Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation

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We present a theoretical description of the scattering of a Gaussian beam by a spherical, homogeneous, and isotropic particle. This theory handles particles with arbitrary size and nature having any location relative to the Gaussian beam. The formulation is based on the Bromwich method and closely follows Kerker's formulation for plane-wave scattering. It provides expressions for the scattered intensities, the phase angle, the cross sections, and the radiation pressure.

1. INTRODUCTION

The scattering theory for a spherical, isotropic, homogeneous, nonmagnetic particle illuminated by a plane wave was described in classical papers by Lorenz,[1,2] Mie,[3] and Debye.[4] It is called the Lorenz–Mie theory (LMT) in this paper.

There has been, for several years, a new interest in the topic, owing to the following facts, among others:

(1) Numerical computations in the LMT framework were usually laborious. Nevertheless, much progress was made recently because of the increase in computer and algorithm efficiencies. Another important improvement was the design by Lentz[5] of a new algorithm to compute Bessel-function ratios involved in the LMT. This algorithm was used to build the so-called SUPERMIDI computer program without practical size or refractive-index limitations. Results were produced,[6–10] and some of them were confirmed by other authors.[11,12] We also mention that computations can be run on standard microcomputers,[13] enabling us to produce extensive numerical results not only in the framework of the LMT but also for what we call the localized approximation to the generalized Lorenz–Mie theory (GLMT) described in this section.

(2) Computational interest is often motivated by the need for sizing techniques with emphasis on the development of systems permitting simultaneous measurements of the velocity and the size of discrete particles transported in flows.[14–24] In this field, Ref. 25 can be considered a significant sample of recent developments in optical measurement methods.

(3) There was also a recent renewal of interest connected with optical levitation experiments performed in the United States and in France, at the Institut d'Optique with potential application to thermonuclear fusion[26–29] and also at Rouen University[30] with pure-scattering motivations.

Nevertheless, new theoretical and experimental advances depend on the ability to handle a more general problem than the pure LMT, namely, the case in which the scatterer is illuminated by a laser beam, working, for instance, in the fundamental TEM_{00} mode, with the prospect (among other possibilities) of later producing a general theory of the visibility technique[14,15] or of the phase Doppler method[31] in laser Doppler anemometry.

The problem of Mie theory generalization was considered by Tam and Corriveau,[32] Morita *et al.*,[33] and Tsai and Pogorzelski[34] and, in 1983, by Kim and Lee,[35] who presented theoretical approaches and numerical results of various extents.

This scattering problem was also considered early by our team. We first examined the case of a sphere arbitrarily located on the axis of a Gaussian beam, neglecting the field components parallel to the direction of propagation. At this time, we used the terminology of axisymmetric light profile to designate a Gaussian beam so described. This terminology has now been replaced by order L^- of approximation; the precise status of this description is discussed later in this section and in Subsection 4.A.2. The results of the theory were limited furthermore to expressions for scattered fields and intensities. Results were announced without any demonstration in 1980 (Ref. 10) and also mentioned in Refs. 18, 36, and 37. Demonstrations were published in a 1980 dissertation[38] and appeared later in an archival journal.[39] The formulation is similar to that of the LMT, as expressed by Kerker[40]; the only difference is the introduction of an infinite set of new coefficients g_n that describe the non-plane-wave character of the incident beam and are all equal to 1 in the framework of the pure LMT. The same problem was discussed again by using a more precise description of the Gaussian beam, called order L of approximation, accounting for axial and transverse field components.[41] Formulas were established not only for the scattered fields and intensities but also for the phase angle between perpendicular polarizations, the cross sections and efficiency factors, and the radiation pressure. However, the generalization of the LMT was not fully completed because the off-axis location of the scattering sphere was not considered. Again, the formulation is similar to that of Kerker except for the appearance of a set of coefficients g_n. At order L of approximation, these coefficients are only slightly more complicated than those at order L^-. Formulas at order L^- can be derived readily as a special case of the formulas at order L. The exact order of approximation of the theory, linked to the paraxial description of

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the incident beam, was discussed extensively in Ref. 42 for both order L and order L^-. The conclusion is that, at order L, significant differences between our theory and an ideal rigorous one and/or perfect experiments can be expected only for exotic situations. More precisely, relative inconsistencies for the scattered fields are $O(s^2)$, where s is a fundamental dimensionless parameter equal to the waist radius w_0 divided by the so-called diffraction or spreading length l ($l = kw_0^2$, where k is the wave number of the incident beam). The highest possible value for s is about 0.15 in the case of the narrowest possible waist ($w_0 = \lambda$, λ being the wavelength), but s is usually much smaller. For example, let us assume a wavelength equal to 0.5 μm and $w_0 = 50$ μm (typical values). These conditions lead to $s \simeq 10^{-3}$ and $s^2 \simeq 10^{-6}$, a small value indeed.

The treatments in Refs. 39, 41, and 42 were purely formal (no numerical results). Numerical results were given and discussed in Refs. 43–47. The correcting scattering coefficients g_n are rather complicated to compute, particularly because they contain quadratures of highly oscillating functions. It was shown, however, in Ref. 43 that the values of the g_n's can be evaluated by using an argument that relies on the principle of localization, leading to a simple exponential expression for each g_n. The introduction of the correcting scattering coefficients, calculated by means of this argument, into the above-described GLMT leads to a simple formulation that we call the localized approximation to the GLMT. This formulation is of particular interest because of its simplicity, its ability to handle easily a large variety of incident beams (not restricted to Gaussian beams), and its straightforward implementation by means of a computer program for the LMT. Strictly speaking, the current localized approximation is only an approximation of the GLMT to order L^-. However, insofar as the order L^- is an approximation of the order L, the localized approximation can also be considered an approximation of the GLMT to order L. In Ref. 43, results from the localized approximation were compared with (1) numerical results obtained for a sphere having a refractive index equal to 1001 in the framework of a Rayleigh–Gans approach, with implantation of the Gaussian character of the laser beam, leading to a satisfactory agreement; (2) numerical results obtained by several authors using other approaches to the Gaussian beam scattering problem,[34,48] leading again to satisfactory agreements; and (3) experimental results from old optical levitation experiments.[30] For this last comparison, numerical results obtained by using the localized approximation approach experimental data satisfactorily, an impossible task in the framework of the pure LMT. In Ref. 44, g_n values computed by using (1) the localized approximation and (2) the exact g_n formulas at order L^- were compared, leading to an agreement of better than 1%. Finally, our results in this field of research were reviewed concisely in Ref. 45, including applications to laser optical sizing plus another satisfactory comparison between localized approximation results and results of more-recent optical levitation experiments performed in our laboratory.

The formal discussion of Ref. 42 and the above-mentioned comparisons of various theoretical results and also of theoretical results with experimental results provided us with evidence of the value of our approach to the Gaussian beam scattering problem and encourage us to complete our formal construction.

The present paper is devoted consequently to a theory of light scattering from a sphere located arbitrarily in a Gaussian beam. The beam is described at order L of approximation, and the scattering theory is based on a Bromwich formulation.

2. BROMWICH FORMULATION

We must solve the Maxwell equations accounting for the boundary conditions defined by the scattering problem under consideration. The Bromwich formulation[49,50] enables us to obtain special solutions satisfying the boundary conditions in special curvilinear coordinate systems (base e_1, e_2, e_3) defined by the following properties:

(1) The system is orthogonal.
(2) $e_1 = 1$.
(3) The ratio e_2/e_3 does not depend on the first coordinate.

Such is the case for the spherical coordinate system (r, θ, φ) used in Fig. 1.

We consider an electromagnetic sinusoidal wave varying in time as $e^{i\omega t}$, where ω is the angular frequency and the components of a vector $\mathbf{V}$ are designated by V_r, V_θ, and V_φ in a local coordinate system at point P. The letter V stands for E (electric field) or H (magnetic field).

In the Bromwich formulation, the solution of the Maxwell equations is written as the sum of two special solutions with the proviso that the boundary conditions must be satisfied. These special solutions are the transverse magnetic (TM) wave, for which $H_r = 0$, and the transverse electric (TE) wave, for which $E_r = 0$. The TM and TE fields are deduced from Bromwich scalar potentials (BSP's), U_{TM} and U_{TE}, respectively. Any BSP U complies with the equation

$$\frac{\partial^2 U}{\partial r^2} + k^2 U + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2} = 0, \tag{1}$$

in which k is the wave number,

$$k = \omega(\mu\epsilon)^{1/2} = M\frac{\omega}{c}, \tag{2}$$

where μ and ϵ are the permeability and the permittivity of the medium, respectively; M is its complex refractive index; and c is the speed of the light.

The BSP

$$U = r\Psi_n^1(kr)P_n^m(\cos\theta)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(m\varphi) \tag{3}$$

is a solution of Eq. (1). $\Psi_n^1(kr)$ is the spherical Bessel function defined by

$$\Psi_n^1(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{n+(1/2)}(kr), \tag{4}$$

where $J_{n+(1/2)}(kr)$ is a Bessel function of half-integer order.

In the present paper, the associated Legendre polynomial $P_n^m(\cos\theta)$ is defined as

$$P_n^m(\cos\theta) = (-1)^m(\sin\theta)^m \frac{\mathrm{d}^m P_n(\cos\theta)}{[\mathrm{d}(\cos\theta)]^m}, \tag{5}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of order n.

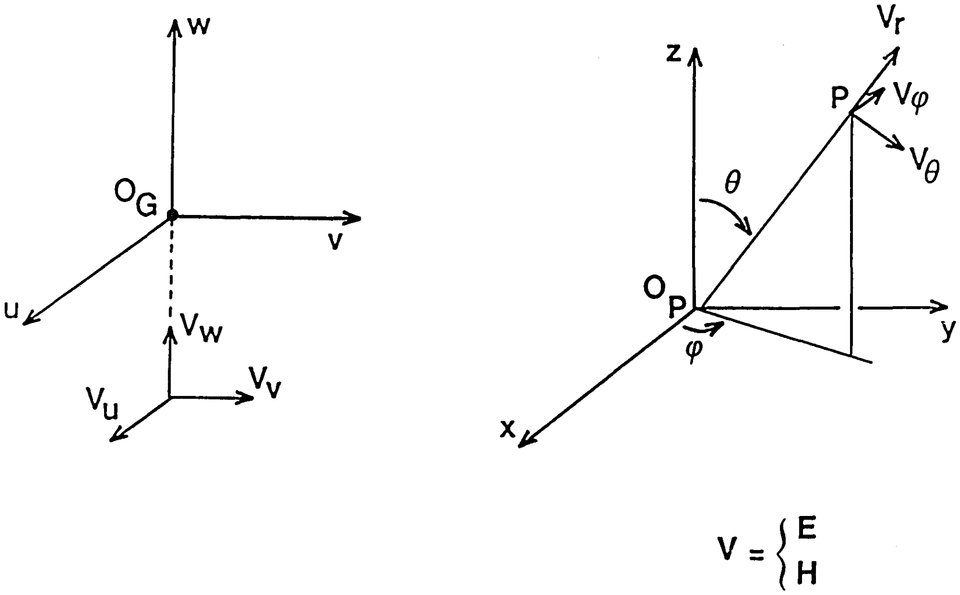

Fig. 1. Coordinate systems for the beam description (beam center O_G) and for the scattered fields (scatterer center O_P; scattered field computed at point P).

We introduce the Ricatti–Bessel functions,

$$\Psi_n(kr) = kr\Psi_n^1(kr), \tag{6}$$

$$\xi_n(kr) = \Psi_n(kr) + i(-1)^n\left(\frac{\pi kr}{2}\right)^{1/2} J_{-n-(1/2)}(kr), \tag{7}$$

where $J_{-n-(1/2)}(kr)$ is a Bessel function of negative half-integer order.

The BSP

$$U = \xi_n(kr)P_n^m(\cos\theta)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(m\varphi) \tag{8}$$

is also a solution of Eq. (1).

The use of either Ψ_n or ξ_n depends on boundary conditions, since $\Psi_n(kr)$ remains finite for $r = 0$ whereas $\xi_n(kr)$ tends to a spherical-wave description for $r \to \infty$.

From the definition of the TM and TE waves, we obtain

$$H_{r,\mathrm{TM}} = E_{r,\mathrm{TE}} = 0. \tag{9}$$

When the BSP's are determined, the other field components are obtained from the following relations:

$$E_{r,\mathrm{TM}} = \frac{\partial^2 U_{\mathrm{TM}}}{\partial r^2} + k^2 U_{\mathrm{TM}}, \tag{10}$$

$$E_{\theta,\mathrm{TM}} = \frac{1}{r}\frac{\partial^2 U_{\mathrm{TM}}}{\partial r\partial\theta}, \tag{11}$$

$$E_{\varphi,\mathrm{TM}} = \frac{1}{r\sin\theta}\frac{\partial^2 U_{\mathrm{TM}}}{\partial r\partial\varphi}, \tag{12}$$

$$H_{\theta,\mathrm{TM}} = \frac{i\omega\epsilon}{r\sin\theta}\frac{\partial U_{\mathrm{TM}}}{\partial\varphi}, \tag{13}$$

$$H_{\varphi,\mathrm{TM}} = -\frac{i\omega\epsilon}{r}\frac{\partial U_{\mathrm{TM}}}{\partial\theta}, \tag{14}$$

$$E_{\theta,\mathrm{TE}} = -\frac{i\omega\mu}{r\sin\theta}\frac{\partial U_{\mathrm{TE}}}{\partial\varphi}, \tag{15}$$

$$E_{\varphi,\mathrm{TE}} = \frac{i\omega\mu}{r}\frac{\partial U_{\mathrm{TE}}}{\partial\theta}, \tag{16}$$

$$H_{r,\mathrm{TE}} = \frac{\partial^2 U_{\mathrm{TE}}}{\partial r^2} + k^2 U_{\mathrm{TE}}, \tag{17}$$

$$H_{\theta,\mathrm{TE}} = \frac{1}{r}\frac{\partial^2 U_{\mathrm{TE}}}{\partial r\partial\theta}, \tag{18}$$

$$H_{\varphi,\mathrm{TE}} = \frac{1}{r\sin\theta}\frac{\partial^2 U_{\mathrm{TE}}}{\partial r\partial\varphi}. \tag{19}$$

3. THE SCATTERING PROBLEM

The center of the scatterer (with a diameter d and a complex refractive index M relative to the surrounding nonabsorbing medium) is located at the point O_P of a Cartesian coordinate system $O_P xyz$ (Fig. 1). It is illuminated by a Gaussian beam; the middle of the beam waist is located at the point O_G. An accessory Cartesian coordinate system $O_G uvw$ is used, with $O_G u$ parallel to $O_P x$ and similar conditions for the other axes. The incident wave propagates from the negative w to the positive w. The wave is described basically by using Cartesian incident-field components E_u, H_v, E_w, and H_w; the other components are identically zero. The coordinates of O_G in the system $O_P xyz$ are (x_0, y_0, z_0).

