OBJECTIVES

After studying this chapter, you should be able to understand :

- coordinates, distance between two points, section formula, area of triangle, collinearity of three points, locus of a point
- straight line and different forms of straight lines and their applications in solving problems
- circle, tangent and normal and solution of problems
- ellipse, parabola, tangent and normal to parabola and problems based on these concepts.

15·0. INTRODUCTION

The credit for bringing out this new branch of geometry goes to the French mathematician Renatus Cartesius (1596—1650) popularly known as Rene Descartes and it is after his name that it is sometimes called as Cartesian Geometry.

Coordinate Geometry is that branch of geometry in which two real numbers, called coordinates, are used to indicate the position of a point in a plane. The main contribution of coordinate geometry is that it has enabled the integration of algebra and geometry. This is evident from the fact that algebraic methods are employed to represent and prove the fundamental properties of geometrical theorems. Equations are also employed to represent the various geometric figures. It is because of these features that the coordinate geometry is considered to be a more powerful tool of analysis than the Euclidian Geometry. It is on this consideration that somtimes it is described as Analytical Geometry.

Before we come to the basic concept of coordinates it is necessary to say a word about the directed line.

15·1. DIRECTED LINE

A directed line is a straight line with number units positive, zero and negative. The point of origin is the number 0. The arrow indicates

Now, the coordinates of the mid-point of the join of the two points $(-1, 5)$ and $(7, 3)$ will be

$$\left(\frac{-1+7}{2}, \frac{2+3}{2}\right), \text{ i.e., } (3, 4).$$

15·5. DISTANCE BETWEEN TWO POINTS

The distance, say d, between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(\text{diff. of abscissae})^2+(\text{diff. of ordinates})^2}$$

Since we take the square of the two differences, we may designate any of the points as (x_1, y_1) and the other (x_2, y_2).

In order to prove the above formula, let us take two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as shown in the following diagram.

The vertical dotted lines PB and QC are perpendiculars from P and Q on the x-axis, and PR is the perpendicular from P on QC. Then

$$PR=BC=OC-OB=x_2-x_1$$

and $QR=QC-RC=y_2-y_1$

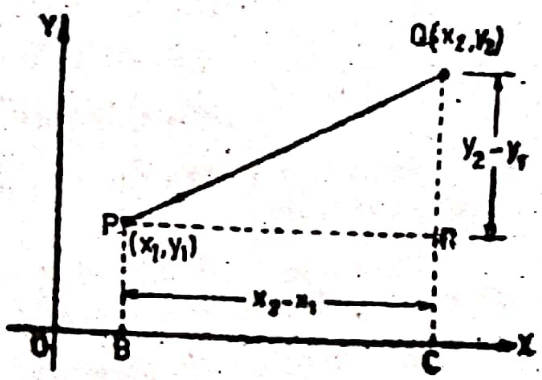


Fig. 6.

From the right angled triangle PRQ, right angled at R, we have by the Pythagoras theorem

$$PQ^2=PR^2+QR^2$$

$\Rightarrow$ $$d^2=(x_2-x_1)^2+(y_2-y_1)^2$$

$\Rightarrow$ $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

It may be noted that the above formula will be valid for points taken in other three quadrants as well.

Also the distance of a point $P(x_1, y_1)$ from the origin is

$$d=\sqrt{(x_1-0)^2+(y_1-0)^2}=\sqrt{x_1^2+y_1^2}$$

Thus, the distance between two points say $(4, -1)$ and $(7, 3)$ is

$$d=\sqrt{(7-4)^2+(3+1)^2}=\sqrt{3^2+4^2}=\sqrt{25}=5 \text{ units.}$$

Example 2. (*a*) *Show that the points* $(6, 6)$, $(2, 3)$ *and* $(4, 7)$ *are the vertices of a right-angled triangle.*

(*b*) *Prove that the points* (4, 3), (7, −1) *and* (9, 3) *are the vertices of an isosceles triangle.*

Solution. (*a*) Let *A*, *B*, *C* be the points (6, 6), (2, 3) and (4, 7) respectively, then

$$AB^2=[(6-2)^2+(6-3)^2]=16+9=25$$
$$BC^2=[(2-4)^2+(3-7)^2]=4+16=20$$
$$CA^2=[(4-6)^2+(7-6)^2]=4+1=5$$

$\therefore \quad AB^2=BC^2+CA^2$

$\Rightarrow \quad \angle ABC=1$ right angle

Hence the points $A(6, 6)$, $B(2, 3)$ and $C(4, 7)$ are the vertices of a right angled triangle.

