# Electromagnetic interaction with two eccentric spheres

**Fabio Mangini,[1,]* Nicola Tedeschi,[1] Fabrizio Frezza,[1] and Ari Sihvola[2]**

[1]*Department of Information Engineering, Electronics and Telecommunications, "La Sapienza" University of Rome, Via Eudossiana 18, 00184 Roma, Italy*

[2]*Department of Radio Science and Engineering, Aalto University, Otakaari 5A, 00076 Espoo, Finland*

**Corresponding author: mangini@diet.uniroma1.it*

Received November 12, 2013; revised February 11, 2014; accepted February 14, 2014;
posted February 14, 2014 (Doc. ID 201087); published March 19, 2014

In this paper, we consider the interaction of an electromagnetic field with two eccentric spheres. We propose a quasi-static approach in order to calculate the scattered field and the polarizability and the effective permittivity of the eccentric spheres. We analyze the behavior of the scattering parameters as a function of the dimension and position of the spherical inclusions. Moreover, we consider the case of plasmonic spheres and study the behavior of the plasmon resonances for different reciprocal positions of the two spheres. © 2014 Optical Society of America

*OCIS codes:* (260.2110) Electromagnetic optics; (260.5740) Resonance; (290.5850) Scattering, particles; (050.2065) Effective medium theory.

http://dx.doi.org/10.1364/JOSAA.31.000783

## 1. INTRODUCTION

Analysis of the scattering by spherical objects with conducting or dielectric inclusions is a classical topic in electromagnetic research. The first result, after the works by Mie and Lorentz on the electromagnetic scattering by a spherical object, can be considered the work on the scattering by two concentric spheres [1], and that on the scattering by a perfectly conducting sphere embedded in a dielectric one [2]. This problem, of great interest from both a theoretical and a practical point of view, has been deeply studied, also in the case of lossy and active media [3]. Moreover, the analogous problem in acoustics has been studied, too [4,5].

The scattering problem by a spherical object with an eccentric inclusion has been studied in the literature as a generalization of the scattering by two concentric spheres. Quasi-static analysis of the interaction of an electric field with a sphere with a spherical eccentric inclusion has been presented in [6]. Moreover, the dynamic Mie–Lorentz-based approach to the problem has been investigated, and there are many results on it in the literature [7–11]. The approach followed in these works is based on Mie scattering by a sphere with the application of the addition theorem to consider spheres with different centers [12–14]. However, the problem has been addressed with other methods, e.g., with the mode-matching technique and with the Green's function approach [15,16]. Furthermore, many generalizations to the problem have been attempted. The case of shaped-beam incidence has been studied with a generalization of the Mie–Lorentz method in [17–19]. The case of an arbitrary number of spherical inclusions has been considered, too, [20,21]. Finally, the case of irregular inclusions has been presented [22]. It can be useful to note that the two-dimensional case of eccentric cylindrical inclusions embedded in infinitely long cylinders has also been studied in the literature [23,24].

The numerous cited works are motivated by the large number of applications of the scattering problem by one or several inclusions in a spherical object. For example, many applications have been proposed in order to study the electromagnetic scattering by the human brain [25,26]. Moreover, the eccentric sphere model has been applied to understand the role of the inhomogeneities in the heart [27]. Furthermore, in recent works, the scattering by single-nuclear cells has been solved with an eccentric-sphere model [28]. Another important field of application of the eccentric-sphere scattering problem is related to nanoparticle fabrication and application. Anisotropic nanostructures, in recent years, have attracted increasing attention, because they have shown superior characteristics with respect to the isotropic structures [29–32].

In this paper, we propose a quasi-static analysis of an eccentric spherical inclusion embedded in a dielectric sphere in order to deduce the properties of the scatterer as a homogenized sphere. Effective-medium analysis of a composite sphere has previously been attempted in [33]. We consider a similar situation, but our purpose is to understand the role of the inclusion position on the effective-medium parameters; i.e., we want to understand how the position and the dimensions of the spherical inclusion affect the scattered field. Moreover, we are interested in the determination of the plasmon resonances when the spheres have plasmonic behavior. The plasmon resonances of a plasmonic sphere with an eccentric inclusion have been studied in the literature and play an important role in the metamaterial design [27,34–36]. Moreover, plasmonic nanostructures with anisotropic behaviors have been proposed for biomedical applications to realize artificial molecules [37].

In Section 2, we present the quasi-static model of an eccentric spherical inclusion embedded in a dielectric sphere, and we obtain an analytical expression for the polarizability and for the effective permittivity of the sphere with the eccentric inclusion. In Section 3, we validate the quasi-static results through comparisons with finite-element-method-based commercial software. Moreover, we present a parametric study to clarify the role of the inclusion position and dimension on the polarizability. Furthermore, we present an analysis of the plasmon resonances of the structure. Finally, in Section 4, conclusions are drawn.

## 2. QUASI-STATIC INTERACTION

We start our study by considering a dielectric sphere with an eccentric spherical perfect electric conductor (PEC) inclusion; see Fig. 1. We consider the center of the host sphere on the origin $O$ of a Cartesian coordinate system. We call $R_s$ the radius of the external sphere and $R_c$ the radius of the internal sphere; the latter one is situated at a distance $d$ from the center of the host sphere. The centers of the two spheres are located on the $z$ axis. Since the system is in azimuthal symmetry, the potential function will be independent of $\varphi$, i.e., the angle between the projection of the position vector on the $(x, y)$ plane and the $x$ axis. We consider a secondary reference frame with the origin $O'$ on the center of the internal sphere.

The medium where the external sphere is posed will be considered as a vacuum, i.e., with permittivity, permeability, and conductivity equal to $\varepsilon_0$, $\mu_0$, and $\sigma_0 = 0$, respectively. On the other hand, the properties of the material of the external sphere are the following: $\varepsilon_s$, $\mu_s$, and $\sigma_s$.

