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Topological Galois theory

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ABSTRACT

We introduce an abstract topos-theoretic framework for building Galois-type theories in a variety of different mathematical contexts; such theories are obtained from representations of certain atomic two-valued toposes as toposes of continuous actions of a topological group. Our framework extends Grothendieck's theory of Galois categories and allows to build Galois-type equivalences in new contexts, such as for example graph theory and finite group theory.

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1. Introduction

The present work provides a general framework, based on Topos Theory, for building Galois-type theories in a variety of different mathematical contexts.

Most notably, we identify a set of necessary and sufficient conditions on a category (resp. on a small category) for it to be equivalent to the category of continuous actions (resp. of continuous non-empty transitive actions) of a topological group on discrete sets. We also intrinsically characterize the categories which can be represented as full subcategories of categories of non-empty transitive actions of a topological group, and describe an elementary process for 'completing' them so as to make them equivalent to such categories of actions.

We show in particular that many classical categories can be naturally embedded into Galois-type categories; for instance, this is the case for the category of finite linear orders and embeddings, the category of finite graphs and embeddings, the category of finite Boolean algebras and injective homomorphisms, or the category of finite groups and injective homomorphisms.

In order to illustrate our main results, we briefly review the classical (infinite) Galois theory and its categorical interpretation.

Let $F \subseteq L$ be a Galois extension, not necessarily finite-dimensional. The group $Aut_F(L)$ of automorphisms of L which fix F can be naturally made into a topological group by endowing it with the so-called *Krull topology*, that is the topology in which the subgroups of $Aut_F(L)$ consisting of the automorphisms which fix a given finite family of elements of L form a basis of open neighborhoods of the identity.

The fundamental theorem of infinite Galois provides an order-reversing bijective correspondence between the intermediate extensions K and the closed subgroups of the Galois group $Aut_F(L)$, which restricts to a bijective correspondence between the *finite intermediate extensions* and the *open subgroups* of $Aut_F(L)$. The correspondence assigns to any intermediate extension K the subgroup of $Aut_F(L)$ consisting of the automorphisms which fix K, and conversely associates to any closed subgroup C of $Aut_F(L)$ the field extension of F consisting of all the elements of L which are fixed by every automorphism in C. In fact, the correspondence between finite extensions and open subgroups can be seen as the 'kernel' of the extended correspondence between the intermediate extensions and the closed subgroups, as the latter can be immediately obtained from the former by observing that, on the one hand, every intermediate extension can be expressed as the union of all its finite sub-extensions, and on the other, every closed subgroup of $Aut_F(L)$ is equal to the intersection of all the open subgroups containing it. The advantage of restricting our attention to this 'kernel' of the correspondence is that, as it was already observed by Grothendieck in [19], it admits a natural categorical interpretation, as a duality between the category $\mathcal{L}_F^L$ of intermediate extensions and field homomorphisms between them and the category $\mathbf{Cont}_t(Aut_F(K))$ of transitive (non-empty) continuous actions of the topological group $Aut_F(L)$ over a discrete (finite) set.

The starting point of the present work is the observation that the above-mentioned equivalence ${\mathcal{L}_F^L}^{\text{op}} \simeq \mathbf{Cont}_t(Aut_F(K))$ naturally extends to an equivalence of toposes

$$\mathbf{Sh}({\mathcal{L}_F^L}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_F(K)),$$

where $\mathbf{Cont}(Aut_F(K))$ is the topos of continuous actions (over discrete sets) of the topological group $Aut_F(K)$ and J_{at} is the atomic topology on ${\mathcal{L}_F^L}^{\text{op}}$ (recall that the *atomic topology*, on a category satisfying the property that any pair of arrows with common codomain can be completed to a commutative square, is the Grothendieck topology whose covering sieves are exactly the non-empty ones). Indeed, $\mathbf{Cont}_t(Aut_F(K))$ is a J_{can}-dense subcategory of $\mathbf{Cont}(Aut_F(K))$, where J_{can} is the canonical topology on $\mathbf{Cont}(Aut_F(K))$, and the Grothendieck topology induced by J_{can} on $\mathbf{Cont}_t(Aut_F(K))$ coincides with the atomic topology on $\mathbf{Cont}_t(Aut_F(K))$ (notice that the objects of the subcategory $\mathbf{Cont}_t(Aut_F(K))$ are precisely the atoms of the topos $\mathbf{Cont}(Aut_F(K))$), whence the Comparison Lemma yields an equivalence

$$\mathbf{Sh}(\mathbf{Cont}_t(Aut_F(K)), J_{at}) \simeq \mathbf{Cont}(Aut_F(K)).$$

This observation suggests that a natural context for building analogues of classical Galois theory in different mathematical domains could be provided by equivalences of toposes of the form

$$\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{D}}(u)),$$

where $\mathcal{C}$ is a category satisfying the *amalgamation property* (i.e. the property that every pair or arrows with common domain can be completed to a commutative square), J_{at} is the atomic topology on it, $\mathcal{D}$ is a category in which $\mathcal{C}$ embeds, u is an object of $\mathcal{D}$ and $Aut_{\mathcal{D}}(u)$ is the group of automorphisms of u in $\mathcal{D}$ endowed with a topology in which the collection of subgroups of the form $\{f : u \cong u \mid f \circ \chi = \chi\}$ for arrows $\chi : c \to u$ in $\mathcal{C}$ forms a basis of open neighborhoods of the identity. Notice that, by Diaconescu's theorem, such an equivalence must be induced by a functor $F : \mathcal{C}^{\text{op}} \to \mathbf{Cont}(Aut_{\mathcal{D}}(u))$ – necessarily taking values in the subcategory $\mathbf{Cont}_t(Aut_{\mathcal{D}}(u))$ of $\mathbf{Cont}(Aut_{\mathcal{D}}(u))$ on the transitive (non-empty) continuous $Aut_{\mathcal{D}}(u)$-actions (since the images of representable functors on $\mathcal{C}$ via the associated sheaf functor $a_{J_{at}} : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$ are atoms of the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$). In light of classical Galois theory, it is natural to require this functor to be the one sending any object of $\mathcal{C}$ to the set $Hom_{\mathcal{D}}(c, u)$, with the obvious action of $Aut_{\mathcal{D}}(u)$, and any arrow $f : d \to c$ of $\mathcal{C}$ to the $Aut_{\mathcal{D}}(u)$-equivariant map $- \circ f : Hom_{\mathcal{D}}(c, u) \to Hom_{\mathcal{D}}(d, u)$.

Some conditions on $\mathcal{C}$, $\mathcal{D}$ and u are necessary in order for such an equivalence to exist:

(i) $\mathcal{C}$ must be non-empty and satisfy the *joint embedding property*, i.e. the property that for any pair of objects $a, b \in \mathcal{C}$ there exists an object $c \in \mathcal{C}$ and arrows $a \to c$ and $b \to c$ in $\mathcal{C}$; indeed, any topos of the form $\mathbf{Cont}(G)$ for a topological group G is atomic and two valued, and it is easy to see that a topos of the form $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$, for a non-empty category $\mathcal{C}$ satisfying the amalgamation property, is (atomic and) two-valued if and only if $\mathcal{C}$ satisfies the joint embedding property;

(ii) The object u must satisfy the property that any object c of $\mathcal{C}$ admits an arrow to u in $\mathcal{D}$ (u is $\mathcal{C}$-universal) and the action of $Aut_{\mathcal{D}}(u)$ on each $Hom_{\mathcal{D}}(c, u)$ is transitive (u is $\mathcal{C}$-ultrahomogeneous).

The following theorem, which we prove in section 3, shows that if $\mathcal{C} \hookrightarrow \mathcal{D}$ is the canonical embedding of $\mathcal{C}$ into its ind-completion Ind-$\mathcal{C}$, then the two above-mentioned conditions are also sufficient.

Theorem. *Let $\mathcal{C}$ be a small non-empty category satisfying the amalgamation and joint embedding properties, and let u be a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object in* Ind-$\mathcal{C}$. *Then the collection $\mathcal{I}_{\mathcal{C}}$ of sets of the form $\mathcal{I}_{\chi} := \{f : u \cong u \mid f \circ \chi = \chi\}$, for an arrow $\chi : c \to u$ from an object c of $\mathcal{C}$ to u defines an algebraic base for the group of automorphisms of u in* Ind-$\mathcal{C}$, *and, denoting by $Aut_{\mathcal{C}}(u)$ the resulting topological group, we have an equivalence of toposes*

$$\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(u))$$

induced by the functor $F : \mathcal{C}^{\text{op}} \to \mathbf{Cont}_t(Aut_{\mathcal{C}}(u))$ which sends any object c of $\mathcal{C}$ to the set $Hom_{\text{Ind-}\mathcal{C}}(c, u)$ (equipped with the obvious action by $Aut_{\mathcal{C}}(u)$) and any arrow $f : c \to d$ in $\mathcal{C}$ to the $Aut_{\mathcal{C}}(u)$-equivariant map $- \circ f : Hom_{\text{Ind-}\mathcal{C}}(d, u) \to Hom_{\text{Ind-}\mathcal{C}}(c, u)$.

On the other hand, Theorem 4.3 provides sufficient conditions, valid for any embedding $\mathcal{C} \hookrightarrow \mathcal{D}$, for such an equivalence to exist in the case the topology J_{at} is subcanonical.

Given an equivalence

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(u))$$

induced by a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}(Aut_{\mathcal{C}}(u))$ as in the theorem above, one can wonder under which conditions F is full and faithful. The following result, which we prove in section 4, gives an answer to this question.

Proposition. *Under the hypotheses of the above theorem, the following conditions are equivalent:*

(i) *Every arrow $f : d \to c$ in $\mathcal{C}$ is a strict monomorphism (in the sense that for any arrow $g : e \to c$ such that $h \circ g = k \circ g$ whenever $h \circ f = k \circ f$, g factors uniquely through f);*

(ii) *The functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{C}}(u))$ which sends any object c of $\mathcal{C}$ to the set $Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$ (endowed with the obvious action by $Aut_{\mathcal{C}}(u)$) and any arrow $f : c \to d$ in $\mathcal{C}$ to the $Aut_{\mathcal{C}}(u)$-equivariant map*

$$- \circ f : Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(d, u) \to Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$$

is full and faithful;

(iii) *For any objects $c, d \in \mathcal{C}$ and any arrows $\chi : c \to u$ and $\xi : d \to u$ in* Ind-$\mathcal{C}$, $\mathcal{I}_{\xi} \subseteq \mathcal{I}_{\chi}$ *(that is, for any automorphism f of u, $f \circ \xi = \xi$ implies $f \circ \chi = \chi$) if and only if there exists a unique arrow $f : c \to d$ in $\mathcal{C}$ such that $\chi = \xi \circ f$:*

$$\begin{tikzcd} c \arrow[r, "\chi"] \arrow[d, dashed, "f"'] & u \\ d \arrow[ur, "\xi"'] & \end{tikzcd}$$

If any of these conditions is satisfied, F yields an essentially full and faithful functor from the opposite of the slice category $\mathcal{C}/u$ to the poset category of open subgroups of the group $Aut_{\mathcal{C}}(c)$; in other words, the following Galois-type property holds: for any arrows $\chi : c \to u$ and $\xi : d \to u$ from objects c and d of $\mathcal{C}$ to u, if all the automorphisms of u which fix χ also fix ξ then ξ factors uniquely through χ in Ind-$\mathcal{C}$. We shall speak in this context of a *concrete Galois theory*. As we show in section 5, examples of such theories abound; for instance, there exists a concrete Galois theory for the category of finite groups and injective homomorphisms between them.

Under the assumption that $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}_t(Aut_{\mathcal{C}}(u))$ is full and faithful, it is natural to wonder under which conditions F is also essentially surjective and hence yields a Galois-type correspondence between the isomorphism classes of objects of the slice category $\mathcal{C}/u$ and the open subgroups of the topological group $Aut_{\mathcal{C}}(u)$. We shall speak in this case of a *standard Galois theory*. By considering the invariant notion of atom in terms of the two toposes related by the equivalence, we obtain in section 4 a purely elementary necessary and sufficient condition on the category $\mathcal{C}$ for the associated functor F to be essentially surjective, which can be interpreted as a form of 'elimination of imaginaries' for $\mathcal{C}$ in the sense of Model Theory; also, we describe for a general $\mathcal{C}$ its completion $\mathcal{C}_{at}$ with respect to 'imaginaries', for which a standard Galois equivalence holds.

Our criteria are widely applicable in practice: besides Fraïssé's construction in Model Theory and its generalizations (cf. Chapter 7 of [21] and [6]), which constitute primary means for constructing universal ultrahomogeneous objects, there are other natural means for obtaining equivalences of toposes of the kind considered above: exploiting the theory of special models for atomic complete theories, Grothendieck's Galois Theory or the representation theory of Grothendieck toposes (cf. section 6).

The scope of applicability of this framework is much broader than that of Grothendieck's theory of Galois categories (cf. [19]), as well as of its infinitary generalization established in [29]. Indeed, the latter corresponds to choosing categories $\mathcal{C}$ which are pretoposes (finitary in the case of [19] or infinitary in the case of [29]) and the automorphisms groups arising from such constructions are always topologically prodiscrete (and in the finitary context profinite).

Also, interpreting Galois-type equivalences in terms of different representations of a given topos paves the way for a systematic use of topos-theoretic invariants for transferring notions and results across the two sides, according to the methodologies 'toposes as bridges' of [11]:

$$\begin{tikzcd}[column sep=small] & \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(u)) \arrow[dl, dashed, no head] \arrow[dr, dashed, no head] & \\ \mathcal{C}^{\mathrm{op}} & & Aut_{\mathcal{C}}(u) \end{tikzcd}$$

In fact, this technique is applied to derive the main results of section 4.

From a logical point of view, the 'Galois-type' theories (i.e., the geometric theories classified by a topos of continuous actions of a topological group) are maximal with respect to the natural (Heyting algebra) ordering between geometric theories over a given signature defined in [10] ($\mathbb{T} \leq \mathbb{T}'$ if and only if $\mathbb{T}'$ is a quotient of $\mathbb{T}$, i.e. every geometric sequent which is provable in $\mathbb{T}$ is also provable in $\mathbb{T}'$); in other words, they do not have any proper quotients over their signature. Conversely, by the results in [7] and [10] (cf. also [3]), any maximal theory which is finitary or written over a countable signature and has a model in **Set** is a Galois-type theory. The question thus naturally arises of whether there is a canonical way for extending an arbitrary geometric theory to a Galois-type theory. The results of [6], combined with those of the present paper, show that the answer to this question is positive for a very large class of theories. Specifically, for any theory of presheaf type $\mathbb{T}$ such that its category of finitely presentable models satisfies the amalgamation property, if there exists a homogeneous $\mathbb{T}$-model in **Set** and the theory $\mathbb{T}$ is finitary or written over a countable signature, there is a completion of the theory of homogeneous $\mathbb{T}$-models which is consistent (i.e., has a model in **Set**) and hence is a Galois-type theory extending $\mathbb{T}$.

This paper can be considered as a companion to [16], where, following [24], a localic Galois theory for atomic connected toposes is developed. Nonetheless, it should be noted that there are important differences between the localic and the topological setting, the main ones being the following:

(i) Whilst any localic group gives naturally rise to a topological group, there are topological groups which do not arise in this way (in fact, it is not even true that every sober topological group can be regarded as a localic group, due to the fact that the canonical functor from the category of topological spaces to that of locales does not preserve finite limits);
(ii) Whilst every atomic connected topos can be represented (by the main result in [16]) as the topos of continuous actions of the localic automorphism group of *any* of its points, an analogous representation in terms of the topological automorphism group of the point holds only if the point satisfies some special properties, as it is shown in the paper.

The plan of the paper is as follows.

In section 2 we investigate topological groups in relation to the associated categories of continuous actions on discrete sets. We define the notion of algebraic base $\mathcal{B}$ of a topological group G and obtain a representation theorem for the topos $\mathbf{Cont}(G)$ in terms of $\mathcal{B}$. We then characterize, in terms of algebraic bases, the topological groups G which are isomorphic to the group of automorphisms of the canonical point of the associated topos $\mathbf{Cont}(G)$.

In section 3, we prove the above-mentioned representation theorem. We first establish a logical formulation of it, and then proceed to translate it in categorical terms.

