# Maß- und Wahrscheinlichkeitstheorie Übersicht

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## Contents



## 1 Erstes Kapitel

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# 2 Ungelöste Fragen

## 2.1 WS11/12 Februar

### 2.1.1 Aufgabe 1

Zeigen Sie, dass $P(\mathbb{N})$ die kleinste $\sigma$-Algebra auf der Menge $\mathbb{N}$ der natürlichen Zahlen ist, die von allen endlichen Teilmengen von natürlichen Zahlen erzeugt ist.

Sei $A_i \in \mathbb{N}$ die Menge aller endlichen Teilmengen von $\mathbb{N}$ mit $i \in \mathbb{N}$ Elementen, dann ist $\bigcup_{i=0}^{\infty} A_i$ die Menge aller endlichen Teilmengen von $\mathbb{N}$. Sei $E := A$ und $A_i^C = \mathbb{N} \setminus A_i$.

$$\sigma(E) = \{\Omega, \emptyset, A, A^C\} = P(\mathbb{N})$$

(i) $\Omega \in P(\mathbb{N})$

(ii) $A \in P(\mathbb{N}) \Rightarrow A^C \in P(\mathbb{N})$

(iii) $(A_i)_{i\in\mathbb{N}} \subset P(\mathbb{N}) \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in P(\mathbb{N})$

$\Rightarrow \sigma(E) = P(\mathbb{N} = \Omega)$ ist $\sigma$-Algebra (trivial da $P(\mathbb{N})$ per Definition eine $\sigma$-Algebra auf $\Omega$ ist).

Ist $\sigma(E)$ aber auch die kleinste $\sigma$-Algebra die $E$ enthält?

Satz 2.11 aus Skript: $\sigma(E)$ von $E$ erzeugte $\sigma$-Algebra
$\Rightarrow \sigma(E)$ ist kleinste $\sigma$-Algebra die $E$ enthält.
$\Rightarrow \sigma(E) = P(\mathbb{N})$ ist kleinste $\sigma$-Algebra die von allen endlichen Teilmengen von $\mathbb{N}$ erzeugt wird.

## 2.2 WS11/12 April alle

## 2.3 One Thousand Exercises in Probability

- 7.9.5

# 3 Sigma-Fields

## 3.1 Definition

1. $\Omega \in A$
2. $A \in A \Rightarrow A^C \in A$
3. $(A_n) \subset A \Rightarrow \cup A_n \in A$

The countable/co-countable $\sigma$-field. Let $\Omega = \mathbb{R}$
$\mathbb{Z}$ : $B = \{A \subset \mathbb{R} : A$ is countable$\} \cup \{A \subset \mathbb{R} : A^C$ is countable$\}$ is a $\sigma$-field

(M1) $\Omega \in B$ (since $\Omega^C = \emptyset$ is countable)

(M2) $A \in B$ implies $A^C \in B$

(M3) $A_i \in B$ implies $\bigcap_{i=1}^{\infty} A_i \in B$

### 3.2 Intersections of Sigma-Algebras

Man Beweise: Sei $\Omega$ eine Menge, sei $I$ eine Indexmenge und für jedes $i \in I$ sei $A_i$ eine $\sigma$-Algebra auf $\Omega$. Dann ist auch

$$\cap A_i := \{A \subset \Omega | A \in A_i \forall_i \in I\}$$

eine $\sigma$-Algebra auf $\Omega$.

1. $\Omega \in A_i \forall_i \in I \Rightarrow \Omega \in \cap A_i$
2. $A \in \cap A_i \Rightarrow A \in A_i \forall i \in I \Rightarrow A^C \in \cap A_i$
3. $A_n \in \cap A_i \forall n \in \mathbb{N} \Rightarrow A_n \in A_i \forall_{i,n} \Rightarrow \cup A_n \in A_i \Rightarrow \cup A_n \in \cap A_i$

$\Rightarrow \cap A_i$ ist $\sigma$-Algebra

### 3.3 Minimal Sigma-Algebras

Let $C$ be a collection of subsets of $\Omega$. The $\sigma$-field generated by $C$, denoted $\sigma(C)$, is a *minimal* $\sigma$-field satisfying

(a) $\sigma(C) \supset C$

(b) If $B'$ is some other $\sigma$-field containing $C$, then $B' \supset \sigma(C)$

Given a class $C$ of subsets of $\Omega$, there is a unique minimal $\sigma$-field containing $C$.

**Proof**: Let

$$\aleph = \{B : B \text{ is a } \sigma - \text{field}, B \supset C\}$$

be the set of all $\sigma$-fields containing $C$. Then $\aleph \neq \varnothing$ since $P(\Omega) \in \aleph$. Let

$$B^{\daleth} = \bigcap_{B \in \aleph} B.$$

Since each class $B \in \aleph$ is a $\sigma$-field, so is $B^{\daleth}$. Since $B \in \aleph$ implies $B \supset C$, we have $B^{\daleth} \supset C$. We claim $B^{\daleth} = \sigma(C)$. We checked $B^{\daleth} \supset C$ and, for minimality, note that if $B'$ is a $\sigma$-field such that $B' \supset C$, then $B' \in \aleph$ and hence $B^{\daleth} \subset B'$.

Let $\Omega = \{1, 2, \dots, 7\}$ and $E = \{\{1, 2\}, \{6\}\}$ then

$$\sigma(E) = \{\varnothing, \{1, 2\}, \{3, 4, 5, 6, 7\}, \{6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 6\}, \{3, 4, 5, 7\}, \Omega\}$$

Let $\Omega$ be set and $A \subset \Omega$. If $E = \{A\}$ then

$$\sigma(E) = \{\varnothing, A, A^C, \Omega\}$$

### 3.4 Inverse Maps

*If $B'$ is a $\sigma$-field of subsets of $\Omega'$, then $X^{-1}(B')$ is a $\sigma$-field of subsets of $\Omega$*

**Proof**:

(M1) Since $\Omega' \in B'$, we have

$$X^{-1}(\Omega') = \Omega \in X^{-1}(B')$$

(M2) If $A' \in B'$, then $(A')^C \in B'$, and so if $X^{-1}(A') \in X^{-1}(B')$ we have

$$X^{-1}((A')^C) = (X^{-1}(A'))^C \in X^{-1}(B')$$

(M3) If $X^{-1}(B'_n) \in X^{-1}(B')$ then since $\bigcup_n B'_n \in B'$

$$\bigcup_n X^{-1}(B'_n) = X^{-1}\left(\bigcup_n B'_n\right) \in X^{-1}(B')$$

*If $C'$ is a class of subsets of $\Omega'$ then*

$$X^{-1}(\sigma(C')) = \sigma(X^{-1}(C'))$$

$\mathbb{Z}$ : $f(A_1) : \{B \subset A_2 : f^{-1}(B) \in A_1\}$ $\sigma$-Algebra auf $\Omega_2$

(M1) $\varnothing \in f(A_1) \Rightarrow \Omega_2 = \varnothing^C \in f(A_1)$

(M2) Sei $B \in f(A_2)$
$f^{-1}(B) \in A_1 \Rightarrow (f^{-1}(B_i))^C \in A_1 \Rightarrow f^{-1}(B^C) \in A_1 \Rightarrow B^C \in f(A_1)$

(M3) Sei $B_i \in f(A_1)$
$f^{-1}(B_i) \in A_1 \Rightarrow \bigcup_{i \in \mathbb{N}} f^{-1}(B_i) \in A_1 \Rightarrow f^{-1}(\bigcup_{i \in \mathbb{N}}) \in A_1 \Rightarrow \bigcup_{i \in \mathbb{N}} B_i \in f(A_1)$

## 4 Measures

Let $A$ be a $\sigma$-field on $\Omega$. $\mu$ is a measure if

$$\mu : A \rightarrow [0, \infty]$$

such that

(M1) $\mu(\varnothing) = 0$

(M2) For disjoint $A_n$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

### 4.1 Probability Measures

#### 4.1.1 Definition

(M1) $\mathbb{P}(A) \geq 0 \, \forall \, A \in B$

(M2) $\mathbb{P}$ is $\sigma$-additive for disjoint Events $A_n$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

(M3) $\mathbb{P}(\Omega) = 1$

### 4.2 Measurability

- Seien $(\Omega_1, A_1), (\Omega_2, A_2)$ zwei Messräume. $X$ ist $A_1 - A_2$-mb. falls

$$\boxed{X^{-1}(A) = \{\omega : X(\omega) \in A\} \in A_1 \forall A \in A_2}$$

- Das **Urbild** $X^{-1}(A_2) := \{X^{-1}(A), A \in A_2\}$ ist kleinste $\sigma$-Algebra bzgl. derer X mb. ist ($\sigma(X) := X^{-1}(A_2)$)
- Sei $E$ ein **Erzeuger** von $A_2$, dann ist $X$ $A_1 - A_2$-mb. falls $X^{-1}(E) \in A_1 \forall E \in E$

### 4.3 Image Measure

Sei $(\Omega, A, \mu)$ ein Maßraum, $(\Omega', A')$ ein Messraum und

$$T : (\Omega, A) \to (\Omega', A')$$

Das durch

$$\mu'(A') = \mu(T^{-1}(A')) \forall A' \in A'$$

definierte Maß $\mu'$ auf $(\Omega', A')$ heißt **Bildmaß** von $\mu$ unter $T$.

