No.

Date

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$\boxed{1}$ (1) $f(x)=\log x$, $f'(x)=\frac{1}{x}$, $f''(x)=-\frac{1}{x^2}$, $f'''(x)=\frac{2}{x^3}$, $\cdots$ $\underline{f^{(n)}(x)=(-1)^{n-1}\frac{(n-1)!}{x^n}}$ //

(実際 帰納法で $n=1$ は $f'(x)=\frac{1}{x}$ で成立
$n=k$ を仮定すると
$f^{(k+1)}(x)=\frac{d}{dx}f^{(k)}(x)=\frac{d}{dx}(-1)^{k-1}\frac{(k-1)!}{x^k}=(-1)^{k-1}(k-1)!(-k)x^{-(k+1)}=(-1)^k\frac{k!}{x^{k+1}}$ で成立)

(2) $f(x)=a^x$, $f'(x)=a^x\log a$, $f''(x)=a^x(\log a)^2$, $\cdots$ $\underline{f^{(n)}(x)=a^x(\log a)^n}$ //

(実際 帰納法で $n=1$ は成立.
$n=k$ 仮定すると $f^{(k+1)}(x)=\frac{d}{dx}a^x(\log a)^k=a^x(\log a)^{k+1}$.)

(3) $f(x)=x^2e^x$, $f'(x)=2xe^x+x^2e^x$ $f''(x)=2e^x+2xe^x+f'(x)=2e^x+4xe^x+x^2e^x$

$f'''(x)=2e^x+2e^x+2xe^x+f''(x)$ より 次が成り立ちそう.

$f^{(n)}(x)=2(n-1)e^x+2xe^x+f^{(n-1)}(x)$

実際 $n=k$ 仮定すると

$f^{(k+1)}(x)=\frac{d}{dx}f^{(k)}(x)=2(k-1)e^x+2e^x+2xe^x+f^{(k)}(x)$
$=2ke^x+2xe^x+f^{(k)}(x)$

で $n=k+1$ も成立. あとはこの漸化式を解けばよい.

$f^{(0)} \xrightarrow{g(1)} f^{(1)} \xrightarrow{g(2)} f^{(2)} \xrightarrow{g(3)} f^{(3)} \cdots f^{(n-1)} \xrightarrow{g(n)} f^{(n)}$

$g(n)=f^{(n)}(x)-f^{(n-1)}(x)=2(n-1)e^x+2xe^x$ とすると $(n\geq 1)$

$f^{(n)}(x)=f^{(0)}(x)+\sum_{k=1}^{n}g(k)$
$=x^2e^x+2xe^x\cdot n+2e^x\sum_{k=1}^{n}(k-1)$
$=x^2e^x+2nxe^x+2e^x\cdot\left(\frac{1}{2}n(n+1)-n\right)=\underline{n(n-1)e^x+2nxe^x+x^2e^x}$

(4) $f(x)=\frac{1}{x^2-1}=\frac{1}{(x-1)(x+1)}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$f'(x)=\frac{1}{2}\left(-\frac{1}{(x-1)^2}-\left(-\frac{1}{(x+1)^2}\right)\right)$

$f''(x)=\frac{1}{2}\left(\frac{2}{(x-1)^3}-\frac{2}{(x+1)^3}\right)$

(1)を用いると

$f^{(n)}(x)=\frac{1}{2}\left\{(-1)^n\frac{n!}{(x-1)^{n+1}}-(-1)^n\frac{n!}{(x+1)^{n+1}}\right\}$

$\underline{=(-1)^n\frac{n!}{2}\left\{\frac{1}{(x-1)^{n+1}}-\frac{1}{(x+1)^{n+1}}\right\}}$ //

$$\frac{dz}{du} = f_x \frac{dx}{du} + f_y \frac{dy}{du} \qquad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\right)\frac{\partial x}{\partial u} + \left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)\right)\frac{\partial y}{\partial u}$$

② $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(\frac{\partial z}{\partial x}\right)x + \left(\frac{\partial z}{\partial y}\right)y.$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial}{\partial x}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial u} + \left(\frac{\partial}{\partial y}\frac{\partial z}{\partial u}\right)\frac{\partial y}{\partial u}$$

$$= \left\{\left(\frac{\partial^2 z}{\partial x^2}\right)x + \left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial^2 z}{\partial x \partial y}\right)y + \cancel{\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}}\right\}x + \left\{\left(\frac{\partial^2 z}{\partial y \partial x}\right)x + \cancel{\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial y}} + \left(\frac{\partial^2 z}{\partial y^2}\right)y + \left(\frac{\partial z}{\partial y}\right)\right\}y$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(\frac{\partial z}{\partial x}\right)(-y) + \left(\frac{\partial z}{\partial y}\right)x$$

xとyは独立変数なので$\frac{\partial y}{\partial x}$とかは意味が分からん.