In section 4, besides investigating under which conditions the functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}_t(Aut_{\mathcal{D}}(u))$ in the above-mentioned characterization theorem is full and faithful (resp. an equivalence), we describe an elementary process for completing a category $\mathcal{C}$ satisfying this condition to one for which the functor F is an equivalence. We also discuss the relationship between the notions of regular and of strict monomorphism and discuss what can be said about the relationship between $\mathcal{C}$ and $Aut_{\mathcal{C}}(u)$ in case F is not full and faithful (equivalently, not all the arrows of $\mathcal{C}$ are strict monomorphisms).

In section 5 we describe some new examples of ‘concrete Galois theories’ in different mathematical fields obtained by applying the methods developed in the paper: these include a Galois theory for graphs, Boolean algebras, finite groups, and linear orders.

In section 6 we discuss the main methods for obtaining pairs $(\mathcal{C}, u)$ satisfying the hypotheses of our main theorem.

2. Topological groups and their toposes of continuous actions

Let us recall that a topological group is a group G with a topology such that the group operation and the inverse operation are continuous with respect to it; for basic background on topological groups we refer the reader to [15].

2.1. Algebraic bases and dense subcategories of actions

The following well-known result allows to make a given group into a topological group starting from a collection of subsets of the group satisfying particular properties:

Lemma 2.1. *Let G be a group and $\mathcal{B}$ be a collection of subsets N of G containing the neutral element e. Then there exists a topology τ on G having $\mathcal{B}$ as a neighborhood basis of e and making (G, τ) into a topological group if and only if all the following conditions are satisfied:*

(i) For any $N, M \in \mathcal{B}$ there exists $P \in \mathcal{B}$ such that $P \subseteq N \cap M$;
(ii) For any $N \in \mathcal{B}$ there exists $M \in \mathcal{B}$ such that $M^2 \subseteq N$;
(iii) For any $N \in \mathcal{B}$ there exists $M \in \mathcal{B}$ such that $M \subseteq N^{-1}$;
(iv) For any $N \in \mathcal{B}$ and any $a \in G$ there exists $M \in \mathcal{B}$ such that $M \subseteq aNa^{-1}$.

A couple of remarks:

- If the topology τ in the statement of the lemma exists then it is necessarily unique; indeed, if $\mathcal{B}$ is a basis of neighborhoods of e then for any $a \in G$, the collection of sets of the form aN for a set N in $\mathcal{B}$ forms a basis of neighborhoods of a, whence the open sets in τ are exactly the unions of sets of the form aN for $N \in \mathcal{B}$.
  We shall call τ the *topology on G generated by $\mathcal{B}$* and denote it by $\tau^G_{\mathcal{B}}$; we shall denote the resulting topological group by $G_{\mathcal{B}}$.
- If all the subsets in the family $\mathcal{B}$ are subgroups of G then conditions (ii) and (iii) in the statement of the lemma are automatically satisfied.
  This motivates the following definition: we say that a collection $\mathcal{B}$ of subgroups of G is an *algebraic base* for G if it is a basis of neighborhood of e, any intersection of subgroups in $\mathcal{B}$ contains a subgroup in $\mathcal{B}$, and any conjugated of a subgroup in $\mathcal{B}$ lies in $\mathcal{B}$.

Let us denote by $\mathbf{GTop}$ the category of topological groups and continuous group homomorphisms between them. We can construct a category $\mathbf{GTop_b}$ of 'groups paired with algebraic bases' as follows: the objects of $\mathbf{GTop_b}$ are pairs $(G, \mathcal{B})$ consisting of a group G and an algebraic base $\mathcal{B}$ for it, while the arrows $(G, \mathcal{B}) \to (G', \mathcal{B}')$ in $\mathbf{GTop_b}$ are the group homomorphisms $f : G \to G'$ such that for any $V \in \mathcal{B}'$, $f^{-1}(V) \in \mathcal{B}$. We have a functor $F : \mathbf{GTop_b} \to \mathbf{GTop}$ sending to any object $(G, \mathcal{B})$ of $\mathbf{GTop_b}$ the topological group $(G, \tau_{\mathcal{B}}^G)$ and acting accordingly on arrows. On the other hand, any topological group G has a canonical algebraic base C_G, namely the one consisting of all the open subgroups of it; this yields a functor $G : \mathbf{GTop} \to \mathbf{GTop_b}$ sending G to (G, C_G) and acting on arrows in the obvious way. It is easily verified that G is left adjoint to F and $F \circ G \cong 1_{\mathbf{GTop}}$, which allows us to regard $\mathbf{GTop}$ as a full subcategory of $\mathbf{GTop_b}$.

For any topological group G, one can construct a topos $\mathbf{Cont}(G)$, whose objects are the left continuous actions $G \times X \to X$ (where X is endowed with the discrete topology and $G \times X$ with the product topology) and whose arrows are the G-equivariant maps between them. Recall that a left action $\alpha : G \times X \to X$ is continuous if and only if for every $x \in X$ the isotropy subgroup $I_x := \{g \in G \mid \alpha(g, x) = x\}$ is open in G. The topos $\mathbf{Cont}(G)$ is atomic; in fact, its atoms are precisely the non-empty transitive continuous actions. Notice that a non-empty transitive action $\alpha : G \times X \to X$ can be identified with the canonical action $G \times G/I_x \to G/I_x$ on the set G/U of left cosets gI_x of the isotropy group I_x of α at any point $x \in X$; conversely, for any open subgroup U of G, the canonical action of G on the set G/U makes G/U into a non-empty transitive G-set.

For any open subgroups U and V of G, the arrows $G/U \to G/V$ in $\mathbf{Cont}(G)$ can be identified with their action on the trivial coset, that is with the V-cosets of the form aV where a is an element of G such that $U \subseteq a^{-1}Va$. Such an arrow is an isomorphism if and only if $U = a^{-1}Va$; indeed, it is invertible if and only if there exists $b \in G$ such that $ba \in U$, $ab \in V$ and $V \subseteq b^{-1}Ub$; but $ba \in U$ implies that a^{-1} and b are equivalent modulo U and hence $V \subseteq b^{-1}Ub$ is equivalent to $V \subseteq a^{-1^{-1}}Ua^{-1} = aUa^{-1}$. We denote by $\mathbf{Cont}_t(G)$ the full subcategory of $\mathbf{Cont}(G)$ on the G-sets of the form G/U.

Two continuous G-sets X and Y are isomorphic in $\mathbf{Cont}(G)$ if and only if the sets of isotropy subgroups at elements of X and of isotropy subgroups at elements of Y contain exactly the same subgroups as elements, if the latter are considered up to conjugation (by some element of G); in particular, any two transitive continuous G-sets are isomorphic if and only if any two isotropy subgroups of them are conjugate to each other.

There is a natural link between algebraic bases for a topological group and dense subcategories of the associated topos of continuous actions.

Remark 2.2. For any algebraic base for G, the G-sets of the form G/U for $U \in \mathcal{B}$ define a dense full subcategory of $\mathbf{Cont}_t(G)$ (in the sense that for any object of $\mathbf{Cont}_t(G)$ there exists an arrow from a G-set of this form to it) which is closed under isomorphisms. Conversely, any dense full subcategory of $\mathbf{Cont}_t(G)$ which is closed under isomorphisms gives rise to an algebraic base for G, namely the base consisting of the open subgroups U of G such that G/U lies in the subcategory. The algebraic bases for G can be thus identified with the dense full subcategories of $\mathbf{Cont}_t(G)$ which are closed under isomorphisms.

Proposition 2.3. *For any algebraic base $\mathcal{B}$ of a group G, the full subcategory $\mathbf{Cont}_{\mathcal{B}}(G)$ of $\mathbf{Cont}_t(G)$ on the objects of the form G/U for $U \in \mathcal{B}$ satisfies the dual of the amalgama-*

tion property and the dual of the joint embedding property (cf. section 3 for a definition of these notions).

Proof. To prove that $\mathbf{Cont}_{\mathcal{B}}(G)$ satisfies the dual of the amalgamation property it suffices to recall for any atomic topos $\mathcal{E}$ and any separating set of atoms of $\mathcal{E}$, the full subcategory $\mathcal{A}$ of $\mathcal{E}$ on this set of atoms satisfies the dual of the amalgamation property (this can be proved by observing that the pullback of two arrows between atoms in $\mathcal{E}$ cannot be zero, since the domains of the two arrows are epimorphic images of it, and hence there is an arrow to it from an atom in the separating set).

On the other hand, $\mathbf{Cont}_{\mathcal{B}}(G)$ satisfies the dual of the joint embedding property, as for any two open subgroups U and V of G belonging to $\mathcal{B}$, we have arrows $G/W \to G/U$ and $G/W \to G/V$ for any W in B such that $W \subseteq U \cap V$. $\square$

Now, since the subcategory $\mathbf{Cont}_{\mathcal{B}}(G)$ is dense in $\mathbf{Cont}_t(G)$ and hence in the topos $\mathbf{Cont}(G)$, Grothendieck's Comparison Lemma yields an equivalence

$$\mathbf{Sh}(\mathbf{Cont}_{\mathcal{B}}(G), J_{at}) \simeq \mathbf{Cont}(G),$$

where J_{at} is the atomic topology on $\mathbf{Cont}_{\mathcal{B}}(G)$.

2.2. Complete topological groups

The topos $\mathbf{Cont}(G)$ has a canonical point p_G, namely the geometric morphism $\mathbf{Set} \to \mathbf{Cont}(G)$ whose inverse image is the forgetful functor $\mathbf{Cont}(G) \to \mathbf{Set}$. Let us denote by $Aut(p_G)$ the group of automorphisms of p_G in the category of points of $\mathbf{Cont}(G)$. We have a canonical map $\xi_G : G \to Aut(p_G)$, sending any element $g \in G$ to the automorphism of p_G which acts at each component as multiplication by the element g (this is indeed an automorphism since the naturality conditions hold as the maps in $\mathbf{Cont}(G)$ are G-equivariant).

As shown in [29], for any topological group G, the group $Aut(p_G)$ can intrinsically be endowed with a pro-discrete topology (that is a topology which is a projective limit of discrete topologies) in which the open subgroups are those subgroups of the form $U_{(X,x)}$ for a continuous G-sets X and an element $x \in X$, where $U_{(X,x)}$ denotes the set of automorphisms $\alpha : p_G \cong p_G$ such that $\alpha(X)(x) = x$; the canonical map $\xi_G : G \to Aut(p_G)$ is continuous with respect to this topology.

It is natural to characterize the topological groups G for which the map ξ_G is a bijection (equivalently, a homeomorphism). Following the terminology of [29], we shall call such groups *complete*, and we shall refer to the topological group $Aut(p_G)$ as to the *completion* of G. For any complete group G with an algebraic base $\mathcal{B}$, we can alternatively describe the topology on $Aut(p_G)$ induced by the topology on G via the bijection ξ_G as follows: a basis of open neighborhoods of the identity is given by the sets of the form $\{\alpha : p_G \cong p_G \mid \alpha(G/U)(eU) = eU\}$ for $U \in \mathcal{B}$.

We have the following characterization of complete groups.

Proposition 2.4. *Let G be a topological group with an algebraic base $\mathcal{B}$. Then G is complete if and only if for any assignment $U \to a_U$ of an element $a_U \in G/U$ to any subset $U \in \mathcal{B}$ such that for any G-equivariant map $m : G/U \to G/V$, $m(a_U) = a_V$ (equivalently, such that for any inclusion $U \subseteq V$ of open subgroups in $\mathcal{B}$, $a_U \equiv a_V$ modulo V and for any open subgroup $U \in \mathcal{B}$ and any element $g \in G$ $a_{gUg^{-1}} \equiv a_U$ modulo U), there exists a unique $g \in G$ such that $a_U = gU$ for all $U \in \mathcal{B}$.*

Proof. Since the category $\mathbf{Cont}_t(G)$ is dense in $\mathbf{Cont}_t(G)$, the automorphisms of p_G correspond exactly to the isomorphisms of the flat functor $F : \mathbf{Cont}_t(G) \to \mathbf{Set}$ corresponding to p_G, that is of the forgetful functor. It thus remains to show that giving an automorphism of F corresponds to giving, for each $U \in \mathcal{B}$, an element $a_U \in G/U$ satisfying the hypotheses of the proposition. One direction is obvious; to prove the other one it suffices to observe that for any automorphism χ of F, $\chi_U([g]) = g[\chi_U(e)]$ for any $U \in \mathcal{B}$; this follows as an immediate consequence of the following two remarks:

(1) for any $U, V \in \mathcal{B}$ and any G-equivariant map $\gamma : G/V \to G/U$, we have $\chi_U(\gamma(eV)) = \chi_U(eU)\gamma(eV)$. Indeed, by the naturality of χ, we have $\chi_U(\gamma(eV)) = \gamma(\chi_V(eV))$; but there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$ whence there exists $g \in \chi_W(eW)$ such that $gU = \chi_U(eU)$ and $gV = \chi_V(eV)$, and by the equivariance of γ we have $\gamma(\chi_V(eV)) = \gamma(gV) = \gamma(g(eV)) = g\gamma(eV) = \chi_U(eU)\gamma(eV)$, as required.

(2) The map $G/g^{-1}Ug \to G/U$ sending $e(g^{-1}Ug)$ to gU is well-defined and equivariant. $\square$

Remarks 2.5.

(a) The uniqueness requirement in the statement of the proposition holds if and only if G is *nearly discrete*, that is if and only if the intersection of all the open subgroups of G is equal to the singleton $\{e\}$.
(b) Two notable classes of complete groups are discrete groups (obviously) and profinite groups (by Grothendieck's Galois Theory [19]).
(c) For any group G and algebraic base $\mathcal{B}$ for G, the collection of subsets of the form $\mathcal{I}_{U,x} := \{\alpha : p_G \cong p_G \mid \alpha(G/U)(x) = x\}$ for $x \in G/U$ and $U \in \mathcal{B}$ forms an algebraic base for the group $Aut(p_G)$ of automorphisms of p_G, and, if we consider $Aut(p_G)$ endowed with the resulting topology, the canonical map $\xi_G : G \to Aut(p_G)$ becomes a homomorphism of topological groups which induces a Morita-equivalence $\mathbf{Cont}(\xi_G) : \mathbf{Cont}(G) \simeq \mathbf{Cont}(Aut(p_G))$ between them.

3. A representation theorem

In this section we establish our main representation theorem, which generalizes the representation theorem for coherent Boolean classifying toposes established by A. Blass and A. Ščedrov in [3]. We shall establish our theorem by using logical techniques, and then translate it into categorical language.

3.1. Logical statement

Recall from section D3.4 [23] that, given a geometric theory $\mathbb{T}$ over a signature Σ, a geometric formula-in-context $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-*complete* if the sequent $(\phi \vdash_{\vec{x}} \bot)$ is not provable in $\mathbb{T}$, but for any for any geometric formula $\psi(\vec{x})$ over Σ in the same context, either $(\phi \vdash_{\vec{x}} \psi)$ is provable in $\mathbb{T}$ or $(\phi \wedge \psi \vdash_{\vec{x}} \bot)$ is provable in $\mathbb{T}$.