Sei $(\Omega, A, \mu)$ der Maßraum mit $\Omega := \mathbb{R}$ und der von allen abzählbaren Mengen erzeugten $\sigma$-Algebra $A$, sowie $\mu(A) = 0$ wenn $A$ abzählbar ist und $\mu(A) = 1$ wenn $A^C$ abzählbar ist.

Für $\Omega' := \{0,1\}$ und $A' := P(\Omega')$ wird die Abbildung $T : \Omega \to \Omega'$ definiert durch

$$T(\omega) := \begin{cases} 0, & \text{falls } \omega \text{ rational} \\ 1, & \text{falls } \omega \text{ irrational} \end{cases}$$

Man zeige, dass $T$ $A \to A'$-messbar ist, und bestimmte das Bildmaß $T(\mu)$.

**Antwort:** $T$ ist messbar $\Leftrightarrow T^{-1}(A') \in A \forall A' \in A$
$\Omega' = \{0,1\}$ $A' = \mathbb{P}(\Omega') = \{\emptyset, \{0,1\}, \{0\}, \{1\}\}$

| $A' \subset A'$ | $\emptyset$ | 0 | 1 | $\{0,1\}$ |
|---|---|---|---|---|
| $T^{-1}(A')$ | $\emptyset$ | $\mathbb{Q}$ | $\mathbb{R} \setminus \mathbb{Q}$ | $\Omega = \mathbb{R}$ |

$\Rightarrow T$ $A - A'$-mb

Bildmaß?

$\mu(T^{-1}(\emptyset)) = \mu(\emptyset) = 0$
$\mu(T^{-1}(0)) = \mu(\mathbb{Q}) = 0$
$\mu(T^{-1}(1)) = \mu(\mathbb{R} \setminus \mathbb{Q}) = 1$
$\mu(T^{-1}(\{0,1\})) = \mu(\mathbb{R}) = 1$

## 5 Integration and Expectation

### 5.1 Expectation

$$\boxed{\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P} = \int_\mathbb{R} x f(x) \, dx} \quad (1)$$

$$\mathbb{E}[h(X)] = \int_\mathbb{R} h(x) \, \mathbb{P}_X \, dx = \begin{cases} \int_\mathbb{R} h(x) f(x) \, dx & \text{im abs. stetigen Fall} \\ \sum_{k=1}^{\infty} h(x_k) \, \mathbb{P}[X = x_k] & \text{im diskreten Fall} \end{cases}$$
(2)

Erwartungswert von $e^x$ bei Normalverteilung

$X \sim N(0,1)$, $\mathbb{E}[e^x]$?

$$\begin{aligned} \mathbb{E}[e^x] &= \int_\Omega e^x \, d\mathbb{P} \\ &= \int_\mathbb{R} e^t \, \mathbb{P}_X \, dt \\ &= \int_\mathbb{R} e^t \, d\lambda(t) \\ &= \int_\mathbb{R} e^t \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \\ &= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}+t} \, dt \\ &= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} * e^{\frac{-t^2+2t+1-1}{2}} \, dt \\ &= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-(t^2-2t-1+1)}{2}} \, dt \\ &= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-((t-1)^2-1)}{2}} \, dt \\ &= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-(t-1)^2}{2}+\frac{1}{2}} \, dt \\ &= e^{\frac{1}{2}} \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-(t-1)^2}{2}} \, dt \quad \sim N(1,1) = \text{Dichte} \\ &= e^{\frac{1}{2}} \end{aligned}$$

Varianz von Exponentialverteilter Zufallsvariable

$X \sim \text{Exp}(\lambda)$, $\mathbb{V}[X]$?

$$\mathbb{E}[X] = \int_0^\infty t\lambda e^{-\lambda t} \, dt \overset{PI}{=} -e^{-\lambda t} t! \,|_0^\infty - \int_0^\infty 1(-e^{-\lambda t}) \, dt = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$$

$$
\begin{aligned}
\mathbb{V}[X] = \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] &= \int_0^\infty (t-\frac{1}{\lambda})^2 \lambda e^{-\lambda t}\,dt \\
&= \int_0^\infty t^2 \lambda e^{-\lambda t}\,dt - \frac{2}{\lambda}\int_0^\infty t\lambda e^{-\lambda t}\,dt + \frac{1}{\lambda^2}\int_0^\infty \lambda e^{-\lambda t}\,dt \\
&\overset{PI}{=} -t^2 e^{-\lambda t}\Big|_0^\infty - \int_0^\infty 2te^{-\lambda t}\,dt - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} \\
&= 0 + \frac{2}{\lambda^2} - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} = \frac{1}{\lambda^2}
\end{aligned}
$$

## 5.2 Probability

$$\boxed{\mathbb{P}[A] = \int_A d\mathbb{P} = \mathbb{E}[\mathbb{1}_A]} \tag{3}$$

## 5.3 Distribution Function

$$\boxed{F(x) = \mathbb{P}[(-\infty, x]] = \mathbb{P}[X \le x], \quad x \in \mathbb{R}} \tag{4}$$

## 5.4 Monotone Convergence

If

$$X_n \uparrow X$$

then

$$\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$$

and

$$\mathbb{E}\left[\sum_{i=1}^\infty X_i\right] = \sum_{i=1}^\infty \mathbb{E}[X_i]$$

## 5.5 Dominated Convergence Theorem

If

$$X_n \to X$$

and there exists $Z \in L_1$ such that

$$|X_n| \le Z$$

then

$$\boxed{\mathbb{E}[X_n] \to \mathbb{E}[X] \; and \; \mathbb{E}\left[|X_n - X|\right] \to 0} \tag{5}$$

## 5.6 Integrable Random Variables

Define $\mathbb{E}[X] := \mathbb{E}[X^+] - \mathbb{E}[X^-]$. The set of integrable random variables is denoted by $L_1$:

$$\boxed{L_1 = \{\text{random variables } X \;:\; \mathbb{E}\left[|X|\right] < \infty\}} \tag{6}$$

## 5.7 Properties of Expectation

1. If $X$ is integrable, then
$$\mathbb{P}[X = \pm\infty] = 0$$

2. If $\mathbb{E}[X]$ exists,
$$\mathbb{E}[cX] = c\,\mathbb{E}[X]$$

3. If $X \ge 0$ then $\mathbb{E}[X] \ge 0$ since $X = X^+$. If $X, Y \in L_1$, and $X \le Y$ then
$$\mathbb{E}[X] \le \mathbb{E}[Y]$$

4. Suppose $\{X_n\}$ is a sequence of random variables such that $X_n \in L_1$ for some $n$. If either
$$X_n \uparrow X$$
or
$$X_n \downarrow X$$
then
$$\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$$
or
$$\mathbb{E}[X_n] \downarrow \mathbb{E}[X]$$

5. If $X \in L_1$,
$$\left|\mathbb{E}[X]\right| \le \mathbb{E}\left[|X|\right]$$

6. Variance and Covariance. If $X \in L_2$ then

$$\mathbb{V}[X] := \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 \tag{7}$$

$$\mathrm{Cov}(X,Y) := \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] \tag{8}$$

$$\mathbb{V}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{V}[X_i] + \sum_{i=1}^n \mathrm{Cov}(X_i, X_j) \tag{9}$$

