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Normal modes and quality factors of shielded composite dielectric spherical resonators

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Abstract : Electromagnetic field analysis of shielded composite dielectric spherical resonator in which the dielectric sphere is composed of two concentric spheres with different dielectric materials has been made. Characteristic equations for the $TE_{nm\ell}$ and $TM_{nm\ell}$ modes have been derived. From these characteristic equations, the resonant frequencies and quality factors have been calculated using numerical method. Computations of the resonant frequencies and quality factors have been made for resonators with parameters suitable for the optical and microware regions.

In this paper we have presented the analysis and the results obtained from the numerical computations for shielded composite dielectric spherical resonator in which the dielectric sphere is composed of two concentric spheres with different dielectric materials. Starting from the Maxwell's equations for such a resonators have been derived and resonant frequencies and quality factor's have been calculated for the $TE_{nm\ell}$ and $TM_{nm\ell}$ modes using numerical method.

Keywords : Spherical resonators; spherical dielectric resonators; Eigen modes; quality factors.

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1. Introduction

High quality factor (Q) resonators are important components of low phase noise oscillators for emerging millimeter wave systems such as automotive radar and wireless communication [1]. Miniaturization of many active and passive microwave components, such as oscillators, filters and antennas have been achieved using dielectric resonators. Dielectric resonators have the disadvantage of energy loss by radiation of electromagnetic energy into the surrounding space. Such energy loss by radiation can be reduced by placing the dielectric resonators in metal shields [2]. These dielectric resonators are widely used in mobile telecommunications and optical

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instruments such as optical couplers and filters. Design of such components requires the knowledge of shape and resonant frequency response of dielectric resonators used. Open dielectric resonators radiate strongly into free space at and in the vicinity of their resonant frequencies. Antennas based on dielectric resonators find many applications as radiating elements at microwave and millimeter wave frequencies owing to their small sizes and low losses. In recent years with the advent of new materials with both high permittivities and quality factors as well as low temperature coefficients a progressive effort with miniaturization and stabilization of components for application in microwave and millimeter wave regions, such as oscillators, dielectric loaded resonators, and filters has been developed. Continuous search for new high Q microwave materials and components is one of the main aim of material science and technology. At millimeter-wave frequencies the reduction in the size of the dielectric resonators has focused the attention of the researchers towards new shapes such as spherical [3] and hemispherical resonators [4]. Single shell spherical dielectric Bragg resonators with Q values as high as 10^5 have been constructed [5]. The quality factor Q depends either on conduction and dielectric losses for the closed structures or radiation and dielectric losses for the open structures. In metal shielded structures Q can be expressed in terms of the material properties (loss tangent and surface resistance), the energy filling factors and the geometry factors.

In order to design reliable resonators for applications an accurate determination of resonant frequencies, unloaded quality factors and distributions of fields can be very useful parameters. A number of workers have studied the resonators with the spherical [3,6–16] and hemispherical [4,17] geometries in the past using different approaches. Electromagnetic analysis for open spherical dielectric resonators has been presented earlier [18]. However, the expressions derived by these authors [18] for the TM modes seem to be incorrect. In all the above cases the resonant frequencies and quality factors have been computed for the resonators with parameters suitable for only the microwave region of the electromagnetic spectrum. From the literature survey it seems that neither the normal mode frequencies nor the quality factors in the optical region have been reported for the shielded composite dielectric spherical resonators. This type of shielded dielectric resonators could be inserted into micro-strip, fin lines or wave-guide structures to realize passive circuits (band-stop and band-pass filters) which in turn could be coupled with active circuits to stabilize the resonant frequency of an oscillator [18,19]. Whispering gallery modes (WGMs) of the spherical resonators could be used to perform Michelson-Morley experiment [20,21] and to develop high Q multi-pole filters in a spherical cavity [22]. In this work we present the electromagnetic field analysis for shielded composite dielectric spherical resonators in which the dielectric sphere is composed of two concentric spheres with different dielectric materials. Expressions for the field components, the characteristic equations and quality factors have been derived. The resonant frequencies and quality factors have been computed for the optical and microwave regions for the $TE_{nm\ell}$ and $TM_{nm\ell}$ the modes.

2. Theory

A shielded composite dielectric spherical resonator consists of two concentric spheres *i.e.* solid inner dielectric sphere with radius *a* and permittivity $\in_1$ and outer dielectric spherical shell with radius *b* and permittivity $\in_2$. The outer dielectric spherical shell is shielded by a perfectly conducting metal case of radius *b*. The dielectrics of the two spheres are non-magnetic, *i.e.* $\mu_1 = \mu_0 = \mu_2$, electrically homogeneous and isotropic. Thus, the resonator consists of two dielectric regions, one in the region $0 \le r \le a$ and the other in the region $a \le r \le b$. Following the procedure detailed in our earlier paper [10], the solution of the radial part of the wave equation in the two regions can be written as,

$$X(r) = AJ_{n+\frac{1}{2}}(k_1 r), \qquad for\ \ 0 \le r \le a \tag{1}$$

$$X(r) = BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r), \quad for\ \ a \le r \le b \tag{2}$$

where, $k_1 = \frac{\omega}{c}\sqrt{\in_{r1}} = k_0\sqrt{\in_{r1}}$, $k_2 = \frac{\omega}{c}\sqrt{\in_{r2}} = k_0\sqrt{\in_{r2}}$, $\in_{r1} = \in_1 / \in_0$, $\in_{r2} = \in_2 / \in_0$ ω is angular frequency and *c* is speed of light in vacuum. The symbols $J_\nu(kr)$ and $Y_\nu(kr)$ ($\nu = n + 1/2$ and $k = k_1/k_2$) appearing in the eqs. (1) and (2) are respectively the Bessel functions of the first and second kinds, ν being the order of the functions and the symbols *A*, *B* and *C* are constants. Apart from a normalization constant the complete solution of the wave equation is given by,

$$\psi(r,\theta,\phi) = \frac{A}{\sqrt{k_1 r}}\ J_{n+\frac{1}{2}}(k_1 r)\ P_n^m(\cos\theta)\cos m\phi\,, \quad \text{for } 0 \le r \le a \tag{3}$$

$$\psi(r,\theta,\phi) = \left\{\frac{B}{\sqrt{k_2 r}} J_{n+\frac{1}{2}}(k_2 r) + \frac{C}{\sqrt{k_2 r}} Y_{n+\frac{1}{2}}(k_2 r)\right\} P_n^m(\cos\theta)\cos m\phi\,, \quad \text{for } a \le r \le b \tag{4}$$

where $\psi(r,\theta,\phi)$ is the amplitude of the electromagnetic radiation (electric/magnetic) field; r,θ,ϕ are the co-ordinates in the spherical polar co-ordinates-system, $P_n^m(\cos\theta)$ is the associated Legendre polynomial of order *m* and degree *n* and *J* and *Y* are the Bessel functions (of the order $n + ½$) of the first and second kinds respectively. Using the expressions for $\psi(r,\theta,\phi)$ from the eqs. (3) and (4) one can find out the expressions for the electric and magnetic field components for the $TE_{nm\ell}$ and $TM_{nm\ell}$ the modes separately. The electric and magnetic field components for the $TE_{nm\ell}$

modes in the two regions ($0 \leq r \leq a$ and $a \leq r \leq b$) of the resonator are given below :