Theorem 3.1. *Let $\mathbb{T}$ be an atomic and complete theory, $\mathcal{I} := \{\phi_i(\vec{x_i}) \mid i \in I\}$ be a set of $\mathbb{T}$-complete formulae such that for any $\mathbb{T}$-complete formula $\phi(\vec{x})$ there exists a $\mathbb{T}$-provably functional formula (equivalently, $\mathbb{T}$-provably functional cover) from $\phi_i(\vec{x_i})$ to $\phi(\vec{x})$. Let M be a model of $\mathbb{T}$ in* **Set** *in which each of the $\phi_i(\vec{x_i})$ is realized and such that for any $i \in I$ and any $\vec{a}, \vec{b} \in [[\vec{x_i}.\phi_i]]_M$ there exists an automorphism f of M such that $f(\vec{a}) = \vec{b}$. Then, if we denote by $Aut_{\mathcal{I}}(M)$ the group of ($\mathbb{T}$-model) automorphisms of M, we have that the sets of the form $\mathcal{I}_{\vec{a}} := \{f : M \cong M \mid f(\vec{a}) = \vec{a}\}$, where $\vec{a} \in [[\vec{x_i}.\phi_i]]_M$ for some $i \in I$, form an algebraic base for $Aut_{\mathcal{I}}(M)$ and, if we endow $Aut_{\mathcal{I}}(M)$ with the resulting topology, we have an equivalence*

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Cont}(Aut_{\mathcal{I}}(M))$$

between the classifying topos of $\mathbb{T}$ and the topos of continuous $Aut_{\mathcal{I}}(M)$-sets (where $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the geometric syntactic site of $\mathbb{T}$). This equivalence restricts to an equivalence

$$\mathcal{C}_{\mathbb{T}}^{\mathcal{I}} \simeq \mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{I}}(M)),$$

where $\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}$ is the full subcategory of $\mathcal{C}_{\mathbb{T}}$ on the formulae of the form $\phi_i(\vec{x_i})$ and $\mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{I}}(M))$ is the full subcategory of $\mathbf{Cont}(Aut_{\mathcal{I}}(M))$ on the $Aut_{\mathcal{I}}(M)$-sets isomorphic to one of the form $Aut_{\mathcal{I}}(M)/\mathcal{I}_{\vec{a}}$, which sends any formula $\phi_i(\vec{x_i})$ in $\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}$ to the set $[[\phi_i(\vec{x_i})]]_M$ with the obvious $Aut_{\mathcal{I}}(M)$-action and any $\mathbb{T}$-provably functional formula θ from $\phi_i(\vec{x_i})$ to $\phi_j(\vec{x_j})$ to the $Aut_{\mathcal{I}}(M)$-equivariant map $[[\phi_i(\vec{x_i})]]_M \to [[\phi_j(\vec{x_j})]]_M$ whose graph is the interpretation $[[\theta]]_M$.

Proof. First, let us show that the sets of the form $\mathcal{I}_{\vec{a}}$ form an algebraic base for the group $Aut_{\mathcal{I}}(M)$. For any $\vec{a} \in [[\phi_i(\vec{x_i})]]_M$ and $\vec{b} \in [[\phi_j(\vec{x_j})]]_M$, by the atomicity of $\mathbb{T}$ there exists a $\mathbb{T}$-complete formula $\chi(\vec{x_i}, \vec{x_j})$ such that $M \vDash \chi(\vec{a}, \vec{b})$. By our hypothesis, there exists a $\mathbb{T}$-provably functional formula $\theta(\vec{x_k}, \vec{x_i}, \vec{x_j})$ from a formula of the form $\phi_k(\vec{x_k})$ to $\chi(\vec{x_i}, \vec{x_j})$, from which it follows, χ being $\mathbb{T}$-complete, that there exists $\vec{c} \in [[\phi_k(\vec{x_k})]]_M$ such that $[[\theta]]_M(\vec{c}) = (\vec{a}, \vec{b})$. Then $\mathcal{I}_{\vec{c}} \subseteq \mathcal{I}_{\vec{a}} \cap \mathcal{I}_{\vec{b}}$, since for any automorphism $f : M \cong M$, $f([[\theta]]_M(\vec{c})) = [[\theta]]_M(f(\vec{c}))$. To verify that every conjugate of a subgroup of the form $\mathcal{I}_{\vec{a}}$ is a subgroup of this form, we observe that for any automorphism $h : M \cong M$, $h\mathcal{I}_{\vec{a}}h^{-1} = \{hfh^{-1} \mid f(\vec{a}) = \vec{a}\} = \{hfh^{-1} \mid hfh^{-1}(h(\vec{a})) = h(\vec{a})\}$ and hence by symmetry $h\mathcal{I}_{\vec{a}}h^{-1} = \{s : M \cong M \mid s(h(\vec{a})) = h(\vec{a})\} = \mathcal{I}_{h(\vec{a})}$ (note that, h being an automorphism, $h(\vec{a}) \in [[\phi(\vec{x})]]_M$ if $\vec{a} \in [[\phi(\vec{x})]]_M$).

From these remarks it follows in particular that the generated topology on $Aut_{\mathcal{I}}(M)$ coincides with the *topology of pointwise convergence*, that is the topology whose basis of open neighborhoods of the identity is given by the sets of the form $\{f : M \cong M \mid f(\vec{a}) = \vec{a}\}$ for *any* $\vec{a} \in M$; indeed, as we have seen, any such set contains one of the form $\mathcal{I}_{\vec{c}}$.

Further, our hypotheses imply that M satisfies the more general property that for any $\vec{a}, \vec{b} \in M$ which satisfy exactly the same geometric (equivalently, the same $\mathbb{T}$-complete) formulae over the signature of $\mathbb{T}$, there exists an automorphism $f : M \cong M$ of M such that $f(\vec{a}) = \vec{b}$. Indeed, if $\vec{a}, \vec{b} \in [[\chi(\vec{x})]]_M$ for a $\mathbb{T}$-complete formula $\chi(\vec{x})$ then there exists a $\mathbb{T}$-provably functional formula $\theta(\vec{x_k}, \vec{x})$ from a formula of the form $\phi_k(\vec{x_k})$ to $\chi(\vec{x})$, from which it follows, χ being $\mathbb{T}$-complete, that there exists $\vec{c} \in [[\phi_k(\vec{x_k})]]_M$ such that $[[\theta]]_M(\vec{c}) = (\vec{a})$ and $\vec{d} \in [[\phi_k(\vec{x_k})]]_M$ such that $[[\theta]]_M(\vec{d}) = (\vec{b})$; any automorphism $f : M \cong M$ such that $f(\vec{c}) = \vec{d}$ will thus satisfy $f(\vec{a}) = \vec{b}$. By using similar arguments, one can also prove that every $\mathbb{T}$-complete (equivalently, geometric) formula is satisfied in M.

Now, since the sets $\mathcal{I}_{\vec{a}}$ form an algebraic base for the group $Aut_{\mathcal{I}}(M)$, by the results of section 2 we have an equivalence

$$\mathbf{Cont}(Aut_{\mathcal{I}}(M)) \simeq \mathbf{Sh}(\mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{I}}(M)), J_{at}),$$

where $\mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{I}}(M))$ is the full subcategory of $\mathbf{Cont}(Aut_{\mathcal{I}}(M))$ on the $Aut_{\mathcal{I}}(M)$-sets of the form $Aut_{\mathcal{I}}(M)/\mathcal{I}_{\vec{a}}$; therefore, showing that we have an equivalence $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Cont}(Aut_{\mathcal{I}}(M))$ amounts precisely to proving that we have an equivalence

$$\mathcal{C}_{\mathbb{T}}^{\mathcal{I}} \simeq \mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{I}}(M)),$$

where $\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}$ is the full subcategory of $\mathcal{C}_{\mathbb{T}}$ on the formulae of the form $\phi_i(\vec{x_i})$. To construct such an equivalence, we argue as follows. We have a functor $F : \mathcal{C}_{\mathbb{T}}^{\mathcal{I}} \to \mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{I}}(M))$ sending any $\mathbb{T}$-complete formula $\phi(\vec{x})$ in $\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}$ to the set $[[\phi(\vec{x})]]_M$, equipped with the obvious action by $Aut_{\mathcal{I}}(M)$, and any $\mathbb{T}$-provably functional formula θ from a formula $\phi(\vec{x})$ in $\mathcal{I}$ to a formula $\psi(\vec{y})$ in $\mathcal{I}$ to the $Aut_{\mathcal{I}}(M)$-equivariant map $[[\phi(\vec{x})]]_M \to [[\psi(\vec{y})]]_M$ whose graph coincides with the interpretation $[[\theta]]_M$ (this functor takes indeed values in $\mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{I}}(M))$ by our hypotheses on M). Clearly, F is essentially surjective by definition of the topology on $Aut_{\mathcal{I}}(M)$. To prove that F is faithful, we observe that for any two $\mathbb{T}$-provably functional formulae $\theta_1(\vec{x}, \vec{y})$ and $\theta_2(\vec{x}, \vec{y})$ from a formula $\phi(\vec{x})$ in $\mathcal{I}$ to a formula $\psi(\vec{y})$ in $\mathcal{I}$, if $[[\theta_1(\vec{x}, \vec{y})]]_M = [[\theta_2(\vec{x}, \vec{y})]]_M$ then the sequent $(\phi(\vec{x}) \vdash_{\vec{x}} \exists \vec{y}(\theta_1(\vec{x}, \vec{y}) \wedge \theta_2(\vec{x}, \vec{y})))$ is provable in $\mathbb{T}$ (since the conjunction of the antecedent and the consequent is satisfied in M, $\mathbb{T}$ is complete and $\phi(\vec{x})$ is $\mathbb{T}$-complete); hence θ_1 and θ_2 are provably equivalent in $\mathbb{T}$, that is they are equal as arrows $\phi(\vec{x}) \to \psi(\vec{y})$ in $\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}$. The fact that F is full can be proved as follows. Let $\gamma : [[\phi(\vec{x})]]_M \to [[\psi(\vec{y})]]_M$ be a $Aut_{\mathcal{I}}(M)$-equivariant map, where $\phi(\vec{x})$ and $\psi(\vec{y})$ are formulae in $\mathcal{I}$. Take $(\vec{a}, \vec{b})$ belonging to the graph of this map (notice that this is possible since $\phi(\vec{x})$ is satisfied in M by our hypotheses). By the atomicity of $\mathbb{T}$, there exists a $\mathbb{T}$-complete formula $\theta(\vec{x}, \vec{y})$ such that $(\vec{a}, \vec{b}) \in [[\theta(\vec{x}, \vec{y})]]_M$.

Now, $\theta(\vec{x}, \vec{y})$ is $\mathbb{T}$-provably functional from $\phi(\vec{x})$ to $\psi(\vec{y})$, since γ is equivariant and the action of $Aut_{\mathcal{I}}(M)$ on $[[\phi(\vec{x})]]_M$ is transitive. The sequent $(\theta \vdash_{\vec{x},\vec{y}} \phi(\vec{x}) \wedge \psi(\vec{y}))$ is provable in $\mathbb{T}$ since it is satisfied in M, $\mathbb{T}$ is complete and θ is $\mathbb{T}$-complete. A similar argument shows that the sequent $(\phi(\vec{x}) \vdash_{\vec{x}} \exists \vec{y} \theta(\vec{x}, \vec{y}))$ is provable in $\mathbb{T}$. To conclude the proof of fullness, it remains to show that the sequent $(\theta \wedge \theta[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z})$ is provable in $\mathbb{T}$. Since $\mathbb{T}$ is complete, it suffices to verify that this sequent is satisfied in M. Suppose that $(\vec{a}, \vec{b}), (\vec{a}, \vec{c}) \in [[\theta]]_M$. Since θ is $\mathbb{T}$-complete then, as we observed above, there exists an automorphism $f : M \cong M$ such that f sends $(\vec{a}, \vec{b})$ to $(\vec{a}, \vec{b'})$, in other words such that $f(\vec{a}) = \vec{a}$ and $f(\vec{b}) = \vec{b'}$; but γ is equivariant, whence $\vec{b'} = f(\vec{b}) = f(\gamma(\vec{a})) = \gamma(f(\vec{a})) = \gamma(\vec{a}) = \vec{b}$, as required. This completes the proof. □

The proof of the theorem motivates the following definition: given an atomic (and complete) theory $\mathbb{T}$, a set-based model M of $\mathbb{T}$ is said to be *special* if every $\mathbb{T}$-complete formula is realized in M and for any $\vec{a}, \vec{b} \in M$ which satisfy exactly the same geometric (equivalently, the same $\mathbb{T}$-complete) formulae over the signature of $\mathbb{T}$, there exists an automorphism $f : M \cong M$ of M sending $\vec{a}$ to $\vec{b}$. The theorem thus says that for any special model M of an atomic complete theory $\mathbb{T}$, we have a representation $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Cont}(Aut(M))$ of its classifying topos (where $Aut(M)$ is endowed with the topology of pointwise convergence).

Remarks 3.2.

(a) We showed in [7] that if $\mathbb{T}$ is a complete atomic geometric theory then $\mathbb{T}$ is countably categorical (i.e., any two countable models of it are isomorphic), and its unique (up to isomorphism) countable model M, if it exists, satisfies the hypotheses of the theorem. As a corollary of the theorem, it follows that the classifying topos of $\mathbb{T}$ can be represented as the topos $\mathbf{Cont}(Aut(M))$ of continuous $Aut(M)$-sets, where $Aut(M)$ is the group of automorphisms of M endowed with the topology of pointwise convergence.

(b) The group $Aut_{\mathcal{I}}(M)$ is complete (in the sense of section 2.2). Indeed, by the equivalence of the theorem, the topos $\mathbf{Cont}(Aut_{\mathcal{I}}(M))$ is the classifying topos for $\mathbb{T}$, with the canonical point

$$p_{Aut_{\mathcal{I}}(M)} : \mathbf{Set} \to \mathbf{Cont}(Aut_{\mathcal{I}}(M))$$

corresponding exactly to the $\mathbb{T}$-model M; the canonical map $Aut_{\mathcal{I}}(M) \to Aut(p_{Aut_{\mathcal{I}}(M)})$ is thus a bijection.

(c) Given two Morita-equivalent theories $\mathbb{T}$ and $\mathbb{T}'$, $\mathbb{T}$ is atomic and complete (resp., is atomic and complete and has a special model) if and only if $\mathbb{T}'$ is atomic and complete (resp., is atomic and complete and has a special model). Indeed, the property of atomicity and completeness of a geometric theory can be expressed as the topos-theoretic invariant property of its classifying topos to be atomic and two-valued (cf. [6]), and hence it is stable under Morita-equivalence. On the other hand, by Theorem 3.1 and Remark 3.2(b), if an atomic complete theory $\mathbb{T}$ has a special model then there exists a point p of its classifying topos $\mathcal{E}_{\mathbb{T}} := \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ such that the latter can be represented as the topos of continuous $Aut(p)$-sets where $Aut(p)$ is the group of automorphisms of p endowed with the topology in which the subgroups of the form $U_{e,x} : \{\alpha : p \cong p \mid \alpha(e)(x) = x\}$ for $e \in \mathcal{E}_{\mathbb{T}}$ and $x \in p^*(e)$ form a basis of open neighborhoods of the identity; also, the equivalence $\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Cont}(Aut(p))$ can be described as sending any object e of $\mathcal{E}_{\mathbb{T}}$ to the canonical continuous action of $Aut(p)$ on the set $p^*(e)$. If $\mathbb{T}$ is Morita-equivalent to a geometric theory $\mathbb{T}'$ then we have an equivalence of classifying toposes $\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$ which sends the point p of $\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$ to a point q of $\mathcal{E}_{\mathbb{T}'}$; hence, denoting by $Aut(q)$ the group of automorphisms of q, endowed with the topology in which the subgroups of the form $U'_{e',y} : \{\beta : q \cong q \mid \beta(e')(y) = y\}$ for $e' \in \mathcal{E}_{\mathbb{T}'}$ and $y \in q^*(e')$ form a basis of open neighborhoods of the identity, we have an isomorphism $\tau : Aut(p) \cong Aut(q)$ of topological groups such that the equivalence of classifying toposes $\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}_{\mathbb{T}'}$, composed with the equivalences $\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Cont}(Aut(p))$ and $\mathbf{Cont}(\tau) : \mathbf{Cont}(Aut(p)) \simeq \mathbf{Cont}(Aut(q))$, yields an equivalence $\mathcal{E}_{\mathbb{T}'} \simeq \mathbf{Cont}(Aut(q))$ which sends any object e' of $\mathcal{E}_{\mathbb{T}'}$ to the canonical continuous action of $Aut(q)$ on the set $q^*(e')$. If we denote by N the $\mathbb{T}'$-model in $\mathbf{Set}$ corresponding to the point q of the classifying topos $\mathcal{E}_{\mathbb{T}'}$ of $\mathbb{T}'$, we thus obtain an equivalence $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \simeq \mathbf{Cont}(Aut(N))$ (where $Aut(N)$ is endowed with the topology of pointwise convergence) which sends any $\mathbb{T}'$-complete formula $\phi(\vec{x})$ to the canonical action of $Aut(N)$ on its interpretation $[[\phi(\vec{x})]]_N$ in N. As the notion of atom is a topos-theoretic invariant, this action is necessarily non-empty and transitive; in other words, the model N is special for $\mathbb{T}'$.

