ECE374 SP23 HW2

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Problem 3

Let B and C be languages over $\Sigma = \{0, 1\}$. Define:

$$B \xrightarrow{0} C = \{w \in C \mid \exists x \in B, \#(0, w) = \#(0, x)\}$$

Show that the class of regular languages is closed under the $\xrightarrow{0}$ operation.

Solution

Proof. We show that if B and C are regular, then $B \xrightarrow{0} C$ is regular. Suppose that B and C can be represented by two NFAs, namely $M_B = (Q_B, \Sigma, \delta_B, s_B, A_B)$ and $M_C = (Q_C, \Sigma, \delta_C, s_C, A_C)$.

We first construct a "swapped" NFA to our convenience:

$$M_{BS} = (Q_B, \Sigma, \delta_{BS}, s_B, A_B)$$

, where $\forall q_B \in Q_B$,

- $\delta_{BS}(q_B, 0) = \{\delta_B(q_B, 0)\}$
- $\delta_{BS}(q_B, 1) = \{q_B\}$
- $\delta_{BS}(q_B, \varepsilon) = \{\delta_B(q_B, 1), \delta_B(q_B, \varepsilon)\}$

Since every state has an implicit ε-transition to itself, we are effectively changing the inputs of the self-loops to 1. Correspondingly, all the arcs labelled 1 are now labelled ε.

Such swapping turns B into a purely "counting" machine, where it pauses when seeing a 1 and uses the ε-transitions to "make up" for the moves. In this case, *any string $w \in \Sigma^*$ whose number of 0's equals to that of a string $x \in B$, even if w is not accepted by M_B, will be accepted by M_{BS}*.

We then construct an NFA M' that accepts $B \xrightarrow{0} C$:

$$M' = (Q', \Sigma, \delta', s', A')$$

, where

- $Q' = Q_B \times Q_C$
- $\delta'((q_B, q_C), a) = \delta_{BS}(q_B, a) \times \delta_C(q_C, a), \forall q_B \in Q_B, q_C \in Q_C, a \in \Sigma$
- $s' = (s_B, s_C)$
- $A' = \{q_1 \times q_2 \mid q_1 \in A_B, q_2 \in A_C\}$

M' is also known as the *cross-product machine* of M_{BS} and M_C.

A string ending up in an accepting state (q_1, q_2) has two implications: (1) $q_1 \in A_B$ so that the string has an acceptable number of 0's and (2) $q_2 \in A_C$ so that the string is in the language C. **Q.E.D.**