(i) Inner dielectric spherical region ($0 \leq r \leq a$) :

$$E_r = 0 \tag{5a}$$

$$E_\theta = \frac{mA}{\sqrt{k_1 r}\,\sin\theta}\, J_{n+\frac{1}{2}}(k_1 r) P_n^m(\cos\theta)\sin m\phi \tag{5b}$$

$$E_\phi = \frac{A}{\sqrt{k_1 r}}\, J_{n+\frac{1}{2}}(k_1 r)\frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi \tag{5c}$$

$$H_r = \frac{n(n+1)A}{j\omega\mu_0 r\sqrt{k_1 r}}\, J_{n+\frac{1}{2}}(k_1 r) P_n^m(\cos\theta)\cos m\phi \tag{5d}$$

$$H_\theta = \frac{A}{j\omega\mu_0 r\sqrt{k_1}}\frac{d}{dr}\left\{\sqrt{r}J_{n+\frac{1}{2}}(k_1 r)\right\}\frac{d}{d\theta}\left\{P_n^m(\cos\ \theta)\right\}\cos m\phi \tag{5e}$$

$$H_\phi = -\frac{mA}{j\omega\mu_0\sqrt{k_1}\ r\sin\ \theta}\frac{d}{dr}\left\{\sqrt{r}J_{n+\frac{1}{2}}(k_1 r)\right\} P_n^m(\cos\theta)\frac{d}{d\phi}\sin m\phi \tag{5f}$$

(ii) Outer dielectric spherical region ($a \leq r \leq b$) :

$$E_r = 0 \tag{6a}$$

$$E_\theta = \frac{m}{\sqrt{k_2 r}\,\sin\theta}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\} P_n^m(\cos\theta)\sin m\phi \tag{6b}$$

$$E_\phi = \frac{1}{\sqrt{k_2 r}}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi \tag{6c}$$

$$H_r = \frac{n(n+1)}{j\omega\mu_0 r\sqrt{k_2 r}}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\left\{P_n^m(\cos\theta)\right\}\cos m\phi \tag{6d}$$

$$H_\theta = \frac{1}{j\omega\mu_0 r\sqrt{k_2}}\frac{d}{dr}\left[\sqrt{r}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\right]\frac{d}{d\theta}\left\{P_n^m(\cos\ \theta)\right\}\cos m\phi \tag{6e}$$

$$H_\phi = -\frac{1}{j\omega\mu_0\sqrt{k_2}\,r\sin\theta}\frac{d}{dr}\left[\sqrt{r}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\right]\left\{P_n^m(\cos\theta)\right\}\sin m\phi \tag{6f}$$

Similarly, the electric and magnetic field expressions for the modes are determined and these are given as,

(i) Inner dielectric spherical region (0 $\leq$ r $\leq$ a) :

$$E_r = \frac{n(n+1)A}{j\omega \in_0 \in_{r1} r\sqrt{k_1 r}} J_{n+\frac{1}{2}}(k_1 r)\left\{P_n^m(\cos\ \theta)\right\}\cos m\phi \tag{7a}$$

$$E_\theta = \frac{A}{j\omega \in_0 \in_{r1} r\sqrt{k_1}} \frac{d}{dr}\left\{\sqrt{r}\ J_{n+\frac{1}{2}}(k_1 r)\right\}\frac{d}{d\theta}\{P_n^m(\cos\theta)\}\cos m\phi \tag{7b}$$

$$E_\phi = -\frac{mA}{j\omega \in_0 \in_{r1} \sqrt{k_1}\ r\sin\ \theta} \frac{d}{dr}\left\{\sqrt{r} J_{n+\frac{1}{2}}(k_1 r)\right\}\left\{P_n^m(\cos\ \theta)\right\}\sin m\phi \tag{7c}$$

$$H_r = 0 \tag{7d}$$

$$H_\theta = -\frac{mA}{\sqrt{k_1 r}\sin\theta} J_{n+\frac{1}{2}}(k_1 r) P_n^m(\cos\theta)\sin m\phi \tag{7e}$$

$$H_\phi = -\frac{A}{\sqrt{k_1 r}} J_{n+\frac{1}{2}}(k_1 r)\frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi \tag{7f}$$

(ii) Outer dielectric spherical region (a $\leq$ r $\leq$ b) :

$$E_r = \frac{n(n+1)}{j\omega \in_0 \in_{r2} r\sqrt{k_2 r}}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\left\{P_n^m(\cos\theta)\right\}\cos m\phi \tag{8a}$$

$$E_\theta = \frac{1}{j\omega \in_0 \in_{r2} r\sqrt{k_2}} \frac{d}{dr}\left[\sqrt{r}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\right]\frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi \tag{8b}$$

$$E_\phi = -\frac{m}{j\omega \in_0 \in_{r2} \sqrt{k_2 r}\ \sin\theta}\frac{d}{dr}\left[\sqrt{r}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\right] P_n^m(\cos\theta)\sin m\phi \tag{8c}$$

$$H_r = 0 \tag{8d}$$

$$H_\theta = -\frac{m}{\sqrt{k_2 r}\sin\theta}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\} P_n^m(\cos\theta)\sin m\phi \tag{8e}$$

$$H_\phi = -\frac{1}{\sqrt{k_2 r}}\left\{BJ_{n+\frac{1}{2}}(k_2 r) + CY_{n+\frac{1}{2}}(k_2 r)\right\}\frac{d}{d\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi \tag{8f}$$

3. Characteristic equations for the $TE_{nm\ell}$ and $TM_{nm\ell}$ modes

Once the expressions for the electric and magnetic fields for the $TE_{nm\ell}$ and $TM_{nm\ell}$ modes in the regions $0 \leq r \leq a$ and $a \leq r \leq b$ are known, one can derive the characteristic equations for the $TE_{nm\ell}$ and the $TM_{nm\ell}$ modes, employing the boundary conditions on the dielectric surface ($r = a$) and the boundary conditions on the metallic surface ($r = b$). The tangential components of ***E*** and ***H*** are continuous at the surface ($r = a$) separating the two dielectric media, and the tangential component of the electric field vector is zero at the metallic surface ($r = b$). Using the expressions for the field(s) components (eqs. 5(a-f) and 6(a-f)) and applying the boundary conditions one obtains the following equations,

$$\frac{A}{\sqrt{k_1}} J_{n+\frac{1}{2}}(k_1 a) - \frac{B}{\sqrt{k_2}} J_{n+\frac{1}{2}}(k_2 a) - \frac{C}{\sqrt{k_2}} Y_{n+\frac{1}{2}}(k_2 a) = 0 \tag{9}$$

$$\frac{A}{\sqrt{k_1}} \left\{ J_{n+\frac{1}{2}}(k_1 a) + 2k_1 a\, J'_{n+\frac{1}{2}}(k_1 a) \right\} - \frac{B}{\sqrt{k_2}} \left\{ J_{n+\frac{1}{2}}(k_2 a) + 2k_2 a\, J'_{n+\frac{1}{2}}(k_2 a) \right\}$$

