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Electromagnetic scattering of a plane wave by a radially inhomogeneous sphere in the short wavelength limit

James A. Lock

Physics Department, Cleveland State University, Cleveland, OH, 44115, USA

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ABSTRACT

The formulas of ray theory and Airy theory for scattering of an electromagnetic plane wave by a radially inhomogeneous sphere are obtained from the generalization of exact Lorenz–Mie wave scattering theory using the WKB approximation to determine the radial portion of the partial wave scalar radiation potential, carrying out a Debye series expansion of it, approximating the sum over partial waves by an integral over an associated impact parameter, and approximately evaluating the integral using either the method of stationary phase or mapping it into one of the phase integrals that describe optical caustics. Various features of the results are commented on, and the procedure is extended to the merging of two rainbows in a given Debye series channel in the context of the longitudinal cusp caustic.

1. Introduction

Although the general theory of scattering of an electromagnetic plane wave by a single dielectric spherical particle is relatively well-understood [1–3], the general theory of electromagnetic scattering by a spherical particle having a radially continuously varying refractive index profile $N_1(r)$ has been developed to a lesser extent (see however Section 5.6.1 of [2] and [4–7]). Certain specific refractive index profiles have been studied in detail, such as a Luneburg lens [8–12], and a particle having a single localized refractive index decrease either in the interior [4–6] or at its surface [13–16]. In addition, propagation through gradient index lenses (see [17–20] and Section 5.5 of [2]), and mirages produced by propagation of sunlight through inhomogeneities in the Earth's atmosphere (see Chapter 7 of [21] and [22–24]) have been widely studied. In most of these studies the emphasis has been on determining the ray theory deflection angle as a function of the angle of incidence (see Section 3.2.1 of [25] and [26–28]). Noticeably fewer studies have been concerned with the wave theory scattered intensity as a function of scattering angle. What is usually done in practice for determining the wave scattered intensity for a sphere having a continuously varying $N_1(r)$ is to subdivide it into a finely-stratified collection of concentric spherical shells, and numerically compute scattering by this assembly. A number of robust finely-stratified sphere Mie scattering codes have been developed and published over the years, which have led to many interesting and fruitful results [10,29–36]. This computational approach, however, usually sheds little light on the physical interpretation of these features or on the physical mechanisms that cause them. As an alternative, this paper develops an approximate analytic theory of electromagnetic scattering by a radially inhomogeneous sphere, applicable in the short wavelength limit, whose physical interpretation is much more transparent and which more easily provides insight into the details of the scattering process.

Ray theory, Airy theory, and a generalization of Airy theory to higher-order optical caustics are derived for scattering by a radially inhomogeneous sphere having a refractive index profile $N_1(r)$ that is continuous, and has continuous first and second derivatives. The derivation borrows the WKB approximation from quantum mechanics in order to approximately determine the radial portion of the partial wave scalar radiation potential, from which the total electric and magnetic fields both outside and inside the sphere are obtained. The scattered portion of the exterior field is then isolated, and a Debye series expansion of the partial wave scattering amplitudes is carried out. The sum over partial waves is then approximated by an integral over an effective impact parameter, and the integral is approximately evaluated using either the method of stationary phase, or by mapping it into one of the standard diffraction integrals in the theory of optical caustics.

The approximations to exact electromagnetic wave scattering theory used here are all familiar from their applications in other scattering situations. The partial wave approach for quantum mechanical scattering of a projectile by an infinite-range potential $V(r)$ is standardly used in the low-energy limit where only a few partial waves contribute (see Sections 11.2, 11.3 of [37], Sections 11.5,

E-mail address: j.lock@csuohio.edu

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11.6 of [38], and Sections 8.1a,b of [39]). Its use for high-energy quantum mechanical potential scattering is complicated by the fact that the sum over partial waves usually contains a large number of slowly convergent terms. For this reason, other approaches such as the Born approximation or the impulse approximation are often much more efficient in the high-energy, or equivalently, the short-wavelength regime (see Section 11.4 of [37], Section 11.4 of [38], and Section 8.3 of [39]). The WKB method has long been used in quantum mechanics to approximate bound state wave functions (see Chapter 8 of [37], Chapter 7 of [38], and Section 4.5 of [39]). It has also been used in high-energy scattering to approximate the partial wave scattering phase shifts (see p.410 of [40]). Ford and Wheeler [41] and Berry [42,43] have used the WKB method to derive both ray theory and Airy theory appropriate to scattering of scalar waves by an infinite-range potential in the short-wavelength limit.

Electromagnetic scattering by a particle differs from scattering by an infinite-range potential in that the effective potential corresponding to the particle has a sharp cutoff at its surface. Debye re-expressed the partial wave amplitudes for electromagnetic scattering by a homogeneous cylindrical particle as the sum of an infinite number of terms corresponding to the physical processes of diffraction, external reflection, direct transmission, and transmission following an arbitrary number of internal reflections [44]. This re-expression is known as the Debye series. In this context, quantum mechanical scattering by an infinite-range potential corresponds to the direct transmission term alone. The sharp cutoff at the particle surface is the source of the external and internal reflections. Van der Pol and Bremmer extended the Debye series to electromagnetic scattering by a homogeneous sphere [45]. Continuing along these lines, van de Hulst derived ray theory for scattering by a homogeneous sphere as a short-wavelength approximation to the exact Lorenz-Mie theory by performing a stationary phase evaluation of each of the individual Debye series terms (see Chapter 5 of [46] and Section 12.3 of [1]). Nussenzveig [47–49] and Khare [50] improved and further extended this approach using the complex angular momentum formalism based on the Watson transformation applied to the scattering amplitude.

Although approximations in other wave scattering situations have combined some of these methods of analysis, the present study combines all of them to determine the short-wavelength limit of electromagnetic scattering by a radially inhomogeneous sphere of radius a having the arbitrary refractive index profile $N_1(r)$. This approach provides a first-principles derivation of a number of physically motivated conjectures concerning electromagnetic scattering by a radially inhomogeneous sphere that have previously appeared in the literature [6,12,15,51,52]. To the author's knowledge, such a first-principles derivation of these conjectures has not appeared before.

The body of this study is organized as follows. In Section 2 the transverse-electric-polarized (TE) and transverse-magnetic-polarized (TM) scattered fields are expressed as derivatives of TE and TM partial wave scalar radiation potentials subject to boundary conditions at the sphere surface. In Section 3 the radial portion of the partial wave scalar radiation potentials is determined using the WKB approximation. In Section 4 the scattered portion of the partial wave scalar radiation potentials is isolated and expanded as a Debye series. In Section 5 each term of the Debye series in the short wavelength limit is summed over a large number of partial waves, and is approximated by an integral over an associated impact parameter. When the phase of the integrand is locally parabolic as a function of the impact parameter, the integral is approximately evaluated using the method of stationary phase and provides the contribution of an isolated geometrical ray to the scattered field. Section 6 considers the case when the phase of the integrand is locally cubic rather than quadratic, leading to the Airy theory of the rainbow. For a scattering by a radially inhomogeneous sphere, it can sometimes happen that the phase of the integrand is locally quartic as a function of the impact parameter, rather than being quadratic or cubic. This leads to a pair of rainbows occurring for an individual Debye process, whereas for scattering by a homogeneous sphere, only a single rainbow is possible. This situation is considered in Section 7 where the scattered fields are described by the Pearcey function, characteristic of the longitudinal cusp optical caustic. Lastly, Section 8 provides some final thoughts concerning inhomogeneous sphere scattering in the short wavelength limit.

