

# Optical forces on small magnetodielectric particles

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**Abstract:** We present a study of the optical force on a small particle with both electric and magnetic response, immersed in an arbitrary non-dissipative medium, due to a generic incident electromagnetic field. This permits us to establish conclusions for any sign of this medium refractive index. Expressions for the gradient force, radiation pressure and curl components are obtained for the force due to both the electric and magnetic dipoles excited in the particle. In particular, for the magnetic force we tentatively introduce the concept of curl of the spin angular momentum density of the magnetic field, also expressed in terms of 3D generalizations of the Stokes parameters. From the formal analogy between the conservation of momentum and the optical theorem, we discuss the origin and significance of the electric-magnetic dipolar interaction force; this is done in connection with that of the angular distribution of scattered light and of the extinction cross section.

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## References and links

1. A. Ashkin, "Acceleration and Trapping of Particles by Radiation Pressure," *Phys. Rev. Lett.* **24**, 156–159 (1970).
2. A. Ashkin, J. M. Dziedzic, J. E. Bjorkholm, and S. Chu, "Observation of a single-beam gradient force optical trap for dielectric particles," *Opt. Lett.* **11**, 288–290 (1986).
3. K. C. Neuman and S. M. Block, "Optical trapping," *Rev. Sci. Instrum.* **75**, 2787–2809(2004).
4. L. Novotny and B. Hecht, *Principles of Nano-Optics*, (Cambridge University Press, Cambridge, 2006).
5. P. C. Chaumet and M. Nieto-Vesperinas, "Coupled dipole method determination of the electromagnetic force on a particle over a flat dielectric substrate," *Phys. Rev. B* **61**, 14119–14127 (2000).
6. P. C. Chaumet and M. Nieto-Vesperinas, "Electromagnetic force on a metallic particle in the presence of a dielectric surface," *Phys. Rev. B* **62**, 11185–11191 (2000).
7. P. C. Chaumet, A. Rahmani, and M. Nieto-vesperinas, "Optical trapping and manipulation of nano-objects with an apertureless probe," *Phys. Rev. Lett.* **88**, 123601–123604 (2002).
8. M. Nieto-Vesperinas, P. C. Chaumet, and A. Rahmani, "Near field photonic forces," *Phil. Trans. R. Soc. Lond. A* **362**, 719–737 (2004).
9. P. Verkerk, B. Lounis, C. Salomon, and C. Cohen-Tannoudji, "Dynamics and spatial order of cold cesium atoms in a periodic optical potential," *Phys. Rev. Lett.* **68**, 3861–3864 (1992).

10. P. S. Jessen, C. Gerz, P. D. Lett, W. D. Phillips, S. L. Rolston, R. J. C. Spreeuw, and C. I. Westbrook, "Observation of quantized motion of Rb atoms in an optical field," *Phys. Rev. Lett.* **69**, 49–52 (1992).
11. A. Hemmerich and T. W. Hänsch, "Two-dimensional atomic crystal bound by light," *Phys. Rev. Lett.* **70**, 410–413 (1993).
12. M. M. Burns, J. M. Fournier, and J. A. Golovchenko, "Optical Binding," *Phys. Rev. Lett.* **63**, 1233–1236 (1989).
13. M. M. Burns, J. M. Fournier, and J. A. Golovchenko, "Optical Matter: Crystallization and Binding in Intense Optical Fields," *Science* **249**, 749–754 (1990).
14. P. C. Chaumet and M. Nieto-Vesperinas, "Optical binding of particles with or without the presence of a flat dielectric surface," *Phys. Rev. B* **64**, 035422–035427 (2001).
15. S. A. Tatarkova, A. E. Carruthers, and K. Dholakia, "One-Dimensional Optically Bound Arrays of Microscopic Particles," *Phys. Rev. Lett.* **89**, 283901–283904 (2002).
16. R. Gómez-Medina and J. J. Sáenz, "Unusually Strong Optical Interactions between Particles in Quasi-One-Dimensional Geometries," *Phys. Rev. Lett.* **93**, 243602–243605 (2004).
17. M. Mansuripur, "Radiation pressure and the linear momentum of the electromagnetic field," *Opt. Express* **12**, 5375–5401 (2004).
18. B. A. Kemp, T. M. Grzegorzczak, and J. A. Kong, "Ab initio study of the radiation pressure on dielectric and magnetic media," *Opt. Express* **13**, 9280–9291 (2005).
19. M. Mansuripur, "Radiation pressure and the linear momentum of the electromagnetic field in magnetic media," *Opt. Express* **15**, 13502–13517 (2007).
20. B. A. Kemp, T. M. Grzegorzczak, and J. A. Kong, "Optical momentum transfer to absorbing Mie particles," *Phys. Rev. Lett.* **97**, 1339021–1339024 (2006).
21. B. A. Kemp, T. M. Grzegorzczak, and J. A. Kong, "Lorentz force on dielectric and magnetic particles," *J. Electromagn. Waves Appl.* **20**, 827–839 (2006).
22. A. Lakhtakia, "Radiation pressure efficiencies of spheres made of isotropic, achiral, passive, homogeneous, negative-phase-velocity materials," *Electromagnetics* **28**, 346–353 (2008).
23. A. Lakhtakia and G. W. Mulholland, "On two numerical techniques for light scattering by dielectric agglomerated structures," *J. Res. Nat. Inst. Stand. Technol.* **98**, 699–716 (1993).
24. P. C. Chaumet and A. Rahmani, "Electromagnetic force and torque on magnetic and negative-index scatterers," *Opt. Express* **17**, 2224–2234 (2009).
25. S. Albaladejo, M. I. Marques, M. Laroche, and J. J. Saenz, "Scattering Forces from the Curl of the Spin Angular momentum of a Light Field," *Phys. Rev. Lett.* **102**, 1136021–1136024 (2009).
26. J. D. Jackson, *Classical Electrodynamics*, (3rd edition, John Wiley, New York, 1998).
27. J. R. Arias-Gonzalez and M. Nieto-Vesperinas, "Optical forces on small particles. Attractive and repulsive nature and plasmon-resonance conditions," *J. Opt. Soc. Am. A* **20**, 1201–1209 (2003).
28. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge U.P., Cambridge, 1995).
29. M. Nieto-Vesperinas, *Scattering and Diffraction in Physical Optics*, (2nd edition, World Scientific, Singapore, 2006).
30. N. J. Moore, M. A. Alonso, and C. J. R. Sheppard, "Monochromatic scalar fields with maximum focal irradiance," *J. Opt. Soc. Am. A* **24**, 2057–2064 (2007).
31. N. J. Moore and M. A. Alonso, "Closed-form bases for the description of monochromatic, strongly focused, electromagnetic fields," *J. Opt. Soc. Am. A* **26**, 2211–2218 (2009).
32. G. C. Sherman, "Diffracted wavefields expressible by plane wave expansions containing only homogeneous waves," *Phys. Rev. Lett.* **21**, 761–764 (1968).
33. M. Born and E. Wolf, *Principles of Optics*, 7th edition, Cambridge U.P., Cambridge, 1999.
34. P. Chylek and R. G. Pinnick, "Nonunitarity of the light scattering approximations," *Appl. Opt.* **18**, 1123–1124 (1979).
35. J. A. Lock, J. T. Hodges, and G. Gouesbet, "Failure of the optical theorem for Gaussian-beam scattering by a spherical particle," *J. Opt. Soc. Am.* **12**, 2708–2715 (1995).
36. B. Garcia-Camara, F. Moreno, F. Gonzalez, and J. M. Saiz, "Comment on Experimental Evidence of Zero Forward Scattering by Magnetic Spheres," *Phys. Rev. Lett.* **98**, 179701–179704 (2007).
37. A. Alu and N. Engheta, "Cloaking a Sensor," *Phys. Rev. Lett.* **102**, 233901–233904 (2009).
38. P. C. Chaumet and M. Nieto-Vesperinas, "Time-averaged total force on a dipolar sphere in an electromagnetic field," *Opt. Lett.* **25**, 1065–1067 (2000).
39. B. T. Draine, "The discrete-dipole approximation and its application to interstellar graphite grains," *Ap. J.* **333**, 848–872 (1988).
40. We acknowledge this remark in a private communication from an anonymous reviewer.
41. C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, (John Wiley, New York, 1998).
42. P. Chylek, J. T. Kiehl, and K. W. Ko, "Narrow resonance structure in the Mie scattering characteristics," *Appl. Opt.* **17**, 3019–3021 (1978).
43. L.D Landau, E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd revised edition (Pergamon Press, Oxford, 1984).