The aim is to compute the properties of the scattered light observed at a point P, defined by its (r, θ, φ) spherical coordinates, and some associated quantities. The following steps must be performed:

(1) We write the BSP's, U_{TM} and U_{TE}, for the incident wave (superscript i), the scattered (also called external) wave (superscript s), and the internal wave inside the sphere (superscript sp). We then have six BSP's: U_{TM}^{i}, U_{TE}^{i}, U_{TM}^{s}, U_{TE}^{s}, U_{TM}^{sp}, and U_{TE}^{sp}.

(2) We are particularly interested in the scattered wave defined by U_{TM}^{s} and U_{TE}^{s}. These functions contain an infi-

nite number of unknown coefficients that can be determined by using boundary conditions at the surface of the sphere.

(3) We determine the scattered-field components from the external BSP's with the aid of Eqs. (10)–(19) in both the near and the far fields.

(4) We complete the formulation by deriving formulas for the scattered intensities, the phase angle, the cross sections, and the radiation pressure force components.

The incident wave is a laser beam in its fundamental basic TEM_{00} Gaussian mode. However, the principle of the present generalization is not expected to be affected if other descriptions of the Gaussian beam or other modes are used. Presumably, only the expressions for the coefficients $g_{n,\mathrm{TM}}^m$ and $g_{n,\mathrm{TE}}^m$, which appear in the theory, would be modified.

4. DESCRIPTION OF THE INCIDENT WAVE

A. Cartesian Description of the Incident Wave in the (u, v, w) System

1. Generalities

We must now establish the structure of the incident wave and the expressions for its field components. For that purpose, let us first consider an electromagnetic wave propagating in the w direction such as the electric field E_i given by (E_u, 0, 0). It is then an exercise to show that the Maxwell equations imply that $\partial E_u/\partial u = 0$. Lax *et al.*[51] noted that this fact is the source of a paradox. The paradox arises when, in theoretical studies of the structure of laser beams, it is assumed that $E_i = (E_u, 0, 0)$ and a wave solution is obtained that is Gaussian with respect to u and v in the lowest order, in contradiction with the starting assumption. The paradox was solved by Lax *et al.*,[51] who examined a paraxial approximation and produced a procedure for deriving higher-order corrections. The same problem was examined later by Davis,[52] who presented the theory in a simpler and more appealing way than that of Lax *et al.* Davis's results are our starting point.

2. Field Components

We introduce the small parameter s discussed in Section 1:

$$s = w_0/l = 1/(kw_0), \tag{20}$$

where w_0 is the waist radius and l is the spreading length. The vector potential describing the beam may be expanded in power series of s, according to Davis. At lowest order (order L), we neglect in Davis's formulation all terms with powers of s higher than 1, which leads to

$$E_v = H_u = 0, \tag{21}$$

$$E_u = E_0\Psi_0 \exp(-ikw), \tag{22}$$

$$E_w = -\frac{2Qu}{l} E_u, \tag{23}$$

$$H_v = H_0\Psi_0 \exp(-ikw), \tag{24}$$

$$H_w = -\frac{2Qv}{l} H_v, \tag{25}$$

where E_0 and H_0 are linked by

$$E_0/H_0 = (\mu/\epsilon)^{1/2}. \tag{26}$$

The lowest-order function Ψ_0 is the well-known fundamental mode solution given by

$$\Psi_0 = iQ \exp(-iQh_+^2), \tag{27}$$

where

$$h_+^2 = u_+^2 + v_+^2, \tag{28}$$

$$Q = \frac{1}{i + 2w_+}. \tag{29}$$

u_+, v_+, and w_+ are reduced quantities defined by

$$u_+ = u/w_0, \qquad v_+ = v/w_0, \tag{30}$$

$$w_+ = w/l. \tag{31}$$

More details concerning this description are available in Refs. 41 and 52. In Ref. 41 the field components were written immediately in the (O_Pxyz) system because we had $O_P \equiv O_G$. The order of approximation resulting from this description was discussed extensively in Ref. 42. The point of concern is that Eqs. (21)–(25) constitute an approximation that does not fully satisfy the Maxwell equations. However, when Eqs. (21)–(25) are introduced into the Maxwell equations, we observe directly that the approximation is perfect for the whole space in the limit $s \to 0$, the relative introduced errors being $O(s^2)$. As stated in Section 1, s is a very small parameter with typical values of $s^2 \simeq 10^{-6}$.

The Bromwich formulation is devoted to the resolution of the Maxwell equations. Consequently the development of a Bromwich formulation by using the set of Eqs. (21)–(25) leads inevitably to inconsistencies. In Ref. 41, the inconsistency was described as follows: starting from Cartesian expressions for the incident fields, the components E_r and H_r are obtained readily in the (r, θ, φ) system. From only E_r and H_r the incident BSP's U_{TM}^i and U_{TE}^i are computed; then, from U_{TM}^i and U_{TE}^i, it is possible to derive expressions for the Cartesian incident field components by using the set of Eqs. (10)–(19). The Cartesian incident field components obtained from the process are not equal to the original ones, as they should be if the set of Eqs. (21)–(25) agreed with the Maxwell equations. This point was extensively discussed and, again, the relative introduced inconsistencies were shown to be $O(s^2)$ for the field components. In other words, the formal inconsistency is of no practical importance for numerical results.

At order L^- of approximation discussed in Section 1, we furthermore neglect the fields E_w and H_w [Eqs. (23) and (25)]. In general, this degree of approximation is also good, as evidenced by a formal discussion[41,42] and numerical results,[43–47] but should be restricted to cases in which the scatterer is not located too far away from the beam axis.

B. Cartesian Description of the Incident Wave in the (x, y, z) System

We readily obtain

$$E_y = H_x = 0, \tag{32}$$

$$E_x = E_0\Psi_0 \exp[-ik(z - z_0)], \tag{33}$$

$$E_z = -\frac{2Q}{l}(x - x_0)E_x, \tag{34}$$

$$H_y = H_0\Psi_0 \exp[-ik(z - z_0)], \tag{35}$$

$$H_z = -\frac{2Q}{l}(y - y_0)H_y, \tag{36}$$

where Ψ_0 is given by Eq. (27) with, however, the following complementary relations:

$$h_+^2 = \frac{1}{w_0^2}[(x - x_0)^2 + (y - y_0)^2], \tag{37}$$

$$Q = \frac{1}{i + 2(\zeta - \zeta_0)}, \tag{38}$$

$$\zeta = \frac{z}{l}, \qquad \zeta_0 = \frac{z_0}{l}. \tag{39}$$

C. Description of the Incident Wave in the (r, θ, φ) System

From Eqs. (32)–(36) we readily obtain, by mere projections, the field components in the (r, θ, φ) system:

$$E_r = E_0\Psi_0\left[\cos\varphi \sin\theta\left(1 - \frac{2Q}{l} r\cos\theta\right) + \frac{2Q}{l}x_0\cos\theta\right]\exp(K), \tag{40}$$

$$E_\theta = E_0\Psi_0\left[\cos\varphi\left(\cos\theta + \frac{2Q}{l} r\sin^2\theta\right) - \frac{2Q}{l}x_0\sin\theta\right]\exp(K), \tag{41}$$

$$E_\varphi = -E_0\Psi_0 \sin\varphi \exp(K), \tag{42}$$

$$H_r = H_0\Psi_0\left[\sin\varphi \sin\theta\left(1 - \frac{2Q}{l} r\cos\theta\right) + \frac{2Q}{l}y_0\cos\theta\right]\exp(K), \tag{43}$$

$$H_\theta = H_0\Psi_0\left[\sin\varphi\left(\cos\theta + \frac{2Q}{l} r\sin^2\theta\right) - \frac{2Q}{l}y_0\sin\theta\right]\exp(K), \tag{44}$$

$$H_\varphi = H_0\Psi_0 \cos\varphi \exp(K), \tag{45}$$

with

$$K = -ik(r\cos\theta - z_0). \tag{46}$$

To build the BPS's for the incident wave ($U^{\mathrm{i}}_{\mathrm{TM}}$ and $U^{\mathrm{i}}_{\mathrm{TE}}$), only the knowledge of the components E_r and H_r is required. Furthermore, we must identify where the argument φ appears and express this dependence on φ in terms of the trigonometric sine and cosine functions involved in the general expressions for the BSP's [Eqs. (3) and (8)].

The argument φ appears explicitly in Eqs. (40)–(45) and is also contained in the function Ψ_0, which can be rewritten as

$$\Psi_0 = \Psi_0^0\Psi_0^\varphi, \tag{47}$$

$$\Psi_0^0 = iQ\exp\left(-iQ\frac{r^2\sin^2\theta}{w_0^2}\right)\exp\left(-iQ\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{48}$$

$$\Psi_0^\varphi = \exp\left[\frac{2iQ}{w_0^2} r\sin\theta(x_0\cos\varphi + y_0\sin\varphi)\right], \tag{49}$$

where Ψ_0^0 does not depend on φ.

We modify Ψ_0^φ by replacing the sine and cosine functions by exponentials of imaginary arguments, and then we expand the resulting exponentials and finally expand terms of the form $(a + b)^j$ to obtain

$$\Psi_0^\varphi = \sum^{jp}\Psi_{jp}\exp[i\varphi(j - 2p)], \tag{50}$$

$$\sum^{jp} = \sum_{j=0}^{\infty}\sum_{p=0}^{j}, \tag{51}$$

$$\Psi_{jp} = \left(\frac{iQr\sin\theta}{w_0^2}\right)^j \frac{(x_0 - iy_0)^{j-p}(x_0 + iy_0)^p}{(j - p)!(p)!}. \tag{52}$$

When the center of the scatterer is located on the beam axis ($x_0 = y_0 = 0$), then all the Ψ_{jp} terms in Eq. (50) are zero, except

$$\Psi_{00} = 1. \tag{53}$$

The formulation given in Ref. 47 for Ψ_0^φ is a bit more complicated but equivalent.

D. Radial Components E_r and H_r

From Eqs. (40), (43), (47), and (50), the radial components E_r and H_r are found to be

$$E_r = E_0\frac{F}{2}\left[\sum^{jp}\Psi_{jp}\exp(ij_+\varphi) + \sum^{jp}\Psi_{jp}\exp(ij_-\varphi)\right] + E_0x_0G\sum^{jp}\Psi_{jp}\exp(ij_0\varphi), \tag{54}$$

$$H_r = H_0\frac{F}{2i}\left[\sum^{jp}\Psi_{jp}\exp(ij_+\varphi) - \sum^{jp}\Psi_{jp}\exp(ij_-\varphi)\right] + H_0y_0G\sum^{jp}\Psi_{jp}\exp(ij_0\varphi), \tag{55}$$

in which

$$F = \Psi_0^0\sin\theta\left(1 - \frac{2Q}{l} r\cos\theta\right)\exp(K), \tag{56}$$

$$G = \Psi_0^0\frac{2Q}{l}\cos\theta\exp(K), \tag{57}$$

and

$$j_+ = j + 1 - 2p = j_0 + 1, \tag{58}$$

$$j_- = j - 1 - 2p = j_0 - 1. \tag{59}$$

5. BROMWICH SCALAR POTENTIALS

A. Incident Wave

From the definition of the TE wave, $E_{r,\mathrm{TE}} = 0$; then, from Eq. (10), the electric field is simply

$$E_r = E_{r,\mathrm{TM}} = \frac{\partial^2 U^{\mathrm{i}}_{\mathrm{TM}}}{\partial r^2} + k^2U^{\mathrm{i}}_{\mathrm{TM}}, \tag{60}$$

in which $U^{\mathrm{i}}_{\mathrm{TM}}$ is the TM BSP for the incident wave.

Since the equations are linear and E_r is a sum of terms, $U^{\mathrm{i}}_{\mathrm{TM}}$ can be treated as a sum of corresponding terms. That procedure is more pedagogic and was used in Ref. 47. In the present paper, we shall be more concise and set directly

$$U^{\mathrm{i}}_{\mathrm{TM}} = E_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} C_n^{\mathrm{pw}} g_{n,\mathrm{TM}}^{m} r \Psi_n^1(kr) P_n^{|m|}(\cos\theta)\exp(im\varphi), \tag{61}$$

in which the coefficients C_n^{pw} (superscript pw indicates a plane-wave term) appear in the Bromwich formulation of the LMT (Ref. 39) and are given by

$$C_n^{\mathrm{pw}} = \frac{1}{k} i^{n-1}(-1)^n \frac{2n+1}{n(n+1)}. \tag{62}$$

In Eq. (62) the polynomials P_n^0 identify with the Legendre polynomials P_n.

To find the unknown coefficients $g_{n,\mathrm{TM}}^m$, we substitute for Eq. (62) in the right-hand side of Eq. (61) and identify the left-hand side of Eq. (61) with the E_r value given by Eq. (54), producing an equation for the unknown coefficients. We also use the differential equation for spherical Bessel functions,[50]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2\right)[r\Psi_n^1(kr)] = \frac{n(n+1)}{r}\Psi_n^1(kr). \tag{63}$$

To isolate the unknown coefficients in the resulting equation, we invoke orthogonality relations successively for the exponentials and for the P_n^m terms:

$$\int_0^{2\pi} \exp[i(m-m')\varphi]\mathrm{d}\varphi = 2\pi\delta_{mm'}, \tag{64}$$

$$\int_0^{\pi} P_n^m(\cos\theta)P_l^m(\cos\theta)\sin\theta \mathrm{d}\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nl}. \tag{65}$$

We can then obtain a better-looking expression by means of the orthogonality relations for the Ψ_n^1 terms:

$$\int_0^{\infty} \Psi_n^1(kr)\Psi_m^1(kr)\mathrm{d}(kr) = \frac{\pi}{2(2n+1)}\delta_{nm}. \tag{66}$$

Multiplying the equation by the adequate integral operators to take advantage of Eqs. (64)–(66), we obtain

$$\begin{aligned} g_{n,\mathrm{TM}}^m = {} & \frac{1}{C_n^{\mathrm{pw}}}\frac{(2n+1)^2}{\pi n(n+1)}\frac{(n-|m|)!}{(n+|m|)!} \\ & \times \int_0^{\pi}\int_0^{\infty}\left[\frac{F}{2}\left(\sum_{j_+=m}^{jp}\Psi_{jp} + \sum_{j_-=m}^{jp}\Psi_{jp}\right) + x_0 G \sum_{j_0=m}^{jp}\Psi_{jp}\right] \\ & \times r\Psi_n^1(kr)P_n^{|m|}(\cos\theta)\sin\theta \mathrm{d}\theta \mathrm{d}(kr). \end{aligned} \tag{67}$$

The symbol $\sum_c^{jp}$ designates the sum $\sum^{jp}$ restricted to the condition c.