$$- \frac{C}{\sqrt{k_2}} \left\{ Y_{n+\frac{1}{2}}(k_2 a) + 2k_2 a\, Y'_{n+\frac{1}{2}}(k_2 a) \right\} = 0 \tag{10}$$

$$B J_{n+\frac{1}{2}}(k_2 b) + C Y_{n+\frac{1}{2}}(k_2 b) = 0\,. \tag{11}$$

In order to satisfy the eqs. (9–11) simultaneously the determinant of the coefficients of A, B and C must vanish which leads to the following equation,

$$Y_{n+\frac{1}{2}}(k_2 b) \left[k_2 J_{n+\frac{1}{2}}(k_1 a) J'_{n+\frac{1}{2}}(k_2 a) - k_1 J_{n+\frac{1}{2}}(k_2 a) J'_{n+\frac{1}{2}}(k_1 a) \right]$$

$$- J_{n+\frac{1}{2}}(k_2 b) [k_2 J_{n+\frac{1}{2}}(k_1 a) Y'_{n+\frac{1}{2}}(k_2 a) - k_1 J'_{n+\frac{1}{2}}(k_1 a) Y_{n+\frac{1}{2}}(k_2 a)] = 0\,. \tag{12}$$

By using relations between Bessel functions (J/Y) and the corresponding spherical Bessel functions (j/y) the eq. (12) reduces to the following characteristic equation for the $TE_{nm\ell}$ modes,

$$k_1 j_{n-1}(k_1 a) \{ j_n(k_2 a) y_n(k_2 b) - j_n(k_2 b) y_n(k_2 a) \}$$

$$+ k_2 j_n(k_1 a) \{ y_{n-1}(k_2 a) j_n(k_2 b) - y_n(k_2 b) j_{n-1}(k_2 a) \} = 0\,. \tag{13}$$

Similar to the case of the $TE_{nm\ell}$ modes, one can find out the characteristic equation for the $TM_{nm\ell}$ modes using the boundary conditions and the field expressions in the two regions. Applying the boundary conditions at the surfaces $r = a$ and $r = b$, one gets the following relations from the eqs. 7(a–f) and 8(a–f),

$$\frac{A}{\in_{r1}\sqrt{k_1}}\left\{J_{n+\frac{1}{2}}(k_1a)+2ak_1J'_{n+\frac{1}{2}}(k_1a)\right\}-\frac{B}{\in_{r2}\sqrt{k_2}}\left\{J_{n+\frac{1}{2}}(k_2a)+2ak_2J'_{n+\frac{1}{2}}(k_2a)\right\}$$

$$-\frac{C}{\in_{r2}\sqrt{k_2}}\left\{Y_{n+\frac{1}{2}}(k_2a)+2ak_2Y'_{n+\frac{1}{2}}(k_2a)\right\}=0 \tag{14}$$

$$\frac{A}{\sqrt{k_1}}J_{n+\frac{1}{2}}(k_1a)-\frac{B}{\sqrt{k_2}}J_{n+\frac{1}{2}}(k_2a)-\frac{C}{\sqrt{k_2}}Y_{n+\frac{1}{2}}(k_2a)=0 \tag{15}$$

$$B\left\{J_{n+\frac{1}{2}}(k_2b)+2bk_2J'_{n+\frac{1}{2}}(k_2b)\right\}+C\left\{Y_{n+\frac{1}{2}}(k_2b)+2bk_2Y'_{n+\frac{1}{2}}(k_2b)\right\}=0\,. \tag{16}$$

Solving the eqs. (14)–(16) and using the relations between the Bessel functions and the spherical Bessel functions, the characteristic equation for the $\mathrm{TM}_{\mathrm{nm}\ell}$ modes is given by,

$$\begin{aligned}
&k_2^2ab\in_{r1}\,j_n(k_1a)\{\,j_{n-1}(k_2a)y_{n-1}(k_2b)-j_{n-1}(k_2b)y_{n-1}(k_2a)\}\\
&\quad+k_1k_2ab\in_{r2}\,j_{n-1}(k_1a)\{\,y_n(k_2a)\,j_{n-1}(k_2b)-y_{n-1}(k_2b)\,j_n(k_2a)\}\\
&\quad+nk_2a\in_{r1}\,j_n(k_1a)\{\,y_{n-1}(k_2a)j_n(k_2b)-y_n(k_2b)j_{n-1}(k_2a)\}\\
&\quad+nk_1a\in_{r2}\,j_{n-1}(k_1a)\{\,j_n(k_2a)y_{n-1}(k_2b)-y_{n-1}(k_2b)j_n(k_2a)\}\\
&\quad+nk_2b\in_{r1}\,j_n(k_1a)\{\,y_n(k_2a)j_{n-1}(k_2b)-y_{n-1}(k_2b)\,j_n(k_2a)\}\\
&\quad+nk_2b\in_{r2}\,j_n(k_1a)\{\,j_n(k_2a)\;\;y_{n-1}(k_2b)-j_{n-1}(k_2b)y_n(k_2a)\}\\
&\quad+n^2\in_{r1}\,j_n(k_1a)\{\,j_n(k_2a)y_n(k_2b)-j_n(k_2b)y_n(k_2a)\}\\
&\quad+n^2\in_{r2}\,j_n(k_1a)\{\,y_n(k_2a)j_n(k_2b)-y_n(k_2b)\,j_n(k_2a)\}=0\,.
\end{aligned} \tag{17}$$

4. Stored energy, losses and quality factors (Q)

4.1. Stored energy :

The expressions for the energy W_{1n} stored within the inner dielectric sphere ($0 \le r \le a$) and the energy W_{2n} stored within the outer dielectric spherical shell ($a \le r \le b$) for the $\mathrm{TM}_{\mathrm{nm}\ell}$ modes are given by,

$$W_{1n}=\frac{4n(n+1)}{(2n+1)}\in_0\in_{r1}A^2a^3\left\{j_1^2(k_1a)-j_0(k_1a)j_2(k_1a)\right\} \tag{18}$$

$$W_{2n}=\frac{4n(n+1)}{(2n+1)}\in_0\in_{r2}\Big[B^2r^3\left\{j_1^2(k_2r)-j_0(k_2r)j_2(k_2r)\right\}$$

$$+C^2 r^3 \left\{ y_1^2(k_2r) - y_0(k_2r)y_2(k_2r) \right\}$$

$$+BCr^3 \left\{ 2y_1(k_2r)\, j_1(k_2r) - y_0(k_2r) j_2(k_2r) - y_2(k_2r)\, j_0(k_2r) \right\} \Big]_a^b \tag{19}$$

where n = 1, 2 and 3 for the $TE_{10\ell}$, $TE_{20\ell}$ and $TE_{30\ell}$ modes respectively and B and C used for the $TE_{n0\ell}$, modes in eq. (19) are determined in terms of A using the eqs. (9)–(11) as,

$$B = \frac{j_n(k_1a)y_n(k_2b)}{j_n(k_2a)y_n(k_2b) - j_n(k_2b)y_n(k_2a)}\, A \tag{20}$$

$$C = -\frac{j_n(k_1a)\, j_n(k_2b)}{j_n(k_2a)y_n(k_2b) - j_n(k_2b)y_n(k_2a)}\, A\,. \tag{21}$$

