Maß- und Wahrscheinlichkeitstheorie Übersicht

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2 Erstes Kapitel

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3 Ungelöste Fragen

3.1 WS11/12 Februar

3.1.1 Aufgabe 1

Zeigen Sie, dass $P(\mathbb{N})$ die kleinste σ-Algebra auf der Menge $\mathbb{N}$ der natürlichen Zahlen ist, die von allen endlichen Teilmengen von natürlichen Zahlen erzeugt ist.

Sei $A_i \in \mathbb{N}$ die Menge aller endlichen Teilmengen von $\mathbb{N}$ mit $i \in \mathbb{N}$ Elementen, dann ist $\bigcup_{i=0}^{\infty} A_i$ die Menge aller endlichen Teilmengen von $\mathbb{N}$. Sei $E := A$ und $A_i^C = \mathbb{N} \setminus A_i$.

$$\sigma(E) = \{\Omega, \emptyset, A, A^C\} = P(\mathbb{N})$$

(i) $\Omega \in P(\mathbb{N})$

(ii) $A \in P(\mathbb{N}) \Rightarrow A^C \in P(\mathbb{N})$

(iii) $(A_i)_{i\in\mathbb{N}} \subset P(\mathbb{N}) \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in P(\mathbb{N})$

$\Rightarrow \sigma(E) = P(\mathbb{N} = \Omega)$ ist σ-Algebra (trivial da $P(\mathbb{N})$ per Definition eine σ-Algebra auf Ω ist).

Ist $\sigma(E)$ aber auch die kleinste σ-Algebra die E enthält?

Satz 2.11 aus Skript: $\sigma(E)$ von E erzeugte σ-Algebra
$\Rightarrow \sigma(E)$ ist kleinste σ-Algebra die E enthält.
$\Rightarrow \sigma(E) = P(\mathbb{N})$ ist kleinste σ-Algebra die von allen endlichen Teilmengen von $\mathbb{N}$ erzeugt wird.

3.2 WS11/12 April alle

3.3 One Thousand Exercises in Probability

- 7.9.5

4 Sigma-Fields

4.1 Definition

1. $\Omega \in A$
2. $A \in A \Rightarrow A^C \in A$
3. $(A_n) \subset A \Rightarrow \cup A_n \in A$

The countable/co-countable σ-field. Let $\Omega = \mathbb{R}$
$\mathbb{Z}_C$: $B = \{A \subset \mathbb{R} : A$ is countable$\} \cup \{A \subset \mathbb{R} :$ A^C is countable$\}$ is a σ-field

(M1) $\Omega \in B$ (since $\Omega^C = \emptyset$ is countable)

(M2) $A \in B$ implies $A^C \in B$

(M3) $A_i \in B$ implies $\bigcap_{i=1}^{\infty} A_i \in B$

4.2 Intersections of Sigma-Algebras

Man Beweise: Sei Ω eine Menge, sei I eine Indexmenge und für jedes $i \in I$ sei A_i eine σ-Algebra auf Ω. Dann ist auch

$$\cap A_i := \{A \subset \Omega | A \in A_i \forall_i \in I\}$$

eine σ-Algebra auf Ω.

1. $\Omega \in A_i \forall_i \in I \Rightarrow \Omega \in \cap A_i$
2. $A \in \cap A_i \Rightarrow A \in A_i \forall i \in I \Rightarrow A^C \in \cap A_i$
3. $A_n \in \cap A_i \forall n \in \mathbb{N} \Rightarrow A_n \in A_i \forall_{i,n} \Rightarrow \cup A_n \in A_i \Rightarrow \cup A_n \in \cap A_i$

$\Rightarrow \cap A_i$ ist σ-Algebra

4.3 Minimal Sigma-Algebras

Let C be a collection of subsets of Ω. The σ-field generated by C, denoted $\sigma(C)$, is a *minimal* σ-field satisfying

(a) $\sigma(C) \supset C$

(b) If B' is some other σ-field containing C, then $B' \supset \sigma(C)$

Given a class C of subsets of Ω, there is a unique minimal σ-field containing C.

Proof: Let

$$\aleph = \{B : B \text{ is a } \sigma - \text{field}, B \supset C\}$$

be the set of all σ-fields containing C. Then $\aleph \neq \emptyset$ since $P(\Omega) \in \aleph$. Let

$$B^{\daleth} = \bigcap_{B \in \aleph} B.$$

Since each class $B \in \aleph$ is a σ-field, so is $B^{\daleth}$. Since $B \in \aleph$ implies $B \supset C$, we have $B^{\daleth} \supset C$. We claim $B^{\daleth} = \sigma(C)$. We checked $B^{\daleth} \supset C$ and, for minimality, note that if B' is a σ-field such that $B' \supset C$, then $B' \in \aleph$ and hence $B^{\daleth} \subset B'$.

Let $\Omega = \{1, 2, \ldots, 7\}$ and $E = \{\{1,2\},\{6\}\}$ then

$$\sigma(E) = \{\emptyset, \{1,2\}, \{3,4,5,6,7\}, \{6\}, \{1,2,3,4,5,7\}, \{1,2,6\}, \{3,4,5,7\}, \Omega\}$$

Let Ω be set and $A \subset \Omega$. If $E = \{A\}$ then

$$\sigma(E) = \{\emptyset, A, A^C, \Omega\}$$

4.4 Inverse Maps

If B' is a σ-field of subsets of Ω', then $X^{-1}(B')$ is a σ-field of subsets of Ω

Proof:

(M1) Since $\Omega' \in B'$, we have

$$X^{-1}(\Omega') = \Omega \in X^{-1}(B')$$

(M2) If $A' \in B'$, then $(A')^C \in B'$, and so if $X^{-1}(A') \in X^{-1}(B')$ we have

$$X^{-1}((A')^C) = (X^{-1}(A'))^C \in X^{-1}(B')$$

(M3) If $X^{-1}(B'_n) \in X^{-1}(B')$ then since $\bigcup_n B'_n \in B'$

$$\bigcup_n X^{-1}(B'_n) = X^{-1}\left(\bigcup_n B'_n\right) \in X^{-1}(B')$$

If C' is a class of subsets of Ω' then

$$X^{-1}(\sigma(C')) = \sigma(X^{-1}(C'))$$

$\mathbb{Z}\mathbb{Z}$: $f(A_1) : \{B \subset A_2 : f^{-1}(B) \in A_1\}$ σ-Algebra auf Ω_2

(M1) $\emptyset \in f(A_1) \Rightarrow \Omega_2 = \emptyset^C \in f(A_1)$

(M2) Sei $B \in f(A_2)$
$f^{-1}(B) \in A_1 \Rightarrow (f^{-1}(B_i))^C \in A_1 \Rightarrow f^{-1}(B^C) \in A_1 \Rightarrow B^C \in f(A_1)$

(M3) Sei $B_i \in f(A_1)$
$f^{-1}(B_i) \in A_1 \Rightarrow \bigcup_{i \in \mathbb{N}} f^{-1}(B_i) \in A_1 \Rightarrow f^{-1}(\bigcup_{i \in \mathbb{N}}) \in A_1 \Rightarrow \bigcup_{i \in \mathbb{N}} B_i \in f(A_1)$

5 Measures

Let A be a σ-field on Ω. μ is a measure if

$$\mu : A \to [0, \infty]$$

such that

(M1) $\mu(\emptyset) = 0$

(M2) For disjoint A_n

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

5.1 Probability Measures

5.1.1 Definition

(M1) $\mathbb{P}(A) \geq 0 \,\forall\, A \in B$

(M2) $\mathbb{P}$ is σ-additive for disjoint Events A_n

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

(M3) $\mathbb{P}(\Omega) = 1$

5.2 Measurability

- Seien $(\Omega_1, A_1), (\Omega_2, A_2)$ zwei Messräume. X ist $A_1 - A_2$-mb. falls

$$\boxed{X^{-1}(A) = \{\omega : X(\omega) \in A\} \in A_1 \forall A \in A_2}$$

- Das **Urbild** $X^{-1}(A_2) := \{X^{-1}(A), A \in A_2\}$ ist kleinste σ-Algebra bzgl. derer X mb. ist ($\sigma(X) := X^{-1}(A_2)$)
- Sei E ein **Erzeuger** von A_2, dann ist X $A_1 - A_2$-mb. falls $X^{-1}(E) \in A_1 \,\forall\, E \in E$

5.3 Image Measure

Sei (Ω, A, μ) ein Maßraum, (Ω', A') ein Messraum und

$$T : (\Omega, A) \to (\Omega', A')$$

Das durch

$$\mu'(A') = \mu(T^{-1}(A')) \,\forall A' \in A'$$

definierte Maß μ' auf (Ω', A') heißt **Bildmaß** von μ unter T.