We suppose that the electric field is directed in the $z$ direction in order to obtain an axial-symmetric problem:

$$\boldsymbol{E}_0 = E_0 z_0, \tag{1}$$

where $z_0$ is the unit vector along the $z$ axis.

To solve for the electrostatic potential, we have to consider the Laplace equation in spherical coordinates, its solution being well known in the literature [38]:

$$\Phi(r, \vartheta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( a_{mn} r^n + \frac{b_{mn}}{r^{n+1}} \right) P_n^m(\cos\vartheta) e^{im\varphi}, \tag{2}$$

where $P_n^m(x)$ is the associate Legendre polynomial. This expression of the potential is valid both in the free space and inside the external sphere:

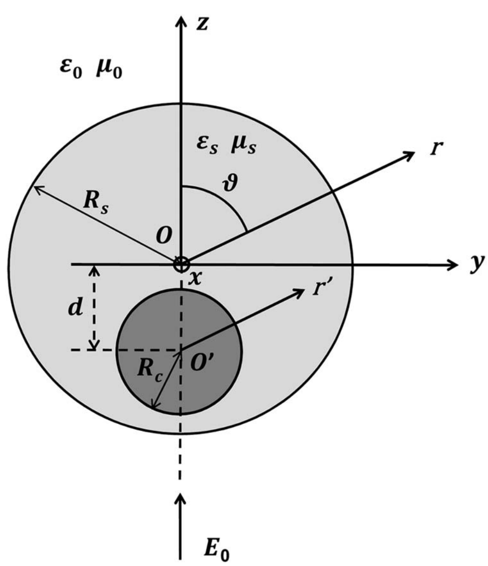


Fig. 1. Geometry of the problem: a dielectric sphere in a vacuum with an eccentric spherical inclusion.

$$\Phi_0(\mathbf{r}) = \sum_{n=0}^{\infty} \left( a_n r^n + \frac{b_n}{r^{n+1}} \right) P_n(\cos\vartheta), \tag{3}$$

$$\Phi_s(\mathbf{r}) = \sum_{n=0}^{\infty} \left( a_n^s r^n + \frac{b_n^s}{r^{n+1}} \right) P_n(\cos\vartheta). \tag{4}$$

Here, $\Phi_0$ is the electric potential in a vacuum, $\Phi_s$ is that inside the external sphere, and $P_n(x)$ is the Legendre polynomial of order $n$. As a consequence of the axial symmetry of the problem, the index $m$ of expression (2) is identically zero in all the series [39]. Note that the internal potential $\Phi_S$ includes negative-power terms, which brings forth singularities at the origin $r = 0$. This problem is avoided by translating the origin of this potential from $O$ to $O'$. In fact, as we will see, after these transformations the discontinuity disappears. The coefficients $a_n$, $b_n$, $a_n^s$, and $b_n^s$ are the unknowns of our problem. In order to determine these coefficients, we have to impose the boundary conditions on each surface:

$$\Phi_0(\mathbf{r}) = \Phi_s(\mathbf{r}) \quad \text{for } r = R_s, \tag{5}$$

$$\varepsilon_0 \frac{\partial \Phi_0(\mathbf{r})}{\partial r} = \varepsilon_s \frac{\partial \Phi_s(\mathbf{r})}{\partial r} \quad \text{for } r = R_s, \tag{6}$$

$$\Phi_s(\mathbf{r}') = U_0 \quad \text{for } r' = R_c, \tag{7}$$

where $U_0$ is the potential on the PEC sphere before the perturbation. We can note that the condition (7) is imposed on the surface of the internal sphere, so the potential $\Phi_s$ is centered in the origin $O'$, in order to impose the condition on $r' = R_c$. Moreover, we must consider the condition at infinity; i.e., at great distance from the object the electric field must be equal to the external one:

$$\lim_{r\to\infty} \nabla \Phi_0(\mathbf{r}) = \mathbf{E}_0. \tag{8}$$

From this condition, we obtain: $a_0 = E_0$ and $a_n = 0$ for $n > 0$. To operate the translation of the potential $\Phi_s$ in Eq. (7), we apply the Morse–Feshbach formulas in Chap. 10 of [40]. These formulas, for a translation along the $z$ axis from $(0, 0, d)$ to (0,0,0), take the following expressions:

$$r'^n P_n^m(\cos\vartheta') e^{im\varphi} = d^n \sum_{\nu=m}^{n} \frac{(-1)^{n-\nu}(n+m)!}{(n-\nu)!(m+\nu)!} \cdot \left(\frac{r}{d}\right)^{\nu} P_\nu(\cos\vartheta) e^{im\varphi}, \tag{9}$$

$$\frac{1}{r'^{n+1}} P_n^m(\cos\vartheta') e^{im\varphi} = \frac{1}{d^{n+1}} \sum_{\nu=m}^{n} \frac{(\nu-m)!}{(n-m)!(\nu-n)!} \cdot \left(\frac{d}{r}\right)^{\nu+1} P_\nu(\cos\vartheta) e^{im\varphi}. \tag{10}$$

The expressions for the translation are given by using the integral representation of the regular and irregular solid harmonics. As we can see from Eq. (9), these formulas transform the regular and irregular spherical harmonics centered in a generic point $(0,0,d)$ into a series of regular and irregular spherical harmonics centered in the origin. Since our problem is axial-symmetric, we have $m=0$ and the formulas can be written as follows [27]:

$$r'^n P_n(\cos\vartheta') = d^n \sum_{\nu=0}^{n} \frac{(-1)^{n-\nu} n!}{(n-\nu)!\nu!}\left(\frac{r}{d}\right)^{\nu} P_\nu(\cos\vartheta), \tag{11}$$

$$\frac{1}{r'^n} P_n(\cos\vartheta') = \frac{1}{d^{n+1}} \sum_{\nu=n}^{\infty} \frac{\nu!}{n!(\nu-n)!}\cdot\left(\frac{d}{r}\right)^{\nu+1} P_\nu(\cos\vartheta). \tag{12}$$