2. Fields scattered by a radially inhomogeneous sphere

Consider a radially inhomogeneous dielectric spherical particle of radius a and having the refractive index profile $N_1(r)$ (region 1) whose center is at the origin of coordinates. It is embedded in an external medium having the refractive index $N_2(r) = 1$ (region 2). The refractive index profile in both regions considered together is denoted here by $N(r)$. A monochromatic electromagnetic plane wave of wavelength λ, wave number $k = 2\pi/\lambda$, angular frequency ω, and field strength E_0 propagates in the $+z$ direction and is linearly polarized in the x direction. The speed of light in the external medium is c, and the time dependence of the fields is taken to be $\exp(-i\omega t)$. The plane wave is incident on the particle and is scattered by it. Maxwell's equations for the total electric and magnetic field in regions 1 or 2 are [4]

$$\nabla \bullet \left[N^2(r)\mathbf{E}\right] = 0 \tag{1a}$$

$$\nabla \bullet c\mathbf{B} = 0 \tag{1b}$$

$$\nabla \times \mathbf{E} = ik\, c\mathbf{B} \tag{1c}$$

$$\nabla \times c\mathbf{B} = -ikN^2(r)\mathbf{E}. \tag{1d}$$

The TE and TM fields satisfying Eqs. (1a)–(1d) may be written in terms of TE and TM scalar radiation potentials Ψ^{TE} and Ψ^{TM} as (see Section 17.5 of [53]).

$$\mathbf{E^{TE}}(r,\theta,\varphi) = -(\mathbf{r} \times \nabla\Psi^{TE}) \tag{2a}$$

$$c\mathbf{B^{TE}}(r,\theta,\varphi) = (i/k)[\nabla \times (\mathbf{r} \times \nabla\Psi^{TE})] \tag{2b}$$

$$\mathbf{E^{TM}}(r,\theta,\varphi) = \{(i/[kN^2(r)]\}[\nabla \times (\mathbf{r} \times \nabla\Psi^{TM})] \tag{2c}$$

$$c\mathbf{B^{TM}}(r,\theta,\varphi) = (\mathbf{r} \times \nabla\Psi^{TM}), \tag{2d}$$

where the scalar radiation potentials satisfy the differential equations

$$\nabla^2\Psi^{TE} + k^2N^2(r)\Psi^{TE} = 0 \tag{3a}$$

$$\nabla^2\Psi^{TM} - 2Q(r)\partial\Psi^{TM}/\partial r - 2[Q(r)/r]\Psi^{TM} + k^2N^2(r)\Psi^{TM} = 0. \tag{3b}$$

The quantity $Q(r)$ is defined in regions 1 and 2 as

$$Q(r) \equiv N\prime(r)/N(r), \tag{4}$$

where the "prime" symbol indicates the derivative of a function with respect to its argument. The solution of Eqs. (3a) and (3b) can be written as

$$\Psi^{TE}(r,\theta,\varphi) = (1/kr)\sum_{n=1}^{\infty}\sum_{m=-n}^{n} g_{n,m}^{TE}F_n^{TE}(r)P_n^{|m|}[\cos(\theta)]\ \exp(im\varphi), \tag{5a}$$

$$\Psi^{TM}(r,\theta,\varphi) = (1/kr)\sum_{n=1}^{\infty}\sum_{m=-n}^{n} g_{n,m}^{TM} N(r) F_n^{TM}(r) P_n^{|m|}[\cos(\theta)] \exp(im\varphi), \tag{5b}$$

where n is the partial wave number, m is the azimuthal mode number, $P_n^{|m|}$ are associated Legendre functions, and $g^{TE}{}_{n,m}$ and $g^{TM}{}_{n,m}$ are beam shape coefficients, which for a plane wave propagating in the $+z$ direction will later be given the values

$$g_{n,1}^{TM} = g_{n,-1}^{TM} = i\, g_{n,1}^{TE} = -i\, g_{n,-1}^{TE} = i^n\,(2n+1)/[n(n+1)]\ , \tag{6}$$

with all other beam shape coefficients being zero. It should be noted that $i^n\,(2n+1)/[n(n+1)]$ is usually pulled out as a separate multiplicative factor multiplying the beam shape coefficients (see p.121 of [1] and Eqs. (III.1) and(III.2) of [54]). The radial functions $F^{TE}{}_n(r)$ and $F^{TM}{}_n(r)$ satisfy the same differential equation [7]

$$F_n''(r) + p^2(r)F_n(r) = 0 \tag{7}$$

with the superscript TE or TM being understood, and where the position-dependent wave number $p(r)$ is given by

$$p(r) = \left[k^2M^2(r) - n(n+1)/r^2\right]^{1/2}, \tag{8}$$

with

$$M^2(r) = N^2(r) \tag{9a}$$

for the TE polarization, and

$$M^2(r) = N^2(r) + \left[Q'(r) - Q^2(r)\right]/k^2 \tag{9b}$$

for the TM polarization. The boundary conditions for the TE radial function are that $F_n(r)$ and $F'_n(r)$ are everywhere continuous. The boundary conditions for the TM radial function are that $N(r)\,F_n(r)$ and $[Q(r)\,F_n(r)+F'_n(r)]/N(r)$ are everywhere continuous. Traditionally, the radial part of Ψ^{TM} has been written somewhat differently so that its differential equation contains both second derivative and first derivative terms [2,4,5,35]. But in the development here it is found to be helpful to consider the same differential equation for the radial function for both polarizations, placing the extra complication for the TM polarization into the position-dependent wave number and the boundary conditions at $r = a$ instead [7]. A similar strategy has been employed for propagation of an electromagnetic plane wave through plane-parallel media (see [55], Section 4.9 of [56], and Section 4.1 of [57]). It should also be noted that for a particle of constant refractive index, the solution of Eq. (7) for both polarizations is a linear combination of Riccati–Bessel and Riccati–Hankel functions (see Eq. (10.3.1) of [58]), which when summed over partial waves, is the contribution of the incoming plane wave and the outgoing scattered wave. Similarly, since one has $N_2(r) = 1$ and $Q_2(r) = 0$ in the present situation, the solution of $F^{TE}{}_n(r)$ and $F^{TM}{}_n(r)$ in region 2 outside the particle is also a linear combination of Riccati–Bessel and Riccati–Hankel functions.

Substituting Eqs. (5a), (5b), (6) into Eqs. (2a), (2c) for plane wave incidence, the total electric field in the $r\to\infty$ far-zone limit is given by

$$\mathbf{E^{TE}}(r,\theta,\varphi) \to (E_0/kr)\sum_{n=1}^{\infty} i^n\{(2n+1)/[n(n+1)]\}F^{TE}{}_n(r) \times[\pi_n(\theta)\cos(\varphi)\mathbf{u}_\theta - \tau_n(\theta)\sin(\varphi)\mathbf{u}_\varphi] \tag{10a}$$

$$\mathbf{E^{TM}}(r,\theta,\varphi) \to (E_0/kr)\sum_{n=1}^{\infty} i^n\{(2n+1)/[n(n+1)]\}F_n^{TM}(r) \times[\tau_n(\theta)\cos(\varphi)\mathbf{u}_\theta - \pi_n(\theta)\sin(\varphi)\mathbf{u}_\varphi], \tag{10b}$$

where the angular functions of Lorenz–Mie theory are

$$\pi_n(\theta) = [1/\sin(\theta)]P_n^1[\cos(\theta)] \tag{11a}$$

$$\tau_n(\theta) = (d/d\theta)P_n^1[\cos(\theta)]. \tag{11b}$$

3. WKB solution of the radial differential equation

The differential equation for $F_n(r)$ in Eq. (7) may be rewritten as a one-dimensional Schrödinger-like equation

$$-d^2F_n/dr^2 + \left\{\left[(n+1/2)^2/r^2\right] - k^2\left[M^2(r) - 1\right]\right\}F_n(r) = k^2F_n(r). \tag{12}$$

The second derivative term on the left side of the equation is the effective kinetic energy, the remainder of the left side is the effective potential energy, and the right side is the effective total energy, where the Langer modification $n(n+1) \approx (n+1/2)^2$ for $n \gg 1$ has been used (see Section 5.1 of [43]). The $k^2[M^2(r) - 1]$ term is analogous to the real potential, and the $(n+1/2)^2/r^2$ term is analogous to the centrifugal potential. In region 2 outside the particle, $F_n(r)$ is the sum of incoming and outgoing traveling waves whose amplitude and phase are position-dependent. In region 1 inside the particle, the classical turning point of the wave motion, r_t, is given implicitly by

$$(n+1/2) = kr_tM_1(r_t). \tag{13}$$

The location of the classical turning point for the TE and TM polarizations differs slightly due to the polarization-dependence of $M_1(r)$ in Eqs. (9a) and (9b). Thus in region 1 inside the particle, $F_n(r)$ again consists of incoming and outgoing traveling waves for $r_t < r \le a$, and is evanescent for $0 \le r < r_t$.