## 5.8 Fatou´s Lemma

If there exists $Z \in L_1$ and $X_n \ge Z$ then

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \le \liminf_{n\to\infty} \mathbb{E}[X_n] \tag{10}$$

and if $X_n \le Z$ then

$$\limsup_{n\to\infty} \mathbb{E}[X_n] \le \mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \tag{11}$$

## 5.9 Fubini Theorem

Let $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$ be a product measure. If $X$ is $B_1 \times B_2$ measurable and integrable with respect to $\mathbb{P}$ then

$$\int_{\Omega_1\times\Omega_2} X\,d\mathbb{P} = \int_{\Omega_1}\int_{\Omega_2} X\,d\mathbb{P}_2\,d\mathbb{P}_1 \tag{12}$$

$$= \int_{\Omega_2}\int_{\Omega_1} X\,d\mathbb{P}_1\,d\mathbb{P}_2 \tag{13}$$

## 5.10 Tonelli

$$\int_{\times_{i=1}^n \Omega_i} f(\omega_1,\dots,\omega_n)\, d\otimes_{i=1}^n \mu_i(\omega_1,\dots,\omega_n) = \int_{\Omega_1}\int_{\Omega_2}\cdots\int_{\Omega_n} f(\omega_1,\dots,\omega_n)\,\mu_n(d\omega_n)\dots\mu_1(d\omega_1)$$

## 5.11 Radon-Nikodym

Sei $(\Omega, A)$ ein Messraum, seien $\mu$ und $\nu$ zwei Maße auf $(\Omega, A)$ so dass

$$d\nu = f\,d\mu$$

für eine $A$-mb Funktion

$$f : \Omega \to \mathbb{R} \text{ mit } f(w) \geq 0\ \forall \omega \in \Omega$$

Dann heisst $f$ **Dichte** oder Dichtefunktion von $\nu$ bzgl. $\mu$.

Seien $\mu$ und $\nu$ Maße auf dem Maßraum $(\Omega, A)$, so dass für jedes $A \in A$ gilt

$$\mu(A) = 0 \quad\Rightarrow\quad \nu(A) = 0$$

Dann sagt man $\nu$ ist absolut stetig bzgl. $\mu$. Notation:

$$\nu \ll \mu$$

**Radon-Nikodym**: Seien $\mu$ und $\nu$ $\sigma$-endliche Maße auf dem Messraum $(\Omega, A)$. Dann sind folgende Aussagen äquivalent:

(i) $\nu$ besitzt eine Dichte bzgl. $\mu$

(ii) $\nu \ll \mu$

Beispiel Normalverteilung

$$dN(\mu,\sigma^2) = f_{\mu,\sigma^2}\,d\lambda \tag{14}$$

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{32} & 0 \leq t < 1 \\ \frac{t^2}{16} & 1 \leq t < 2 \\ \frac{t}{8} + \frac{1}{4} & 2 \leq t < 4 \\ 1 & t \geq 4 \end{cases}$$

$\mathbb{Z}$ : Dichte bzgl. $\lambda + \delta_0 + \delta_1 + \delta_2 + \delta_4$

**Diskreter Teil**: Unstetigkeitsstellen
$\mathbb{P}[x_i] \geq 0\ i = 1,2,3\ \ \alpha_i = \mathbb{P}[x_i],\ x_1 = 1, x_2 = 2, x_3 = 4$

**Absolut stetiger Teil**: $F(t)$ abs. stetig auf $\mathbb{R} \setminus \{1,2,4\}$
d.h. $\mathbb{P}(B) = \int_B d\mathbb{P} = \int_B f(x)\,d\lambda\ \forall\ B \in B, \{1,2,4\} \notin B$
$\mathbb{P}(B) = \mathbb{E}(\mathbb{1}_B) = \int \mathbb{1}_B\,d\mathbb{P} = \int_B d\mathbb{P}$
$F(t) = \int_{-\infty}^t f(t)\,dt \Rightarrow F'(t) = f(t)$

$\Rightarrow F'(t) = f(t) = \frac{1}{32}\mathbb{1}(0 < t < 1) + \frac{1}{8}t\mathbb{1}(1 < t < 2) + \frac{1}{8}\mathbb{1}(2 < f < 4)$

$\Rightarrow \hat{f}(t) = \begin{cases} f(t) & \forall t \in \mathbb{R} \setminus \{1,2,4\} \\ \alpha_j & \forall t = x_j,\ j = 1,2,3 \end{cases}$

$\Rightarrow \hat{\mathbb{P}} \ll \mu$

## 5.12 Transformationssatz für Dichten

Sei $f : \mathbb{R}^p \to \mathbb{R}$, $(x_1,\dots,x_p) \mapsto f(x_1,\dots,x_p)$ die $\lambda^p$-Dichte eines Wahrscheinlichkeitsmaßes $\mathbb{P}_X$. Seien $G, G' \in B^{\otimes p}$ offen und die Abbildung

$$T : G \to G' \tag{15}$$

$$(x_1,\dots,x_p) \mapsto \left(T_1(x_1,\dots,x_p),\dots,T_p(x_1,\dots,x_p)\right) \tag{16}$$

bijektiv und samt $T^{-1}$ messbar und differenzierbar.
Dann gilt für die $\lambda^p$-Dichte $g$ von $T(\mathbb{P}_X)$:

$$\boxed{\begin{aligned} g(y_1,\dots,y_p) &= \left|\det J_{T^{-1}}(y_1,\dots,y_p)\right| \cdot f\left(T^{-1}(y_1,\dots,y_p)\right) \\ &= \left|\det J_T\left(T^{-1}(y_1,\dots,y_p)\right)\right| \cdot f\left(T^{-1}(y_1,\dots,y_p)\right) \end{aligned}}$$

(17)

(18)

Im **eindimensionalen** Fall vereinfacht sich die Dichtetransformationsformel zu

$$\boxed{g(y) = \left|(T^{-1})'(y)\right| \cdot f\left(T^{-1}(y)\right)} \tag{19}$$

Sei $X \sim \text{Exp}$ mit der Dichte $f(x) = \lambda e^{-\lambda x}\mathbb{1}_{(0,\infty)}(x)$.

Die Abbildung

$$T : \ x \mapsto x^2$$

ist bijektiv mit Umkehrfunktion

$$y \mapsto \sqrt{y}$$

Mit Ableitung

$$\frac{dT^{-1}(y)}{dy} = \frac{1}{2}y^{-\frac{1}{2}}$$

Dann ist

$$g(y) = \left|\frac{1}{2}y^{-\frac{1}{2}}\right| \cdot f(\sqrt{x}) = \frac{1}{2}y^{-\frac{1}{2}} \cdot \lambda e^{-\lambda\sqrt{y}}$$

für $y > 0$.

## 5.13 Convolutions

The Convolution $f = f_1 * f_2$ of two densities $f_1$ and $f_2$ is defined by

$$\boxed{f(z) = \int_{-\infty}^{+\infty} f_1(z-y)f_2(y)\,dy} \tag{20}$$

# 6 Conditional Expectation

$$\mathbb{E}[Y \mid X] = \int y \cdot f_{Y \mid X}(y \mid x) dy = \int y \cdot \frac{f_{Y,X}(y,x)}{f_X(x)} dy = \int y \cdot \frac{f_{Y,X}(y,x)}{\int f_{Y,X}(y,x) dy} dy \tag{21}$$

$$\mathbb{E}[X \mid B] = \frac{1}{\mathbb{P}[B]} \int_B X \, d\mathbb{P} = \frac{\mathbb{E}\left[X \cdot \mathbb{1}_B\right]}{\mathbb{P}(B)} \tag{22}$$

$$\mathbb{E}\left[\psi(Y,X) \mid X = x\right] = \int_{\Omega_2} \int_{\Omega_1} \psi(y,x) \, \mathbb{P}^{Y \mid X = x} dy \, \mathbb{P}^X dx \tag{23}$$

## 6.1 Properties of Conditional Expectation

Sei $(\Omega, A, \mathbb{P})$ ein Wahrscheinlichkeitsraum und Seien

$$f : \Omega \to \mathbb{R}, \quad f_1 : \Omega \to \mathbb{R}, \quad f_2 : \Omega \to \mathbb{R}$$

bzgl. $\mathbb{P}$ integrierbare Funktionen. Sei $C$ eine Unter-$\sigma$-Algebra von $A$.
Dann gilt:

