Promotion-induced permutations and web interactions

Ron Cherny

University of Waterloo

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Joint work with Oliver Pechenik and Stephan Pfannerer

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Introduction

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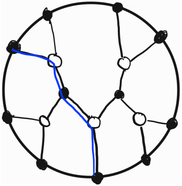

Introduction

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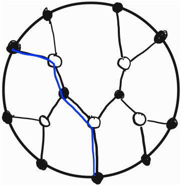


Definition/Theorem (Postnikov 2006)

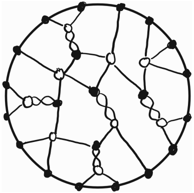
A plabic graph is a **reduced** if it has:

1. no round-trips
2. no trips with essential self-intersections
3. no trips with bad double crossings

Introduction

Definition (Gaetz, Pechenik, Pfannerer, Striker, Swanson 2025+)

An r-**hourglass plabic graph** is a planar bipartite graph embedded in a disc with boundary vertices of degree 1 and internal vertices of degree r, where we allow multiple edges.



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Theorem (Gaetz, Pechenik, Pfannerer, Striker, Swanson 2025+)

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Theorem (Gaetz, Pechenik, Pfannerer, Striker, Swanson 2025+)

There is a bijection between equivalence classes of top fully reduced 4-hourglass plabic graphs and 4-row fluctuating tableaux. Moreover, passing through the bijection, promotion corresponds to rotation and $\mathrm{trip}_\bullet = \mathrm{prom}_\bullet$

Promotion

Start with a standard Young tableau, that is a partition with fillings of that are strictly increasing along the rows and columns

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1	3	6	10
2	4	9	13
5	7	12	15
8	11	14	16

Promotion

•	3	6	10
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Promotion

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Promotion

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8	14	$\bullet$	16

Promotion

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4	7	9	13
5	11	12	15
8	14	16	$\bullet$

Promotion

2	3	6	10
4	7	9	13
5	11	12	15
8	14	16	**17**

Promotion

1	2	5	9
3	6	8	12
4	10	11	14
7	13	15	16

Promotion Permutation

1	3	6	10
2	4	9	13
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$\rightarrow$

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4	10	14	16
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Promotion Permutation

Repeating this process we get the following permutations in one-line notation

$$\begin{array}{lcccccccccccccccc}
\mathrm{prom}_1(T) = & 2 & 9 & 4 & 5 & 13 & 7 & 8 & 3 & 12 & 11 & 1 & 15 & 14 & 6 & 16 & 10 \\
\mathrm{prom}_2(T) = & 7 & 12 & 5 & 11 & 3 & 8 & 1 & 6 & 15 & 14 & 4 & 2 & 16 & 10 & 9 & 13 \\
\mathrm{prom}_3(T) = & 11 & 1 & 8 & 3 & 4 & 14 & 6 & 7 & 2 & 16 & 10 & 9 & 5 & 13 & 12 & 15
\end{array}$$

Promotion Permutation

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$\mathrm{prom}_1(T) =$	2	9	4	5	13	7	8	3	12	11	1	15	14	6	16	10
$\mathrm{prom}_2(T) =$	7	12	5	11	3	8	1	6	15	14	4	2	16	10	9	13
$\mathrm{prom}_3(T) =$	11	1	8	3	4	14	6	7	2	16	10	9	5	13	12	15

Let $\mathrm{prom}_a(T)(i) \equiv j + i - 1 \pmod{n}$, where j moves into the ath column on the ith promotion.

Local Rules

One may think of a standard Young tableau as a growing sequence of partitions.

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Example

For the standard Young tableau

$$\begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau}$$

we correspond to it the sequence

$$\ydiagram{1} \to \ydiagram{2} \to \ydiagram{2,1} \to \ydiagram{2,1,1} \to \ydiagram{2,2,1}$$

Local Rules

Definition

Given partitions $\lambda, \mu, \kappa, \nu$. We say the following diagram

$$\begin{array}{ccc} \kappa & \rightarrow & \mu \\ \uparrow & & \uparrow \\ \lambda & \rightarrow & \nu \end{array}$$

satisfies a **local rule** if $\nu = \text{sort}(\lambda + \mu - \kappa)$

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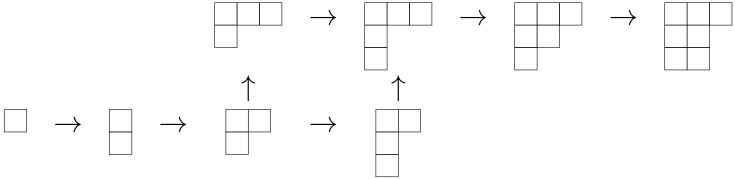
satisfies a **local rule** if $\nu = \text{sort}(\lambda + \mu - \kappa)$

In the context of standard Young tableau, applying the local rule at the *i*th position corresponds to applying the *i*th Bender-Knuth involution.