The solution of Eq. (12) can be accurately approximated in the short wavelength limit, $ka \gg 1$ using the WKB method. The evanescent wave for $0 \le r < r_t$ is approximately, (see Eq. (8.17) of [37]),

$$F_n(r) \approx C_n^{tunnel}|p_1(r)|^{-1/2}\exp\left[\int_{r_t}^{r}|p_1(r')|dr'\right]. \tag{14a}$$

The incoming and outgoing traveling waves inside the particle in region 1 for $r_t < r \le a$ are (see Eq. (8.10) of [37])

$$F_n(r) \approx C_n^{incoming}p_1(r)^{-1/2}\exp\left[-i\int_{r_t}^{r}p_1(r')dr'\right] + C_n^{outgoing}p_1(r)^{-1/2}\exp\left[i\int_{r_t}^{r}p_1(r')dr'\right], \tag{14b}$$

and the incoming and outgoing traveling waves outside the particle in region 2 for $r \ge a$ are

$$F_n(r) \approx I_n^{incoming}p_2(r)^{-1/2}\exp\left\{-i\left[\int_{r_t}^{a}p_1(r')dr' + \int_{a}^{r}p_2(r')dr'\right]\right\} + S_n^{outgoing}p_2(r)^{-1/2}\exp\left\{i\left[\int_{r_t}^{a}p_1(r')dr' + \int_{a}^{r}p_2(r')dr'\right]\right\}. \tag{14c}$$

The integrals in Eqs. (14b) and (14c) keep track of the phase accumulated from r_t to r, the exponential in Eq. (14a) provides the increased damping of the evanescent wave as it penetrates progressively deeper into the classically forbidden region, and the amplitude factor in the denominator of Eqs. (14b) and (14c) reflects the fact that a quantum mechanical particle has a low probability to be found in spatial intervals where its velocity is large (see p.318 of [37]). The quantities C_n^{tunnel}, $C_n^{incoming}$, $C_n^{outgoing}$, and $S_n^{outgoing}$ are unknown constants to be evaluated. The quantity $I_n^{incoming}$ is half of the incoming plane wave amplitude in region 2 whose value will be adjusted later in order to produce a plane wave of electric field strength E_0. Lastly, the quantity $S_n^{outgoing}$ is the partial wave scattering amplitude, without the contribution of diffraction included. The second half of the incoming plane wave amplitude and the diffracted amplitude exactly compensate for each other and will be added in later. For scattering by a homogeneous particle, Eq. (7) is the differential equation for Riccati–Bessel and

Riccati–Hankel functions, and the WKB procedure applied to this case gives the first term of Debye's asymptotic expansion of the Bessel functions (see Eqs. (9.3.7), (9.3.8), (9.3.15), (9.3.16), (9.3.23), and (9.3.24) of [58]).

Applying the WKB method to the general situation here, the coefficients C_n^{tunnel} and $C_n^{outgoing}$ are related to $C_n^{incoming}$ by realizing that for $r \approx r_t$ the effective potential decreases approximately linearly, so that $F_n(r)$ is proportional to an Airy function there (see Section 8.3 of [37]). The boundary condition matching at r_t gives rise to a phase shift of $-\pi/2$ of the outgoing wave with respect to that of the incoming wave. The coefficients $C_n^{incoming}$ and $S_n^{outgoing}$ are then related to $I_n^{incoming}$ by applying the TE or TM boundary conditions at $r = a$. After much algebra in implementing these boundary conditions, one obtains for the TE polarization

$$S_n^{outgoing} = I_n^{incoming} \exp(-2i\Delta_n) \times [-A_n + B_n \exp(i\Omega_n)]/[D_n - E_n \exp(i\Omega_n)] \tag{15}$$

where

$$A_n = \left[p_1(a)p_1^{in}(a)\right]^{1/2} - \left[p_2(a)p_2^{in}(a)\right]^{1/2} \tag{16a}$$

$$B_n = \left[p_1(a)p_1^{out}(a)\right]^{1/2} + \left[p_2(a)p_2^{in}(a)\right]^{1/2} \tag{16b}$$

$$D_n = \left[p_1(a)p_1^{in}(a)\right]^{1/2} + \left[p_2(a)p_2^{out}(a)\right]^{1/2} \tag{16c}$$

$$E_n = \left[p_1(a)p_1^{out}(a)\right]^{1/2} - \left[p_2(a)p_2^{out}(a)\right]^{1/2} \tag{16d}$$

and

$$\left[p_1^{in}(a)\right]^{1/2} = [p_1(a)]^{1/2}\left\{1 - ip'_1(a)/\left[2p_1^2(a)\right]\right\} \tag{17a}$$

$$\left[p_1^{out}(a)\right]^{1/2} = [p_1(a)]^{1/2}\left\{1 + ip'_1(a)/\left[2p_1^2(a)\right]\right\} \tag{17b}$$

$$\left[p_2^{in}(a)\right]^{1/2} = [p_2(a)]^{1/2}\left\{1 - ip'_2(a)/\left[2p_2^2(a)\right]\right\} \tag{17c}$$

$$\left[p_2^{out}(a)\right]^{1/2} = [p_2(a)]^{1/2}\left\{1 + ip'_2(a)/\left[2p_2^2(a)\right]\right\}, \tag{17d}$$

with $p_{1,2}'(a) = [dp_{1,2}(r)/dr]_{r=a}$. The position-dependent wave number of Eq. (8) at the particle surface in region 1 is $p_1(a)$, and is $p_2(a)$ at the surface in region 2. It will turn out that the last term on the right side of Eqs. (17a)–(17d) is a $1/ka$ correction to the first term, and is neglected in the dominant term of ray theory and Airy theory in Eqs. (52) and (63) below. For the TM polarization and taking into account that $N_2(a) = 1$ and $Q_2(a) = 0$ outside the particle, everything is the same as for the TE polarization except that Eqs. (16a)–(16d) are replaced by

$$A_n = \left[p_1(a)p_1^{in}(a)\right]^{1/2}/N_1^2(a) - \left[p_2(a)p_2^{in}(a)\right]^{1/2} + iQ_1(a)/N_1^2(a) \tag{18a}$$

$$B_n = \left[p_1(a)p_1^{out}(a)\right]^{1/2}/N_1^2(a) + \left[p_2(a)p_2^{in}(a)\right]^{1/2} - iQ_1(a)/N_1^2(a) \tag{18b}$$

$$D_n = \left[p_1(a)p_1^{in}(a)\right]^{1/2}/N_1^2(a) + \left[p_2(a)p_2^{out}(a)\right]^{1/2} + iQ_1(a)/N_1^2(a) \tag{18c}$$

$$E_n = \left[p_1(a)p_1^{out}(a)\right]^{1/2}/N_1^2(a) - \left[p_2(a)p_2^{out}(a)\right]^{1/2} - iQ_1(a)/N_1^2(a), \tag{18d}$$

where $N_1(a)$ is the refractive index at the surface of the particle in region 1. The last term in Eqs. (18a)–(18d) is again a $1/ka$ correction to the dominant term, and is neglected in Eqs. (52) and (63). The phase accumulated between r_t and a in region 1 is

$$\Delta_n = \int_{r_t}^{a} p_1(r')dr' = \int_{r_t}^{a} k\, M_1(r')\, dr' \left\{1 - (n+1/2)^2/[kr'M_1(r')]^2\right\}^{1/2} \tag{19a}$$

and

$$\Omega_n = 2\Delta_n - \pi/2. \tag{19b}$$

For convenience, using the new variable

$$u \equiv (n+1/2)/ka, \tag{20}$$

the expression for Δ_n in Eq. (19a) can be rewritten in terms of dimensionless quantities as

$$\Delta(u) = ka\int_{w_t}^{1} (dw/w)\left[w^2M_1^2(w) - u^2\right]^{1/2} \tag{21}$$

with

$$w \equiv r/a. \tag{22}$$

The classical turning point w_t of Eq. (13) is implicitly a function of u via

$$w_t M_1(w_t) = u. \tag{23}$$

It should be noted that for scattering by a homogeneous particle of refractive index N, the integral in Eq. (21) can be evaluated analytically giving

$$\Delta(u) = Nka\left\{\left[1 - (u/N)^2\right]^{1/2} - (u/N)\arcsin\left[1 - (u/N)^2\right]^{1/2}\right\}. \tag{24}$$

Similarly, if the particle were not present in the general situation, the phase accumulated between r_t^0 and a, the new classical turning point corresponding to $M_1(r) = 1$, is

$$\Delta_n^0 = \int_{r_t^0}^{a} p_1^0(r')dr' = \int_{r_t^0}^{a} k\, dr'\left[1 - (n+1/2)^2/(kr')^2\right]^{1/2} \tag{25a}$$

or

$$\Delta^0(u) = ka\left\{\left(1-u^2\right)^{1/2} - u\ \arcsin\left[\left(1-u^2\right)^{1/2}\right]\right\}. \tag{25b}$$