3.2. Categorical formulation

Thanks to the results obtained in [12] on the class of theories of presheaf type, we can recast our main representation theorem in purely categorical language.

To this end, we have to recall from [6] some natural categorical notions.

Definition 3.3. Let $\mathcal{C}$ be a small category.

- $\mathcal{C}$ is said to satisfy the *amalgamation property* (AP) if for every objects $a, b, c \in \mathcal{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathcal{C}$ there exists an object $d \in \mathcal{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathcal{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, dashed, "f'"] \\ c \arrow[r, dashed, "g'"'] & d \end{tikzcd}$$

- $\mathcal{C}$ is said to satisfy the *joint embedding property* (JEP) if for every pair of objects $a, b \in \mathcal{C}$ there exists an object $c \in \mathcal{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathcal{C}$:

$$\begin{tikzcd} & a \arrow[d, dashed, "f"] \\ b \arrow[r, dashed, "g"'] & c \end{tikzcd}$$

- Given a full embedding of categories $\mathcal{C} \hookrightarrow \mathcal{D}$, an object u of $\mathcal{D}$ is said to be $\mathcal{C}$*-homogeneous* if for every objects $a, b \in \mathcal{C}$ and arrows $j : a \to b$ in $\mathcal{C}$ and $\chi : a \to u$ in $\mathcal{D}$ there exists an arrow $\tilde{\chi} : b \to u$ such that $\tilde{\chi} \circ j = \chi$:

$$\begin{tikzcd} a \arrow[r, "\chi"] \arrow[d, "j"'] & u \\ b \arrow[ur, dashed, "\tilde{\chi}"'] & \end{tikzcd}$$

- Given a full embedding of categories $\mathcal{C} \hookrightarrow \mathcal{D}$, an object u of $\mathcal{D}$ is said to be $\mathcal{C}$*-ultrahomogeneous* if for every objects $a, b \in \mathcal{C}$ and arrows $j : a \to b$ in $\mathcal{C}$ and $\chi_1 : a \to u$, $\chi_2 : b \to u$ in $\mathcal{D}$ there exists an isomorphism $\check{j} : u \to u$ such that $\check{j} \circ \chi_1 = \chi_2 \circ j$:

$$\begin{tikzcd} a \arrow[r, "\chi_1"] \arrow[d, "j"'] & u \arrow[d, dashed, "\check{j}"] \\ b \arrow[r, "\chi_2"'] & u \end{tikzcd}$$

- Given a full embedding of categories $\mathcal{C} \hookrightarrow \mathcal{D}$, an object u of $\mathcal{D}$ is said to be $\mathcal{C}$*-universal* if it is $\mathcal{C}$-cofinal, that is for every $a \in \mathcal{C}$ there exists an arrow $\chi : a \to u$ in $\mathcal{D}$:

$$\begin{tikzcd} a \arrow[r, dashed, "\chi"] & u \end{tikzcd}$$

Remarks 3.4.

(a) In the above definition of $\mathcal{C}$-ultrahomogeneous object, one can suppose $j = 1_a$ without loss of generality;

(b) Any object which is both $\mathcal{C}$-ultrahomogeneous and $\mathcal{C}$-universal is $\mathcal{C}$-homogeneous; in particular, any $\mathcal{C}$-ultrahomogeneous and $\mathcal{C}$-universal object of the ind-completion Ind-$\mathcal{C}$ of $\mathcal{C}$ can be identified with an object of the full subcategory $\mathbf{Flat}_{J_{at}}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ of Ind-$\mathcal{C}$ on the J_{at}-continuous flat functors.

Theorem 3.5. *Let $\mathcal{C}$ be a small non-empty category satisfying AP and JEP, and let u be a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object in* Ind-$\mathcal{C}$*. Then the collection $\mathcal{I}_{\mathcal{C}}$ of sets of the form $\mathcal{I}_{\chi} := \{f : u \cong u \mid f \circ \chi = \chi\}$ (for an arrow $\chi : c \to u$ from an object c of $\mathcal{C}$ to u) defines an algebraic base for the group of automorphisms of u in* Ind-$\mathcal{C}$*, and, denoting by $Aut_{\mathcal{C}}(u)$ the resulting topological group, we have an equivalence of toposes*

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(u))$$

induced by the functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}_t(Aut_{\mathcal{C}}(u))$ which sends any object c of $\mathcal{C}$ to the set $Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$ (equipped with the obvious action by $Aut_{\mathcal{C}}(u)$) and any arrow $f : c \to d$ in $\mathcal{C}$ to the $Aut_{\mathcal{C}}(u)$-equivariant map $- \circ f : Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(d, u) \to Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$.

Proof. Let $\mathbb{T}$ be the geometric theory of flat functors on $\mathcal{C}^{\mathrm{op}}$, and $\mathbb{T}'$ its quotient axiomatizing the J_{at}-continuous flat functors on $\mathcal{C}^{\mathrm{op}}$. Then $\mathbb{T}'$ is classified by the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, which is atomic (since $\mathcal{C}$ satisfies the amalgamation property) and two-valued (since $\mathcal{C}$ satisfies the joint embedding property, cf. [6]). Therefore $\mathbb{T}$ is atomic and complete. Every object of $\mathcal{C}$ can be seen as a finitely presentable $\mathbb{T}$-model, and, as proved in [8], the formulae which present these objects are $\mathbb{T}'$-complete and hence satisfy the hypotheses of Theorem 3.1 since the objects of the form $l(c)$, where l is the composite of the Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ with the associated sheaf functor $a_{J_{at}} : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, define a separating set for the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ and hence are dense in the family of its atoms. To conclude, it remains to observe that, for any $c \in \mathcal{C}$, if ϕ_c is the geometric formula which presents the object c then the interpretation of ϕ_c in u, regarded as a $\mathbb{T}$-model, corresponds precisely to the set of arrows from c to u in Ind-$\mathcal{C}$. □

Remarks 3.6.

(a) Theorem 3.1 can be deduced from its categorical counterpart (Theorem 3.5) by taking $\mathcal{C}$ equal to the category $\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}$ and u to be object of Ind-$\mathcal{C} = \mathbf{Flat}(\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}, \mathbf{Set})$ corresponding to the object M of the category $\mathbb{T}$-mod($\mathbf{Set}$) under the Morita-equivalence $\mathbb{T}$-mod($\mathbf{Set}$) $\simeq \mathbf{Flat}_{J_{at}}(\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}, \mathbf{Set})$; notice that the category $\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}$ satisfies the amalgamation property (by our denseness hypothesis on $\mathcal{I}$) and the joint embedding property (since it is complete and classified by the atomic topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{\mathcal{I}}, J_{at})$, cf. [6]) and u, regarded as a flat functor $F : \mathcal{C}_{\mathbb{T}}^{\mathcal{I}} \to \mathbf{Set}$, satisfies the property that for any $\phi(\vec{x})$ in $\mathcal{I}$, $F(\phi(\vec{x})) = [[\phi(\vec{x})]]_M$, which implies that $Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(\phi(\vec{x}), u)$ is isomorphic to $[[\phi(\vec{x})]]_M$.

(b) All the hypotheses of Theorem 3.5 are necessary. Indeed, the amalgamation property on $\mathcal{C}$ is necessary for defining the atomic topology on the opposite of it, while the joint embedding property is necessary because it is equivalent, under the assumption that $\mathcal{C}$ is non-empty, to the condition that the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ is two-valued, which is always satisfied if the latter is equivalent to a topos of continuous actions of a topological group. The condition that u should be $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous is also necessary since it is equivalent to the requirement that the functor F should take values in the subcategory of non-empty transitive actions, which is always the case if F induces an equivalence of toposes $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(u))$ (the composite of the Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ with the associated sheaf functor $a_{J_{at}} : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ takes values in the full subcategory of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ on the atoms, and any equivalence sends atoms to atoms). It is also worth to note that if all arrows in $\mathcal{C}$ are monomorphisms, the existence of a $\mathcal{C}$-universal and $\mathcal{C}$-homogeneous object in Ind-$\mathcal{C}$ implies the fact that $\mathcal{C}$ satisfies the amalgamation property.

A particularly natural context in which Theorems 3.1 and 3.5 can be applied is that of theories of presheaf type.

Corollary 3.7. *Let $\mathbb{T}$ be a theory of presheaf type with the property that its category f.p.$\mathbb{T}$-mod($\mathbf{Set}$) of finitely presentable models satisfies AP and JEP, and let M be a f.p.$\mathbb{T}$-mod($\mathbf{Set}$)-universal and f.p.$\mathbb{T}$-mod($\mathbf{Set}$)-ultrahomogeneous model of $\mathbb{T}$. Then we have an equivalence of toposes*

$$\mathbf{Sh}(\textit{f.p.}\mathbb{T}\text{-}\textit{mod}(\mathbf{Set})^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(M)),$$

where $Aut(M)$ is endowed with the topology of pointwise convergence.

Let $\phi(\vec{x})$ and $\psi(\vec{y})$ be formulae presenting respectively $\mathbb{T}$-models M_ϕ and M_ψ, and let $\vec{a}$ and $\vec{b}$ be elements of $[[\phi(\vec{x})]]_M$ and $[[\psi(\vec{y})]]_M$. If every automorphism of f which fixes $\vec{b}$ fixes $\vec{a}$ then there exists a unique $\mathbb{T}'$-provably functional formula $\theta(\vec{x}, \vec{y})$ from $\phi(\vec{x})$ to $\psi(\vec{y})$ such that $[[\theta]]_M(\vec{a}) = \vec{b}$. □

Remarks 3.8.

(a) The topos $\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\mathrm{op}}, J_{at})$ classifies the homogeneous $\mathbb{T}$-models (in the sense of [6]); in particular, it classifies in $\mathbf{Set}$ the f.p.$\mathbb{T}$-mod($\mathbf{Set}$)-homogeneous objects of the category $\mathbb{T}$-mod($\mathbf{Set}$);

(b) For any theory of presheaf type $\mathbb{T}$ such that its category f.p.$\mathbb{T}$-mod($\mathbf{Set}$) of finitely presentable models satisfies AP, the joint embedding property on f.p.$\mathbb{T}$-mod($\mathbf{Set}$) can always be achieved at the cost of replacing $\mathbb{T}$ with any of its 'connected components' (that is, with any of the quotients of $\mathbb{T}$ axiomatized by the geometric sequents which are valid in all the models belonging to a given connected component of the category f.p.$\mathbb{T}$-mod($\mathbf{Set}$)). In fact, the completions of the geometric theory of homogeneous $\mathbb{T}$-models correspond exactly to the connected components of the category f.p.$\mathbb{T}$-mod($\mathbf{Set}$) (cf. [6]).

4. Concrete Galois theories

Now that we have established an equivalence of classifying toposes, namely that of Theorem 3.5, it is natural to wonder under which conditions this equivalence restricts to an equivalence, or a full embedding, of sites.

To this end, we shall consider this Morita-equivalence in conjunction with appropriate topos-theoretic invariants to construct ‘bridges’ (in the sense of [11]) for connecting the two sides with each other.

4.1. Strict monomorphisms and Galois-type equivalences

By considering the invariant notion of arrow between two given objects in the context of the two representations above, we obtain the following ‘Galois-type’ result.

Proposition 4.1. *Let $\mathcal{C}$ be a small category satisfying the amalgamation and joint embedding properties, and let u be a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object in* Ind-$\mathcal{C}$*. Then the following conditions are equivalent:*

(i) Every arrow $f : d \to c$ in $\mathcal{C}$ is a strict monomorphism (in the sense that for any arrow $g : e \to c$ such that $h \circ g = k \circ g$ whenever $h \circ f = k \circ f$, g factors uniquely through f);
(ii) The functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}_{\mathcal{I}}(Aut_{\mathcal{C}}(u))$ which sends any object c of $\mathcal{C}$ to the set $Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$ (endowed with the obvious action by $Aut_{\mathcal{C}}(u)$) and any arrow $f : c \to d$ in $\mathcal{C}$ to the $Aut_{\mathcal{C}}(u)$-equivariant map

$$- \circ f : Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(d, u) \to Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$$

is full and faithful;
(iii) For any objects $c, d \in \mathcal{C}$ and any arrows $\chi : c \to u$ and $\xi : d \to u$ in Ind-$\mathcal{C}$*, $\mathcal{I}_\xi \subseteq \mathcal{I}_\chi$ (that is, for any automorphism f of u, $f \circ \xi = \xi$ implies $f \circ \chi = \chi$) if and only if there exists a unique arrow $f : c \to d$ in $\mathcal{C}$ such that $\chi = \xi \circ f$:*

$$\begin{tikzcd} c \arrow[r, "\chi"] \arrow[d, dashed, "f"'] & u \\ d \arrow[ur, "\xi"'] & \end{tikzcd}$$

Proof. Condition (i) is exactly equivalent to the condition that J_{at} should be subcanonical (cf. Example C2.1.12(b) in [23]); but this condition is in turn equivalent, by Lemma C2.2.15 in [23], to the requirement that the composite $l : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ of the Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ with the associated sheaf functor $a_{J_{at}} : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ be full and faithful, that is to condition (ii).

To prove the equivalence of property (iii) with the condition for l to be full and faithful, we observe that, given an equivariant map $\gamma : F(c) \to F(d)$, we can describe γ as follows. Take $\xi \in F(c)$ and $\chi = \gamma(\xi)$; then $\mathcal{I}_\xi \subseteq \mathcal{I}_\chi$ and γ can be identified with the map sending $g \circ \xi$ to $g \circ \chi$ for every automorphism g of u. Conversely, if we have two arrows $\chi : c \to u$ and $\xi : d \to u$ in Ind-$\mathcal{C}$ such that $\mathcal{I}_\xi \subseteq \mathcal{I}_\chi$ then the map $F(c) \to F(d)$ sending $g \circ \xi$ to $g \circ \chi$ for every automorphism g of u is well-defined and $Aut_{\mathcal{I}}(u)$-equivariant. To conclude, it suffices to notice that such a map is equal to $F(f)$ for an arrow $f : d \to c$ in $\mathcal{C}$ if and only if $\xi \circ f = \chi$. □

The proposition motivates the following definition: given a small category $\mathcal{C}$ satisfying AP and JEP, and a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object u in Ind-$\mathcal{C}$, we say that the pair $(\mathcal{C}, u)$ defines a *concrete Galois theory* if every arrow of $\mathcal{C}$ is a strict monomorphism.

Remarks 4.2.

(a) A canonical example of categories satisfying the conditions of Proposition 4.1 is given by the subcategories of the syntactic categories of theories $\mathbb{T}$ satisfying the hypotheses of Theorem 3.1 on the $\mathbb{T}$-complete formulae. We shall see other examples of categories satisfying this condition in section 5.

(b) If in Corollary 3.7 all the arrows in f.p.$\mathbb{T}$-mod(**Set**) are strict monomorphisms, the formula θ in the statement of the corollary can be taken to be provably functional in $\mathbb{T}$; in other words it induces a $\mathbb{T}$-model homomorphism $z : M_\psi \to M_\phi$ such that, denoting by $a : M_\phi \to M$ the $\mathbb{T}$-model homomorphism corresponding to the element $\vec{a}$ and by $b : M_\psi \to M$ the $\mathbb{T}$-model homomorphism corresponding to the element $\vec{b}$, we have $b \circ z = a$.

Under the hypotheses that all the arrows in $\mathcal{C}$ are strict monomorphisms, we can construct alternative 'Galois representations' of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, as follows.

Theorem 4.3. *Suppose that $\mathcal{C} \hookrightarrow \mathcal{D}$ is a full embedding of a category $\mathcal{C}$ into a category $\mathcal{D}$ containing a $\mathcal{C}$-ultrahomogeneous and $\mathcal{C}$-universal object u. Suppose that the following two properties are satisfied:*

(a) For any objects $a, b \in \mathcal{C}$ and arrows $\xi : a \to u$ and $\chi : b \to u$ in $\mathcal{D}$ there exists an object c of $\mathcal{C}$, arrows $f : a \to c$ and $g : b \to c$ in $\mathcal{C}$ and an arrow $\epsilon : c \to u$ in $\mathcal{D}$ such that $\epsilon \circ f = \xi$ and $\epsilon \circ g = \chi$;

(b) For any arrows $f, g : a \to b$ in $\mathcal{C}$ and any arrow $\chi : b \to u$, $\chi \circ f = \chi \circ g$ implies $f = g$.