In terms of Ricatti–Bessel functions that are now preferred, Eq. (61) becomes

$$U^{\mathrm{i}}_{\mathrm{TM}} = \frac{E_0}{k}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}} g_{n,\mathrm{TM}}^m \Psi_n(kr)P_n^{|m|}(\cos\theta)\exp(im\varphi). \tag{68}$$

Working similarly with $H_r = H_{r,\mathrm{TE}}$ instead of $E_r = E_{r,\mathrm{TM}}$, we obtain the TE BSP for the incident wave:

$$U^{\mathrm{i}}_{\mathrm{TE}} = \frac{H_0}{k}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}} g_{n,\mathrm{TE}}^m \Psi_n(kr)P_n^{|m|}(\cos\theta)\exp(im\varphi), \tag{69}$$

in which

$$\begin{aligned} g_{n,\mathrm{TE}}^m = {} & \frac{1}{C_n^{\mathrm{pw}}}\frac{(2n+1)^2}{\pi n(n+1)}\frac{(n-|m|)!}{(n+|m|)!} \\ & \times \int_0^{\pi}\int_0^{\infty}\left[\frac{F}{2}\left(-i\sum_{j_+=m}^{jp}\Psi_{jp} + i\sum_{j_-=m}^{jp}\Psi_{jp}\right)\right. \\ & \left. + y_0 G\sum_{j_0=m}^{jp}\Psi_{jp}\right] r\Psi_n^1(kr)P_n^{|m|}(\cos\theta)\sin\theta \mathrm{d}\theta \mathrm{d}(kr). \end{aligned} \tag{70}$$

B. External and Sphere Waves

We refer to the wave scattered by the particle as the external wave and to the wave inside the particle as the sphere wave. The BSP's for the external ($U^{\mathrm{s}}_{\mathrm{TM}}$, $U^{\mathrm{s}}_{\mathrm{TE}}$) and sphere ($U^{\mathrm{sp}}_{\mathrm{TM}}$, $U^{\mathrm{sp}}_{\mathrm{TE}}$) waves are set to be

$$U^{\mathrm{s}}_{\mathrm{TM}} = \frac{-E_0}{k}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}} A_n^m \xi_n(kr)P_n^{|m|}(\cos\theta)\exp(im\varphi), \tag{71}$$

$$U^{\mathrm{s}}_{\mathrm{TE}} = \frac{-H_0}{k}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}} B_n^m \xi_n(kr)P_n^{|m|}(\cos\theta)\exp(im\varphi), \tag{72}$$

$$U^{\mathrm{sp}}_{\mathrm{TM}} = \frac{kE_0}{k_{\mathrm{sp}}^2}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}} E_n^m \Psi_n(k_{\mathrm{sp}}r)P_n^{|m|}(\cos\theta)\exp(im\varphi), \tag{73}$$

$$U^{\mathrm{sp}}_{\mathrm{TE}} = \frac{kH_0}{k_{\mathrm{sp}}^2}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}} D_n^m \Psi_n(k_{\mathrm{sp}}r)P_n^{|m|}(\cos\theta)\exp(im\varphi). \tag{74}$$

k_{sp} is the wave number in the sphere material. The functions $\xi_n(kr)$ are used in Eqs. (71) and (72) to satisfy later the boundary conditions in the limit $r \to \infty$, i.e., to give the components of a spherical wave in this limit.

C. Scattering Coefficients of the External Wave

The scattering coefficients A_n^m and B_n^m of the external wave are determined by writing the tangential continuity of the electric and magnetic fields at the surface of the sphere ($r = d/2$); the field components are obtained from Eqs. (10)–(19), using Eqs. (68), (69), and (71)–(74). We write

$$V^{i}_{\theta,X} + V^{\mathrm{s}}_{\theta,X} = V^{\mathrm{sp}}_{\theta,X}, \tag{75}$$

where V stands for E or H and where X stands for TM or TE. There are four boundary conditions, which lead to

$$M[g_{n,\mathrm{TM}}^m\Psi_n'(\alpha) - A_n^m\xi_n'(\alpha)] = E_n^m\Psi_n'(\beta), \tag{76}$$

$$M^2[g_{n,\mathrm{TE}}^m\Psi_n(\alpha) - B_n^m\xi_n(\alpha)] = D_n^m\Psi_n(\beta), \tag{77}$$

$$[g_{n,\mathrm{TM}}^m\Psi_n(\alpha) - A_n^m\xi_n(\alpha)] = E_n^m\Psi_n(\beta), \tag{78}$$

$$M[g^m_{n,\mathrm{TE}}\Psi'_n(\alpha) - B^m_n\xi'_n(\alpha)] = D^m_n\Psi'_n(\beta), \tag{79}$$

where α is the size parameter $\pi d/\lambda$ (λ is the wavelength in the surrounding medium) and β is $M\alpha$. The prime indicates the derivative of the function for the value of the argument indicated in parentheses. To establish these equations, we assume that the particle is nonmagnetic ($\mu_{\mathrm{sp}}/\mu = 1$) and use the accessory relation, which is valid for a nonmagnetic particle:

$$M = \frac{k_{\mathrm{sp}}}{k} = \left(\frac{\epsilon_{\mathrm{sp}}}{\epsilon}\right)^{1/2}. \tag{80}$$

We obtain the same set [Eqs. (76)–(79)] if we write Eq. (75) with φ instead of θ. This is the reason why the boundary conditions are presented with only θ fields, whereas they should actually involve the θ fields and the φ fields. Equations (76)–(79) are solved easily to obtain

$$A^m_n = a_n g^m_{n,\mathrm{TM}}, \tag{81}$$

$$B^m_n = b_n g^m_{n,\mathrm{TE}}, \tag{82}$$

where a_n and b_n are the usual scattering coefficients of the LMT:

$$a_n = \frac{\Psi_n(\alpha)\Psi'_n(\beta) - M\Psi'_n(\alpha)\Psi_n(\beta)}{\xi_n(\alpha)\Psi'_n(\beta) - M\xi'_n(\alpha)\Psi_n(\beta)}, \tag{83}$$

$$b_n = \frac{M\Psi_n(\alpha)\Psi'_n(\beta) - \Psi'_n(\alpha)\Psi_n(\beta)}{M\xi_n(\alpha)\Psi'_n(\beta) - \xi'_n(\alpha)\Psi_n(\beta)}. \tag{84}$$

6. SCATTERED-FIELD COMPONENTS

A. General Expressions

From the external BSP's the field components of the scattered wave are obtained by using Eqs. (10)–(19). We obtain

$$E_r = -kE_0 \sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C^{\mathrm{pw}}_n a_n g^m_{n,\mathrm{TM}}[\xi''_n(kr) + \xi_n(kr)] \times P^{|m|}_n(\cos\theta)\exp(im\varphi), \tag{85}$$

$$E_\theta = -\frac{E_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C^{\mathrm{pw}}_n[a_n g^m_{n,\mathrm{TM}}\xi'_n(kr)\tau^{|m|}_n(\cos\theta) + mb_n g^m_{n,\mathrm{TE}}\xi_n(kr)\Pi^{|m|}_n(\cos\theta)]\exp(im\varphi), \tag{86}$$

$$E_\varphi = -\frac{iE_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C^{\mathrm{pw}}_n[ma_n g^m_{n,\mathrm{TM}}\xi'_n(kr)\Pi^{|m|}_n(\cos\theta) + b_n g^m_{n,\mathrm{TE}}\xi_n(kr)\tau^{|m|}_n(\cos\theta)]\exp(im\varphi), \tag{87}$$

$$H_r = -kH_0 \sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C^{\mathrm{pw}}_n b_n g^m_{n,\mathrm{TE}}[\xi''_n(kr) + \xi_n(kr)] \times P^{|m|}_n(\cos\theta)\exp(im\varphi), \tag{88}$$

$$H_\theta = \frac{H_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C^{\mathrm{pw}}_n[ma_n g^m_{n,\mathrm{TM}}\xi_n(kr)\Pi^{|m|}_n(\cos\theta) - b_n g^m_{n,\mathrm{TE}}\xi'_n(kr)\tau^{|m|}_n(\cos\theta)]\exp(im\varphi), \tag{89}$$

$$H_\varphi = \frac{iH_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C^{\mathrm{pw}}_n[a_n g^m_{n,\mathrm{TM}}\xi_n(kr)\tau^{|m|}_n(\cos\theta) - mb_n g^m_{n,\mathrm{TE}}\xi'_n(kr)\Pi^{|m|}_n(\cos\theta)]\exp(im\varphi), \tag{90}$$

where we use

$$k = \omega\mu\frac{H_0}{E_0} = \omega\epsilon\frac{E_0}{H_0}. \tag{91}$$

We define generalized Legendre functions according to

$$\tau^k_n(\cos\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}P^k_n(\cos\theta), \tag{92}$$

$$\Pi^k_n(\cos\theta) = \frac{P^k_n(\cos\theta)}{\sin\theta}. \tag{93}$$

For $k = 1$, τ^k_n and Π^k_n are the usual Legendre functions appearing in the LMT, namely, τ_n and Π_n. Note the following correspondence to classical notation: $\tau^1_n = \tau_n$, $\Pi^1_n = \Pi_n$, but $P^0_n = P_n$.

B. Scattered-Field Components in the Far Field

Equations (85)–(90) can be of interest in some applications. An example of near-field computations, although limited to LMT, is given in Ref. 53. However, in most cases, interest is limited to the so-called far field, defined by the inequality $r \gg \lambda$. The exact meaning of this inequality is not a trivial matter.[53] Nevertheless, it leads to an asymptotic expression for the functions $\xi_n(kr)$, which is[40]

$$\xi_n(kr) \to i^{n+1}\exp(-ikr). \tag{94}$$

In this limit, then,

$$\xi''_n(kr) + \xi_n(kr) = 0, \tag{95}$$

leading to

$$E_r = H_r = 0. \tag{96}$$

With the limiting approximation described above, the nonzero field components simplify to

$$E_\theta = \frac{iE_0}{kr}\exp(-ikr)\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\frac{2n+1}{n(n+1)}[a_n g^m_{n,\mathrm{TM}}\tau^{|m|}_n(\cos\theta) + imb_n g^m_{n,\mathrm{TE}}\Pi^{|m|}_n(\cos\theta)]\exp(im\varphi), \tag{97}$$

$$E_\varphi = \frac{-E_0}{kr}\exp(-ikr)\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\frac{2n+1}{n(n+1)}[ma_n g^m_{n,\mathrm{TM}} \times \Pi^{|m|}_n(\cos\theta) + ib_n g^m_{n,\mathrm{TE}}\tau^{|m|}_n(\cos\theta)]\exp(im\varphi), \tag{98}$$

$$H_\varphi = \frac{H_0}{E_0}E_\theta, \tag{99}$$

$$H_\theta = -\frac{H_0}{E_0}E_\varphi. \tag{100}$$

The scattered wave has become a transverse wave.

C. Scattered Intensities

The scattered intensities are computed with the aid of the Poynting theorem,

$$S^+ = \tfrac{1}{2}\,\mathrm{Re}(E_\theta H^*_\varphi - E_\varphi H^*_\theta), \tag{101}$$

where S^+ is a dimensionless quantity produced by using the normalizing condition,

$$\frac{1}{2}\left(\frac{\epsilon}{\mu}\right)^{1/2} E_0^2 = 1. \tag{102}$$

Equation (102) is used throughout the rest of this paper, although we do not mention it again.

By inserting Eqs. (97)–(100) into Eq. (101) and separating the contributions of I_θ^+ and I_φ^+ to S^+, we obtain

$$\begin{pmatrix} I_\theta^+ \\ I_\varphi^+ \end{pmatrix} = \frac{\lambda^2}{4\pi^2 r^2} \begin{pmatrix} |\mathscr{S}_2|^2 \\ |\mathscr{S}_1|^2 \end{pmatrix}, \tag{103}$$

where $\mathscr{S}_2$ and $\mathscr{S}_1$ are defined by

$$E_\theta = \frac{iE_0}{kr} \exp(-ikr)\mathscr{S}_2, \tag{104}$$

$$E_\varphi = -\frac{E_0}{kr} \exp(-ikr)\mathscr{S}_1. \tag{105}$$

D. Phase Angle

The phase angle δ between the components E_θ and E_φ characterizes the state of polarization of the scattered wave and is equal to the phase angle between the functions $\mathscr{E}_\theta$ and $\mathscr{E}_\varphi$ given by

$$\mathscr{E}_\theta = i\mathscr{S}_2 = A_\theta \exp(i\varphi_2), \tag{106}$$

$$\mathscr{E}_\varphi = -\mathscr{S}_1 = A_\varphi \exp(i\varphi_1), \tag{107}$$

where A_θ and A_φ are real numbers. We readily obtain

$$\tan\delta = \tan(\varphi_2 - \varphi_1) = \frac{\mathrm{Re}(\mathscr{S}_1)\mathrm{Re}(\mathscr{S}_2) + \mathrm{Im}(\mathscr{S}_1)\mathrm{Im}(\mathscr{S}_2)}{\mathrm{Im}(\mathscr{S}_1)\mathrm{Re}(\mathscr{S}_2) - \mathrm{Re}(\mathscr{S}_1)\mathrm{Im}(\mathscr{S}_2)}. \tag{108}$$

7. RADIATIVE BALANCE

A. General Discussion

Let us consider a sphere of center O_P (Fig. 1) and arbitrary radius $r \gg \lambda$ surrounding the scatterer. At a point of the sphere, the total fields E_j^t and H_j^t are equal to the incident fields E_j^i and H_j^i plus the scattered fields E_j^s and H_j^s:

$$E_j^t = E_j^i + E_j^s, \tag{109}$$

$$H_j^t = H_j^i + H_j^s. \tag{110}$$

In Sections 4–6 superscripts were omitted for convenience. It is now necessary to reintroduce them.

The component $S_\perp^+$ of the Poynting vector $\mathbf{S}^+$ perpendicular to the surface of the sphere gives the energy flux per unit area and unit of time as

$$S_\perp^+ = \tfrac{1}{2}\mathrm{Re}(E_\theta^t H_\varphi^{t*} - E_\varphi^t H_\theta^{t*}). \tag{111}$$

The energy balance is expressed by the integration of $S_\perp^+$ on the surface of the sphere:

$$-C_{\mathrm{abs}} = +\int_{(S)} S_\perp^+ \mathrm{d}S, \tag{112}$$

where C_{abs} is the absorption cross section of the scatterer located inside the sphere.