The phase accumulated in region 1 from a to r_t and back to a again, including the WKB reflection phase shift of $-\pi/2$ at r_t due to the Airy function matching at the classical turning point, is

$$\Omega(u) = 2\Delta(u) - \pi/2. \tag{26}$$

If the particle were not present, the phase accumulated in region 1 from a to r_t^0 and back to a again, including the $-\pi/2$ reflection phase shift at r_t^0, would be

$$\Omega^0(u) = 2\Delta^0(u) - \pi/2. \tag{27}$$

4. Debye series decomposition of the scattered wave

The partial wave amplitude of the scattered wave in region 2 outside the particle given by Eq. (15) can be expanded in terms of an infinite series, known as the Debye series, which is a sum over the various ways an incoming partial wave can interact with the surface of the spherical particle [44,59]. After simplifying the denominator of Eq. (15) and then expanding it as an infinite series, one obtains after a reasonable amount of algebra

$$S_n^{scattered} = I_n^{incident} \exp(-2i\Delta_n) \left[R_n^{212} + \sum_{p=1}^{\infty} T_n^{21}\left(R_n^{121}\right)^{p-1} T_n^{12} \exp(ip\Omega_n) \right], \quad (28)$$

where the Debye process p in Eq. (28) should not be confused with the position-dependent wave number $p(r)$ of Eq. (8). The amplitude for a partial wave to be transmitted into and out of the spherical particle is

$$T_n^{21}T_n^{12} = (-A_nE_n + B_nD_n)/D_n^2 = 4p_1(a)p_2(a)/D_n^2 \quad \text{for TE}$$
$$= 4p_1(a)p_2(a)/\left[N_1^2(a)D_n^2\right] \quad \text{for TM}. \quad (29a)$$

The amplitude for a partial wave to be reflected at the particle surface from the outside is

$$R_n^{212} = -A_n/D_n, \quad (29b)$$

and the amplitude for a partial wave to be reflected at the particle surface from the inside is

$$R_n^{121} = E_n/D_n. \quad (29c)$$

The $\exp(ip\Omega_n)$ term in Eq. (28) describes the phase accumulated during p-1 internal reflections as the traveling wave in region 1 goes back and forth from a to r_t and back to a again a total of p times.

The WKB approximation to the to the oscillatory wave in the far-zone of region 2 also requires the evaluation of the phase accumulated in region 2 between a and $r\rightarrow\infty$ after the scattered wave leaves the particle,

$$\int_a^r p_2(r')dr' = kr - (n+1/2)\pi/2 - \Delta_n^0 = (ka)w - kau\,\pi/2 - \Delta^0(u). \quad (30)$$

Substituting this along with Eq. (28) into Eq. (14c), and with suitable adjustment of the value of $I_n^{incident}$ in order to produce an incident plane wave with field strength E_0, gives

$$F_n(r) \rightarrow \psi_n(kr) - (1/2)\zeta_n^{(1)}(kr)\{1 - R_n^{212}\exp\left(-i\Omega_n^0\right) - \sum_{p=1}^{\infty} T_n^{21}\left(R_n^{121}\right)^{p-1}T_n^{12}\exp\left[i\left(p\Omega_n - \Omega_n^0\right)\right]\} \quad (31)$$

where ψ_n is a Riccati–Bessel function and $\zeta^{(1)}{}_n$ is a Riccati–Hankel function of the first kind. The factor $\exp(-i\Omega_n{}^0)$ is the phase accumulated by the externally reflected partial wave, and $\exp[i(p\Omega_n-\Omega_n{}^0)]$ is the total phase accumulated by the partial wave participating in the $p\geq1$ Debye process. This result, substituted into Eqs. (10a) and (10b) and using the far-zone limit of the Riccati–Bessel and Riccati–Hankel functions (see Eqs. (9.2.1)–(9.2.3) of [58]), gives the incident and far-zone scattered electric fields for each polarization.

5. Stationary phase evaluation of the scattered electric field

The sum over partial waves in Eqs. (10a) and (10b) for the p term of the Debye series can be approximately evaluated in the short wavelength limit using the following five steps. First, the Lorenz–Mie angular functions of Eqs. (11a) and (11b) are approximated by (see p.212 of [1])

$$\tau_n(\theta) \approx \left\{2(n+1/2)^3/[\pi\,\sin(\theta)]\right\}^{1/2}\cos[(n+1/2)\theta - \pi/4] + O\left[(n+1/2)^{1/2}\right] \quad (32a)$$

$$\pi_n(\theta) \approx \left\{2(n+1/2)/\left[\pi\,\sin^3(\theta)\right]\right\}^{1/2}\sin[(n+1/2)\theta - \pi/4] \quad (32b)$$

for $0 \ll \theta \ll \pi$ and $n \gg 1$. The angular function $\pi_n(\theta)$ and the $O[(n+1/2)^{1/2}]$ correction to the dominant $O[(n+1/2)^{3/2}]$ portion of $\tau_n(\theta)$ are neglected, since retaining them would provide only $1/ka$ corrections to the dominant portion of the scattered fields. Second, the scattering angle θ, which is confined to the interval $0\leq\theta\leq\pi$, is related to the deflection angle Θ via

$$\Theta = 2\pi t + q\theta \quad (33)$$

where $t=0,1,2,\ldots$ and $q=\pm1$.

Third, each p term of the Debye series of the partial wave scattering amplitude S_n is considered separately in Eq. (28). Fourth, the sum over the partial waves is approximated by an integral over the associated impact parameter u of Eq. (20). The scattered electric field for the p term of the Debye series for either polarization is then

$$\mathbf{E_p^{scattered}}(r,\theta,\varphi) \approx K\,\mathbf{\Pi}\int_{-\infty}^{\infty} du\,A_p(u)\exp\left\{i\left[\Phi_p^+(u,\Theta) + \Phi_p^-(u,\Theta)\right]\right\}. \quad (34)$$

The u-independent polarization vector is given by

$$\mathbf{\Pi} = -\cos(\varphi)\mathbf{u}_\theta \quad \text{for TM}$$
$$= \sin(\varphi)\mathbf{u}_\varphi \quad \text{for TE}, \quad (35)$$

the slowly varying u-dependent amplitude factor is

$$A_p(u) = u^{1/2}T^{21}(u)\left[R^{121}(u)\right]^{p-1}T^{12}(u), \quad (36)$$

the rapidly varying u-dependent phase

$$\Phi_p^{\pm}(u,\Theta) = 2p\Delta(u) - 2\Delta^0(u) \pm qukа\Theta \quad (37)$$

is a function of both u and the deflection angle Θ, and the u-independent complex constant K is

$$K = [\exp(ikr)/(kr)](ka)\,[2\pi\,\sin(\theta)]^{-1/2} \times\exp[i(-p+2\mp2qt\mp1/2)\pi/2]. \quad (38)$$

The $\mp$ sign in Eq. (38) follows the convention of the $\pm$ sign in Eq. (37).

Instead of taking the point of view that Θ is an independent variable in Eq. (37), it is often useful to consider it to as a function of a new independent variable $\sin(\theta_i)$ via the expression

$$\Theta^B[\sin(\theta_i)] = -\pi + 2\theta_i + 2p\,\sin(\theta_i) \times\int_{w_t}^1 (dw/w)\left[w^2M_1{}^2(w) - \sin^2(\theta_i)\right]^{-1/2} \quad (39)$$

The superscript B denotes that Eq. (39) gives the deflection angle of a geometrical ray with the angle of incidence θ_i, obtained from Bouguer's law [25–28]. Bouguer's law is a generalization of Snell's law that takes into account the fact that the ray path inside the sphere is curved since the refractive index continually changes along it (see Fig. 1). It should be noted that for a homogeneous particle of refractive index N, the interior ray path is a series of straight lines and the integral in Eq. (39) can be evaluated analytically to give

$$\Theta^B = (p-1)\pi + 2\theta_i - 2p\theta_t, \quad (40)$$

where θ_t is related to θ_i by Snell's law. For future reference, it should also be noted that for the first two terms on the right side of Eq. (37), one has

$$(d/du)\left[-2p\Delta(u) + 2\Delta^0(u)\right] = 2pkau\int_{w_t}^1 (dw/w) \times\left[w^2M_1^2(w) - u^2\right]^{-1/2} - 2ka\arcsin\left[\left(1-u^2\right)^{1/2}\right]. \quad (41)$$

