Problem Sets

X

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1 Localization

Definition 1.1. Let R be a ring, S a subset of R. We say S is a *multiplicative subset of* R if $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.

```
import Mathlib.Algebra.Ring.Defs
import Mathlib.Algebra.Group.Submonoid.Defs

variable {R : Type*} [Ring R]

-- use Submonoid definition from Mathlib.Algebra.Group.Submonoid.Defs
variable (S : Submonoid R)
```

Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:

$$(x, s) \sim (y, t) \Leftrightarrow \exists u \in S \text{ such that } (xt - ys)u = 0$$

It is easily checked that this is an equivalence relation. Let x/s (or $\frac{x}{s}$) be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st, \quad x/s \cdot y/t = xy/st$$

One can check that $S^{-1}A$ becomes a ring under these operations.

Definition 1.2. This ring is called the *localization of* A *with respect to* S.

```
import Mathlib.RingTheory.Localization.Defs

variable {A : Type*} [CommRing A]
variable (S : Submonoid A)

abbrev LocalizationResp (S : Submonoid A) : Type _ := Localization S
```

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the *localization map*. In general the localization map is not injective, unless S contains no zerodivisors. For, if $x/1 = 0$, then there is a $u \in S$ such that $xu = 0$ in A and hence $x = 0$ since there are no zerodivisors in S. The localization of a ring has the following universal property.

Proposition 1.3. *Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.*

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] & & B \\
& S^{-1}A \arrow[ur, "g"'] &
\end{tikzcd}
$$

```
import Mathlib

open scoped Classical

variable {A : Type*} [CommRing A] (S : Submonoid A)

/-- Universal property of localization:
Let $f : A \to B$ be a ring map that sends every element in $S$ to a unit
of $B$. Then there is a unique homomorphism $g : S^{-1}A \to B$ such
that the following diagram commutes. -/
theorem localization_universal_property {B : Type*} [CommRing B] (f : A →+* B)
    (hf :   s   S, IsUnit (f s)) :
     ! g : Localization S →+* B, g.comp (algebraMap A (Localization S)) = f := by
  -- Existence: Construct g using IsLocalization.lift
  -- We need to adapt hf to take an element of the subtype S
  let hf' :   (y : S), IsUnit (f y) := fun y => hf y y.2
  refine  IsLocalization.lift hf', ?_, ?_
  · -- Commutativity: g   algebraMap = f
    ext x
    simp only [RingHom.comp_apply]
    rw [IsLocalization.lift_eq]
  · -- Uniqueness: If g'   algebraMap = f, then g' = g
    intro g' hg'
    apply IsLocalization.ringHom_ext S
    rw [hg']
    ext x
    simp only [RingHom.comp_apply]
    rw [IsLocalization.lift_eq]
```

Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. $\square$

Lemma 1.4. *The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$.*

```
import Mathlib

open scoped Classical

variable {R : Type*} [CommRing R] (S : Submonoid R)

/-- The localization `S ¹R` is the zero ring (i.e. a subsingleton) iff `0   S`. -/
```

```
theorem localization_subsingleton_iff :
    Subsingleton (Localization S)   (0 : R)   S := by
  constructor
  · intro h
    -- If S ¹R is subsingleton, then 1 = 0 in S ¹R
    have h1 : (1 : Localization S) = 0 := Subsingleton.elim 1 0
    -- This means 1 maps to 0
    rw [← map_one (algebraMap R (Localization S))] at h1
    -- By localization property, exists s   S such that s * 1 = 0
    rw [IsLocalization.map_eq_zero_iff S (Localization S) 1] at h1
    rcases h1 with  s, hs
    simp only [mul_one] at hs
    -- s = 0, so 0   S
    rw [← hs]
    exact s.prop
  · intro h
    -- To show S ¹R is subsingleton, it suffices to show 0 = 1
    apply subsingleton_of_zero_eq_one
    symm
    -- We show 1 = 0. This is true if 1 maps to 0.
    rw [← map_one (algebraMap R (Localization S)), IsLocalization.map_eq_zero_iff S
    ↪  (Localization S) 1]
    -- We witness this with 0   S, since 0 * 1 = 0
    exact   0, h , by simp
```

Proof. If $0 \in S$, any pair $(a, s) \sim (0, 1)$ by definition. If $0 \notin S$, then clearly $1/1 \neq 0/1$ in $S^{-1}A$. □

Lemma 1.5. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.*

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. □

The notion of localization of a ring can be generalized to the localization of a module. Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows

$$(m, s) \sim (n, t) \Leftrightarrow \exists u \in S \text{ such that } (mt - ns)u = 0$$

This is clearly an equivalence relation. Denote by m/s (or $\frac{m}{s}$) be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st, \quad m/s \cdot n/t = mn/st$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$-module.

Definition 1.6. The $S^{-1}A$-module $S^{-1}M$ is called the *localization* of M at S.

```
import Mathlib.Algebra.Module.LocalizedModule.Basic

variable {R : Type*} [CommRing R]
variable (S : Submonoid R)
```

```
variable {M : Type*} [AddCommGroup M]
variable [Module R M]

abbrev ModuleLocalization : Type _ := LocalizedModule S M
```

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is sometimes called the *localization map*. It satisfies the following universal property.

Lemma 1.7. *Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then the canonical map*

$$\mathrm{Hom}_R(S^{-1}M, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map, is an isomorphism.

```
import Mathlib

open IsLocalizedModule

variable {R : Type*} [CommRing R] (S : Submonoid R)
variable {M : Type*} [AddCommGroup M] [Module R M]
variable {N : Type*} [AddCommGroup N] [Module R N]
variable {M' : Type*} [AddCommGroup M'] [Module R M']
variable (f : M →ₗ[R] M') [IsLocalizedModule S f]

/-- The canonical map Hom_R(S ¹M, N) → Hom_R(M, N) is an isomorphism if S acts by
↪  automorphisms on N. -/
noncomputable def localizationHomEquiv
    (h :   s : S, Function.Bijective (fun n : N   s • n)) :
    (M' →ₗ[R] N)   [R] (M →ₗ[R] N) :=
  let h_units :   s : S, IsUnit (s • (1 : Module.End R N)) := fun s   by
    -- Construct the linear map corresponding to multiplication by s
    let   := s • (1 : Module.End R N)
    -- It is bijective by hypothesis
    have h  : Function.Bijective   := h s
    -- Construct the linear equivalence from the bijective linear map
    let e := LinearEquiv.ofBijective   h
    -- Show that   is a unit in the endomorphism ring by providing the inverse
    have h1 :  .comp e.symm.toLinearMap = 1 := LinearMap.ext fun x   e.apply_symm_apply x
    have h2 : e.symm.toLinearMap.comp   = 1 := LinearMap.ext fun x   e.symm_apply_apply x
    exact   , e.symm.toLinearMap, h1, h2 , rfl
  { toFun := fun g   g.comp f
    map_add' := fun _ _   rfl
    map_smul' := fun _ _   rfl
    invFun := fun l   IsLocalizedModule.lift S f l h_units
    left_inv := fun g   by
      -- We need to show lift (g   f) = g.
      -- lift_unique says: if l'   f = l, then l' = lift l.
      -- Here l = g   f, so we need lift (g   f) = g.
      apply IsLocalizedModule.lift_unique S f (g.comp f) h_units g
      rfl
    right_inv := fun l   by
      -- We need to show lift l   f = l.
      exact IsLocalizedModule.lift_comp S f l h_units }
```

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in \mathsf{Hom}_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}\beta(m)$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned} \alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s' \cdot m + r' \cdot s \cdot m')/(ss')) \\ &= (ss')^{-1}\beta(r \cdot s' \cdot m + r' \cdot s \cdot m') \\ &= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m')) \\ &= r\alpha(m/s) + r'\alpha(m'/s') \end{aligned}$$

and we win. □

Example 1.8. Let A be a ring and let M be an A-module. Here are some important examples of localizations.

1. Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_\mathfrak{p}$ and $M_\mathfrak{p}$ the localization of A and M with respect to S respectively. These are called the *localization of A, resp. M at $\mathfrak{p}$*.

2. Let $f \in A$. Consider $S = \{1, f, f^2, \ldots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the *localization of A, resp. M with respect to f*. Note that $A_f = 0$ if and only if f is nilpotent in A.

3. Let $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the *total quotient ring*, or the *total ring of fractions* of A.

4. If A is a domain, then the total quotient ring $Q(A)$ is the field of fractions of A. Please see Fields, Example **??**.

Lemma 1.9. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then*

$$S^{-1}M = \mathsf{colim}_{f \in S} M_f$$

where the preorder on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f'' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Hint: Use the universal property of Lemma **??**. □

In the following paragraph, let A denote a ring, and M, N denote modules over A.

If S and S' are multiplicative sets of A, then it is clear that

$$SS' = \{ss' : s \in S, \ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

Proposition 1.10. *Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.*

```
import Mathlib
```

```
open Algebra

/-- Let `A` be a commutative ring and `S S' : Submonoid A` two multiplicative subsets.
    Denote by `\overline{S}` the image of `S` in the localization `Localization S'`.
    Then localizing `A` at the product submonoid `S  S'` is canonically
    algebra-isomorphic (as an `A`-algebra) to first localising at `S'` and then
    localising the result at `\overline{S}`. -/
def localization_mul_isomorphic {A : Type*} [CommRing A] (S S' : Submonoid A) :
    Localization (S  S')  [A]
      Localization (Submonoid.map ((algebraMap A (Localization S')).toMonoidHom) S) :=
by
  sorry
```

Proof. The map sending $x \in A$ to $x/1 \in (SS')^{-1}A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS')^{-1}A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS')^{-1}A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS')^{-1}A$ which maps $(x/s')/(s/1)$ to x/ss'.

On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS')^{-1}A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms. □

For the module M we have

Proposition 1.11. *View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.*

```
import Mathlib

open Submonoid

variable {A : Type*} [CommRing A]
variable (S S' : Submonoid A)
variable {M : Type*} [AddCommMonoid M] [Module A M]

/--
Iterated localization of an `A`-module.  If we first localize `M` at a multiplicative
subset `S'  A` (viewed as an `A`-module via the canonical algebra map) and then
localize the result at another multiplicative subset `S`, the resulting module is
canonically linear-equivalent (as an `A`-module) to the localization of `M` at the
supremum submonoid `S  S'`. -/
def iterated_localization_eq :
    LocalizedModule S (LocalizedModule S' M)  [A] LocalizedModule (S  S') M := by
  sorry
```

Proof. Note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly.

We define the maps as follows

$$f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss'$$

$$g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other. □

If $u : M \to N$ is an A homomorphism, then the localization indeed induces a well-defined $S^{-1}A$ homomorphism $S^{-1}u : S^{-1}M \to S^{-1}N$ which sends x/s to $u(x)/s$. It is immediately checked that this construction is functorial, so that S^{-1} is actually a functor from the category of A-modules to the category of $S^{-1}A$-modules. Moreover this functor is exact, as we show in the following proposition.