Replacing Eqs. (11) and (12) into the potential $\Phi_s$ in Eq. (4), inverting the sum order and interchanging the indices of the summations $n$ and $\nu$, we obtain [26,27]

$$\Phi_s(\mathbf{r}') = \sum_{n=0}^{\infty} P_n(\cos\vartheta')\Bigg[\left(\frac{r'}{d}\right)^n \sum_{\nu=n}^{\infty} a_\nu^s (d)^\nu \gamma_{\nu n} + \left(\frac{d}{r'}\right)^{n+1} \sum_{\nu=0}^{n} \frac{b_\nu^s}{(d)^{\nu+1}} \lambda_{\nu n}\Bigg] \tag{13}$$

with

$$\gamma_{\nu n} = \frac{(-1)^{\nu-n}\nu!}{(\nu-n)!n!}, \tag{14}$$

$$\lambda_{\nu n} = \frac{n!}{\nu!(n-\nu)!}. \tag{15}$$

Equation (13) is the expression of the electric potential inside the external sphere centered in the origin $O'$. We can see that this expression does not present discontinuities anymore, because the point $\boldsymbol{r}'=\mathbf{0}$ is not anymore in the domain of the potential $\Phi_s(\boldsymbol{r}')$. Let us consider now the orthogonality relation for the Legendre polynomials [41]:

$$\int_{-\pi}^{\pi} P_m(\cos\vartheta) P_n(\cos\vartheta)\mathrm{d}\vartheta = \frac{2}{2n+1}\delta_{mn}. \tag{16}$$

By inserting expressions (3), (4), and (13) in the boundary conditions (5)–(7), and by using Eq. (16), we can write the following equations:

$$-E_a R_s^n + \frac{b_n}{R_s^{n+1}} = a_n^s R_s^n + \frac{b_n^s}{R_s^{n+1}}, \tag{17}$$

$$-nE_a R_s^{n-1} - (n+1)\frac{b_n}{R_s^{n+2}} = \frac{\epsilon_s}{\epsilon_0}\left[n a_n^s R_s^{n-1} - (n+1)\frac{b_n^s}{R_s^{n+2}}\right], \tag{18}$$

$$\frac{R_c^n}{(-d)^n} \sum_{\nu=n}^{\infty} a_\nu^s \gamma_{\nu n} (-d)^\nu + \frac{(-d)^{n+1}}{R_c^{n+1}} \sum_{\nu=0}^{n} b_\nu^s \frac{\lambda_{\nu n}}{(-d)^{\nu+1}} = U_0. \tag{19}$$

The solution of this system of equations gives the unknowns of the problem, i.e., $b_n$, $a_n^s$, and $b_n^s$.

Let us next consider the case in which the inclusion core is not a perfect conductor but a dielectric with relative permittivity $\varepsilon_c$. In this case the analysis is the same as in the above, but with another potential $\Phi_c$ connected to the electric field inside the spherical inclusion, which can be written as follows:

$$\Phi_c(\mathbf{r}') = \sum_{n=0}^{\infty} a_n^c r'^n P_n(\cos\vartheta'). \tag{20}$$

This new potential is centered in the origin $O'$. It is interesting to note that in the expression of this potential there are not the irregular harmonics, because the origin is contained in its domain. The new boundary conditions are the following:

$$\Phi_0(\mathbf{r}) = \Phi_s(\mathbf{r}) \quad \text{for } r = R_s, \tag{21}$$

$$\varepsilon_0 \frac{\partial\Phi_0(\mathbf{r})}{\partial r} = \varepsilon_s \frac{\partial\Phi_s(\mathbf{r})}{\partial r} \quad \text{for } r = R_s, \tag{22}$$

$$\Phi_s(\mathbf{r}') = \Phi_c(\mathbf{r}') \quad \text{for } r' = R_c, \tag{23}$$

$$\varepsilon_s \frac{\partial\Phi_s(\mathbf{r}')}{\partial r'} = \varepsilon_c \frac{\partial\Phi_c(\mathbf{r}')}{\partial r'} \quad \text{for } r' = R_c. \tag{24}$$

Inserting the expression of the potentials (3), (4), and (20) and by using the representation (13) and the orthogonality property (16), the following equations can be obtained:

$$\frac{b_n}{R^{2n+1}} - a_n^s - \frac{b_n^s}{R^{2n+1}} = -E_0\delta_{1n}, \tag{25}$$

$$\frac{\varepsilon_0}{\varepsilon_s} N_n \frac{b_n}{R^{2n+1}} - a_n^s + N_n \frac{b_n^s}{R^{2n+1}} = -\frac{\varepsilon_0}{\varepsilon_s} E_0 \delta_{1n}, \tag{26}$$

$$\sum_{\nu=n}^{\infty}[a_\nu^s(-d)^{\nu-n}\gamma_{\nu n}] + \sum_{\nu=1}^{n}\left[b_\nu^s \frac{(-d)^{n-\nu}}{R_c^{2n+1}}\lambda_{\nu n}\right] - a_n^c = 0, \tag{27}$$

$$\frac{\varepsilon_s}{\varepsilon_c}\sum_{\nu=n}^{\infty}[a_\nu^s(-d)^{\nu-n}\gamma_{\nu n}] - \frac{\varepsilon_s}{\varepsilon_c} N_n \sum_{\nu=1}^{n}\left[b_\nu^s \frac{(-d)^{n-\nu}}{R_c^{2n+1}}\lambda_{\nu n}\right] - a_n^c = 0. \tag{28}$$

The solution of this system of equations gives the unknowns of the problem, i.e., $b_n$, $a_n^s$, $b_n^s$, and $a_n^c$.