From Eqs. (109)–(111), the right-hand side of Eq. (112) is the sum of three terms:

$$\mathcal{T}^{\mathrm{i}} = \int_0^\pi \int_0^{2\pi} \frac{1}{2}\mathrm{Re}(E_\theta^i H_\varphi^{i*} - E_\varphi^i H_\theta^{i*}) r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi, \tag{113}$$

$$\mathcal{T}^{\mathrm{s}} = \int_0^\pi \int_0^{2\pi} \frac{1}{2}\mathrm{Re}(E_\theta^s H_\varphi^{s*} - E_\varphi^s H_\theta^{s*}) r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi, \tag{114}$$

$$\mathcal{T}^{\mathrm{is}} = \int_0^\pi \int_0^{2\pi} \frac{1}{2}\mathrm{Re}(E_\theta^i H_\varphi^{s*} + E_\theta^s H_\varphi^{i*} - E_\varphi^i H_\theta^{s*} - E_\varphi^s H_\theta^{i*}) \times r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi. \tag{115}$$

The first integral $\mathcal{T}^{\mathrm{i}}$ corresponds to an energy balance of the nonperturbed incident fields. Since the surrounding medium is nonabsorbing, $\mathcal{T}^{\mathrm{i}}$ must be zero (see Subsection 7.B). The second integral $\mathcal{T}^{\mathrm{s}}$ corresponds to an energy balance of the scattered fields. Thus it is equal to the scattering cross section C_{sca} of the particle. The third integral is then found to be minus the extinction cross section:

$$\mathcal{T}^{\mathrm{is}} = +\int_{(S)} S_\perp^+ \mathrm{d}S - \mathcal{T}^{\mathrm{i}} - \mathcal{T}^{\mathrm{s}} = -C_{\mathrm{abs}} - C_{\mathrm{sca}} = -C_{\mathrm{ext}}. \tag{116}$$

B. Incident-Field Balance

In this subsection, we check the supposition that $\mathcal{T}^{\mathrm{i}} = 0$ with the expression of the electromagnetic field derived from BSP's.

From the BSP's for the incident fields, $U_{\mathrm{TM}}^{\mathrm{i}}$ and $U_{\mathrm{TE}}^{\mathrm{i}}$ [Eqs. (68) and (69)] and Eqs. (10)–(19), the incident-field components E_θ^i, E_φ^i, H_θ^i, and H_φ^i are found to be

$$E_\theta^i = \frac{E_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}}(g_{n,\mathrm{TM}}^m \Psi_n' \tau_n^{|m|} + m g_{n,\mathrm{TE}}^m \Psi_n \Pi_n^{|m|}) \times \exp(im\varphi), \tag{117}$$

$$E_\varphi^i = \frac{E_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}}(im g_{n,\mathrm{TM}}^m \Psi_n' \Pi_n^{|m|} + i g_{n,\mathrm{TE}}^m \Psi_n \tau_n^{|m|}) \times \exp(im\varphi), \tag{118}$$

$$H_\theta^i = \frac{H_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}}(-m g_{n,\mathrm{TM}}^m \Psi_n \Pi_n^{|m|} + g_{n,\mathrm{TE}}^m \Psi_n' \tau_n^{|m|}) \times \exp(im\varphi), \tag{119}$$

$$H_\varphi^i = \frac{H_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} C_n^{\mathrm{pw}}(-i g_{n,\mathrm{TM}}^m \Psi_n \tau_n^{|m|} + im g_{n,\mathrm{TE}}^m \Psi_n' \Pi_n^{|m|}) \times \exp(im\varphi), \tag{120}$$

where the arguments (kr) and $(\cos\theta)$ are now omitted for convenience.

By inserting Eqs. (117)–(120) into Eq. (113) and integrating with respect to φ, using Eq. (64), we obtain

$$\mathcal{T}^{\mathrm{i}} = 2\pi\,\mathrm{Re}\Big\{ i \sum_{p=-\infty}^{+\infty} \sum_{n=|p|\neq 0}^{\infty} \sum_{m=|p|\neq 0}^{\infty} C_n^{\mathrm{pw}} C_m^{\mathrm{pw}*} \times [I_1(g_{n,\mathrm{TM}}^p g_{m,\mathrm{TM}}^{p*} \Psi_n' \Psi_m - g_{n,\mathrm{TE}}^p g_{m,\mathrm{TE}}^{p*} \Psi_m' \Psi_n) + p I_2(g_{n,\mathrm{TE}}^p g_{m,\mathrm{TM}}^{p*} \Psi_n \Psi_m - g_{n,\mathrm{TM}}^p g_{m,\mathrm{TE}}^{p*} \Psi_m' \Psi_n')] \Big\}, \tag{121}$$

in which I_1 and I_2 are two integrals with respect to θ, given by (Appendix A)

$$I_1 = \int_0^{\pi} (\tau_n^{|p|}\tau_m^{|p|} + p^2\Pi_n^{|p|}\Pi_m^{|p|})\sin\theta d\theta, \tag{122}$$

which is

$$I_1 = \frac{2n(n+1)}{2n+1}\frac{(n+|p|)!}{(n-|p|)!}\delta_{nm} \tag{123}$$

and

$$I_2 = \int_0^{\pi} (\Pi_n^{|p|}\tau_m^{|p|} + \Pi_m^{|p|}\tau_n^{|p|})\sin\theta d\theta = 0 \qquad \text{if } p \neq 0. \tag{124}$$

Substituting for Eqs. (123) and (124) in Eq. (121), we obtain (with A_{np} real numbers)

$$\mathcal{T}^{\mathrm{i}} = \mathrm{Im}\left[\sum_{p=-\infty}^{+\infty}\sum_{n=|p|\neq 0}^{\infty} A_{np}|C_n^{\mathrm{pw}}|^2(|g_{n,\mathrm{TM}}^p|^2 - |g_{n,\mathrm{TE}}^p|^2)\Psi_n\Psi_n'\right]$$
$$= 0. \tag{125}$$

This result is important for the internal consistency of the theory. The energy balance of the incident fields is found to be correct (leading to $\mathcal{T}^{\mathrm{i}} = 0$, as predicted in Subsection 7.A), although the components of the original field [Eqs. (21)–(25) or (32)–(36)] do not fully satisfy the Maxwell equations. The reason is that the process of establishing the incident BSP's $U_{\mathrm{TM}}^{\mathrm{i}}$ and $U_{\mathrm{TE}}^{\mathrm{i}}$ leads to a kind of remodeling of the description of the incident Gaussian beam in agreement with Maxwell's equations. Starting from the fields E_r^{i}, H_r^{i}, E_θ^{i}, H_θ^{i}, E_φ^{i}, and H_φ^{i} obtained from the incident BSP's, we can obtain new expressions for E_x, H_x, E_y, H_y, E_z, and H_z. These new expressions would not agree with the original set [Eqs. (32)–(36)]. They could be considered to represent a remodeled Cartesian description of the incident beam. For the case in which $x_0 = y_0 = z_0 = 0$, the comparison between the original expressions for $E_x, \ldots, H_z$ and the new expressions obtained from the incident BSP's was discussed extensively in Ref. 42.

C. Scattering Cross Section

The scattering cross section can be computed from $\mathcal{T}^{\mathrm{s}}$ [Eq. (114)] or, equivalently, from

$$C_{\mathrm{sca}} = \int_0^{\pi}\int_0^{2\pi} (I_\theta^+ + I_\varphi^+)r^2\sin\theta d\theta d\varphi, \tag{126}$$

where the intensities I_θ^+ and I_φ^+ are given by Eq. (103).

Integrating over φ, using Eq. (64), and then over θ, using Eqs. (123) and (124), we obtain

$$C_{\mathrm{sca}} = \frac{\lambda^2}{\pi}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\frac{2n+1}{n(n+1)}\frac{(n+|m|)!}{(n-|m|)!}$$
$$\times (|a_n|^2|g_{n,\mathrm{TM}}^m|^2 + |b_n|^2|g_{n,\mathrm{TE}}^m|^2). \tag{127}$$

D. Extinction Cross Section

From Eqs. (115) and (116) we obtain

$$C_{\mathrm{ext}} = \int_0^{\pi}\int_0^{2\pi}\frac{1}{2}\mathrm{Re}(E_\varphi^{\mathrm{i}}H_\theta^{\mathrm{s}*} + E_\varphi^{\mathrm{s}}H_\theta^{\mathrm{i}*} - E_\theta^{\mathrm{i}}H_\varphi^{\mathrm{s}*} - E_\theta^{\mathrm{s}}H_\varphi^{\mathrm{i}*})$$
$$\times r^2\sin\theta d\theta d\varphi. \tag{128}$$

The incident-field components (superscript i) are given by Eqs. (117)–(120), and the scattered-field components are given by Eqs. (97)–(100), in which the superscript s is omitted for convenience.

Integrating over φ, using Eq. (64), and rearranging with C_n^{pw} [Eq. (62)], we obtain

$$C_{\mathrm{ext}} = 2\pi\,\mathrm{Re}\Bigg(\sum_{p=-\infty}^{+\infty}\sum_{n=|p|\neq 0}^{\infty}\sum_{m=|p|\neq 0}^{\infty} C_n^{\mathrm{pw}}C_m^{\mathrm{pw}*}$$
$$\times \{I_1[(-i)^m\exp(ikr)(a_m^* g_{n,\mathrm{TM}}^p g_{m,\mathrm{TM}}^{p*}\Psi_n' - ib_m^* g_{n,\mathrm{TE}}^p g_{m,\mathrm{TE}}^{p*}\Psi_n)$$
$$+ i^n\exp(-ikr)(ia_n g_{n,\mathrm{TM}}^p g_{m,\mathrm{TM}}^{p*}\Psi_m + b_n g_{n,\mathrm{TE}}^p g_{m,\mathrm{TE}}^{p*}\Psi_m')]$$
$$+ pI_2[(-i)^m\exp(ikr)(a_m^* g_{n,\mathrm{TE}}^p g_{m,\mathrm{TM}}^{p*}\Psi_n - ib_m^* g_{n,\mathrm{TM}}^p g_{m,\mathrm{TE}}^{p*}\Psi_n')$$
$$- i^n\exp(-ikr)(ia_n\Psi_m' g_{n,\mathrm{TM}}^p g_{m,\mathrm{TE}}^{p*} + b_n\Psi_m g_{n,\mathrm{TE}}^p g_{m,\mathrm{TM}}^{p*})]\}\Bigg). \tag{129}$$

Integrating over θ, using Eqs. (123) and (124), we obtain

$$C_{\mathrm{ext}} = \frac{4\pi}{k^2}\mathrm{Re}\Bigg\{\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\frac{2n+1}{n(n+1)}\frac{(n+|m|)!}{(n-|m|)!}$$
$$[(-i)^n\exp(ikr)(a_n^*\Psi_n'|g_{n,\mathrm{TM}}^m|^2 - ib_n^*\Psi_n|g_{n,\mathrm{TE}}^m|^2)$$
$$+ i^n\exp(-ikr)(ia_n\Psi_n|g_{n,\mathrm{TM}}^m|^2 + b_n\Psi_n'|g_{n,\mathrm{TE}}^m|^2)]\Bigg\}. \tag{130}$$

Since the radius of the sphere surrounding the particle is arbitrary, let us take it in the limit $r \rightarrow \infty$. We can then use the limit expression[54]

$$\Psi_n(kr) \rightarrow \tfrac{1}{2}[(-i)^{n+1}\exp(ikr) + i^{n+1}\exp(-ikr)]. \tag{131}$$

Equation (130) and relation (131) lead to the final expression for C_{ext}:

$$C_{\mathrm{ext}} = \frac{\lambda^2}{\pi}\mathrm{Re}\Bigg[\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\frac{2n+1}{n(n+1)}\frac{(n+|m|)!}{(n-|m|)!}$$
$$\times (a_n|g_{n,\mathrm{TM}}^m|^2 + b_n|g_{n,\mathrm{TE}}^m|^2)\Bigg]. \tag{132}$$

8. RADIATION PRESSURE

The reduced momentum of the wave is equal to the ratio of the reduced Poynting vector S^+ over the speed of light c. The radiation pressure force exerted by the beam on the scatterer is proportional to the net momentum removed from the incident beam. The x, y, and z components of the reduced radiation pressure force $\mathbf{F}^+$ can be characterized by three pressure cross sections, $C_{\mathrm{pr},x}$, $C_{\mathrm{pr},y}$ and $C_{\mathrm{pr},z}$, which are examined separately in this section.

A. Longitudinal Radiation Pressure (*z* Direction)

In the z direction, the pressure cross section $C_{\mathrm{pr},z}$ can be defined as

$$C_{\mathrm{pr},z} = cF_z^+ = (\overline{\cos\theta})C_{\mathrm{ext}} - (\overline{\cos\theta})C_{\mathrm{sca}}, \tag{133}$$

where the mean value (denoted by an overbar) indicates an integration of the reduced Poynting vector weighted by $\cos\theta$.

The first term is the forward momentum removed from the beam, and the second is minus the forward momentum given by the scatterer to the scattered wave. These terms identify with the expressions $(-\mathcal{T}^{is}\,\overline{\cos\theta})$ and $(-\mathcal{T}^{s}\,\overline{\cos\theta})$, respectively, from Subsection 7.A. There is no radiation pressure from the integral $\mathcal{T}^{i}$, since the unperturbed field does not leave any momentum to the scatterer.

In our previous paper[41] the first term was set equal to C_{ext}. This was certainly a good approximation, since only a waist location of the particle was considered; i.e., we assumed that the wave front on the scatterer was (nearly) a plane, which permitted us to use the same formulation as given by van de Hulst.[55] In the present paper, in which arbitrary location of the particle is considered, this assumption must be relaxed.