Then the category $\mathcal{C}$ satisfies AP and JEP, the sets of the form $\mathcal{I}_\chi := \{f : u \cong u \mid f \circ \chi = \chi\}$ define an algebraic base for the group $Aut_{\mathcal{D}}(u)$ of automorphisms of u in $\mathcal{D}$ and we *have a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}(Aut_{\mathcal{D}}(u))$ taking values in the subcategory of transitive $Aut_{\mathcal{D}}(u)$-actions which sends every object of $\mathcal{C}$ to the set $Hom_{\mathcal{D}}(c, u)$, endowed with the obvious action of $Aut_{\mathcal{D}}(u)$, and any arrow $f : d \to c$ of $\mathcal{C}$ to the $Aut_{\mathcal{D}}(u)$-equivariant map $Hom_{\mathcal{D}}(-, f) : Hom_{\mathcal{D}}(c, u) \to Hom_{\mathcal{D}}(d, u)$. Moreover, the following conditions are equivalent:*

(i) All the arrows of $\mathcal{C}$ are strict monomorphisms;
(ii) The functor F is full and faithful and induces an equivalence

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{D}}(u)).$$

Proof. The fact that $\mathcal{C}$ satisfies the amalgamation property follows from the $\mathcal{C}$-universality of u and properties (a) and (b), while the fact that $\mathcal{C}$ satisfies the joint embedding property follows from property (a). The fact that the sets $\mathcal{I}_\chi$ form an algebraic base for $Aut_{\mathcal{D}}(u)$ also follows from property (a).

The fact that (ii) implies (i) follows as in the proof of Proposition 4.1. To prove that (i) implies (ii) it suffices to show that the functor F is full and faithful, since if this is the case we have a full and faithful functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}(Aut_{\mathcal{D}}(u))$ whose image can be identified with the dense subcategory of $\mathbf{Cont}(Aut_{\mathcal{D}}(u))$ corresponding to the algebraic base $\mathcal{I}_{\mathcal{C}}$ as in Remark 2.2, and hence the toposes of sheaves $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ and $\mathbf{Cont}(Aut_{\mathcal{D}}(u))$ on the two categories with respect to the atomic topology are equivalent. Let us thus suppose that $\xi : a \to u$ and $\chi : b \to u$ are arrows in $\mathcal{D}$ from objects of $\mathcal{C}$ such that $\mathcal{I}_\chi \subseteq \mathcal{I}_\xi$. By property (a) there exist arrows $f : a \to c$ and $g : b \to c$ in $\mathcal{C}$ and an arrow $\epsilon : c \to u$ in $\mathcal{D}$ such that $\epsilon \circ f = \xi$ and $\epsilon \circ g = \chi$. By property (b), χ factors (uniquely) through ξ if and only if g factors (uniquely) through f. Since all the arrows of $\mathcal{C}$ are strict monomorphisms, it is equivalent to verify that for any pair of arrows $h, k : b \to e$, if $h \circ g = k \circ g$ then $h \circ f = k \circ f$. Let us therefore suppose that $h \circ g = k \circ g$. Since u is $\mathcal{C}$-homogeneous, there exist arrows $\alpha : e \to u$ and $\beta : e \to u$ in $\mathcal{D}$ such that $\alpha \circ h = \epsilon$ and $\beta \circ k = \epsilon$. Now, since u is $\mathcal{C}$-ultrahomogeneous, there exists an automorphism $j : u \cong u$ of u in $\mathcal{D}$ such that $\beta = j \circ \alpha$. We have that $j \circ \chi = j \circ \epsilon \circ g = j \circ \alpha \circ h \circ g = \beta \circ h \circ g = \beta \circ k \circ g = \epsilon \circ g = \chi$. Therefore, as $\mathcal{I}_\chi \subseteq \mathcal{I}_\xi$, we have that $j \circ \xi = \xi$. But $j \circ \xi = j \circ \alpha \circ h \circ f = \beta \circ h \circ f$, and $\xi = \beta \circ k \circ f$; hence $\beta \circ (h \circ f) = \beta \circ (k \circ f)$, from which it follows, by condition (b), that $h \circ f = k \circ f$, as required. □

Remarks 4.4.

(a) Under the hypothesis that all the arrows of $\mathcal{C}$ are strict monomorphisms, condition (b) in the statement of the theorem is necessary; indeed, the fact that the functor F induces an equivalence of toposes implies that for any arrows $\chi : c \to u$ and $\xi : d \to u$ there exists at most one factorization of χ through ξ. Now, if for two arrows $f, g : a \to b$ in $\mathcal{C}$ and an arrow $\chi : b \to u$ we have $\chi \circ f = \chi \circ g$, f and g are two factorizations of $\chi \circ f = \chi \circ g$ through χ and hence $f = g$.

(b) In the case all the arrows of $\mathcal{C}$ are strict monomorphisms, Theorem 3.5 can be deduced as the particular case of Theorem 4.3 when $\mathcal{C} \hookrightarrow \mathcal{D}$ is the embedding $\mathcal{C} \hookrightarrow \text{Ind-}\mathcal{C}$.

Let us now discuss the relationship between a category $\mathcal{D}$ with an object u as in the hypotheses of Theorem 4.3 and the ind-completion Ind-$\mathcal{C}$ of the category $\mathcal{C}$.

In [23] (section C4.2), the following criterion for a full embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ of categories to be of the form $\mathcal{C} \hookrightarrow \text{Ind-}\mathcal{C}$ is given: the category $\mathcal{D}$ has all (small) filtered colimits, every object of $\mathcal{C}$ is finitely presentable in $\mathcal{D}$ (in the sense that the corresponding representable hom functor preserves small filtered colimits) and every object of $\mathcal{D}$ can be expressed as a (small) filtered colimit of objects of $\mathcal{C}$. The embedding $\mathcal{C} \hookrightarrow \text{Ind-}\mathcal{C}$ can also be characterized as the filtered-colimit completion of the category $\mathcal{C}$.

Notice that if $\mathcal{C} \hookrightarrow \mathcal{D}$ is isomorphic to $\mathcal{C} \hookrightarrow \text{Ind-}\mathcal{C}$ then we have an equivalence $\mathcal{D} \simeq \text{Ind-}\mathcal{C} \simeq \mathbf{Flat}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ sending any object d of $\mathcal{D}$ to the flat functor $Hom_{\mathcal{D}}(-, d) : \mathcal{C}^{\text{op}} \to \mathbf{Set}$, and hence the three properties in the following proposition (expressing the fact that the functor $Hom_{\mathcal{D}}(-, d) : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is flat, i.e. that its category of elements is filtered) hold:

Proposition 4.5. *Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be a full embedding isomorphic to $\mathcal{C} \hookrightarrow \text{Ind-}\mathcal{C}$. Then the following three properties hold:*

(i) Every object d of $\mathcal{D}$ is the colimit of the canonical diagram on $\mathcal{C}/d$ in $\mathcal{D}$;
(ii) For any object d of $\mathcal{D}$ and any arrows $\xi : a \to d$ and $\chi : b \to d$ in $\mathcal{D}$ from objects a and b of $\mathcal{C}$, there exists an object c of $\mathcal{C}$, arrows $f : a \to c$ and $g : b \to c$ of $\mathcal{C}$ and an arrow $\epsilon : c \to d$ in $\mathcal{D}$ such that $\epsilon \circ f = \xi$ and $\epsilon \circ g = \chi$;
(iii) For any arrows $f, g : a \to b$ in $\mathcal{C}$, arrows $\xi : a \to u$ in $\mathcal{D}$ and $\chi : b \to u$ such that $\chi \circ f = \xi$ and $\chi \circ g = \xi$, there exists an arrow $h : b \to c$ in $\mathcal{C}$ and an arrow $\epsilon : c \to u$ in $\mathcal{D}$ such that $\epsilon \circ h = \chi$ and $h \circ f = h \circ g$. $\square$

Remark 4.6. From the theorem it follows that the two conditions in the statement of Theorem 4.3 are automatically satisfied if the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ is (isomorphic to) the canonical embedding $\mathcal{C} \hookrightarrow \text{Ind-}\mathcal{C}$.

The following proposition provides some convenient criteria for identifying ind-completions.

Proposition 4.7.

(i) Let $\mathbb{T}$ be a theory of presheaf type. Then the category $\mathbb{T}$-mod($\mathbf{Set}$) can be identified as the ind-completion of the category f.p.$\mathbb{T}$-mod($\mathbf{Set}$); in particular, the embedding f.p.$\mathbb{T}$-mod($\mathbf{Set}$) $\hookrightarrow$ $\mathbb{T}$-mod($\mathbf{Set}$) satisfies the three conditions in the statement of Proposition 4.5 above.

(ii) Let $\mathbb{T}$ be a geometric theory over a signature which does not contain any relation symbols, and let $\mathcal{C}$ be a set of finitely generated $\mathbb{T}$-models such that every $\mathbb{T}$-model can be expressed as a directed union of models in $\mathcal{C}$. Then we have $\mathbb{T}\text{-}mod_i(\mathbf{Set}) \simeq \text{Ind-}\tilde{\mathcal{C}}$, where $\mathbb{T}\text{-}mod_i(\mathbf{Set})$ is the category of $\mathbb{T}$-models in $\mathbf{Set}$ and injective homomorphisms between them and $\tilde{\mathcal{C}}$ is the category of $\mathbb{T}$-models in $\mathcal{C}$ and injective homomorphisms between them.

(iii) Let $\mathbb{T}$ be a finitary first-order theory and $\mathbb{T}\text{-}mod_e(\mathbf{Set})$ be the category of $\mathbb{T}$-models in $\mathbf{Set}$ and elementary embeddings between them. If $\mathcal{C}$ is a set of finitely generated $\mathbb{T}$-models in $\mathbf{Set}$ such that every $\mathbb{T}$-model can be expressed as a filtered colimit in $\mathbb{T}\text{-}mod_e(\mathbf{Set})$ of models in $\mathcal{C}$, we have $\mathbb{T}\text{-}mod_e(\mathbf{Set}) \simeq \text{Ind-}\tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}}$ is the category of models in $\mathcal{C}$ and elementary embeddings between them.

Proof. (i) If $\mathbb{T}$ is a theory of presheaf type then $\mathbb{T}$ is classified by the presheaf topos $[\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$, and hence by Diaconescu's equivalence we have that $\mathbb{T}\text{-mod}(\mathbf{Set}) \simeq \mathbf{Flat}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, \mathbf{Set})$.

(ii) The theory $\mathbb{T}'$ obtained from $\mathbb{T}$ by adding a binary predicate and the coherent sequents asserting that this predicate is complemented to the equality relation is clearly geometric and hence its category of models, which coincides with the category of $\mathbb{T}$-models in $\mathbf{Set}$ and injective homomorphisms between them, has filtered colimits (cf. Lemma D2.4.9 in [23]). Now, if the signature of $\mathbb{T}$ does not contain any relation symbols, any finitely generated $\mathbb{T}$-model A is finitely presentable; indeed, for any $\mathbb{T}$-model M equal to the directed colimit $colim(M_i)$ of a diagram M_i of models, any homomorphism $f : A \to M$ factors, as a function, through a canonical embedding $M_i \hookrightarrow M$ (indeed, A being finitely generated, it suffices to choose i so that the image under f of any of the generators of A belongs to M_i), and such factorization must be a $\mathbb{T}$-model homomorphism since the signature of $\mathbb{T}$ does not contain any relation symbol and the arrows $M_i \to M$ are injective. The above-mentioned characterization for ind-completions thus applies.

(iii) As observed at p. 887 of the second volume of [23], the category $\mathbb{T}\text{-mod}_e(\mathbf{Set})$ has all small filtered colimits. The proof of condition (ii) shows that any factorization $A \to M_i$ of a $\mathbb{T}$-model homomorphism $f : A \to M$ from a finitely generated $\mathbb{T}$-model A to a model $M \cong colim(M_i)$ is a $\mathbb{T}$-model homomorphism; indeed, the fact that it respects function symbols follows from the injectivity of the morphisms in $\mathbb{T}\text{-mod}_e(\mathbf{Set})$, while the fact that it respects relation symbols follows from the fact that such maps are elementary embeddings and hence reflect the relations over the signature of $\mathbb{T}$. □

Given a category $\mathcal{D}$ with an object u as in the hypotheses of Theorem 4.3, we can naturally obtain an object $\tilde{u}$ of Ind-$\mathcal{C}$, as follows. The hypotheses of Theorem 4.3 ensure that the category $\mathcal{C}/u$ is filtered; we can thus consider the colimit $\tilde{u}$ in Ind-$\mathcal{C}$ of the canonical functor $\mathcal{C}/u \to \mathcal{C}$. Let us show that $\tilde{u}$ is a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object. Let us denote by $J_{(c,\xi)} : c \to \tilde{u}$ the canonical colimit arrows in Ind-$\mathcal{C}$. The fact that u is $\mathcal{C}$-universal implies the existence, for any object c of $\mathcal{C}$, of an arrow $\xi : c \to u$

in $\mathcal{D}$; hence for every object c of $\mathcal{C}$ we have an arrow $c \to \tilde{u}$ in Ind-$\mathcal{C}$, namely $J_{(c,\xi)}$. To prove that $\tilde{u}$ is $\mathcal{C}$-ultrahomogeneous, we first observe that any arrow $r : c \to \tilde{u}$ in Ind-$\mathcal{C}$ from an object c of $\mathcal{C}$ to $\tilde{u}$ is of the form $J_{(c,\xi)}$ for some object $(c, \xi) \in \mathcal{C}/u$. As c is finitely presentable in Ind-$\mathcal{C}$, we have a factorization of r as $J_{(d,\chi)} \circ f$ for some arrow $f : c \to d$ in $\mathcal{C}$; but $(c, \xi \circ f)$ is an object of $\mathcal{C}/u$, and f is an arrow $(c, \xi \circ f) \to (d, \chi)$ in $\mathcal{C}/u$, whence $r = J_{(d,\chi)} \circ f = J_{(c,\xi\circ f)}$. Now, given two arrows $J_{(c,\xi_1)}, J_{(c,\xi_2)} : c \to \tilde{u}$ in Ind-$\mathcal{C}$, by the $\mathcal{C}$-ultrahomogeneity of u we have an automorphism $j : u \cong u$ of u in $\mathcal{D}$ such that $j \circ \xi_1 = \xi_2$. By the universal property of the colimit $\tilde{u}$, there exists an automorphism $\tilde{j} : \tilde{u} \cong \tilde{u}$ of $\tilde{u}$ in Ind-$\mathcal{C}$ such that $\tilde{j} \circ J_{(c,\xi_1)} = J_{(c,\xi_2)}$. Indeed, the arrows $J_{(d,j\circ\chi)} : d \to \tilde{u}$ (for $(d, \chi) \in \mathcal{C}/u$) clearly form a cocone in Ind-$\mathcal{C}$ on $\mathcal{C}/u$ with vertex $\tilde{u}$; we set $\tilde{j} : \tilde{u} \to \tilde{u}$ equal to the unique arrow s in Ind-$\mathcal{C}$ such that $s \circ J_{(d,\chi)} = J_{(d,j\circ\chi)}$ for any object (d, χ) of $\mathcal{C}/u$. The arrow $\tilde{j}$ is an automorphism because $\widetilde{j^{-1}}$ is an inverse for it, and it clearly satisfies the required property. This completes the proof that $\tilde{u}$ is $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous. Therefore the object $\tilde{u}$ satisfies the hypotheses of Theorem 3.5, and we have an equivalence

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(\tilde{u})),$$

which, combined with the equivalence of Theorem 4.3, exhibits the topological group $Aut_{\mathcal{C}}(\tilde{u})$ as the completion (in the sense of section 2.2) of the topological group $Aut_{\mathcal{D}}(u)$.

Summarizing, we have the following result.