(*b*) We know that the property of an isosceles triangle is that two of its sides are equal.

Using the distance formula, we have

$$AB=\sqrt{(4-7)^2+(3+1)^2}=\sqrt{9+16}=5$$
$$BC=\sqrt{(7-9)^2+(-1-3)^2}=\sqrt{4+16}=2\sqrt{5}$$
$$AC=\sqrt{(9-4)^2+(3-3)^2}=\sqrt{25}=5$$

Since two of the sides, *i.e.*, *AB* and *AC* are equal, the triangle is an isosceles triangle.

Example 3. *Prove that the quadrilateral with vertices* (2, −1), (3, 4), (−2, 3) *and* (−3, −2) *is a rhombus.*

(*b*) *Show that the points* (4, −5), (8, 1), (14, −3) *and* (10, −9) *are the vertices of a square.*

Solution. (*a*) Let $A(2, -1)$, $B(3, 4)$, $C(-2, 3)$ and $D(-3, -2)$ be the four vertices of the quadrilateral

$$AB=\sqrt{(2-3)^2+(-1-4)^2}=\sqrt{26}$$
$$BC=\sqrt{[3-(-2)]^2+(4-3)^2}=\sqrt{26}$$
$$CD=\sqrt{[(-2)-(-3)]^2+[3-(-2)]^2}=\sqrt{26}$$
$$DA=\sqrt{[(-3)-(2)]^2+[(-2)-(-1)]^2}=\sqrt{26}$$
$$AC=\sqrt{(2+2)^2+(-1-3)^2}=\sqrt{32},\ BD=\sqrt{(3+3)^2+(4+2)^2}=\sqrt{72}$$

$\Rightarrow \quad AB=BC=CD=DA,\ AC\neq BD$

$\Rightarrow \quad ABCD$ is a rhombus.

(*b*) Left as an exercise for the student.

Example 4. *Prove that* (−2, −1), (1, 0), (4, 3) *and* (1, 2) *are the vertices of a parallelogram.*

Solution. Let $A(-2, -1)$, $B(1, 0)$, $C(4, 3)$ and $D(1, 2)$ be the vertices of a quadrilateral.

Then the mid-point of $AC=\left(\frac{-2+4}{2}, \frac{-1+3}{2}\right)=(1, 1)$ ...(1)

and the mid-point of $BD=\left(\frac{1+1}{2}, \frac{0+2}{2}\right)=(1, 1)$ ...(2)

From (1) and (2), we conclude that AC and BD bisect each other at the same point (1, 1) and hence the quadrilateral $ABCD$ is a parallelogram.

Example 5. *Find the coordinates of the circumcentre of a triangle whose coordinates are (3, −2), (4, 3) and (−6, 5). Hence find the circumradius.*

Solution. Let $A(3, -2)$, $B(4, 3)$ and $C(-6, 5)$ be the vertices of the triangle and $P(x, y)$ be the circumcentre.

$\therefore$ By definition $PA=PB=PC \Rightarrow PA^2=PB^2=PC^2$

Now by the distance formula :

$$PA^2=(x-3)^2+(y+2)^2=x^2+y^2-6x+4y+13$$
$$PB^2=(x-4)^2+(y-3)^2=x^2+y^2-8x-6y+25$$
$$PC^2=(x+6)^2+(y-5)^2=x^2+y^2+12x-10y+61$$