Let us now examine successively each term on the right-hand side of Eq. (133), beginning with the second term, which can be written either as $(-\mathcal{T}^{s}\,\overline{\cos\theta})$ or as an integral involving the reduced scattered intensities,

$$(\overline{\cos\theta})C_{\text{sca}} = \int_0^{\pi}\int_0^{2\pi}(I_\theta^{+} + I_\varphi^{+})r^2\sin\theta\cos\theta\,\mathrm{d}\theta\mathrm{d}\varphi. \tag{134}$$

Integrating over φ, using Eq. (64), we obtain

$$\begin{aligned}(\overline{\cos\theta})C_{\text{sca}} = \frac{2\pi}{k^2}\sum_{p=-\infty}^{+\infty}\sum_{n=|p|\neq 0}^{\infty}\sum_{m=|p|\neq 0}^{\infty}\frac{2n+1}{n(n+1)}\frac{2m+1}{m(m+1)}\\ \times[I_3(a_n a_m^{*} g_{n,\text{TM}}^{p} g_{m,\text{TM}}^{p*} + b_n b_m^{*} g_{n,\text{TE}}^{p} g_{m,\text{TE}}^{p*})\\ + ipI_4(b_n a_m^{*} g_{n,\text{TE}}^{p} g_{m,\text{TM}}^{p*} - a_n b_m^{*} g_{n,\text{TM}}^{p} g_{m,\text{TE}}^{p*})],\end{aligned} \tag{135}$$

in which (see Appendix B)

$$\begin{aligned}I_3 &= \int_0^{\pi}(\tau_n^{|p|}\tau_m^{|p|} + p^2\Pi_n^{|p|}\Pi_m^{|p|})\cos\theta\sin\theta\mathrm{d}\theta\\ &= \frac{2(n-1)(n+1)(n+|p|)!}{(2n-1)(2n+1)(n-1-|p|)!}\delta_{m,n-1}\\ &\quad + \frac{2(m-1)(m+1)(m+|p|)!}{(2m-1)(2m+1)(m-1-|p|)!}\delta_{n,m-1},\end{aligned} \tag{136}$$

$$\begin{aligned}I_4 &= \int_0^{\pi}(\tau_n^{|p|}\Pi_m^{|p|} + \tau_m^{|p|}\Pi_n^{|p|})\cos\theta\sin\theta\mathrm{d}\theta\\ &= \frac{2(n+|p|)!}{2n+1(n-|p|)!}\delta_{nm}.\end{aligned} \tag{137}$$

The θ integration then leads to

$$\begin{aligned}(\overline{\cos\theta})C_{\text{sca}} = -\frac{2\lambda^2}{\pi}\sum_{n=1}^{\infty}\sum_{p=-n}^{+n}p\frac{2n+1}{n^2(n+1)^2}\frac{(n+|p|)!}{(n-|p|)!}\\ \times\mathrm{Re}(ia_n b_n^{*} g_{n,\text{TM}}^{p} g_{n,\text{TE}}^{p*}) - \frac{1}{(n+1)^2}\\ \times\frac{(n+1+|p|)!}{(n-|p|)!}\mathrm{Re}(a_n a_{n+1}^{*} g_{n,\text{TM}}^{p} g_{n+1,\text{TM}}^{p*}\\ + b_n b_{n+1}^{*} g_{n,\text{TE}}^{p} g_{n+1,\text{TE}}^{p*}).\end{aligned} \tag{138}$$

The expression for the first term on the right-hand side of Eq. (133) is somewhat different:

$$\begin{aligned}(\overline{\cos\theta})C_{\text{ext}} = \int_0^{\pi}\int_0^{2\pi}\frac{1}{2}\mathrm{Re}(E_\varphi^{i}H_\theta^{s*} + E_\varphi^{s}H_\theta^{i*} - E_\theta^{i}H_\varphi^{s*} - E_\theta^{s}H_\varphi^{i*})\\ \times r^2\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}\varphi.\end{aligned} \tag{139}$$

Equation (139) is exactly the same as Eq. (128) except for the presence of $\cos\theta$. By following the same procedure as in Subsection 7.D, we obtain an equation that is analogous to Eq. (129), where the θ integrals I_1 and I_2 are replaced by I_3 and I_4 [Eqs. (136) and (137)], respectively.

We then perform the θ integration [Eqs. (136) and (137)] and rearrange the subscripts to obtain

$$\begin{aligned}(\overline{\cos\theta})C_{\text{ext}} = 2\pi\,\mathrm{Re}\Bigg\{\sum_{p=-\infty}^{+\infty}\sum_{n=|p|\neq 0}^{\infty}\frac{2n(n+2)}{(2n+1)(2n+3)}\\ \times\frac{(n+1+|p|)!}{(n-|p|)!}C_n^{\text{pw}}C_{n+1}^{\text{pw}*}[(-i)^{n+1}\exp(ikr)(\Psi_n' - i\Psi_n)\\ \times(a_{n+1}^{*} g_{n,\text{TM}}^{p} g_{n+1,\text{TM}}^{p*} + b_{n+1}^{*} g_{n,\text{TE}}^{p} g_{n+1,\text{TE}}^{p*}) + i^n\exp(-ikr)\\ \times(\Psi_{n+1}' + i\Psi_{n+1})(a_n g_{n,\text{TM}}^{p} g_{n+1,\text{TM}}^{p*} + b_n g_{n,\text{TE}}^{p} g_{n+1,\text{TE}}^{p*})]\\ + \sum_{p=-\infty}^{+\infty}\sum_{n=|p|\neq 0}^{\infty}\frac{2p}{2n+1}\frac{(n+|p|)!}{(n-|p|)!}|C_n^{\text{pw}}|^2 i^n\exp(-ikr)\\ \times(\Psi_n - i\Psi_n')(a_n g_{n,\text{TM}}^{p} g_{n,\text{TE}}^{p*} - b_n g_{n,\text{TE}}^{p} g_{n,\text{TM}}^{p*})\Bigg\};\end{aligned} \tag{140}$$

then, by assuming that the spherical surface used for integration has a large radius, we can replace the Ricatti–Bessel functions by their asymptotic expressions (131). By using also Eq. (62) and rearranging the summations, we obtain

$$\begin{aligned}(\overline{\cos\theta})C_{\text{ext}} = \frac{\lambda^2}{\pi}\sum_{n=1}^{\infty}\sum_{p=-n}^{+n}\Bigg\{\frac{1}{(n+1)^2}\frac{(n+1+|p|)!}{(n-|p|)!}\\ \times\mathrm{Re}[(a_n + a_{n+1}^{*})g_{n,\text{TM}}^{p} g_{n+1,\text{TM}}^{p*} + (b_n + b_{n+1}^{*})g_{n,\text{TE}}^{p} g_{n+1,\text{TE}}^{p*}]\\ - p\frac{2n+1}{n^2(n+1)^2}\frac{(n+|p|)!}{(n-|p|)!}\mathrm{Re}[i(a_n + b_n^{*})g_{n,\text{TM}}^{p} g_{n,\text{TE}}^{p*}]\Bigg\}.\end{aligned} \tag{141}$$

Substracting Eq. (138) from Eq. (141) gives the final expression for the pressure cross section $C_{\text{pr},z}$:

$$\begin{aligned}C_{\text{pr},z} = \frac{\lambda^2}{\pi}\sum_{n=1}^{\infty}\sum_{p=-n}^{+n}\Bigg\{\frac{1}{(n+1)^2}\frac{(n+1+|p|)!}{(n-|p|)!}\\ \times\mathrm{Re}[(a_n + a_{n+1}^{*} - 2a_n a_{n+1}^{*})g_{n,\text{TM}}^{p} g_{n+1,\text{TM}}^{p*}\\ + (b_n + b_{n+1}^{*} - 2b_n b_{n+1}^{*})g_{n,\text{TE}}^{p} g_{n+1,\text{TE}}^{p*}]\\ + p\frac{2n+1}{n^2(n+1)^2}\frac{(n+|p|)!}{(n-|p|)!}\\ \times\mathrm{Re}[i(2a_n b_n^{*} - a_n - b_n^{*})g_{n,\text{TM}}^{p} g_{n,\text{TE}}^{p*})]\Bigg\}.\end{aligned} \tag{142}$$

B. Transverse Radiation Pressure (x and y Directions)

The pressure cross section in the x direction is defined by an expression similar to that in Eq. (133) but with a weighting coefficient of $\sin\theta\cos\varphi$ instead of $\cos\theta$:

$$C_{\mathrm{pr},x} = cF_x^+ = (\overline{\sin\theta\cos\varphi})C_{\mathrm{ext}} - (\overline{\sin\theta\cos\varphi})C_{\mathrm{sca}}. \quad (143)$$

We compute the second term on the right-hand side:

$$(\overline{\sin\theta\cos\varphi})C_{\mathrm{sca}} = \int_0^\pi \int_0^{2\pi} (I_\theta^+ + I_\varphi^+) r^2 \sin^2\theta \cos\varphi \mathrm{d}\theta \mathrm{d}\varphi. \quad (144)$$

Replacing I_θ^+ and I_φ^+ by their expressions and integrating with respect to φ, using

$$\int_0^{2\pi} \cos\varphi \exp(ik\varphi)\exp(-ik'\varphi)\mathrm{d}\varphi = \pi(\delta_{k',k+1} + \delta_{k,k'+1}), \quad (145)$$

we obtain

$$(\overline{\sin\theta\cos\varphi})C_{\mathrm{sca}} = \frac{\lambda^2}{4\pi} \sum_{p=-\infty}^{+\infty} \sum_{n=|p|\neq 0}^{\infty} \sum_{m=|p+1|\neq 0}^{\infty} \frac{2n+1}{n(n+1)} \times \frac{2m+1}{m(m+1)} [I_5 \mathrm{Re}(U_{nm}^p) + I_6 \mathrm{Re}(V_{nm}^p)], \quad (146)$$

in which

$$U_{nm}^p = a_n a_m^* g_{n,\mathrm{TM}}^p g_{m,\mathrm{TM}}^{p+1*} + b_n b_m^* g_{n,\mathrm{TE}}^p g_{m,\mathrm{TE}}^{p+1*}, \quad (147)$$

$$V_{nm}^p = ib_n a_m^* g_{n,\mathrm{TE}}^p g_{m,\mathrm{TM}}^{p+1*} - ia_n b_m^* g_{n,\mathrm{TM}}^p g_{m,\mathrm{TE}}^{p+1*}, \quad (148)$$

and (see Appendix C)

$$I_5 = \int_0^\pi (\tau_n^{|p|}\tau_m^{|p+1|} + p(p+1)\Pi_n^{|p|}\Pi_m^{|p+1|}) \sin^2\theta \mathrm{d}\theta$$

$$= \begin{cases} \dfrac{2}{(2n+1)(2m+1)} \dfrac{(m+p+1)!}{(m-p-1)!} [(n-1)(n+1)\delta_{n,m+1} - (m-1)(m+1)\delta_{m,n+1}], & p \geqslant 0 \\ \dfrac{2}{(2n+1)(2m+1)} \dfrac{(n-p)!}{(n+p)!} [(m-1)(m+1)\delta_{m,n+1} - (n-1)(n+1)\delta_{n,m+1}], & p < 0 \end{cases}, \quad (149)$$

$$I_6 = \int_0^\pi (p\Pi_n^{|p|}\tau_m^{|p+1|} + (p+1)\Pi_m^{|p+1|}\tau_n^{|p|}) \sin^2\theta \mathrm{d}\theta$$

$$= \begin{cases} \dfrac{2}{2n+1} \dfrac{(n+p+1)!}{(n-p-1)!} \delta_{nm}, & p \geqslant 0 \\ \dfrac{-2}{2n+1} \dfrac{(n-p)!}{(n+p)!} \delta_{nm}, & p < 0 \end{cases}. \quad (150)$$

The θ integration then leads to

$$(\overline{\sin\theta\cos\varphi})C_{\mathrm{sca}} = \frac{\lambda^2}{\pi} \left\{ \sum_{p=0}^{\infty} \sum_{n=|p|\neq 0}^{\infty} \sum_{m=|p+1|\neq 0}^{\infty} \frac{(m+p+1)!}{(m-p-1)!} \times \left[\mathrm{Re}(U_{nm}^p)\left(\frac{1}{n^2}\delta_{n,m+1} - \frac{1}{m^2}\delta_{m,n+1}\right) + \mathrm{Re}(V_{nm}^p)\frac{2n+1}{n^2(n+1)^2}\delta_{nm}\right] + \sum_{p=-1}^{-\infty} \sum_{n=|p|\neq 0}^{\infty} \sum_{m=|p+1|\neq 0}^{\infty} \frac{(n-p)!}{(n+p)!} \times \left[\mathrm{Re}(U_{nm}^p)\left(\frac{1}{m^2}\delta_{m,n+1} - \frac{1}{n^2}\delta_{n,m+1}\right) - \mathrm{Re}(V_{nm}^p)\frac{2n+1}{n^2(n+1)^2}\delta_{nm}\right]\right\}, \quad (151)$$

which can be written more concisely as

$$(\overline{\sin\theta\cos\varphi})C_{\mathrm{sca}} = \frac{\lambda^2}{\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1\neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \times \left[\mathrm{Re}(U_{mn}^{p-1} + U_{nm}^{-p})\left(\frac{1}{m^2}\delta_{m,n+1} - \frac{1}{n^2}\delta_{n,m+1}\right) + \frac{2n+1}{n^2(n+1)^2}\delta_{nm}\mathrm{Re}(V_{mn}^{p-1} - V_{nm}^{-p})\right]. \quad (152)$$

The first term on the right-hand side of Eq. (143) is Eq. (139) with $\cos\theta$ replaced by $\sin\theta\cos\varphi$:

$$(\overline{\sin\theta\cos\varphi})C_{\mathrm{ext}} = \int_0^\pi \int_0^{2\pi} \frac{1}{2}\mathrm{Re}(E_\varphi^{\mathrm{i}}H_\theta^{\mathrm{s}*} + E_\varphi^{\mathrm{s}}H_\theta^{\mathrm{i}*} - E_\theta^{\mathrm{i}}H_\varphi^{\mathrm{s}*} - E_\theta^{\mathrm{s}}H_\varphi^{\mathrm{i}*}) r^2 \sin^2\theta \cos\varphi \mathrm{d}\theta \mathrm{d}\varphi. \quad (153)$$

The φ integration is done by using Eq. (145), and the resulting expression can be rearranged to obtain

$$(\overline{\sin\theta\cos\varphi})C_{\mathrm{ext}} = \frac{\lambda^2}{4\pi}\mathrm{Re} \sum_{p=-\infty}^{+\infty} \sum_{n=|p|\neq 0}^{\infty} \sum_{m=|p+1|\neq 0}^{\infty} \frac{2n+1}{n(n+1)} \times \frac{2m+1}{m(m+1)} \{I_5[i^n \exp(-ikr)(\Psi_n' + i\Psi_n)(a_m g_{m,\mathrm{TM}}^{p+1} g_{n,\mathrm{TM}}^{p*} + b_m g_{m,\mathrm{TE}}^{p+1} g_{n,\mathrm{TE}}^{p*}) + i^m \exp(-ikr)(\Psi_m' + i\Psi_m)(a_n g_{n,\mathrm{TM}}^p g_{m,\mathrm{TM}}^{p+1*} + b_n g_{n,\mathrm{TE}}^p g_{m,\mathrm{TE}}^{p+1*})] + I_6[i^{m-1}\exp(-ikr)(\Psi_m' + i\Psi_m) \times (a_n g_{n,\mathrm{TM}}^p g_{m,\mathrm{TE}}^{p+1*} - b_n g_{n,\mathrm{TE}}^p g_{m,\mathrm{TM}}^{p+1*}) + i^{n-1}\exp(-ikr) \times (\Psi_n' + i\Psi_n)(a_m g_{m,\mathrm{TM}}^{p+1} g_{n,\mathrm{TE}}^{p*} - b_m g_{m,\mathrm{TE}}^{p+1} g_{n,\mathrm{TM}}^{p*})]\}. \quad (154)$$