44. V. G. Veselago, "The electrodynamics of substances with simultaneous negative values of  $\epsilon$  and  $\mu$ ," *Sov. Phys. Usp.* **10**, 509–514 (1968).
45. V. Wong and M. Ratner, "Gradient and nongradient contributions to plasmon-enhanced optical forces on silver nanoparticles," *Phys. Rev. B* **73**, 0754161–0754166 (2006).
46. S. M. Barnett, "Optical angular-momentum flux," *J. Opt. B: Quantum Semiclass. Opt.* **4**, S7–S16 (2002).
47. L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, "Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes," *Phys. Rev. A* **45**, 8185–8189 (1992).
48. Notice that in ref. [25], the equation giving  $L_{Se}$  has the opposite sign. However, the actual force (in S.I. units) is exactly the same as the one derived here in Gaussian units.
49. In paraxial beams, the *helicity* is given by the projection of the spin on the propagation direction. If the helicity is not uniform, the curl of the spin would be, in general, different from zero. The spin curl term would then be relevant for optical beams with non-uniform helicity. However, a non-uniform spin density does not imply a non-uniform helicity: we may have non-zero curl force even if the helicity is zero, as it is the case for the standing waves with p-polarized beams discussed in [25]. (Yiqiao Tang, Harvard University, private communication).
50. T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, "Degree of polarization for optical near fields," *Phys. Rev. E* **66**, 0166151–0166158 (2002).

## 1. Introduction

Light carries energy and both linear and angular momenta that can be transferred to atoms, molecules and particles. Demonstration of levitation and trapping of micron-sized particles by radiation pressure dates back to 1970 and the experiments reported by Ashkin and co-workers [1]. Light forces on small particles are usually described as the sum of two terms: the dipole or gradient force and the radiation pressure or scattering force [2–4, 6–8]: In analogy with electrostatics, small particles develop an electric (magnetic) dipole moment in response to the light electric (magnetic) field. The induced dipole is then drawn by field intensity gradients which compete with radiation pressure due to momentum transferred from the photons in the beam. By fashioning proper optical field gradients it is possible to trap and manipulate small dielectric particles with optical tweezers [2, 3] or create atomic arrays in optical lattices [9, 11]. Intense optical fields can also induce significant forces between particles [12–16].

Some previous work focused on optical forces on macroscopic media, either with electric [17] or magnetic response [18, 19], or particles with electric response [20]. Radiation pressure forces on dielectric and magnetic particles under plane wave incidence have been computed for both small cylinders [21] and spheres [22, 23]. The total force on a dipolar magnetodielectric particle has been shown [24] to have a similarity with that previously obtained for electric dipoles. For an electric dipole, a third component of the force has been recently reported [25] to constitute an additional non-conservative contribution to the scattering force, proportional to the curl of the spin angular momentum density of the electric field. The magnetic counterpart of this interaction remains to be addressed. Moreover, in the presence of both electric and magnetic responses, the force presents an additional term proportional to the cross product of the electric and magnetic dipoles [24]. Nevertheless, the relevance and physical origin of this electric-magnetic dipolar interaction term for a single particle has not yet been discussed.

In this paper we analyze in detail the different contributions to the optical force on an electric and magnetic small particle, immersed in an arbitrary non-dissipative medium, due to an external field. We intend to provide a comprehensive view of the principles governing this force. In Section 2 we consider the general results for the optical forces on a dipolar magnetic particle in an electromagnetic field. As an alternative proof of the expression obtained in [24] which was derived in the near field, we calculate the Maxwell stress tensor flux across a spherical surface in the far zone enclosing the particle. The advantage of this approach is that it shows the formal analogy, both in terms of mathematical expressions and of their derivations, between the forces obtained from the momentum conservation law, and the optical theorem stemming from the law of energy conservation. This is seen in Section 3, and has not been previously established as far as we know. This analogy provides a framework to illustrate the relevance

of the different contributions to the force as a result of the electric and magnetic interaction, which is carried out in 3.1. Then, as discussed in Section 3.2, the electric-magnetic dipolar interaction term of the force is explained in analogy with the contribution of the interference between the fields radiated by the electric and magnetic dipoles to the angular distribution of scattered intensity, or differential scattering cross section, in comparison with that of the total scattering and extinction cross sections. As an example, in Section 3.4 we illustrate the three force components acting on a particle which contributes to the scattering with the first electric and magnetic Mie coefficients, in terms of which the polarizabilities are generalized. We also discuss the limiting case for long wavelengths of the small perfectly conducting sphere. In addition, we address the consequences that the sign and value of the constitutive parameters of the exterior medium have upon the radiation pressure on the particle, in particular confirming its reversal in a negative-index medium.

The optical forces on magnetic small particles in arbitrary fields, and their decomposition into gradient, scattering and curl components are discussed in Section 4. The force component due to the electric-magnetic dipolar interaction also contributes to both the radiation pressure, or scattering force, and to the gradient force. In analogy with electric dipoles, the optical forces on a magnetic dipole are shown to present a non-conservative contribution proportional to the curl of the spin angular momentum density of the magnetic field. These curl forces can also be expressed in terms of the three-dimensional Stokes parameters as discussed in Sec 4.1.

## 2. Force on a small particle with electric and magnetic response to an electromagnetic wave

We consider a dipolar particle embedded in a non-dissipative medium with relative dielectric permittivity  $\epsilon$  and magnetic permeability  $\mu$ , subjected to an incident electromagnetic field whose electric and magnetic vectors are  $\mathbf{E}^{(i)}$  and  $\mathbf{B}^{(i)}$ , respectively. The total time-averaged electromagnetic force acting on the particle is [5, 26]:

$$\langle \mathbf{F} \rangle = \frac{1}{8\pi} \Re \left\{ \int_S \left[ \epsilon (\mathbf{E} \cdot \mathbf{s}) \mathbf{E}^* + \mu^{-1} (\mathbf{B} \cdot \mathbf{s}) \mathbf{B}^* - \frac{1}{2} (\epsilon |\mathbf{E}|^2 + \mu^{-1} |\mathbf{B}|^2) \mathbf{s} \right] dS \right\}, \quad (1)$$

where  $\Re$  stands for real part,  $dS$  denotes the element of any surface  $S$  that encloses the particle. The fields in Eq. (1) are total fields, namely the sum of the incident and scattered (re-radiated) fields:  $\mathbf{E}^{(i)} + \mathbf{E}^{(r)}$ ,  $\mathbf{B}^{(i)} + \mathbf{B}^{(r)}$ .  $\mathbf{s}$  is its local outward unit normal. A time dependence  $\exp(-i\omega t)$  is assumed throughout. For a small particle, within the range of validity of the dipolar approximation (see [27]), the scattered field corresponds to that radiated by the induced electric and magnetic dipole moments,  $\mathbf{p}$  and  $\mathbf{m}$ , respectively. In this case, Eq. (1) leads to the expression

$$\langle \mathbf{F} \rangle = \frac{1}{2} \Re \left\{ \mathbf{p}(\nabla \otimes \mathbf{E}^{(i)*}) + \mathbf{m}(\nabla \otimes \mathbf{B}^{(i)*}) - \frac{2k^4}{3} \sqrt{\frac{\mu}{\epsilon}} (\mathbf{p} \times \mathbf{m}^*) \right\}. \quad (2)$$

Equation (2) represents the generalization of the result of [24] for the time-averaged force on a particle immersed in an arbitrary medium with refractive index:  $n = \sqrt{\epsilon\mu}$ . The wavenumber is  $k = n\omega/c$ ,  $\omega$  being the frequency. The symbol  $\otimes$  represents the dyadic product so that the matrix operation:  $\mathbf{W}(\nabla \otimes \mathbf{V})$  has elements  $W_j \partial_i V_j$  for  $i, j = 1, 2, 3$ . All variables in Eq. (2) are evaluated at a point  $\mathbf{r} = \mathbf{r}_0$  in the particle. The first term of Eq. (2) is the force  $\langle \mathbf{F}_e \rangle$  exerted by the incident field on the induced electric dipole and was obtained in [38], the second and third terms  $\langle \mathbf{F}_m \rangle$  and  $\langle \mathbf{F}_{e-m} \rangle$  are the force on the induced magnetic dipole and the force due to the interaction between both dipoles, respectively; they were derived in [24].