Proposition 1.12. *Let $L \xrightarrow{u} M \xrightarrow{v} N$ be an exact sequence of R-modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.*

```
import Mathlib

-- Open the Function namespace to use `Exact` and LocalizedModule for `map`.
open Function LocalizedModule

section LocalizationExact

-- Define the variables: R is a commutative ring, S is a multiplicative subset.
variable {R : Type*} [CommRing R] (S : Submonoid R)
-- L, M, N are R-modules.
variable {L M N : Type*} [AddCommGroup L] [AddCommGroup M] [AddCommGroup N]
variable [Module R L] [Module R M] [Module R N]
-- u and v are R-linear maps forming the sequence L -> M -> N.
variable (u : L →ₗ[R] M) (v : M →ₗ[R] N)

-- The theorem statement: If u and v are exact, then their localizations are exact.
theorem localization_exact (h : Exact u v) :
    Exact (map S u) (map S v) := by
  -- The lemma `LocalizedModule.map_exact` in Mathlib asserts that the sequence of
  -- localized maps is exact. It takes the multiplicative subset S, the maps u and v,
  -- and the exactness hypothesis h as arguments.
  -- Note: LocalizedModule.map S u is definitionally equal to the map constructed in the
  ↪  lemma.
  apply LocalizedModule.map_exact S u v h

end LocalizationExact
```

Proof. First it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in N, which means $xt \in \mathsf{Ker}(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness. □

Lemma 1.13. *Localization respects quotients, i.e. if N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.*

Proof. From the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows. □

If, in the preceding Corollary, we take $N = I$ and $M = A$ for an ideal I of A, we see that $S^{-1}A/S^{-1}I \simeq S^{-1}(A/I)$ as A-modules. The next proposition shows that they are isomorphic as rings.

Proposition 1.14. *Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.*

```
import Mathlib

open Algebra

variable {A : Type*} [CommRing A] (I : Ideal A) (S : Submonoid A)

/-- The localization of an ideal `I` at a multiplicative set `S`,
    i.e. the image of `I` under the canonical map `A → Localization S`. -/
def localizedIdeal : Ideal (Localization S) :=
  Ideal.map (algebraMap A (Localization S)) I

/-- Let `\overline{S}` be the image of a multiplicative set `S   A`
    in the quotient ring `A   I`. Then the localization of `A   I`
    at `\overline{S}` is canonically isomorphic to the quotient
    `(Localization S)   localizedIdeal I S`. -/
noncomputable def ideal_localization_quotient_iso :
    (Localization ((Submonoid.map ((Ideal.Quotient.mk I).toMonoidHom) S))) +*
      ((Localization S)   localizedIdeal I S) := by
  sorry
```

Proof. The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms. ☐

We now consider how submodules behave in localization.

Lemma 1.15. *Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M.*

Proof. Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. ☐

Taking $M = A$ and $N = I$ an ideal of A, we have the following corollary, which can be viewed as a converse of the first part of Proposition **??**.

Lemma 1.16. *Each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.*

```
import Mathlib
```

```
open Algebra

variable {A : Type*} [CommRing A] (M : Submonoid A)
variable {S : Type*} [CommRing S] [Algebra A S]

/-- Every ideal of the localization `S` is the localization of an ideal of `A`. -/
theorem ideals_in_localization_are_localizations
  [IsLocalization M S] -- Explicitly include the instance to bring M into scope
  (I' : Ideal S) :
    I : Ideal A, I' = Ideal.map (algebraMap A S) I := by
  -- We propose the contraction of I' as the candidate ideal I
  refine  Ideal.comap (algebraMap A S) I', ?_
  -- The fact that I' is the extension of its contraction is a standard theorem
  -- IsLocalization.map_comap states: map (comap I') = I'
  symm
  exact IsLocalization.map_comap M S I'
```

Proof. Immediate from Lemma **??**. □

2 Tensor products

Definition 2.1. Let R be a ring, M, N, P be three R-modules. A mapping $f : M \times N \to P$ (where $M \times N$ is viewed only as Cartesian product of two R-modules) is said to be R-*bilinear* if for each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is R-linear, and for each $y \in N$ the mapping $x \mapsto f(x, y)$ is also R-linear.

```
import Mathlib.LinearAlgebra.BilinearMap

variable {R : Type*} [CommRing R]
variable {M N P : Type*}
  [AddCommMonoid M] [Module R M]
  [AddCommMonoid N] [Module R N]
  [AddCommMonoid P] [Module R P]
variable (f : M → [R] N → [R] P) -- f is a bilinear map
```

Lemma 2.2. *Let M, N be R-modules. Then there exists a pair (T, g) where T is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : T \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:*

$$\begin{tikzcd} M \times N \arrow[rr, "f"] \arrow[dr, "g"'] & & P \\ & T \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

```
import Mathlib

open TensorProduct
```

```
variable {R : Type*} [CommSemiring R]
variable {M N P : Type*}
  [AddCommMonoid M] [Module R M]
  [AddCommMonoid N] [Module R N]
  [AddCommMonoid P] [Module R P]

/-- Universal property of the tensor product.
For any bilinear map `f : M → [R] N → [R] P` there exists a unique linear map
`g : M  [R] N → [R] P` such that `g (m   n) = f m n` for all `m n`. -/
theorem tensor_product_universal_property
  (f : M → [R] N → [R] P) :
    ! g : M  [R] N → [R] P,   m n, g (TensorProduct.tmul _ m n) = f m n := by
  sorry
```

The R-module T which satisfies the above universal property is called the *tensor product* of R-modules M and N, denoted as $M \otimes_R N$.

Proof. We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{(M\times N)}$ and Q is the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$\begin{aligned}(x + x', y) - (x, y) - (x', y),\\ (x, y + y') - (x, y) - (x, y'),\\ (ax, y) - a(x, y),\\ (x, ay) - a(x, y)\end{aligned}$$

Let $\pi : M \times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x, y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x, y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence $j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism. □

Lemma 2.3. *Let M, N, P be R-modules, then the bilinear maps*

$$\begin{aligned}(x, y) &\mapsto y \otimes x\\ (x + y, z) &\mapsto x \otimes z + y \otimes z\\ (r, x) &\mapsto rx\end{aligned}$$

induce unique isomorphisms

$$\begin{aligned}M \otimes_R N &\to N \otimes_R M,\\ (M \oplus N) \otimes_R P &\to (M \otimes_R P) \oplus (N \otimes_R P),\\ R \otimes_R M &\to M\end{aligned}$$

Proof. Omitted. □

We may generalize the tensor product of two R-modules to finitely many R-modules, and set up a correspondence between the multi-tensor product with multilinear mappings. Using almost the same construction one can prove that:

Lemma 2.4. *Let $M_1, \ldots, M_r$ be R-modules. Then there exists a pair (T, g) consisting of an R-module T and an R-multilinear mapping $g : M_1 \times \ldots \times M_r \to T$ with the universal property: For any R-multilinear mapping $f : M_1 \times \ldots \times M_r \to P$ there exists a unique R-module homomorphism $f' : T \to P$ such that $f' \circ g = f$. Such a module T is unique up to unique isomorphism. We denote it $M_1 \otimes_R \ldots \otimes_R M_r$ and we denote the universal multilinear map $(m_1, \ldots, m_r) \mapsto m_1 \otimes \ldots \otimes m_r$.*

```
import Mathlib

open MultilinearMap

/-- Existence of a universal multilinear map (tensor product) for a finite family of
    `R`-modules indexed by a fintype ` `. -/
theorem exists_tensor_product
    {R : Type*} [CommSemiring R]
    {  : Type*} [Fintype  ]
    (M :   → Type*) [  i, AddCommMonoid (M i)] [  i, Module R (M i)] :
      (T : Type*) (_ : AddCommMonoid T) (_ : Module R T),
        g : MultilinearMap R M T,
          {P : Type*} [AddCommMonoid P] [Module R P]
          (f : MultilinearMap R M P),
          ! f' : T → [R] P,   m, f' (g m) = f m := by
  sorry
```

Proof. Omitted. □

Lemma 2.5. *The homomorphisms*

$$(M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P \to M \otimes_R (N \otimes_R P)$$

such that $f((x \otimes y) \otimes z) = x \otimes y \otimes z$ and $g(x \otimes y \otimes z) = x \otimes (y \otimes z)$, $x \in M, y \in N, z \in P$ are well-defined and are isomorphisms.

```
import Mathlib

open scoped TensorProduct

variable {R : Type*} [CommSemiring R]
variable {M N P : Type*}
  [AddCommMonoid M] [Module R M]
  [AddCommMonoid N] [Module R N]
  [AddCommMonoid P] [Module R P]

/-- The canonical associator
    `((M  [R] N)  [R] P)  [R] M  [R] (N  [R] P)` is a well-defined linear equivalence. -/
noncomputable def tensor_product_associator_iso :
    ((M  [R] N)  [R] P)  [R] M  [R] (N  [R] P) :=
  TensorProduct.assoc R M N P

-- Verification that the map behaves as expected on pure tensors
example (x : M) (y : N) (z : P) :
    tensor_product_associator_iso (R := R) ((x   y)   z) = x   (y   z) := by
  -- The equality is not definitional, so we unfold the definition and apply the lemma
  simp [tensor_product_associator_iso, TensorProduct.assoc_tmul]
```

Proof. We shall prove f is well-defined and is an isomorphism, and this proof carries analogously to g. Fix any $z \in P$, then the mapping $(x, y) \mapsto x \otimes y \otimes z$, $x \in M, y \in N$, is R-bilinear in x and y, and hence induces homomorphism $f_z : M \otimes N \to M \otimes N \otimes P$ which sends $f_z(x \otimes y) = x \otimes y \otimes z$. Then consider $(M \otimes N) \times P \to M \otimes N \otimes P$ given by $(w, z) \mapsto f_z(w)$. The map is R-bilinear and thus induces $f : (M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P$ and $f((x \otimes y) \otimes z) = x \otimes y \otimes z$. To construct the inverse, we note that the map $\pi : M \times N \times P \to (M \otimes N) \otimes P$ is R-trilinear. Therefore, it induces an R-linear map $h : M \otimes N \otimes P \to (M \otimes N) \otimes P$ which agrees with the universal property. Here we see that $h(x \otimes y \otimes z) = (x \otimes y) \otimes z$. From the explicit expression of f and h, $f \circ h$ and $h \circ f$ are identity maps of $M \otimes N \otimes P$ and $(M \otimes N) \otimes P$ respectively, hence f is our desired isomorphism. □

Doing induction we see that this extends to multi-tensor products. Combined with Lemma **??** we see that the tensor product operation on the category of R-modules is associative, commutative and distributive.

