frac{(N-1)\pi}{N}\right) \end{pmatrix} = \begin{pmatrix} 2\psi_1^F \\ 1+\cos\left(\frac{\pi}{N}\right)\psi_2^F \\ \vdots \end{pmatrix}$

$= \begin{pmatrix} \vdots \\ \psi^F \\ \vdots \\ \vdots \\ \psi^F \\ \vdots \end{pmatrix} \odot \begin{pmatrix} 2 \\ \vdots \\ 1+\cos\left(\frac{N-1}{N}\pi\right) \\ 2 \\ \vdots \\ 1+\cos\left(\frac{N-1}{N}\pi\right) \end{pmatrix} = \begin{pmatrix} 2\psi_1^F \\ \left(1+\cos\left(\frac{\pi}{N}\right)\right)\psi_2^F \\ \vdots \\ \left(1+\cos\left(\frac{N-1}{N}\pi\right)\right)\psi_N^F \\ 2\psi_1^F \\ 1+\cos\left(\frac{\pi}{N}\right)\psi_2^F \\ \vdots \\ 1+\cos\left(\frac{N-1}{N}\pi\right)\psi_N^F \end{pmatrix} //$

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