We next obtain Eq. (2) in a way alternative to that reported in [24] which involved the near field expansion of the scattered field. This is done by integration of Eq. (1) in the far zone. In fact we know that according to the theorem of momentum conservation from which Eq. (1) derives,

the same result Eq. (2) should be obtained by choosing in Eq. (1) any surface  $S$  enclosing the particle. The advantage of this approach is that it shows the formal analogy both between the mathematical expression of the force and of the optical theorem, as well as between their respective derivations. This helps to obtain an interpretation of the third term of Eq. (2), as shown next.

In the far (radiation) zone the scattered fields take on the limiting forms [26],

$$\mathbf{B}^{(r)}(\mathbf{r}) = k^2 \left( \mu (\mathbf{s} \times \mathbf{m}) \times \mathbf{s} + \sqrt{\frac{\mu}{\epsilon}} \mathbf{s} \times \mathbf{p} \right) \frac{e^{ikr}}{r}, \quad \mathbf{E}^{(r)}(\mathbf{r}) = \frac{1}{n} \mathbf{B}^{(r)}(\mathbf{r}) \times \mathbf{s} \quad (3)$$

showing the typical behaviour of radiation fields, transverse to the radius vector  $r \equiv r \mathbf{s}$ . Choosing  $S$  to be a sphere around the dipoles, Eq. (1) can be written as

$$\langle \mathbf{F} \rangle = -\frac{n}{c} \int_S \left\{ \langle \mathbf{S} \rangle - \langle \mathbf{S}^{(i)} \rangle \right\} dS. \quad (4)$$

where  $\langle \mathbf{S} \rangle = c/(8\pi) \Re \{ \mathbf{E} \times \mathbf{H}^* \}$  is the time-averaged Poynting vector, with

$$\begin{aligned} \langle \mathbf{S} \rangle - \langle \mathbf{S}^{(i)} \rangle &= \frac{c}{8\pi} \mu^{-1} \Re \left\{ \mathbf{E}^{(r)} \times \mathbf{B}^{(i)*} + \mathbf{E}^{(i)} \times \mathbf{B}^{(r)*} + \mathbf{E}^{(r)} \times \mathbf{B}^{(r)*} \right\} \\ &= \frac{c}{8\pi n} \Re \left\{ \epsilon \mathbf{E}^{(i)} \cdot \mathbf{E}^{(r)*} + \mu^{-1} \mathbf{B}^{(r)} \cdot \mathbf{B}^{(i)*} + \frac{1}{2} \left( \epsilon |\mathbf{E}^{(r)}|^2 + \mu^{-1} |\mathbf{B}^{(r)}|^2 \right) \right\} \mathbf{s} \\ &\quad - \frac{c}{8\pi n} \Re \left\{ \epsilon \left( \mathbf{E}^{(i)} \cdot \mathbf{s} \right) \mathbf{E}^{(r)*} + \mu^{-1} \left( \mathbf{B}^{(i)*} \cdot \mathbf{s} \right) \mathbf{B}^{(r)} \right\}. \end{aligned} \quad (5)$$

Equation (4) yields a simple conceptual definition of the optical force on any scattering object as it represents the rate of loss of electromagnetic momentum. In other words, from the conservation law of momentum, the force given by Eq. (4) represents the rate at which the momentum is being “absorbed” by the particle.

In order to retrieve Eq. (2), let us express the incident field as a decomposition of plane wave components [28–31],

$$\mathbf{E}^{(i)}(\mathbf{r}) = \int_{\mathcal{D}} \mathbf{e}^{(i)}(\mathbf{u}) \exp(ik\mathbf{u} \cdot \mathbf{r}) d\Omega, \quad (6)$$

$$\mathbf{B}^{(i)}(\mathbf{r}) = \int_{\mathcal{D}} \mathbf{b}^{(i)}(\mathbf{u}) \exp(ik\mathbf{u} \cdot \mathbf{r}) d\Omega, \quad (7)$$

$$\nabla \cdot \mathbf{B}^{(i)}(\mathbf{r}) = 0, \quad (8)$$

$$\nabla \cdot \mathbf{E}^{(i)}(\mathbf{r}) = ik \int_{\mathcal{D}} \left( \mathbf{e}^{(i)}(\mathbf{u}) \cdot \mathbf{u} \right) \exp(ik\mathbf{u} \cdot \mathbf{r}) d\Omega = 0. \quad (9)$$

In these equations,  $\mathbf{u}$  is a unit vector characterizing the propagation direction of each plane wave component; the integration is done in the unit sphere  $\mathcal{D}$  whose solid angle element is  $d\Omega$ . Equation (9) follows from the *source-free* condition [29, 32]. By taking the sphere  $S$  in Eq. (4) so large that  $k|r - r_0| \rightarrow \infty$ , introducing Eq. (6) and Eq. (7) into Eq. (4), using Jones’ lemma based on the principle of the stationary phase, (see Appendix XII of [33]), and the source-free condition, Eq. (8)–Eq. (9), the terms that do not become zero after integration in Eq. (4) reduce to

$$\langle \mathbf{F} \rangle = -\frac{1}{8\pi} \Re \int_S \left\{ \epsilon \mathbf{E}^{(i)*} \cdot \mathbf{E}^{(r)} + \mu^{-1} \mathbf{B}^{(i)*} \cdot \mathbf{B}^{(r)} + \frac{1}{2} \left( \epsilon |\mathbf{E}^{(r)}|^2 + \mu^{-1} |\mathbf{B}^{(r)}|^2 \right) \right\} \mathbf{s} dS. \quad (10)$$

The first two terms of Eq. (10), coming from the interference between the incident and radiated fields, yield the electric and magnetic dipolar forces, respectively:

$$\langle \mathbf{F}_e \rangle + \langle \mathbf{F}_m \rangle = -\frac{k}{8\pi} \Re \left\{ \int_{\mathcal{D}} i \mathbf{u} \left( \epsilon \mathbf{e}^{(i)*}(\mathbf{u}) \cdot \mathbf{p} + \mu^{-1} \mathbf{b}^{(i)*}(\mathbf{u}) \cdot \mathbf{m} \right) d\Omega \right\} \quad (11)$$

$$= \frac{1}{2} \Re \left\{ \mathbf{p} \left( \nabla \otimes \mathbf{E}^{(i)*} \right)_{r=r_0} + \mathbf{m} \left( \nabla \otimes \mathbf{B}^{(i)*} \right)_{r=r_0} \right\}. \quad (12)$$

The third and fourth terms of Eq. (10) correspond to

$$\langle \mathbf{F}_{e-m} \rangle = -\frac{n}{c} \int_S \langle \mathbf{S}^{(r)} \rangle dS = -\frac{k^4}{3} \sqrt{\frac{\mu}{\epsilon}} \Re \{ \mathbf{p} \times \mathbf{m}^* \}, \quad (13)$$

$\mathbf{S}^{(r)}$  denoting the Poynting vector of the scattered fields. Equation (13) represents an electric-magnetic dipolar interaction between both induced dipoles. Obviously, the sum of Eq. (12) and Eq. (13) coincides with Eq. (2).