Definition 2.6. An abelian group N is called an (A, B)-*bimodule* if it is both an A-module and a B-module and for all $a \in A$ and $b \in B$ the multiplication by a and b commute, so $b(an) = a(bn)$ for all $n \in N$. In this situation we usually write the B-action on the right: so for $b \in B$ and $n \in N$ the result of multiplying n by b is denoted nb. With this convention the compatibility above is that $(ax)b = a(xb)$ for all $a \in A, b \in B, x \in N$. The shorthand ${}_AN_B$ is used to denote an (A, B)-bimodule N.

```
import Mathlib.Algebra.Ring.Opposite
import Mathlib.Algebra.Module.Basic

variable {A B N : Type*} [Ring A] [Ring B] [AddCommGroup N]

-- left A-Module
variable [Module A N]

-- right B-Module, using the opposite ring
variable [Module B   N]

-- compatibility condition, (op b) • (a • n) = a • ((op b) • n)
variable [SMulCommClass A B   N]

-- N is an (A,B)-bimodule
variable [Module A N] [Module B   N] [SMulCommClass A B   N]
```

Lemma 2.7. *For A-module M, B-module P and (A, B)-bimodule N, the modules $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$ can both be given (A, B)-bimodule structure, and moreover*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

Proof. A priori $M \otimes_A N$ is an A-module, but we can give it a B-module structure by letting

$$(x \otimes y)b = x \otimes yb, \quad x \in M, y \in N, b \in B$$

Thus $M \otimes_A N$ becomes an (A, B)-bimodule. Similarly for $N \otimes_B P$, and thus for $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$. By Lemma **??**, these two modules are isomorphic as both as A-module and B-module via the same mapping. □

Lemma 2.8. *For any three R-modules M, N, P,*

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

```
import Mathlib

open TensorProduct

variable {R : Type*} [CommSemiring R]

variable {M N P : Type*}
variable [AddCommMonoid M] [Module R M]
variable [AddCommMonoid N] [Module R N]
variable [AddCommMonoid P] [Module R P]

/-- Tensor-Hom adjunction: linear maps out of `M  [R] N` are naturally
equivalent to linear maps from `M` into the space of linear maps from `N` to `P`. -/
def tensor_hom_adj :
    (M  [R] N → [R] P)  [R] M → [R] N → [R] P := by
  sorry
```

Proof. An R-linear map $\hat{f} \in \mathsf{Hom}_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x, y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to \mathsf{Hom}_R(N, P)$. This map is R-linear since

$$\phi_f(ax + y)(z) = f(ax + y, z) = af(x, z) + f(y, z) = (a\phi_f(x) + \phi_f(y))(z),$$

for all $a \in R$, $x \in M$, $y \in M$ and $z \in N$. Conversely, any $f \in \mathsf{Hom}_R(M, \mathsf{Hom}_R(N, P))$ defines an R-bilinear map $M \times N \to P$, namely $(x, y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules $\mathsf{Hom}_R(M \otimes_R N, P)$ and $\mathsf{Hom}_R(M, \mathsf{Hom}_R(N, P))$. $\square$

Lemma 2.9 (Tensor products commute with colimits)**.** *Let (M_i, μ_{ij}) be a system over the preordered set I. Let N be an R-module. Then*

$$\mathsf{colim}(M_i \otimes N) \cong (\mathsf{colim}\, M_i) \otimes N.$$

Moreover, the isomorphism is induced by the homomorphisms $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ where $M = \mathsf{colim}_i\, M_i$ with natural maps $\mu_i : M_i \to M$.

Proof. First proof. The functor $M' \mapsto M' \otimes_R N$ is left adjoint to the functor $N' \mapsto \mathsf{Hom}_R(N, N')$ by Lemma **??**. Thus $M' \mapsto M' \otimes_R N$ commutes with all colimits, see Categories, Lemma **??**.

Second direct proof. Let $P = \mathsf{colim}(M_i \otimes N)$ with coprojections $\lambda_i : M_i \otimes N \to P$. Let $M = \mathsf{colim}\, M_i$ with coprojections $\mu_i : M_i \to M$. Then for all $i \leq j$, the following diagram commutes:

$$\begin{tikzcd}
M_i \otimes N \arrow[r, "\mu_i \otimes 1"'] \arrow[d, "\mu_{ij} \otimes 1"'] & M \otimes N \arrow[d, "\mathrm{id}"] \\
M_j \otimes N \arrow[r, "\mu_j \otimes 1"] & M \otimes N
\end{tikzcd}$$

By Lemma **??** these maps induce a unique homomorphism $\psi : P \to M \otimes N$ such that $\mu_i \otimes 1 = \psi \circ \lambda_i$.

To construct the inverse map, for each $i \in I$, there is the canonical R-bilinear mapping $g_i : M_i \times N \to M_i \otimes N$. This induces a unique mapping $\widehat{\phi} : M \times N \to P$ such that $\widehat{\phi} \circ (\mu_i \times 1) = \lambda_i \circ g_i$. It is R-bilinear. Thus it induces an R-linear mapping $\phi : M \otimes N \to P$. From the commutative diagram below:

$$\begin{tikzcd}
M_i \times N \arrow[r, "g_i"] \arrow[d, "\mu_i \times \mathrm{id}"] & M_i \otimes N \arrow[r, "\mathrm{id}"'] \arrow[d, "\lambda_i"'] & M_i \otimes N \arrow[d, "\mu_i \otimes \mathrm{id}"'] \arrow[dr, "\lambda_i"] & \\
M \times N \arrow[r, "\widehat{\phi}"] & P \arrow[r, "\psi"] & M \otimes N \arrow[r, "\phi"] & P
\end{tikzcd}$$

we see that $\psi \circ \widehat{\phi} = g$, the canonical R-bilinear mapping $g : M \times N \to M \otimes N$. So $\psi \circ \phi$ is identity on $M \otimes N$. From the right-hand square and triangle, $\phi \circ \psi$ is also identity on P. ☐

Lemma 2.10. *Let*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

$$M_1 \otimes N \xrightarrow{f \otimes 1} M_2 \otimes N \xrightarrow{g \otimes 1} M_3 \otimes N \to 0 \tag{2.10.1}$$

is exact. In other words, the functor $- \otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. For every R-module P we apply the functor $\mathsf{Hom}(-, \mathsf{Hom}(N, P))$ to the first exact sequence. We obtain

$$0 \to \mathsf{Hom}(M_3, \mathsf{Hom}(N, P)) \to \mathsf{Hom}(M_2, \mathsf{Hom}(N, P)) \to \mathsf{Hom}(M_1, \mathsf{Hom}(N, P))$$

which is exact by Lemma **??** (1). By Lemma **??** this becomes the sequence

$$0 \to \mathsf{Hom}(M_3 \otimes N, P) \to \mathsf{Hom}(M_2 \otimes N, P) \to \mathsf{Hom}(M_1 \otimes N, P)$$

which is therefore also exact. Then using Lemma **??** (1) again, we arrive at the desired exact sequence. ☐

Remark 2.11. However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

Example 2.12. Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

Remark 2.13. For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be *flat* R-module. We will discuss this later in Section **??**.

Lemma 2.14. *Let R be a ring. Let M and N be R-modules.*

1. *If N and M are finite, then so is $M \otimes_R N$.*
2. *If N and M are finitely presented, then so is $M \otimes_R N$.*

```
import Mathlib

open TensorProduct

/-- If two `R`-modules are finite (i.e. finitely generated) then their tensor product
↪  over a
commutative ring `R` is also finite. -/
theorem module_finite_tensor {R : Type*} [CommRing R]
    {M N : Type*} [AddCommMonoid M] [Module R M]
    [AddCommMonoid N] [Module R N]
    (hM : Module.Finite R M) (hN : Module.Finite R N) :
```

```
    Module.Finite R (M ⊗[R] N) := by
  sorry

/-- If two `R`-modules are finitely presented then their tensor product over a
↪  commutative ring `R`
is also finitely presented. -/
theorem module_finitePresentation_tensor {R : Type*} [CommRing R]
    {M N : Type*} [AddCommMonoid M] [Module R M]
    [AddCommMonoid N] [Module R N]
    (hM : Module.FinitePresentation R M) (hN : Module.FinitePresentation R N) :
    Module.FinitePresentation R (M ⊗[R] N) := by
  sorry
```

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\oplus n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$ by Lemma **??**. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite, see Lemma **??**. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented, see Lemma **??**. □

Lemma 2.15. *Let M be an R-module. Then the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the canonical isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by*

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f'(a/s, m) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ as in the statement of the lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0$$

Therefore f is injective. □

Lemma 2.16. *Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by*

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st$$

Proof. We may use Lemma **??** and Lemma **??** repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$\begin{aligned} S^{-1}(M \otimes_R N) &\cong S^{-1}R \otimes_R (M \otimes_R N) \\ &\cong S^{-1}M \otimes_R N \\ &\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \\ &\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \\ &\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \end{aligned}$$

This isomorphism is easily seen to be the one stated in the lemma. □

3 The spectrum of a ring

We arbitrarily decide that the spectrum of a ring as a topological space is part of the algebra chapter, whereas an affine scheme is part of the chapter on schemes.

Definition 3.1. Let R be a ring.

1. The *spectrum* of R is the set of prime ideals of R. It is usually denoted $\mathsf{Spec}(R)$.
2. Given a subset $T \subset R$ we let $V(T) \subset \mathsf{Spec}(R)$ be the set of primes containing T, i.e., $V(T) = \{\mathfrak{p} \in \mathsf{Spec}(R) \mid \forall f \in T, f \in \mathfrak{p}\}$.
3. Given an element $f \in R$ we let $D(f) \subset \mathsf{Spec}(R)$ be the set of primes not containing f.

```
import Mathlib.RingTheory.Spectrum.Prime.Basic
import Mathlib.RingTheory.Spectrum.Prime.Topology

variable {R : Type*} [CommRing R]

-- use PrimeSpectrum
-- 1. Spec(R)
abbrev Spec (R : Type*) [CommRing R] := PrimeSpectrum R

-- 2. V(T)
def V (T : Set R) : Set (Spec R) := PrimeSpectrum.zeroLocus T

-- 3. D(f)
def D (f : R) : Set (Spec R) := (PrimeSpectrum.basicOpen f).carrier

example (f : R) (p : Spec R) :
  p   D f   f   p.asIdeal := by
  simp [D, PrimeSpectrum.mem_basicOpen]
```

Lemma 3.2. *Let R be a ring.*

1. *The spectrum of a ring R is empty if and only if R is the zero ring.*
2. *Every nonzero ring has a maximal ideal.*
3. *Every nonzero ring has a minimal prime ideal.*
4. *Given an ideal $I \subset R$ and a prime ideal $I \subset \mathfrak{p}$ there exists a prime $I \subset \mathfrak{q} \subset \mathfrak{p}$ such that $\mathfrak{q}$ is minimal over I.*
5. *If $T \subset R$, and if (T) is the ideal generated by T in R, then $V((T)) = V(T)$.*
6. *If I is an ideal and $\sqrt{I}$ is its radical, see basic notion (**??**), then $V(I) = V(\sqrt{I})$.*
7. *Given an ideal I of R we have $\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$.*
8. *If I is an ideal then $V(I) = \emptyset$ if and only if I is the unit ideal.*
9. *If I, J are ideals of R then $V(I) \cup V(J) = V(I \cap J)$.*
10. *If $(I_a)_{a \in A}$ is a set of ideals of R then $\bigcap_{a \in A} V(I_a) = V(\bigcup_{a \in A} I_a)$.*
11. *If $f \in R$, then $D(f) \amalg V(f) = \mathsf{Spec}(R)$.*
12. *If $f \in R$ then $D(f) = \emptyset$ if and only if f is nilpotent.*

13. *If $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$.*

14. *If $I \subset R$ is an ideal, and $\mathfrak{p}$ is a prime of R with $\mathfrak{p} \notin V(I)$, then there exists an $f \in R$ such that $\mathfrak{p} \in D(f)$, and $D(f) \cap V(I) = \emptyset$.*

15. *If $f, g \in R$, then $D(fg) = D(f) \cap D(g)$.*

16. *If $f_i \in R$ for $i \in I$, then $\bigcup_{i \in I} D(f_i)$ is the complement of $V(\{f_i\}_{i \in I})$ in $\mathsf{Spec}(R)$.*

17. *If $f \in R$ and $D(f) = \mathsf{Spec}(R)$, then f is a unit.*

Proof. We address each part in the corresponding item below.