Again, we replace the Ricatti–Bessel functions by their asymptotic expressions (131) and assume that the spherical surface on which the integration is performed has a large radius. We also use Eqs. (149) and (150) for I_5 and I_6 and introduce the following notation:

$$S_{nm}^p = (a_n + a_m^*) g_{n,\mathrm{TM}}^p g_{m,\mathrm{TM}}^{p+1*} + (b_n + b_m^*) g_{n,\mathrm{TE}}^p g_{m,\mathrm{TE}}^{p+1*}, \quad (155)$$

$$T_{nm}^p = -i(a_n + b_m^*) g_{n,\mathrm{TM}}^p g_{m,\mathrm{TE}}^{p+1*} + i(b_n + a_m^*) g_{n,\mathrm{TE}}^p g_{m,\mathrm{TM}}^{p+1*}. \quad (156)$$

Thus Eq. (154) becomes

$$(\overline{\sin\theta\cos\varphi})C_{\text{ext}} = \frac{\lambda^2}{2\pi}\left\{\sum_{p=0}^{\infty}\sum_{n=|p|\neq 0}^{\infty}\sum_{m=|p+1|\neq 0}^{\infty}\frac{(m+p+1)!}{(m-p-1)!}\right.$$
$$\times\left[\mathrm{Re}(S_{nm}^{p})\left(\frac{1}{n^2}\delta_{n,m+1}-\frac{1}{m^2}\delta_{m,n+1}\right)\right.$$
$$\left.+\mathrm{Re}(T_{nm}^{p})\frac{2m+1}{m^2(m+1)^2}\delta_{nm}\right]$$
$$+\sum_{p=-1}^{-\infty}\sum_{n=|p|\neq 0}^{\infty}\sum_{m=|p+1|\neq 0}^{\infty}\frac{(n-p)!}{(n+p)!}$$
$$\times\left[\mathrm{Re}(S_{nm}^{p})\left(\frac{1}{m^2}\delta_{m,n+1}-\frac{1}{n^2}\delta_{n,m+1}\right)\right.$$
$$\left.\left.-\mathrm{Re}(T_{nm}^{p})\frac{2n+1}{n^2(n+1)^2}\delta_{nm}\right]\right\}, \tag{157}$$

which can be rewritten more concisely by rearranging the subscripts and superscripts:

$$(\overline{\sin\theta\cos\varphi})C_{\text{ext}} = \frac{\lambda^2}{2\pi}\sum_{p=1}^{\infty}\sum_{n=p}^{\infty}\sum_{m=p-1\neq 0}^{\infty}\frac{(n+p)!}{(n-p)!}$$
$$\times\left[\mathrm{Re}(S_{mn}^{p-1}+S_{nm}^{-p})\left(\frac{1}{m^2}\delta_{m,n+1}-\frac{1}{n^2}\delta_{n,m+1}\right)\right.$$
$$\left.+\mathrm{Re}(T_{mn}^{p-1}-T_{nm}^{-p})\frac{2n+1}{n^2(n+1)^2}\delta_{nm}\right]. \tag{158}$$

By substracting Eq. (152) from Eq. (158), we obtain the pressure cross section:

$$C_{\text{pr},x} = cF_x^+ = \frac{\lambda^2}{2\pi}\sum_{p=1}^{\infty}\sum_{n=p}^{\infty}\sum_{m=p-1\neq 0}^{\infty}\frac{(n+p)!}{(n-p)!}$$
$$\times\left[\mathrm{Re}(S_{mn}^{p-1}+S_{nm}^{-p}-2U_{mn}^{p-1}-2U_{nm}^{-p})\right.$$
$$\times\left(\frac{1}{m^2}\delta_{m,n+1}-\frac{1}{n^2}\delta_{n,m+1}\right)+\frac{2n+1}{n^2(n+1)^2}\delta_{nm}$$
$$\left.\times\mathrm{Re}(T_{mn}^{p-1}-T_{nm}^{-p}-2V_{mn}^{p-1}+2V_{nm}^{-p})\right]. \tag{159}$$

For the reduced pressure radiation force component F_y^+, we have an expression similar to Eq. (143):

$$C_{\text{pr},y} = cF_y^+ = (\overline{\sin\theta\sin\varphi})C_{\text{ext}} - (\overline{\sin\theta\sin\varphi})C_{\text{sca}}. \tag{160}$$

The second term on the right-hand side is

$$(\overline{\sin\theta\sin\varphi})C_{\text{sca}} = \int_0^{\pi}\int_0^{2\pi}(I_\theta^+ + I_\varphi^+)r^2\sin^2\theta\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi. \tag{161}$$

Integration with respect to φ is performed by using

$$\int_0^{2\pi}\sin\varphi\exp(ik\varphi)\exp(-ik'\varphi)\mathrm{d}\varphi = i\pi(\delta_{k,k'+1}-\delta_{k',k+1}). \tag{162}$$

When the integration with respect to θ is performed also, the result for $(\overline{\sin\theta\sin\varphi})C_{\text{sca}}$ is found to be equal to the right-hand sides of Eqs. (151) and (152) with Re replaced by Im. The same remark holds for the term $(\overline{\sin\theta\sin\varphi})C_{\text{ext}}$, which can be written, after φ and θ integrations, as Eqs. (157) and (158) with Re replaced by Im. Thus the y pressure cross section can be expressed in the same form as Eq. (159):

$$C_{\text{pr},y} = cF_y^+ = \frac{\lambda^2}{2\pi}\sum_{p=1}^{\infty}\sum_{n=p}^{\infty}\sum_{m=p-1\neq 0}^{\infty}\frac{(n+p)!}{(n-p)!}$$
$$\times\left[\mathrm{Im}(S_{mn}^{p-1}+S_{nm}^{-p}-2U_{mn}^{p-1}-2U_{nm}^{-p})\right.$$
$$\times\left(\frac{1}{m^2}\delta_{m,n+1}-\frac{1}{n^2}\delta_{n,m+1}\right)+\frac{2n+1}{n^2(n+1)^2}\delta_{nm}$$
$$\left.\times\mathrm{Im}(T_{mn}^{p-1}-T_{nm}^{-p}-2V_{mn}^{p-1}+2V_{nm}^{-p})\right]. \tag{163}$$

9. ANOTHER FORMULATION

To set a bridge between the present formulation and the results obtained previously in Refs. 39, 41, and 42 and with the LMT, another formulation is required.

Let S_2 and S_1 be the amplitude functions given by the following relations[39,41,42]:

$$S_1 = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}g_n[a_n\Pi_n(\cos\theta)+b_n\tau_n(\cos\theta)], \tag{164}$$

$$S_2 = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}g_n[a_n\tau_n(\cos\theta)+b_n\Pi_n(\cos\theta)], \tag{165}$$

where the coefficients g_n are given by

$$g_n = \frac{k(2n+1)}{i^{n-1}(-1)^n\pi n(n+1)}\int_0^{\pi}\int_0^{\infty}Fr\Psi_n^1(kr)P_n^1(\cos\theta)$$
$$\times\sin\theta\mathrm{d}\theta\mathrm{d}(kr). \tag{166}$$

These coefficients appeared in the previous versions of our generalization.[39,41,42]

We then establish that

$$\mathcal{S}_2 = (\cos\varphi)S_2 + \mathcal{S}'_2, \tag{167}$$

$$\mathcal{S}_1 = i(\sin\varphi)S_1 + \mathcal{S}'_1, \tag{168}$$

with $\mathcal{S}_2$ and $\mathcal{S}_1$ given by Eqs. (104) and (105) and

$$\mathcal{S}'_2 = \sum_{n=1}^{\infty}\sum_{\substack{m=-n\\|m|\neq 1}}^{+n}\frac{2n+1}{n(n+1)}[a_n g_{n,\mathrm{TM}}^{m}\tau_n^{|m|}(\cos\theta)$$
$$+imb_n g_{n,\mathrm{TE}}^{m}\Pi_n^{|m|}(\cos\theta)]\exp(im\varphi)+\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}$$
$$\times\{a_n\tau_n(\cos\theta)[(\cos\varphi)G_{n,\mathrm{TM}}^{+}+i(\sin\varphi)G_{n,\mathrm{TM}}^{-}]$$
$$+b_n\Pi_n(\cos\theta)[i(\cos\varphi)G_{n,\mathrm{TE}}^{-}-(\sin\varphi)G_{n,\mathrm{TE}}^{+}]\}, \tag{169}$$

$$\mathcal{S}'_1 = \sum_{n=1}^{\infty}\sum_{\substack{m=-n\\|m|\neq 1}}^{+n}\frac{2n+1}{n(n+1)}[ma_n g_{n,\mathrm{TM}}^{m}\Pi_n^{|m|}(\cos\theta)$$
$$+ib_n g_{n,\mathrm{TE}}^{m}\tau_n^{|m|}(\cos\theta)]\exp(im\varphi)+\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}$$
$$\times\{a_n\Pi_n(\cos\theta)[(\cos\varphi)G_{n,\mathrm{TM}}^{-}+i(\sin\varphi)G_{n,\mathrm{TM}}^{+}]$$
$$+b_n\tau_n(\cos\theta)[i(\cos\varphi)G_{n,\mathrm{TE}}^{+}-(\sin\varphi)G_{n,\mathrm{TE}}^{-}]\}, \tag{170}$$

in which

$$G^{+}_{n,\mathrm{TM}} = g^{1}_{n,\mathrm{TM}} + g^{-1}_{n,\mathrm{TM}} - g_n, \tag{171}$$

$$G^{+}_{n,\mathrm{TE}} = g^{1}_{n,\mathrm{TE}} + g^{-1}_{n,\mathrm{TE}}, \tag{172}$$

$$G^{-}_{n,\mathrm{TM}} = g^{1}_{n,\mathrm{TM}} - g^{-1}_{n,\mathrm{TM}}, \tag{173}$$

$$G^{-}_{n,\mathrm{TE}} = g^{1}_{n,\mathrm{TE}} - g^{-1}_{n,\mathrm{TE}} + ig_n. \tag{174}$$

From Eqs. (167) and (168), the expressions for the scattered intensities become

$$\begin{pmatrix} I^{+}_{\theta} \\ I^{+}_{\varphi} \end{pmatrix} = \begin{pmatrix} I^{+\mathrm{L}}_{\theta} \\ I^{+\mathrm{L}}_{\varphi} \end{pmatrix} + \begin{pmatrix} I^{+\mathrm{C}}_{\theta} \\ I^{+\mathrm{C}}_{\varphi} \end{pmatrix} + \begin{pmatrix} I^{+\mathrm{S}}_{\theta} \\ I^{+\mathrm{S}}_{\varphi} \end{pmatrix}, \tag{175}$$

with

$$\begin{pmatrix} I^{+\mathrm{L}}_{\theta} \\ I^{+\mathrm{L}}_{\varphi} \end{pmatrix} = \frac{\lambda^2}{4\pi^2 r^2}\begin{pmatrix} i_2 \cos^2\varphi \\ i_1 \sin^2\varphi \end{pmatrix}, \tag{176}$$

where the intensity functions i_j are $|S_j|^2$ and

$$\begin{pmatrix} I^{+\mathrm{C}}_{\theta} \\ I^{+\mathrm{C}}_{\varphi} \end{pmatrix} = \frac{2\lambda^2}{4\pi^2 r^2}\begin{bmatrix} \cos\varphi\, \mathrm{Re}(S_2 \mathscr{S}_2'^{*}) \\ \sin\varphi\, \mathrm{Re}(iS_1 \mathscr{S}_1'^{*}) \end{bmatrix}, \tag{177}$$

$$\begin{pmatrix} I^{+\mathrm{S}}_{\theta} \\ I^{+\mathrm{S}}_{\varphi} \end{pmatrix} = \frac{\lambda^2}{4\pi^2 r^2}\begin{pmatrix} |\mathscr{S}_2'|^2 \\ |\mathscr{S}_1'|^2 \end{pmatrix}. \tag{178}$$

The first terms (superscript L) have the structure of the term appearing in the LMT and are called the leader terms or the LMT terms. The last terms contain $\mathscr{S}_2'$ and $\mathscr{S}_1'$, which must be added to the LMT terms to produce $\mathscr{S}_2$ and $\mathscr{S}_1$ [Eqs. (167) and (168)] and are called the secondary terms (superscript S). The second terms involve a coupling between the leader and secondary terms and are called the cross terms (superscript C).

For the phase angle δ, we put

$$\tan\delta_0 = \frac{\mathrm{Re}(S_1)\mathrm{Im}(S_2) - \mathrm{Re}(S_2)\mathrm{Im}(S_1)}{\mathrm{Re}(S_1)\mathrm{Re}(S_2) + \mathrm{Im}(S_1)\mathrm{Im}(S_2)}, \tag{179}$$

where the right-hand side is formally identical to the expression for the phase-angle tangent in the pure LMT. By inserting Eqs. (167) and (168) into Eq. (108), we find the link between δ and δ_0:

$$\tan\delta = \tan(\delta_0 + \delta_1) + \tan\delta_2, \tag{180}$$

where the expressions for δ_1 and δ_2 are not given, since they are not essential in the present discussion.

Similarly, the expressions for the cross sections and the radiation pressure force components in the GLMT are found to be the sum of the L, C, and S terms, where the L terms have the structure appearing in the LMT, except for the presence of the coefficients g_n. These leader terms are given in Ref. 41.