### 3. Optical theorem and forces on a dipolar particle

The question of energy conservation has been recurrently addressed and debated as regards small particles [34, 35], specially in connection with magnetic particles that produce zero-forward scattering intensity [36, 37]. It is thus relevant to explore the formal analogy between the force as momentum “absorption” rate and the optical theorem expressing the conservation of electromagnetic energy. From the Poynting’s theorem [26, 33], the rate  $-\mathcal{W}^{(a)}$  at which energy is being absorbed by the particle is given by

$$-\mathcal{W}^{(a)} = \int_S \left\{ \langle \mathbf{S} \rangle - \langle \mathbf{S}^{(i)} \rangle \right\} \cdot \mathbf{s} dS \quad (14)$$

$$= \frac{c}{8\pi n} \Re \left\{ \int_S \epsilon \mathbf{E}^{(i)*} \cdot \mathbf{E}^{(r)} + \mu^{-1} \mathbf{B}^{(i)*} \cdot \mathbf{B}^{(r)} + \frac{1}{2} (\epsilon |\mathbf{E}^{(r)}|^2 + \mu^{-1} |\mathbf{B}^{(r)}|^2) dS \right\} \quad (15)$$

The right hand side of Eq. (15) is analogous to that of Eq. (10) except for the scalar product by  $\mathbf{s}$  in the integrand and, as such, using the same calculation procedure leading to Eq. (10), we get the optical theorem for an arbitrary field:

$$-\mathcal{W}^{(a)} = -\frac{\omega}{2} \Im \left\{ \mathbf{p} \cdot \mathbf{E}^{(i)*}(r_0) \right\} - \frac{\omega}{2} \Im \left\{ \mathbf{m} \cdot \mathbf{B}^{(i)*}(r_0) \right\} + \frac{c k^4}{n 3} \left\{ \epsilon^{-1} |\mathbf{p}|^2 + \mu |\mathbf{m}|^2 \right\}. \quad (16)$$

The first two terms of Eq. (16), coming from the interference between the incident and radiated fields, are the energy analogue of the electric and magnetic dipolar forces given by Eq. (12). The third and fourth terms of Eq. (16) that come from the integral of the third and fourth terms of Eq. (15), now yield the rate  $\mathcal{W}^{(s)}$  at which the energy is being scattered, which together with the left hand side of this equation contributes to the rate of energy extinction by the particle  $\mathcal{W}^{(a)} + \mathcal{W}^{(s)}$ :

$$\mathcal{W}^{(a)} + \mathcal{W}^{(s)} = \frac{\omega}{2} \Im \left\{ \mathbf{p} \cdot \mathbf{E}^{(i)*}(r_0) \right\} + \frac{\omega}{2} \Im \left\{ \mathbf{m} \cdot \mathbf{B}^{(i)*}(r_0) \right\}. \quad (17)$$

Analogously as with the rate of scattered energy, the electric-magnetic dipolar interaction term of the force Eq. (13) corresponds to the rate at which momentum is being scattered by the particle. We shall explore in some detail this analogy in order to illustrate the physical origin of

$\langle \mathbf{F}_{e-m} \rangle$ . We notice that the power density of the scattered field can be written as the sum of two terms

$$\begin{aligned} \langle \mathbf{S}^{(r)} \rangle dS = & \frac{c}{8\pi n} k^4 \left( \epsilon^{-1} |\mathbf{p} \times \mathbf{s}|^2 + \mu |\mathbf{m} \times \mathbf{s}|^2 \right) \mathbf{s} d\Omega \\ & + \frac{c}{4\pi n} k^4 \sqrt{\frac{\mu}{\epsilon}} \Re \left\{ (\mathbf{s} \times \mathbf{p}) \cdot \mathbf{m}^* \right\} \mathbf{s} d\Omega, \end{aligned} \quad (18)$$

where the second term of Eq. (18) corresponds to the interference between the electric and magnetic dipolar fields. After integration over the closed surface  $S$ , that second term does not contribute to the radiated power, while it is the only contribution to the electric-magnetic dipolar interaction term of the force in Eq. (13). Namely,  $\langle \mathbf{F}_{e-m} \rangle$  comes from the interference between the fields radiated by  $\mathbf{p}$  and  $\mathbf{m}$ .

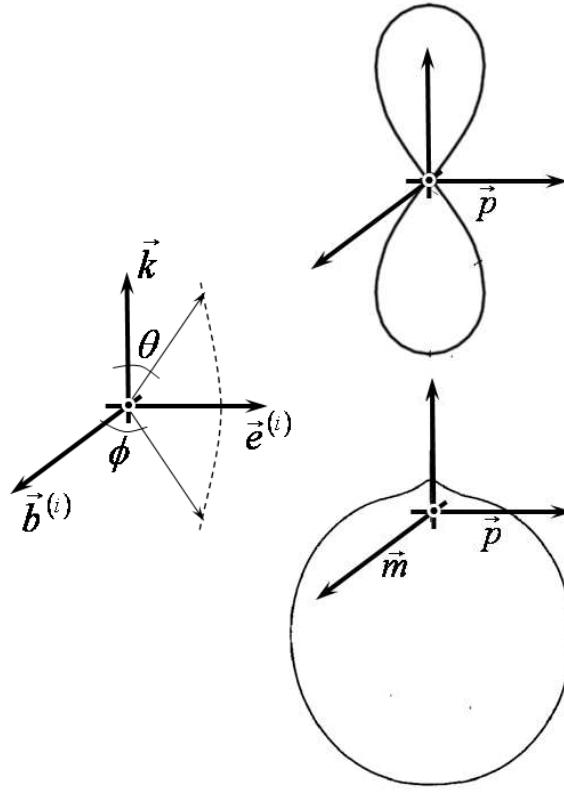


Fig. 1. Scattering by a dipolar particle. Left: Scattering geometry. Top right: Angular  $\theta$ -distribution of scattered light, (integrated over the azimuthal angle  $\phi$ ), by an electric dipolar particle. Bottom right: The same for a small perfectly conducting spherical particle (which has both electric and magnetic induced dipoles). The asymmetry of the radiation from this particle leads to the interaction force term  $\langle F_{e-m} \rangle$  (see text).

### 3.1. Forces on small dielectric magnetic spheres for plane wave incidence.

In order to illustrate the relevance of the different terms in the optical forces, we shall next consider the force from a plane wave  $(\mathbf{E}^{(i)} = \mathbf{e}^{(i)} e^{i\mathbf{k} \cdot \mathbf{r}}, \mathbf{B}^{(i)} = \mathbf{b}^{(i)} e^{i\mathbf{k} \cdot \mathbf{r}}, \text{ with } \mathbf{e}^{(i)} = 1/(kn) \{ \mathbf{b}^{(i)} \times \mathbf{k} \})$ ,



on a small dielectric and magnetic spherical particle characterized by its electric and magnetic polarizabilities  $\alpha_e$  and  $\alpha_m$ . When the induced dipole moments are expressed in terms of the incident field, i.e.

$$\mathbf{p} = \alpha_e \mathbf{e}^{(i)}; \quad \mathbf{m} = \alpha_m \mathbf{b}^{(i)}, \quad (19)$$

Then, on considering that for an incident plane wave, the incident power density is given by  $(c/n)\epsilon|\mathbf{e}^{(i)}|^2/(8\pi) = (c/n)\mu^{-1}|\mathbf{b}^{(i)}|^2/(8\pi)$ , Eq. (17) becomes after normalization

$$\sigma^{(ext)} = 4\pi k \Im [\epsilon^{-1} \alpha_e] + 4\pi k \Im [\mu \alpha_m] = \sigma^{(a)} + \sigma^{(s)}. \quad (20)$$

$\sigma^{(ext)}$ ,  $\sigma^{(a)}$  and  $\sigma^{(s)}$  being the particle extinction, absorption and scattering cross sections, respectively. Notice that the extinction cross section can be written as the sum of “electric” ( $\sigma_e^{(ext)} = 4\pi k \Im [\epsilon^{-1} \alpha_e]$ ) and “magnetic” ( $\sigma_m^{(ext)} = 4\pi k \Im [\mu \alpha_m]$ ) contributions,  $\sigma^{(ext)} = \sigma_e^{(ext)} + \sigma_m^{(ext)}$ .  $\sigma^{(s)}$ , the scattering cross section, is given by

$$\sigma^{(s)} = \left( \frac{n}{c} \frac{8\pi}{\epsilon|\mathbf{e}^{(i)}|^2} \right) \int_S \langle \mathbf{S}^{(r)} \rangle \cdot \mathbf{s} dS = \int_S \left( \frac{d\sigma^{(s)}}{d\Omega} \right) d\Omega, \quad (21)$$

where  $d\sigma^{(s)}/d\Omega$  is the differential scattering cross section [26, 41], defined as the energy scattered per unit time into a solid angle about a direction  $\Omega$ , which may be specified by two angles, the scattering angle  $\theta$  ( $\mathbf{s} \cdot \mathbf{k} = k \cos \theta$ ) and the azimuthal angle  $\phi$  (see Fig. 1):

$$\begin{aligned} \frac{d\sigma^{(s)}}{d\Omega} = & k^4 |\epsilon^{-1} \alpha_e|^2 (\sin^2 \phi + \cos^2 \theta \cos^2 \phi) \\ & + k^4 |\mu \alpha_m|^2 (\cos^2 \phi + \cos^2 \theta \sin^2 \phi) \\ & + 2k^4 \frac{\mu}{\epsilon} 2\Re\{\alpha_e \alpha_m^*\} \cos \theta. \end{aligned} \quad (22)$$

After integration over the solid angle, the last term, coming from the interference between radiated fields, cancels out and  $\sigma^{(s)}$  is given by:

$$\sigma^{(s)} = \frac{8\pi}{3} k^4 \left( |\epsilon^{-1} \alpha_e|^2 + |\mu \alpha_m|^2 \right), \quad (23)$$

i.e. the scattering cross section is given by the sum of the electric and magnetic scattering cross sections.