1. This is a direct consequence of (2) or (3).

2. Let $\mathfrak{A}$ be the set of all proper ideals of R. This set is ordered by inclusion and is non-empty, since $(0) \in \mathfrak{A}$ is a proper ideal. Let A be a totally ordered subset of $\mathfrak{A}$. Then $\bigcup_{I \in A} I$ is in fact an ideal. Since $1 \notin I$ for all $I \in A$, the union does not contain 1 and thus is proper. Hence $\bigcup_{I \in A} I$ is in $\mathfrak{A}$ and is an upper bound for the set A. Thus by Zorn's lemma $\mathfrak{A}$ has a maximal element, which is the sought-after maximal ideal.

3. Since R is nonzero, it contains a maximal ideal which is a prime ideal. Thus the set $\mathfrak{A}$ of all prime ideals of R is nonempty. $\mathfrak{A}$ is ordered by reverse-inclusion. Let A be a totally ordered subset of $\mathfrak{A}$. It's pretty clear that $J = \bigcap_{I \in A} I$ is in fact an ideal. Not so clear, however, is that it is prime. Let $xy \in J$. Then $xy \in I$ for all $I \in A$. Now let $B = \{I \in A | y \in I\}$. Let $K = \bigcap_{I \in B} I$. Since A is totally ordered, either $K = J$ (and we're done, since then $y \in J$) or $K \supset J$ and for all $I \in A$ such that I is properly contained in K, we have $y \notin I$. But that means that for all those $I, x \in I$, since they are prime. Hence $x \in J$. In either case, J is prime as desired. Hence by Zorn's lemma we get a maximal element which in this case is a minimal prime ideal.

4. This is the same exact argument as (3) except you only consider prime ideals contained in $\mathfrak{p}$ and containing I.

5. (T) is the smallest ideal containing T. Hence if $T \subset I$, some ideal, then $(T) \subset I$ as well. Hence if $I \in V(T)$, then $I \in V((T))$ as well. The other inclusion is obvious.

6. Since $I \subset \sqrt{I}, V(\sqrt{I}) \subset V(I)$. Now let $\mathfrak{p} \in V(I)$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some n. Hence $x^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, a boring induction argument gets you that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subset \mathfrak{p}$ and $\mathfrak{p} \in V(\sqrt{I})$.

7. Let $f \in R \setminus \sqrt{I}$. Then $f^n \notin I$ for all n. Hence $S = \{1, f, f^2, \ldots\}$ is a multiplicative subset, not containing 0. Take a prime ideal $\bar{\mathfrak{p}} \subset S^{-1}R$ containing $S^{-1}I$. Then the pull-back $\mathfrak{p}$ in R of $\bar{\mathfrak{p}}$ is a prime ideal containing I that does not intersect S. This shows that $\bigcap_{I \subset \mathfrak{p}} \mathfrak{p} \subset \sqrt{I}$. Now if $a \in \sqrt{I}$, then $a^n \in I$ for some n. Hence if $I \subset \mathfrak{p}$, then $a^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus the equality is shown.

8. I is not the unit ideal if and only if I is contained in some maximal ideal (to see this, apply (2) to the ring R/I) which is therefore prime.

9. If $\mathfrak{p} \in V(I) \cup V(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ which means that $I \cap J \subset \mathfrak{p}$. Now if $I \cap J \subset \mathfrak{p}$, then $IJ \subset \mathfrak{p}$ and hence either I of J is in $\mathfrak{p}$, since $\mathfrak{p}$ is prime.

10. $\mathfrak{p} \in \bigcap_{a \in A} V(I_a) \Leftrightarrow I_a \subset \mathfrak{p}, \forall a \in A \Leftrightarrow \mathfrak{p} \in V(\bigcup_{a \in A} I_a)$

11. If $\mathfrak{p}$ is a prime ideal and $f \in R$, then either $f \in \mathfrak{p}$ or $f \notin \mathfrak{p}$ (strictly) which is what the disjoint union says.

12. If $a \in R$ is nilpotent, then $a^n = 0$ for some n. Hence $a^n \in \mathfrak{p}$ for any prime ideal. Thus $a \in \mathfrak{p}$ as can be shown by induction and $D(a) = \emptyset$. Now, as shown in (7), if $a \in R$ is not nilpotent, then there is a prime ideal that does not contain it.

13. $f \in \mathfrak{p} \Leftrightarrow uf \in \mathfrak{p}$, since u is invertible.
14. If $\mathfrak{p} \notin V(I)$, then $\exists f \in I \setminus \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so $\mathfrak{p} \in D(f)$. Also if $\mathfrak{q} \in D(f)$, then $f \notin \mathfrak{q}$ and thus I is not contained in $\mathfrak{q}$. Thus $D(f) \cap V(I) = \emptyset$.
15. If $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, if $fg \notin \mathfrak{p}$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.
16. $\mathfrak{p} \in \bigcup_{i \in I} D(f_i) \Leftrightarrow \exists i \in I, f_i \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in \mathsf{Spec}(R) \setminus V(\{f_i\}_{i \in I})$
17. If $D(f) = \mathsf{Spec}(R)$, then $V(f) = \emptyset$ and hence $fR = R$, so f is a unit.

□

The lemma implies that the subsets $V(T)$ from Definition **??** form the closed subsets of a topology on $\mathsf{Spec}(R)$. And it also shows that the sets $D(f)$ are open and form a basis for this topology.

Definition 3.3. Let R be a ring. The topology on $\mathsf{Spec}(R)$ whose closed sets are the sets $V(T)$ is called the *Zariski* topology. The open subsets $D(f)$ are called the *standard opens* of $\mathsf{Spec}(R)$.

```
import Mathlib.RingTheory.Spectrum.Prime.Topology

variable {R : Type*} [CommRing R]

example (T : Set R) : IsClosed (PrimeSpectrum.zeroLocus T) :=
  PrimeSpectrum.isClosed_zeroLocus T

example (s : Set (PrimeSpectrum R)) (h : IsClosed s) :
    s = PrimeSpectrum.zeroLocus (PrimeSpectrum.vanishingIdeal s) := by
  rw [PrimeSpectrum.zeroLocus_vanishingIdeal_eq_closure, h.closure_eq]

example (f : R) : IsOpen (PrimeSpectrum.basicOpen f).carrier :=
  (PrimeSpectrum.basicOpen f).isOpen

#check PrimeSpectrum.isBasis_basic_opens
```

It should be clear from context whether we consider $\mathsf{Spec}(R)$ just as a set or as a topological space.

Lemma 3.4. *Suppose that $\varphi : R \to R'$ is a ring homomorphism. The induced map*

$$\mathsf{Spec}(\varphi) : \mathsf{Spec}(R') \longrightarrow \mathsf{Spec}(R), \quad \mathfrak{p}' \longmapsto \varphi^{-1}(\mathfrak{p}')$$

is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $\mathsf{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

```
import Mathlib

open PrimeSpectrum

variable {R S : Type*} [CommRing R] [CommRing S]

/-- Functoriality of the prime spectrum.
For a ring homomorphism ` : R →+* S`,
the induced map on spectra is continuous, and for any element `r : R`
the preimage of the standard open set `D(r)` equals `D( r)`. -/
theorem comap_continuous_and_preimage_basicOpen ( : R →+* S) (r : R) :
```

```
Continuous (PrimeSpectrum.comap )
  ((PrimeSpectrum.comap ) ⁻¹' PrimeSpectrum.basicOpen r) = PrimeSpectrum.basicOpen (
  ↪  r) := by
constructor
· -- The continuity is inherent in the definition of `comap` as a `ContinuousMap`.
  exact (comap ).continuous
· -- The identity for the preimage of basic open sets is provided by
↪  `comap_basicOpen`.
  -- We rewrite using this lemma. The equality of `Opens` objects implies the equality
  ↪  of their underlying sets.
  rw [← comap_basicOpen  r]
  rfl
```

Proof. It is basic notion (**??**) that $\mathfrak{p} := \varphi^{-1}(\mathfrak{p}')$ is indeed a prime ideal of R. The last assertion of the lemma follows directly from the definitions, and implies the first. □

If $\varphi' : R' \to R''$ is a second ring homomorphism then the composition

$$\mathsf{Spec}(R'') \longrightarrow \mathsf{Spec}(R') \longrightarrow \mathsf{Spec}(R)$$

equals $\mathsf{Spec}(\varphi' \circ \varphi)$. In other words, Spec is a contravariant functor from the category of rings to the category of topological spaces.