Similarly, the expressions for the energy W'_{1n} stored within the inner dielectric sphere ($0 \le r \le a$) and the energy W'_{2n} stored within the outer dielectric spherical shell ($a \le r \le b$) for the $TM_{nm\ell}$ modes are given by,

$$W'_{1n} = \frac{4n(n+1)}{(2n+1)}\, \mu_0 A^2 a^3 \left\{ j_1^2(k_1a) - j_0(k_1a) j_2(k_1a) \right\} \tag{22}$$

$$W'_{2n} = \frac{4n(n+1)}{(2n+1)}\, \mu_0 \Big[B^2 r^3 \left\{ j_1^2(k_2r) - j_0(k_2r)\, j_2(k_2r) \right\}$$

$$+C^2 r^3 \left\{ y_2^2(k_2r) - y_0(k_2r)y_2(k_2r) \right\}$$

$$+BCr^3 \left\{ 2y_1(k_2r)\, j_1(k_2r) - y_0(k_2r) j_2(k_2r) - j_0(k_2r)\, y_2(k_2r) \right\} \Big]_a^b \tag{23}$$

where n = 1, 2 and 3 for the $TM_{10\ell}$, $TM_{20\ell}$ and $TM_{30\ell}$ modes respectively and B and C used for the modes in eqs. (22) and (23) are determined in terms of A using the eqs. (14)–(16) as,

$$B = \frac{j_n(k_1a)\left\{ y_n(k_2b) + bk_2 y'_n(k_2b) \right\}}{j_n(k_2a)\left\{ y_n(k_2b) - bk_2\; y'_n(k_2b) \right\} - y_n(k_2a)\left\{ j_n(k_2b) \;+\; bk_2 j'_n(k_2b) \right\}}\, A \tag{24}$$

$$C = -\frac{j_n(k_1a)\left\{ j_n(k_2b) + bk_2\; j'_n(k_2b) \right\}}{j_n(k_2a)\left\{ y_n(k_2b) \;+\; bk_2\; j'_n(k_2b) \right\} - y_n(k_2a)\left\{ j_n(k_2b) + bk_2\; j'_n(k_2b) \right\}}\, A \tag{25}$$

where j'_n and y'_n are the first derivatives of j_n and y_n, respectively, with respect to their respective arguments.

4.2. Losses :

Due to the lossy natures of the dielectrics and metallic shield there is energy loss associated with the composite shielded spherical resonator. Losses in the composite dielectric spherical shell resonator in the reasons $0 \le r \le a$ and $a \le r \le b$ can be evaluated as explained earlier [5]. The expression for the metallic loss (P_{mn}) due to a finite conductivity (σ) of the metallic shield in the shielded composite dielectric spherical resonator for the $TE_{n0\ell}$ modes is given by,

$$P_{mn} = \frac{8n(n+1)}{(2n+1)} \sqrt{\frac{1}{2\sigma\omega^3\mu_0^3}} k_2^2 b^2 \left\{ B\, j_1'(k_2 b) + C\, y_1'(k_2 b) \right\}^2 . \tag{26}$$

The dielectric loss (P_{dn}) in the shielded composite dielectric spherical resonator for the $TE_{n0\ell}$ modes can be determined from the relation,

$$P_{dn} = \omega\,(W_{1n} + W_{2n}) \tan\delta \tag{27}$$

where, $\tan\delta$ is the loss tangent of the dielectric medium. For quartz the value of loss tangent is 10^{-4} in the optical region while its value is given by $\nu/40000$ (where ν is the frequency in GHz) [15]. Similarly, the expression for the energy loss (P'_{mn}) on the metal surface for the modes of the sphere is given as,

$$P'_{mn} = \frac{8n(n+1)}{(2n+1)}\, b^2 \sqrt{\frac{\omega\ \mu_0}{2\sigma}} \left\{ Bj_1(k_2 b) + Cy_1(k_2 b) \right\}^2 . \tag{28}$$

Finally, by using the eqs. (22) and (23) the expression for the dielectric loss (P'_{dn}) for the $TM_{n0\ell}$ modes is calculated to be,

$$P'_{dn} = \omega\,(W'_{1n} + W'_{2n}) \tan\delta . \tag{29}$$

4.3. Quality factor (Q) :

The expression of the quality factor (Q_n) for the $TM_{n0\ell}$ modes is obtained as,

$$Q_n = \frac{1}{\tan\delta + g f_n(j)} \tag{30}$$

where n = 1, 2 and 3 for the $TE_{10\ell}$, $TE_{20\ell}$ and $TE_{30\ell}$ modes respectively and g and $f_n(j)$ are given by the relations,

$$g = \sqrt{\frac{2}{\sigma\omega^5\mu_0^3 \in_0^2}} \tag{31}$$

$$f_n(j) = \frac{k_2^2 b^2 \left\{ Bj_n'(k_2 b) + Cy'(k_2 b) \right\}^2}{\left\{ \begin{array}{l} \in_{r1} A^2 a^3 \left\{ j_n^2(k_1 a) - j_{n-1}(k_1 a) j_{n+1}(k_1 a) \right\} + \in_{r2} \Big[B^2 r^3 \left\{ j_n^2(k_2 r) \right. \\ \left. - j_{n-1}(k_2 r) j_{n+1}(k_2 r) \right\} + C^2 r^3 \left\{ y_n^2(k_2 r) - y_{n-1}(k_2 r) y_{n+1}(k_2 r) \right\} + BCr^3 \\ \left\{ 2y_n(k_2 r) j_n(k_2 r) - y_{n-1}(k_2 r) j_{n+1}(k_2 r) - y_{n+1}(k_2 r) j_{n-1}(k_2 r) \right\} \Big]_a^b \end{array} \right\}} . \tag{32}$$

Similarly, the expression of the quality factor (Q'_n) for the $TM_{n0\ell}$ mode is obtained as,

$$Q'_n = \frac{1}{\tan\delta + g' f'_n(j)} \tag{33}$$

where n = 1, 2, 3, for the $TM_{10\ell}$, $TM_{20\ell}$, and $TM_{30\ell}$ modes respectively and g' and $f'_n(j)$ are given by,

$$g' = \sqrt{\frac{2}{\sigma\omega\mu_0}} \tag{34}$$

$$f'_n(j) = \frac{b^2\left\{Bj_n(k_2b) + Cy_n(k_2b)\right\}^2}{\left\{\begin{array}{l}\left[A^2a^3\left\{j_n^2(k_1a) - j_{n-1}(k_1a)j_{n+1}(k_1a)\right\}\right] + \left[B^2r^3\left\{j_n^2(k_2r)\right.\right.\\ \left.-j_{n-1}(k_2r).j_{n+1}(k_2r)\right\} + C^2r^3\left\{y_n^2(k_2r) - y_{n-1}(k_2r)y_{n+1}(k_2r)\right\} + BCr^3\\ \left.\left\{2y_n(k_2r)j_n(k_2r) - y_{n-1}(k_2r)j_{n+1}(k_2r) - j_{n-1}(k_2r)y_{n+1}(k_2r)\right\}\right]_a^b\end{array}\right\}}. \tag{35}$$