Returning to the optical forces on dipolar particles, for plane wave incidence the total force is given by

$$\begin{aligned} \langle \mathbf{F} \rangle &= \langle \mathbf{F}_e \rangle + \langle \mathbf{F}_m \rangle + \langle \mathbf{F}_{e-m} \rangle \\ &= \frac{\mathbf{k}}{2} \Im \left\{ \mathbf{p} \cdot \mathbf{e}^{(i)*} + \mathbf{m} \cdot \mathbf{b}^{(i)*} \right\} - \frac{n}{c} \int_S \langle \mathbf{S}^{(r)} \rangle dS \end{aligned} \quad (24)$$

$$= \frac{\epsilon}{8\pi} |\mathbf{e}^{(i)}|^2 \left\{ \sigma^{(ext)} \frac{\mathbf{k}}{k} - \int_S \left( \frac{d\sigma^{(s)}}{d\Omega} \right) \mathbf{s} d\Omega \right\}, \quad (25)$$

where we have made use of Eq. (18), Eq. (20) and Eq. (21). The first term in Eq. (25) corresponds to the sum of radiation pressures for a pure electric and a pure magnetic dipole. The second term,  $\langle \mathbf{F}_{e-m} \rangle$ , is the time-averaged scattered momentum rate, and we shall see below that it also contributes to radiation pressure. Due to the presence of the vector  $\mathbf{s}$  in the integral, only the interference term, i.e. the last term, of the differential scattering cross section Eq. (22) gives a non-zero contribution. The total force is then given by

$$\langle \mathbf{F} \rangle = \frac{\mathbf{k}}{2} \epsilon |\mathbf{e}^{(i)}|^2 \left\{ \frac{\Im \alpha_e}{\epsilon} + \mu \Im \alpha_m - \frac{2k^3}{3} \frac{\mu}{\epsilon} (\Re \alpha_e \Re \alpha_m + \Im \alpha_e \Im \alpha_m) \right\}. \quad (26)$$



### 3.2. Electric-magnetic dipolar interaction contribution of the force

Let us discuss in more detail the physical origin of the electric-magnetic dipolar interaction contribution to the optical force. The last term of Eq. (26) due to the electric-magnetic dipolar interaction component  $\langle \mathbf{F}_{e-m} \rangle$  of the force has a sign that depends on whether the interference factor  $\Re[\alpha_e \alpha_m^*]$  from both dipoles is positive or negative, i.e., on whether they are in, or in opposition of, phase. In the first case,  $\langle \mathbf{F}_{e-m} \rangle$  is negative, and thus opposes to the radiation pressure  $\langle \mathbf{F}_e \rangle + \langle \mathbf{F}_m \rangle$  of the pure dipoles. However, in the second case, the electric and magnetic polarizabilities have different sign, like for a metallic particle (see below), and  $\langle \mathbf{F}_{e-m} \rangle$  sums to that radiation pressure. It is interesting that this behavior of the force is linked with that of the differential scattering cross section Eq. (22), namely, when  $\alpha_e$  and  $\alpha_m$  are in phase, the scattering is mainly forward, whereas, there is predominance in the back-scattering region when they are in opposition of phase. For pure dielectric or pure magnetic particles, the force is simply proportional to the extinction cross section as expected from the concept of radiation pressure. However, it should be noticed that the electric-magnetic dipolar interaction, while leading to the term of the force  $\langle \mathbf{F}_{e-m} \rangle$ , and to the angular distribution of scattered intensity, it does not contribute, as is well known [26], to the extinction or scattering cross-sections. The contribution of these cross terms to the electric-magnetic dipolar interaction force is then connected with the asymmetry that they produce in the angular distribution of scattered power (differential cross section). For instance, for a perfectly conducting sphere, the predominance of the angular distribution of scattered power in the backscattering direction is well known [26] (see Fig. 1 and the discussion in Section 3.4 below).

As a summary of this discussion, it can be concluded that the electric-magnetic dipolar interaction contribution to the total force is a consequence of the non-isotropic radiation from the particle. This anisotropy arising as a consequence of the interference between the emitted fields, due to the coherent superposition of the radiating electric and magnetic induced dipoles.

### 3.3. Forces on Rayleigh particles

For a small spherical particle of radius  $a$ , with constants  $\epsilon_p$  and  $\mu_p$ , and in the Rayleigh limit ( $ka \ll 1$ ,  $n_p ka \ll 1$ ,  $n_p \equiv \sqrt{\epsilon_p \mu_p}/n$ ), we adopt Draine's expression [39] for the electric polarizability:

$$\alpha_e = \alpha_e^{(0)} \left( 1 - i \frac{2}{3\epsilon} k^3 \alpha_e^{(0)} \right)^{-1}, \quad (27)$$

$\alpha_e^{(0)}$  being the static polarizability,

$$\alpha_e^{(0)} = \epsilon a^3 \frac{\epsilon_p - \epsilon}{\epsilon_p + 2\epsilon}. \quad (28)$$

For sufficiently small radius, Draine's correction is not significant.

In absence of absorption,  $\alpha_e^{(0)}$  is a real quantity and Eq. (27) fulfills the lossless version of the optical theorem for dielectric particles,

$$\sigma_e^{(ext)} = 4\pi k \Im [\epsilon^{-1} \alpha_e] = \frac{8\pi k^4}{3\epsilon^2} |\alpha_e|^2 \approx \frac{8\pi k^4}{3\epsilon^2} |\alpha_e^{(0)}|^2. \quad (29)$$

Equation (27), and Eq. (17) and Eq. (20) of the optical theorem, convey the following expression for the magnetic polarizability:

$$\alpha_m = \alpha_m^{(0)} \left( 1 - i \frac{2}{3} \mu k^3 \alpha_m^{(0)} \right)^{-1}, \quad (30)$$

$\alpha_m^{(0)}$  being the static polarizability:

$$\alpha_m^{(0)} = \mu^{-1} a^3 \frac{\mu_p - \mu}{\mu_p + 2\mu}. \quad (31)$$

Again, in absence of absorption we have

$$\sigma_m^{(ext)} = 4\pi k \Im[\mu \alpha_m] = \frac{8\pi k^4 \mu^2}{3} |\alpha_m|^2 \approx \frac{8\pi k^4 \mu^2}{3} |\alpha_m^{(0)}|^2. \quad (32)$$

The polarizabilities expressed in the forms Eq. (27) and Eq. (30) hold the lossless optical theorem  $\sigma^{(ext)} = \sigma_e^{(ext)} + \sigma_m^{(ext)} = \sigma^{(s)}$ .