1. $\mathbb{E}\left[f \mid C\right] \in L_1(\Omega, A, \mathbb{P})$
2. $\mathbb{E}\left[\mathbb{E}\left[f \mid C\right]\right] = \mathbb{E}\left[f\right]$
3. $f$ ist $C$-messbar $\Rightarrow \mathbb{E}\left[f \mid C\right] = f$ $\mathbb{P}$-f.s.
4. $f = g\mathbb{P}$-f.s. $\Rightarrow$ $\mathbb{E}\left[f \mid C\right] = \mathbb{E}\left[g \mid C\right]$ $\mathbb{P}$-f.s.
5. $f = \text{const} = \alpha$ $\Rightarrow$ $\mathbb{E}\left[f \mid C\right] = \alpha\mathbb{P}$-f.s.
6. Wenn $X_i$ iid sind, dann ist
$$\mathbb{E}\left[X \mid \sum_{i=1}^n X_i\right] = \frac{\sum_{i=1}^n X_i}{n}$$
also z.b. $X, Y \sim \text{Exp}(\lambda)$, dann ist
$$\mathbb{E}[X \mid X + Y] \overset{iid}{=} \frac{X+Y}{2}$$
7. Für $\alpha_1, \alpha_2 \in \mathbb{R}$ ist $\mathbb{E}\left[\alpha_1 f_1 + \alpha_2 f_2 \mid C\right] = \alpha_1 \mathbb{E}\left[f_1 \mid C\right] + \alpha_2 \mathbb{E}\left[f_2 \mid C\right]$
8. $f_1 \le f_2\mathbb{P}$-f.s. $\Rightarrow$ $\mathbb{E}\left[f_1 \mid C\right] \le \mathbb{E}\left[f_2 \mid C\right]$
9. $\left|\mathbb{E}\left[f \mid C\right]\right| \le \mathbb{E}\left[\left|f\right| \mid C\right]$
10. **Monotone convergence.** If $X \in L_1$, $0 \le X_n \uparrow X$, then
$$\mathbb{E}\left[X_n \mid C\right] \uparrow \mathbb{E}\left[X \mid C\right]$$
11. Monotone convergence implies the **Fatou Lemma**. If $0 \le X_n \in L_1$, then
$$\mathbb{E}\left[\liminf_{n\to\infty} X_n \mid C\right] \le \liminf_{n\to\infty} \mathbb{E}\left[X_n \mid C\right]$$
and while if $X_n \le Z \in L_1$, then
$$\mathbb{E}\left[\limsup_{n\to\infty} X_n \mid C\right] \ge \limsup_{n\to\infty} \mathbb{E}\left[X_n \mid C\right]$$
12. Fatou implies **dominated convergence**. If $X_n \in L_1$, $\left|X_n\right| \le Z \in L_1$ and $X_n \to X_\infty$, then
$$\mathbb{E}\left[\lim_{n\to\infty} X_n \mid C\right] \overset{a.s.}{=} \lim_{n\to\infty} \mathbb{E}\left[X_n \mid C\right]$$

## 6.2 Glättungseigenschaften

## 6.3 Bedingte Dichten

$$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{24}$$

## 6.4 Bedingte Wahrscheinlichkeiten

$$\begin{aligned}
\mathbb{P}[A] &= \int_A d\mathbb{P} = \mathbb{E}\left[\mathbb{1}_A\right] \\
\mathbb{P}[A \mid C] &= \mathbb{E}\left[\mathbb{1}_A \mid C\right] \\
\mathbb{P}[A \mid T] &= \mathbb{E}\left[\mathbb{1}_A \mid T\right] \\
\mathbb{P}[A \mid T = t] &= \mathbb{E}\left[\mathbb{1}_A \mid T = t\right] \\
\mathbb{P}[X \in A \mid T = t] &= \int_A f_{X \mid Y}(x \mid y) \, dx
\end{aligned}$$

## 6.5 Examples

Let $X$ and $Y$ be jointly continious random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{if } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}[X + Y \mid X < Y]$:

$$\begin{aligned}
\mathbb{P}[X < Y] &= \int_{-\infty}^{\infty} \int_x^{\infty} \left(f_{X,Y}(x,y)\right) dy \, dx \\
&= \int_0^{\infty} \int_x^{\infty} e^{-x-y} \, dy \, dx \\
&= \int_0^{\infty} e^{-2x} \, dx = \frac{1}{2}
\end{aligned}$$

Next,

$$\mathbb{E}\left[\mathbb{1}_{(X<Y)}(X+Y)\right] = \int_{-\infty}^{\infty}\int_{x}^{\infty}\left((x+y)f_{X,Y}(x,y)\right)dy\,dx$$
$$= \int_{0}^{\infty}\int_{x}^{\infty}(x+y)e^{-x-y}\,dy\,dx$$
$$= \int_{0}^{\infty}(2x+1)e^{-2x}\,dx = 1$$

It follows that

$$\mathbb{E}[X+Y\,|\,X<Y] = \frac{\mathbb{E}\left[\mathbb{1}_{(X<Y)}(X+Y)\right]}{\mathbb{P}[X<Y]} = \frac{1}{1/2} = 2$$

$X,Y$ haben gemeinsame Dichte $f_{X,Y}(x,y) = xe^{-x(y+1)}\cdot\mathbb{1}_{R^2}(x,y)$. Gesucht: $\mathbb{E}[Y\,|\,X=x]$

$$f_X(x) = \int f_{X,Y}(x,y)\,dy$$
$$= \int xe^{-x(y+1)}\cdot\mathbb{1}_{\mathbb{R}_+^2}(x,y)\,dy$$
$$= \int_0^{\infty} xe^{-x(y+1)}\cdot\mathbb{1}_{\mathbb{R}_+}(x,y)\,dy$$
$$= e^{-x}\underbrace{\int_0^{\infty} xe^{-xy}\cdot\mathbb{1}_{\mathbb{R}_+}(x)\,dy}_{\text{Dichte einer Exp. Vert.=1}}$$
$$= e^{-x}\cdot\mathbb{1}_{R_+}(x)$$

$$\mathbb{E}[Y\,|\,X=x] = \int y\cdot f_{Y\,|\,X}(y\,|\,x)\,dx$$
$$= \int y\cdot\frac{f_{X,Y}(x,y)}{f_X(x)}\,dx$$

Seien $X,Y$ Zufallsvariablen mit gemeinsamer Dichte $f_{X,Y}(x,y) = x(y-x)e^{-y}$ und $0\le x\le y<\infty$.
Geben Sie $\mathbb{E}[Y\,|\,X]$ an.
Tip: (Merhfache) partielle Integration

$f_{Y\,|\,X}(y\,|\,x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

$$\Rightarrow f_X(x) = \int_x^{\infty} f_{X,Y}(x,y)\,dy$$
$$= \int_x^{\infty} x(y-x)e^{-y}\,dy$$
$$= \int_x^{\infty} xye^{-y}\,dy - \int_x^{\infty} x^2e^{-y}\,dy$$
$$= x[-e^{-y}(y+1)]_x^{\infty} - x^2[-e^{-y}]_x^{\infty}$$
$$= x[0+e^{-x}(x+1)] - x^2[0+e^{-x}]$$
$$= xe^{-x}(x+1) - x^2e^{-x}$$
$$= x^2e^{-x} + xe^{-x} - x^2e^{-x}$$
$$= xe^{-x}$$

$$\mathbb{E}[Y\,|\,X] = \int_x^{\infty} yf_{Y\,|\,X}(y\,|\,x)\,dy$$
$$= \int_x^{\infty} y\frac{x(y-x)e^{-y}}{xe^{-x}}\,dy$$
$$= \int_x^{\infty} y(y-x)e^{x-y}\,dy$$
$$= \int_x^{\infty} y^2e^{x-y} - yxe^{x-y}\,dy$$
$$= e^x\int_x^{\infty} y^2e^{-y}\,dy - xe^x\int_x^{\infty} ye^{-y}\,dy$$
$$= e^x[-y^2e^{-y}|_x^{\infty} + \int_x^{\infty} 2ye^{-y}\,dy] - xe^x[-e^{-y}(y+1)]_x^{\infty}$$
$$= e^x[x^2e^{-x} + 2[-e^{-y}(y+1)]_x^{\infty}] - xe^x[e^{-x}(x+1)]$$
$$= e^xx^2e^{-x} + 2e^{-x}(x+1)e^x - xe^xe^{-x}(x+1)$$
$$= 2+x$$

# 7 Martingales

For integrable random variables $\{X_n, n\ge 0\}$ and $\sigma$-fields $\{B_n, n\ge 0\}$ which are sub $\sigma$-fields of $B$, $\{(X_n,B_n), n\ge 0\}$ is a **martingale** if

(M1) Information accumulates, i.e. $A_n \subset A_{n+1}$

(M2) $X_n$ is adapted in the sense that for each $n$, $X_n \in B_n$; that, $X_n$ is $B_n$-measureable.