5. Computations of the resonant frequencies and quality factors

Similar to the resonators discussed earlier [10] the characteristic equations for the $TE_{nm\ell}$ and $TM_{nm\ell}$ modes in the present case are also transcendental in nature and hence, to solve these one has to use numerical methods. A mode with a given value of n is n+1 fold degenerate *i.e.* (n+1) modes have the same frequency, *e.g.* $TE_{10\ell}$ and $TE_{11\ell}$ have the same resonant frequency for a given value of ℓ. We have determined five roots (= 1–5) for each of the characteristic eqs. (13) and (17) for the dielectric materials with $\in_{r1}$ = 4.0 and $\in_{r2}$ = 3.78 ($\in_{r1}$ = 4.0 for Corning glass and $\in_{r1}$ = 3.78 for quartz) and a = 0.1–0.9 μm and b = 1.0–10.0 μm. The resonant frequencies for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes (n = 1–3, ℓ = 1–5) have been determined for a = 0.1–0.9 μm and fixed b = 1.0 μm.

6. Results and discussion

The resonant frequencies for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes for the different values of radius a of the inner dielectric sphere with fixed b (1.0 μm) are collected in Tables 1 and 2 and the corresponding curves are plotted in Figures 1 and 2 respectively. Likewise, the resonant frequencies for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes for the different values of radius b of the outer dielectric sphere with fixed a (0.1 μm) are collected in Tables 3 and 4 with the corresponding curves drawn in Figures 3 and 4 respectively. With the help of eqs. (30) and (33) we have calculated the quality factors for the $TE_{10\ell}$ and $TM_{10\ell}$ modes with a = 0.1 μm and b = 1.0–10.0 μm. The values

Table 1. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$Hz) with a for $TE_{10\ell}$ modes.

a (μm)	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
0.1	1.10	1.89	2.68	3.45	4.22
0.2	1.10	1.89	2.67	3.42	4.21
0.3	1.10	1.88	2.65	3.41	4.21
0.4	1.10	1.87	2.65	3.41	4.19
0.5	1.09	1.86	2.65	3.49	4.19
0.6	1.09	1.86	2.64	3.38	4.17
0.7	1.08	1.86	2.63	3.37	4.17
0.8	1.07	1.85	2.63	3.36	4.16
0.9	1.07	1.84	2.62	3.34	4.15

b = 1.0 μm, $\in_{r1}$ = 4.0, $\in_{r2}$ = 3.78.

Table 2. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$Hz) with a for $TM_{10\ell}$ modes.

a (μm)	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
0.1	0.67	1.50	2.28	3.06	3.83
0.2	0.67	1.49	2.28	3.05	3.82
0.3	0.67	1.49	2.27	3.04	3.80
0.4	0.67	1.48	2.26	3.03	3.79
0.5	0.64	1.48	2.25	3.02	3.79
0.6	0.66	1.48	2.24	3.01	3.78
0.7	0.66	1.47	2.24	3.00	3.76
0.8	0.65	1.46	2.23	2.98	3.75
0.9	0.65	1.46	2.22	2.98	3.74

b = 1.0 μm, $\in_{r1}$ = 4.0, $\in_{r2}$ = 3.78.

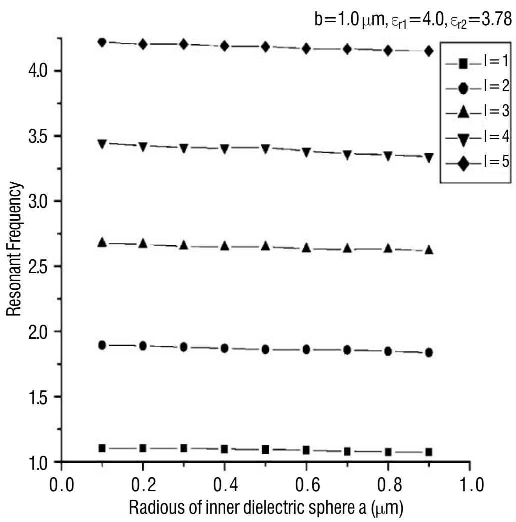

Figure 1. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$ Hz) with a (μm) for $TE_{10\ell}$ modes.

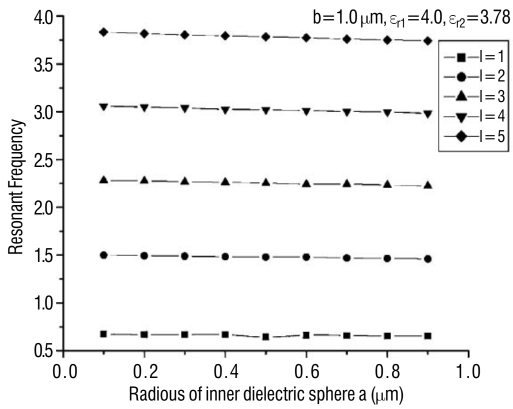

Figure 2. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$ Hz) with a (μm) for $TM_{10\ell}$ modes.

of quality factors for the $TE_{10\ell}$ and $TM_{10\ell}$ modes for the different values of radius b of the outer dielectric sphere with fixed a (0.1 μm) are collected in Tables 5 and 6. The variations of quality factor with a for the $TE_{10\ell}$ and $TM_{10\ell}$ modes are shown in Figures 5 and 6 respectively. When the outer radius (b) of the resonator increases the quality factor (Q) of the resonator increases monotonically.

The resonant frequencies have also been calculated with the parameters suitable for the microwave region as given in Ref. [18]. The resonant frequencies have been calculated by taking $\in_{r1}$ = 36.0 and $\in_{r2}$ = 1, a = 0.1–0.9 mm and b = 1.0–10.0

Table 3. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$Hz) with b for $TE_{10\ell}$ modes.

b (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
1.0	1.10	1.89	2.68	3.45	4.22
2.0	0.55	0.94	1.34	1.72	2.11
3.0	0.37	0.63	0.89	1.15	1.41
4.0	0.27	0.47	0.67	0.86	1.05
5.0	0.22	0.38	0.53	0.69	0.84
6.0	0.18	0.31	0.44	0.57	0.70
7.0	0.16	0.27	0.38	0.49	0.60
8.0	0.13	0.23	0.33	0.43	0.52
9.0	0.12	0.21	0.30	0.38	0.47
10.0	0.11	0.19	0.27	0.34	0.42

a = 0.1 μm, $\in_{r1}$ = 4.0, $\in_{r2}$ = 3.78.