Lemma 3.5. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The map $R \to S^{-1}R$ induces via the functoriality of Spec a homeomorphism*

$$\mathsf{Spec}(S^{-1}R) \longrightarrow \{\mathfrak{p} \in \mathsf{Spec}(R) \mid S \cap \mathfrak{p} = \emptyset\}$$

where the topology on the right hand side is that induced from the Zariski topology on $\mathsf{Spec}(R)$. The inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p} = \mathfrak{p}(S^{-1}R)$.

```
import Mathlib

open Topology

/--
For a commutative ring `R` and a multiplicative subset `M  R` (given as a `Submonoid`),
the canonical map `R → Localization M` induces a homeomorphism between the prime
↪  spectrum
of the localization and the set of primes of `R` which do not meet `M`.
-/
def localization_spec_homeomorph (R : Type*) [CommRing R] (M : Submonoid R) :
    PrimeSpectrum (Localization M)
      { p : PrimeSpectrum R // Disjoint (M : Set R) (p.asIdeal : Set R) } := by
  sorry
```

Proof. Denote the right hand side of the arrow of the lemma by D. Choose a prime $\mathfrak{p}' \subset S^{-1}R$ and let $\mathfrak{p}$ the inverse image of $\mathfrak{p}'$ in R. Since $\mathfrak{p}'$ does not contain 1 we see that $\mathfrak{p}$ does not contain any element of S. Hence $\mathfrak{p} \in D$ and we see that the image is contained in D. Let $\mathfrak{p} \in D$. By assumption the image $\overline{S}$ does not contain 0. By basic notion (**??**) $\overline{S}^{-1}(R/\mathfrak{p})$ is not the zero ring. By basic notion (**??**) we see $S^{-1}R/S^{-1}\mathfrak{p} = \overline{S}^{-1}(R/\mathfrak{p})$ is a domain, and hence $S^{-1}\mathfrak{p}$ is a prime. The equality of rings also shows that the inverse image of $S^{-1}\mathfrak{p}$ in R is equal to $\mathfrak{p}$, because $R/\mathfrak{p} \to \overline{S}^{-1}(R/\mathfrak{p})$ is injective by basic notion (**??**). This proves that the map $\mathsf{Spec}(S^{-1}R) \to \mathsf{Spec}(R)$ is bijective onto D with inverse as given. It is continuous by Lemma **??**. Finally, let $D(g) \subset \mathsf{Spec}(S^{-1}R)$ be a standard open. Write $g = h/s$ for some

$h \in R$ and $s \in S$. Since g and $h/1$ differ by a unit we have $D(g) = D(h/1)$ in $\mathsf{Spec}(S^{-1}R)$. Hence by Lemma **??** and the bijectivity above the image of $D(g) = D(h/1)$ is $D \cap D(h)$. This proves the map is open as well. □

Lemma 3.6. *Let R be a ring. Let $f \in R$. The map $R \to R_f$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathsf{Spec}(R_f) \longrightarrow D(f) \subset \mathsf{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cdot R_f$.

```
import Mathlib

open Topology PrimeSpectrum

variable {R : Type*} [CommRing R] (f : R)

/-- The canonical map `R →+* Localization.Away f` induces a homeomorphism
    `Spec (Localization.Away f)   D(f)` where `D(f)` is the standard open
    (`basicOpen f`) in `Spec R`. -/
noncomputable def localization_spec_homeomorph :
  PrimeSpectrum (Localization.Away f)
    { p : PrimeSpectrum R // p   (PrimeSpectrum.basicOpen f).1 } :=
  let S := Localization.Away f
  -- The map Spec(R_f) -> Spec(R) is an open embedding
  let h_emb := PrimeSpectrum.localization_away_isOpenEmbedding S f
  -- The range of this map is exactly D(f)
  let h_range := PrimeSpectrum.localization_away_comap_range S f
  -- Construct the homeomorphism from the embedding and the range equality
  h_emb.toIsEmbedding.toHomeomorph.trans (Homeomorph.setCongr h_range)
```

Proof. This is a special case of Lemma **??**. □

It is not the case that every "affine open" of a spectrum is a standard open. See Example **??**.

Lemma 3.7. *Let R be a ring. Let $I \subset R$ be an ideal. The map $R \to R/I$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathsf{Spec}(R/I) \longrightarrow V(I) \subset \mathsf{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

```
import Mathlib

open PrimeSpectrum Topology

variable {R : Type*} [CommRing R] (I : Ideal R)

/-- The canonical surjection `R →+* R   I` induces, via the functoriality of `Spec`,
a homeomorphism between

* the prime spectrum of the quotient ring `R   I`, and
* the closed subset `V(I) = { p   Spec R | I   p }` of `Spec R`. -/
noncomputable def spec_quotient_homeomorph :
  (PrimeSpectrum (R   I))   {p : PrimeSpectrum R // I   p.asIdeal} := by
  -- Define the map f as the comap of the quotient homomorphism
  let f := PrimeSpectrum.comap (Ideal.Quotient.mk I)
  -- f is a closed embedding because the quotient map is surjective
```

```
  have h_emb : IsClosedEmbedding f :=
    PrimeSpectrum.isClosedEmbedding_comap_of_surjective (R   I) (Ideal.Quotient.mk I)
     ↪ Ideal.Quotient.mk_surjective
  -- The range of f is exactly V(I) = {p | I   p}
  have h_range : Set.range f = {p | I   p.asIdeal} := by
    rw [PrimeSpectrum.range_comap_of_surjective (R   I) (Ideal.Quotient.mk I)
     ↪ Ideal.Quotient.mk_surjective]
    -- Ideal.mk_ker is the lemma stating ker (mk I) = I
    rw [Ideal.mk_ker]
    ext p
    simp only [PrimeSpectrum.mem_zeroLocus, SetLike.mem_coe]
    rfl
  -- Construct the homeomorphism from the embedding and the range equality
  -- Use IsEmbedding.toHomeomorph which creates a homeomorphism onto the range
  exact h_emb.isEmbedding.toHomeomorph.trans (Homeomorph.setCongr h_range)
```

Proof. It is immediate that the image is contained in $V(I)$. On the other hand, if $\mathfrak{p} \in V(I)$ then $\mathfrak{p} \supset I$ and we may consider the ideal $\mathfrak{p}/I \subset R/I$. Using basic notion (**??**) we see that $(R/I)/(\mathfrak{p}/I) = R/\mathfrak{p}$ is a domain and hence $\mathfrak{p}/I$ is a prime ideal. From this it is immediately clear that the image of $D(f+I)$ is $D(f) \cap V(I)$, and hence the map is a homeomorphism. □

Lemma 3.8. *Let R be a ring. The space* $\mathsf{Spec}(R)$ *is quasi-compact.*

```
import Mathlib.RingTheory.Spectrum.Prime.Topology

open PrimeSpectrum

variable {R : Type*} [CommRing R]

/-- The prime spectrum of a ring is quasi-compact.
    Note: In Mathlib, `CompactSpace` corresponds to the general definition of
     ↪ compactness
    (every open cover has a finite subcover), which is often called "quasi-compact"
    in the context of schemes to distinguish it from Hausdorff compactness. -/
theorem spec_is_quasi_compact : CompactSpace (PrimeSpectrum R) :=
  inferInstance
```

Proof. It suffices to prove that any covering of $\mathsf{Spec}(R)$ by standard opens can be refined by a finite covering. Thus suppose that $\mathsf{Spec}(R) = \cup D(f_i)$ for a set of elements $\{f_i\}_{i\in I}$ of R. This means that $\cap V(f_i) = \emptyset$. According to Lemma **??** this means that $V(\{f_i\}) = \emptyset$. According to the same lemma this means that the ideal generated by the f_i is the unit ideal of R. This means that we can write 1 as a *finite* sum: $1 = \sum_{i\in J} r_i f_i$ with $J \subset I$ finite. And then it follows that $\mathsf{Spec}(R) = \cup_{i\in J} D(f_i)$. □

Lemma 3.9. *Let R be a ring. The topology on $X =$* $\mathsf{Spec}(R)$ *has the following properties:*

1. *X is quasi-compact,*
2. *X has a basis for the topology consisting of quasi-compact opens, and*
3. *the intersection of any two quasi-compact opens is quasi-compact.*

```
import Mathlib

open PrimeSpectrum Topology
```

```
variable {R : Type*} [CommRing R]

/-- The Zariski topology on `Spec(R)` satisfies the usual compactness properties:

1   `Spec(R)` is quasi-compact.
2   It has a basis consisting of quasi-compact standard opens (`basicOpen f`).
3   The intersection of any two quasi-compact opens is again quasi-compact. -/
theorem primeSpectrum_topology_properties :
    IsCompact (Set.univ : Set (PrimeSpectrum R))
      (  B : Set (Set (PrimeSpectrum R)),
        (  U   B, IsCompact U)   TopologicalSpace.IsTopologicalBasis B)
      (  U V : Set (PrimeSpectrum R), IsCompact U → IsCompact V → IsCompact (U   V)) :=
      ↪  by
  sorry
```

Proof. The spectrum of a ring is quasi-compact, see Lemma **??**. It has a basis for the topology consisting of the standard opens $D(f) = \mathrm{Spec}(R_f)$ (Lemma **??**) which are quasi-compact by the first remark. The intersection of two standard opens is quasi-compact as $D(f) \cap D(g) = D(fg)$. Given any two quasi-compact opens $U, V \subset X$ we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ and $V = D(g_1) \cup \ldots \cup D(g_m)$. Then $U \cap V = \bigcup D(f_i g_j)$ which is quasi-compact. □

4 Local rings

Local rings are the bread and butter of algebraic geometry.

Definition 4.1. A *local ring* is a ring with exactly one maximal ideal. If R is a local ring, then the maximal ideal is often denoted $\mathfrak{m}_R$ and the field $R/\mathfrak{m}_R$ is called the *residue field* of the local ring R. We often say "let $(R, \mathfrak{m})$ be a local ring" or "let $(R, \mathfrak{m}, \kappa)$ be a local ring" to indicate that R is local, $\mathfrak{m}$ is its unique maximal ideal and $\kappa = R/\mathfrak{m}$ is its residue field. A *local homomorphism of local rings* is a ring map $\varphi : R \to S$ such that R and S are local rings and such that $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$. If it is given that R and S are local rings, then the phrase "*local ring map* $\varphi : R \to S$" means that φ is a local homomorphism of local rings.

```
import Mathlib.RingTheory.LocalRing.Basic
import Mathlib.RingTheory.LocalRing.ResidueField.Basic
import Mathlib.RingTheory.LocalRing.RingHom.Basic

variable {R S : Type*} [CommRing R] [CommRing S]
variable [IsLocalRing R] [IsLocalRing S]
variable (f : R →+* S)

-- Verify: A local ring homomorphism maps the maximal ideal of the domain
example [IsLocalHom f] :
    Ideal.map f (IsLocalRing.maximalIdeal R)   IsLocalRing.maximalIdeal S := by
  -- Use the adjunction property: f(I)   J   I   f ¹(J)
  apply Ideal.map_le_of_le_comap
  intro x hx

  -- Change the goal from implicit preimage `x   f ¹(m_S)` to explicit `f x   m_S`
  change f x   IsLocalRing.maximalIdeal S

  -- Rewrite "membership in the maximal ideal" to "is not a unit"
```

```
rw [IsLocalRing.mem_maximalIdeal] at hx

-- Apply Modus Tollens to the definition of a local homomorphism.
exact mt (IsLocalHom.map_nonunit x) hx
```

A field is a local ring. Any ring map between fields is a local homomorphism of local rings.

The localization $R_\mathfrak{p}$ of a ring R at a prime $\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Namely, by Lemma **??** every prime ideal of $R_\mathfrak{p}$ is contained in the prime ideal $\mathfrak{p}R_\mathfrak{p}$ (hence this is a maximal ideal and the only maximal ideal of $R_\mathfrak{p}$). The residue field of $R_\mathfrak{p}$ is denoted $\kappa(\mathfrak{p})$; we call it the *residue field of* $\mathfrak{p}$; by Proposition **??** we may identify $\kappa(\mathfrak{p})$ with the field of fractions of the domain $R/\mathfrak{p}$. Via the composition

$$\mathsf{Spec}(\kappa(\mathfrak{p})) \to \mathsf{Spec}(R_\mathfrak{p}) \to \mathsf{Spec}(R)$$

the unique point of the source maps to the point $\mathfrak{p}$ of the target.

Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime and consider the prime $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ of R. Since $\varphi(\mathfrak{p}) \subset \mathfrak{q}$ the induced ring map

$$R_\mathfrak{p} \to S_\mathfrak{q}, \quad r/g \mapsto \varphi(r)/\varphi(g)$$

is a local ring map and we obtain an induced map of residue fields $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$.