10. SPECIAL CASES

We now specify the formulation to some special cases of interest and consider first the case in which the center of the scattering sphere is located on the axis of the incident beam ($x_0 = y_0 = 0$). The formulation then simplifies dramatically because

$$\Psi_{jp}(x_0 = y_0 = 0) = \delta_j^0. \tag{181}$$

Consequently, all the coefficients $g^{m}_{n,\mathrm{TM}}$ and $g^{m}_{n,\mathrm{TE}}$ become equal to 0, except for $|m| = 1$. For these last coefficients, we find

$$g^{1}_{n,\mathrm{TM}} = g^{-1}_{n,\mathrm{TM}} = \frac{1}{2} g_n, \tag{182}$$

$$g^{1}_{n,\mathrm{TE}} = -g^{-1}_{n,\mathrm{TE}} = -\frac{i}{2} g_n, \tag{183}$$

leading to [Eqs. (171)–(174)]

$$G^{+}_{n,\mathrm{TM}} = G^{+}_{n,\mathrm{TE}} = G^{-}_{n,\mathrm{TM}} = G^{-}_{n,\mathrm{TE}} = 0. \tag{184}$$

Hence

$$\mathscr{S}_2' = \mathscr{S}_1' = 0. \tag{185}$$

The scattered intensities are given consequently by the leader terms, since any cross or secondary terms are now identical to zero:

$$\begin{bmatrix} I^{+}_{\theta}(x_0 = y_0 = 0) \\ I^{+}_{\varphi}(x_0 = y_0 = 0) \end{bmatrix} = \frac{\lambda^2}{4\pi^2 r^2}\begin{bmatrix} i_2(x_0 = y_0 = 0)\cos^2\varphi \\ i_1(x_0 = y_0 = 0)\sin^2\varphi \end{bmatrix}. \tag{186}$$

For the phase angle, we find that

$$\delta_1 = \delta_2 = 0, \tag{187}$$

$$\tan\delta = \tan\delta_0. \tag{188}$$

The scattering cross section becomes

$$C_{\mathrm{sca}} = C^{\mathrm{L}}_{\mathrm{sca}} = \frac{\lambda^2}{2\pi}\sum_{n=1}^{\infty}(2n+1)|g_n|^2[|a_n|^2 + |b_n|^2]. \tag{189}$$

The extinction cross section becomes

$$C_{\mathrm{ext}} = C^{\mathrm{L}}_{\mathrm{ext}} = \frac{\lambda^2}{2\pi}\mathrm{Re}\sum_{n=1}^{\infty}(2n+1)|g_n|^2(a_n + b_n). \tag{190}$$

For the radiation pressure force components, we have

$$F^{+}_{x} = F^{\mathrm{L}+}_{x} = F^{+}_{y} = F^{\mathrm{L}+}_{y} = 0 \tag{191}$$

and

$$C_{\mathrm{pr},z} = (\overline{\cos\theta})C^{\mathrm{L}}_{\mathrm{ext}} - (\overline{\cos\theta})C^{\mathrm{L}}_{\mathrm{sca}}, \tag{192}$$

leading to

$$\begin{aligned} C_{\mathrm{pr},z} = \frac{\lambda^2}{2\pi}\sum_{n=1}^{\infty} & \frac{2n+1}{n(n+1)}|g_n|^2\mathrm{Re}(a_n + b_n - 2a_n b_n^{*}) \\ & + \frac{n(n+2)}{n+1}\mathrm{Re}[g_n g^{*}_{n+1}(a_n + b_n + a^{*}_{n+1} \\ & + b^{*}_{n+1} - 2a_n a^{*}_{n+1} - 2b_n b^{*}_{n+1})]. \end{aligned} \tag{193}$$

In all the above expressions of this section, the coefficients g_n reduce to

$$\begin{aligned} g_n = \frac{2n+1}{\pi n(n+1)}\frac{1}{(-1)^n i^{n-1}} & \int_0^{\pi}\int_0^{\infty} iQ\exp\left(-iQ\frac{r^2\sin^2\theta}{w_0^2}\right) \\ & \times \exp(ikz_0)\exp(-ikr\cos\theta)\left(1 - \frac{2Q}{l}r\cos\theta\right) \\ & \times \Psi_n(kr)P_n^1(\cos\theta)\sin^2\theta\, \mathrm{d}\theta\, \mathrm{d}(kr). \end{aligned} \tag{194}$$

If we except the pressure cross section, the above formulation with $x_0 = y_0 = 0$ is identical to the one of the LMT with the addition of the g_n coefficients. The expression for the pressure cross section differs from the one of the LMT that is due to the wave-front curvature, which is discussed in Subsection 8.A. Furthermore, if $z_0 = 0$, as assumed in Refs. 41 and 42, the g_n terms reduce to

$$g_n = \frac{1}{i^{n-1}(-1)^n \pi} \frac{2n+1}{n(n+1)} \int_0^\pi \int_0^\infty (\sin^2\theta) f \exp(-ikr\cos\theta) \times \Psi_n(kr) P_n^1(\cos\theta)\mathrm{d}\theta\mathrm{d}(kr), \tag{195}$$

where the radial basic function f is given by

$$f = iQ \exp\left(-iQ\frac{r^2\sin^2\theta}{w_0^2}\right)\left(1 - \frac{2Q}{l} r\cos\theta\right), \tag{196}$$

in agreement with the results obtained in Refs. 41 and 42 except for the pressure cross section $C_{\mathrm{pr},z}$, which was given with an approximation in Ref. 41.

At the order of approximation L^- (discussed in Ref. 42), the radial basic function simplifies to

$$f = iQ \exp\left(-iQ\frac{r^2\sin^2\theta}{w_0^2}\right), \tag{197}$$

in agreement with the results obtained in Ref. 39. Finally, if $w_0 \to \infty$, the incident beam becomes a plane wave.

Accordingly, we find that all the Ψ_{jp} and g_n^m terms are zero except

$$\Psi_{00} = 1, \tag{198}$$

$$g_{n,\mathrm{TM}}^1 = g_{n,\mathrm{TM}}^{-1} = \frac{1}{2}, \tag{199}$$

$$g_{n,\mathrm{TE}}^1 = -g_{n,\mathrm{TE}}^{-1} = -\frac{i}{2}. \tag{200}$$

With Eqs. (198)–(200) it is an exercise to show that we recover the usual expressions for the LMT, although some algebra is required for $C_{\mathrm{pr},z}$. The LMT has become a special case of our GLMT, as it should for correctness.

11. CONCLUSION

We have built a theoretical description of the scattering of a Gaussian beam by a spherical homogeneous and isotropic particle. This formulation, which is called the GLMT, is valid regardless of the size and nature of the scatterer and its location relative to the beam. The beam is modeled as a Gaussian beam at order L; i.e., the beam description complies with Maxwell's equations at first order in the whole space. The scattering problem is solved by use of the Bromwich method.

The GLMT provides a theoretical tool that may be of direct interest and application, for example, in particle sizing and optical levitation.

We have already designed a so-called localized approximation to the GLMT that permits easy numerical computations for scattering by particles located on the beam axis. The next step of our work will be turned toward the generalization of the localized approximation to off-axis location of the scatterer. We also plan to discuss the physical meaning of a few results of the GLMT in a forthcoming paper.

APPENDIX A

The purpose of this appendix is to determine

$$I_1 = \int_0^\pi (\tau_n^k \tau_m^k + k^2 \Pi_n^k \Pi_m^k)\sin\theta\mathrm{d}\theta, \tag{A1}$$

$$I_2 = \int_0^\pi (\tau_n^k \Pi_m^k + \Pi_n^k \tau_m^k)\sin\theta\mathrm{d}\theta. \tag{A2}$$

For I_2, we use the definitions of τ_n^k and Π_n^k [Eqs. (92) and (93)] to obtain

$$I_2 = \int_0^\pi (P_m^k \mathrm{d}P_n^k + P_n^k \mathrm{d}P_m^k). \tag{A3}$$

We partially integrate the first term, leading to

$$I_2 = P_m^k(-1)P_n^k(-1) - P_m^k(1)P_n^k(1). \tag{A4}$$

However,[56]

$$P_n^k(\pm 1) = 0, \tag{A5}$$

leading to

$$I_2 = 0. \tag{A6}$$

For I_1, we obtain

$$I_1 = I_1^1 + I_1^2 = \int_0^\pi \left(\frac{\mathrm{d}P_n^k}{\mathrm{d}\theta}\frac{\mathrm{d}P_m^k}{\mathrm{d}\theta}\right)\sin\theta\mathrm{d}\theta + \int_0^\pi k^2 \frac{P_n^k}{\sin\theta}\frac{P_m^k}{\sin\theta}\sin\theta\mathrm{d}\theta. \tag{A7}$$

We integrate I_1^1 partially and use the associated Legendre equation,[56]

$$\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\sin^2\theta\frac{\mathrm{d}P_n^k(\cos\theta)}{\mathrm{d}(\cos\theta)} + \left[n(n+1) - \frac{k^2}{\sin^2\theta}\right]P_n^k(\cos\theta) = 0, \tag{A8}$$

to obtain

$$I_1^1 = m(m+1)\int_0^\pi P_n^k(\cos\theta)P_m^k(\cos\theta)\sin\theta\mathrm{d}\theta - k^2\int_0^\pi \frac{P_n^k(\cos\theta)}{\sin\theta}\frac{P_m^k(\cos\theta)}{\sin\theta}\sin\theta\mathrm{d}\theta, \tag{A9}$$

leading to

$$I_1 = m(m+1)\int_0^\pi P_n^k(\cos\theta)P_m^k(\cos\theta)\sin\theta\mathrm{d}\theta. \tag{A10}$$

The integral in Eq. (A10) is standard[56]:

$$\int_0^\pi P_n^k(\cos\theta)P_m^k(\cos\theta)\sin\theta\mathrm{d}\theta = \frac{2}{2n+1}\frac{(n+k)!}{(n-k)!}\delta_{nm}. \tag{A11}$$

Hence we write

$$I_1 = \frac{2m(m+1)(m+k)!}{(2m+1)(m-k)!}\delta_{nm}. \tag{A12}$$

APPENDIX B

The purpose of this appendix is to determine

$$I_3 = \int_0^\pi (\tau_n^k \tau_m^k + k^2 \Pi_n^k \Pi_m^k)\cos\theta \sin\theta d\theta, \tag{B1}$$

$$I_4 = \int_0^\pi (\tau_n^k \Pi_m^k + \tau_m^k \Pi_n^k)\cos\theta \sin\theta d\theta. \tag{B2}$$

For I_4, we use the definitions of τ_n^k and Π_n^k [Eqs. (92) and (93)] and obtain

$$I_4 = -\int_0^\pi P_n^k \frac{dP_m^k}{d(\cos\theta)} \cos\theta \sin\theta d\theta - \int_0^\pi P_m^k \frac{dP_n^k}{d(\cos\theta)} \cos\theta \sin\theta d\theta. \tag{B3}$$

By partially integrating the first integral in Eq. (B3) and rearranging, we find that I_4 is exactly the standard integral [Eq. (A11)].

For I_3, we obtain

$$I_3 = \int_0^\pi k^2 P_n^k \frac{P_m^k}{\sin\theta} \cos\theta d\theta + \int_0^\pi \frac{dP_n^k}{d\theta}\frac{dP_m^k}{d\theta} \cos\theta \sin\theta d\theta = I_3^1 + I_3^2. \tag{B4}$$

We partially integrate I_3^2 and rearrange terms to obtain a new expression for I_3, and then we use the associated Legendre equation (A8) in the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP_m^k}{d\theta} + m(m+1)P_m^k = \frac{k^2 P_m^k}{\sin^2\theta}, \tag{B5}$$

leading to

$$I_3 = m(m+1)\int_0^\pi P_n^k P_m^k \sin\theta\cos\theta d\theta - \int_0^\pi P_n^k\left[\sin^2\theta \frac{dP_m^k}{d(\cos\theta)}\right]\sin\theta d\theta. \tag{B6}$$

We need the following relations among the associated Legendre polynomials[56]:

$$\sin^2\theta \frac{dP_m^k}{d(\cos\theta)} = -k(\cos\theta)P_m^k - (\sin\theta)P_m^{k+1} \tag{B7}$$

$$(\sin\theta)P_m^{k+1} = (m-k)(\cos\theta)P_m^k - (m+k)P_{m-1}^k. \tag{B8}$$

We note that, in Ref. 56, there is a misprint for Eq. (B8). We obtain

$$I_3 = m(m+2)\int_0^\pi P_n^k P_m^k \cos\theta \sin\theta d\theta - (m+k) \times \int_0^\pi P_n^k P_{m-1}^k \sin\theta d\theta. \tag{B9}$$

The second integral on the right-hand side of Eq. (B9) is standard [Eq. (A11)].

For the first integral, called A, we integrate partially and rearrange. We use Eq. (B7) and then Eq. (B8) to obtain the standard form [Eq. (A11)], leading to

$$A = \frac{n+k}{2+n+m}\frac{2}{2m+1}\frac{(m+k)!}{(m-k)!}\delta_{m,n-1} + \frac{m+k}{2+n+m}\frac{2}{2n+1}\frac{(n+k)!}{(n-k)!}\delta_{n,m-1} \tag{B10}$$

and finally to

$$I_3 = \frac{2(n-1)(n+1)}{(2n-1)(2n+1)}\frac{(n+k)!}{(n-1-k)!}\delta_{m,n-1} + \frac{2(m-1)(m+1)}{(2m-1)(2m+1)}\frac{(m+k)!}{(m-1-k)!}\delta_{n,m-1}. \tag{B11}$$

APPENDIX C

The purpose of this appendix is to determine

$$I_5 = \int_0^\pi (\tau_n^{|p|}\tau_m^{|p+1|} + p(p+1)\Pi_n^{|p|}\Pi_m^{|p+1|})\sin^2\theta d\theta, \tag{C1}$$

$$I_6 = \int_0^\pi (p\tau_m^{|p+1|}\Pi_n^{|p|} + (p+1)\tau_n^{|p|}\Pi_m^{|p+1|})\sin^2\theta d\theta, \tag{C2}$$

and we partially follow the method given by Kim and Lee.[35]

For I_6, we first consider the case in which $p > 0$:

$$I_6(p>0) = \int_0^\pi [p\Pi_n^p\tau_m^{p+1} + (p+1)\tau_n^p\Pi_m^{p+1}]\sin^2\theta d\theta. \tag{C3}$$

We replace Π_n^k and τ_m^k by their expressions in terms of P_n^k [Eqs. (92) and (93)], and we partially integrate the second integral to obtain

$$I_6(p>0) = -\int_0^\pi P_n^p\left[\sin\theta\frac{dP_m^{p+1}}{d\theta} + (p+1)(\cos\theta)P_m^{p+1}\right]d\theta. \tag{C4}$$

We use[35]

$$(2l+1)\sin\theta\frac{dP_l^m}{d\theta} = l(l-m+1)P_{l+1}^m - (l+1)(l+m)P_{l-1}^m, \tag{C5}$$

$$(2l+1)(\cos\theta)P_l^m = (l+m)P_{l-1}^m + (l-m+1)P_{l+1}^m, \tag{C6}$$

to obtain

$$I_6(p>0) = \frac{(p-m)(m+p+1)}{2m+1}\int_0^\pi P_n^p(P_{m+1}^{p+1} - P_{m-1}^{p+1})d\theta. \tag{C7}$$

However, with the definition that we use for P_n^m [Eq. (5)], we obtain

$$(2l+1)(\sin\theta)P_l^m = P_{l-1}^{m+1} - P_{l+1}^{m+1}, \tag{C8}$$

leading to

$$I_6(p>0) = (m-p)(m+p+1)\int_0^\pi P_n^p P_m^p \sin\theta d\theta. \tag{C9}$$

The integral in Eq. (C9) is standard [Eq. (A11)]. The expression for the case in which $p < 0$ can be reduced to that for the case in which $p > 0$. The case in which $p = 0$ requires a direct integration, which is easy to do. We then obtain Eq. (150).