Theorem 4.8. *Let $(\mathcal{C} \hookrightarrow \mathcal{D}, u)$ be a pair satisfying the hypotheses of Theorem 4.3. Then there is a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object $\tilde{u}$ in* Ind-$\mathcal{C}$ *and equivalences of toposes*

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \simeq \mathbf{Cont}(Aut_{\mathcal{D}}(u)) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(\tilde{u})). \qquad \square$$

Remark 4.9. The theorem shows in particular that there can be many different Morita-equivalent topological groups (i.e., topological groups having equivalent toposes of continuous actions).

4.1.1. Regular and strict monomorphisms

Let $\mathcal{C}$ be a small category. Recall that an arrow $f : d \to c$ in $\mathcal{C}$ is said to be a *strict monomorphism* if for any arrow $g : e \to c$ such that $h \circ g = k \circ g$ whenever $h \circ f = k \circ f$, g factors uniquely through f; in other words, f is the limit of the (possibly large) diagram consisting of all the pairs (h, k) which coequalize f. The dual concept of strict epimorphism was introduced by Grothendieck in [2] (Exposé I, 10.2 and 10.3).

An arrow f is said to be a *regular monomorphism* in a category $\mathcal{C}$ if its cokernel pair exists and f is the equalizer of it. Clearly, for any arrow f of a category $\mathcal{C}$ having a cokernel pair in $\mathcal{C}$, f is a regular monomorphism if and only if it is a strict monomorphism. This shows that the notion of strict monomorphism represents a natural generalization of the notion of regular monomorphism to categories in which cokernel pairs of arrows do not necessarily exist.

The following result enlightens the relationships between strict monomorphisms in different categories.

Proposition 4.10.

(i) Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be an embedding of a small category $\mathcal{C}$ into a category $\mathcal{D}$ satisfying the following properties:
   (a) For any pair of composable arrows g, h in $\mathcal{D}$ with domains and codomains in $\mathcal{C}$, if $g \circ h$ belongs to $\mathcal{C}$ then h belongs to $\mathcal{C}$;
   (b) For any objects a, b of $\mathcal{C}$, d of $\mathcal{D}$ and arrows $\xi : a \to d, \chi : b \to d$ in $\mathcal{D}$, there exists an object c of $\mathcal{C}$, arrows $f : a \to c$ and $g : b \to c$ in $\mathcal{C}$ and an arrow $\epsilon : c \to d$ in $\mathcal{D}$ such that $\epsilon \circ f = \xi$ and $\epsilon \circ g = \chi$;
   (c) For any arrows $f, g : a \to b$ in $\mathcal{C}$, object d of $\mathcal{D}$ and arrow $\chi : b \to d$, $\chi \circ f = \chi \circ g$ implies $f = g$.

   Then any arrow of $\mathcal{C}$ which is a strict monomorphism in $\mathcal{D}$ is a strict monomorphism in $\mathcal{C}$.

   In particular (cf. point (ii) below), if $\mathcal{C}$ is a small category with all arrows monic then for any arrow f in $\mathcal{C}$, f is a strict monomorphism in $\mathcal{C}$ if and only if it is a strict monomorphisms in Ind-$\mathcal{C}$.

(ii) Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be a full embedding. If any object of $\mathcal{D}$ can be written as a colimit of objects of $\mathcal{D}$ (that is, of a diagram on $\mathcal{D}$ which factors through the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$) then every strict monomorphism in $\mathcal{C}$ is a strict monomorphism in $\mathcal{D}$.

(iii) If the domains A of cokernel pairs of arrows in $\mathcal{C}$ exist in $\mathcal{D}$ and are limits in $\mathcal{D}$ of the canonical diagram on $A/\mathcal{C}$ then if f is a regular monomorphism in $\mathcal{D}$, f is a strict monomorphism in $\mathcal{C}$.

Proof. (i) This immediately follows from the fact that the two conditions in the statement of part (i) of the proposition ensure that the limit in $\mathcal{D}$ over the diagram formed by the pairs of arrows which coequalize f in $\mathcal{D}$, if it exists, coincides with the limit in $\mathcal{C}$ over the diagram formed by the pairs of arrows which coequalize f in $\mathcal{C}$; in other words, if f is a strict monomorphism in $\mathcal{D}$ then it is a strict monomorphism in $\mathcal{C}$.

For any small category $\mathcal{C}$ whose arrows are all monic, the embedding $\mathcal{C} \hookrightarrow$ Ind-$\mathcal{C}$ is full and satisfies property (a) since for every object d of Ind-$\mathcal{C}$ the category $\mathcal{C}/d$ satisfies the joint embedding property (it being filtered), and property (b) since the category $\mathcal{C}/d$ satisfies the weak coequalizer property (it being filtered) and every morphism in $\mathcal{C}$ is monic.

(ii) Suppose that every object d of $\mathcal{D}$ is the colimit of a diagram $D : \mathcal{I} \to \mathcal{D}$ which factors through the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$, and let $f : a \to b$ be a strict monomorphism in $\mathcal{C}$. To prove that f is a strict monomorphism in $\mathcal{D}$, suppose that $g : d \to b$ is an arrow in $\mathcal{D}$ such that any pair of arrows in $\mathcal{D}$ which coequalize f coequalize g. Consider the composites $g_i := D(i) \to b$ of the canonical colimit arrows $\xi_i : D(i) \to d$ with g; these arrows are in $\mathcal{C}$, as the embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ is full, and clearly each of them satisfies the property of being coequalized by any pair of arrows in $\mathcal{C}$ which coequalizes f. Therefore, as f is a strict monomorphism, for every $i \in \mathcal{I}$ there exists a unique arrow $\gamma_i : D(i) \to a$ in $\mathcal{C}$ such that $f \circ \gamma_i = g_i$ (for all $i \in \mathcal{I}$). Now, by the uniqueness of the factorization in the definition of strict monomorphism, the arrows $\gamma_i : D(i) \to a$ (for $i \in \mathcal{I}$) define a conone on D with vertex a and hence induce a unique arrow $\gamma : d \to a$ such that $\gamma \circ \xi_i = \gamma_i$ and therefore $f \circ \gamma = g$. The uniqueness of a factorization of g through f in $\mathcal{D}$ follows from the uniqueness of the factorization of each of the arrows g_i through f.

(iii) Suppose that $f : d \to c$ is an arrow of $\mathcal{C}$ which is a regular monomorphism in $\mathcal{D}$ with cokernel pair $\xi_1, \xi_2 : c \to K_f$, and that $g : e \to c$ is an arrow in $\mathcal{C}$ such that any pair of arrows in $\mathcal{C}$ which coequalizes f coequalizes g; we want to prove that g factors uniquely through f. Let us denote by $\chi_1, \chi_2 : c \to K_g$ the cokernel pair of g. Then for any arrow $s : K_f \to a$ to an object a of $\mathcal{C}$ there exists a unique arrow $t_s : K_g \to a$ such that $t_s \circ \chi_1 = s \circ \xi_1$ and $t_s \circ \chi_2 = s \circ \xi_2$. The arrows $t_s : K_g \to a$ form a cone with vertex K_g on the diagram D on $K_f/\mathcal{C}$ with vertex K_f and legs $s : K_f \to a$ (indeed, if $z \circ s = s'$ for some arrows $s' : K_f \to b$ in $\mathcal{D}$ and $z : a \to b$ in $\mathcal{C}$ then $z \circ t_s = t_{s'}$), and hence, since by our hypothesis D is limiting, there exists a unique arrow $y : K_g \to K_f$ such that for any $s : K_f \to a$ (with $a \in \mathcal{C}$), $s \circ y = t_s$; also, the universal property of the limit K_f ensures that $y \circ \chi_1 = \xi_1$ and $y \circ \chi_2 = \xi_2$. Therefore, as f is the equalizer of ξ_1 and ξ_2, g factors uniquely through f. □

The question now naturally arises of how one can obtain embeddings $\mathcal{C} \hookrightarrow \mathcal{D}$ satisfying the hypotheses of the proposition. In this respect, the following result is useful.

Proposition 4.11.

(i) Let $\mathcal{C}$ be a small category, Ind-$\mathcal{C}$ *its ind-completion and $\mathcal{A}$ a subcategory of* Ind-$\mathcal{C}$ *whose objects are exactly the objects of* Ind-$\mathcal{C}$ *and which satisfy the following properties:*
   (a) The composite of two arrows $g \circ h$ belongs to $\mathcal{A}$ then h belongs to $\mathcal{A}$,
   (b) $\mathcal{A}$ is closed under filtered colimits in Ind-$\mathcal{C}$*, and*
   (c) any object of Ind-$\mathcal{C}$ *can be written as a filtered colimit of a diagram in $\mathcal{C}$ with values in $\mathcal{A}$.*

   Then, denoting by $\mathcal{C}_{\mathcal{A}}$ the full subcategory of $\mathcal{A}$ on the objects of $\mathcal{C}$, we have $\mathcal{A} \simeq$ Ind-$\mathcal{C}_{\mathcal{A}}$*.*

(ii) Let $\mathcal{C}$ be a small category, $\mathcal{B}$ a full subcategory of $\mathcal{C}$ and $\mathcal{D}$ be a full subcategory of its ind-completion Ind-$\mathcal{C}$*. If $\mathcal{B} \subseteq \mathcal{D}$, $\mathcal{D}$ is closed in* Ind-$\mathcal{C}$ *under filtered colimits and every object of $\mathcal{D}$ can be written as a filtered colimit of objects in $\mathcal{B}$ then $\mathcal{D} \simeq$* Ind-$\mathcal{B}$*.*

Proof. (i) We use the following classical criterion for an embedding to be an ind-completion (cf. section C4.2 in [23]): for a full embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ to be isomorphic to the embedding $\mathcal{C} \hookrightarrow$ Ind-$\mathcal{C}$ it is necessary and sufficient that the category $\mathcal{D}$ has all (small) filtered colimits, every object of $\mathcal{C}$ is finitely presentable in $\mathcal{D}$ and every object of $\mathcal{D}$ can be expressed as a (small) filtered colimit of objects of $\mathcal{C}$. The first condition is satisfied by condition (b), the third condition is satisfied by condition (c). The validity of the second condition follows from the fact that condition (a) ensures that every finitely presentable object in Ind-$\mathcal{C}$ is finitely presentable also an object of $\mathcal{A}$.

(ii) This equally follows from the same criterion for ind-completions, in light of the fact that every object of $\mathcal{B}$ is finitely presentable in $\mathcal{D}$. □

Remarks 4.12.

(a) Condition (i) in the proposition is satisfied by the subcategory $\mathcal{A}$ of Ind-$\mathcal{C}$ consisting of the objects of Ind-$\mathcal{C}$ and strict monomorphisms between them;
(b) For any geometric theory $\mathbb{T}$ such that the category $\mathbb{T}$-mod(**Set**) can be expressed as Ind-$\mathcal{C}$ of a full subcategory $\mathcal{C}$ of finitely presentable $\mathbb{T}$-models, the subcategory $\mathbb{T}$-mod$_i$(**Set**) of $\mathbb{T}$-mod(**Set**) consisting of the $\mathbb{T}$-models in **Set** and injective $\mathbb{T}$-model homomorphisms between them satisfies condition (i) (obviously) and condition (ii) (since $\mathbb{T}$-mod$_i$(**Set**) is closed in $\mathbb{T}$-mod(**Set**) under filtered colimits); condition (iii) holds if and only if every $\mathbb{T}$-model can be expressed as a directed union of models in $\mathcal{C}$.

Proposition 4.13. *Let $\mathbb{T}$ be a geometric theory whose category $\mathbb{T}$-mod(**Set**) can be expressed as* Ind-$\mathcal{C}$*, for a full subcategory $\mathcal{C}$ of finitely presentable $\mathbb{T}$-models (for instance, a theory of presheaf type $\mathbb{T}$) and such that every $\mathbb{T}$-model can be expressed as a directed union of models in $\mathcal{C}$. Then $\mathbb{T}$-mod$_i$(**Set**) $\simeq$* Ind-$\mathcal{C}_i$*, where $\mathcal{C}_i$ is the category whose objects are the objects of $\mathcal{C}$ and whose arrows are the injective $\mathbb{T}$-model homomorphisms between them, and the category $\mathcal{C}_{sm}$ of objects of $\mathcal{C}$ and arrows between them which are strict monomorphisms in $\mathbb{T}$-mod(**Set**) has the property that every arrow in it is a strict monomorphism.*

Proof. This immediately follows from Proposition 4.11(i) in view of Remark 4.12(b) and by Proposition 4.10(i). □

Remark 4.14. This proposition can be profitably applied in connection with theories $\mathbb{T}$ whose category $\mathbb{T}$-mod(**Set**) of set-based models possesses cokernel pairs of arrows (for instance, for algebraic theories $\mathbb{T}$) and for any arrow in $\mathcal{C}_i$ its cokernel pair lies in $\mathbb{T}$-mod$_i$(**Set**); indeed, in this case the subcategory $\mathcal{C}_{sm}$ coincides with the category whose objects are the objects of $\mathcal{C}$ and whose arrows are the injective $\mathbb{T}$-model homomorphisms between them which are regular monomorphisms in the category $\mathbb{T}$-mod(**Set**). In fact, in many algebraic categories (e.g., the category of vector spaces and linear maps, the category of groups, the category of Boolean algebras, etc.), all monomorphisms are regular, and monomorphisms in categories of models of algebraic theories are precisely the injective model homomorphisms (cf. Proposition 3.4.1 in [4]). The problem of whether the cokernel pair of a monomorphism, provided that it exists, is again a pair of monomorphisms, is non-trivial in general, and a considerable amount of literature has been written on this subject (cf. for instance [22,25–27,32,31]).

Finally, we note that if $\mathbb{T}$ is a theory of presheaf type then in studying the property of an arrow in f.p.$\mathbb{T}$-mod($\mathbf{Set}$) to be a strict monomorphism one can exploit the syntactic characterization of finitely presentable $\mathbb{T}$-models as the finitely presented models of $\mathbb{T}$ (cf. [8]), which provides a (natural) bijective correspondence between the arrows $M \to N$ in f.p.$\mathbb{T}$-mod($\mathbf{Set}$) from such a model M presented by a formula $\phi(\vec{x})$ to any (finitely presentable) $\mathbb{T}$-model N and the elements of the interpretation $[[\phi(\vec{x})]]_N$ of $\phi(\vec{x})$ in N.

4.2. The general case

We have seen in the last section that an equivalence of toposes

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(u))$$

provided by Theorem 3.5 restricts to a 'Galois-type' categorical equivalence of $\mathcal{C}^{\mathrm{op}}$ with a full subcategory of $\mathbf{Cont}(Aut_{\mathcal{C}}(u))$ if and only if the topology J_{at} on the category $\mathcal{C}$ is subcanonical (equivalently, all the arrows in $\mathcal{C}$ are strict monomorphisms). Still, it is natural to ask what one can say about the $Aut_{\mathcal{C}}(u)$-equivariant maps $Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u) \to Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(d, u)$ (for objects c, d of $\mathcal{C}$) in the general case.

Clearly, the equivalence of Theorem 3.5 sends, for each $c \in \mathcal{C}$, the object $a_{J_{at}}(yc)$ of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ to the $Aut_{\mathcal{C}}(u)$-set $Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$, where yc is the representable functor associated to c and $a_{J_{at}}$ is the associated sheaf functor. Therefore the arrows $Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u) \to Hom_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$ in $\mathbf{Cont}(Aut_{\mathcal{C}}(c))$ correspond to the arrows $a_{J_{at}}(yc) \to a_{J_{at}}(yd)$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. To characterize such arrows in more explicit terms we shall undertake a general analysis valid for any site.