Now $PA^2=PB^2$

$\Rightarrow x^2+y^2-6x+4y+13=x^2+y^2-8x-6y+25$

$\Rightarrow 2x+10y=12$

$\Rightarrow x+5y=6$ ...(1)

and $PB^2=PC^2$

$\Rightarrow x^2+y^2-8x-6y+25=x^2+y^2+12x-10y+61$

$\Rightarrow -20x+4y=36$

$\Rightarrow -5x+y=9$ ...(2)

Solving (1) and (2), we get

$$x=-\frac{3}{2}, \quad y=\frac{3}{2}$$

$\therefore$ The circumcentre P is $\left(-\frac{3}{2}, \frac{3}{2}\right)$

Now the circumradius of $\triangle ABC$ is PA or PB or PC. Therefore

$$PA=\sqrt{\left(-\frac{3}{2}-3\right)^2+\left(\frac{3}{2}+2\right)^2}$$

$$=\sqrt{\frac{81}{4}+\frac{49}{4}}=\sqrt{\frac{130}{4}}$$

or $$PA=\frac{\sqrt{130}}{2}$$

Let the coordinates of P be $(a, 0)$. We have by the section formula

$$a=\frac{k+2}{k+1} \text{ and } 0=\frac{5+9k}{k+1}$$

From the latter equation, we have $k=-\frac{5}{9}$

Substituting this value in the former equation, we get

$$a=\frac{-\frac{5}{9}+2}{-\frac{5}{9}+1}=\frac{13}{4}$$

Since the ratio of division is negative, the division is external, *i.e.*, P divides AB externally in the ratio of $5:9$ and the coordinates of the point of division are $\left(\frac{13}{4}, 0\right)$.

Example 10. *If $(-3, 4)$ is the centroid of the triangle whose vertices are $(6, 2)$, $(x, 3)$, $(0, y)$, find x and y.*

Solution. Using the centroid formula, we have

$$-3=\frac{6+x+0}{3} \Rightarrow x=-15$$

and

$$4=\frac{2+3+y}{3} \Rightarrow y=7$$

15·9. AREA OF A TRIANGLE

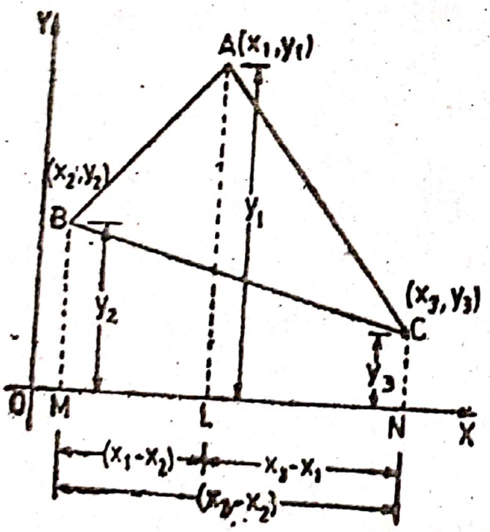


Fig. 9.

We can find out the area of a triangle with the vertices given.

For this let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the coordinates of the vertices of the triangle ABC. From A, B, C draw perpendiculars AL, BM and CN on the x-axis.

As is clear from the figure, area of $\triangle ABC$

$=$Area of trapezium $ABML$

$+$Area of trapezium $ALNC-$Area of trapezium $BMNC$

Since the area of the trapezium

$=\frac{1}{2}$ [Sum of the parallel sides]

$\times$[Perpendicular distance between the parallel sides]

Hence the area of the ΔABC can be given as

$$\Delta ABC=\tfrac{1}{2}(BM+AL)ML+\tfrac{1}{2}(AL+CN)LN-\tfrac{1}{2}(BM+CN)MN$$

$$=\tfrac{1}{2}(y_2+y_1)(x_1-x_2)+\tfrac{1}{2}(y_1+y_3)(x_3-x_1)-\tfrac{1}{2}(y_3+y_2)(x_3-x_2)$$

$$=\tfrac{1}{2}[x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3] \qquad ...(1)$$

The above on simplification can take the following form :

$$=\tfrac{1}{2}[(x_1y_2-x_1y_3)+(x_2y_3-x_2y_1)+(x_3y_1-x_3y_2)] \qquad ...(2)$$

$$=\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)] \qquad ...(3)$$

Remarks 1. The sign of the area of the triangle is positive or negative as the arrangement of vertices are counter-clockwise or clockwise as shown below.