In the Rayleigh limit and in the presence of absorption by the particle, the real and imaginary parts of the polarizability are proportional to  $a^3$  and the total force is mainly given by the first two terms of Eq. (26), (i.e.  $\langle \mathbf{F}_e \rangle + \langle \mathbf{F}_m \rangle \sim a^3$ ). However, in absence of absorption (and assuming  $\varepsilon_p \neq -2\varepsilon$  and  $\mu_p \neq -2\mu$ ),  $\Re \alpha \sim a^3$  while  $\Im \alpha \sim a^3 (ka)^3 \ll \Re \alpha$ , and the three terms of the force scale with  $\sim a^6$ . For instance, for a non-absorbing dielectric magnetic sphere, using Eq. (27)–Eq. (31) this force becomes :

$$\langle \mathbf{F} \rangle \approx \frac{k^4 a^6}{3} \frac{\mathbf{k}}{k} \varepsilon |\mathbf{e}^{(i)}|^2 \left\{ \left[ \frac{\varepsilon_p - \varepsilon}{\varepsilon_p + 2\varepsilon} \right]^2 + \left[ \frac{\mu_p - \mu}{\mu_p + 2\mu} \right]^2 - \frac{\varepsilon_p - \varepsilon}{\varepsilon_p + 2\varepsilon} \frac{\mu_p - \mu}{\mu_p + 2\mu} \right\}. \quad (33)$$

It should be remarked that Eq. (33) leads to a value of the same order of magnitude as that obtained from the use of the Lorenz-Mie coefficients [22,23,40]. It has been suggested that near a surface plasmon resonance, where  $\varepsilon_p \sim -2\varepsilon$  or/and  $\mu_p \sim -2\mu$ , the radiation pressure can be very high [22]. However, it is worth mentioning that near the resonance there is always some absorption and Eq. (33) does not apply. Even in the hypothetical case of a non-absorbing resonance,  $\alpha_{e/m}^{(0)}$  diverges and the polarizability is then a pure imaginary quantity. The force would then be given by Eq. (26) with  $\Re \alpha_{e/m} \sim 0$  and  $\varepsilon^{-1} \Im \alpha_e$  or/and  $\mu \Im \alpha_m$  replaced by  $3/(2k^3)$ .

### 3.4. Forces beyond the Rayleigh limit

The discussion above can be extended beyond the Rayleigh limit for arbitrary values of  $|kan_p|$  as long as the particle polarizabilities are given by the first two Mie coefficients  $a_1$  and  $b_1$ , (cf. Eq. (4.56) and Eq. (4.57) of [41]):

$$\alpha_e = i \frac{3\varepsilon}{2k^3} a_1, \quad (34)$$

$$\alpha_m = i \frac{3}{2\mu k^3} b_1. \quad (35)$$

These electric and magnetic polarizabilities can still be expressed in the forms Eq. (27) and Eq. (30),  $\alpha_e^{(0)}$  and  $\alpha_m^{(0)}$  now being

$$\alpha_e^{(0)} = \frac{3\varepsilon}{2k^3} \frac{\mu n_p^2 j_1(n_p x) [x j_1(x)]' - \mu_p j_1(x) [n_p x j_1(n_p x)]'}{\mu n_p^2 j_1(n_p x) [x y_1(x)]' - \mu_p y_1(x) [n_p x j_1(n_p x)]'}, \quad (36)$$

$$\alpha_m^{(0)} = \frac{3}{2\mu k^3} \frac{\mu_p j_1(n_p x) [x j_1(x)]' - \mu j_1(x) [n_p x j_1(n_p x)]'}{\mu_p j_1(n_p x) [x y_1(x)]' - \mu y_1(x) [n_p x j_1(n_p x)]'}. \quad (37)$$

$x = ka$  and  $j_1(x)$ ,  $y_1(x)$  stand for the first order spherical Bessel functions. Using these Mie polarizabilities in Eq. (26), one obtains the first dipolar terms of the Mie expansion of the

radiation pressure on a sphere under plane wave illumination [33, 42]. In this connection, it should be remarked that in the small particle limit, *approximations should be made starting from Eq. (34) and Eq. (35)*, or equivalently: Eq. (27) and Eq. (30) with Eq. (36) and Eq. (37); this involves to use the complex spherical Hankel functions  $h_1^{(1)}$  in the denominators of  $a_1$  and  $b_1$ , (cf. Eq. (4.56) and Eq. (4.57) of [41]). When this is done, one obtains the correct polarizabilities: Eq. (27) and Eq. (30) with Eq. (28) and Eq. (31). *Otherwise*, if as commonly made, one directly employs directly Eq. (36) and Eq. (37), which is equivalent to use only the function  $y_1^{(1)}$  in the above mentioned denominators, *one is led to a result inconsistent with the optical theorem, as pointed out in [42]*.

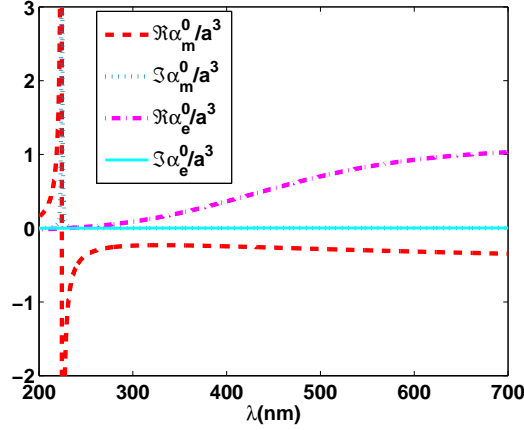


Fig. 2. Normalized real and imaginary parts of both electric and magnetic static polarizabilities, Eq. (36) and Eq. (37), in terms of the wavelength, for a 100nm highly conducting spherical particle in a host medium with  $\epsilon = \mu = 1$ . The peaks of both real and imaginary parts of the magnetic polarizability are about 30 a.u

As a specific example, let us consider the case of a highly conducting particle characterized by  $\mu_p = 1$  and a standard permittivity [33]

$$\epsilon_p(\omega) = 1 + i \frac{4\pi\sigma}{\omega} \quad (38)$$

where  $\sigma$  is the conductivity. Figure 2 shows the real and imaginary parts of the static electric and magnetic polarizabilities Eq. (36) and Eq. (37) for a 100 nm radius metallic particle with  $\sigma = 5 \cdot 10^{19} \text{ s}^{-1}$ . The poles of these two coefficients show in Fig. 2 the Mie resonances of these polarizabilities, describing the excitation of the *localized surface electric and magnetic plasmon* of the particle. On the other hand, Fig. 3 shows the radiation-reaction polarizabilities Eq. (34) and Eq. (35). Notice that a peak in the imaginary part of the polarizabilities corresponds to a maximum in the extinction and, correspondingly, in the radiation pressure terms.

Figure 4 illustrates the contributions to the total force,  $|\langle \mathbf{F}_e \rangle|$ ,  $|\langle \mathbf{F}_m \rangle|$  and  $|\langle \mathbf{F}_{e-m} \rangle|$ , obtained from Eqs. (26) and Eq. (34)–Eq. (38) for the conducting nanoparticle as a function of the wavelength  $\lambda$ . Notice the peaks due to the excitation of both the electric and magnetic localized surface plasmon, described by the  $a_1$  and  $b_1$  Mie resonances, respectively. The magnetic and the electric-magnetic dipolar interaction force components are smaller, but not at all negligible as compared to the electric one. It is worth noticing that in the region where the real part of both electric and magnetic polarizabilities is positive, the electric-magnetic dipolar interaction

term is *negative* and the scattered light is mainly forward, as indicated by the sign of  $\Re[\alpha_e \alpha_m^*]$ . At longer wavelengths, however, the real part of the magnetic polarizability is negative and  $\langle \mathbf{F}_{e-m} \rangle$  is positive.