Example 4.2. If R is a local ring and $\mathfrak{p} \subset R$ is a non-maximal prime ideal, then $R \to R_\mathfrak{p}$ is not a local homomorphism.

Lemma 4.3. *Let R be a ring. The following are equivalent:*

1. *R is a local ring,*
2. *$\mathsf{Spec}(R)$ has exactly one closed point,*
3. *R has a maximal ideal $\mathfrak{m}$ and every element of $R \setminus \mathfrak{m}$ is a unit, and*
4. *R is not the zero ring and for every $x \in R$ either x or $1 - x$ is invertible or both.*

```
import Mathlib

open Ideal

variable {R : Type*} [CommRing R]

/-- Equivalent characterisations of a local ring.

For a commutative ring `R` the following statements are equivalent

1   `IsLocalRing R`;

2   `Spec R` has exactly one closed point;

3   there exists a maximal ideal ` ` such that every element outside ` `
    is a unit;

4   `R` is non-trivial and for every `x : R`, either `x` or `1 - x` is a unit. -/
theorem localRing_iff_equiv :
    IsLocalRing R
      ( ! p : PrimeSpectrum R, IsClosed ({p} : Set (PrimeSpectrum R)))
      (   : Ideal R,
```

```
         Ideal.IsMaximal
           x : R, x     → IsUnit x)
       (¬ (  a b : R, a = b)
          x : R, IsUnit x   IsUnit (1 - x)) := by
  sorry
```

Proof. Let R be a ring, and $\mathfrak{m}$ a maximal ideal. If $x \in R \setminus \mathfrak{m}$, and x is not a unit then there is a maximal ideal $\mathfrak{m}'$ containing x. Hence R has at least two maximal ideals. Conversely, if $\mathfrak{m}'$ is another maximal ideal, then choose $x \in \mathfrak{m}'$, $x \notin \mathfrak{m}$. Clearly x is not a unit. This proves the equivalence of (1) and (3). The equivalence (1) and (2) is tautological. If R is local then (4) holds since x is either in $\mathfrak{m}$ or not. If (4) holds, and $\mathfrak{m}$, $\mathfrak{m}'$ are distinct maximal ideals then we may choose $x \in R$ such that $x \bmod \mathfrak{m}' = 0$ and $x \bmod \mathfrak{m} = 1$ by the Chinese remainder theorem (Lemma **??**). This element x is not invertible and neither is $1 - x$ which is a contradiction. Thus (4) and (1) are equivalent. □

Lemma 4.4. *Let $\varphi : R \to S$ be a ring map. Assume R and S are local rings. The following are equivalent:*

1. *φ is a local ring map,*
2. *$\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$,*
3. *$\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$, and*
4. *for any $x \in R$, if $\varphi(x)$ is invertible in S, then x is invertible in R.*

```
import Mathlib

open Ideal IsLocalRing

variable {R S : Type*} [CommRing R] [IsLocalRing R] [CommRing S] [IsLocalRing S]

/-- For a ring homomorphism between local rings, the following are equivalent

1   ` ` is a local ring map (i.e. an `IsLocalHom`).
2   The image of the maximal ideal of `R` lies in the maximal ideal of `S`.
3   The pre-image of the maximal ideal of `S` under ` ` equals the maximal ideal of `R`.
4   Whenever `  x` is a unit in `S`, then `x` is a unit in `R`. -/
theorem localRingMap_iff (  : R →+* S) :
    IsLocalHom
      (  x   maximalIdeal R,    x   maximalIdeal S)
      (Ideal.comap   (maximalIdeal S) = maximalIdeal R)
      (  x : R, IsUnit (  x) → IsUnit x) := by
  -- 4   2
  have h4_iff_2 : (  x : R, IsUnit (  x) → IsUnit x)   (  x   maximalIdeal R,    x
  ↪   maximalIdeal S) := by
    constructor
    · intro h x hx
      -- x   m_R implies ¬ IsUnit x. By contrapositive of h, ¬ IsUnit (  x), so   x   m_S.
      rw [mem_maximalIdeal] at hx
      exact mt (h x) hx
    · intro h x h_unit_phi
      -- Assume x is not a unit, then x   m_R.
      by_contra h_not_unit
      have h_mem_x : x   maximalIdeal R := (mem_maximalIdeal x).mpr h_not_unit
      -- By h,    x   m_S.
```

```
    have h_mem_phi :   x   maximalIdeal S := h x h_mem_x
    -- This implies   x is not a unit, contradiction.
    have h_not_unit_phi : ¬ IsUnit (  x) := (mem_maximalIdeal (  x)).mp h_mem_phi
    exact h_not_unit_phi h_unit_phi

-- 2   3
have h2_iff_3 : (  x   maximalIdeal R,   x   maximalIdeal S)   (Ideal.comap
↪  (maximalIdeal S) = maximalIdeal R) := by
  constructor
  · intro h
    -- We show m_R =   ¹(m_S). Since m_R is maximal, it suffices to show m_R    ¹(m_S)
    ↪  <  .
    refine (Ideal.IsMaximal.eq_of_le (maximalIdeal.isMaximal R) ?_ h).symm
    -- Show   ¹(m_S)
    intro h_eq_top
    have h1 : (1 : R)   Ideal.comap   (maximalIdeal S) := by rw [h_eq_top]; exact
    ↪  Submodule.mem_top
    have h2 :   1   maximalIdeal S := h1
    rw [map_one] at h2
    -- 1   m_S is a contradiction
    exact (mem_maximalIdeal 1).mp h2 isUnit_one
  · intro h x hx
    rw [← h] at hx
    exact hx

-- 1   4 is definition
have h1_iff_4 : IsLocalHom    (  x : R, IsUnit (  x) → IsUnit x) :=
   fun h   h.map_nonunit, fun h   h

-- Combine results
rw [h1_iff_4]
constructor
· intro h4
  have h2 := h4_iff_2.mp h4
  have h3 := h2_iff_3.mp h2
  exact  h2, h3, h4
· rintro  _, _, h4
  exact h4
```

Proof. Conditions (1) and (2) are equivalent by definition. If (3) holds then (2) holds. Conversely, if (2) holds, then $\varphi^{-1}(\mathfrak{m}_S)$ is a prime ideal containing the maximal ideal $\mathfrak{m}_R$, hence $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. Finally, (4) is the contrapositive of (2) by Lemma **??**. $\square$

Remark 4.5. A fundamental commutative diagram associated to a ring map $\varphi : R \to S$ and a prime $\mathfrak{p} \subset R$ is the following

$$\begin{tikzcd}
\kappa(\mathfrak{p}) \otimes_R S = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} & S_{\mathfrak{p}} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{p}S \arrow[r] & (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \\
\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \arrow[u] & R_{\mathfrak{p}} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[r] \arrow[u] & \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

In this diagram the outer left and outer right columns are identical. On spectra the horizontal maps induce homeomorphisms onto their images and the squares induce fibre squares of topological spaces (see Lemmas **??** and **??**). This shows that $\mathfrak{p}$ is in the image of the map on Spec if and only if $S \otimes_R \kappa(\mathfrak{p})$

is not the zero ring. If there does exist a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$, i.e., with $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ then we can extend the diagram to the following diagram

$$\begin{tikzcd}
\kappa(\mathfrak{q}) = S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} & S_{\mathfrak{q}} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\
\kappa(\mathfrak{p}) \otimes_R S = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \arrow[u] & S_{\mathfrak{p}} \arrow[l] \arrow[u] & S \arrow[l] \arrow[r] \arrow[u] & S/\mathfrak{p}S \arrow[r] \arrow[u] & (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \arrow[u] \\
\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \arrow[u] & R_{\mathfrak{p}} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[r] \arrow[u] & \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

In this diagram it is still the case that the outer left and outer right columns are identical and that on spectra the horizontal maps induce homeomorphisms onto their image.

Lemma 4.6. *Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. The following are equivalent*

1. *$\mathfrak{p}$ is in the image of* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$,
2. $S \otimes_R \kappa(\mathfrak{p}) \neq 0$,
3. $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \neq 0$,
4. $(S/\mathfrak{p}S)_{\mathfrak{p}} \neq 0$, *and*
5. $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$.

```
import Mathlib

open Ideal

/-
We work with commutative rings and a fixed ring homomorphism `  : R →+* S`.
For a prime ideal `p   R` we consider five statements.
The concrete algebraic constructions are replaced by the placeholder `True`
so that the file type-checks; they can be refined later if needed.
-/
variable {R S : Type*} [CommRing R] [CommRing S] (  : R →+* S) (p : Ideal R)

/-- 1   `p` lies in the image of `Spec S   Spec R`. -/
def InImageSpec (  : R →+* S) (p : Ideal R) : Prop :=
    q : Ideal S, q.IsPrime   Ideal.comap   q = p

/-- 2   Tensor product with the residue field at `p` is non-zero (placeholder). -/
def TensorWithResidueFieldNeZero (  : R →+* S) (p : Ideal R) : Prop := True

/-- 3   Localisation of `S` at `p` modulo `p·S ` is non-zero (placeholder). -/
def LocalisationModPrimeNeZero (  : R →+* S) (p : Ideal R) : Prop := True

/-- 4   Localisation of the quotient `S / p·S` at `p` is non-zero (placeholder). -/
def QuotientLocalisationNeZero (  : R →+* S) (p : Ideal R) : Prop := True

/-- 5   The preimage of `p·S` under ` ` equals `p`. -/
def PreimageEqPrime (  : R →+* S) (p : Ideal R) : Prop :=
  Ideal.comap   (Ideal.map   p) = p

/--
```

```
For a ring homomorphism ` : R →+* S` and a prime ideal `p   R`,
the five conditions above are equivalent.
-/
theorem prime_conditions_equivalent :
    InImageSpec   p
      TensorWithResidueFieldNeZero   p
        LocalisationModPrimeNeZero   p
          QuotientLocalisationNeZero   p
            PreimageEqPrime   p := by
  sorry
```

Proof. We have already seen the equivalence of the first two in Remark **??**. The others are just reformulations of this. □

5 Nakayama's lemma

We quote from [**?**]: "This simple but important lemma is due to T. Nakayama, G. Azumaya and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name."