In our case, because of the axial symmetry, the electric field $E_0$ generates a dipole perturbation in the sphere's surroundings; then the series can be stopped at the first order, and the polarizability can be written as

$$\alpha = 4\pi\varepsilon_0 \frac{b_1}{E_0}. \tag{29}$$

Actually, there are also other perturbations, i.e., higher-order multipoles, in the surroundings. However, the dipole is the most important, and the only one remaining in the far field. For this reason, in the following, we will use this formula to compute the polarizability.

From the polarizability, we can calculate the effective permittivity of the two eccentric spheres with the formula [42]

$$\varepsilon_{\mathrm{eff}} = \varepsilon_0 + \frac{\frac{\alpha}{V}}{1 - \frac{\alpha}{3\varepsilon_0 V}}, \tag{30}$$

where $V$ is the volume of the external sphere.

## 3. VALIDATION AND RESULTS

In this section, we present some numerical results obtained with a MATLAB code. To demonstrate the validity of the model, we provide comparisons between such results and simulations implemented on the commercial software Comsol Multiphysics, based on the finite-element method.

To start, let us discuss the convergence of the series in Eq. (3), where the potential is represented in terms of spherical harmonics. In Fig. 2, the coefficients of the expansion series are shown as a function of the order. Four different cases of the eccentricity of the internal sphere are shown in the case of an eccentric PEC sphere inside a dielectric sphere ($\varepsilon_s = 2.25$). As can be seen, the series converges very rapidly for a small eccentricity. However, when the internal sphere moves further away from the center, the convergence becomes slower.

We consider the case of an external sphere with a ratio between the radius and the wavelength equal to $10^{-4}$, i.e., $R_s = 10^{-4}\lambda$. The total electric field outside the sphere is the negative gradient of the electric potential $\Phi_0$. In Fig. 3, the electric field along a line $z = -1.5R_s$ and in the interval $x = [-R_s; 0]$ is shown, in the case of an internal sphere with radius $R_c = R_s/2$, posed at a distance from the origin $d = R_c/2$. The external sphere has a relative dielectric permittivity $\varepsilon_s = 2.25$ (glass), and the internal sphere is of a PEC. In Fig. 4, the components of the electric field are shown in the same configuration of Fig. 3, but with a dielectric inclusion with relative permittivity $\varepsilon_c = 5$. These comparisons show excellent agreement between the results of our code and the simulations performed with the commercial software, both in the PEC and dielectric-core cases. In particular, these results were computed using only nine terms in the potential electrostatic expansion. Due to the rapid convergence, our code turns out to be about 100 times faster than the commercial software, when operated on the same machine. It is noteworthy that the field behavior in these two cases is very similar. However, there is a difference in the amplitude of the fields: in the PEC case the scattered electric field is greater than in the one of the dielectric case. This is in agreement with the observation that the polarizabilities of the composite spheres in these two cases are rather close to each other: $\alpha_{\mathrm{PEC}} = 1.2677$ and $\alpha_{\mathrm{Diel}} = 1.0012$. As can be seen, the behavior of the electric field is approximately the same in the two cases of PEC or dielectric inclusion.

At this point, we try to understand the behavior of the dielectric sphere with a spherical inclusion in terms of scattering efficiency. We take as a reference parameter the polarizability of the object. In Fig. 5, the polarizability of a dielectric sphere with a dielectric inclusion, normalized with respect to the volume of the external sphere, is shown as a function of the distance between the centers of the two spheres; different values of the radius of the spherical inclusion are considered. The dielectric permittivities of the spheres are the same as in Fig. 4. First of all, we have to note that the interval of variation of the distance depends on the radius of the inclusion. In fact, we ask that the inclusion is always totally embedded in the external sphere. We can see from Fig. 5 that the polarizability is almost independent of the distance, but it is strongly dependent on the radius of the internal sphere. The result can be explained by considering the fact that the polarizability involves the integral of the fields internal to the object, and it is reasonable that this integral is more affected by the dimension of the inclusion than by its position. Moreover, since the core permittivity is larger than the sphere permittivity, $\varepsilon_c > \varepsilon_s$, it is natural that a larger core results in a larger polarizability.

The discussion on the polarizability concerns an integral property of the interaction between the electric field and the object. We saw that this integral interaction is not affected by the distance. However, if we consider the local behavior of the electric field, we expect that it is strongly affected by the sphere position. As an example, let us consider the case $R_c = R_s/2$. In Fig. 6, the axial component of the electric field in a point on the positive part of the $z$ axis at a distance $-1.5R_s$ from the center of the external sphere is shown as a function

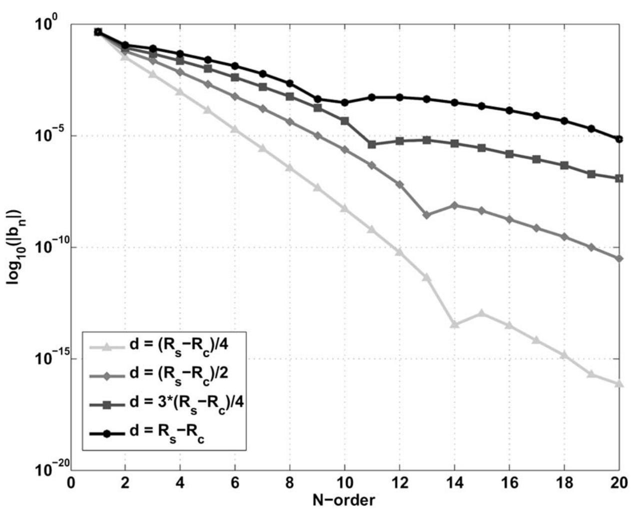


Fig. 2. Coefficients of the expansion series in Eq. (3) as a function of the order $N$, for different eccentricities of the internal sphere in the case of an eccentric PEC sphere inside a dielectric sphere ($\varepsilon_s = 2.25$).