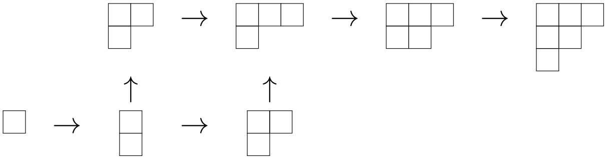
For example, here we have



Where we see

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \Rightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

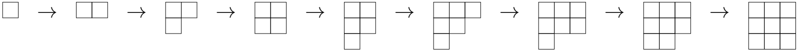
On the other hand we have

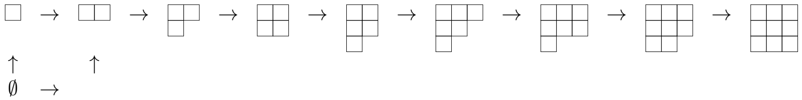


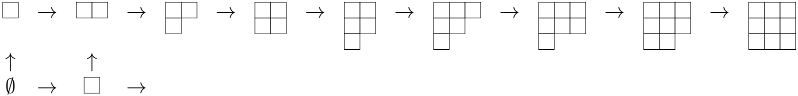
Where the tableau stays the same since we cannot swap 3 and 4.

1	3	4
2	5	
6		

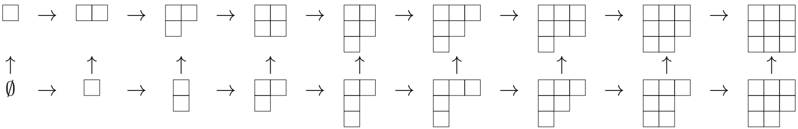
1	2	6
3	4	7
5	8	9

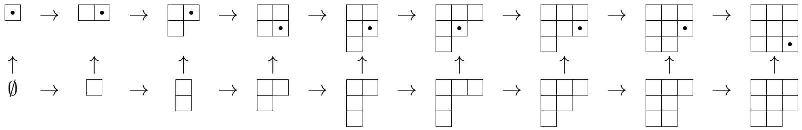


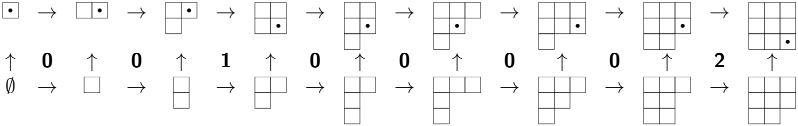




$$
\begin{tikzcd}
\ydiagram{1} \arrow[r] & \ydiagram{2} \arrow[r] & \ydiagram{2,1} \arrow[r] & \ydiagram{2,2} \arrow[r] & \ydiagram{2,2,1} \arrow[r] & \ydiagram{3,2,1} \arrow[r] & \ydiagram{3,3,1} \arrow[r] & \ydiagram{3,3,2} \arrow[r] & \ydiagram{3,3,3} \\
\emptyset \arrow[u] \arrow[r] & \ydiagram{1} \arrow[u] \arrow[r] & \ydiagram{1,1} \arrow[u] \arrow[r] & {}
\end{tikzcd}
$$






0
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0
0
0
0
2
∅

$$
\begin{array}{ccccccccccccccccccccc}
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 21 & \rightarrow & 22 & \rightarrow & 221 & \rightarrow & 321 & \rightarrow & 331 & \rightarrow & 332 & \rightarrow & 333 & & \\
 & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{1} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{2} & \uparrow & \mathbf{0} & \\
 & & 0 & \rightarrow & 1 & \rightarrow & 11 & \rightarrow & 21 & \rightarrow & 211 & \rightarrow & 311 & \rightarrow & 321 & \rightarrow & 322 & \rightarrow & 332 & \rightarrow & 333
\end{array}
$$