(M3) $\mathbb{E}\left[|X_n|\right] < \infty$

(M4) $\mathbb{E}\left[X_{n+1}\,|\,B_n\right] \overset{a.s.}{=} X_n$

Sub-Martingal $\le$
Martingal bzgl. $(A_t)_{t\in T}$ $:\Longleftrightarrow$ $\forall s\le t:\; X_s = \mathbb{E}\left[X_t\,|\,A_s\right]$, $\mathbb{P}-f.s.$
Super-Martingal $\ge$

(25)

(26)

(27)

## 7.1 Properties

1. $(X_t)_{t\in T}$ sei ein Martingal bzgl. $(A_t)_{t\in T}$ mit $X_t \in L_p\ \forall t\in T\ (1\le p<\infty)$. Dann ist $\left(|X_t|^p\right)_{t\in T}$ ein Sub-Martingal bzgl. $(A_t)_{t\in T}$

2. Für jedes $c\in\mathbb{R}$ und Sub-Martingal $(X_t)_{t\in T}$ ist auch $(\max\{c,X_t\})_{t\in T}$ ein Sub-Martingal bzgl. $(A_t)_{t\in T}$. Insbesondere ist mit $c=0$ dann auch $(X_t^+)_{t\in T}$ ein Sub-Martingal.

3. Ist $(X_t)_{t\in T}$ ein Super-Martingal bzgl. $(A_t)_{t\in T}$, so ist $(X_t^-)_{t\in T}$ ein Sub-Martingal bzgl. $(A_t)_{t\in T}$. Zur Erinnerung: $X_t^- := -\min\{0, X_t\}$.

## 7.2 Stopping Times

A mapping $\nu : \Omega \mapsto \bar{\mathbb{N}}$ is a stopping time if

$$[\nu = n] \in B_n, \quad \forall n \in \mathbb{N} \tag{28}$$

## 7.3 Martingaldifferenzfolgen

Sei $\eta_n \in L(\Omega, A, \mathbb{P})$, $n \in \mathbb{N}$, mit $A_n := \sigma(\eta_1, \dots, \eta_n)$ und $a \in \mathbb{R}$ beliebig.
Definiere

$$X_1 := \eta_1 - a \text{ und } X_{n+1} := X_n + \eta_{n+1} - \mathbb{E}\left[\eta_{n+1} \,|\, A_n\right] \quad (n \geq 1)$$

Dann gilt

$$\begin{aligned}
\mathbb{E}\left[X_{n+1} \,|\, A_n\right] &= \mathbb{E}\left[X_n \,|\, A_n\right] + \mathbb{E}\left[\eta_{n+1} \,|\, A_n\right] - \mathbb{E}\left[\mathbb{E}\left[\eta_{n+1} \,|\, A_n\right] \,|\, A_n\right] \\
&= X_n + \mathbb{E}\left[\eta_{n+1} \,|\, A_n\right] - \mathbb{E}\left[\eta_{n+1} \,|\, A_n\right] \\
&= X_n
\end{aligned}$$

Das heißt, die Folge $(X_n)_{n\in\mathbb{N}}$ bildet ein Martingal.

Ist umgekehrt $(X_n)_{n\in\mathbb{N}}$ als Martingal vorausgesetzt und definiert man

$$\eta_1 := X_1 \qquad \eta_n := X_n - X_{n-1} \quad (n \geq 2)$$

dann gilt

$$\begin{aligned}
\mathbb{E}\left[\eta_{n+1} \,|\, \eta_1, \dots, \eta_n\right] &= \mathbb{E}\left[X_{n+1} - X_n \,|\, \eta_1, \dots, \eta_n\right] \\
&= \mathbb{E}\left[X_{n+1} - X_n \,|\, X_1, \dots, X_n\right] \\
&= \mathbb{E}\left[X_{n+1} \,|\, X_1, \dots, X_n\right] - X_n \\
&= 0
\end{aligned}$$

Daher ist eine Folge reeler integrierbarer Zufallsvariablen $(\eta_n)_{n\in\mathbb{N}}$ heißt **Martingaldifferenzfolge**, falls

$$\boxed{\mathbb{E}\left[\eta_{n+1} \,|\, \eta_1, \dots \eta_n\right] = 0 \quad \mathbb{P}\text{-f.s.}, \quad \forall n \in \mathbb{N}} \tag{29}$$

## 7.4 Examples

Seien $Z_1, \dots, Z_n$ unabhängig und identisch verteilt (iid) mit $Z_i \sim N(0,1)$ und
$F_n = \sigma(Z_1, \dots, Z_n)$ eine Filtration. Ferner sei $X_n := \exp\left(\sum_{i=1}^n (Z_i - c)\right), n \in \mathbb{N}, c \in \mathbb{R}$.
Für welche Werte $c$ ist $(X_n)_{n\in\mathbb{N}}$ ein Martingal, Submartingal bzw. Supermartingal bzgl. $(F_n)$?
Bitte begründen Sie Ihre Schritte kurz!

- $X_n$ ist $F_n$-mb. da Komposition aus $Z_i$ und $F_n = \sigma(Z_1, \dots, Z_n)$

- $F_n$ ist Filtration (Information komm hinzu) $\Rightarrow$ $F_n \subset F_{n+1}$ $\forall n$

- $\mathbb{E}\left[|X_n|\right] < \infty$? (ist $Z_n$ integrierbar?)

$$\begin{aligned}
\mathbb{E}\left[|X_n|\right] &= \mathbb{E}\left[\exp\left(\sum_{i=1}^n Z_i - c\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^n \exp\left(Z_i - c\right)\right] \\
&\overset{\text{iid}}{=} \left(\mathbb{E}\left[\exp(Z - c)\right]\right)^n \\
&= \left(\int_{\mathbb{R}} \exp(z - c) \, d\mathbb{P}_Z\right)^n \\
&= \left(\int_{\mathbb{R}} \exp(z - c) \frac{1}{\sqrt{2\pi}} \exp\left(-\left(\frac{z^2}{2}\right)\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} + z - c\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2 + 2z - 2c}{2}\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2 + 2z - 2c + 1 - 1}{2}\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-((z^2 - 1)^2 - 1 + 2c)}{2}\right) dz\right)^n \\
&= \left(e^{\frac{1}{2} - c} \cdot \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(z-1)^2}{2}\right) dz}_{\sim N(1,1) = 1}\right)^n \\
&= (e^{\frac{1}{2} - c})^n \\
&= e^{n\left(\frac{1}{2} - c\right)} < \infty
\end{aligned}$$

- Martingaleigenschaft: $\mathbb{E}\left[X_{n+1}\,|\,F_n\right] \overset{\text{f.s.}}{=} X_n$?

$$\begin{aligned}\mathbb{E}\left[X_{n+1}\,|\,F_n\right] &= \mathbb{E}\left[X_n\cdot\exp\left(Z_{n+1}-c\right)|\,F_n\right]\\ \left(X_n \text{ ist } F_n\text{-mb.}\right) \Rightarrow &= X_n\cdot\mathbb{E}\left[\exp\left(Z_{n+1}-c\right)|\,F_n\right]\\ &\overset{\text{iid}}{=} X_n\cdot\mathbb{E}\left[\exp\left(Z_{n+1}-c\right)\right]\\ &= X_n\cdot e^{\frac{1}{2}-c}\\ &= X_n \text{ für } c=\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\Rightarrow\ X_n \text{ Martingal für } c &= \frac{1}{2}\\ X_n \text{ Super-Martingal für } c &> \frac{1}{2}\\ X_n \text{ Sub-Martingal für } c &< \frac{1}{2}\end{aligned}$$

**Martingales and smoothing**. Suppose $X\in L_1$ and $\{B_n, n\geq 0\}$ is an increasing family of sub $\sigma$-fields of $B$. Define for $n\geq 0$

$$X_n := \mathbb{E}\left[X\,|\,B_n\right]$$

Then

$$\{(X_n, B_n), n\geq 0\}$$

is a martingale:

$$\begin{aligned}\mathbb{E}\left[X_{n+1}\,|\,B_n\right] &= \mathbb{E}\left[\mathbb{E}\left[X\,|\,B_{n+1}\right]|\,B_n\right]\\ &= \mathbb{E}\left[X\,|\,B_n\right] \qquad \text{(smoothing)}\\ &= X_n\end{aligned}$$

**Martingales and sums of independent random variables**. Suppose that $\{Z_n, n\geq 0\}$ is an independent sequence of integrable random variables satisfying for $n\geq 0$, $\mathbb{E}\left[Z_n\right]=0$. Set $X_0=0$, $X_n=\sum_{i=1}^n Z_i$, $n\geq 1$, and $B_n := \sigma(Z_0,\dots,Z_n)$.