Table 4. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$Hz) with b for $TM_{10\ell}$ modes.

b (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
1.0	0.67	1.50	2.28	3.06	3.83
2.0	0.33	0.75	1.14	1.53	1.92
3.0	0.22	0.50	0.76	1.02	1.28
4.0	0.17	0.37	0.57	0.76	0.96
5.0	0.13	0.30	0.45	0.61	0.76
6.0	0.11	0.25	0.38	0.51	0.64
7.0	0.10	0.21	0.33	0.43	0.54
8.0	0.08	0.19	0.28	0.38	0.48
9.0	0.07	0.16	0.25	0.34	0.43
10.0	0.07	0.15	0.22	0.31	0.38

a = 0.1 μm, $\in_{r1}$ = 4.0, $\in_{r2}$ = 3.78.

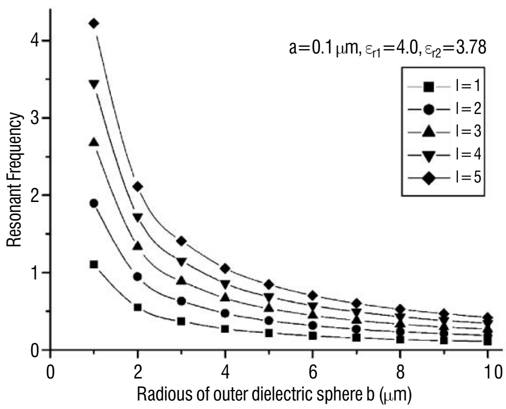

Figure 3. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$ Hz) with b (μm) for $TE_{10\ell}$ modes.

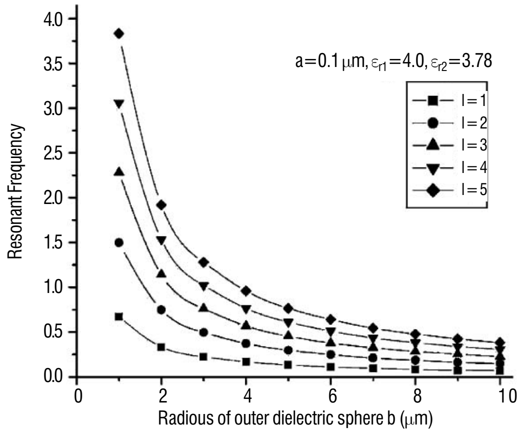

Figure 4. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-14}$ Hz) with b (μm) for $TM_{10\ell}$ modes.

mm. The loss tangent has been taken to be $\nu/40000$ [18] (where ν is the frequency in GHz). The resonant frequencies for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes (n = 1–3, ℓ = 1–5) have been determined for a = 0.1–0.9 mm and fixed b (1.0 mm). The values of resonant frequencies for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes for different values of a with b (1.0 mm) are collected in Tables 7 and 8 with the corresponding curves given in Figures 7 and 8 respectively. It is found that when the inner radius a is increased from 0.1 mm to 0.9 mm with fixed b = 1.0 mm, the resonant frequency decreases and it lies in the range ~35–730 GHz for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes (n = 1–3, b = 1–5). The quality factors for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes have also been calculated by taking $\in_{r1}$ = 36.0 and $\in_{r2}$ = 1.0, a = 0.1–0.9 mm and fixed b = 1.0 mm. The values of quality factors for $TE_{n0\ell}$ and $TM_{n0\ell}$ modes for different value of radius a of inner

Table 5. Variation of *Q*-factor with *b* for $TE_{10\ell}$ modes.

b (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
1.0	157	205	242	275	303
2.0	223	289	340	385	422
3.0	267	352	417	466	513
4.0	324	406	475	540	589
5.0	346	454	526	601	652
6.0	401	491	581	654	716
7.0	409	559	617	698	773
8.0	517	592	658	746	828
9.0	570	595	704	790	864
10.0	483	699	733	828	919

a = 0.1 μm, $\in_{r1}$ = 4.0, $\in_{r2}$ = 3.78.

Table 6. Variation of *Q*-factor with *b* for $TM_{10\ell}$ modes.

b (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
1.0	90	173	120	256	287
2.0	129	244	308	357	400
3.0	156	303	377	434	486
4.0	183	349	421	501	558
5.0	202	396	492	558	627
6.0	236	425	541	602	673
7.0	245	494	568	668	744
8.0	309	490	632	695	785
9.0	348	609	638	739	822
10.0	284	566	718	777	867

a = 0.1 μm, $\in_{r1}$ = 4.0, $\in_{r2}$ = 3.78.

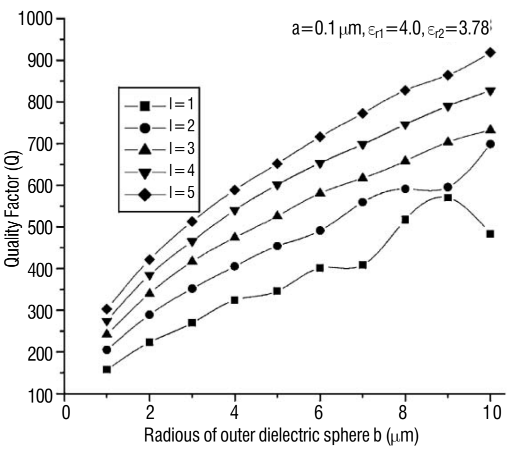

Figure 5. Variation of *Q*-factor with *b* (μm) for $TE_{10\ell}$ modes.

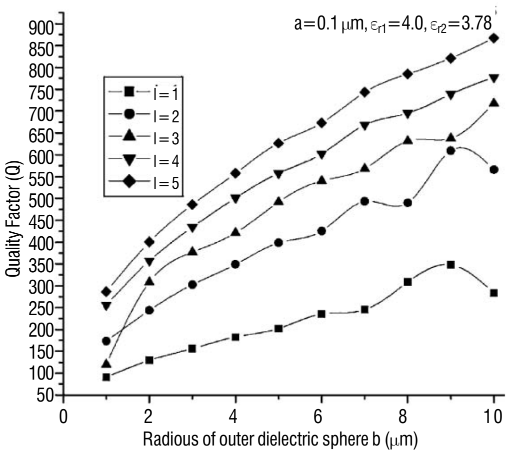

Figure 6. Variation of *Q*-factor with *b* (μm) for $TM_{10\ell}$ modes.

dielectric sphere with *b* = 1.0 mm are collected in Tables 9 and 10 respectively. The variations of quality factor with *b* for the above modes are shown in Figures 9 and 10. It is to noted be here that the quality factor increases with increasing inner radius *a* of the sphere for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes (*n* = 1–3, ℓ = 1–5). It is observed that when the inner radius is increased from 0.1 mm to 0.9 mm with fixed *b* = 1.0 mm, the quality factor increases and its magnitudes lies in the range ~50–1000 for $TE_{n0\ell}$ modes and $TM_{n0\ell}$ modes (*n* = 1–3, ℓ = 1–5).