As a mathematical aside, Eq. (41) relied on the fact that the contribution to $d\Delta/du$ resulting from the u-dependence of w_t in the lower limit of integration in Eq. (21) vanishes. This turns out

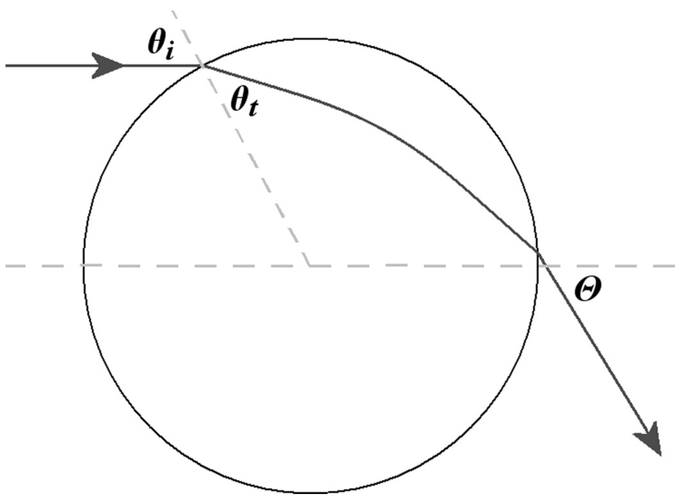

Fig. 1. Curved ray path in a radially inhomogeneous sphere having the refractive index $N_1(r)$. The angle of incidence of a ray is θ_i, the transmission angle θ_t is given by Snell's law applied at the sphere surface, and the deflection angle is Θ.

to be true as long as $d[w_t\ M_1(w_t)]/dw_t \neq 0$, or in other words, as long as the refractive index does not decrease too fast, which would cause the phenomenon of orbiting to occur (see Section 5.24 of [60], Section 3.10 of [61], and Section 5.4 of [62]). For example, using Eqs. (4) and (9c) for the TE polarization, the avoidance of orbiting requires that $|Q(w_t)| < w_t$.

Fifth, either $\Phi_p{}^{+}(u,\Theta)$ or $\Phi_p{}^{-}(u,\Theta)$ is usually locally quadratic in u, having a relative maximum or minimum stationary point at $u = u_{sp}$, for which

$$\left(\partial\Phi_p^{\pm}/\partial u\right)_{usp} = 0, \tag{42}$$

$$\left(\partial^2\Phi_p^{\pm}/\partial u^2\right)_{usp} \neq 0, \tag{43a}$$

and

$$s = -\mathrm{sgn}\left(\partial^2\Phi_p^{\pm}/\partial u^2\right)_{usp}, \tag{43b}$$

where

$$\begin{aligned} \mathrm{sgn}(x) &= 1 \quad \text{if}\, x > 0 \\ &= -1 \quad \text{if}\, x < 0. \end{aligned} \tag{44}$$

The stationary phase condition is a localized region of constructive interference in the integrand, which when integrated over the entire range of u in Eq. (34), dominates the value of integral. The other phase does not have any stationary points in u, and is thus dominated by destructive interference. It gives a negligible contribution when integrated over the entire range of u in Eq. (34). The phase Φ_p in Eq. (34) is then Taylor series expanded in powers of $(u - u_{sp})$ about the point u_{sp} keeping terms up to and including the quadratic term. The stationary phase evaluation of the resulting integral is then (see Section 8.2 of [63] and Section 4.2a of [64])

$$\int_{-\infty}^{\infty} du\ A_p(u)\exp[i\Phi_p(u,\Theta)] \approx A_p(u_{sp})\exp[i\Phi_p(u_{sp},\Theta_{sp})]$$
$$(2\pi/ka)^{1/2}\exp(-is\pi/4)\left|\partial^2\Phi_p/\partial u^2\right|_{usp}^{-1/2} \tag{45}$$

and is interpreted as the contribution of a geometrical ray whose angle of incidence is

$$u_{sp} = \sin(\theta_i). \tag{46}$$

Using Eqs. (21), (25b) and (37), the stationary phase condition of Eq. (42) is

$$\pm q\Theta^B[(\sin(\theta_i)] = 2pu\int_{w_t}^{1}(dw/w)\left[w^2M_1^2(w) - u^2\right]^{-1/2}$$
$$-2\arcsin[\left(1 - u^2\right)^{1/2}]. \tag{47}$$

Assuming $\pm q = 1$, Eq. (47) is the same function of u that Eq. (39) was of $\sin(\theta_i)$. Thus

$$\partial\Phi_p^{\pm}/\partial u = ka\left[\Theta - \Theta^B(u)\right] \tag{48a}$$

$$\partial^2\Phi_p^{\pm}/\partial u^2 = -ka\left(d\Theta^B/du\right). \tag{48b}$$

The stationary phase deflection angle then corresponds to that predicted by Bouguer's law,

$$\Theta = \Theta^B[\sin(\theta_i)]. \tag{49}$$

Similarly, Eq. (43b) can be written as

$$s = \mathrm{sgn}\left\{\left[d\Theta^B/d\sin(\theta_i)\right]_{sp}\right\}. \tag{50}$$

As a special case, a homogeneous sphere of refractive index N gives

$$d\Theta^B/d\sin(\theta_i) = [2/\cos(\theta_i)]\{1 - p\cos(\theta_i)/[N\cos(\theta_t)]\}. \tag{51}$$

Finally, combining Eqs. (34)–(36), (38) and (45), the stationary phase approximation to the ray theory far-zone scattered electric field for the p Debye process for a radially inhomogeneous sphere is

$$\begin{aligned} \mathbf{E}_{\mathbf{p}}^{\mathbf{scattered}}(r,\theta,\varphi) &= E_0\mathbf{\Pi}[\exp(ikr)/(kr)](ka)[\sin(\theta_i)/\sin(\theta)]^{1/2} \\ &\times\left|d\Theta^B/d\sin(\theta_i)\right|^{-1/2}T^{21}(\theta_i)\left[R^{121}(\theta_i)\right]^{p-1}T^{12}(\theta_i) \\ &\times\exp\left\{ika\left[L^B(\sin\theta_i) - 2\right]\right\}\exp[i(-p+2-2t-q/2-s/2)\pi/2]. \end{aligned} \tag{52}$$

The length of the curved ray path from the entrance plane of the particle to its exit plane, normalized to the sphere radius a, is

$$\begin{aligned} L^B[\sin(\theta_i)] &= 2 - 2\cos(\theta_i) + 2p\int_{w_t}^{1}(dw/w)w^2M_1^2(w) \\ &\times\left[w^2M_1^2(w) - \sin^2(\theta_i)\right]^{-1/2}, \end{aligned} \tag{53a}$$

and is related to Eq. (39) by

$$\left\{dL^B/d[\sin(\theta_i)]\right\} = \sin(\theta_i)\left\{d\Theta^B/d[\sin(\theta_i)]\right\}. \tag{53b}$$

For scattering by a homogeneous sphere, the integral in Eq. (53a) may again be analytically evaluated giving

$$L^B[\sin(\theta_i)] = 2 - 2\cos(\theta_i) + 2pN\cos(\theta_t). \tag{54}$$

The superscript B again denotes that the quantity is identical to that obtained using Bouguer's law. Hereafter, the superscript B will be omitted for simplicity. The term $\exp\{ika[L(\sin\theta_i)-2]\}$ is called the path-length phase of the scattered electric field, and $\exp[i(-p+2-2t-q/2-s/2)\pi/2]$ is called the non-path-length phase. The leading term in the expansion of the transmission and reflection amplitudes in Eq. (52) for each polarization is identical to the flat interface Fresnel transmission and reflection coefficients (see Eqs. (4.34), (4.35), (4.40), (4.41) of [65]),

$$T^{21,TE}(\theta_i)T^{12,TE}(\theta_i) = 4N_1(a)\cos(\theta_i)\cos(\theta_t)/[N_1(a)\cos(\theta_t) + \cos(\theta_i)]^2 \tag{55a}$$

$$R^{121,TE}(\theta_i) = [N_1(a)\cos(\theta_t) - \cos(\theta_i)]/[N_1(a)\cos(\theta_t) + \cos(\theta_i)] \tag{55b}$$

$$T^{21,TM}(\theta_i)T^{12,TM}(\theta_i) = 4N_1(a)\cos(\theta_i)\cos(\theta_t)/[\cos(\theta_t) + N_1(a)\cos(\theta_i)]^2 \tag{55c}$$

$$R^{121,TM}(\theta_i) = [\cos(\theta_t) - N_1(a)\cos(\theta_i)]/[\cos(\theta_t) + N_1(a)\cos(\theta_i)] \tag{55d}$$

where refraction of the ray at the surface of the sphere is given by Snell's law,

$$N_1(a)\,\sin(\theta_t) = \sin(\theta_i). \tag{56}$$

Corrections to the O(ka) ray theory scattered electric field of Eq. (52), are at most O(1), and are small in the short wavelength limit. If the refractive index of the radially inhomogeneous sphere smoothly merges into the external refractive index so that $N_1(a) = 1$, then to leading order $R^{121,TE}(\theta_i) = R^{121,TM}(\theta_i) = 0$. The $p \geq 2$ Debye process will not occur to leading order, leaving only diffraction and transmission [9]. The $p = 1$ version of Eq. (52) for the scattered field, without the inclusion of the Fresnel coefficients, agrees with the amplitude for scalar waves to be transmitted through an infinite-range potential, that was obtained using the WKB method and the stationary phase approximation (see Eq. (64) of [60]). It also agrees with the scattered scalar wave amplitude of Eq. (27) of [51] obtained using flux conservation.