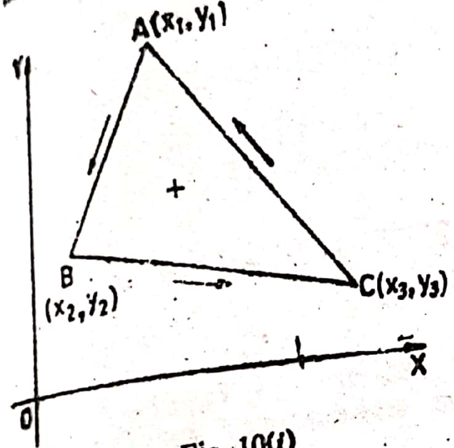


Fig. 10(*i*)

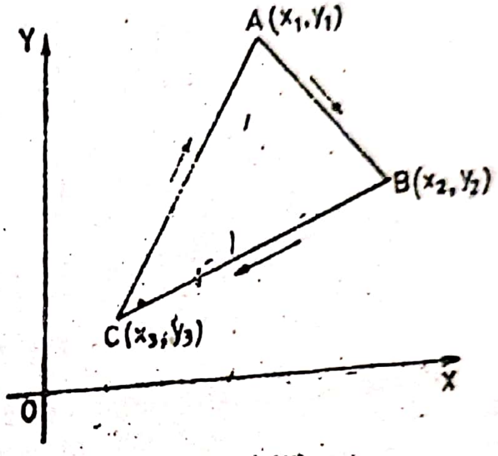


Fig. 10(*ii*)

The proper area formula is therefore given by

$$\text{Area of } \Delta=\pm\tfrac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3)$$

2. *Aid to memory.* The formula for area of ΔABC can be conveniently remembered by using the memory chart in the following form :

$$\text{Area of a } \Delta=\tfrac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix}$$

(where dotted arrow is preceded by a minus sign and the other one by a plus sign).

Example 11. *Find the area of the triangle whose vertices are (2, 3), (5, 7), (−3, 4).*

Solution. Using the above aid to memory, *viz.*,

$$\tfrac{1}{2}\begin{bmatrix} 2 & 5 & -3 & 2 \\ 3 & 7 & 4 & 3 \end{bmatrix}, \text{ we get}$$

$$\text{Area of } \Delta=\tfrac{1}{2}[2.7-3.5+5.4-7.(-3)+(-3).3-4.2]$$

$$=\tfrac{1}{2}(14-15+20+21-9-8)$$

$$=11{\cdot}5 \text{ sq. units}$$

It should be noted that the vertices are taken anti-clockwise and therefore, the result is positive. If we take in the reverse order placing $(-3, 4)$ as (x_1, y_1), the result will be negative.

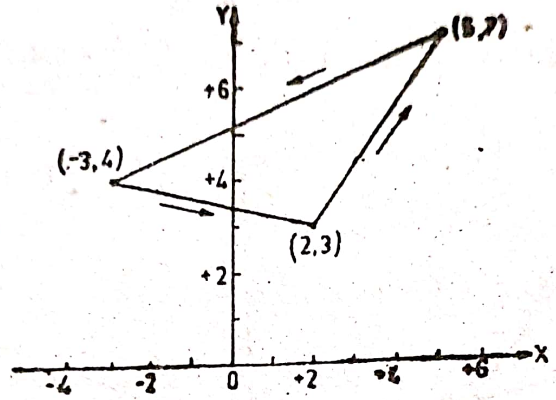


Fig. 11.