Then $\{(X_n, B_n), n\geq 0\}$ is a martingale since $\{(Z_n, B_n), n\geq 0\}$ is a fair sequence.

Es sei $(X_t)_{t\in\mathbb{N}}$ eine Folge von unabhängigen und identisch verteilten Zufallsvariablen mit $\mathbb{E}\left[X_1\right]=1$. Zeigen Sie, dass der stochastische Prozess $(Z_t, t\in\mathbb{N})$ mit

$$Z_t = \prod_{s=1}^t X_s$$

ein Martingal bezüglich der kanonischen Filtration $\sigma(X_1, X_2, \dots)$ ist.

Es gilt für jedes $t\in\mathbb{N}$:

$$\begin{aligned}\mathbb{E}\left[Z_{t+1}\,|\,A_t\right] &= \mathbb{E}\left[\prod_{i=1}^{t+1} X_i\,|\,\sigma(X_1,\dots,X_t)\right]\\ &= \mathbb{E}\left[\prod_{i=1}^{t} X_i\,|\,\sigma(X_1,\dots,X_t)\right]\cdot\mathbb{E}\left[X_{t+1}\,|\,\sigma(X_1,\dots,X_t)\right]\\ &= \prod_{i=1}^t X_i\cdot\mathbb{E}\left[X_{t+1}\right] = \prod_{i=1}^t X_i = Z_t\end{aligned}$$

Es sei $(X_t)_{t\in\mathbb{N}}$ eine Folge von unabhängigen und identisch verteilten Zufallsvariablen mit $\mathbb{E}\left[X_1\right]=0$ und $\mathbb{E}\left[X_1^2\right]=\sigma^2$. Weiter sei $S_t=\sum_{s=1}^t X_s$. Zeigen Sie, dass der stochastische Prozess $(Z_t, t\in\mathbb{N})$ mit

$$Z_t = S_t^2 - t\sigma^2$$

ein Martingal bezüglich der kanonischen Filtration $\sigma(X_1, X_2, \dots)$ ist.

Es gilt für jedes $t\in\mathbb{N}$:

$$\begin{aligned}\mathbb{E}\left[Z_{t+1}\,|\,A_t\right] &= \mathbb{E}\left[S_{t+1}^2-(t+1)\sigma^2\,|\,\sigma(X_1,\dots,X_t)\right]\\ &= \mathbb{E}\left[S_t^2+2S_t^2X_{t+1}+X_{t+1}^2\,|\,\sigma(X_1,\dots,X_t)\right]-(t+1)\sigma^2\\ &= S_t^2+\mathbb{E}\left[X_{t+1}^2\right]-(t+1)\sigma^2 = S_t^2-t\sigma^2 = Z_t\end{aligned}$$

# 8 Convergence

## 8.1 Almost Sure Convergence

We say that a statement about random elements hold *almost surely* if there exists an event $A\in B$ with $\mathbb{P}\left[A\right]=0$ such that the statement holds if $w\in A^C$.

$$\boxed{\forall\epsilon>0:\ \ \mathbb{P}\left[\limsup_{n\to\infty}\left|X_n-X\right|>\epsilon\right]=0} \tag{30}$$

Oder kurz

$$X_n \overset{n\to\infty}{\longrightarrow} X \ \ \mathbb{P}-\text{f.s.}$$

Let $\{X_r :\geq 1\}$ be independent Poisson variables with respective parameters $\lambda_r : r\geq 1$. Show that $\sum_{r=1}^\infty X_r$ converges or diverges almost surely according as $\sum_{r=1}^\infty \lambda_r$

The partial sum $S_n=\sum_{r=1}^n X_r$ is Poisson-distributed with parameters $\sigma_n=\sum_{r=1}^n\lambda_r$. For fixed $x$, the event $\{S_n\leq x\}$ is decreasing in $n$, whence by Lemma 1.3.5, if $\sigma_n\to\sigma<\infty$ and $x$ is non-negative integer.

$$\mathbb{P}\left[\sum_{r=1}^\infty X_r\leq x\right] = \lim_{n\to\infty}\mathbb{P}\left[S_n\leq x\right] = \sum_{j=0}^x \frac{e^{-\sigma}\sigma^j}{j!}$$

Hence if $\sigma < \infty$, $\sum_{r=1}^{\infty} X_r$ converges to a Poisson random variable. On the other hand, if $\sigma_n \to \infty$ then $e^{-\sigma_n} \sum_{j=0}^{x} \frac{\sigma_n^j}{j!} \to 0$, giving that $\mathbb{P}\left[\sum_{r=1}^{\infty} X_r > x\right] = 1$ for all $x$, and therefore the sum diverges with probability 1, as required.

### 8.1.1 Kolmogorov Convergence Criterion

If
$$\sum_{i=1}^{\infty} \mathbb{V}\left[X_i\right] < \infty$$
then
$$\sum_{i=1}^{\infty} \left(X_i - \mathbb{E}\left[X_i\right]\right)$$
converges almost surely.

## 8.2 Convergence in Probability

$X_n \xrightarrow{P} X$ if for $\forall\, \epsilon > 0$

$$\boxed{\lim_{n\to\infty} \mathbb{P}\left[\left|X_n - X\right| \geq \epsilon\right] = 0} \tag{31}$$

Sei $(X_n)_{n\in\mathbb{N}}$ eine Folge unabhängiger Zufallsvariablen, welche Exp(1)-verteilt sind.
Zeigen Sie, dass $n^{\alpha} \cdot \min_{k\leq n} X_k$ stochastisch gegen Null konvergiert für alle $\alpha < 1, n \in \mathbb{N}$.

$$\begin{aligned}
\forall \epsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}\left[\left|n^{\alpha} \min_{k\leq n} X_k\right| \geq \epsilon\right] = 0 \iff n^{\alpha} \min_{k\leq n} X_k \xrightarrow{\mathbb{P}} 0 \\
&= \lim_{n\to\infty} \mathbb{P}\left[\min_{k\leq n} X_k \geq \frac{\epsilon}{n^{\alpha}}\right] \\
&= \lim_{n\to\infty} \mathbb{P}\left[\bigcap_{1\leq k\leq n} \{\omega : X_k(\omega)\} \geq \frac{\epsilon}{n^{\alpha}}\right] \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \mathbb{P}\left[X_k \geq \frac{\epsilon}{n^{\alpha}}\right] \\
&\overset{\text{iid}}{=} \lim_{n\to\infty} \left(\mathbb{P}\left[X_1 \geq \frac{\epsilon}{n^{\alpha}}\right]\right)^n \\
&\overset{\text{Exp}(1)}{=} \lim_{n\to\infty} \left(e^{-\frac{\epsilon}{n^{\alpha}}}\right)^n = 0
\end{aligned}$$

## 8.3 $L_p$ Convergence

$X \in L_p$ means $\mathbb{E}\left[|X|^p\right] < \infty$. A sequence $\{X_n\}$ of random variables converges in $L_p$ to $X$, written
$$X_n \xrightarrow{L_p} X$$
if
$$\boxed{\mathbb{E}\left[\left|X_n - X\right|^p\right] \to 0} \tag{32}$$
as $n \to \infty$.
It follows that if $X_n \xrightarrow{L_p} X$ then $\mathbb{E}\left[\left|X_n^p\right|\right] \to \mathbb{E}\left[\left|X^p\right|\right]$

Suppose $\{X_n\}$ is an iid sequence of random variables with $\mathbb{E}\left[X_n\right] = \mu$ and $\mathbb{V}\left[X_n\right] = \sigma^2$. Then
$$\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n} \xrightarrow{L_2} \mu,$$

since
$$\begin{aligned}
\left(\mathbb{E}\left[\frac{S_n}{n} - \mu\right]\right)^2 &= \frac{1}{n^2}\left(\mathbb{E}\left[S_n - n\mu\right]\right)^2 \\
&= \frac{1}{n^2} \mathbb{V}\left[S_n\right] \\
&= \frac{n\sigma^2}{n^2} \to 0.
\end{aligned}$$

Suppose $X_n \xrightarrow{L_1} X$. Show that $\mathbb{E}\left[X_n\right] \to \mathbb{E}\left[X\right]$. Is the converse true?