Sei (Ω, A, μ) der Maßraum mit $\Omega := \mathbb{R}$ und der von allen abzählbaren Mengen erzeugten σ-Algebra A, sowie $\mu(A) = 0$ wenn A abzählbar ist und $\mu(A) = 1$ wenn A^C abzählbar ist.

Für $\Omega' := \{0, 1\}$ und $A' := P(\Omega')$ wird die Abbildung $T : \Omega \to \Omega'$ definiert durch

$$T(\omega) := \begin{cases} 0, & \text{falls } \omega \text{ rational} \\ 1, & \text{falls } \omega \text{ irrational} \end{cases}$$

Man zeige, dass T $A \to A'$-messbar ist, und bestimmte das Bildmaß $T(\mu)$.

Antwort: T ist messbar $\Leftrightarrow T^{-1}(A') \in A \forall A' \in A$
$\Omega' = \{0, 1\}$ $A' = \mathbb{P}(\Omega') = \{\varnothing, \{0, 1\}, \{0\}, \{1\}\}$

$A' \subset A'$	$\varnothing$	0	1	$\{0,1\}$
$T^{-1}(A')$	$\varnothing$	$\mathbb{Q}$	$\mathbb{R}\setminus\mathbb{Q}$	$\Omega = \mathbb{R}$

$\Rightarrow T$ $A - A'$-mb

Bildmaß?

$\mu(T^{-1}(\varnothing)) = \mu(\varnothing) = 0$
$\mu(T^{-1}(0)) = \mu(\mathbb{Q}) = 0$
$\mu(T^{-1}(1)) = \mu(\mathbb{R} \setminus \mathbb{Q}) = 1$
$\mu(T^{-1}(\{0, 1\})) = \mu(\mathbb{R}) = 1$

6 Integration and Expectation

6.1 Expectation

$$\boxed{\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P} = \int_\mathbb{R} x f(x)\, dx} \tag{1}$$

$$\mathbb{E}[h(X)] = \int_\mathbb{R} h(x)\, \mathbb{P}_X \, dx = \begin{cases} \int_\mathbb{R} h(x) f(x)\, dx & \text{im abs. stetigen Fall} \\ \sum_{k=1}^{\infty} h(x_k)\, \mathbb{P}\left[X = x_k\right] & \text{im diskreten Fall} \end{cases} \tag{2}$$

Erwartungswert von e^x bei Normalverteilung

$X \sim N(0, 1)$, $\mathbb{E}\left[e^x\right]$?

$$\begin{aligned}
\mathbb{E}\left[e^x\right] &= \int_\Omega e^x \, d\mathbb{P} \\
&= \int_\mathbb{R} e^t \mathbb{P}_X \, dt \\
&= \int_\mathbb{R} e^t \, d\lambda(t) \\
&= \int_\mathbb{R} e^t \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \\
&= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}+t} \, dt \\
&= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} * e^{\frac{-t^2+2t+1-1}{2}} \, dt \\
&= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-(t^2-2t-1+1)}{2}} \, dt \\
&= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-((t-1)^2-1)}{2}} \, dt \\
&= \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-(t-1)^2}{2}+\frac{1}{2}} \, dt \\
&= e^{\frac{1}{2}} \int_\mathbb{R} \frac{1}{\sqrt{2\pi}} e^{\frac{-(t-1)^2}{2}} \, dt \quad \sim N(1,1) = \text{Dichte} \\
&= e^{\frac{1}{2}}
\end{aligned}$$

Varianz von Exponentialverteilter Zufallsvariable

$X \sim \mathrm{Exp}(\lambda)$, $\mathbb{V}[X]$?

$$\mathbb{E}[X] = \int_0^\infty t\lambda e^{-\lambda t}\,dt \overset{PI}{=} -e^{-\lambda t}t!|_0^\infty - \int_0^\infty 1(-e^{-\lambda t})\,dt = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$$

$$\begin{aligned}
\mathbb{V}[X] = \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] &= \int_0^\infty (t-\frac{1}{\lambda})^2\lambda e^{-\lambda t}\,dt \\
&= \int_0^\infty t^2\lambda e^{-\lambda t}\,dt - \frac{2}{\lambda}\int_0^\infty t\lambda e^{-\lambda t}\,dt + \frac{1}{\lambda^2}\int_0^\infty \lambda e^{-\lambda t}\,dt \\
&\overset{PI}{=} -t^2 e^{-\lambda t}|_0^\infty - \int_0^\infty 2te^{-\lambda t}\,dt - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} \\
&= 0 + \frac{2}{\lambda^2} - \frac{2}{\lambda^2} + \frac{1}{\lambda^2} = \frac{1}{\lambda^2}
\end{aligned}$$

6.2 Probability

$$\boxed{\mathbb{P}[A] = \int_A d\mathbb{P} = \mathbb{E}[\mathbb{1}_A]} \tag{3}$$

6.3 Distribution Function

$$\boxed{F(x) = \mathbb{P}[(-\infty, x]] = \mathbb{P}[X \le x], \quad x \in \mathbb{R}} \tag{4}$$

6.4 Monotone Convergence

If

$$X_n \uparrow X$$

then

$$\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$$

and

$$\mathbb{E}\left[\sum_{i=1}^\infty X_i\right] = \sum_{i=1}^\infty \mathbb{E}[X_i]$$

6.5 Dominated Convergence Theorem

If

$$X_n \to X$$

and there exists $Z \in L_1$ such that

$$|X_n| \le Z$$

then

$$\boxed{\mathbb{E}[X_n] \to \mathbb{E}[X] \; and \; \mathbb{E}\left[|X_n - X|\right] \to 0} \tag{5}$$

6.6 Integrable Random Variables

Define $\mathbb{E}[X] := \mathbb{E}[X^+] - \mathbb{E}[X^-]$. The set of integrable random variables is denoted by L_1:

$$\boxed{L_1 = \{\text{random variables } X \,:\, \mathbb{E}\left[|X|\right] < \infty\}} \tag{6}$$

6.7 Properties of Expectation

1. If X is integrable, then
$$\mathbb{P}[X = \pm\infty] = 0$$
2. If $\mathbb{E}[X]$ exists,
$$\mathbb{E}[cX] = c\,\mathbb{E}[X]$$
3. If $X \ge 0$ then $\mathbb{E}[X] \ge 0$ since $X = X^+$. If $X, Y \in L_1$, and $X \le Y$ then
$$\mathbb{E}[X] \le \mathbb{E}[Y]$$
4. Suppose $\{X_n\}$ is a sequence of random variables such that $X_n \in L_1$ for some n. If either
$$X_n \uparrow X$$
or
$$X_n \downarrow X$$
then
$$\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$$
or
$$\mathbb{E}[X_n] \downarrow \mathbb{E}[X]$$
5. If $X \in L_1$,
$$\left|\mathbb{E}[X]\right| \le \mathbb{E}\left[|X|\right]$$
6. Variance and Covariance. If $X \in L_2$ then

$$\mathbb{V}[X] := \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \tag{7}$$

$$\mathrm{Cov}(X,Y) := \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] \tag{8}$$

$$\mathbb{V}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{V}[X_i] + \sum_{i=1}^n \mathrm{Cov}(X_i, X_j) \tag{9}$$

6.8 Fatou´s Lemma

If there exists $Z \in L_1$ and $X_n \ge Z$ then

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \le \liminf_{n\to\infty} \mathbb{E}[X_n] \tag{10}$$

and if $X_n \le Z$ then

$$\limsup_{n\to\infty} \mathbb{E}[X_n] \le \mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \tag{11}$$

