ECE374 SP23 HW8

Contributors

Zhirong Chen (zhirong4)

Ziyuan Chen (ziyuanc3)

Problem 4

SAT reductions.

(a) Stingy SAT. Given a set of clauses (each a disjunction of literals) and an integer k, find a satisfying assignment in which *at most* k variables are True, if such an assignment exists. Prove that *Stingy SAT* is NP-hard.

(b) Double SAT. The problem asks whether a given satisfiability problem has *at least* two different satisfying assignments. Prove that *Double SAT* is NP-hard.

Solution

(a)

To prove that *Stingy SAT* is NP-hard, we need to (1) prove that *Stingy SAT* is an NP problem, and (2) reduce the SAT problem to a *Stingy SAT* problem.

Pre-condition. Let (F, k) be an instance of *Stingy SAT*, where F is an instance of SAT with k variables. We need to show that a set of assignments X can make (F, k) a real solution in polynomial time, which proves that *Stingy SAT* is an NP problem.

Target. We need to create a certificate that converts SAT to *Stingy SAT*, i.e., X is the solution of F if and only if X is the solution of (F, k). $\text{SAT} \Leftrightarrow \text{Stingy SAT}$.

Proof of adequacy. If X is the solution of F, then at most k variables are True in X. Therefore, X satisfies (F, k), which means X is the solution of (F, k).

Proof of necessity. Assuming X is the solution of (F, k), it implies that X assigns at most k variables to True, and also satisfies F. Hence, X is the solution of F. ■

(b)

We can demonstrate that *Double SAT* is NP-hard by providing a reduction from SAT. Given an instance ϕ of SAT, which is a CNF formula containing n variables $x_1, x_2, \ldots, x_n$, we construct a new variable x_{n+1}. We then define $\psi = \phi \land (x_{n+1} \lor \neg x_{n+1})$ as the corresponding instance of *Double SAT*.

We claim that ϕ has a satisfying assignment (is an instance of SAT) if and only if ψ has at least two satisfying assignments (is an instance of *Double SAT*).

If. If ϕ has a satisfying assignment f, we can obtain two distinct satisfying assignments for ψ by extending f with $x_{n+1} = \text{True}$ and $x_{n+1} = \text{False}$, respectively.

Only if. On the other hand, if ψ has at least two satisfying assignments, then the restriction of any of them to the set $x_1, x_2, \ldots, x_n$ is a satisfying assignment for ϕ. ■