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0 & \to & 1 & \to & 2 & \to & 21 & \to & 22 & \to & 221 & \to & 321 & \to & 331 & \to & 332 & \to & 333 & & \\
 & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{1} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{2} & \uparrow & & \\
 & & 0 & \to & 1 & \to & 11 & \to & 21 & \to & 211 & \to & 311 & \to & 321 & \to & 322 & \to & 332 & \to & 333 \\
 & & & \mathbf{0} & \uparrow & \mathbf{1} & \uparrow & \mathbf{0} & \uparrow & \mathbf{2} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & & \\
 & & & & 0 & \to & 1 & \to & 2 & \to & 21 & \to & 31 & \to & 32 & \to & 321 & \to & 331 & \to & \cdots
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 & & 0 & \to & 1 & \to & 11 & \to & 21 & \to & 211 & \to & 311 & \to & 321 & \to & 322 & \to & 332 & \to & 333 \\
 & & & \mathbf{0} & \uparrow & \mathbf{1} & \uparrow & \mathbf{0} & \uparrow & \mathbf{2} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & \mathbf{0} & \uparrow & & \\
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 & & & & & & & & & & & \mathbf{0} & \uparrow & \mathbf{1} & \uparrow & \mathbf{2} & \uparrow & \mathbf{0} & \uparrow & & \\
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 & & & & & & & & & & & & & & & & & \mathbf{0} & \uparrow & & \\
 & & & & & & & & & & & & & & & & & & 0 & \to & \cdots
\end{array}
$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Properties of Promotion Permutations

Let T be a standard Young tableau of shape $r \times c$ with length $n = rc$. Then for $1 \leq a < r$ and $1 \leq i \leq n$ we have

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- $\mathrm{prom}_a(T) = \mathrm{prom}_{r-a}(T)^{-1}$
- $\mathrm{prom}_a(T)(i) < i$ if and only if i appears in the top a rows of T
- For $r = 2, 3, 4$ the known bijections to webs satisfy

$$\mathrm{trip}_\bullet(\mathit{Web}(T)) = \mathrm{prom}_\bullet(T)$$

A Nice Characterization

Theorem (C, Pechenik, Pfannerer)

Let T be a standard Young tableau of shape $3 \times c$. Suppose $\mathrm{prom}_1(T)$ is a product of disjoint cycles, then the corresponding web for T is a non-crossing set partition of block size 3.

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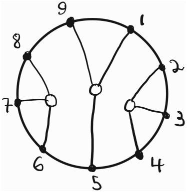
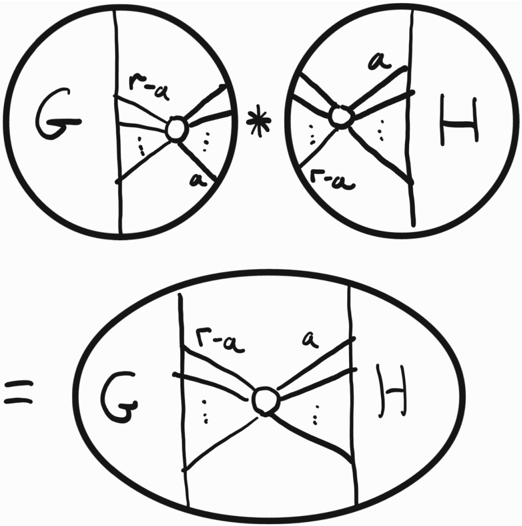
1	2	6
3	5	7
4	8	9

$\leftrightarrow$



Gluing Webs



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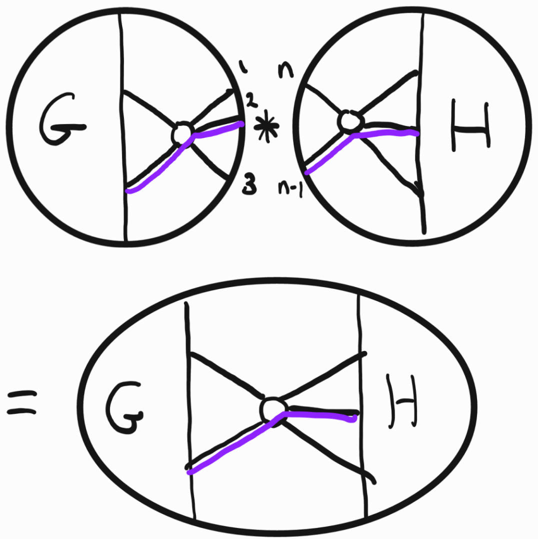