6.9 Fubini Theorem

Let $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$ be a product measure. If X is $B_1 \times B_2$ measurable and integrable with respect to $\mathbb{P}$ then

$$\int_{\Omega_1\times\Omega_2} X\,d\mathbb{P} = \int_{\Omega_1}\int_{\Omega_2} X\,d\mathbb{P}_2\,d\mathbb{P}_1 \tag{12}$$
$$= \int_{\Omega_2}\int_{\Omega_1} X\,d\mathbb{P}_1\,d\mathbb{P}_2 \tag{13}$$

6.10 Tonelli

$$\int_{\times_{i=1}^n \Omega_i} f(\omega_1,\dots,\omega_n)\,d\otimes_{i=1}^n \mu_i(\omega_1,\dots,\omega_n) = \int_{\Omega_1}\int_{\Omega_2}\cdots\int_{\Omega_n} f(\omega_1,\dots,\omega_n)\,\mu_n(d\omega_n)\dots\mu_1(d\omega_1)$$

6.11 Radon-Nikodym

Sei (Ω, A) ein Messraum, seien μ und ν zwei Maße auf (Ω, A) so dass

$$d\nu = f\,d\mu$$

für eine A-mb Funktion

$$f : \Omega \to \mathbb{R} \text{ mit } f(w) \geq 0\ \forall \omega \in \Omega$$

Dann heisst f **Dichte** oder Dichtefunktion von ν bzgl. μ.

Seien μ und ν Maße auf dem Maßraum (Ω, A), so dass für jedes $A \in A$ gilt

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

Dann sagt man ν ist absolut stetig bzgl. μ. Notation:

$$\nu \ll \mu$$

Radon-Nikodym: Seien μ und ν σ-endliche Maße auf dem Messraum (Ω, A). Dann sind folgende Aussagen äquivalent:

(i) ν besitzt eine Dichte bzgl. μ

(ii) $\nu \ll \mu$

Beispiel Normalverteilung

$$dN(\mu,\sigma^2) = f_{\mu,\sigma^2}\,d\lambda \tag{14}$$

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{32} & 0 \leq t < 1 \\ \frac{t^2}{16} & 1 \leq t < 2 \\ \frac{t}{8} + \frac{1}{4} & 2 \leq t < 4 \\ 1 & t \geq 4 \end{cases}$$

ZZ: Dichte bzgl. $\lambda + \delta_0 + \delta_1 + \delta_2 + \delta_4$

Diskreter Teil: Unstetigkeitsstellen
$\mathbb{P}[x_i] \geq 0\ i = 1,2,3\ \ \alpha_i = \mathbb{P}[x_i],\ x_1 = 1, x_2 = 2, x_3 = 4$

Absolut stetiger Teil: $F(t)$ abs. stetig auf $\mathbb{R} \setminus \{1,2,4\}$
d.h. $\mathbb{P}(B) = \int_B d\mathbb{P} = \int_B f(x)\,d\lambda\ \forall B \in B, \{1,2,4\} \notin B$
$\mathbb{P}(B) = \mathbb{E}(\mathbb{1}_B) = \int \mathbb{1}_B\,d\mathbb{P} = \int_B d\mathbb{P}$
$F(t) = \int_{-\infty}^t f(t)\,dt \Rightarrow F'(t) = f(t)$
$\Rightarrow F'(t) = f(t) = \frac{1}{32}\mathbb{1}(0 < t < 1) + \frac{1}{8}t\mathbb{1}(1 < t < 2) + \frac{1}{8}\mathbb{1}(2 < f < 4)$

$$\Rightarrow \hat{f}(t) = \begin{cases} f(t) & \forall t \in \mathbb{R}\setminus\{1,2,4\} \\ \alpha_j & \forall t = x_j,\ j = 1,2,3 \end{cases}$$

$$\Rightarrow \hat{\mathbb{P}} \ll \mu$$

6.12 Transformationssatz für Dichten

Sei $f : \mathbb{R}^p \to \mathbb{R}$, $(x_1,\dots,x_p) \mapsto f(x_1,\dots,x_p)$ die λ^p-Dichte eines Wahrscheinlichkeitsmaßes $\mathbb{P}_X$. Seien $G, G' \in B^{\otimes p}$ offen und die Abbildung

$$T : G \to G' \tag{15}$$
$$(x_1,\dots,x_p) \mapsto \left(T_1(x_1,\dots,x_p),\dots,T_p(x_1,\dots,x_p)\right) \tag{16}$$

bijektiv und samt T^{-1} messbar und differenzierbar.
Dann gilt für die λ^p-Dichte g von $T(\mathbb{P}_X)$:

$$\boxed{\begin{aligned} g(y_1,\dots,y_p) &= \left|\det J_{T^{-1}}(y_1,\dots,y_p)\right| \cdot f\left(T^{-1}(y_1,\dots,y_p)\right) \\ &= \left|\det J_T\left(T^{-1}(y_1,\dots,y_p)\right)\right| \cdot f\left(T^{-1}(y_1,\dots,y_p)\right) \end{aligned}}$$

(17)

(18)

Im **eindimensionalen** Fall vereinfacht sich die Dichtetransformationsformel zu

$$\boxed{g(y) = \left|(T^{-1})'(y)\right| \cdot f\left(T^{-1}(y)\right)} \tag{19}$$

Sei $X \sim \text{Exp}$ mit der Dichte $f(x) = \lambda e^{-\lambda x}\mathbb{1}_{(0,\infty)}(x)$.

Die Abbildung

$$T : x \mapsto x^2$$

ist bijektiv mit Umkehrfunktion

$$y \mapsto \sqrt{y}$$

Mit Ableitung

$$\frac{dT^{-1}(y)}{dy} = \frac{1}{2}y^{-\frac{1}{2}}$$

Dann ist

$$g(y) = \left|\frac{1}{2}y^{-\frac{1}{2}}\right| \cdot f(\sqrt{x}) = \frac{1}{2}y^{-\frac{1}{2}} \cdot \lambda e^{-\lambda\sqrt{y}}$$

für $y > 0$.

6.13 Convolutions

The Convolution $f = f_1 * f_2$ of two densities f_1 and f_2 is defined by

$$\boxed{f(z) = \int_{-\infty}^{+\infty} f_1(z-y) f_2(y)\, dy} \tag{20}$$

7 Conditional Expectation

$$\mathbb{E}[Y \mid X] = \int y \cdot f_{Y \mid X}(y \mid x) dy = \int y \cdot \frac{f_{Y,X}(y,x)}{f_X(x)} dy = \int y \cdot \frac{f_{Y,X}(y,x)}{\int f_{Y,X}(y,x) dy} dy \tag{21}$$

$$\mathbb{E}[X \mid B] = \frac{1}{\mathbb{P}[B]} \int_B X \, d\mathbb{P} = \frac{\mathbb{E}[X \cdot \mathbb{1}_B]}{\mathbb{P}(B)} \tag{22}$$

$$\mathbb{E}\left[\psi(Y,X) \mid X = x\right] = \int_{\Omega_2} \int_{\Omega_1} \psi(y,x) \, \mathbb{P}^{Y \mid X = x} dy \, \mathbb{P}^X dx \tag{23}$$

7.1 Properties of Conditional Expectation

Sei $(\Omega, A, \mathbb{P})$ ein Wahrscheinlichkeitsraum und Seien

$$f : \Omega \to \mathbb{R}, \quad f_1 : \Omega \to \mathbb{R}, \quad f_2 : \Omega \to \mathbb{R}$$

bzgl. $\mathbb{P}$ integrierbare Funktionen. Sei C eine Unter-σ-Algebra von A.
Dann gilt:

1. $\mathbb{E}\left[f \mid C\right] \in L_1(\Omega, A, \mathbb{P})$
2. $\mathbb{E}\left[\mathbb{E}\left[f \mid C\right]\right] = \mathbb{E}\left[f\right]$
3. f ist C-messbar $\Rightarrow \mathbb{E}\left[f \mid C\right] = f$ $\mathbb{P}$-f.s.
4. $f = g \mathbb{P}$-f.s. $\Rightarrow$ $\mathbb{E}\left[f \mid C\right] = \mathbb{E}\left[g \mid C\right]$ $\mathbb{P}$-f.s.
5. $f = \text{const} = \alpha$ $\Rightarrow$ $\mathbb{E}\left[f \mid C\right] = \alpha \mathbb{P}$-f.s.
6. Wenn X_i iid sind, dann ist
$$\mathbb{E}\left[X \mid \sum_{i=1}^n X_i\right] = \frac{\sum_{i=1}^n X_i}{n}$$
also z.b. $X, Y \sim \text{Exp}(\lambda)$, dann ist
$$\mathbb{E}[X \mid X+Y] \overset{iid}{=} \frac{X+Y}{2}$$
7. Für $\alpha_1, \alpha_2 \in \mathbb{R}$ ist $\mathbb{E}\left[\alpha_1 f_1 + \alpha_2 f_2 \mid C\right] = \alpha_1 \mathbb{E}\left[f_1 \mid C\right] + \alpha_2 \mathbb{E}\left[f_2 \mid C\right]$
8. $f_1 \leq f_2 \mathbb{P}$-f.s. $\Rightarrow$ $\mathbb{E}\left[f_1 \mid C\right] \leq \mathbb{E}\left[f_2 \mid C\right]$
9. $\left|\mathbb{E}\left[f \mid C\right]\right| \leq \mathbb{E}\left[\left|f\right| \mid C\right]$
10. **Monotone convergence**. If $X \in L_1$, $0 \leq X_n \uparrow X$, then
$$\mathbb{E}\left[X_n \mid C\right] \uparrow \mathbb{E}\left[X \mid C\right]$$
11. Monotone convergence implies the **Fatou Lemma**. If $0 \leq X_n \in L_1$, then
$$\mathbb{E}\left[\liminf_{n\to\infty} X_n \mid C\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[X_n \mid C\right]$$
and while if $X_n \leq Z \in L_1$, then
$$\mathbb{E}\left[\limsup_{n\to\infty} X_n \mid C\right] \geq \limsup_{n\to\infty} \mathbb{E}\left[X_n \mid C\right]$$
12. Fatou implies **dominated convergence**. If $X_n \in L_1$, $\left|X_n\right| \leq Z \in L_1$ and $X_n \to X_\infty$, then
$$\mathbb{E}\left[\lim_{n\to\infty} X_n \mid C\right] \overset{a.s.}{=} \lim_{n\to\infty} \mathbb{E}\left[X_n \mid C\right]$$

7.2 Glättungseigenschaften

7.3 Bedingte Dichten

$$\boxed{f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}} \tag{24}$$

7.4 Bedingte Wahrscheinlichkeiten

$$\begin{aligned}
\mathbb{P}[A] &= \int_A d\mathbb{P} = \mathbb{E}\left[\mathbb{1}_A\right] \\
\mathbb{P}[A \mid C] &= \mathbb{E}\left[\mathbb{1}_A \mid C\right] \\
\mathbb{P}[A \mid T] &= \mathbb{E}\left[\mathbb{1}_A \mid T\right] \\
\mathbb{P}[A \mid T = t] &= \mathbb{E}\left[\mathbb{1}_A \mid T = t\right] \\
\mathbb{P}[X \in A \mid T = t] &= \int_A f_{X \mid Y}(x \mid y) \, dx
\end{aligned}$$

7.5 Examples

Let X and Y be jointly continious random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}[X+Y \mid X<Y]$:

$$\begin{aligned}\mathbb{P}[X<Y] &= \int_{-\infty}^{\infty}\int_{x}^{\infty}\left(f_{X,Y}(x,y)\right)dy\,dx\\ &= \int_{0}^{\infty}\int_{x}^{\infty} e^{-x-y}\,dy\,dx\\ &= \int_{0}^{\infty} e^{-2x}\,dx = \frac{1}{2}\end{aligned}$$

Next,

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{(X<Y)}(X+Y)\right] &= \int_{-\infty}^{\infty}\int_{x}^{\infty}\left((x+y)f_{X,Y}(x,y)\right)dy\,dx\\ &= \int_{0}^{\infty}\int_{x}^{\infty}(x+y)e^{-x-y}\,dy\,dx\\ &= \int_{0}^{\infty}(2x+1)e^{-2x}\,dx = 1\end{aligned}$$

It follows that

$$\mathbb{E}[X+Y \mid X<Y] = \frac{\mathbb{E}\left[\mathbb{1}_{(X<Y)}(X+Y)\right]}{\mathbb{P}[X<Y]} = \frac{1}{1/2} = 2$$

X,Y haben gemeinsame Dichte $f_{X,Y}(x,y) = xe^{-x(y+1)} \cdot \mathbb{1}_{R^2}(x,y)$. Gesucht: $\mathbb{E}[Y \mid X=x]$

$$\begin{aligned}f_X(x) &= \int f_{X,Y}(x,y)\,dy\\ &= \int xe^{-x(y+1)}\cdot \mathbb{1}_{\mathbb{R}_+^2}(x,y)\,dy\\ &= \int_0^{\infty} xe^{-x(y+1)}\cdot \mathbb{1}_{\mathbb{R}_+}(x,y)\,dy\\ &= e^{-x}\underbrace{\int_0^{\infty} xe^{-xy}\cdot \mathbb{1}_{\mathbb{R}_+}(x)\,dy}_{\text{Dichte einer Exp. Vert.=1}}\\ &= e^{-x}\cdot \mathbb{1}_{R_+}(x)\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Y \mid X=x] &= \int y\cdot f_{Y\mid X}(y\mid x)\,dx\\ &= \int y\cdot \frac{f_{X,Y}(x,y)}{f_X(x)}\,dx\end{aligned}$$

Seien X,Y Zufallsvariablen mit gemeinsamer Dichte $f_{X,Y}(x,y) = x(y-x)e^{-y}$ und $0 \le x \le y < \infty$.
Geben Sie $\mathbb{E}[Y \mid X]$ an.
Tip: (Merhfache) partielle Integration

$f_{Y\mid X}(y\mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

$$\begin{aligned}\Rightarrow\quad f_X(x) &= \int_x^{\infty} f_{X,Y}(x,y)\,dy\\ &= \int_x^{\infty} x(y-x)e^{-y}\,dy\\ &= \int_x^{\infty} xye^{-y}\,dy - \int_x^{\infty} x^2e^{-y}\,dy\\ &= x[-e^{-y}(y+1)]_x^{\infty} - x^2[-e^{-y}]_x^{\infty}\\ &= x[0+e^{-x}(x+1)] - x^2[0+e^{-x}]\\ &= xe^{-x}(x+1) - x^2e^{-x}\\ &= x^2e^{-x} + xe^{-x} - x^2e^{-x}\\ &= xe^{-x}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Y \mid X] &= \int_x^{\infty} yf_{Y\mid X}(y\mid x)\,dy\\ &= \int_x^{\infty} y\frac{x(y-x)e^{-y}}{xe^{-x}}\,dy\\ &= \int_x^{\infty} y(y-x)e^{x-y}\,dy\\ &= \int_x^{\infty} y^2e^{x-y} - yxe^{x-y}\,dy\\ &= e^x\int_x^{\infty} y^2e^{-y}\,dy - xe^x\int_x^{\infty} ye^{-y}\,dy\\ &= e^x[-y^2e^{-y}|_x^{\infty} + \int_x^{\infty} 2ye^{-y}\,dy] - xe^x[-e^{-y}(y+1)]_x^{\infty}\\ &= e^x[x^2e^{-x} + 2[-e^{-y}(y+1)]_x^{\infty}] - xe^x[e^{-x}(x+1)]\\ &= e^xx^2e^{-x} + 2e^{-x}(x+1)e^x - xe^xe^{-x}(x+1)\\ &= 2+x\end{aligned}$$

8 Martingales

For integrable random variables $\{X_n, n\ge 0\}$ and σ-fields $\{B_n, n\ge 0\}$ which are sub σ-fields of B, $\{(X_n,B_n), n\ge 0\}$ is a **martingale** if

(M1) Information accumulates, i.e. $A_n \subset A_{n+1}$

(M2) X_n is adapted in the sense that for each n, $X_n \in B_n$; that, X_n is B_n-measureable.