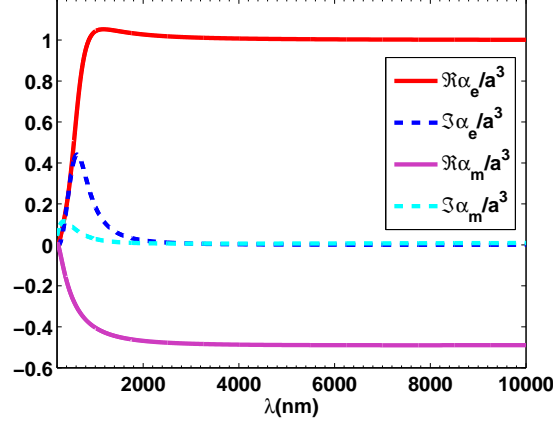


Fig. 3. Normalized real and imaginary parts of both electric and magnetic radiation-reaction polarizabilities, Eq. (34) and Eq. (35) in terms of the wavelength, for a 100nm highly conducting spherical particle in a host medium with  $\epsilon = \mu = 1$ . As  $\lambda$  becomes very large, the real part of the electric and magnetic polarizability tends to the perfect conductor values: 1 and  $-1/2$  a.u

At longer wavelengths, we reach the interesting “*small perfectly conducting sphere*” limit where  $ka \ll 1 \ll |kan_p|$ . As discussed in Ref. [43], this limit corresponds to the scattering by a perfectly reflecting dipolar sphere into which neither the electric nor the magnetic field penetrates. In this limit, we can apply Eq. (33) with an effective  $|\epsilon_p| \rightarrow \infty$  and  $\mu_p \rightarrow 0$  (see Ref. [26,43]). This is of particular interest since then the particle electric and magnetic polarizabilities have the same order of magnitude, differing by a factor -2:  $\alpha_e^{(0)} = \epsilon a^3$ ,  $\alpha_m^{(0)} = -a^3/(2\mu)$ , with  $\sigma^{(ext)} = \sigma^{(s)} \approx 10\pi k^4 a^6/3$ . The differential cross section is then given by [43]

$$\frac{d\sigma^{(s)}}{d\Omega} = k^4 a^6 \left( \frac{5}{4} - \left( \cos^2 \phi + \frac{1}{4} \sin^2 \phi \right) \sin^2 \theta - \cos \theta \right) \quad (39)$$

which, for unpolarized light (after averaging over the azimuthal angle  $\phi$ ), gives

$$\frac{d\sigma^{(s)}}{d\Omega} = \frac{5}{8} k^4 a^6 \left( 1 + \cos^2 \theta - \frac{8}{5} \cos \theta \right). \quad (40)$$

Now the angular distribution of scattered intensity is highly asymmetrical about the plane  $\theta = \pi/2$ , the scattering is mainly backward as the sign of  $\Re[\alpha_e \alpha_m^*]$  is negative, [cf. Eq. (22)], the ratio of forward and backward scattered intensities being 1/9, (see Fig. 1 and Sections 9.6 and 9.2 in [26] and [43], respectively). The three terms of the force Eq. (33) then become

$$\langle \mathbf{F} \rangle = \frac{k^4 a^6}{3} \frac{\mathbf{k}}{k} \epsilon |\mathbf{e}^{(i)}|^2 \left\{ 1 + \frac{1}{4} + \frac{1}{2} \right\} = \frac{\mathbf{k}}{k} \frac{7k^4 a^6}{12} \epsilon |\mathbf{e}^{(i)}|^2. \quad (41)$$

In this case, the highly asymmetric angular distribution of scattered intensity (cf. Fig. 1) leads to an electric-magnetic dipolar interaction contribution to the force which is half the electric radiation pressure, and twice the pure magnetic contribution.

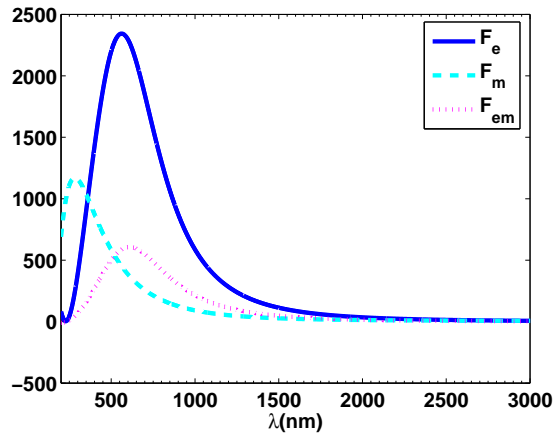


Fig. 4. The three components of the total radiation pressure versus the wavelength for the same particle as in Figs. 2 and 3, in vacuum. Forces are normalized to the incident field intensity:  $|\mathbf{e}^{(i)}|^2$ . Notice that as  $\lambda$  grows, these components tend to zero, [cf. Eq. (41)].

### 3.5. Some remarks on radiation pressure forces in negative index media

Equation (26) and Eq. (33) show that the sign of the electric and magnetic dipole components of the radiation pressure depend on the sign of the constitutive parameters of the exterior medium. In particular in a hypothetical *negative index* fluid with negligible dissipation,  $\epsilon$ ,  $\mu$  and  $n$  are negative and the sign of the radiation pressure components  $\langle \mathbf{F}_e \rangle$  and  $\langle \mathbf{F}_m \rangle$  reverses, confirming the conclusion of [44] for  $\langle \mathbf{F}_e \rangle$ , which now can be made extensive to  $\langle \mathbf{F}_m \rangle$ . On the other hand, the sign of the electric-magnetic dipolar interaction component  $\langle \mathbf{F}_{e-m} \rangle$ , being related to interference effects between both dipoles, [see Eq. (26) and Eq. (33)] depends on the signs of  $\epsilon_p - 2|\epsilon|$  and  $\mu_p - 2|\mu|$ . Hence the resultant force will depend on the weight of this component with respect to that of the other two components. For example, in the case of a perfectly conducting sphere [cf. Eq. (41)] this component of the radiation pressure also reverses its sign in a negative index medium. On the other hand, if the sphere had negative constitutive parameters, the electric and magnetic dipole forces would not change sign [cf. Eq. (33)], however the electric-magnetic dipolar interaction force would again depend on the contrast between these parameters and those of the exterior medium.

## 4. Optical forces on dipolar particles in arbitrary fields. Gradient, scattering and curl components

The three components of the force on the electric-magnetic dipole particle given by Eq. (2):  $\langle \mathbf{F} \rangle = \langle \mathbf{F}_e \rangle + \langle \mathbf{F}_m \rangle + \langle \mathbf{F}_{e-m} \rangle$  may also be written in terms of the polarizabilities [cf. Eq. (19)] as:

$$\langle \mathbf{F}_e \rangle = \frac{1}{4} \Re \alpha_e \nabla |\mathbf{E}|^2 + \frac{k}{2n} \Im \alpha_e \Re (\mathbf{E} \times \mathbf{B}^*) + \frac{1}{2} \Im \alpha_e \Im [(\mathbf{E}^* \cdot \nabla) \mathbf{E}]. \quad (42)$$

$$\langle \mathbf{F}_m \rangle = \frac{1}{4} \Re \alpha_m \nabla |\mathbf{B}|^2 + \frac{k}{2} n \Im \alpha_m \Re (\mathbf{E} \times \mathbf{B}^*) + \frac{1}{2} \Im \alpha_m \Im [(\mathbf{B}^* \cdot \nabla) \mathbf{B}]. \quad (43)$$

$$\begin{aligned}
\langle \mathbf{F}_{e-m} \rangle &= -\frac{k^4}{3} \sqrt{\frac{\mu}{\epsilon}} \left\{ \Re(\alpha_e \alpha_m^*) \Re(\mathbf{E} \times \mathbf{B}^*) - \Im(\alpha_e \alpha_m^*) \Im(\mathbf{E} \times \mathbf{B}^*) \right\} \\
&= -\frac{k^4}{3} \sqrt{\frac{\mu}{\epsilon}} \Re(\alpha_e \alpha_m^*) \Re(\mathbf{E} \times \mathbf{B}^*) + \frac{k^3}{3} \mu \Im(\alpha_e \alpha_m^*) \left[ \frac{1}{2} \nabla |\mathbf{E}|^2 - \Re[(\mathbf{E}^* \cdot \nabla) \mathbf{E}] \right]. \quad (44)
\end{aligned}$$

In passing from the first to the second line of Eq. (44) the following equality has been used:  $k \Im(\mathbf{E} \times \mathbf{B}^*) = \frac{1}{2} \nabla |\mathbf{E}|^2 - \Re[(\mathbf{E}^* \cdot \nabla) \mathbf{E}]$ .

Equation (42) for  $\langle \mathbf{F}_e \rangle$  is already well known for an electric-dipole particle [5], [27], [45] and its three terms can be rewritten as [25]

$$\langle \mathbf{F}_e \rangle = \frac{4\pi \Re\{\alpha_e\}}{\epsilon} \nabla \langle U_e \rangle + \sigma_e^{ext} \left\{ \frac{n}{c} \langle \mathbf{S} \rangle \right\} - \sigma_e^{ext} \left\{ \frac{c}{n} \nabla \times \langle \mathbf{L}_{Se} \rangle \right\} \quad (45)$$

where

$$\langle U_e \rangle = \frac{1}{2} \frac{\epsilon}{8\pi} |\mathbf{E}|^2. \quad (46)$$

$$\langle \mathbf{S} \rangle = \frac{1}{2} \frac{c}{4\pi\mu} \mathbf{E} \times \mathbf{B}^*. \quad (47)$$

$$\langle \mathbf{L}_{Se} \rangle = \frac{1}{2} \frac{\epsilon}{8\pi\omega i} \mathbf{E}^* \times \mathbf{E} \quad (48)$$

are the time-averaged electric energy density, Poynting vector and *electric spin density* of the optical field, respectively.