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(\frac{\partial}{\partial x}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial v} + \left(\frac{\partial}{\partial y}\frac{\partial z}{\partial v}\right)\frac{\partial y}{\partial v}$$

$$= \left\{\left(\frac{\partial^2 z}{\partial x^2}\right)(-y) + \left(\frac{\partial z}{\partial x}\right)\left(-\frac{\partial y}{\partial x}\right) + \left(\frac{\partial^2 z}{\partial x \partial y}\right)x + \left(\frac{\partial z}{\partial y}\right)\right\}(-y)$$

$$+ \left\{\left(\frac{\partial^2 z}{\partial y \partial x}\right)(-y) + \left(\frac{\partial z}{\partial x}\right)(-1) + \left(\frac{\partial^2 z}{\partial y^2}\right)x + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial y}\right)\right\}x.$$

よって

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = x^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + y^2\left(\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2}\right)$$

$$+ xy\left(\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y \partial x} - \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y \partial x}\right)$$

$$+ x\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} - \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial x}{\partial y}\right) + y\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial y}{\partial x}\right)$$

$$= \boxed{(x^2+y^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)} + \cancel{x\frac{\partial z}{\partial y}\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right) + y\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right)} \quad //$$

2変数関数の連鎖律

$z = f(x, y)$ で x, y は u, v の関数

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}y\right)$$

$$= \left(\frac{\partial}{\partial u}\frac{\partial z}{\partial x}\right)x + \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial}{\partial u}\frac{\partial z}{\partial y}y + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u}x + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial u}x\right) + \frac{\partial z}{\partial x}x$$

$$+ \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}y + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial u}y\right) + \frac{\partial z}{\partial y}y$$

$$= x^2\left(\frac{\partial^2 z}{\partial x^2}\right) + 2\frac{\partial^2 z}{\partial x \partial y}(x^2+y^2) + y^2\left(\frac{\partial^2 z}{\partial y^2}\right) + \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial \theta}y.$$

同様に

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}(-y) + \frac{\partial z}{\partial y}x\right)$$

$$= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial v} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial v}\right)(-y) + \frac{\partial z}{\partial x}\left(-\frac{\partial y}{\partial v}\right)$$

$$+ \left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial v} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial v}\right)x + \frac{\partial z}{\partial y}\frac{\partial x}{\partial v}$$

$$= y^2\left(\frac{\partial^2 z}{\partial x^2}\right) + \frac{\partial^2 z}{\partial x \partial y}(-xy - xy) + x^2\left(\frac{\partial^2 z}{\partial y^2}\right) - \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}(-y)$$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (x^2+y^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

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$\boxed{3}$ (1) $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

ここで $x = r\cos\theta,\ y = r\sin\theta\ (0 \leq r,\ 0 \leq \theta \leq 2\pi)$ とすると

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r dr d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = 2\pi\left(0+\frac{1}{2}\right) = \pi$$

よって $\underline{I = \sqrt{\pi}}$ //

(2) $J = \int_0^{\infty}\int_0^{\infty} (ax^2+by^2) e^{-(ax^2+by^2)} dxdy$ とすると

$$J = \int_0^{\infty}\int_0^{\infty} ax^2 e^{-ax^2} e^{-by^2} dxdy + \int_0^{\infty}\int_0^{\infty} by^2 e^{-ax^2} e^{-by^2} dxdy$$

$$= \int_0^{\infty} e^{-by^2} dy \int_0^{\infty} ax^2 e^{-ax^2} dx + \int_0^{\infty} e^{-ax^2} dx \int_0^{\infty} by^2 e^{-by^2} dy$$

ここで(1)より $\int_0^{\infty} e^{-x^2} dx = \frac{I}{2} = \frac{\sqrt{\pi}}{2}$

$x = \sqrt{a}z$ とすると

$$\frac{\sqrt{\pi}}{2} = \int_0^{\infty} e^{-az^2}\sqrt{a}\, dz \quad より \quad \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

また, $\int_0^{\infty} ax^2 e^{-ax^2} dx = \int_0^{\infty} -\frac{x}{2}\left(e^{-ax^2}\right)' dx = \left[-\frac{x}{2}e^{-ax^2}\right]_0^{\infty} + \int_0^{\infty} \frac{1}{2} e^{-ax^2} dx$

$$= \frac{1}{2}\int_0^{\infty} e^{-ax^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{a}}.$$

以上を用いて

$$J = \frac{1}{2}\sqrt{\frac{\pi}{b}}\cdot\frac{1}{4}\sqrt{\frac{\pi}{a}} + \frac{1}{2}\sqrt{\frac{\pi}{a}}\cdot\frac{1}{4}\sqrt{\frac{\pi}{b}} = \underline{\frac{\pi}{4\sqrt{ab}}}$$ //