ECE374 SP23 HW9

Contributors

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Problem 1

A *strongly independent set* is a subset of vertices S in a graph G such that for any two vertices in S, there is no path of length two in G. Prove that *Strongly Independent Set* is NP-hard.

Solution

We reduce the *strongly independent set* problem to the *independent set* problem.

Reduction. For an arbitrary *undirected* graph $G = (V, E)$, we construct a new *undirected* graph G' by

- adding a new vertex on every edge of G, and
- connecting two "edge" vertices if the edges share a common vertex.

Formally,

$$\begin{aligned}
G' &= (V', E'), \text{ where} \\
V' &= V \cup \{v_{ij} \mid (i, j) \in E\} \\
E' &= \{(i, v_{ij}), (j, v_{ij}) \mid (i, j) \in E\} \cup \{(v_{ik}, v_{jk}) \mid (i, k), (j, k) \in E\}
\end{aligned}$$

We claim that G has an *independent set* if and only if G' has a *strongly independent set*.

If. Suppose G has an *independent set* S.

- Then the path any two vertices $i, j \in S$ has length at least 2 in G, say $i \to k \to j$.
- Therefore, the path between i and j in G' has length at least 3 -- that is, $i \to v_{ik} \to v_{jk} \to j$.

This confirms that G' has a *strongly independent set*.

Only if. Suppose G' has a *strongly independent set* S'. We can swap every "edge" vertex in S' with a "vertex" vertex such that S' remains a *strongly independent set* in G'.

- Formally, if $v_{ij} \in S'$, then we delete v_{ij} and add i ***or*** j to S', say i.
- Suppose the minimum distance to v_{ij} from $\forall u \in S' - v_{ij}$ is d.
- Every path from u to i must go through one of the "edge" vertices connected to v_{ij}, say v_{ik}.
  - If the path goes through v_{ij}, then the path length is at least $d + 1$ (append $v_{ij} \to i$)
  - If the path does not go through v_{ij}, then the path length is at least d (change $v_{ik} \to v_{ij}$ to $v_{ik} \to i$)

In this way, S' is still a *strongly independent set* in G'. Meanwhile, all the "vertex" vertices in S' are now present in S! They constitute an *independent set* in G. ■