ECE374 SP23 HW8

Contributors

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Problem 3

The low-degree spanning tree problem. Given a graph G and an integer k, does G contain a spanning tree such that all vertices in the tree have degree *at most* k?

(a) Prove that the low-degree spanning tree problem is NP-hard with a reduction from Hamiltonian path.

(b) Consider **the high-degree spanning tree problem.** Given a graph G and an integer k, does G contain a spanning tree whose highest degree vertex is *at least* k? Give an efficient algorithm to solve this problem and analyze its time complexity.

Solution

(Adapted from the solution to Problem 3, Lab 20)

(a)

We prove the **low-degree spanning tree problem** is NP-hard by a reduction from the **undirected Hamiltonian path problem**.

Given an arbitrary graph G, let H be the graph obtained by attaching a "fan" of $k-2$ edges (with $k-2$ new vertices) to every vertex of G. I claim that G has a Hamiltonian path if and only if H has a low-degree spanning tree.

If. Suppose G has a Hamiltonian path P.

- Let T be the spanning tree of H obtained by adding every "fan edge" in H to P.
- Every vertex $v \in H$ is either a leaf of T or a vertex of P.
- If $v \in P$, then $\deg_P(v) \le 2$, and therefore $\deg_T(v) = \deg_P(v) + (k-2) \le k$.

We conclude that H has a low-degree spanning tree.

Only if. Suppose H has a low-degree spanning tree T.

- The leaves l of T are the vertices of H with degree 1, and $l \in H - G$.
- Let P be the subtree of T obtained by deleting all the leaves.
- P is a spanning tree of G, and for every vertex $v \in P$, we have $\deg_P(v) = \deg_T(v) - (k-2) \le 2$.

We conclude that P is a Hamiltonian path in G. ■

(Page 3 of this source provides another proof without constructing the "fan." k is assumed to be greater than 2.)

We claim that G has a Hamiltonian path if and only if it has a spanning tree with vertex degree *at most* 2.

If. If G contains a Hamiltonian path, this path must be a spanning tree since the path visits every node, and a path trivially is a tree.

Only if. It is easy to see that such a spanning tree is a Hamiltonian path. Since it has degree *at most* 2, it cannot branch; since it is spanning, only two vertices can have degree < 2. ■

(b)

Pseudocode.

- Count the degree of each vertex in G.
- If all vertices have degrees *lower than* k, return False.
- Select any vertex v with degree *higher than* k as the root of the high-degree spanning tree. Include all edges incident to v in the tree.
- Perform a depth-first search on G starting at v and its children.
- Check if all vertices have been visited (in case there are multiple connected components). If so, return True.

Runtime analysis. Checking the degrees takes $O(n)$ time. The DFS takes $O(n)$ time. Total runtime is $O(n)$.