For a general radially inhomogeneous sphere, both the ray theory and Airy theory scattered fields are expressed in terms of two related fundamental integrals over the refractive index, $\Theta[\sin(\theta_i)]$ of Eq. (39) and $L[\sin(\theta_i)]$ of Eq. (53a). If $N_1(r)$ is such that these two integrals can be evaluated analytically, ray and Airy theory are also expressible analytically. As was mentioned a number of times above, this is the case for a homogeneous sphere. It also is true for a modified Luneburg lens whose refractive index profile is

$$N_1(r) = \left[2\beta - \gamma (r/a)^2\right]^{1/2}, \tag{57}$$

where β and γ are constants [11]. Analytical evaluation of these two fundamental integrals is also possible for a number of other refractive index profiles [66]. Otherwise, Eqs. (39) and (53a) must be evaluated numerically in order to obtain the scattered fields.

6. Airy theory for a scattering by a radially inhomogeneous sphere

For scattering by a homogeneous sphere with $p \geq 2$, it is well known that one of the phase functions, $\Phi_p{}^+(u,\Theta)$ or $\Phi_p{}^-(u,\Theta)$, of Eq. (37) is locally cubic in u for a certain range of deflection angles Θ. The same phenomenon is also known to occur for scattering by a radially inhomogeneous sphere for $p \geq 1$ [6,8–12,26–28]. The cubic phase function has both a relative maximum and a relative minimum which correspond to the contribution of two supernumerary rays. As the deflection angle Θ is varied, the relative maximum and minimum of $\Phi_p{}^+(u,\Theta)$ or $\Phi_p{}^-(u,\Theta)$ approach each other and coalesce when the impact parameter u reaches that of the Descartes rainbow ray

$$u_R = \sin\left(\theta_i^R\right). \tag{58a}$$

The Descartes rainbow angle

$$\Theta^R = \Theta\left[\sin\left(\theta_i^R\right)\right] \tag{58b}$$

is an extremum of the deflection angle, giving

$$\left[\partial^2\Phi_p^{\pm}/\partial u^2\right]_{uR} = -ka[\mathrm{d}\Theta/\mathrm{d}\sin(\theta_i)]_R = 0. \tag{58c}$$

In addition, we define the cubic phase front parameter h by

$$2kah \equiv \left|\partial^3\Phi_p^{\pm}/\partial u^3\right|_{uR} = ka\left|\mathrm{d}^2\Theta/\mathrm{d}\sin(\theta_i)^2\right|_R \neq 0. \tag{58d}$$

For scattering by a homogeneous sphere of refractive index N, one obtains

$$\cos\left(\theta_i^R\right) = \left(N^2-1\right)^{1/2}/\left(p^2-1\right)^{1/2} \tag{59}$$

from Eq. (58c) and

$$h = \sin\left(\theta_i^R\right)\left[N^3\cos^3\left(\theta_t^R\right) - p\cos^3\left(\theta_i^R\right)\right]/\left[N^3\cos^3\left(\theta_i^R\right)\cos^3\left(\theta_t^R\right)\right]$$
$$= \left(p^2-1\right)^2\left(p^2-N^2\right)^{1/2}/\left[p^2\left(N^2-1\right)^{3/2}\right] \tag{60}$$

from Eq. (58d).

As was the case in Eq. (45), the amplitude $A_p(u)$ in Eq. (34) is slowly varying over the extended constructive interference region of u, and can approximately be brought outside the integral. The rapidly varying phase $\exp[\mathrm{i}\Phi_p{}^+(u,\Theta)]$ or $\exp[\mathrm{i}\Phi_p{}^-(u,\Theta)]$ is then Taylor series expanded in powers of $(u\text{-}u_R)$ about the point u_R, keeping terms up to and including the cubic term. The coefficients of the linear and cubic terms are evaluated using Eqs. (48a) and (58d), respectively. Since the coefficient of the quadratic term is zero from Eq. (58c), the resulting phase integral is the integral representation of the Airy function (see Eq. (10.4.32) of [58])

$$\mathrm{Ai}(W) = 3^{1/3}/2\pi\int_{-\infty}^{\infty}\mathrm{d}u\,\exp\left[\mathrm{i}\left(u^3 + 3^{1/3}Wu\right)\right], \tag{61}$$

with

$$W = -\sigma\,(ka)^{2/3}\left(\Theta - \Theta^R\right)/h^{1/3} \tag{62a}$$

$$\sigma = \mathrm{sgn}\left[\mathrm{d}^2\Theta/\mathrm{d}\sin(\theta_i)^2\right]_R. \tag{62b}$$

The scattered electric field of the p Debye process in the vicinity of the rainbow then becomes

$$\begin{aligned}\mathbf{E}_\mathbf{p}^{\mathbf{scattered}}(r,\theta,\varphi) &= E_0\,\mathbf{\Pi}[\exp(\mathrm{i}kr)/(kr)](ka)^{7/6}\\ &\times\left[2\pi\sin\left(\theta_i^R\right)/\sin\left(\theta^R\right)\right]^{1/2}h^{-1/3}T^{21}\left(\theta_i^R\right)\\ &\times\left[R^{121}\left(\theta_i^R\right)\right]^{p-1}T^{12}\left(\theta_i^R\right)\mathrm{Ai}(W)\\ &\times\exp\left\{\mathrm{i}ka\left[L\left(\sin\theta_i^R\right) - 2\right]\right\}\exp[\mathrm{i}(-p+2-2t-q/2)\pi/2].\end{aligned} \tag{63}$$

The coalescence of the two supernumerary rays at Θ^R produces a relatively wide region of constructive interference, leading to a flatter than normal phase front. This causes an enhancement to the scattered field of O[$(ka)^{1/6}$] above the O(ka) geometrical optics background. Rainbows for which $\sigma = -1$ are called relative maximum bows, and those for which $\sigma = +1$ are called relative minimum bows because they correspond to a relative maximum or minimum, respectively, of $\Theta(\theta_i)$ at $\theta_i{}^R$. Only relative minimum bows occur for scattering by a homogeneous sphere with $p \geq 2$. But for $p = 1$ scattering by a radially inhomogeneous sphere, relative maximum bows are known to occur as well, often in association with a nearby relative minimum bow [6,12]. In addition, the far-zone non-path-length phase of the upper supernumerary ray (i.e. $\theta_i > \theta_i{}^R$) of a relative minimum bow is $\pi/2$ less than that of the lower supernumerary ray (i.e. $\theta_i < \theta_i{}^R$), while for a relative maximum bow the non-path-length phase of the upper supernumerary ray is $\pi/2$ greater. The TE polarization version of Eq. (63) for $p = 1$ without the inclusion of the Fresnel coefficients agrees with Eq. (25) of [41], Eq. (17) of [42], and Eq. (6.31) of [43], which were obtained for scalar waves undergoing rainbow scattering in a continuous infinite-range potential. The value for h of Eq. (58d) also agrees with Eq. (7) of [52] which was obtained by an analogy to the homogeneous sphere result of Eq. (60).