(M3) $\mathbb{E}\left[|X_n|\right] < \infty$

(M4) $\mathbb{E}\left[X_{n+1} \mid B_n\right] \overset{a.s.}{=} X_n$

$$\boxed{\begin{array}{c}\text{Sub-Martingal} \leq \\ \text{Martingal bzgl. } (A_t)_{t\in T} :\iff \forall s \leq t: \; X_s = \mathbb{E}\left[X_t \mid A_s\right], \; \mathbb{P}-\text{f.s.} \\ \text{Super-Martingal} \geq \end{array}}$$

(25)

(26)

(27)

8.1 Properties

1. $(X_t)_{t\in T}$ sei ein Martingal bzgl. $(A_t)_{t\in T}$ mit $X_t \in L_p \, \forall \, t \in T \, (1 \leq p < \infty)$. Dann ist $\left(\left|X_t\right|^p\right)_{t\in T}$ ein Sub-Martingal bzgl. $(A_t)_{t\in T}$

2. Für jedes $c \in \mathbb{R}$ und Sub-Martingal $(X_t)_{t\in T}$ ist auch $(\max\{c, X_t\})_{t\in T}$ ein Sub-Martingal bzgl. $(A_t)_{t\in T}$. Insbesondere ist mit $c = 0$ dann auch $(X_t^+)_{t\in T}$ ein Sub-Martingal.

3. Ist $(X_t)_{t\in T}$ ein Super-Martingal bzgl. $(A_t)_{t\in T}$, so ist $(X_t^-)_{t\in T}$ ein Sub-Martingal bzgl. $(A_t)_{t\in T}$. Zur Erinnerung: $X_t^- := -\min\{0, X_t\}$.

8.2 Stopping Times

A mapping $\nu : \Omega \mapsto \bar{\mathbb{N}}$ is a stopping time if

$$[\nu = n] \in B_n, \quad \forall n \in \mathbb{N} \tag{28}$$

8.3 Martingaldifferenzfolgen

Sei $\eta_n \in L(\Omega, A, \mathbb{P})$, $n \in \mathbb{N}$, mit $A_n := \sigma(\eta_1, \ldots, \eta_n)$ und $a \in \mathbb{R}$ beliebig.
Definiere

$$X_1 := \eta_1 - a \text{ und } X_{n+1} := X_n + \eta_{n+1} - \mathbb{E}\left[\eta_{n+1} \mid A_n\right] \;\; (n \geq 1)$$

Dann gilt

$$\begin{aligned}\mathbb{E}\left[X_{n+1} \mid A_n\right] &= \mathbb{E}\left[X_n \mid A_n\right] + \mathbb{E}\left[\eta_{n+1} \mid A_n\right] - \mathbb{E}\left[\mathbb{E}\left[\eta_{n+1} \mid A_n\right] \mid A_n\right] \\ &= X_n + \mathbb{E}\left[\eta_{n+1} \mid A_n\right] - \mathbb{E}\left[\eta_{n+1} \mid A_n\right] \\ &= X_n\end{aligned}$$

Das heißt, die Folge $(X_n)_{n\in\mathbb{N}}$ bildet ein Martingal.

Ist umgekehrt $(X_n)_{n\in\mathbb{N}}$ als Martingal vorausgesetzt und definiert man

$$\eta_1 := X_1 \qquad \eta_n := X_n - X_{n-1} \;\; (n \geq 2)$$

dann gilt

$$\begin{aligned}\mathbb{E}\left[\eta_{n+1} \mid \eta_1, \ldots, \eta_n\right] &= \mathbb{E}\left[X_{n+1} - X_n \mid \eta_1, \ldots, \eta_n\right] \\ &= \mathbb{E}\left[X_{n+1} - X_n \mid X_1, \ldots, X_n\right] \\ &= \mathbb{E}\left[X_{n+1} \mid X_1, \ldots, X_n\right] - X_n \\ &= 0\end{aligned}$$

Daher ist eine Folge reeler integrierbarer Zufallsvariablen $(\eta_n)_{n\in\mathbb{N}}$ heißt **Martingaldifferenzfolge**, falls

$$\boxed{\mathbb{E}\left[\eta_{n+1} \mid \eta_1, \ldots \eta_n\right] = 0 \quad \mathbb{P}\text{-f.s.}, \;\; \forall n \in \mathbb{N}} \tag{29}$$

8.4 Examples

Seien $Z_1, \ldots, Z_n$ unabhängig und identisch verteilt (iid) mit $Z_i \sim N(0,1)$ und
$F_n = \sigma(Z_1, \ldots, Z_n)$ eine Filtration. Ferner sei $X_n := \exp\left(\sum_{i=1}^n (Z_i - c)\right), n \in \mathbb{N}, c \in \mathbb{R}$.
Für welche Werte c ist $(X_n)_{n\in\mathbb{N}}$ ein Martingal, Submartingal bzw. Supermartingal bzgl. (F_n)?
Bitte begründen Sie Ihre Schritte kurz!

- X_n ist F_n-mb. da Komposition aus Z_i und $F_n = \sigma(Z_1, \ldots, Z_n)$

- F_n ist Filtration (Information komm hinzu) $\Rightarrow$ $F_n \subset F_{n+1} \, \forall n$

- $\mathbb{E}\left[\left|X_n\right|\right] < \infty$? (ist Z_n integrierbar?)

$$\begin{aligned}
\mathbb{E}\left[\left|X_n\right|\right] &= \mathbb{E}\left[\exp\left(\sum_{i=1}^{n} Z_i - c\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{n}\exp\left(Z_i - c\right)\right] \\
&\overset{\text{iid}}{=} \left(\mathbb{E}\left[\exp(Z-c)\right]\right)^n \\
&= \left(\int_{\mathbb{R}} \exp(z-c)\, d\mathbb{P}_Z\right)^n \\
&= \left(\int_{\mathbb{R}} \exp(z-c)\,\frac{1}{\sqrt{2\pi}}\exp\left(-\left(\frac{z^2}{2}\right)\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2} + z - c\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-z^2 + 2z - 2c}{2}\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-z^2 + 2z - 2c + 1 - 1}{2}\right) dz\right)^n \\
&= \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-((z^2-1)^2 - 1 + 2c)}{2}\right) dz\right)^n \\
&= \left(e^{\frac{1}{2}-c}\cdot \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(z-1)^2}{2}\right) dz}_{\sim N(1,1)=1}\right)^n \\
&= (e^{\frac{1}{2}-c})^n \\
&= e^{n\left(\frac{1}{2}-c\right)} < \infty
\end{aligned}$$

- Martingaleigenschaft: $\mathbb{E}\left[X_{n+1} \mid F_n\right] \overset{\text{f.s.}}{=} X_n$?

$$\begin{aligned}
\mathbb{E}\left[X_{n+1} \mid F_n\right] &= \mathbb{E}\left[X_n \cdot \exp\left(Z_{n+1} - c\right) \mid F_n\right] \\
\left(X_n \text{ ist } F_n\text{-mb.}\right) \Rightarrow &= X_n \cdot \mathbb{E}\left[\exp\left(Z_{n+1} - c\right) \mid F_n\right] \\
&\overset{\text{iid}}{=} X_n \cdot \mathbb{E}\left[\exp\left(Z_{n+1} - c\right)\right] \\
&= X_n \cdot e^{\frac{1}{2}-c} \\
&= X_n \text{ für } c = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ X_n \text{ Martingal für } c &= \frac{1}{2} \\
X_n \text{ Super-Martingal für } c &> \frac{1}{2} \\
X_n \text{ Sub-Martingal für } c &< \frac{1}{2}
\end{aligned}$$

Martingales and smoothing. Suppose $X \in L_1$ and $\{B_n, n \geq 0\}$ is an increasing family of sub σ-fields of B. Define for $n \geq 0$

$$X_n := \mathbb{E}\left[X \mid B_n\right]$$

Then

$$\{(X_n, B_n), n \geq 0\}$$

is a martingale:

$$\begin{aligned}
\mathbb{E}\left[X_{n+1} \mid B_n\right] &= \mathbb{E}\left[\mathbb{E}\left[X \mid B_{n+1}\right] \mid B_n\right] \\
&= \mathbb{E}\left[X \mid B_n\right] \quad \text{(smoothing)} \\
&= X_n
\end{aligned}$$

Martingales and sums of independent random variables. Suppose that $\{Z_n, n \geq 0\}$ is an independent sequence of integrable random variables satisfying for $n \geq 0$, $\mathbb{E}\left[Z_n\right] = 0$. Set $X_0 = 0$, $X_n = \sum_{i=1}^{n} Z_i$, $n \geq 1$, and $B_n := \sigma(Z_0, \ldots, Z_n)$.