For I_5, we first consider the case in which $p > 0$:

$$I_5(p > 0) = \int_0^\pi [\tau_n^p \tau_m^{p+1} + p(p+1)\Pi_n^p \Pi_m^{p+1}]\sin^2\theta d\theta. \quad \text{(C10)}$$

We replace τ_n^k and Π_m^k by their expressions in terms of P_n^k and then partially integrate the first integral. We then use Eq. (C5) twice and then Eq. (C6) to obtain

$$\begin{aligned} I_5(p > 0) = \int_0^\pi P_n^p \Bigg\{ & p(p+1)P_m^{p+1} + \frac{m(m-p)(m+1)(m+2)}{(2m+1)(2m+3)} \\ & \times (P_m^{p+1} - P_{m+2}^{p+1}) + \frac{mp(m-p)}{(2m+1)(2m+3)} \\ & \times [(m+1)P_m^{p+1} + (m+2)P_{m+2}^{p+1}] \\ & + \frac{m(m-1)(m+1)(m+p+1)}{(2m-1)(2m+1)} (P_m^{p+1} - P_{m-2}^{p+1}) \\ & - \frac{p(m+1)(m+p+1)}{(2m-1)(2m+1)} \\ & \times [mP_m^{p+1} + (m-1)P_{m-2}^{p+1}] \Bigg\} d\theta. \end{aligned} \quad \text{(C11)}$$

We then use Eq. (C8) and find that Eq. (C11) involves only the standard integral [Eq. (A11)]. The expression for the case in which $p < 0$ can be reduced to the case in which $p > 0$. The case in which $p = 0$ is treated independently. We finally obtain Eq. (149).

REFERENCES AND NOTES

1. L. Lorenz, "Lysbevaegelsen i og uden for en haf plane lysbölger belyst Kulge," Vidensk. Selk. Skr. **6**, 1–62 (1890).
2. L. Lorenz, "Sur la lumière réfléchie et réfractée par une sphère transparente," in *Oeuvres Scientifiques de L. Lorenz, revues et annotées par H. Valentiner* (Librairie Lehmann et Stage, Copenhagen, 1898), pp. 405–529.
3. G. Mie, "Beiträge zur Optik Trüber Medien, speziell Kolloidaler Metallösungen," Ann. Phys. **25**, 377–452 (1908).
4. P. Debye, "Der Lichtdruck auf Kugeln von Beliebigem Material," Ann. Phys. **30**, 57–136 (1909).
5. W. J. Lentz, "Generating Bessel functions in Mie scattering calculations using continued fractions," Appl. Opt. **15**, 668–671 (1976).
6. G. Gréhan and G. Gouesbet, "Mie theory calculations: new progress, with emphasis on particle sizing," Appl. Opt. **18**, 3489–3493 (1979).
7. G. Gréhan, G. Gouesbet, and C. Rabasse, "Monotonic relationships between scattered powers and diameters in Lorenz–Mie theory, for simultaneous velocimetry and sizing of single particles," Appl. Opt. **20**, 796–799 (1981).
8. A. Ungut, G. Gréhan, and G. Gouesbet, "Comparisons between geometrical optics and Lorenz–Mie theory," Appl. Opt. **20**, 2911–2918 (1981).
9. G. Gouesbet, G. Gréhan, and B. Maheu, "Single scattering characteristics of volume elements in coal clouds," Appl. Opt. **22**, 2038–2050 (1983).
10. G. Gréhan, G. Gouesbet, and C. Rabasse, "The computer program SUPERMIDI for Lorenz–Mie theory and the research of one to one relationships for particle sizing," in *Proceedings of the Symposium on Long Range and Short Range Optical Velocity Measurements*, H. J. Pfeifer, ed. (Institut Francoallemand de St. Louis, St. Louis, France, 1980), pp. XVI.1–VI.107.
11. Y. Kakui, J. Hirono, A. Nishimoto, M. Nango, and Y. Kanno, Bull. Electrotech. Lab. Jpn. **46**, 640 (1982).
12. A. R. Jones, "Calculations of the ratio of complex Ricatti–Bessel functions for Mie scattering," J. Phys. D **16**, 149–152 (1983).
13. F. Corbin, G. Gréhan, G. Gouesbet, and B. Maheu, "Interaction between a sphere and a Gaussian beam: computations on a microcomputer," Part. Part. Syst. Character. (to be published).
14. W. M. Farmer, "Measurements of particle size, number density and velocity using laser interferometer," Appl. Opt. **11**, 2603–2612 (1972).
15. W. M. Farmer, "Visibility of large spheres observed with a laser velocimeter: a simple model," Appl. Opt. **19**, 3660–3666 (1980).
16. A. J. Yule, N. A. Chigier, S. Atakan, and A. Ungut, "Particle size and velocity measurement by laser anemometry," J. Energy **1**, 220–228 (1977).
17. D. Holve and S. A. Self, "Optical particle sizing for *in situ* measurements," Part 1, Appl. Opt. **18**, 1632–1645 (1979); Part 2, Appl. Opt. **18**, 1646–1652 (1979).
18. G. Gouesbet and G. Gréhan, "Advances in quasi-elastic scattering of light with emphasis on simultaneous measurements of velocities and sizes of particles embedded in flows," presented at the American Institute of Aeronautics and Astronautics 16th Thermophysics Conference, Palo Alto, Calif., June 23–25, 1981.
19. D. Allano, G. Gouesbet, G. Gréhan, and D. Lisiecki, "Comparative measurements of calibrated droplets using Gabor holography and corrected top-hat laser beam sizing with discussion of simultaneous velocimetry," in *Proceedings of the International Symposium on Applications of Laser Doppler Anemometry to Fluid Mechanics*, D. F. G. Durão, ed. (Lisbon University, Lisbon, 1982), pp. 6.1.1–6.1.10.
20. D. Allano, G. Gouesbet, G. Gréhan, and D. Lisiecki, "Droplet sizing using a 'top-hat' laser beam technique," J. Phys. D **17**, 43–58 (1984).
21. G. Gouesbet, G. Gréhan, and R. Kleine, "Simultaneous optical measurement of velocity and size of individual particles in flows," in *Proceedings of the Second International Symposium on Applications of Laser Anemometry to Fluid Mechanics*, D. F. G. Durão, ed. (Lisbon University, Lisbon, 1984), pp. 18.2.1–18.2.7.
22. G. Gréhan, G. Gouesbet, R. Kleine, V. Renz, and I. Wilhelmi, "Corrected laser beam techniques for simultaneous velocimetry and sizing of particles in flows and applications," in *Proceedings of the Third International Symposium on Application of Laser Anemometry to Fluid Mechanics*, D. F. G. Durão, ed. (Lisbon University, Lisbon, 1986), pp. 20.5.1–20.5.57.
23. G. Gouesbet, P. Gougeon, G. Gréhan, J. N. Letoulouzan, N. Lhuissier, B. Maheu, and M. E. Weill, "Laser optical sizing from 100 Å to 1 mm diameter, and from 0 to 10 kg/m^3 concentration," presented at the American Institute of Aeronautics and Astronautics 20th Thermophysics Conference, Williamsburg, Va., June 19–21, 1985.
24. G. Gréhan and G. Gouesbet, "Simultaneous measurements of velocities and sizes of particles in flows using a combined system incorporating a top-hat beam technique," Appl. Opt. **25**, 3527–3538 (1986).
25. G. Gouesbet and G. Gréhan, eds., *Proceedings of the International Symposium on Optical Particle Sizing: Theory and Practice* (Plenum, New York, 1988).
26. A. Ashkin, "Acceleration and trapping of particles by radiation pressure," Phys. Rev. Lett. **24**, 156–159 (1970).
27. P. Chylek, V. Ramaswamy, A. Ashkin, and J. Dziedzic, "Simultaneous determination of refractive index and size of spherical dielectric particles from light scattering data," Appl. Opt. **22**, 2302–2307 (1983).
28. G. Roosen, "A theoretical and experimental study of the stable equilibrium positions of spheres levitated by two horizontal laser beams," Opt. Commun. **21**, 189–194 (1977).
29. G. Roosen, "La lévitation optique de sphère," Can. J. Phys. **57**, 1260–1279 (1979).
30. G. Gréhan and G. Gouesbet, "Optical levitation of a single particle to study the theory of the quasi-elastic scattering of light," Appl. Opt. **19**, 2485–2487 (1980).
31. W. D. Bachalo and M. J. Houser, "An instrument for two-components velocity and particle size measurement," in *Proceedings of the Third International Symposium on Applications of Laser Anemometry to Fluid Mechanics*, D. F. G. Durão, ed. (Lisbon University, Lisbon, 1986), pp. 18.3.1–18.3.67.
32. W. G. Tam and R. Corriveau, "Scattering of electromagnetic beams by spherical objects," J. Opt. Soc. Am. **68**, 763–767 (1978).
33. N. Morita, T. Tanaka, T. Yamasaki, and Y. Nakanishi, "Scattering of a beam wave by a spherical object," IEEE Trans. Antennas Propag. **AP-16**, 724–727 (1968).

34. W. C. Tsai and R. J. Pogorzelski, "Eigenfunction solution of the scattering of beam radiation fields by spherical objects," J. Opt. Soc. Am. **65,** 1457–1463 (1975).
35. J. S. Kim and S. S. Lee, "Scattering of laser beams and the optical potential well for a homogeneous sphere," J. Opt. Soc. Am. **73,** 303–312 (1983).
36. G. Gouesbet and G. Gréhan, "The quasi-elastic scattering of light: A lecture with emphasis on particulate diagnosis," in *Proceedings of the NATO Workshop on Soot in Combustion Systems and Its Toxic Properties*, J. Lahaye and G. Prado, eds. (Plenum, New York, 1981), pp. 395–412.
37. G. Gouesbet, "Optical sizing with emphasis on simultaneous measurements of velocities and sizes of particles embedded in flows," presented at the XVth International Symposium on Heat and Mass Transfer, Dubrovnik, Yugoslavia, September 5–9, 1983 [in R. I. Soloukhin and N. H. Afgan, eds., *Measurement Techniques for Heat Mass Transfer* (Springer-Verlag, Berlin, 1983), pp. 27–39].
38. G. Gréhan, "Nouveaux progrès en théorie de Lorenz–Mie. Application la mesure de diamètres de particules dans des écoulements," thèse de troisième cycle (Rouen University, Rouen, France, 1980).
39. G. Gouesbet and G. Gréhan, "Sur la généralisation de la théorie de Lorenz–Mie," J. Opt. **13,** 97–103 (1982).
40. M. Kerker, *The Scattering of Light and Other Electromagnetic Radiation* (Academic New York, 1969).
41. G. Gouesbet, G. Gréhan, and B. Maheu, "Scattering of a Gaussian beam by a Mie scatter center using a Bromwich formalism," J. Opt. **16,** 83–93 (1985).
42. G. Gouesbet, B. Maheu, and G. Gréhan, "The order of approximation in a theory of the scattering of a Gaussian beam by a Mie scatter center," J. Opt. **16,** 239–247 (1985).
43. G. Gréhan, B. Maheu, and G. Gouesbet, "Scattering of laser beams by Mie scatter centers: numerical results using a localized approximation," Appl. Opt. **25,** 3539–3548 (1986).
44. B. Maheu, G. Gréhan, G. Gouesbet, "Generalized Lorenz–Mie theory: first exact values and comparisons with the localized approximation," Appl. Opt. **26,** 23–25 (1987). Note that, although it was not stated explicitly in Refs. 44–46, the approximation $iQ \simeq 1$ was included in the computations in those references. Values obtained without this approximation are published in Ref. 47 and compare well with those in Refs. 44–46.
45. G. Gréhan, B. Maheu, and G. Gouesbet, "Localized approximation to the generalized Lorenz–Mie theory and its application to optical sizing," presented at the International Congress on Applications of Lasers and Electro-Optics, Arlington, Va., November 10–13, 1986.
46. B. Maheu, G. Gréhan, and G. Gouesbet, "Diffusion de la lumière par une sphère dans le cas d'un faisceau d'extension finie: 1ère partie-théorie de Lorenz–Mie généralisée: les coefficients g_n et leur calcul numérique," J. Aerosol. Sci. **19,** 47–53 (1988).
47. B. Maheu, G. Gréhan, and G. Gouesbet, "Laser beam scattering by individual particles: numerical results and applications to optical sizing," presented at the Symposium on Optical Particle Sizing: Theory and Practice, Mt. St. Aignan, France, May 12–15, 1987 [in G. Gouesbet and G. Gréhan, eds., *Optical Particle Sizing: Theory and Practice* (Plenum, New York, 1988), pp. 77–88].
48. C. W. Yeh, S. Colak, and P. W. Barber, "Scattering of sharply focused beams by arbitrarily shaped dielectric particles: an exact solution," Appl. Opt. **21,** 4426–4433 (1982).
49. T. J. Bromwich, "Electromagnetic waves," Philos. Mag. **38,** 143–164 (1919).
50. P. Poincelot, *Précis d'Électromagnétisme Théorique* (Masson, Paris, 1963).
51. M. Lax, W. H. Louisell, and W. B. McKnight, "From Maxwell to paraxial optics," Phys. Rev. A **11,** 1365–1370 (1975).
52. L. W. Davis, "Theory of electromagnetic beams," Phys. Rev. A **19,** 1177–1179 (1979).
53. F. Slimani, G. Gréhan, G. Gouesbet, and D. Allano, "Near-field Lorenz–Mie theory and its application to microholography," Appl. Opt. **23,** 4140–4148 (1984).
54. L. Brillouin, "The scattering cross-sections of spheres for electromagnetic scattering," J. Appl. Phys. **20,** 1110–1125 (1949).
55. H. C. van de Hulst, *Light Scattering by Small Particles* (Wiley, New York, 1957).
56. L. Robin, *Fonctions Sphériques de Legendre et Fonctions Sphéroïdales* (Gauthier-Villars, Paris, 1957–1959), Vols. 1–3.