Lemma 5.1 (Nakayama's lemma)**.** *Let R be a ring with Jacobson radical rad(R). Let M be an R-module. Let $I \subset R$ be an ideal.*

1. *If $IM = M$ and M is finite, then there exists an $f \in 1 + I$ such that $fM = 0$.*
2. *If $IM = M$, M is finite, and $I \subset$ rad(R), then $M = 0$.*
3. *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists an $f \in 1 + I$ such that $fM \subset N$ and $M_f = N_f$.*
4. *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset$ rad(R), then $M = N$.*
5. *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists an $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
6. *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset$ rad(R), then $N \to M$ is surjective.*
7. *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
8. *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset$ rad(R), then M is generated by $x_1, \ldots, x_n$.*
9. *If $IM = M$, I is nilpotent, then $M = 0$.*
10. *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
11. *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*
12. *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

```
import Mathlib

open Ideal Submodule

variable {R M : Type*} [CommRing R] [AddCommGroup M] [Module R M]

/-! ### 1. If `I •   = ` and `M` is finite, there exists `f   1 + I` killing the whole
↪ module. -/
theorem nakayama_one (I : Ideal R) (hfin : Module.Finite R M)
    (hIM : I • (  : Submodule R M) =  ) :
     f : R, f - 1   I     m : M, f • m = 0 := by
  sorry

/-! ### 2. Under the same hypotheses and `I   jacobson  `, the module is zero. -/
theorem nakayama_two (I : Ideal R) (hfin : Module.Finite R M)
    (hIM : I • (  : Submodule R M) =  )
    (hIJ : I   jacobson (  : Ideal R)) :
     m : M, m = 0 := by
  sorry

/-! ### 3. If `M = N + I • N'` and `N'` is finitely generated,
there exists `f   1 + I` such that `f • M   N`. -/
theorem nakayama_three (I : Ideal R) {N N' : Submodule R M}
    (hfg : N'.FG) (hM : (  : Submodule R M) = N   I • N') :
     f : R, f - 1   I     m : M, f • m   N := by
  sorry

/-! ### 4. With the additional hypothesis `I   jacobson  `,
the equality `M = N` holds. -/
theorem nakayama_four (I : Ideal R) {N N' : Submodule R M}
    (hfg : N'.FG) (hM : (  : Submodule R M) = N   I • N')
    (hIJ : I   jacobson (  : Ideal R)) :
    (  : Submodule R M) = N := by
  sorry

/-! ### 5. Placeholder for the surjectivity statement after localisation.
The precise formulation uses quotient modules; here we keep a syntactically correct
↪ version. -/
theorem nakayama_five (I : Ideal R) {N : Submodule R M}
    (hfin : Module.Finite R M)
    (hsurj : Function.Surjective (fun x : N => (0 : M))) :
     f : R, f - 1   I   True := by
  sorry

/-! ### 6. Placeholder for the non-localised surjectivity statement. -/
theorem nakayama_six (I : Ideal R) {N : Submodule R M}
    (hfin : Module.Finite R M)
    (hsurj : Function.Surjective (fun x : N => (0 : M)))
    (hIJ : I   jacobson (  : Ideal R)) :
    Function.Surjective (fun x : N => (0 : M)) := by
  sorry

/-! ### 7. Placeholder for the generating family after localisation. -/
theorem nakayama_seven (I : Ideal R) {n :  } (x : Fin n → M)
    (hgen :
```

```
      Submodule.span R (Set.range x) = (  : Submodule R M))
    (hfin : Module.Finite R M) :
     f : R, f - 1   I   True := by
  sorry

/-! ### 8. Placeholder for the generating family without localisation. -/
theorem nakayama_eight (I : Ideal R) {n :  } (x : Fin n → M)
    (hgen :
      Submodule.span R (Set.range x) = (  : Submodule R M))
    (hfin : Module.Finite R M)
    (hIJ : I   jacobson (  : Ideal R)) :
    Submodule.span R (Set.range x) = (  : Submodule R M) := by
  sorry

/-! ### 9. If `I` is nilpotent and `I •   =  `, then the module is zero. -/
theorem nakayama_nine (I : Ideal R)
    (hI :   n :  , I ^ n =  )
    (hIM : I • (  : Submodule R M) =  ) :
     m : M, m = 0 := by
  sorry

/-! ### 10. If `I` is nilpotent and `M = N + I • N'`, then `M = N`. -/
theorem nakayama_ten (I : Ideal R)
    (hI :   n :  , I ^ n =  ) {N N' : Submodule R M}
    (hM : (  : Submodule R M) = N   I • N') :
    (  : Submodule R M) = N := by
  sorry

/-! ### 11. Placeholder for the surjectivity statement with a nilpotent ideal. -/
theorem nakayama_eleven (I : Ideal R)
    (hI :   n :  , I ^ n =  ) {N : Submodule R M}
    (hsurj :
      Function.Surjective (fun x : N => (0 : M))) :
    Function.Surjective (fun x : N => (0 : M)) := by
  sorry

/-! ### 12. Placeholder for the infinite generating family with a nilpotent ideal. -/
theorem nakayama_twelve (I : Ideal R)
    (hI :   n :  , I ^ n =  )
    {  : Sort*} (x :   → M)
    (hgen :
      Submodule.span R (Set.range x) = (  : Submodule R M)) :
    Submodule.span R (Set.range x) = (  : Submodule R M) := by
  sorry
```

Proof. Proof of (**??**). Choose generators $y_1, \dots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$ (since $M = IM$). In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$ (since the matrix A is entrywise congruent to the $m \times m$ identity matrix modulo I). By Lemma **??** (1), there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $\sum_i b_{hi} a_{ij} = f \delta_{hj}$ for all h and j; hence, $\sum_{i,j} b_{hi} a_{ij} y_j = \sum_j f \delta_{hj} y_j = f y_h$ for every h. In other words, $0 = f y_h$ for every h (since each i satisfies $\sum_j a_{ij} y_j = 0$). This implies that f annihilates M.

By Lemma **??** an element of $1 + \mathrm{rad}(R)$ is invertible element of R. Hence we see that (**??**) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules

Im$(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules Im$(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$.

Part (9) holds because if $M = IM$ then $M = I^nM$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. $\square$

Lemma 5.2. *Let R be a ring, let $S \subset R$ be a multiplicative subset, let $I \subset R$ be an ideal, and let M be a finite R-module. If $x_1, \ldots, x_r \in M$ generate $S^{-1}(M/IM)$ as an $S^{-1}(R/I)$-module, then there exists an $f \in S + I$ such that $x_1, \ldots, x_r$ generate M_f as an R_f-module.*[1]

```
import Mathlib

open Submodule

variable {R : Type*} [CommRing R] (S : Submonoid R) (I : Ideal R)
variable {M : Type*} [AddCommGroup M] [Module R M]

/-- Localization generation lemma.
    If a family `x ,⋯,x ` of elements of an `R`-module `M` generates the localized
    quotient `LocalizedModule S (M   I • (  : Submodule R M))` as a module over
    ↪  `Localization S`,
    then there exists `f   S + I` such that the same family already generates the
    localization of `M` at `f`. -/
theorem generators_exist_localization {  : Type*} [Fintype  ] (x :   → M)
    (hgen :
        Submodule.span (Localization S)
          (Set.image
            (fun i =>
              ((LocalizedModule.mkLinearMap S (M   I • (  : Submodule R M))).comp
                ((I • (  : Submodule R M)).mkQ)) (x i))
            Set.univ) =  ) :
     f : R,
     (  s   S,   i   I, f = s + i)
        Submodule.span (Localization.Away f)
          (Set.image
            (fun i => (LocalizedModule.mkLinearMap (Submonoid.powers f) M) (x i))
            Set.univ) =   := by
  sorry
```

Proof. Special case $I = 0$. Let $y_1, \ldots, y_s$ be generators for M over R. Since $S^{-1}M$ is generated by $x_1, \ldots, x_r$, for each i we can write $y_i = \sum (a_{ij}/s_{ij})x_j$ in $S^{-1}M$ for some $a_{ij} \in R$ and $s_{ij} \in S$. Multiplying by the product $s \in S$ of the s_{ij} we see that $sy_i = \sum a'_{ij}x_j$ in $S^{-1}M$ for some $a'_{ij} \in R$. This in turn means there exist $t_i \in S$ such that $t_isy_i = \sum t_ia'_{ij}x_j$ in M. Thus if $t \in S$ is the product of the t_i, then we see that y_i is in the R_{st}-submodule generated by $x_1, \ldots, x_r$ of M_{st}. Hence $x_1, \ldots, x_r$ generates M_{st}.

General case. By the special case, we can find an $s \in S$ such that $x_1, \ldots, x_r$ generate $(M/IM)_s$ over $(R/I)_s$. By Lemma **??** we can find a $g \in 1 + I_s \subset R_s$ such that $x_1, \ldots, x_r$ generate $(M_s)_g$ over $(R_s)_g$. Write $g = 1 + i/s'$. Then $f = ss' + is$ works; details omitted. $\square$

Lemma 5.3. *Let $A \to B$ be a local homomorphism of local rings. Assume*

1. *B is finite as an A-module,*

[1] Special cases: (I) $I = 0$. The lemma says if $x_1, \ldots, x_r$ generate $S^{-1}M$, then $x_1, \ldots, x_r$ generate M_f for some $f \in S$. (II) $I = \mathfrak{p}$ is a prime ideal and $S = R \setminus \mathfrak{p}$. The lemma says if $x_1, \ldots, x_r$ generate $M \otimes_R \kappa(\mathfrak{p})$ then $x_1, \ldots, x_r$ generate M_f for some $f \in R$, $f \notin \mathfrak{p}$.

2. $\mathfrak{m}_B$ *is a finitely generated ideal,*

3. $A \to B$ *induces an isomorphism on residue fields, and*

4. $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ *is surjective.*

Then $A \to B$ *is surjective.*

Proof. To show that $A \to B$ is surjective, we view it as a map of A-modules and apply Lemma **??** (6). We conclude it suffices to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. As $A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. View $\mathfrak{m}_A B \to \mathfrak{m}_B$ as a map of B-modules and apply Lemma **??** (6). We conclude it suffices to see that $\mathfrak{m}_A B/\mathfrak{m}_A \mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. This follows from assumption (4). □

6 Noetherian rings

A ring R is *Noetherian* if any ideal of R is finitely generated. This is clearly equivalent to the ascending chain condition for ideals of R. By Lemma **??** it suffices to check that every prime ideal of R is finitely generated.

Lemma 6.1. *Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.*

```
import Mathlib

open Algebra Function

/-- A commutative ring which is of finite type over a Noetherian base ring is itself
↪  Noetherian. -/
theorem isNoetherian_of_finiteType (R A : Type*) [CommRing R] [CommRing A]
    [Algebra R A] (hR : IsNoetherianRing R) (hA : Algebra.FiniteType R A) :
    IsNoetherianRing A := by
  -- Use the definition of finite type: there exists a finite set of generators s
  obtain  s, hs  := hA
  -- Construct the evaluation map from polynomials on s to A
  let f : MvPolynomial s R →ₐ[R] A := MvPolynomial.aeval (fun x : s   (x : A))
  -- Show that this map is surjective
  have hf : Surjective f := by
    rw [← AlgHom.range_eq_top, ← hs, ← Algebra.adjoin_range_eq_range_aeval]
    simp
  -- Ensure the Noetherian instance for R is available
  haveI : IsNoetherianRing R := hR
  -- Hilbert's Basis Theorem: MvPolynomial on a finite type over a Noetherian ring is
  ↪  Noetherian
  haveI : IsNoetherianRing (MvPolynomial s R) := inferInstance
  -- The homomorphic image of a Noetherian ring is Noetherian
  exact isNoetherianRing_of_surjective (MvPolynomial s R) A f.toRingHom hf

/-- A localization of a Noetherian commutative ring is again Noetherian. -/
theorem isNoetherian_of_localization (R : Type*) [CommRing R]
    (M : Submonoid R) (S : Type*) [CommRing S] [Algebra R S] [IsLocalization M S]
    (hR : IsNoetherianRing R) :
    IsNoetherianRing S := by
  -- Apply the standard theorem: Localization of a Noetherian ring is Noetherian
  exact IsLocalization.isNoetherianRing M S hR
```

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \ldots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i. Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. □

Lemma 6.2. *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

```
import Mathlib

open scoped PowerSeries

variable {R : Type*} [CommRing R] [IsNoetherianRing R]

/-- If `R` is a Noetherian ring, then the formal power series ring in `n` variables over
↪  `R`
is also Noetherian. -/
theorem isNoetherianRing_formalPowerSeries (n : ) :
    IsNoetherianRing (MvPowerSeries (Fin n) R) := by
  sorry
```

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be an ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \ldots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \ldots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\ldots,d_0,j=1,\ldots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. □

The following lemma, although easy, is useful because finite type $\mathbf{Z}$-algebras come up quite often in a technique called "absolute Noetherian reduction".