The resonant frequencies and quality factors have also been calculated by taking $\in_{r1}$ = 36.0 and $\in_{r2}$ = 1.0, *a* = 0.72 mm and *b* =1.0–10.0 mm. The values of resonant frequencies for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes for the different values of *b* with *a* = 0.72 mm are collected in Tables 11 and 12 respectively. The graphical plots corresponding to the above Tables are shown in Figures 11 and 12. It is observed that

Table 7. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$Hz) with *a* for the $TE_{10\ell}$ modes.

a (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
0.1	212	248	374	487	540
0.2	122	218	253	365	406
0.3	87	163	231	258	332
0.4	63	124	185	241	273
0.5	52	100	149	198	246
0.6	44	85	125	166	207
0.7	40	74	108	143	179
0.8	37	66	96	127	158
0.9	36	62	89	116	143

b = 1.0 mm, $\in_{r1}$ = 36.0, $\in_{r2}$ = 1.0.

Table 8. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$Hz) with *a* for the $TM_{10\ell}$ modes.

a (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
0.1	130	283	346	427	551
0.2	125	173	257	308	400
0.3	108	124	199	253	295
0.4	84	107	153	213	263
0.5	67	96	126	173	222
0.6	56	85	108	146	187
0.7	47	75	96	127	161
0.8	41	65	87	114	143
0.9	34	57	80	105	130

b = 1.0 mm, $\in_{r1}$ = 36.0, $\in_{r2}$ = 1.0.

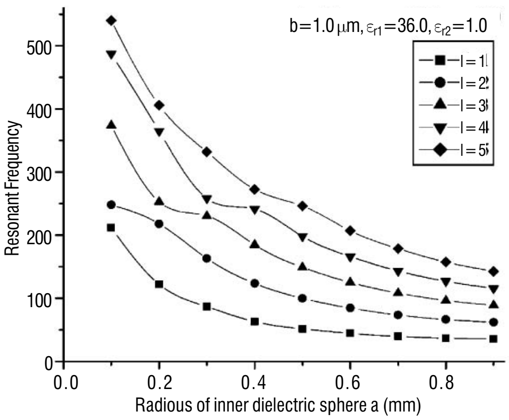

Figure 7. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$ Hz) with *a* (mm) for $TE_{10\ell}$ modes.

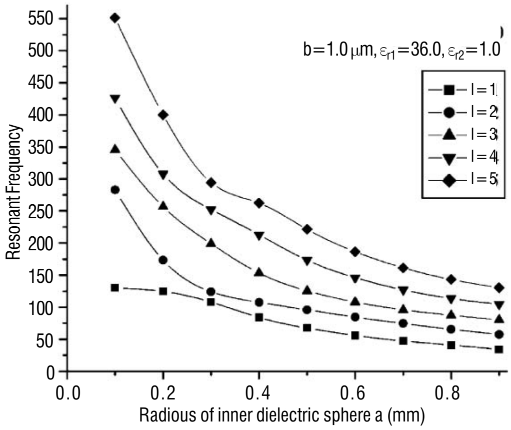

Figure 8. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$ Hz) with *a* (mm) for $TM_{10\ell}$ modes.

when the outer radius *b* increases from 1.0 mm to 10.0 mm with *a* = 0.72 mm, the resonant frequency decreases gradually and lies in the range ~10–200 GHz for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes (*n* = 1–3, ℓ = 1–5). The values of quality factors for the $TE_{n0\ell}$ and $TM_{n0\ell}$ modes for different values of radius *b* of the outer dielectric sphere with fixed *a* = 0.72 mm are collected in Tables 13 and 14 with the corresponding variations shown in Figures 13 and 14 respectively. It is observed that when the outer radius *b* is increased from 1.0 mm to 10.0 mm with fixed *a* = 0.72 mm, the quality factor also increases and lies in the range ~200–1800 for the $TE_{n0\ell}$ modes and in the range 200–2700 for the $TM_{n0\ell}$ modes (*n* = 1–3, ℓ = 1–5).

A perusal of the Tables 1–2 shows that the resonant frequencies for the different modes lie in the range 1 × 10^{14} – 4 × 10^{14} Hz which is the near IR region of the electromagnetic spectrum. The maximum value of the frequency is practically

Table 9. Variation of *Q*-factor with *a* for the $TE_{10\ell}$ modes.

a (μm)	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
0.1	184	161	106	82	74
0.2	327	179	158	109	98
0.3	482	245	171	153	121
0.4	627	323	216	165	144
0.5	753	399	268	202	162
0.6	845	469	319	241	193
0.7	891	533	367	279	224
0.8	818	574	408	313	253
0.9	820	574	429	338	277

b = 1.0 mm, ϵ_{r1} = 36.0, ϵ_{r2} = 1.0.

Table 10. Variation of *Q*-factor with *a* for the $TM_{10\ell}$ Modes.

a (μm)	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
0.1	287	139	116	93	72
0.2	301	230	154	130	99
0.3	358	311	200	156	135
0.4	470	349	260	187	150
0.5	587	401	315	230	180
0.6	706	441	362	273	214
0.7	826	497	397	312	247
0.8	941	562	424	344	277
0.9	1050	614	441	361	299

b = 1.0 mm, ϵ_{r1} = 36.0, ϵ_{r2} = 1.0.

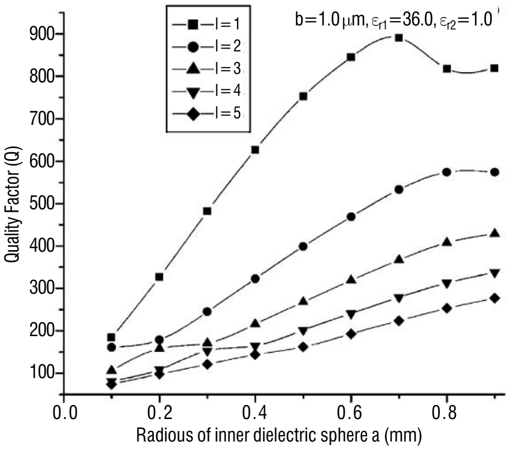

Figure 9. Variation of *Q*-factor with *a* (mm) for $TE_{10\ell}$ modes.

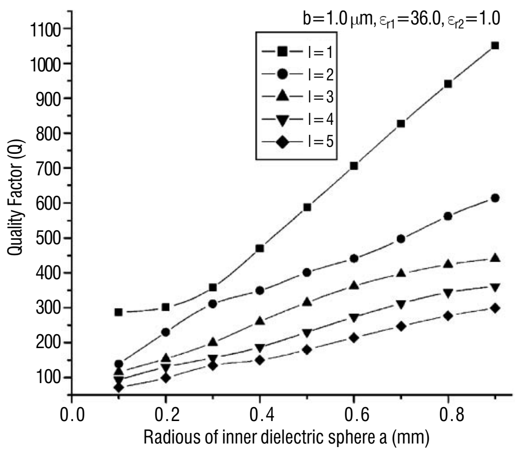

Figure 10. Variation of *Q*-factor with *a* (mm) for $TM_{10\ell}$ modes.

determined by the radius (*b*) of the outer sphere as long as the permittivities of the two dielectric materials are not too different. In order to have the resonant frequencies in the optical region the size of the outer sphere (*b*) should lie in the range 10^{-1}–10^{-2} μm. The Tables 3–4 show that as the radius of the outer sphere (*b*) increases from 1 μm to 10 μm the resonant frequency decreases to lie in the IR region of the spectrum. From the Tables 5–6 it is clear that the quality factor (*Q*) for the higher frequency (near IR) range is of the order of 10^2 whereas for the lower frequency (IR) range it is of the order of 10^3.