Let $(\mathcal{C}, J)$ be a site, $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding and $a_J : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ be the associated sheaf functor. Let C and C' be objects of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Suppose that $r : a_J(C) \to a_J(C')$ is an arrow in $\mathbf{Sh}(\mathcal{C}, J)$. Then, the unit of the adjunction between the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor a_J provides an arrow $\eta_{C'} : C \to a_J(C')$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, while the arrow r corresponds, via this adjunction, to an arrow $r_a : C \to a_J(C')$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Let us consider the pullback

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & C' \arrow[d, "\eta_{C'}"] \\ C \arrow[r, "r_a"'] & a_J(C') \end{tikzcd}$$

in the topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Notice that $a_J(\alpha)$ is an isomorphism, while $a_J(\beta)$ is isomorphic to r. The arrow r can thus be identified with the pair (α, β); conversely, any pair of arrows $(\alpha : A \to C, \beta : A \to C')$ in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ with common domain with the property that $a_J(\alpha)$ is an isomorphism gives rise to an arrow $a_J(C) \to a_J(C')$ in $\mathbf{Sh}(\mathcal{C}, J)$, namely $a_J(\beta) \circ a_J(\alpha)^{-1}$; in fact, this correspondence can be made into an reflection of the discrete category $Hom_{\mathbf{Sh}(\mathcal{C},J)}(a_J(C), a_J(C'))$ on the set of arrows $a_J(C) \to a_J(C')$ in $\mathbf{Sh}(\mathcal{C}, J)$ into the category $\mathcal{K}_{C,C'}$ having as objects such pairs (α, β) and as arrows $(\alpha, \beta) \to (\alpha', \beta')$ the arrows $s : A \to A'$ in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ such that $\alpha' \circ s = \alpha$ and $\beta' \circ s = \beta$ (where $dom(\alpha) = dom(\beta) = A$ and $dom(\alpha') = dom(\beta') = A'$).

Now, as observed in [9], we can give an explicit condition on a morphism $\gamma : F \to G$ in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ for it to be sent by the associated sheaf functor a_J to an isomorphism. Indeed, $a_J(\gamma)$ is an isomorphism if and only if its image $Im(\gamma) \rightarrowtail G$ is J-dense and the equalizer of its kernel pair is J-dense; and the condition for a monomorphism $A \rightarrowtail E$ in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ to be J-dense can be explicitly described as follows: for any $x \in E(c)$, $\{f : d \to c \mid E(f)(x) \in A(d)\} \in J(c)$.

Notice that the objects of the category $\mathcal{K}_{C,C'}$ can be described in more elementary terms, that is in terms of discrete fibrations with codomain $\mathcal{C}$, by using Grothendieck's construction. Specifically, for any objects A and E of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, the natural transformations $A \to E$ are in bijective correspondence with the functors $F : \int A \to \int E$ such that $\pi_E \circ F = \pi_A$, where $\int A$ and $\int E$ are respectively the category of elements of the presheaf A and of the presheaf E and π_A and π_E are the two canonical projections to $\mathcal{C}$. Conversely, any discrete fibration $P : \mathcal{B} \to \mathcal{C}$ with codomain $\mathcal{C}$ corresponds to a presheaf on $\mathcal{C}$ whose category of elements is isomorphic to $\mathcal{B}$, and any morphism of discrete fibrations of $\mathcal{C}$ corresponds to a natural transformation between the corresponding presheaves (whose corresponding functor between the associated categories of elements is isomorphic to the given morphism). The point of view of discrete fibrations is natural in this context because the property of a morphism $\gamma : A \to E$ to be sent by the associated sheaf functor a_J to an isomorphism can be naturally expressed in terms of the associated morphism of fibrations. In particular, if C and C' are representable functors yc and yc', the discrete fibrations corresponding to them are respectively the canonical projections $p_c : \mathcal{C}/c \to \mathcal{C}$ and $p_{c'} : \mathcal{C}/c' \to \mathcal{C}$. The existence of a pair (α, β) satisfying the above conditions translates into the existence of a discrete fibration $r : \mathcal{I} \to \mathcal{C}$ and two morphisms of fibrations $a : r \to p_c$ and $b : r \to p_{c'}$ such that a satisfies the (elementary condition equivalent to the) condition that its associated presheaf be sent by a_J to an isomorphism.

4.3. Atoms and transitive actions

Let us now consider, in the context of an equivalence

$$\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(u))$$

provided by Theorem 3.5, the notion of atom of a topos.

Let us first characterize the atoms of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. Let A be an atom of this topos. Since the $l(c)$ (for $c \in \mathcal{C}$) form a separating set for this topos, there is an arrow $e : l(c) \to A$ for some $c \in \mathcal{C}$; A being an atom, e is an epimorphism, so it is the coequalizer of its kernel pair $S \rightarrowtail l(c) \times l(c)$. We can suppose, without loss of generality, S to be of the form $a_{J_{at}}(R)$ where R is a subobject of $\mathcal{C}(c, -) \times \mathcal{C}(c, -)$ in $[\mathcal{C}, \mathbf{Set}]$; in fact, we can suppose R, without loss of generality, to be an equivalence relation on $\mathcal{C}(c, -)$. Indeed, consider the language with just one binary relation symbol K and the structure for its language in the topos $[\mathcal{C}, \mathbf{Set}]$ obtained by interpreting K as R; then one can write down a geometric formula over this language whose interpretation I_Z in this structure is precisely the equivalence relation on $\mathcal{C}(c, -)$ generated by R; since the associated sheaf functor $a_{J_{at}}$ is a geometric functor, $a_{J_{at}}(I_Z)$ is isomorphic to the equivalence relation generated by $a_{J_{at}}(R)$, that is, to $a_{J_{at}}(R)$ itself, since $a_{J_{at}}(R)$ is an equivalence relation on $l(c)$.

Notice that R can be thought of as a set R of pairs of arrows in $\mathcal{C}$ with common domain c and common codomain such that for any $(f, g) \in R$ and any $h : \mathit{cod}(f) = \mathit{cod}(g) \to \mathit{cod}(h)$ in $\mathcal{C}$, $(h \circ f, h \circ g) \in R$. Let us denote by (f, g) the set of pairs of arrows of the form $(h \circ f, h \circ g)$, for an arrow $h \in \mathcal{C}$; then (f, g) is a subobject of $\mathcal{C}(c, -) \times \mathcal{C}(c, -)$ in $[\mathcal{C}, \mathbf{Set}]$, and $a_{J_{at}}(f, g) \rightarrowtail l(c) \times l(c)$ can be identified with the monic part of the epi-mono factorization in the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ of the arrow $< l(f), l(g) >: l(\mathit{cod}(f)) = l(\mathit{cod}(g)) \to l(c) \times l(c)$. Therefore we can regard $a_{J_{at}}(R)$ as the union in $\mathrm{Sub}(l(c) \times l(c))$ of the epi-mono factorizations of all the arrows $< l(f), l(g) >$ for all the pairs $(f, g) \in R$. Notice that $a_{J_{at}}(R)$ corresponds under the equivalence $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathit{Aut}_{\mathcal{C}}(u))$ to the ($\mathit{Aut}_{\mathcal{C}}(u)$-equivariant) subset of $\mathit{Hom}_{\text{Ind-}\mathcal{C}}(c, u) \times \mathit{Hom}_{\text{Ind-}\mathcal{C}}(c, u)$ consisting of the collection R_t of pairs of the form $(\chi \circ f, \chi \circ g)$ where $\chi \in \mathit{Hom}_{\text{Ind-}\mathcal{C}}(\mathit{cod}(f), u)$ and $(f, g) \in R$. The quotient in $\mathbf{Cont}(\mathit{Aut}_{\mathcal{C}}(u))$ by this equivalence relation on $\mathit{Hom}_{\text{Ind-}\mathcal{C}}(c, u)$ thus corresponds, under the equivalence $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\mathit{Aut}_{\mathcal{C}}(u))$, to the quotient in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ of $l(c)$ by the relation $a_{J_{at}}(R)$ (notice that R_t is indeed an equivalence relation since this concept is a topos-theoretic invariant).

Now, let us characterize the atoms in $\mathbf{Cont}(\mathit{Aut}_{\mathcal{C}}(u))$. These are exactly, up to isomorphism, the $\mathit{Aut}_{\mathcal{C}}(u)$-sets of the form $\mathit{Aut}_{\mathcal{C}}(u)/U$, where U is an open subgroup of $\mathit{Aut}_{\mathcal{C}}(u)$. Since $\mathcal{I}_{\mathcal{C}}$ is an algebraic base for $\mathit{Aut}_{\mathcal{C}}(u)$, any such U contains an open subgroup of the form $\mathcal{I}_{\xi}$ for some $\xi : c \to u$, so we have an epimorphism $\mathit{Aut}_{\mathcal{C}}(u)/\mathcal{I}_{\xi} \to \mathit{Aut}_{\mathcal{C}}(u)/U$. From the preceding remarks, we thus deduce that there exists an equivalence relation R on $\mathcal{C}(c, -)$ in $[\mathcal{C}, \mathbf{Set}]$, with the property that U is equal to the set $\{z \in \mathit{Aut}_{\mathcal{C}}(u) \mid (\xi, z \circ \xi) \in R_t\}$. Notice that, by definition of R_t, $(\xi, z \circ \xi) \in R_t$ if and only if there exists $(f, g) \in R$ and $\chi : \mathit{cod}(f) \to u$ such that $\xi = \chi \circ f$ and $z \circ \xi = \chi \circ g$.

Summarizing, we have the following result.

Proposition 4.15. *Under the hypotheses of Theorem 3.5, for any subset U of $\mathit{Aut}_{\mathcal{C}}(u)$, U is an open subgroup of $\mathit{Aut}_{\mathcal{C}}(u)$ if and only if there exists and object c of $\mathcal{C}$ and an equivalence relation R on $\mathcal{C}(c, -)$ in $[\mathcal{C}, \mathbf{Set}]$ such that $U = \{z \in \mathit{Aut}_{\mathcal{C}}(u) \mid z \circ \chi \circ f = \chi \circ g$ for some $(f, g) \in R$ and $\chi : \mathit{cod}(f) \to u\}$. More precisely, for any $\xi : c \to u$, a subset $U \subseteq \mathcal{I}_{\xi}$ is an open subgroup of $\mathit{Aut}_{\mathcal{C}}(u)$ if and only if there exists an equivalence* *relation R on $\mathcal{C}(c,-)$ in $[\mathcal{C},\mathbf{Set}]$ such that $U = \{z \in Aut_{\mathcal{C}}(u) \mid \xi = \chi \circ f$ and $z \circ \xi = \chi \circ g$ for some $(f,g) \in R$ and $\chi : cod(f) \to u\}$.* $\square$

Let us now suppose that the topology J_{at} on $\mathcal{C}$ is subcanonical, i.e. that all the arrows in $\mathcal{C}$ are strict monomorphisms. It is natural to wonder whether we can give an elementary description of the dual $\mathcal{C}_{at}$ of the full subcategory of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ on its atoms, and of the canonical embedding $i_{at} : \mathcal{C} \hookrightarrow \mathcal{C}_{at}$. We shall call $\mathcal{C}_{at}$ the *atomic completion* of the category $\mathcal{C}$, and we shall say that $\mathcal{C}$ is *atomically complete* if the embedding i_{at} is an equivalence.

Let us denote by $\mathbf{At}$ the category whose objects are the categories $\mathcal{D}$ with the amalgamation property and in which all arrows are strict monomorphisms, and whose morphisms $\mathcal{D} \to \mathcal{D}'$ are the morphisms of sites $(\mathcal{D}, J_{at}) \to (\mathcal{D}', J_{at})$; let us denote by $\mathbf{AtC}$ the full subcategory of $\mathbf{At}$ on the atomically complete categories. Then the assignment $\mathcal{C} \to \mathcal{C}_{at}$ can be made into a functor $C : \mathbf{At} \to \mathbf{AtC}$; indeed, for any arrow $f : \mathcal{C} \to \mathcal{D}$ in $\mathbf{At}$, $\mathbf{Sh}(f^{op})^* : \mathbf{Sh}(\mathcal{C}^{^{\mathrm{op}}}, J_{at}) \simeq \mathbf{Sh}(\mathcal{C}_{at}^{^{\mathrm{op}}}, J_{at}) \to \mathbf{Sh}(\mathcal{D}^{^{\mathrm{op}}}, J_{at}) \simeq \mathbf{Sh}(\mathcal{D}_{at}^{^{\mathrm{op}}}, J_{at})$ restricts to an arrow $\mathcal{C}_{at} \to \mathcal{D}_{at}$ in $\mathbf{AtC}$, since it preserves epimorphisms and every quotient of an atom is an atom. In fact, C defines a left adjoint to the canonical inclusion $\mathbf{AtC} \hookrightarrow \mathbf{At}$, making $\mathbf{AtC}$ into a reflective subcategory of $\mathbf{At}$.

To define explicitly an extension $\mathcal{C}_e$ of the category $\mathcal{C}$ in $\mathbf{At}$ which is equivalent to $\mathcal{C}_{at}$, we observe that any atom of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ is isomorphic to a quotient yc/R for an equivalence relation R on yc in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, where $y : \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ is the Yoneda embedding. Notice that an equivalence relation R on yc in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ is an equivalence relation R on yc in $[\mathcal{C}, \mathbf{Set}]$ which is J_{at}-closed as a subobject of $yc \times yc$. Concretely, an equivalence relation R on yc in $[\mathcal{C}, \mathbf{Set}]$ can be identified with a function which assigns to each object e of $\mathcal{C}$ an equivalence relation R_e on the set $Hom_{\mathcal{C}}(c, e)$ in such a way that for any arrow $h : e \to e'$ in $\mathcal{C}$ and any $(\chi, \xi) \in R_e$, $(h \circ \chi, h \circ \xi) \in R_{e'}$; R is an equivalence relation in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ if it is moreover J_{at}-closed, that is if it is equal to its J_{at}-closure $\overline{R}^{J_{at}}$, which is given by: for each $e \in \mathcal{C}$ $\overline{R}_e^{J_{at}} = \{(\chi, \xi) \mid$ for some $h : e \to e'$, $(h \circ \chi, h \circ \xi) \in R_{e'}\}$.

This suggests that a natural choice for the objects of the category $\mathcal{C}_e$ is given by the pairs (c, R), where c is an object of $\mathcal{C}$ and R is a J_{at}-closed equivalence relation on yc. In order to have an equivalence $\mathcal{C}_e \simeq \mathcal{C}_{at}$ given by the assignment $(c, R) \to a_{J_{at}}(yc/R)$, we need to be able to the define the arrows in $\mathcal{C}_e$ (as well as their composition) in such a way that the arrows $(c', R') \to (c, R)$ in $\mathcal{C}_e$ correspond bijectively with the arrows $a_{J_{at}}(yc/R) \to a_{J_{at}}(yc'/R')$ in the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. To an arrow $z : a_{J_{at}}(yc/R) \to a_{J_{at}}(yc'/R')$ we can associate the following set of data: z corresponds, by the universal property of $a_{J_{at}}$ as a left adjoint, to a unique arrow $\tilde{z} : yc/R \to a_{J_{at}}(yc'/R')$ in $[\mathcal{C}, \mathbf{Set}]$; since z is an epimorphism, it is isomorphic to the coequalizer of its kernel pair, which can be identified with an equivalence relation P on yc containing R, and we have a unique isomorphism $w : a_{J_{at}}(yc/P) \to a_{J_{at}}(yc'/R')$ such that $w \circ a_{J_{at}}(\pi_{R,P}) = z$, where $\pi_{R,P} : yc/R \to yc/P$ is the canonical projection arrow. Notice that the equivalence relations on yc/R are in bijection with the equivalence relations on yc which contain

R, via the correspondence which sends any equivalence relation on yc/R to its pullback along the canonical arrow $\pi_R : yc \to yc/R$, and any equivalence relation $z : Z \rightarrowtail yc \times yc$ on yc which contains R to the image $\tilde{Z}$ of the composite arrow $(\pi_R \times \pi_R) \circ z$.

As in section 4.2, let us consider the pullback

$$\begin{tikzcd} Q \arrow[r, "\beta"] \arrow[d, "\alpha"] & yc'/R' \arrow[d, "\eta_{yc'/R'}"] \\ yc/P \arrow[r, "\tilde{w}"'] & a_{J_{at}}(yc'/R'), \end{tikzcd}$$

in the topos $[\mathcal{C}, \mathbf{Set}]$, where $\tilde{w}$ is the composite of w with the unit $yc/P \to a_{J_{at}}(yc/P)$ and $\eta_{yc'/R'}$ is the unit $yc'/R' \to a_{J_{at}}(yc'/R')$; clearly, both $a_{J_{at}}(\alpha)$ and $a_{J_{at}}(\beta)$ are isomorphisms, and we have that $a_{J_{at}}(w) = a_{J_{at}}(\beta) \circ a_{J_{at}}(\alpha)^{-1}$. Therefore the set of data consisting of the triple (P, α, β) completely determines our original arrow z, which can be recovered as $a_{J_{at}}(\pi_{R,P}) \circ a_{J_{at}}(\beta) \circ a_{J_{at}}(\alpha)^{-1}$.