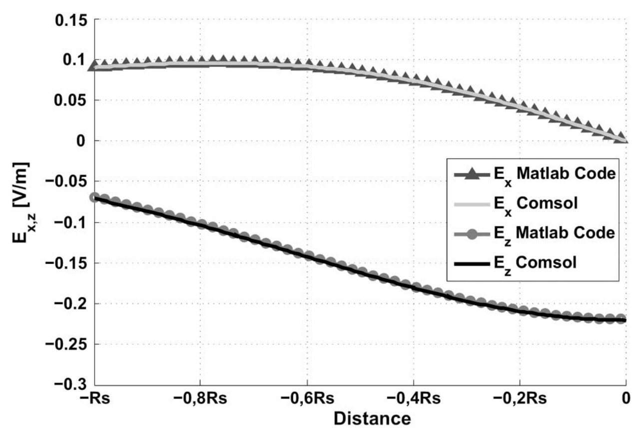


Fig. 3. Real parts of the electric field components computed along a line of coordinates $(x, z) = ([-R_s; 0], -1.5R_s)$ in the case of an eccentric PEC sphere inside a dielectric sphere ($\varepsilon_s = 2.25$).

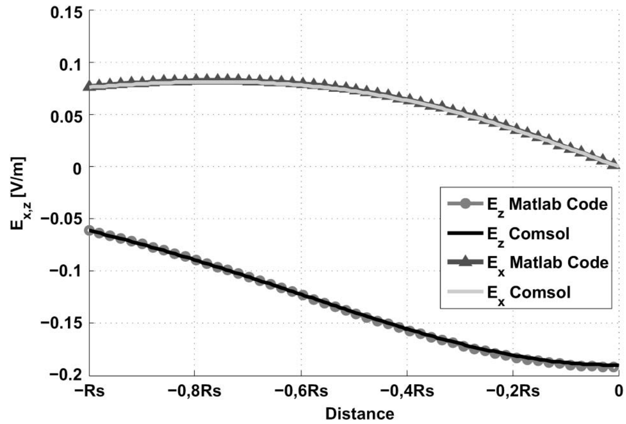


Fig. 4. Real parts of the electric field components computed along a line of coordinates $(x, z) = ([-R_s; 0], -1.5R_s)$ in the case of an eccentric dielectric sphere inside a dielectric sphere. The relative permittivity of the external sphere is $\varepsilon_s = 2.25$, and that of the inclusion is $\varepsilon_c = 5$.

of the distance between the centers of the two spheres. In this case, we can see that the field is strongly affected by the position of the inclusion. As we expected, the field is stronger when the inclusion is nearer to the boundary of the external sphere, and it is weaker when the two spheres are concentric.

Another parameter of interest is the effective permittivity of the eccentric spheres. Like the polarizability, the effective permittivity is also an integral parameter, so we expect that it is not strongly affected by the position of the inclusion. However, it is strongly affected by the inclusion radius.

To show this dependence, let us consider the case of two internally tangential spheres, with the radius of the internal sphere varying from zero to the radius of the external sphere; see Fig. 7. The permittivity of this object must vary all the way from $\varepsilon_s$ to $\varepsilon_c$. To emphasize this effect, we consider the case of a PEC inclusion. Therefore, increasing the inclusion's radius, the permittivity must go to infinity. In Fig. 8, the effective permittivity as a function of the radius of the inclusion is shown. We can see that the effective permittivity grows extremely slowly with the inclusion's radius until $R_c = 0.7R_s$. Beyond

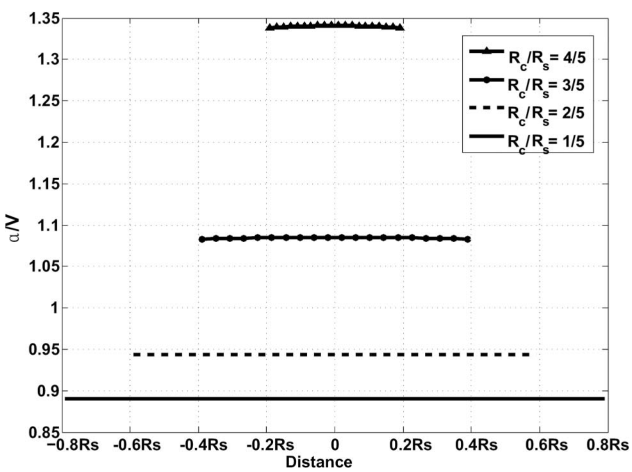


Fig. 5. Polarizability of a sphere of radius $R_s$ with a spherical inclusion of radius $R_c$ as a function of the distance between the centers of the spheres. The permittivities of the two spheres are the same as in Fig. 4. Different values of the ratio $R_c/R_s$ are considered.

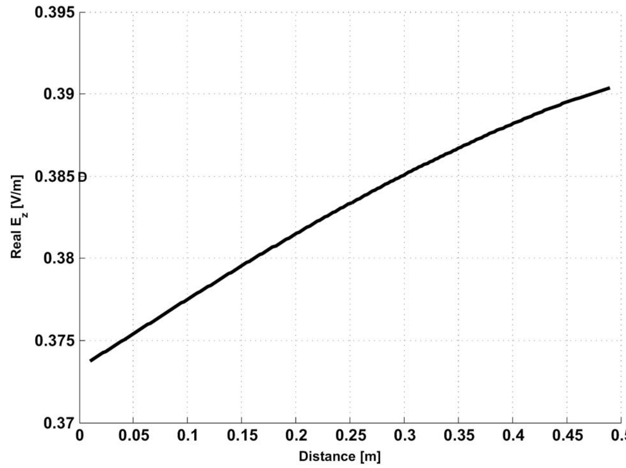


Fig. 6. Real part of the electric field component along $z$ on a point of the positive $z$ axis at a distance $-1.5R_s$ from the center of the external sphere as a function of the distance between the centers of the two spheres. The parameters of the two spheres are the same as in Fig. 5.

that, the increase becomes more nonlinear and tends to infinity.