Then $\{(X_n, B_n), n \geq 0\}$ is a martingale since $\{(Z_n, B_n), n \geq 0\}$ is a fair sequence.

Es sei $(X_t)_{t\in\mathbb{N}}$ eine Folge von unabhängigen und identisch verteilten Zufallsvariablen mit $\mathbb{E}\left[X_1\right] = 1$. Zeigen Sie, dass der stochastische Prozess $(Z_t, t \in \mathbb{N})$ mit

$$Z_t = \prod_{s=1}^{t} X_s$$

ein Martingal bezüglich der kanonischen Filtration $\sigma(X_1, X_2, \ldots)$ ist.

Es gilt für jedes $t \in \mathbb{N}$:

$$\begin{aligned}
\mathbb{E}\left[Z_{t+1} \mid A_t\right] &= \mathbb{E}\left[\prod_{i=1}^{t+1} X_i \mid \sigma(X_1, \ldots, X_t)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{t} X_i \mid \sigma(X_1, \ldots, X_t)\right] \cdot \mathbb{E}\left[X_{t+1} \mid \sigma(X_1, \ldots, X_t)\right] \\
&= \prod_{i=1}^{t} X_i \cdot \mathbb{E}\left[X_{t+1}\right] = \prod_{i=1}^{t} X_i = Z_t
\end{aligned}$$

Es sei $(X_t)_{t\in\mathbb{N}}$ eine Folge von unabhängigen und identisch verteilten Zufallsvariablen mit $\mathbb{E}\left[X_1\right] = 0$ und $\mathbb{E}\left[X_1^2\right] = \sigma^2$. Weiter sei $S_t = \sum_{s=1}^{t} X_s$. Zeigen Sie, dass der stochastische Prozess $(Z_t, t \in \mathbb{N})$ mit

$$Z_t = S_t^2 - t\sigma^2$$

ein Martingal bezüglich der kanonischen Filtration $\sigma(X_1, X_2, \ldots)$ ist.

Es gilt für jedes $t \in \mathbb{N}$:

$$\begin{aligned}
\mathbb{E}\left[Z_{t+1} \mid A_t\right] &= \mathbb{E}\left[S_{t+1}^2 - (t+1)\sigma^2 \mid \sigma(X_1, \ldots, X_t)\right] \\
&= \mathbb{E}\left[S_t^2 + 2S_t^2 X_{t+1} + X_{t+1}^2 \mid \sigma(X_1, \ldots, X_t)\right] - (t+1)\sigma^2 \\
&= S_t^2 + \mathbb{E}\left[X_{t+1}^2\right] - (t+1)\sigma^2 = S_t^2 - t\sigma^2 = Z_t
\end{aligned}$$

9 Convergence

9.1 Almost Sure Convergence

We say that a statement about random elements hold *almost surely* if there exists an event $A \in B$ with $\mathbb{P}[A] = 0$ such that the statement holds if $\omega \in A^C$.

$$\boxed{\forall \epsilon > 0: \quad \mathbb{P}\left[\limsup_{n\to\infty} \left|X_n - X\right| > \epsilon\right] = 0} \tag{30}$$

Oder kurz

$$X_n \overset{n\to\infty}{\longrightarrow} X \ \ \mathbb{P}-\text{f.s.}$$

Let $\{X_r :\geq 1\}$ be independent Poisson variables with respective parameters $\lambda_r : r \geq 1$. Show that $\sum_{r=1}^{\infty} X_r$ converges or diverges almost surely according as $\sum_{r=1}^{\infty} \lambda_r$

The partial sum $S_n = \sum_{r=1}^{n} X_r$ is Poisson-distributed with parameters $\sigma_n = \sum_{r=1}^{n} \lambda_r$. For fixed x, the event $\{S_n \leq x\}$ is decreasing in n, whence by Lemma 1.3.5, if $\sigma_n \to \sigma < \infty$ and x is non-negative integer.

$$\mathbb{P}\left[\sum_{r=1}^{\infty} X_r \leq x\right] = \lim_{n\to\infty} \mathbb{P}\left[S_n \leq x\right] = \sum_{j=0}^{x} \frac{e^{-\sigma}\sigma^j}{j!}$$

Hence if $\sigma < \infty$, $\sum_{r=1}^{\infty} X_r$ converges to a Poisson random variable. On the other hand, if $\sigma_n \to \infty$ then $e^{-\sigma_n}\sum_{j=0}^{x} \frac{\sigma_n^j}{j!} \to 0$, giving that $\mathbb{P}\left[\sum_{r=1}^{\infty} X_r > x\right] = 1$ for all x, and therefore the sum diverges with probability 1, as required.

9.1.1 Kolmogorov Convergence Criterion

If

$$\sum_{i=1}^{\infty} \mathbb{V}\left[X_i\right] < \infty$$

then

$$\sum_{i=1}^{\infty} \left(X_i - \mathbb{E}\left[X_i\right]\right)$$

converges almost surely.

9.2 Convergence in Probability

$X_n \overset{P}{\to} X$ if for $\forall\, \epsilon > 0$

$$\boxed{\lim_{n\to\infty} \mathbb{P}\left[\left|X_n - X\right| \geq \epsilon\right] = 0} \tag{31}$$

Sei $(X_n)_{n\in\mathbb{N}}$ eine Folge unabhängiger Zufallsvariablen, welche Exp(1)-verteilt sind.
Zeigen Sie, dass $n^\alpha \cdot \min_{k\leq n} X_k$ stochastisch gegen Null konvergiert für alle $\alpha < 1, n \in \mathbb{N}$.

$$\begin{aligned}
\forall \epsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}\left[\left|n^\alpha \min_{k\leq n} X_k\right| \geq \epsilon\right] = 0 \iff n^\alpha \min_{k\leq n} X_k &\overset{\mathbb{P}}{\longrightarrow} 0 \\
&= \lim_{n\to\infty} \mathbb{P}\left[\min_{k\leq n} X_k \geq \frac{\epsilon}{n^\alpha}\right] \\
&= \lim_{n\to\infty} \mathbb{P}\left[\bigcap_{1\leq k\leq n} \{\omega : X_k(\omega)\} \geq \frac{\epsilon}{n^\alpha}\right] \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \mathbb{P}\left[X_k \geq \frac{\epsilon}{n^\alpha}\right] \\
&\overset{\text{iid}}{=} \lim_{n\to\infty} \left(\mathbb{P}\left[X_1 \geq \frac{\epsilon}{n^\alpha}\right]\right)^n \\
&\overset{\text{Exp}(1)}{=} \lim_{n\to\infty} \left(e^{-\frac{\epsilon}{n^\alpha}}\right)^n = 0
\end{aligned}$$

9.3 L_p Convergence

$X \in L_p$ means $\mathbb{E}\left[|X|^p\right] < \infty$. A sequence $\{X_n\}$ of random variables converges in L_p to X, written

$$X_n \overset{L_p}{\to} X$$

if

$$\boxed{\mathbb{E}\left[\left|X_n - X\right|^p\right] \to 0} \tag{32}$$

as $n \to \infty$.
It follows that if $X_n \overset{L_p}{\to} X$ then $\mathbb{E}\left[\left|X_n^p\right|\right] \to \mathbb{E}\left[\left|X^p\right|\right]$

Suppose $\{X_n\}$ is an iid sequence of random variables with $\mathbb{E}\left[X_n\right] = \mu$ and $\mathbb{V}\left[X_n\right] = \sigma^2$. Then

$$\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n} \overset{L_2}{\to} \mu,$$

since

$$\begin{aligned}
\left(\mathbb{E}\left[\frac{S_n}{n} - \mu\right]\right)^2 &= \frac{1}{n^2}\left(\mathbb{E}\left[S_n - n\mu\right]\right)^2 \\
&= \frac{1}{n^2}\mathbb{V}\left[S_n\right] \\
&= \frac{n\sigma^2}{n^2} \to 0.
\end{aligned}$$

Suppose $X_n \overset{L_1}{\to} X$. Show that $\mathbb{E}\left[X_n\right] \to \mathbb{E}[X]$. Is the converse true?