7. Two-rainbow coalescence and the longitudinal cusp caustic

As was mentioned above, numerical studies have shown that when the refractive index of a radially inhomogeneous sphere has a relatively localized and sharp decrease in the interior as in Fig. 2, it is possible that the $p = 1$ deflection angle Θ as a function of $\sin(\theta_i)$ can exhibit a pair of bows, a relative maximum bow followed by a relative minimum bow (see Figs. 3 and 4). For a given $N_1(r)$, let S be a measure of the magnitude of the slope of the localized refractive index decrease. In situations for which a pair of bows is able to occur, they will occur when S is relatively large. As S decreases, the two rainbows approach each other, coalesce,

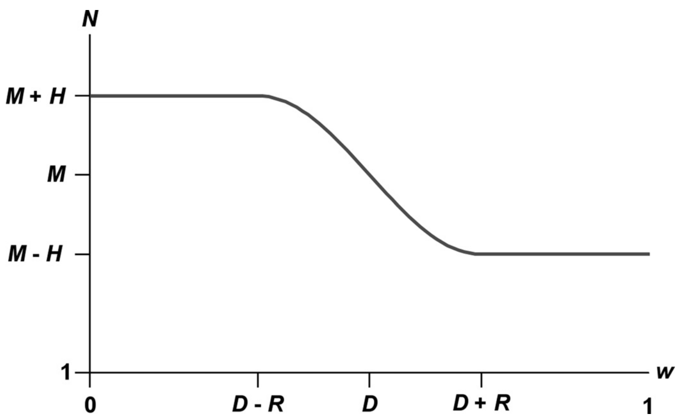

Fig. 2. Refractive index profile of Eq. (11) of [6] that exhibits a strong localized decrease in the particle interior. The magnitude of the constant slope of the decrease is S.

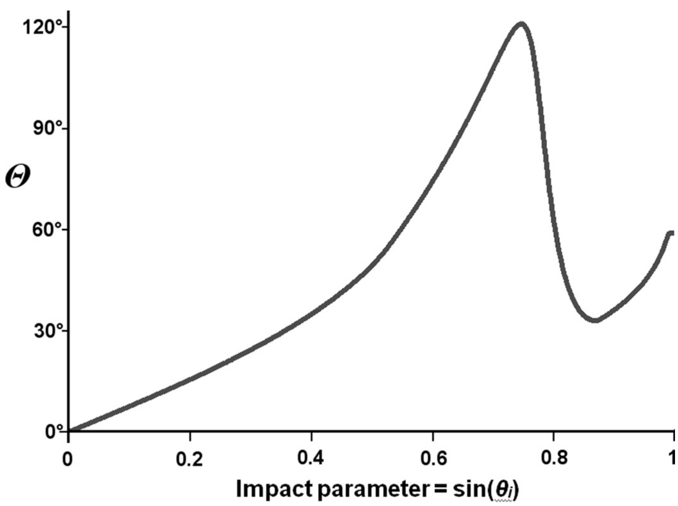

Fig. 3. Ray theory deflection angle Θ as a function of $\sin(\theta_i)$ from Eq. (39) for $p=1$ and the refractive index profile of Fig. 2 with $D=0.5$, $R=0.2$, $M=1.5$, and $H=0.25$.

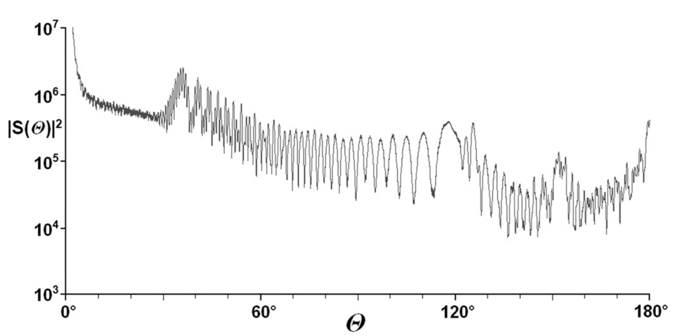

Fig. 4. Wave theory TE scattered intensity as a function of the scattering angle θ for an incident wavelength of $\lambda = 0.65\,\mu$m, sphere radius $a = 100\,\mu$m, and the refractive index profile of Fig. 2. Rainbows in the $p=1$ Debye channel occur at $\theta \approx 34°$, 120°, and a large number of supernumeraries lie between them. The scattered fields were computed using multi-layer Mie theory, and the result was "sun-smoothed" by convolving the resulting intensity with a 0.5° diameter to remove the high-frequency interference structure between the various Debye series terms.

and are extinguished. For yet smaller S, the deflection angle becomes a monotonically increasing function of $\sin(\theta_i)$. This behavior is a signal that the two bows should not be considered as separate entities, but are instead parts of a more extensive scattering feature. The relative maximum and minimum bows were conjectured in [6,12] to be the two curved fold caustic arms of a longitudinal

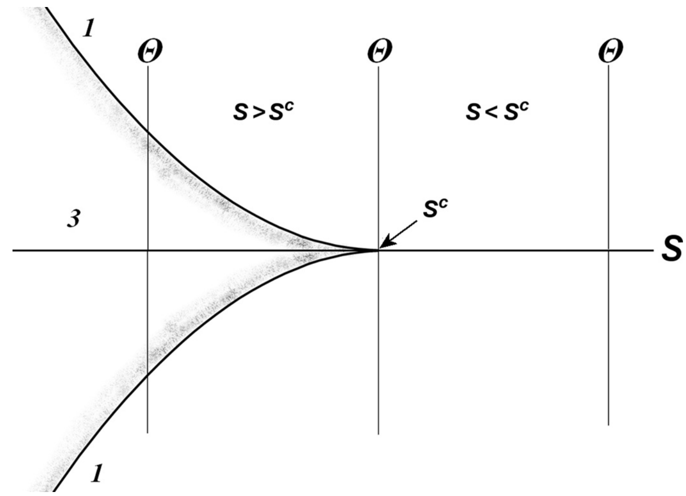

Fig. 5. Longitudinal cusp caustic as a function of Θ and S, with S increasing to the left in the figure. The number of participating rays inside and outside the caustic is listed, and the shaded region indicates the region where ray interference occurs.

cusp caustic that are joined together at the cusp point [67,68]. This is illustrated in Fig. 5. The longitudinal cusp caustic is a function of two independent variables, which were assumed in [6,12] to be Θ and S.

The mathematical justification for this conjecture is as follows. Consider Taylor series expanding the phase $\Phi_p{}^{\pm}$ of Eq. (37) in powers of $(u - u_C)$ about the point u_C which is given by

$$\left(\partial^3\Phi_p/\partial u^3\right)_{uC} = -ka\left[d^2\Theta/\partial\sin(\theta_i)^2\right]_C = 0. \tag{64}$$

The effective impact parameter of the corresponding ray is

$$u_C = \sin\left(\theta_i^C\right), \tag{65a}$$

and its deflection angle is given by Bouguer's law

$$\Theta^C = \Theta\left[\sin\left(\theta_i^C\right)\right]. \tag{65b}$$

If $\Phi_p{}^+(u,\Theta)$ or $\Phi_p{}^-(u,\Theta)$ is approximately locally quartic in u, there will always be a single value of u_C, which may weakly depend on S, for which Eq. (64) is satisfied. When S is relatively large so that the deflection angle exhibits both a relative maximum and a relative minimum bow, the impact parameter $\sin(\theta_i{}^C)$ lies roughly midway between the impact parameters of the two bows, and Θ^C lies roughly midway between their deflection angles, as is evident in Fig. 3. The cusp caustic parameter H is defined by

$$kaH \equiv \left|\partial^4\Phi_p^{\pm}/\partial^4 u\right|_{uC} = ka\left|d^3\Theta/d\sin(\theta_i)^3\right|_C \neq 0. \tag{65c}$$

Although it has not been explicitly mentioned thus far, the deflection angle of Eq. (39) for a fixed value of $\sin(\theta_i)$ depends on $N(r)$. This is evident in Eq. (40) for a homogeneous sphere, as well as for quantities derived from Θ such as in Eqs. (51), (59) and (60). This detail was only of passing interest in ray theory and Airy theory where the primary focus was determining the scattered intensity as a function of Θ for a particle with a fixed $N_1(r)$. But for the longitudinal cusp, this detail takes on an added importance since the complete cusp morphology is exhibited only as both Θ and S are varied. Thus, the numerical value of $[d\Theta/d\sin(\theta_i)]_C$ will in general depend on S and be nonzero. But if it were to vanish when $S = S^C$, i.e.

$$[d\Theta/d\sin(\theta_i)]_{C,SC} = 0, \tag{66}$$

Eq. (66) along with Eq. (64) corresponds to the coalescence and extinguishing of the relative maximum and minimum bows at the cusp point. As a practical matter, not all refractive index profiles $N_1(r)$ will be able to simultaneously satisfy Eqs. (64) and (66). One