Until now, we have analyzed dielectric and conducting spheres. Let us next focus on plasmonic composite structures. This kind of structure is widely used in the metamaterial design especially because of the presence of plasmon resonances and localization of light. We want to study the effect of the eccentricity on the position and magnitude of these resonances. With the term "plasmon resonance," we refer to the high value of the polarizability of an inclusion with a certain negative value of permittivity. This condition is known in the literature as the Fröhlich condition [41]. The effect of a dielectric shell on a plasmonic sphere or of a dielectric core into a plasmonic sphere on the behavior of the plasmon resonance has been studied [43–45]. In particular, a dielectric sphere with a plasmonic core has a single plasmon resonance, but for values of the relative permittivity less than −2. On the other hand, a dielectric sphere with a plasmonic coating has two different plasmon resonances, one for a relative permittivity of the shell less than −2 and the other for a relative permittivity larger than −2, but always less than 0. It is important to note that the materials with negative permittivity are inherently dispersive, and a function over the negative range of permittivities translates into a function over plasmonic

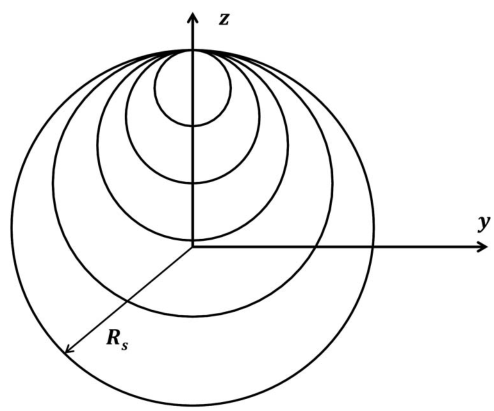


Fig. 7. Geometry of two internally tangential spheres with different values of the radius of the internal sphere.

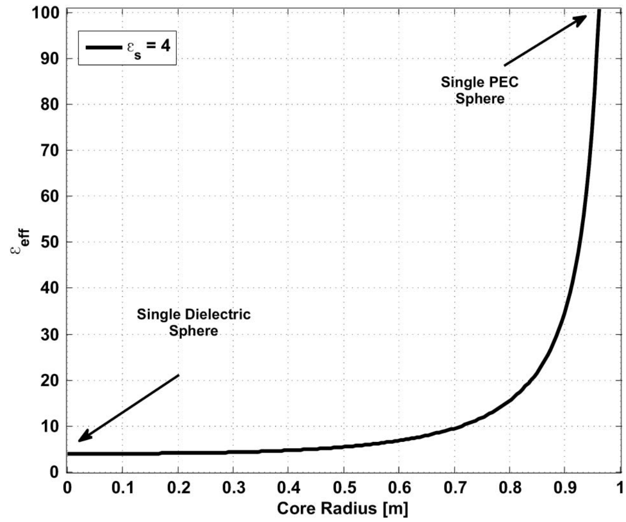


Fig. 8. Effective permittivity of two internally tangential spheres as a function of the radius of the internal sphere. The parameters of the two spheres are the same as in Fig. 5.

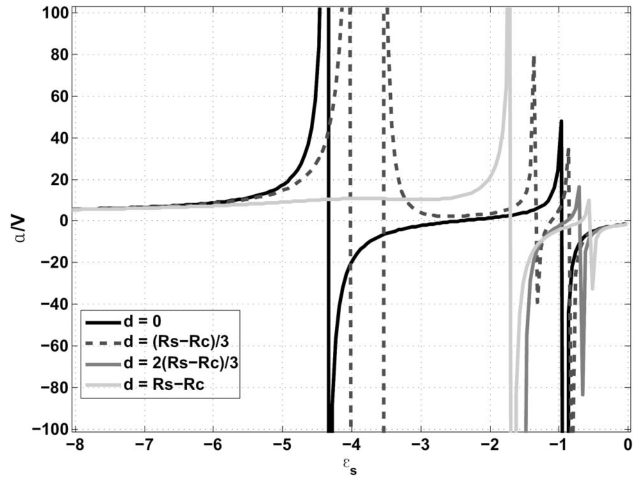


Fig. 10. Normalized polarizability of two eccentric spheres, with an internal sphere with radius $R_c = R_s/2$ and a relative permittivity $\varepsilon_c = 4$, as a function of the relative permittivity $\varepsilon_s$ of the external sphere.

frequencies. Therefore, talking about different values of permittivity can be interpreted as considering different values of frequency.

First of all, we study the case of a dielectric sphere with an eccentric plasmonic inclusion, as a generalization of a dielectric sphere with a plasmonic core. In Fig. 9, we can see the normalized polarizability, for an external sphere with relative permittivity equal to $\varepsilon_s = 4$ and a spherical inclusion with radius $R_c = R_s/2$, as a function of the relative permittivity $\varepsilon_c$ of the inclusion. We see that, increasing the eccentricity $d$ of the inclusion, the plasmon resonance decreases in magnitude, and it vanishes for the maximum value of $d$. Therefore, we find that a dielectric sphere with a plasmonic eccentric inclusion does not present plasmon resonances for high values of the eccentricity.

Now, we are going to study the case of two eccentric spheres with the external sphere with a plasmonic behavior, and with a dielectric inclusion. In Fig. 10, the normalized polarizability of two eccentric spheres of this kind is shown for different values of the distance between the centers of the spheres. The radius of the internal sphere is $R_c = R_s/2$, and its relative permittivity is $\varepsilon_c = 4$. We can see that for $d = 0$, we have two resonances, as expected from the literature results. By increasing $d$, we can see that the first resonance, i.e., the one at lower permittivity, decreases in amplitude until it disappears for the highest value of $d$. On the other hand, the second resonance increases in amplitude with $d$. Since the first resonance happens at the outer surface, and the second one takes place at the shell-core boundary, this result implies that the resonance at the outer surface does not take place anymore for the highest value of $d$. Therefore, we can conclude that two eccentric spheres, with a plasmonic coating, tend to have only one plasmon resonance for a value of the relative permittivity of the shell larger than −2, exactly the opposite behavior of a dielectric sphere with a concentric plasmonic core.