We have that

$$\left|\mathbb{E}\left[X_n\right] - \mathbb{E}[X]\right| = \left|\mathbb{E}\left[X_n - X\right]\right| \leq \mathbb{E}\left[\left|X_n - X\right|\right] \overset{n\to\infty}{\longrightarrow} 0$$

The converse is clearly false. If each X_n takes the values ± 1, each with probability $\frac{1}{2}$, then $\mathbb{E}\left[X_n\right] = 0$, but $\mathbb{E}\left[\left|X_n - 0\right|\right] = 1$.

$$\text{Z}: X_n \xrightarrow{L_2} X \Rightarrow \mathbb{V}\left[X_n\right] \to \mathbb{V}\left[X\right]$$

$\mathbb{E}\left[X_n^2\right] \to \mathbb{E}\left[X^2\right]$ and $X_n \xrightarrow{L_1} X$. Therefore $\mathbb{E}\left[X_n\right] \to \mathbb{E}\left[X\right]$. Thus $\mathbb{V}\left[X_n\right] = \mathbb{E}\left[X_n^2\right] - \mathbb{E}\left[X_n\right]^2 \to \mathbb{V}\left[X\right]$.

9.4 Convergence in Distribution (Weak Convergence)

$$\lim_{n\to\infty} \mathbb{E}\left[f \circ X_n\right] = \mathbb{E}\left[f \circ X\right] \iff \int f \circ X_n \, d\mathbb{P} \overset{n\to\infty}{\longrightarrow} \int f \circ X \, d\mathbb{P}$$
$$\iff \int f \, d\mathbb{P}_{X_n} \overset{n\to\infty}{\longrightarrow} \int f \, d\mathbb{P}_X$$

(33)

(34)

Let $\{X_n, n \geq 1\}$ be iid with common unit exponential distribution

$$\mathbb{P}\left[X_n > x\right] = e^{-x}, \quad x > 0$$

Set $M_n = \vee_{i=1}^n X_i$ for $n \geq 1$. Then

$$M_n - \ln n \Rightarrow Y,$$

where

$$\mathbb{P}\left[Y \leq x\right] = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R} \qquad (35)$$

To prove **??**, note that for $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}\left[M_n - \ln n \leq x\right] &= \mathbb{P}\left[\bigcap_{i=1}^n (X_i \leq x + \ln n)\right] \\
&= (1 - e^{-(x+\ln n)})^n \\
&= \left(1 - \frac{e^{-x}}{n}\right)^n \to \exp\left(-e^{-x}\right)
\end{aligned}$$

Let $X_1, X_2, \ldots, X_n$ be i.i.d. Cauchy. Show that $M_n = \max X_i$ is such that $\pi M_n / n$ converges in distribution, the limiting distribution function being given by $F(x) = e^{-1/x}$ if $x \geq 0$.

We have that

$$\mathbb{P}\left[M_m \leq xn/\pi\right] = \left\{\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{xn}{\pi}\right)\right\}^n = \left\{1 - \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{xn}\right)\right\}^n$$

if $x > 0$, by elementary trigonometry. Now $\tan^{-1} y = y + o(y)$ as $y \to 0$, and therefore

$$\mathbb{P}\left[M_m \leq xn/\pi\right] = \left(1 - \frac{1}{xn} + o(n^{-1})\right)^n \to e^{-1/x} \quad \text{as } n \to \infty$$

9.4.1 Extreme Value Distributions

$\{X_n, n \geq 1\}$ idd with common distribution F. The Extreme observation among the first n is

$$M_n := \vee_{i=1}^n X_i.$$

Suppose there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^n(a_n x + b_n) = \mathbb{P}\left[\frac{M_n - b_n}{a_n} \leq x\right] \xrightarrow{D} G(x), \qquad (36)$$

where the limit distribution G is proper and non-degenerate. Then G is the type of one of the following extreme value distributions:

1. $\Phi_\alpha(x) = \exp\left(-x^{-\alpha}\right), \quad X > 0, \quad \alpha > 0,$
2. $\Psi_\alpha(x) = \begin{cases} \exp\left(-(x)^\alpha\right), & x < 0, \quad \alpha > 0 \\ 1 & x > 0, \end{cases}$
3. $\Lambda(x) = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R}$

The statistical significance is the following. The types of the three extreme value distributions can be united as a one parameter family indexed by shape parameter $\gamma \in \mathbb{R}$:

$$G_\gamma(x) = \exp\left(-(1+\gamma x)^{-1/x}\right), \quad 1 + \gamma x > 0 \qquad (37)$$

where we interpret the case of $\gamma = 0$ as

$$G_0 = \exp\left(-e^{-x}\right) \quad x \in \mathbb{R}$$

9.5 Implications

$$L_p - \text{Konvergenz} \Rightarrow L_q - \text{Konvergenz}\,(q \leq p) \Rightarrow \text{stoc}$$

(38)

sowie

$$\text{fast sichere Konvergenz} \Rightarrow \text{stochastische Konvergenz}$$

(39)

X_i i.i.d., $\mathbb{E}\left[X_i\right] = \mu$, $\mathbb{V}\left[X_i\right] < \infty$. Show that

$$\binom{n}{2}^{-1} \sum_{1\leq i\leq j\leq n} X_i X_j \xrightarrow{\mathbb{P}} \mu^2, \quad n \to \infty$$

$$\binom{n}{2}^{-1} \sum_{1\leq i\leq j\leq n} X_i X_j = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 - \frac{1}{n(n-1)}\sum_{i=1}^n X_i^2$$

Now $n^{-1}\sum_{i=1}^n X_i \xrightarrow{D} \mu$ by law of large numbers $\Rightarrow$ $n^{-1}\sum_{i=1}^n X_i \xrightarrow{\mathbb{P}} \mu$ (see **??**). It follows that $(n^{-1}\sum_{i=1}^n X_i)^2 \xrightarrow{\mathbb{P}} \mu^2$. Since if $c_n \to c$ and $X_n \xrightarrow{\mathbb{P}} X$ then $c_n X_n \xrightarrow{\mathbb{P}} cX$. So

$$\frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 \xrightarrow{\mathbb{P}} \mu^2$$

and

$$\frac{1}{n(n-1)}\sum_{i=1}^{n} X_i^2 \xrightarrow{\mathbb{P}} 0.$$

The result follows from the fact that If $X_n \xrightarrow{\mathbb{P}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$ then $X_n + Y_n \xrightarrow{\mathbb{P}} X + Y$.

