ECE374 SP23 HW9

Contributors

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Problem 5

Are the following problems decidable or undecidable? If the language is decidable, explain why, and if it's not, prove it

(a) $L_{5a} = \left\{\langle M\rangle \mid \text{ M is a TM that accepts the string 1011 in } |1011|^6 \text{ steps}\right\}$

(b) $L_{5b} = \left\{\langle M\rangle \mid \text{ M is a TM that does not accept any strings in } |w|^6 \text{ steps}\right\}$

Solution

(a)

L_{5a} is decidable. We can simulate the TM M with a counter. If the counter reaches $|1011|^6$ before the TM halts, we reject the TM, otherwise we accept it.

$$
\begin{array}{l}
\text{Decide}L_{5a}(M) \\
\quad \text{counter} \leftarrow 0 \\
\quad \text{Initialize } M \text{ with input } 1011 \\
\quad \textbf{while } \text{counter} < |1011|^6 \\
\quad\quad \text{state} = \text{Take a step in } M \\
\quad\quad \textbf{if } \text{ state} = \text{ACCEPT} \\
\quad\quad\quad \textbf{return } \text{True} \\
\quad\quad \textbf{if } \text{ state} = \text{REJECT} \\
\quad\quad\quad \textbf{return } \text{False} \\
\quad\quad \text{counter} \leftarrow \text{counter} + 1 \\
\quad \textbf{return } \text{False}
\end{array}
$$

(b)

L_{5b} is undecidable.

Suppose there is an algorithm $\text{Decide}L_{5b}$ that correctly decides the language L_{5b}. Then we can solve the halting problem as follows:

$$
\begin{array}{l}
\text{DecideHalt}(\langle M, w\rangle) \\
\quad \text{Encode the following Turing machine } M' : \\
\quad\quad M'(x) \\
\quad\quad\quad \text{run } M \text{ on input } w \text{ for at most } |x|^6 + 1 \text{ steps} \\
\quad\quad\quad \textbf{if } M \text{ accepts } w \text{ in } |x|^6 \text{ steps} \\
\quad\quad\quad\quad \textbf{return } \text{True} \\
\quad\quad\quad \textbf{return } \text{False}
\end{array}
$$

$$
\begin{array}{l}
\textbf{if } \text{Decide}L_{5b}(\langle M' \rangle) \\
\quad \textbf{return } \text{False} \\
\textbf{else} \\
\quad \textbf{return } \text{True}
\end{array}
$$

We prove this reduction correct as follows.

If. Suppose M accepts input w in $|x|^6$ steps.

- Then M' accepts an input string x in $|x|^6$ steps (during the simulation of $\langle M, w\rangle$).
- So $\text{Decide}L_{5b}$ rejects the encoding $\langle M' \rangle$.
- So DecideHalt correctly accepts the encoding $\langle M, w\rangle$.

Only if. Suppose M does not halt on input w.

- Then M' does not accept any input string x.
- So $\text{Decide}L_{5b}$ accepts the encoding $\langle M' \rangle$.
- So DecideHalt correctly rejects the encoding $\langle M, w\rangle$.