It is worth noticing that, although the *angular momentum density* of a light field is a well defined quantity [26], its separation in terms of an *orbital* and a *spin angular momentum* is not clear except within the paraxial approximation [46, 47]. In this later case, the density of spin angular momentum can be linked to the expression given in Eq. (48) [48]. Bearing this in mind, and following [25], we tentatively associate for arbitrary 3D fields the curl term in the electric force to the curl of the spin density of the electric field. It was pointed out, that this additional non-conservative contribution to the scattering force is relevant for optical beams with non-uniform electric helicity [49].

The force on the magnetic dipole  $\langle \mathbf{F}_m \rangle$ , Eq. (43), has an analogous decomposition in terms of a gradient component (first term), a radiation pressure, or scattering, component (second term, which is the only one contributing in the case of a plane wave) and a third term that depends on the polarization and is, analogously to its electric counterpart Eq. (48), linked to the vector  $\mathbf{B} \times \mathbf{B}^*$  as a curl:

$$\frac{1}{2} \Im \alpha_m \Im[(\mathbf{B}^* \cdot \nabla) \mathbf{B}] = -i \frac{\Im \alpha_m}{4} \nabla \times (\mathbf{B} \times \mathbf{B}^*), \quad (49)$$

where the identities  $\nabla \times (\mathbf{B} \times \mathbf{B}^*) = \mathbf{B}(\nabla \cdot \mathbf{B}^*) - \mathbf{B}^*(\nabla \cdot \mathbf{B}) + (\mathbf{B}^* \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{B}^*$  and  $\nabla \cdot \mathbf{B} = 0$  have been used.

In analogy with the electric dipole force, we can then write

$$\langle \mathbf{F}_m \rangle = \frac{4\pi \Re\{\alpha_m\}}{\epsilon} \nabla \langle U_m \rangle + \sigma_m^{ext} \left\{ \frac{n}{c} \langle \mathbf{S} \rangle \right\} - \sigma_m^{ext} \left\{ \frac{c}{n} \nabla \times \langle \mathbf{L}_{Sm} \rangle \right\} \quad (50)$$

where

$$\langle U_m \rangle = \frac{1}{2} \frac{1}{8\pi\mu} |\mathbf{B}|^2; \quad (51)$$

and we tentatively define

$$\langle \mathbf{L}_{Sm} \rangle = \frac{1}{2} \frac{1}{8\pi\mu\omega i} \mathbf{B}^* \times \mathbf{B} \quad (52)$$

as the *spin density of angular momentum* of the magnetic field [49]. While circular polarized plane waves have the same uniform electric and magnetic spins, in general, this is not the case, as it can be shown, for counter-propagating incident beams as discussed next.

As seen from the above, the radiation pressure component of the electric dipole and of the magnetic dipole forces only differ by the corresponding polarizability. However both the gradient and the curl components have complementary behaviors. For instance, for linear polarization, since  $\mathbf{E}$  and  $\mathbf{B}$  are in the complementary polarization of each other, (say TE or TM), the effect of the gradient and curl components of the electric force  $\langle \mathbf{F}_e \rangle$  for one polarization, (say TE), is the same (apart from the polarizability factors) as that of the corresponding quantities of the magnetic force  $\langle \mathbf{F}_m \rangle$  for the complementary polarization, (say TM). In particular, results as those obtained in [25] for four counterpropagating incident waves behave in complementary ways for the electric and magnetic dipole forces. On the other hand, [cf. Eq. (44)], *the electric-magnetic dipolar interaction force component  $\langle \mathbf{F}_{e-m} \rangle$  also contributes, even though with different weight of the polarizabilities, to the radiation pressure and gradient forces.* In particular, for an incident plane wave, Eq. (44) shows that the whole component  $\langle \mathbf{F}_{e-m} \rangle$  is radiation pressure, and its sign depends on that of  $\Re[\alpha_e \alpha_m^*]$ .

#### 4.1. Spin contribution and Stokes parameters

The tentative introduction of a 3D generalization of the spin density of the field beyond the paraxial approximation, contributing to the third term of  $\langle \mathbf{F}_e \rangle$  and  $\langle \mathbf{F}_m \rangle$  [cf. Eq. (45) and Eq. (50)] as discussed above, suggests the possibility of expressing this term by means of the 3D Stokes parameters. This has an analogy with the generalization to 3D [50] of the 2D Stokes parameters employed for plane waves or optical beams propagating along a well characterized direction [33].

The 3D Stokes parameters that we need, are [50]:

$$\begin{aligned} s_2 &= (3/2)i(\Phi_{xy} - \Phi_{yx}), \\ s_5 &= (3/2)i(\Phi_{xz} - \Phi_{zx}), \\ s_7 &= (3/2)i(\Phi_{yz} - \Phi_{zy}), \end{aligned} \quad (53)$$

where  $\Phi_{ij}^e = E_i^* E_j$  and  $\Phi_{ij}^m = B_i^* B_j$ ,  $i, j = x, y, z$ , which the superscript now used for either the electric or magnetic vector quantities.

In terms of these parameters it is straightforward to obtain the following representation for the last term of Eq. (42) and Eq. (43), [or of their equivalent: Eq. (45) and Eq. (50)]:

$$\frac{1}{2} \Im \alpha_e \Im [(\mathbf{E}^* \cdot \nabla) \mathbf{E}] = \frac{1}{6} \Im \alpha_e \nabla \times (s_7^e, -s_5^e, s_2^e), \quad (54)$$

$$\frac{1}{2} \Im \alpha_m \Im [(\mathbf{B}^* \cdot \nabla) \mathbf{B}] = \frac{1}{6} \Im \alpha_m \nabla \times (s_7^m, -s_5^m, s_2^m). \quad (55)$$

## 5. Conclusion

We have analyzed the different contributions to the optical force on a small dipolar magnetic particle, immersed in an arbitrary non-dissipative medium, with both electric and magnetic response to an arbitrary external electromagnetic field. In the light of the formal analogy of the



conservation laws from which the photonic force and the rate of dissipated energy derive, we have discussed the origin and physical significance of the electric-magnetic dipolar interaction component that appears in the force, when both the electric and magnetic dipoles are excited, as a result of their interference; also illustrating the weight of this cross term with respect to those of the pure dipoles, showing the comparable magnitude of them both, also beyond the Rayleigh limit. A numerical example has been given to estimate the magnitude of the electric-magnetic dipolar interaction component of the force versus those pure electric and magnetic components. A conducting sphere whose scattering is described by its first electric and magnetic Mie coefficients was considered. By introducing a generalization of the polarizabilities in terms of these coefficients, we have shown the magnetic partial wave response and the corresponding magnetic polarizability. At large wavelengths, the results for this particle tend, as they should, to those of the classical perfectly conducting small sphere and show regions where the magnitude of those three components is comparable. This opens the way to future research on some composite metallic or dielectric particles of high refractive index, that should present substantial response to the magnetic field of an electromagnetic wave in the infrared and visible regions.

We have illustrated the effect of the constitutive parameters of the medium in which the particle is immersed, as regards the radiation pressure, in particular confirming its reversal in negative-index media, even though the sign of the total force, including that of the electric-magnetic dipolar interaction, may or may not change depending on the value of the relative constitutive parameters.

Further, we have obtained a decomposition for the magnetic force term as a gradient, scattering and curl components, in analogy to pure electric dipole forces. In this form, we have derived a non-conservative contribution of the magnetic term as proportional to the curl of a tentatively introduced spin angular momentum density of the magnetic field. Also, the electric-magnetic dipolar interaction component contributes to both the radiation pressure and gradient forces. These results show the complementary contribution of the electric and magnetic fields to these forces, depending on the polarization. Finally, we have expressed those curl forces in terms of three-dimensional generalizations of the Stokes parameters for the electric and magnetic vectors.

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