We have that
$$\left|\mathbb{E}\left[X_n\right] - \mathbb{E}\left[X\right]\right| = \left|\mathbb{E}\left[X_n - X\right]\right| \leq \mathbb{E}\left[\left|X_n - X\right|\right] \xrightarrow{n\to\infty} 0$$
The converse is clearly false. If each $X_n$ takes the values $\pm 1$, each with probability $\frac{1}{2}$, then $\mathbb{E}\left[X_n\right] = 0$, but $\mathbb{E}\left[\left|X_n - 0\right|\right] = 1$.

Zz: $X_n \xrightarrow{L_2} X \Rightarrow \mathbb{V}\left[X_n\right] \to \mathbb{V}\left[X\right]$

$\mathbb{E}\left[X_n^2\right] \to \mathbb{E}\left[X^2\right]$ and $X_n \xrightarrow{L_1} X$. Therefore $\mathbb{E}\left[X_n\right] \to \mathbb{E}\left[X\right]$. Thus $\mathbb{V}\left[X_n\right] = \mathbb{E}\left[X_n^2\right] - \mathbb{E}\left[X_n\right]^2 \to \mathbb{V}\left[X\right]$.

## 8.4 Convergence in Distribution (Weak Convergence)

$$\boxed{\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[f \circ X_n\right] = \mathbb{E}\left[f \circ X\right] &\iff \int f \circ X_n \, d\mathbb{P} \xrightarrow{n\to\infty} \int f \circ X \, d\mathbb{P} \\
&\iff \int f \, d\mathbb{P}_{X_n} \xrightarrow{n\to\infty} \int f \, d\mathbb{P}_X
\end{aligned}}$$

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Let $\{X_n, n \geq 1\}$ be iid with common unit exponential distribution
$$\mathbb{P}\left[X_n > x\right] = e^{-x}, \quad x > 0$$
Set $M_n = \vee_{i=1}^{n} X_i$ for $n \geq 1$. Then
$$M_n - \ln n \Rightarrow Y,$$

where

$$\mathbb{P}[Y \le x] = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R} \tag{35}$$

To prove ??, note that for $x \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}[M_n - \ln n \le x] &= \mathbb{P}\left[\bigcap_{i=1}^{n}(X_i \le x + \ln n)\right] \\ &= (1 - e^{-(x+\ln n)})^n \\ &= \left(1 - \frac{e^{-x}}{n}\right)^n \to \exp\left(-e^{-x}\right)\end{aligned}$$

Let $X_1, X_2, \dots, X_n$ be i.i.d. Cauchy. Show that $M_n = \max X_i$ is such that $\pi M_n / n$ converges in distribution, the limiting distribution function being given by $F(x) = e^{-1/x}$ if $x \ge 0$.

We have that

$$\mathbb{P}[M_m \le xn/\pi] = \left\{\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{xn}{\pi}\right)\right\}^n = \left\{1 - \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{xn}\right)\right\}^n$$

if $x > 0$, by elementary trigonometry. Now $\tan^{-1} y = y + o(y)$ as $y \to 0$, and therefore

$$\mathbb{P}[M_m \le xn/\pi] = \left(1 - \frac{1}{xn} + o(n^{-1})\right)^n \to e^{-1/x} \quad \text{as} \, n \to \infty$$

### 8.4.1 Extreme Value Distributions

$\{X_n, n \ge 1\}$ idd with common distribution $F$. The Extreme observation among the first $n$ is

$$M_n := \vee_{i=1}^n X_i.$$

Suppose there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) = \mathbb{P}\left[\frac{M_n - b_n}{a_n} \le x\right] \xrightarrow{D} G(x), \tag{36}$$

where the limit distribution $G$ is proper and non-degenerate. Then $G$ is the type of one of the following extreme value distributions:

1. $\Phi_\alpha(x) = \exp\left(-x^{-\alpha}\right), \;\; X > 0, \;\; \alpha > 0,$
2. $\Psi_\alpha(x) = \begin{cases}\exp\left(-(x)^\alpha\right), & x < 0, \;\; \alpha > 0 \\ 1 & x > 0,\end{cases}$
3. $\Lambda(x) = \exp\left(-e^{-x}\right), \;\; x \in \mathbb{R}$

The statistical significance is the following. The types of the three extreme value distributions can be united as a one parameter family indexed by shape parameter $\gamma \in \mathbb{R}$:

$$G_\gamma(x) = \exp\left(-(1+\gamma x)^{-1/x}\right), \;\; 1 + \gamma x > 0 \tag{37}$$

where we interpret the case of $\gamma = 0$ as

$$G_0 = \exp\left(-e^{-x}\right) \quad x \in \mathbb{R}$$

## 8.5 Implications

$$L_p - \text{Konvergenz} \;\Rightarrow\; L_q - \text{Konvergenz}\,(q \le p) \;\Rightarrow\; \text{stoc}$$

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sowie

$$\text{fast sichere Konvergenz} \;\Rightarrow\; \text{stochastische Konvergenz}$$

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$X_i$ i.i.d., $\mathbb{E}[X_i] = \mu$, $\mathbb{V}[X_i] < \infty$. Show that

$$\binom{n}{2}^{-1} \sum_{1 \le i \le j \le n} X_i X_j \xrightarrow{\mathbb{P}} \mu^2, \;\; n \to \infty$$

$$\binom{n}{2}^{-1} \sum_{1 \le i \le j \le n} X_i X_j = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n} X_i^2$$

Now $n^{-1}\sum_{i=1}^n X_i \xrightarrow{D} \mu$ by law of large numbers $\Rightarrow$ $n^{-1}\sum_{i=1}^n X_i \xrightarrow{\mathbb{P}} \mu$ (see ??). It follows that $(n^{-1}\sum_{i=1}^n X_i)^2 \xrightarrow{\mathbb{P}} \mu^2$. Since if $c_n \to c$ and $X_n \xrightarrow{\mathbb{P}} X$ then $c_n X_n \xrightarrow{\mathbb{P}} cX$. So

$$\frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 \xrightarrow{\mathbb{P}} \mu^2$$

and

$$\frac{1}{n(n-1)}\sum_{i=1}^{n} X_i^2 \xrightarrow{\mathbb{P}} 0.$$

The result follows from the fact that If $X_n \xrightarrow{\mathbb{P}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$ then $X_n + Y_n \xrightarrow{\mathbb{P}} X + Y$.

