ECE374 SP23 HW3

Contributors

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Problem 5

Prove this language is not regular by providing a fooling set. Be sure to prove that the fooling set is (1) valid and (2) infinite.

$$L_{P5} = \{w \mid |w| = \lceil k\sqrt{k} \rceil, k \in \mathbf{N}\}$$

Hint. $F = \{0^{m^6} \mid m \geq 1\}$ is a fooling set that works. Also, the difference between consecutive strings in the language, $\lceil (k+1)^{1.5} \rceil - \lceil k^{1.5} \rceil$, is bounded above and below by

$$1.5\sqrt{k} - 1 \leq \lceil (k+1)^{1.5} \rceil - \lceil k^{1.5} \rceil \leq 1.5\sqrt{k} + 3$$

All that's left is to prove that F is a fooling set for L_{P5}.

Solution

Proof. Picking values k that satisfy $k = m^4$ for some integer m constructs a set

$$F = \{0^{k\sqrt{k}} \mid k = m^4, m \in \mathbf{N}^+\} = \{0^{m^6} \mid m \in \mathbf{N}^+\}$$

that contains all-0 strings of length m^6. We now prove that F is a fooling set for L_{P5}.

We pick two elements $x = 0^{m^6}$ and $y = 0^{n^6}, m < n$ from F. Let z be the ***smallest*** suffix to append to x and y that makes $xz \in L_{P5}$ and $yz \notin L_{P5}$. Then $|xz| = (m+1)^6$ and

$$|xz| - |x| = |z| = \lceil (m+1)^6 \rceil - \lceil m^6 \rceil \leq 1.5m^2 + 3 \tag{1}$$

We proof by contradiction. ***Assume*** that $yz \in L_{P5}$. Then $|yz| \geq \lceil (n+1)^6 \rceil$ and

$$|yz| - |y| = |z| = \lceil (n+1)^6 \rceil - \lceil n^6 \rceil \geq 1.5n^2 - 1 \tag{2}$$

Taking (1) and (2) together, we have

$$1.5n^2 - 1 \leq |z| \leq 1.5m^2 + 3 \tag{3}$$

However, since $m < n$, we have

$$\begin{aligned} m &\leq n - 1 \\ 1.5m^2 &\leq 1.5(n-1)^2 \\ 1.5m^2 + 3 &\leq 1.5(n-1)^2 + 3 \\ 1.5m^2 + 3 &\leq 1.5(n^2 - 2n + 1) + 3 \\ 1.5m^2 + 3 &\leq (1.5n^2 - 1) + (5.5 - 3n) \end{aligned} \tag{4}$$

Considering $1 \leq m < n$ and taking (4) together, we have

$$\begin{aligned} 5.5 - 3n \leq 0 \\ 1.5m^2 + 3 \leq 1.5n^2 - 1 \end{aligned} \tag{5}$$

(3) and (5) contradict each other! Thus, our assumption is false, $yz \notin L_{P5}$, z can distinguish x and y, and F is a fooling set for L_{P5}. Moreover, since there are infinitely many natural numbers, F is an infinite set, and therefore L_{P5} is not regular. **Q.E.D.**