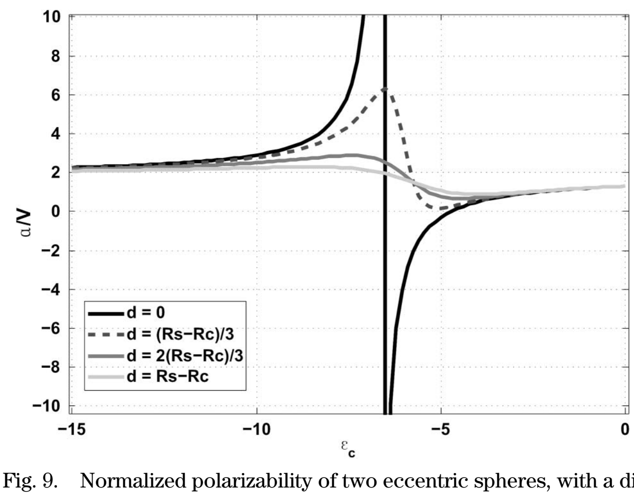


Fig. 9. Normalized polarizability of two eccentric spheres, with a dielectric external sphere with relative permittivity $\varepsilon_s = 4$, and an internal sphere with radius $R_c = R_s/2$, as a function of the relative permittivity $\varepsilon_c$ of the internal sphere.

## 4. CONCLUSION

In this paper, we studied the behavior of an eccentric composite core-shell structure in the presence of an external electric field. The spheres are considered with different values of the radii, relative permittivities, and reciprocal positions. A quasi-static analysis has been developed in order to obtain the scattering behavior and, in particular, the polarizability and the effective permittivity of the spheres. Only the axial excitation has been considered, and not the case in which the incident electric field is perpendicular to the axis that contains the centers of the spheres. A comparison with a commercial software has been presented to validate the results. We found that the polarizability of the two eccentric spheres is affected mostly by the radius of the internal sphere and weakly by the internal sphere's position. On the other hand, we demonstrated that this dependence is only on the integral behavior, while the local behavior of the electric field strongly depends on the position of the internal sphere. Moreover, we analyzed the behavior of the effective permittivity in the particular case of internally tangent spheres. Finally, we considered the case of plasmonic spheres and we analyzed the plasmon resonances. We found that for two eccentric spheres with a plasmonic inclusion, the plasmon resonance disappears for high values of eccentricity. Moreover, in the case of two eccentric spheres with a plasmonic coating, one of the two plasmon resonances disappears, and the other one is increased in magnitude.