Gluing Webs Through Tableaux

1	2	4
3	5	7
6	8	11
9	10	14
12	13	15

1	4
2	6
3	7
5	9
8	10

$\Longrightarrow$

1	2	4	14
3	5	7	16
6	8	11	17
9	10	15	19
12	13	18	20

Gluing Webs Through Tableaux

Theorem (C, Pechenik, Pfannerer)

Let T, S be rectangular standard Young tableaux with the same number of rows. Suppose T and S can be glued, then promotion permutation of the gluing follows the pattern of the trip permutations

Gluing Webs Through Tableaux

Theorem (C, Pechenik, Pfannerer)

Let T, S be rectangular standard Young tableaux with the same number of rows. Suppose T and S can be glued, then promotion permutation of the gluing follows the pattern of the trip permutations

Moreover, if the glued tableaux have corresponding hourglass plabic webs that satisfy $\mathrm{trip}_\bullet = \mathrm{prom}_\bullet$ and promotion equates to rotation, then their gluing also satisfies these properties

Consequences of Gluing

Gaetz, Pechenik, Pfannerer, Striker and Swanson also constructed a web framework for 2 column rectangular tableaux (2025+).

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Now with this gluing operation we may repeatedly glue 2 column tableaux to get a web framework for tableaux satisfying $\mathrm{col}_i < \mathrm{col}_{i+2}$ for all i.

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1	2
3	4
5	8
6	9
7	10

1	3
2	4
5	7
6	9
8	10

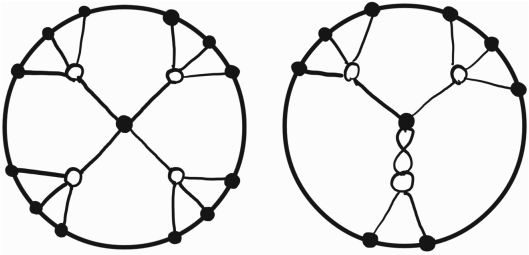
1	4
2	5
3	6
7	8
9	10

$\Longrightarrow$

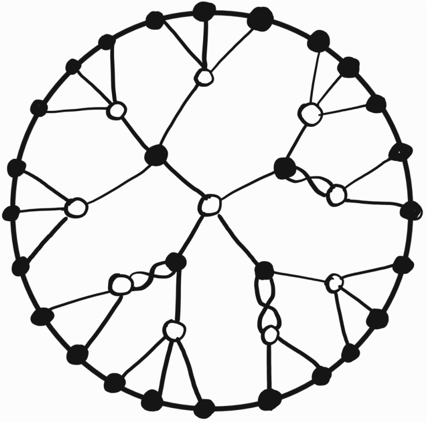
1	2	8	14
3	4	9	15
5	10	12	16
6	11	17	18
7	13	19	20

Consequences of Gluing

Using the following building blocks, one may glue them recursively to create all tree webs

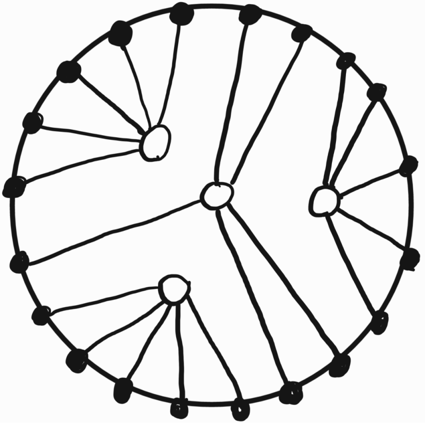


Consequences of Gluing



New Fuß-Catalan objects

We can specialize these forest webs to non-crossing set partitions of fixed block size, which gives us a natural bijection into rectangular standard Young tableaux in the plabic-hourglass framework.




THANK
YOU

1	3	10	16
2	4	11	17
5	8	12	18
6	9	13	19
7	14	15	20