ECE374 SP23 HW9

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Problem 2

Consider a decision problem X defined as follows:

- **Input:** A directed graph $G = (V, E)$ and a positive integer k
- **Output:** True if there exists a subset $V' \subseteq V$ with $|V'| \leq k$ such that deleting the vertices in V' and their adjacent edges from G leaves a DAG, False otherwise.

Show that X is NP-complete.

Solution

For an arbitrary *undirected* graph H, we construct a directed graph G by swapping every undirected edge (u, v) with two directed edges (u, v) and (v, u).

We reduce the decision problem X to the *vertex cover problem* and claim that $X(G, k) = \text{True}$ iff H has a vertex cover of size at most k.

If. Suppose $X(G, k) = \text{True}$,

- which means that we can delete at most k vertices from G to form a DAG.
- *Proof by contradiction.* Suppose these deleted vertices does not form a vertex cover of H. Then the edges left out in H will "translate" into loops ($u \to v \to u$) in G, which is not a DAG.

Therefore, the deleted vertices of size at most k form a vertex cover of H.

Only if. Suppose H has a vertex cover V' of size at most k.

- Then deleting V' *and their adjacent edges* from G leaves a number of isolated vertices, which collectively form a DAG.

Therefore, $X(G, k) = \text{True}$. $\blacksquare$