Example 12. *The vertices of a triangle ABC are $A(5, 2)$, $B(-9, -3)$ and $C(-3, -5)$, D, E, F are respectively the mid-points of BC, CA and AB. Prove that*

$$\Delta ABC=4\ \Delta DEF$$

Solution. Area of ΔABC is

$$\Delta ABC=\tfrac{1}{2}[5\{-3-(-5)\}+(-9)\{-5-2\}+(-3)\{2-(-3)\}]$$

$$=\tfrac{1}{2}[10+63-15]=29 \text{ sq. units} \qquad ...(1)$$

Also the coordinates of D, E and F are

$$D=\left[\frac{(-9)+(-3)}{2}, \frac{(-3)+(-5)}{2}\right]=(-6, -4)$$

$$E=\left[\frac{-3+5}{2}, \frac{-5+2}{2}\right]=\left(1, -\frac{3}{2}\right)$$

$$F=\left[\frac{-9+5}{2}, \frac{-3+2}{2}\right]=(-2, -\tfrac{1}{2})$$

$$\therefore \quad \Delta DEF=\tfrac{1}{2}[(-6)\{-\tfrac{3}{2}-(-\tfrac{1}{2})\}+1\{-\tfrac{1}{2}-(-4)\}+(-2)\{-4-(-\tfrac{3}{2})\}]$$

$$=\frac{1}{2}\left[6+\frac{7}{2}+5\right]=\frac{29}{4} \text{ sq. units} \qquad ...(2)$$

From (1) and (2), we conclude that

$$\Delta ABC=4\Delta DEF$$

Example 13. *Prove that the triangle formed by the points $A(8, -10)$, $B(7, -3)$, $C(0, -4)$ is a right angled triangle.*

Solution. We know that in a right angled triangle

$$AC^2 = AB^2 + BC^2$$

Also distance between two points P and Q is

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \quad AB^2 = (7-8)^2 + (-3+10)^2 = 50$$

$$BC^2 = (0-7)^2 + (-4+3)^2 = 50$$

$$AC^2 = (8-0)^2 + (-10+4)^2 = 100$$

$$AC^2 = AB^2 + BC^2$$

or and hence the triangle is a right angled triangle.

15·10. COLLINEARITY OF THREE POINTS

In case the three points of a triangle are in a straight line, they are called collinear. The area of such a triangle is equal to zero as indicated below :

$$\text{Area of } \Delta = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) = 0$$

Example 14. *Show that the following points are collinear :*

$P(3, -2)$, $Q(-1, 1)$, $R(-5, 4)$.

Solution.

Memory Chart :

$$\Delta = \frac{1}{2}\begin{bmatrix} 3 & -1 & -5 & 3 \\ -2 & 1 & 4 & -2 \end{bmatrix}$$

$$\text{Area of } \Delta PQR = \tfrac{1}{2}[(3)(1) - (-2)(-1) + (-1)(4) - (1)(-5) + (-5)(-2) - (4)(3)]$$

$$= \tfrac{1}{2}[3 - 2 - 4 + 5 + 10 - 12] = \tfrac{1}{2}[0] = 0$$

$\therefore$ Hence the points P, Q and R are collinear.

Example 15. *Find the area of the triangle with vertices $A(3, 1)$, $B(2k, 3k)$ and $C(k, 2k)$. Show that the three distinct points A, B and C are collinear when $k = -2$.* [I.C.W.A., December 1990]

Solution. Area of the ΔABC

$$= \tfrac{1}{2}[3(3k - 2k) + 2k(2k - 1) + k(1 - 3k)]$$

$$= \tfrac{1}{2}(3k + 4k^2 - 2k + k - 3k^2)$$

$$= \tfrac{1}{2}(k^2 + 2k).$$

$\therefore$ The three points A, B and C will be collinear if

$$\tfrac{1}{2}(k^2+2k)=0$$

$$\Rightarrow \qquad k=0 \qquad \text{or} \qquad k=-2.$$

15·11. AREA OF A QUADRILATERAL

In case of a quadrilateral, it is possible to split it into two triangles and then add the area of them, *i.e.*,

$$\text{Quadrilateral } ABCD = \Delta ABC + \Delta ADC$$

$$= \tfrac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]$$
$$+\tfrac{1}{2}[(x_1y_3-x_3y_1)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$$
$$= \tfrac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$$

The area of the quadrilateral with 4 sides can be found out by extending the same aid to memory illustrated for a triangle.