### 8.5.1 Converse Implications

(a) If $X_n \xrightarrow{D} c$, where $c$ is constant, then $X_n \xrightarrow{\mathbb{P}} c$

(b) If $X_n \xrightarrow{\mathbb{P}} X$ and $\mathbb{P}\left[|X_n| \le k\right] = 1$ for all $n$ and some $k$, then $X_n \xrightarrow{L_p} X$ for all $p \ge 1$

(c) If $\mathbb{P}\left[|X_n - X| > \epsilon\right]$ satisfies $\sum_n \mathbb{P}\left[|X_n - X| > \epsilon\right] < \infty$ for all $\epsilon > 0$, then $X_n \xrightarrow{\text{a.s.}} X$

### 8.5.2 Slutsky´s Theorem

$$X_n \xrightarrow{D} X, \; A_n \xrightarrow{\mathbb{P}} a \text{ and } B_n \xrightarrow{\mathbb{P}} b \;\Rightarrow\; A_n + B_n \cdot X_n \xrightarrow{D} a + b * X$$

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# 9 Appendix

## 9.1 Stammfunktionen

$$\int \frac{1}{x}\,dx = \ln|x| + c$$
$$\int e^x\,dx = e^x + c$$
$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$$
$$\int a^x \ln a\,dx = a^x + c$$
$$\int \ln x\,dx = x\ln x - x$$
$$\int \sin(x)\,dx = -\cos(x) + c$$
$$\int \cos(x)\,dx = \sin(x) + c$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$
$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$
$$\int xe^{-ax}\,dx = \frac{-e^{-ax}}{a^2}(ax+1)$$
$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2-2ax+2)$$
$$\int_0^\infty x^2ae^{-ax}\,dx = -x^2e^{-ax}|_0^\infty + \int_0^\infty 2xe^{-ax}\,dx = 0 + \frac{2}{a^2}$$
$$\int x^ne^{ax}\,dx = \frac{1}{a}x^ne^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$
$$\int \frac{1}{1+e^{ax}}\,dx = \frac{1}{a}\ln\frac{e^{ax}}{1+e^{ax}}$$
$$\int \frac{1}{b+ce^{ax}}\,dx = \frac{x}{b} - \frac{1}{ab}\ln\left|b+ce^{ax}\right|$$
$$\int \frac{e^{ax}}{b+ce^{ax}}\,dx = \frac{1}{ac}\ln\left|b+ce^{ax}\right|$$

### 9.1.1 Beispiele

- ??
- ??

## 9.2 Partielle Integration

$$\boxed{\int_a^b f'(x)\cdot g(x)\,dx = [f(x)\cdot g(x)]_a^b - \int_a^b f(x)\cdot g'(x)\,dx} \tag{41}$$

## 9.3 Sets and Events

### 9.3.1 De Morgan

$$\left(\bigcup_i A_i\right)^C = \bigcap_i A_i^C$$
$$\left(\bigcap_i A_i\right)^C = \bigcup_i A_i^C$$

### 9.3.2 Limits of Sets

- $\inf_{k\geq n} A_k := \bigcap_{k=n}^\infty A_k,\ \sup_{k\geq n} A_k := \bigcup_{k=n}^\infty A_k$
- $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty A_k$
- $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k$
- If $\liminf_{n\to\infty} B_n = \limsup_{n\to\infty} B_n = B$ then we say $B_n \to B$
- $\limsup_{n\to\infty} A_n = [A_n\, i.o.]$

### 9.3.3 Borel-Cantelli Lemma

Let $\{A_n\}$ be any events. If

$$\sum_n \mathbb{P}\left[A_n\right] < \infty$$

then

$$\mathbb{P}\left[A_n\, i.o.\right] = \mathbb{P}\left[\limsup_{n\to\infty} A_n\right] = 0$$

Let $X_n \sim \mathrm{Exp}(1)$

$$\mathbb{Z}: \quad \mathbb{P}\left[\limsup_{n\to\infty}\frac{X_n}{\log n} = 1\right] = 1$$

Evidently

$$\mathbb{P}\left[\frac{X_n}{\log n} \geq 1+\epsilon\right] = \frac{1}{n^{1+\epsilon}}, \quad \text{for } |\epsilon| \leq 1$$

By the Borel-Cantelli lemmas, the events $A_n = \{X_n/\log n \geq 1+\epsilon\}$ occur a.s. infinitely often for $-1 < \epsilon \leq 0$, and a.s. only finitely often for $\epsilon > 0$.

### 9.3.4 Borel Zero-One Law

If $\{A_n\}$ is a sequence of independent events, then

$$\mathbb{P}\left[A_n\, i.o.\right] = \begin{cases} 0, & \mathit{iff} \sum_n \mathbb{P}\left[A_n\right] < \infty \\ 1, & \mathit{iff} \sum_n \mathbb{P}\left[A_n\right] = \infty \end{cases}$$

## 9.4 Inequalities

### 9.4.1 Markov

$$\boxed{\mathbb{P}\left[|X| \geq \lambda\right] \leq \frac{\mathbb{E}\left(|X|\right)}{\lambda}} \tag{42}$$

### 9.4.2 Chebychev

$$\boxed{\mathbb{P}\left[|X - \mathbb{E}(X)| \geq \lambda\right] \leq \frac{\mathbb{V}\left[X\right]}{\lambda^2}} \tag{43}$$

### 9.4.3 Kolmogorov

$$\boxed{\mathbb{P}\left[\max_{1\leq k\leq n} \left|X_k\right| \geq \lambda\right] \leq \frac{\mathbb{V}(X_n)}{\lambda^2} = \frac{1}{\lambda^2} \sum_{k=1}^{n} \mathbb{V}\left[X_k\right]} \tag{44}$$

### 9.4.4 Schwartz

$X, Y \in L_2$ then

$$\boxed{\left|\mathbb{E}\left[XY\right]\right| \leq \mathbb{E}\left[|XY|\right] \leq \sqrt{\mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]}} \tag{45}$$

### 9.4.5 Hölder

Suppose $p, q$ satisfy

$$p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$$

and that

$$\mathbb{E}\left[|X|^p\right] < \infty,\ \mathbb{E}\left[|X|^q\right] < \infty$$

then

$$\boxed{\left|\mathbb{E}\left[XY\right]\right| \leq \mathbb{E}\left[|XY|\right] \leq \left(\mathbb{E}\left[|X|^p\right]\right)^{1/p} \left(\mathbb{E}\left[|Y|^q\right]\right)^{1/q}} \tag{46}$$

### 9.4.6 Minkowski

For $1 \leq p < \infty$, assume $X, Y \in L_p$. Then $X + Y \in L_p$ and

$$\boxed{\|X+Y\|_p \leq \|X\|_p + \|Y\|_p} \tag{47}$$

### 9.4.7 Jensen

Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is convex and $\mathbb{E}\left[|X|\right] < \infty$ and $\mathbb{E}\left[|f(X)|\right] < \infty$. Then

$$\boxed{\mathbb{E}\left[f(X)\right] \geq f(\mathbb{E}\left[X\right])} \tag{48}$$

A special case is

$$\boxed{\mathbb{E}\left[X^2\right] \geq (\mathbb{E}\left[X\right])^2} \tag{49}$$

If $f$ is concave, the inequality reverses.

## 9.5 Stochastics

### 9.5.1 Law of Large Numbers

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu} \tag{50}$$

### 9.5.2 Central Limit Theorem

$$\boxed{\mathbb{P}\left[\frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}} \leq x\right] \to N(x) := \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du} \tag{51}$$

## 9.6 Extema and Order Statistics

### 9.6.1 Minima

Seien $X_1, X_2, \ldots$ iid auf $[0,1]$ Gleichverteilt. Gegen welche Verteilung konvergiert $n \cdot \min_{1\leq k\leq n} X_k$ schwach?

$$
\begin{aligned}
\mathbb{P}\left[n\cdot \min_{1\le k\le n} < c\right] &= 1-\mathbb{P}\left[n\cdot \min_{1\le k\le n} \ge c\right]\\
&= 1-\mathbb{P}\left[\bigcap_{1\le k\le n}\left\{\omega : X_k(\omega)\ge \frac{c}{n}\right\}\right]\\
&= 1-\left(\mathbb{P}\left[X\ge \frac{c}{n}\right]\right)^n\\
&= 1-\left(\int \mathbb{1}_{x\ge \frac{c}{n}}(x)\cdot \frac{1}{1-0}\,dx\right)^n\\
&= 1-\left(\int_{\frac{c}{n}}^{1} dx\right)^n\\
&= 1-\left(1-\frac{c}{n}\right)^n\\
\xrightarrow[n\to\infty]{\lim}\ 1-e^c
\end{aligned}
$$

Konvergiert gegen ZV die Exp(1) verteilt ist.

### 9.6.2 Maxima

$$
\begin{aligned}
\mathbb{P}\left[\max_{1\le k\le n} X_k < c\right] &= \mathbb{P}\left[\bigcap_{1\le k\le n}\left\{\omega : X_k(\omega) < c\right\}\right]\\
&= \prod_{k=1}^{n}\mathbb{P}\left[X_k < c\right]\\
&= \left(\mathbb{P}\left[X_1 < c\right]\right)^n
\end{aligned}
$$