For the microwave region the resonant frequencies in the range 10^2–10^3 GHz can be achieved by choosing the radius (*b*) of the outer sphere in the range 0.1–1.0 mm. The *Q* values obtained here are of the order of those obtained for spherical resonators in cavities [19,23]. Micro-machined hemispherical cavity resonators with

Table 11. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$Hz) with b for the $TE_{10\ell}$ modes.

b (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
1.0	39	72	106	140	174
2.0	34	68	102	126	141
3.0	34	66	78	103	133
4.0	34	55	69	96	106
5.0	33	44	67	77	102
6.0	33	37	62	69	90
7.0	30	34	53	68	77
8.0	26	34	46	64	70
9.0	23	33	42	58	68
10.0	21	33	37	53	66

a = 0.72 mm, $\in_{r1}$ = 36.0, $\in_{r2}$ = 1.0.

Table 12. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$Hz) with b for the $TE_{10\ell}$ modes.

b (μm)	ℓ = 1	ℓ = 2	ℓ = 3	ℓ = 4	ℓ = 5
1.0	46	73	94	124	157
2.0	46	56	85	116	134
3.0	40	49	79	90	119
4.0	32	48	66	84	102
5.0	25	47	55	78	87
6.0	22	50	49	69	83
7.0	18	40	48	61	78
8.0	16	35	47	54	70
9.0	14	32	46	50	66
10.0	13	29	43	48	58

a = 0.72 mm, $\in_{r1}$ = 36.0, $\in_{r2}$ = 1.0.

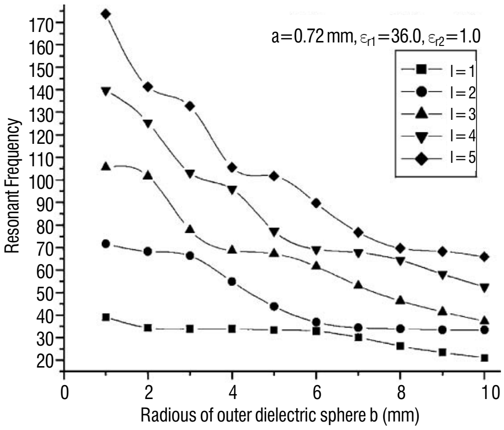

Figure 11. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$ Hz) with b (mm) for $TE_{10\ell}$ modes.

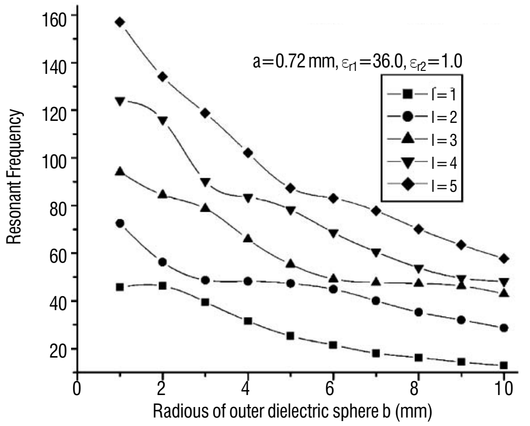

Figure 12. Variation of resonant frequency ($\nu_{10\ell} \times 10^{-9}$ Hz) with b (mm) for $TM_{10\ell}$ modes.

radii ~2 mm are also reported to have frequencies ~10^2 GHz and Q values ~ [24]. In the present case as the permittivity of the inner sphere is much larger ($36\varepsilon_0$) as compared to the outer one (ε_0), the variation of the inner radius (a) causes large variation in the resonant frequency (Tables 7–8), whereas the variation of b has insignificant effect on the magnitude of the resonant frequency (Tables 11–12). Similar to the case of the optical range (Tables 5–6) the Q value increases with the increasing b and has magnitude in the range 10^2–10^3. Contrary to the case of optical region where variation of a has little effect on the quality factor and resonant frequency owing to the closer magnitudes of the permittivities of the two dielectrics, in this case, however, the quality factor (Q) depends on the inner radius (a) strongly due to widely differing permittivities of the two spheres.

Table 13. Variation of *Q*-factor with *b* for the $TE_{10\ell}$ modes.

b (μm)	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
1.0	893	544	376	285	230
2.0	1149	585	392	311	281
3.0	1175	597	495	387	299
4.0	1177	694	580	410	377
5.0	1192	861	592	505	392
6.0	1193	1027	636	575	439
7.0	1259	1150	742	589	511
8.0	1435	1176	844	613	570
9.0	1600	1193	931	674	585
10.0	1779	1190	1036	747	603

a = 0.72 mm, $\in_{r1}$ = 36.0, $\in_{r2}$ = 1.0.

Table 14. Variation of *Q*-factor with *b* for the $TE_{10\ell}$ modes.

b (μm)	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
1.0	826	511	403	319	253
2.0	844	653	471	342	290
3.0	927	811	500	435	335
4.0	1204	827	588	477	384
5.0	1498	843	698	505	454
6.0	1598	874	798	577	480
7.0	2031	964	835	645	509
8.0	2052	1113	844	724	564
9.0	2398	1197	853	793	620
10.0	2655	1391	911	827	681

a = 0.72 mm, $\in_{r1}$ = 36.0, $\in_{r2}$ = 1.0.

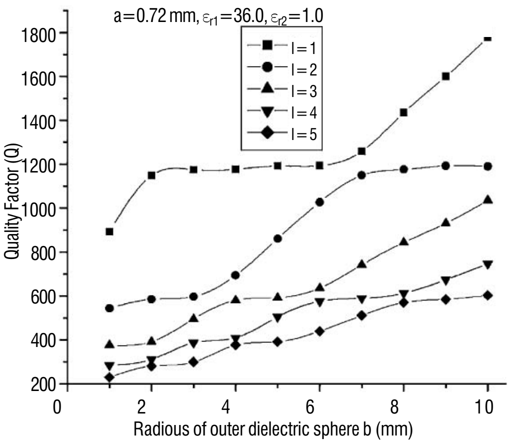

Figure 13. Variation of *Q*-factor with *b* (mm) for $TE_{10\ell}$ modes.

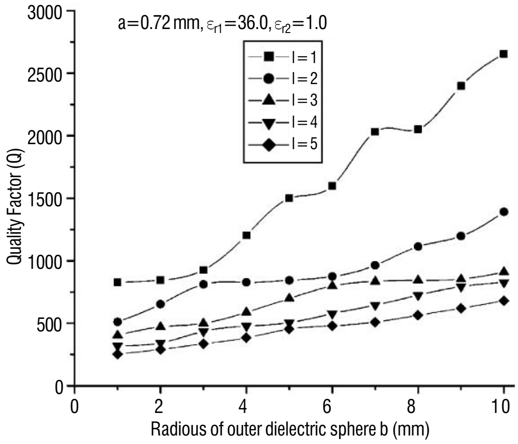

Figure 14. Variation of *Q*-factor with *b* (mm) for $TM_{10\ell}$ modes.

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