$$\text{Area of quad.} = \pm\tfrac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$$

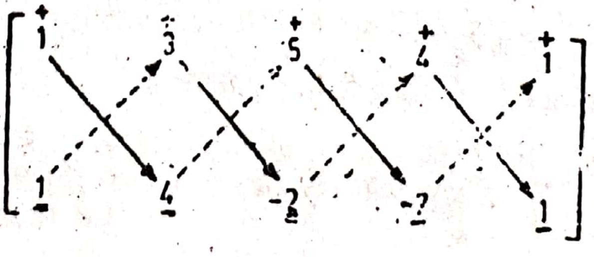
The same as given above because the middle two terms cancel out. The area is indicated by the $\pm$ sign. The sign of area will be positive if the vertices are taken counter clockwise and negative if taken clockwise.

Example 16. *Find the area of a quadrilateral whose vertices are*

$$A(1, 1)\ ;\ B(3, 4)\ ;\ C(5, -2) \text{ and } D(4, -7)$$

Solution. Let us make use of the above formula and the memory chart for the sake of convenience.

$=\frac{1}{2}$



$$\text{Area of quad.} = \tfrac{1}{2}[\{(4)-(3)\}+\{(-6)-(20)\}+\{(-35)-(-8)\}+\{(4)-(-7)\}]$$

$$= \frac{1}{2}[1+(-26)+(-27)+11] = -\frac{41}{2} \text{ sq. units.}$$

EXERCISE (I)

1. Find the distance between the following pair of points :

(*i*) $(0, 0)$; (p, q), (*ii*) $(9, -1)$; $(-2, 10)$, (*iii*) $(\frac{2}{3}, -\frac{1}{2})$; $(-\frac{1}{3}, -\frac{1}{2})$

(*iv*) $(1+\sqrt{2}, 2)$; $(1, 1-\sqrt{2})$, (*v*) $(at_1^2, 2at_1)$; $(at_2^2, 2at_2)$

(*vi*) $(a \cos \alpha, a \sin \alpha)$, $(a \cos \beta, a \sin \beta)$.

2. (*a*) If the point $(a, 3)$ is at a distance of $\sqrt{5}$ units from the point $(2, a)$, find a.

(*b*) What will be the values of x if the distance between $(x, -4)$ and $(-8, 2)$ be 10 ?

(*c*) If the distance between the points $(a, -5)$ and $(2, a)$ is 13, find a.

3. Show that the points

$(1, -1)$, $(-1, 1)$ and $(-\sqrt{3}, -\sqrt{3})$

are the vertices of equilateral triangle.

4. The points $(3, 4)$ and $(-2, 3)$ form with another point (x, y) an equilateral triangle. Find x and y.

5. Prove that the triangle with vertices at the points $(0, 3)$, $(-2, 1)$ and $(-1, 4)$ is right angled.

6. Show that the triangle whose vertices are $(1, 10)$, $(2, 1)$, and $(-7, 0)$ is an isosceles triangle. Find also the altitude of this triangle.

7. Prove that the points

$$\left(\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right), \left(-\frac{\sqrt{3}a}{2}, \frac{a}{2}\right), \left(-\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right)$$

and $\left(\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$

are the vertices of a square.

8. Show that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the vertices of a rectangle.

9. Prove that the following points are the vertices of a parallelogram

(*i*) $(2, 1)$, $(5, 2)$, $(6, 4)$ and $(3, 3)$

(*ii*) $(0, 0)$, $(a, 0)$, $(a+b, c)$, (b, c)

10. Find the coordinates of the circumcentre of a triangle whose coordinates are $(7, -1)$, $(5, 1)$ and $(-3, -7)$. Hence find the circumradius.

11. If $(-3, 2)$; $(1, -2)$ and $(5, 6)$ are the mid-points of the sides of a triangle, find the coordinates of the vertices of the triangle.

12. Prove that the point $(3, 3)$ is equidistant from $(0, -1)$, $(-2, 3)$, $(6, 7)$ and $(8, 3)$. Find this distance and show that the point is the intersection of the diagonals of a rectangle formed by the four points.