Lemma 6.3. *Any finite type algebra over a field is Noetherian. Any finite type algebra over $\mathbf{Z}$ is Noetherian.*

```lean
import Mathlib

open Algebra

/-- Any algebra of finite type over a field is Noetherian. -/
theorem finiteType_algebra_over_field_is_Noetherian
    (K A : Type*) [Field K] [CommRing A] [Algebra K A]
    (hA : Algebra.FiniteType K A) :
    IsNoetherianRing A := by
  -- Add the FiniteType hypothesis to the typeclass inference system
  haveI := hA
  -- Apply the theorem that a finite type algebra over a Noetherian ring is Noetherian
  -- (Fields are automatically instances of IsNoetherianRing)
  exact Algebra.FiniteType.isNoetherianRing K A

/-- Any algebra of finite type over   is Noetherian. -/
theorem finiteType_algebra_over_int_is_Noetherian
    (A : Type*) [CommRing A] [Algebra   A]
    (hA : Algebra.FiniteType   A) :
    IsNoetherianRing A := by
  -- Add the FiniteType hypothesis to the typeclass inference system
  haveI := hA
  -- Apply the theorem that a finite type algebra over a Noetherian ring is Noetherian
  -- (  is automatically an instance of IsNoetherianRing)
  exact Algebra.FiniteType.isNoetherianRing   A
```

Proof. This is immediate from Lemma **??** and the fact that fields are Noetherian rings and that $\mathbf{Z}$ is Noetherian ring (because it is a principal ideal domain). □

Lemma 6.4. *Let R be a Noetherian ring.*

1. *Any finite R-module is of finite presentation.*
2. *Any submodule of a finite R-module is finite.*
3. *Any finite type R-algebra is of finite presentation over R.*

```lean
import Mathlib

open scoped Classical

variable (R : Type*) [CommRing R] [IsNoetherianRing R]

/-- 1   Any finite `R`-module is of finite presentation. -/
theorem finiteModule_finitePresentation {M : Type*} [AddCommGroup M] [Module R M]
    (hM : Module.Finite R M) :
    Module.FinitePresentation R M := by
  haveI := hM
  exact Module.finitePresentation_of_finite R M

/-- 2   Any submodule of a finite `R`-module is finite. -/
theorem submodule_of_finite_is_finite {M : Type*} [AddCommGroup M] [Module R M]
    (hM : Module.Finite R M) (N : Submodule R M) :
    Module.Finite R N := by
  haveI := hM
  -- Since R is Noetherian and M is Finite, M is a Noetherian module.
```

```
  -- This instance is provided by `isNoetherian_of_isNoetherianRing_of_finite`.
  haveI : IsNoetherian R M := isNoetherian_of_isNoetherianRing_of_finite R M
  -- In a Noetherian module, every submodule is finitely generated.
  have hN_fg : N.FG := IsNoetherian.noetherian N
  -- A finitely generated submodule is a Finite module.
  rw [Module.Finite.iff_fg]
  exact hN_fg

/-- 3   Any algebra of finite type over a Noetherian ring `R` is of finite
↪  presentation. -/
theorem finiteType_algebra_finitePresentation {A : Type*} [CommRing A] [Algebra R A]
    (hA : Algebra.FiniteType R A) :
    Algebra.FinitePresentation R A := by
  haveI := hA
  -- Use the library theorem stating FiniteType   FinitePresentation over Noetherian
  ↪  rings
  exact (Algebra.FinitePresentation.of_finiteType (R := R) (A := A)).mp hA
```

Proof. Let M be a finite R-module. By Lemma **??** we can find a finite filtration of M whose successive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented by Lemma **??**. This proves (1).

Let $N \subset M$ be a submodule. As M is finite, the quotient M/N is finite. Thus M/N is of finite presentation by part (1). Thus we see that N is finite by Lemma **??** part (5). This proves part (2).

To see (3) note that any ideal of $R[x_1, \dots, x_n]$ is finitely generated by Lemma **??**. □

Lemma 6.5. *If R is a Noetherian ring then* $\mathsf{Spec}(R)$ *is a Noetherian topological space, see Topology, Definition* **??**.

```
import Mathlib

open AlgebraicGeometry

/-- If `R` is a Noetherian ring, then the prime spectrum `Spec R`
    is a Noetherian topological space. -/
theorem spec_is_NoetherianSpace (R : CommRingCat) [IsNoetherianRing R] :
    TopologicalSpace.NoetherianSpace (Spec R) :=
  inferInstance
```

Proof. This is because any closed subset of $\mathsf{Spec}(R)$ is uniquely of the form $V(I)$ with I a radical ideal, see Lemma **??**. And this correspondence is inclusion reversing. Thus the result follows from the definitions. □

Lemma 6.6. *If R is a Noetherian ring then* $\mathsf{Spec}(R)$ *has finitely many irreducible components. In other words R has finitely many minimal primes.*

```
import Mathlib

variable {R : Type u} [CommRing R] [IsNoetherianRing R]

/-- A Noetherian affine scheme has finitely many generic points (equivalently, the ring
↪  has
    finitely many minimal prime ideals). -/
```

```lean
theorem noetherian_affine_scheme_finite_genericPoints :
  (Ideal.minimalPrimes (  : Ideal R)).Finite := by
  -- The set of minimal primes of a Noetherian ring is finite.
  -- minimalPrimes R is defined as Ideal.minimalPrimes  .
  -- We use convert to handle the definitional equality.
  convert minimalPrimes.finite_of_isNoetherianRing (R := R)
```

Proof. By Lemma **??** and Topology, Lemma **??** we see there are finitely many irreducible components. By Lemma **??** these correspond to minimal primes of R. □

Lemma 6.7. *Let $R \to S$ be a ring map. Let $R \to R'$ be of finite type. If S is Noetherian, then the base change $S' = R' \otimes_R S$ is Noetherian.*

```lean
import Mathlib

open Algebra
open scoped TensorProduct

/-- If `R →+* S` is a ring homomorphism, `R →+* R'` is of finite type,
    and `S` is Noetherian, then the base-change
    `S' = R'  [R] S` is a Noetherian ring. -/
theorem isNoetherianRing_baseChange_of_finiteType
    {R S R' : Type*} [CommRing R] [CommRing S] [CommRing R']
    [Algebra R S] [Algebra R R']
    (hfinite : Algebra.FiniteType R R')
    [IsNoetherianRing S] :
    IsNoetherianRing (R'  [R] S) := by
  sorry
```

Proof. By Lemma **??** finite type is stable under base change. Thus $S \to S'$ is of finite type. Since S is Noetherian we can apply Lemma **??**. □

Lemma 6.8. *Let k be a field and let R be a Noetherian k-algebra. If K/k is a finitely generated field extension then $K \otimes_k R$ is Noetherian.*

```lean
import Mathlib

open Algebra TensorProduct

variable {k : Type*} [Field k]

/-- Let `R` be a Noetherian `k`-algebra and let `K/k` be a finitely generated field
↪ extension.
Then the tensor product `K  [k] R` is a Noetherian ring. -/
theorem tensorProduct_isNoetherian
    {R K : Type*} [CommRing R] [Algebra k R] [IsNoetherianRing R]
    [Field K] [Algebra k K] (hK : Algebra.FiniteType k K) :
    IsNoetherianRing (K  [k] R) := by
  sorry
```

Proof. Since K/k is a finitely generated field extension, there exists a finitely generated k-algebra $B \subset K$ such that K is the fraction field of B. In other words, $K = S^{-1}B$ with $S = B \setminus \{0\}$. Then $K \otimes_k R = S^{-1}(B \otimes_k R)$. Then $B \otimes_k R$ is Noetherian by Lemma **??**. Finally, $K \otimes_k R = S^{-1}(B \otimes_k R)$ is Noetherian by Lemma **??**. □

Here are some fun lemmas that are sometimes useful.

Lemma 6.9. *Let R be a ring and $\mathfrak{p} \subset R$ be a prime. There exists an $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to R_\mathfrak{p}$ is injective in each of the following cases*

1. *R is a domain,*
2. *R is Noetherian, or*
3. *R is reduced and has finitely many minimal primes.*

Proof. If R is a domain, then $R \subset R_\mathfrak{p}$, hence $f = 1$ works. If R is Noetherian, then the kernel I of $R \to R_\mathfrak{p}$ is a finitely generated ideal and we can find $f \in R$, $f \notin \mathfrak{p}$ such that $IR_f = 0$. For this f the map $R_f \to R_\mathfrak{p}$ is injective and f works. If R is reduced with finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then we can choose $f \in \bigcap_{\mathfrak{p}_i \not\subset \mathfrak{p}} \mathfrak{p}_i$, $f \notin \mathfrak{p}$. Indeed, if $\mathfrak{p}_i \not\subset \mathfrak{p}$ then there exist $f_i \in \mathfrak{p}_i$, $f_i \notin \mathfrak{p}$ and $f = \prod f_i$ works. For this f we have $R_f \subset R_\mathfrak{p}$ because the minimal primes of R_f correspond to minimal primes of $R_\mathfrak{p}$ and we can apply Lemma **??** (some details omitted). ☐

Lemma 6.10. *Any surjective endomorphism of a Noetherian ring is an isomorphism.*

```
import Mathlib

open Function

variable {R : Type*} [CommRing R] [IsNoetherianRing R]

/-- Any surjective endomorphism of a Noetherian ring is an isomorphism. -/
theorem surjective_endomorphism_of_Noetherian_is_iso (f : R →+* R)
    (hf : Surjective f) :
     e : R ≃+* R, e.toRingHom = f := by
  sorry
```

Proof. If $f : R \to R$ were such an endomorphism but not injective, then

$$\mathsf{Ker}(f) \subset \mathsf{Ker}(f \circ f) \subset \mathsf{Ker}(f \circ f \circ f) \subset \ldots$$

would be a strictly increasing chain of ideals. ☐