9.5.1 Converse Implications

(a) If $X_n \xrightarrow{D} c$, where c is constant, then $X_n \xrightarrow{\mathbb{P}} c$

(b) If $X_n \xrightarrow{\mathbb{P}} X$ and $\mathbb{P}\left[\left|X_n\right| \leq k\right] = 1$ for all n and some k, then $X_n \xrightarrow{L_p} X$ for all $p \geq 1$

(c) If $\mathbb{P}\left[\left|X_n - X\right| > \epsilon\right]$ satisfies $\sum_n \mathbb{P}\left[\left|X_n - X\right| > \epsilon\right] < \infty$ for all $\epsilon > 0$, then $X_n \xrightarrow{\text{a.s.}} X$

9.5.2 Slutsky´s Theorem

$$\boxed{X_n \xrightarrow{D} X,\ A_n \xrightarrow{\mathbb{P}} a \text{ and } B_n \xrightarrow{\mathbb{P}} b \ \Rightarrow\ A_n + B_n \cdot X_n \xrightarrow{D} a + b * X} \tag{40}$$

10 Appendix

10.1 Stammfunktionen

$$\int \frac{1}{x}\,dx = \ln|x| + c$$
$$\int e^x\,dx = e^x + c$$
$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$$
$$\int a^x \ln a\,dx = a^x + c$$
$$\int \ln x\,dx = x\ln x - x$$
$$\int \sin(x)\,dx = -\cos(x) + c$$
$$\int \cos(x)\,dx = \sin(x) + c$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$
$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$
$$\int xe^{-ax}\,dx = \frac{-e^{-ax}}{a^2}(ax+1)$$
$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$
$$\int_0^\infty x^2ae^{-ax}\,dx = -x^2e^{-ax}|_0^\infty + \int_0^\infty 2xe^{-ax}\,dx = 0 + \frac{2}{a^2}$$
$$\int x^ne^{ax}\,dx = \frac{1}{a}x^ne^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$
$$\int \frac{1}{1+e^{ax}}\,dx = \frac{1}{a}\ln\frac{e^{ax}}{1+e^{ax}}$$
$$\int \frac{1}{b+ce^{ax}}\,dx = \frac{x}{b} - \frac{1}{ab}\ln\left|b+ce^{ax}\right|$$
$$\int \frac{e^{ax}}{b+ce^{ax}}\,dx = \frac{1}{ac}\ln\left|b+ce^{ax}\right|$$

10.1.1 Beispiele

- **??**
- **??**

10.2 Partielle Integration

$$\boxed{\int_a^b f'(x)\cdot g(x)\,dx = [f(x)\cdot g(x)]_a^b - \int_a^b f(x)\cdot g'(x)\,dx} \tag{41}$$

10.3 Sets and Events

10.3.1 De Morgan

$$\left(\bigcup_i A_i\right)^C = \bigcap_i A_i^C$$
$$\left(\bigcap_i A_i\right)^C = \bigcup_i A_i^C$$

10.3.2 Limits of Sets

- $\inf_{k\geq n} A_k := \bigcap_{k=n}^{\infty} A_k, \ \sup_{k\geq n} A_k := \bigcup_{k=n}^{\infty} A_k$
- $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$
- $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$
- If $\liminf_{n\to\infty} B_n = \limsup_{n\to\infty} B_n = B$ then we say $B_n \to B$
- $\limsup_{n\to\infty} A_n = [A_n \, i.o.]$

10.3.3 Borel-Cantelli Lemma

Let $\{A_n\}$ be any events. If

$$\sum_n \mathbb{P}\left[A_n\right] < \infty$$

then

$$\mathbb{P}\left[A_n \, i.o.\right] = \mathbb{P}\left[\limsup_{n\to\infty} A_n\right] = 0$$

Let $X_n \sim \text{Exp}(1)$

$$\text{Z}: \quad \mathbb{P}\left[\limsup_{n\to\infty}\frac{X_n}{\log n} = 1\right] = 1$$

Evidently

$$\mathbb{P}\left[\frac{X_n}{\log n} \geq 1+\epsilon\right] = \frac{1}{n^{1+\epsilon}}, \quad \text{for } |\epsilon| \leq 1$$

By the Borel-Cantelli lemmas, the events $A_n = \{X_n/\log n \geq 1+\epsilon\}$ occur a.s. infinitely often for $-1 < \epsilon \leq 0$, and a.s. only finitely often for $\epsilon > 0$.

10.3.4 Borel Zero-One Law

If $\{A_n\}$ is a sequence of independent events, then

$$\mathbb{P}\left[A_n \, i.o.\right] = \begin{cases} 0, & \textit{iff} \sum_n \mathbb{P}\left[A_n\right] < \infty \\ 1, & \textit{iff} \sum_n \mathbb{P}\left[A_n\right] = \infty \end{cases}$$

10.4 Inequalities

10.4.1 Markov

$$\boxed{\mathbb{P}\left[|X| \geq \lambda\right] \leq \frac{\mathbb{E}\left(|X|\right)}{\lambda}} \tag{42}$$

10.4.2 Chebychev

$$\boxed{\mathbb{P}\left[|X-\mathbb{E}(X)| \geq \lambda\right] \leq \frac{\mathbb{V}\left[X\right]}{\lambda^2}} \tag{43}$$

10.4.3 Kolmogorov

$$\boxed{\mathbb{P}\left[\max_{1\leq k\leq n}\left|X_k\right| \geq \lambda\right] \leq \frac{\mathbb{V}(X_n)}{\lambda^2} = \frac{1}{\lambda^2}\sum_{k=1}^{n}\mathbb{V}\left[X_k\right]} \tag{44}$$

10.4.4 Schwartz

$X, Y \in L_2$ then

$$\boxed{\left|\mathbb{E}\left[XY\right]\right| \leq \mathbb{E}\left[|XY|\right] \leq \sqrt{\mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]}} \tag{45}$$

10.4.5 Hölder

Suppose p, q satisfy

$$p > 1, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1$$

and that

$$\mathbb{E}\left[|X|^p\right] < \infty, \ \mathbb{E}\left[|X|^q\right] < \infty$$

then

$$\boxed{\left|\mathbb{E}\left[XY\right]\right| \leq \mathbb{E}\left[|XY|\right] \leq \left(\mathbb{E}\left[|X|^p\right]\right)^{1/p}\left(\mathbb{E}\left[|Y|^q\right]\right)^{1/q}} \tag{46}$$

10.4.6 Minkowski

For $1 \le p < \infty$, assume $X, Y \in L_p$. Then $X+Y \in L_p$ and

$$\boxed{\|X+Y\|_p \le \|X\|_p + \|Y\|_p} \tag{47}$$

10.4.7 Jensen

Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is convex and $\mathbb{E}\left[|X|\right] < \infty$ and $\mathbb{E}\left[\left|f(X)\right|\right] < \infty$. Then

$$\boxed{\mathbb{E}\left[f(X)\right] \ge f(\mathbb{E}\left[X\right])} \tag{48}$$

A special case is

$$\boxed{\mathbb{E}\left[X^2\right] \ge (\mathbb{E}\left[X\right])^2} \tag{49}$$

If f is concave, the inequality reverses.

10.5 Stochastics

10.5.1 Law of Large Numbers

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu} \tag{50}$$

10.5.2 Central Limit Theorem

$$\boxed{\mathbb{P}\left[\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \le x\right] \to N(x) := \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du} \tag{51}$$

10.6 Extema and Order Statistics

10.6.1 Minima

Seien $X_1, X_2, \dots$ iid auf $[0,1]$ Gleichverteilt. Gegen welche Verteilung konvergiert $n \cdot \min_{1\le k\le n} X_k$ schwach?

$$\begin{aligned}
\mathbb{P}\left[n \cdot \min_{1\le k\le n} < c\right] &= 1 - \mathbb{P}\left[n \cdot \min_{1\le k\le n} \ge c\right] \\
&= 1 - \mathbb{P}\left[\bigcap_{1\le k\le n} \left\{\omega : X_k(\omega) \ge \frac{c}{n}\right\}\right] \\
&= 1 - \left(\mathbb{P}\left[X \ge \frac{c}{n}\right]\right)^n \\
&= 1 - \left(\int \mathbb{1}_{x \ge \frac{c}{n}}(x) \cdot \frac{1}{1-0}\, dx\right)^n \\
&= 1 - \left(\int_{\frac{c}{n}}^{1} dx\right)^n \\
&= 1 - \left(1 - \frac{c}{n}\right)^n \\
&\overset{\lim}{\underset{n\to\infty}{\longrightarrow}} 1 - e^c
\end{aligned}$$

Konvergiert gegen ZV die Exp(1) verteilt ist.

10.6.2 Maxima

$$\begin{aligned}
\mathbb{P}\left[\max_{1\le k\le n} X_k < c\right] &= \mathbb{P}\left[\bigcap_{1\le k\le n} \left\{\omega : X_k(\omega) < c\right\}\right] \\
&= \prod_{k=1}^{n} \mathbb{P}\left[X_k < c\right] \\
&= \left(\mathbb{P}\left[X_1 < c\right]\right)^n
\end{aligned}$$