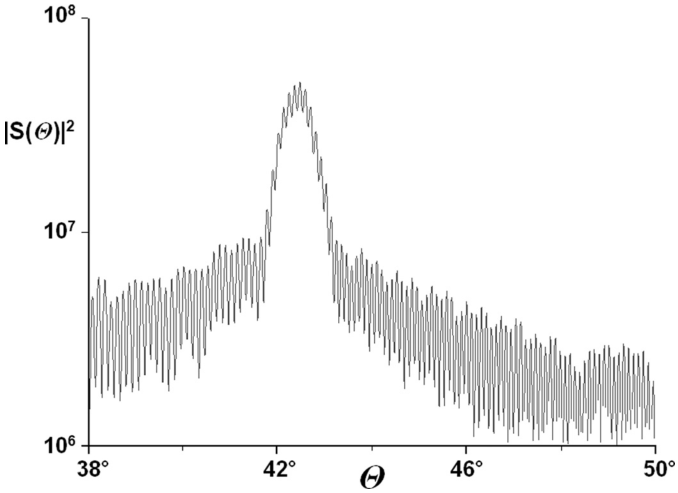

Fig. 6. Wave theory TE scattered intensity as a function of the scattering angle θ for an incident wavelength of $\lambda = 0.65\,\mu$m, sphere radius $a = 200\,\mu$m, and the refractive index profile of Fig. 2 with $D = 0.3333$, $R = 0.32$, $M = 1.558$, and $H = 0.142$, corresponding to $S = S^C$. The large isolated scattering enhancement centered on $\theta \approx 42.4°$ without the appearance of a surrounding supernumerary interference pattern is the signature of the coalescence of the relative maximum and minimum bows of Fig. 4.

must resort to numerical experimentation, as was done in [6,12], to discern whether or not these conditions can be simultaneously satisfied, and two rainbows in a given Debye channel p can coalesce and be extinguished.

When such a behavior can occur, the slowly-varying amplitude factor $A_p(u)$ may again be approximately brought outside the integral in Eq. (34). The Taylor series expansion of $\exp[i\Phi_p{}^{+}(u,\Theta)]$ or $\exp[i\Phi_p{}^{-}(u,\Theta)]$ up to and including the quartic term, along with Eqs. (64) and (65c), results in the Pearcey integral [69]

$$\mathrm{Pe}(U,V) = \int_{-\infty}^{\infty} du \exp\left[i\left(u^4 + Uu^2 + Vu\right)\right] \tag{67}$$

where

$$U = -6^{1/2}(ka)^{1/2}|d\Theta/d(\sin\theta_i)|_C/H, \tag{68a}$$

$$V = 24^{1/4}(ka)^{3/4}\left(\Theta^C - \Theta\right)/H^{1/4}. \tag{68b}$$

The scattered electric field of the p Debye process in the vicinity of the cusp point then becomes

$$\begin{aligned}\mathbf{E}_{\mathbf{p}}^{\mathbf{scattered}}(r,\theta,\varphi) &= E_0\boldsymbol{\Pi}[\exp(ikr)/(kr)](ka)^{5/4} \\ &\times \left[\sin\left(\theta_i^C\right)/\sin\left(\theta^C\right)\right]^{1/2} H^{-1/4} \\ &\times \left(24^{1/2}/2\pi\right)^{1/2} T^{21}\left(\theta_i^C\right)\left[R^{121}\left(\theta_i^C\right)\right]^{p-1} T^{12}\left(\theta_i^C\right)\mathrm{Pe}^*(U,V) \\ &\times \exp\left\{ika\left[L\left(\sin\theta_i^C\right) - 2\right]\right\}\exp[i(-p+2-2t-q/2)\pi/2].\end{aligned} \tag{69}$$

As is seen in Eqs. (68a) and (68b), the cusp point Pe(0,0) occurs when $\Theta = \Theta^C$ and $[d\Theta/d\sin(\theta_i)]_{C,SC} = 0$. Three rays contribute to the scattered electric field at each point inside the longitudinal cusp, and only one ray contributes at each point outside. At the cusp point, the three contributing rays constructively interfere, giving an especially wide region of near-constant phase that produces an enhancement in the scattered field of $O[(ka)^{1/4}]$ above the $O(ka)$ geometrical optics background. This is evident in Fig. 6.

The fact that each of the two contributing bows consists of two supernumerary rays, but only three rays contribute for points inside the longitudinal cusp, can be explained as follows. When S is sufficiently large, the deflection angle axis crosses the cusp twice, first at the relative minimum bow $\Theta^R{}_{min}$, and then at the relative maximum bow $\Theta^R{}_{max}$, with $\Theta^R{}_{max} > \Theta^R{}_{min}$. This is shown in

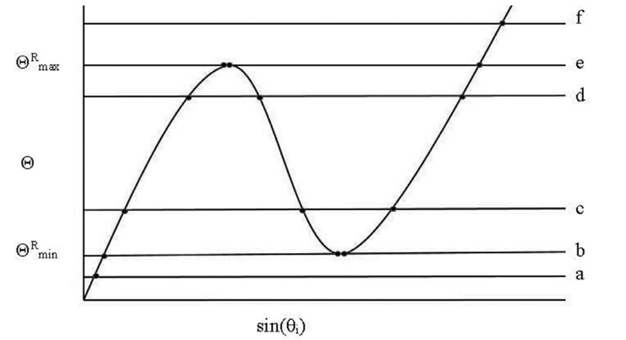

Fig. 7. Deflection angle Θ as a function of $\sin(\theta_i)$ exhibiting a relative maximum bow at $\Theta^R{}_{max}$, a relative minimum bow at $\Theta^R{}_{min}$. The evolution of the supernumerary rays is illustrated by the intersections of the deflection angle graph with the lines a–f.

Fig. 5. When $\Theta \langle \Theta^R{}_{min}$ (line a in Fig. 7), the only contributing ray at each Θ outside the cusp is the lower supernumerary ray of the relative maximum bow. When $\Theta = \Theta^R{}_{min}$ (line b), the lower and upper supernumerary rays of the relative minimum bow are created, and for $\Theta \rangle \Theta^R{}_{min}$ (line c), they join the lower supernumerary ray of the relative maximum bow inside the cusp. As Θ continues to increase (line d), the lower supernumerary ray of the relative minimum bow migrates toward the lower supernumerary ray of the relative maximum bow and takes on the role of the upper supernumerary ray of the relative maximum bow. When $\Theta = \Theta^R{}_{max}$ (line e), the two supernumerary rays of the relative maximum bow coalesce and are extinguished, leaving only the upper supernumerary ray of the relative minimum bow to contribute at each deflection angle when $\Theta > \Theta^R{}_{max}$ outside the cusp (line f). When $S = S^C$, the deflection angle axis crosses the longitudinal cusp only at the cusp point, as is shown in Fig. 5. When S decreases further, the Θ axis no longer cuts the longitudinal cusp, and only one isolated ray contributes at each Θ, as was the case in Eq. (52) and qualitatively resembles scattering by a homogeneous sphere. Similar creation and extinction of ray pairs has been observed in the reflection of sunlight off a rippled water surface [70,71].

8. Summary

Most of the formulas concerning ray theory and Airy theory for scattering of a plane wave by a radially inhomogeneous sphere have long been known, based on a combination of physical intuition, analogy, and approximate mathematical approaches. For example, there have been many applications of the well-known formula for the deflection of a ray by a radially inhomogeneous sphere as a function of its angle of incidence, based on Bouguer's law. But fewer applications, most notably rainbow refractometry, have been concerned with various features of the scattered intensity as a function of deflection angle. In this study, all the results of ray theory and Airy theory are obtained as a series of short-wavelength approximations applied to the generalization of exact Lorenz–Mie theory for electromagnetic wave scattering by a radially inhomogeneous sphere.

The basic approach followed here has long been used in short-wavelength scattering of scalar quantum mechanical waves by a potential of infinite range. What has been done here is to borrow this procedure and adapt it to (i) the vector nature and polarization-dependence of electromagnetic waves, and (ii) to scattering by a particle whose refractive index is cut off, either sharply or smoothly, at its radius. This approach provides a first-principles derivation of radially inhomogeneous sphere ray theory and Airy theory, and permits further generalization phenomena based on optical caustics, such as the merging of two rainbows in a given

Debye series channel in the context of the longitudinal cusp caustic.

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