## REFERENCES


1. A. Aden and M. Kerker, "Scattering of electromagnetic waves from two concentric spheres," J. Appl. Phys. **22**, 1242–1246 (1951).
2. J. Rheinstein, "Scattering of electromagnetic waves from dielectric coated conducting spheres," IEEE Trans. Antennas Propag. **12**, 334–340 (1964).
3. J. S. Kim and J. K. Chang, "Light scattering by two concentric optically active spheres: I. General theory," J. Korean Phys. Soc. **45**, 352–365 (2004).
4. I. Arnaoudov and G. Venkov, "The transmission acoustic scattering problem for bi-spheres in low frequency regime," Rend. Istit. Mat. Univ. Trieste **XXXVIII**, 73–93 (2006).
5. J. McNew, R. Lavarello, and W. D. O'Brien, Jr., "Sound scattering from two concentric fluid spheres," J. Acoust. Soc. Am. **122**, 2968–2975 (2007).
6. R. Dempsey, T. J. Gallagher, and B. K. P. Scaife, "Computation of the potential distribution for a composite dielectric sphere with a noncentral spherical inclusion polarized by a static uniform external field," in *10th International Conference on Conduction and Breakdown in Dielectric Liquids* (1990), pp. 243–247.
7. J. G. Fikioris and N. K. Uzunoglu, "Scattering from an eccentrically stratified dielectric sphere," J. Opt. Soc. Am. **69**, 1359–1366 (1979).
8. F. Borghese, P. Denti, R. Saija, and O. I. Sindoni, "Optical properties of spheres containing a spherical eccentric inclusion," J. Opt. Soc. Am. A **9**, 1327–1335 (1992).
9. J. A. Roumeliotis, N. B. Kakogiannos, and J. D. Kanellopoulos, "Scattering from a sphere of small radius embedded into a dielectric one," IEEE Trans. Microwave Theor. Tech. **43**, 155–168 (1995).
10. K. A. Fuller, "Scattering and absorption cross section of compounded spheres. III. Spheres containing arbitrarily located spherical inhomogeneities," J. Opt. Soc. Am. A **12**, 893–904 (1995).
11. G. Gouesbet and G. Grhan, "Generalized Lorenz-Mie theory for a sphere with an eccentrically located spherical inclusion," J. Mod. Opt. **47**, 821–837 (2000).
12. R. Nozawa, "Bipolar expansion of screened Coulomb potentials, Helmholtz' solid harmonics, and their addition theorems," J. Math. Phys. **7**, 1841–1860 (1966).
13. M. J. Caola, "Solid harmonics and their addition theorems," J. Phys. A **11**, L23–L25 (1978).
14. K. Sasihithlu and A. Narayanaswamy, "Convergence of vectorial spherical wave expansion method applied to near-field radiative transfer," Opt. Express **19**, A772–A785 (2011).
15. N. C. Skaropoulos, M. P. Ioannidou, and D. P. Chrissoulidis, "Indirect mode-matching solution to scattering from a dielectric sphere with an eccentric inclusion," J. Opt. Soc. Am. A **11**, 1859–1866 (1994).
16. A. P. Moneda and D. P. Chrissoulidis, "Dyadic Green's function of a sphere with an eccentric spherical inclusion," J. Opt. Soc. Am. A **24**, 1695–1703 (2007).
17. G. Han, Y. Han, J. Liu, and Y. Zhang, "Scattering of an eccentric sphere arbitrarily located in a shaped beam," J. Opt. Soc. Am. B **25**, 2064–2072 (2008).
18. B. Yan, X. Han, and K. F. Ren, "Scattering of a shaped beam by a spherical particle with an eccentric spherical inclusion," J. Opt. Pure Appl. Opt. **11**, 015705 (2009).
19. F. Vervelidou and D. Chrissoulidis, "Scattering of a pulsed wave by a sphere with an eccentric spherical inclusion," J. Opt. Soc. Am. A **29**, 605–616 (2012).
20. F. Borghese, P. Denti, and R. Saija, "Optical properties of spheres containing several spherical inclusions," Appl. Opt. **33**, 484–493 (1994).
21. M. P. Ioannidou and D. P. Chrissoulidis, "Electromagnetic-wave scattering by a sphere with multiple spherical inclusions," J. Opt. Soc. Am. A **19**, 505–512 (2002).
22. G. Videen, D. Ngo, P. Chýlek, and R. G. Pinnick, "Light scattering from a sphere with an irregular inclusion," J. Opt. Soc. Am. A **12**, 922–928 (1995).
23. J. A. Roumeliotis, J. G. Fikioris, and G. P. Gounaris, "Electromagnetic scattering from an eccentrically coated infinite metallic cylinder," Appl. Opt. **51**, 4488–4493 (1980).
24. A. A. Kishk, R. P. Parrikar, and A. Z. Elsherbeni, "Electromagnetic scattering from an eccentric multilayered circular cylinder," IEEE Trans. Antennas Propag. **40**, 295–303 (1992).
25. J. W. Meijs and M. J. Peters, "The EEG and MEG, using a model of eccentric spheres to describe the head," IEEE Trans. Biomed. Eng. **BME-34**, 913–920 (1987).
26. B. N. Cuffin, "Eccentric sphere models of the head," IEEE Trans. Biomed. Eng. **38**, 871–878 (1991).
27. Y. Rudy, "The eccentric spheres model as the basis for a study of the role of geometry and inhomogeneities in electrocardiography," IEEE Trans. Biomed. Eng. **BME-26**, 392–399 (1979).
28. B. Miu and W. Yawei, "Scattering analysis for eccentric-sphere model of single-nuclear cell," in *2011 Symposium on Photonics and Optoelectronics (SOPO)* (2011), pp. 1–4.
29. S. Xing, Y. Feng, Y. Y. Tay, T. Chen, J. Xu, M. Pan, J. He, H. H. Hng, Q. Yan, and H. Chen, "Reducing the symmetry of bimetallic Au@Ag nanoparticles by exploiting eccentric polymer shells," J. Am. Chem. Soc. **132**, 9537–9539 (2010).
30. Z. W. Seh, S. Liu, S. Y. Zhang, M. S. Bharathi, H. Ramanarayan, M. Low, K. W. Shah, Y. W. Zhang, and M. Y. Han, "Anisotropic growth of titania onto various gold nanostructures: synthesis, theoretical understanding, and optimization for catalysis," Angew. Chem. Int. Ed. **50**, 10140–10143 (2011).
31. J. Geng, K. Li, K. Y. Pu, D. Ding, and B. Liu, "Conjugated polymer and gold nanoparticle co-loaded PLGA nanocomposites with eccentric internal nanostructure for dual-modal targeted cellular imaging," Small **8**, 2421–2429 (2012).
32. W. P. Li, V. Shanmugam, C. C. Huang, G. D. Huang, Y. K. Huang, S. H. Chiu, and C. S. Yeh, "Eccentric inorganic-polymeric nanoparticles formation by thermal induced cross-linked esterification and conversion of eccentricity to raspberry-like Janus," Chem. Commun. **49**, 1609–1611 (2013).
33. P. Chýlek and G. Videen, "Scattering by a composite sphere and effective medium approximations," Opt. Commun. **146**, 15–20 (1998).
34. B. Yun, Z. Wang, G. Hu, and Y. Cui, "Theoretical studies on the near field properties of non-concentric core–shell nanoparticle dimers," Opt. Commun. **283**, 2947–2952 (2010).
35. O. Pea-Rodríguez and U. Pal, "Enhanced plasmonic behavior of bimetallic (Ag-Au) multilayered spheres," Nanoscale Res. Lett. **6**, 279–283 (2011).
36. C. Liu, C. C. Mi, and B. Q. Li, "The plasmon resonance of a multilayered gold nanoshell and its potential bioapplications," IEEE Trans. Nanotechnol. **10**, 797–805 (2011).
37. H. Wang, D. W. Brandl, P. Nordlander, and N. J. Halas, "Plasmonic nanostructures: artificial molecules," Acc. Chem. Res. **40**, 53–62 (2007).
38. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, 1941).
39. J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Hamilton Printing Company, 1998).
40. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part II* (McGraw-Hill, 1953).
41. C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles* (Wiley, 1940).
42. A. Sihvola, *Electromagnetic Mixing Formulas and Applications* (The Institution of Electrical Engineers, 1999).
43. A. Sihvola, "Character of surface plasmons in layered spherical structures," Progress Electromagn. Res. **62**, 317–331 (2006).
44. Y. Zeng, Q. Wu, and D. H. Werner, "Electrostatic theory for designing lossless negative permittivity metamaterials," Opt. Express **35**, 1431–1433 (2010).
45. U. K. Chettiar and N. Engheta, "Internal homogenization: effective permittivity of a coated sphere," Opt. Express **20**, 22976–22986 (2012).