ECE374 SP23 HW9

Contributors

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Problem 4

Let $T = \{\langle M \rangle \mid \text{M is a TM that accepts } w^R \text{ whenever it accepts } w\}$. Show that T is undecidable.

Solution

Suppose there is an algorithm DecideT that correctly decides the language T. Then we can solve the halting problem as follows:

$$
\begin{array}{l}
\text{DecideHalt}(\langle M, w \rangle) \\
\quad \text{Encode the following Turing machine } M' : \\
\quad\quad M'(x) \\
\quad\quad\quad \text{if } x = 01 \\
\quad\quad\quad\quad \textbf{return } \text{True} \\
\quad\quad\quad \text{run } M \text{ on input } w \\
\quad\quad\quad \textbf{return } \text{True} \\
\quad \textbf{if } \text{DecideT}(\langle M' \rangle) \\
\quad\quad \textbf{return } \text{True} \\
\quad \textbf{else} \\
\quad\quad \textbf{return } \text{False}
\end{array}
$$

We prove this reduction correct as follows.

If. Suppose M halts on input w.

- Then M' accepts every input string x.
- In particular, M' accepts w^R whenever it accepts w.
- So $\text{DecideT}(\langle M' \rangle)$ accepts the encoding $\langle M' \rangle$.
- So DecideHalt correctly accepts the encoding $\langle M, w \rangle$.

Only if. Suppose M does not halt on input w.

- Then M' diverges on every input string x *except* 01.
- In particular, M' does not accept w^R whenever it accepts w -- a counterexample is $w = 01$ and $w^R = 10$.
- So $\text{DecideT}(\langle M' \rangle)$ rejects the encoding $\langle M' \rangle$.
- So DecideHalt correctly rejects the encoding $\langle M, w \rangle$.

Therefore, DecideHalt is correct. But this contradicts the fact that the halting problem is undecidable. Thus, DecideT is incorrect, and T is undecidable. $\blacksquare$