Notice that, since the above square is a pullback, the arrow $< \alpha, \beta >: Q \to yc/P \times yc'/R'$ is a monomorphism. Now, for any subobject $< \delta, \gamma >: Q' \rightarrowtail yc/P \times yc'/R'$ such that $a_{J_{at}}(\delta)$ and $a_{J_{at}}(\gamma)$ are isomorphisms, the composite $a_{J_{at}}(\gamma) \circ a_{J_{at}}(\delta)^{-1}$ yields an arrow $a_{J_{at}}(yc/P) \to a_{J_{at}}(yc'/R')$ in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$ and hence an arrow $a_{J_{at}}(\pi_{R,P}) \circ a_{J_{at}}(\gamma) \circ a_{J_{at}}(\delta)^{-1}$. Therefore, we can identify the arrows $a_{J_{at}}(yc/R) \to a_{J_{at}}(yc'/R')$ in the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$ with the triples (P, α, β) consisting of a J_{at}-closed equivalence relation P on c containing R and a subobject $< \alpha, \beta >: Q \to yc/P \times yc'/R'$ such that $a_{J_{at}}(\alpha)$ and $a_{J_{at}}(\beta)$ are isomorphisms, modulo the equivalence relation $\simeq$ given by: $(P, \alpha, \beta) \simeq (P, \delta, \gamma)$ if and only if $a_{J_{at}}(\beta) \circ a_{J_{at}}(\alpha)^{-1} = a_{J_{at}}(\gamma) \circ a_{J_{at}}(\delta)^{-1}$. This equivalence relation $\simeq$ admits an elementary description not involving the functor $a_{J_{at}}$. Indeed, let us consider the pullback

$$\begin{tikzcd} B_{\beta,\gamma} \arrow[r, "p_\beta"] \arrow[d, "p_\gamma"] & Q \arrow[d, "\beta"] \\ Q' \arrow[r, "\gamma"'] & yc'/R' \end{tikzcd}$$

in the topos $[\mathcal{C}, \mathbf{Set}]$. This diagram is sent by the associated sheaf functor $a_{J_{at}}$ to a pullback in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$; therefore, both the arrows $a_{J_{at}}(p_\beta)$ and $a_{J_{at}}(p_\gamma)$ are isomorphisms (as they are pullbacks of isomorphisms) and the condition that $a_{J_{at}}(\beta) \circ a_{J_{at}}(\alpha)^{-1} = a_{J_{at}}(\gamma) \circ a_{J_{at}}(\delta)^{-1}$ can be formulated as the existence of a factorization of $< a_{J_{at}}(\alpha)^{-1}, a_{J_{at}}(\delta)^{-1} >$ through $< a_{J_{at}}(p_\beta), a_{J_{at}}(p_\gamma) >$; but this is in turn equivalent to the condition that $a_{J_{at}}(\alpha \circ p_\beta) = a_{J_{at}}(\delta \circ p_\gamma)$, which admits an elementary formulation (since it is equivalent to the assertion that the equalizer of $\alpha \circ p_\beta$ and $\delta \circ p_\gamma$ is a J_{at}-dense monomorphism – cf. section 4.2).

We thus define an arrow $(c', R') \to (c, R)$ in $\mathcal{C}_e$ as a triple (P, α, β) consisting of a J_{at}-closed equivalence relation P on c containing R and a subobject $< \alpha, \beta >: Q \rightarrowtail$

$yc/P \times yc'/R'$ such that $a_{J_{at}}(\alpha)$ and $a_{J_{at}}(\beta)$ are isomorphisms modulo the equivalence relation $\simeq$ defined above.

It remains to understand how to define the composition of arrows in $\mathcal{C}_e$ in such a way to make the assignments $(c, R) \to a_{J_{at}}(yc/R)$ and $((P, \alpha, \beta) : (c', R') \to (c, R)) \to a_{J_{at}}(\pi_{R,P}) \circ a_{J_{at}}(\beta) \circ a_{J_{at}}(\alpha)^{-1}$ into a (full and faithful) functor $E : \mathcal{C}_e \to \mathcal{C}_{at}$. Given two (representatives of) arrows $(P, \alpha, \beta) : (c', R') \to (c, R)$ and $(P', \alpha', \beta') : (c'', R'') \to (c', R')$ in $\mathcal{C}_e$ corresponding respectively to arrows $u : a_{J_{at}}(yc/R) \to a_{J_{at}}(yc'/R')$ and $v : a_{J_{at}}(yc'/R') \to a_{J_{at}}(yc''/R'')$, the kernel pair of the composite arrow $v \circ u$ can be identified with the pullback of the kernel pair of the arrow v along the arrow $u \times u$; but the kernel pair of v can be identified with the equivalence relation on yc'/R' corresponding to P', while u is equal to the arrow $a_{J_{at}}(\pi_{R,P}) \circ a_{J_{at}}(\beta) \circ a_{J_{at}}(\alpha)^{-1}$.

$$
\begin{tikzcd}
P \arrow[dr, shift left] \arrow[dr, shift right] & & & \\
R \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & yc \arrow[r] \arrow[ddr] & a_{J_{at}}(yc/R) \arrow[dd, "u"] \arrow[ddr, "a_{J_{at}}(\pi_{R,P})"] & \\
P' \arrow[dr, shift left] \arrow[dr, shift right] & & & \\
R' \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & yc' \arrow[r] \arrow[dr] & a_{J_{at}}(yc'/R') \arrow[d, "v"] \arrow[dr, "a_{J_{at}}(\pi_{R',P'})"] & a_{J_{at}}(yc/P) \arrow[l, "\cong"'] \\
 & yc'' \arrow[r] & a_{J_{at}}(yc''/R'') & a_{J_{at}}(yc'/P') \arrow[l, "\cong"']
\end{tikzcd}
$$

Let us thus define P'' to be the pullback along the canonical arrow $yc \to yc/R$ of the equivalence relation on yc/R generated by the arrow $h : H \to yc/R \times yc/R$ defined by the following diagram, where the upper and the lower square are pullbacks in $[\mathcal{C}, \mathbf{Set}]$:

$$
\begin{tikzcd}
H \arrow[r, "h"] \arrow[d, "r"] & yc/R \times yc/R \arrow[d, "\pi_{R,P} \times \pi_{R,P}"] \\
L \arrow[r, "<\alpha \circ \chi, \beta \circ \xi>"] & yc/P \times yc/P \\
L \arrow[u, "1_L"] \arrow[r, "<\chi, \xi>"] \arrow[d, "s"] & Q \times Q \arrow[u, "\alpha \times \alpha"'] \arrow[d, "\beta \times \beta"] \\
\tilde{P}' \arrow[r] & yc'/R' \times yc'/R'.
\end{tikzcd}
$$

The relation P'' will be the first component of a triple representing the composition of the two arrows $(P, \alpha, \beta) : (c', R') \to (c, R)$ and $(P', \alpha', \beta') : (c'', R'') \to (c', R')$ in the category $\mathcal{C}_e$. To define the remaining two components, we observe that the arrow $m : a_{J_{at}}(yc/P'') \to a_{J_{at}}(yc''/R'')$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ corresponding to the composition $v \circ u$ can be expressed as the composition of arrows $a_{J_{at}}(\beta') \circ a_{J_{at}}(\alpha')^{-1} \circ q$, where q is the canonical arrow from the coequalizer of $a_{J_{at}}(h)$ to the coequalizer of $a_{J_{at}}(\tilde{P}')$. Now, if we denote by $b_1 : yc/P'' \to C$ (resp. by $b_2 : C' \to C$, by $b_3 : C' \to yc'/P'$) the canonical arrows between the coequalizers of h and $< \alpha \circ \chi, \beta \circ \xi >$ (resp. of $< \chi, \xi >$ and $< \alpha \circ \chi, \beta \circ \xi >$, of $< \chi, \xi >$ and $\tilde{P}'$) then the arrow q is equal to the composite $a_{J_{at}}(b_3) \circ a_{J_{at}}(b_2)^{-1} \circ a_{J_{at}}(b_1)$. Notice that $a_{J_{at}}(b_2)$ is an isomorphism since $a_{J_{at}}(\alpha)$ is an isomorphism, and $a_{J_{at}}(b_3)$ is an isomorphism since $a_{J_{at}}(\beta)$ is an isomorphism; as q is an isomorphism (by definition of P''), it follows that $a_{J_{at}}(b_1)$ is also an isomorphism. Now, consider the pullbacks

$$\begin{tikzcd} M \arrow[r, "a"] \arrow[d, "c"] & C' \arrow[d, "b_2"] \\ yc/P'' \arrow[r, "b_1"'] & C \end{tikzcd}$$

and

$$\begin{tikzcd} N \arrow[r, "d"] \arrow[d, "b"] & M \arrow[d, "a"] \\ W \arrow[r, "y"'] \arrow[d, "x"] & C' \arrow[d, "b_3"] \\ Q' \arrow[r, "\alpha'"'] & yc'/P' \end{tikzcd}$$

in the topos $[\mathcal{C}, \mathbf{Set}]$.

Notice that $a_{J_{at}}(c)$ is an isomorphism since $a_{J_{at}}(b_2)$ is; similarly, $a_{J_{at}}(y)$ and $a_{J_{at}}(d)$ are isomorphisms as they are pullbacks of isomorphisms. Let us set $\alpha''_p = c \circ d$ and $\beta''_p = \beta' \circ x \circ b$, and verify that $m = a_{J_{at}}(\beta''_p) \circ a_{J_{at}}(\alpha''_p)^{-1}$. We have $m = a_{J_{at}}(\beta') \circ a_{J_{at}}(\alpha')^{-1} \circ q = a_{J_{at}}(\beta') \circ a_{J_{at}}(\alpha')^{-1} \circ a_{J_{at}}(b_3) \circ a_{J_{at}}(b_2)^{-1} \circ a_{J_{at}}(b_1) = a_{J_{at}}(\beta') \circ a_{J_{at}}(x) \circ a_{J_{at}}(y)^{-1} \circ a_{J_{at}}(a) \circ a_{J_{at}}(c)^{-1} = a_{J_{at}}(\beta') \circ a_{J_{at}}(x) \circ a_{J_{at}}(b) \circ a_{J_{at}}(d)^{-1} \circ a_{J_{at}}(c)^{-1} = a_{J_{at}}(\beta''_p) \circ a_{J_{at}}(\alpha''_p)^{-1}$, as required. We define $< \alpha'', \beta'' >$ as the image of $< \alpha''_p, \beta''_p >$ in $[\mathcal{C}, \mathbf{Set}]$; clearly, $a_{J_{at}}(\beta'') \circ a_{J_{at}}(\alpha'')^{-1} = a_{J_{at}}(\beta''_p) \circ a_{J_{at}}(\alpha''_p)^{-1} = m$, and hence (P'', α'', β'') defines an arrow $(c'', R) \to (c, R)$ in $\mathcal{C}_e$ which is the composition of the arrows $(P, \alpha, \beta) : (c', R') \to (c, R)$ and $(P', \alpha', \beta') : (c'', R'') \to (c', R')$ in $\mathcal{C}_e$.

Note that the definition of the composition operation in the category $\mathcal{C}_e$ is completely elementary. We thus have a category $\mathcal{C}_e$ (the categorical axioms are easily verified at this point) and a functor $\mathcal{C}_e \to \mathcal{C}_{at}$ which is full and faithful and essentially surjective, i.e. a (weak) equivalence. The following theorem records this fact.

Theorem 4.16. *Let $\mathcal{C}$ be a category in* $\mathbf{At}$*. Then the category $\mathcal{C}_e$ defined above is (weakly) equivalent to the atomic completion $\mathcal{C}_{at}$ of $\mathcal{C}$.* $\square$

Note that the definition of the category $\mathcal{C}_e$ is relatively complicated since, in order to be equivalent to $\mathcal{C}_{at}$, one has to characterize in 'elementary terms' (i.e., in a way which involves only constructions in the presheaf topos $[\mathcal{C}, \mathbf{Set}]$, and not the associated sheaf functor $a_{J_{at}}$) *all* the arrows $a_{J_{at}}(yc/R) \to a_{J_{at}}(yc'/R')$ in the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. If instead one restricts to the arrows $a_{J_{at}}(yc/R) \to a_{J_{at}}(yc'/R')$ of the form $a_{J_{at}}(h)$ for an arrow $h : yc/R \to yc'/R'$ in $[\mathcal{C}, \mathbf{Set}]$, one obtains a subcategory $\mathcal{C}_r$ admitting a simpler description. Let us define the objects of $\mathcal{C}_r$ as the pairs (c, R), where c is an object of $\mathcal{C}$ and R is a J_{at}-closed equivalence relation on yc, and the arrows $(c', R') \to (c, R)$ in $\mathcal{C}_r$ to be the arrows $f : c \to c'$ in $\mathcal{C}$ such that the arrow $yf : yc \to yc'$ factorizes through the projection arrows $yc \to yc/R$ and $yc' \to yc'/R'$, modulo the equivalence relation which identifies two arrows $f, f' : c \to c'$ when the images of the factorizations of yf and yf' through the projection arrows under the associated sheaf functor $a_{J_{at}}$ are equal (note that this condition admits an elementary formulation). Notice that to every object c of $\mathcal{C}$ we can associate an object of $\mathcal{C}_e$, namely the pair (c, R_c), where R_c is the trivial relation on yc, and every arrow $h : c' \to c$ in $\mathcal{C}$ gives canonically rise to an arrow $(c', R_{c'}) \to (c, R_c)$ in $\mathcal{C}_e$, given by the triple (P, α, β), where P is the kernel pair of yh in $[\mathcal{C}, \mathbf{Set}]$, α is the identity on yc/P and β is equal to the factorization of $yh : yc \to yc'$ through the canonical projection $yc \to yc/P$. In fact, these assignments yield a functor $\mathcal{C} \to \mathcal{C}_r$ which, composed with the canonical embedding $\mathcal{C}_r \hookrightarrow \mathcal{C}_e$, yields a functor $\mathcal{C} \to \mathcal{C}_e$ which is isomorphic to the canonical functor $\mathcal{C} \to \mathcal{C}_{at}$.

The following result gives a criterion for recognizing atomically complete categories.

Recall that an equivalence relation in a topos $[\mathcal{C}, \mathbf{Set}]$ on an object E can be seen as a subfunctor $\mathcal{C} \to \mathbf{Set}$ of $E \times E$ sending to every object c of $\mathcal{C}$ an equivalence relation of $E(c)$. Indeed, the concept of equivalence relation can be formalized within geometric logic and the evaluation functors $ev_c : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ (for $c \in \mathcal{C}$) are geometric and jointly conservative.

Theorem 4.17. *Let $\mathcal{C}$ be a category in* $\mathbf{At}$*. Then the following conditions are equivalent:*

(i) *$\mathcal{C}$ is atomically complete;*
(ii) *For every object c of $\mathcal{C}$ and any equivalence relation R on yc in $[\mathcal{C}, \mathbf{Set}]$, there exists an arrow $m : d \to c$ such that for any arrows $f, g : c \to e$ in $\mathcal{C}$, $f \circ m = g \circ m$ if and only if there exists $h : e \to e'$ such that $(h \circ f, h \circ g) \in R_{e'}$;*
(iii) *The category $\mathcal{C}$ has equalizers, for any object c of $\mathcal{C}$ there exist arbitrary intersections of subobjects of c in $\mathcal{C}$, and for any pair of arrows $h, k : c \to e$ in $\mathcal{C}$ with equalizer $m : d \rightarrowtail c$ we have that for any pair of arrows $l, n : c \to e'$, $l \circ m = n \circ m$ if and only*

if there exists an arrow $s : e' \to e''$ such that $(s \circ l, s \circ n)$ belongs to the equivalence relation on $Hom_{\mathcal{C}}(c, e'')$ generated by the relation consisting of the pairs of the form $(t \circ h, t \circ k)$ for an arrow $t : e \to e''$.

Proof. Let us first prove the equivalence between (i) and (ii).

Let A be an atom of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. As the objects of the form $l(c)$ form a separating set for the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, there exists an arrow (in fact, an epimorphism) $l(c) \to A$ for some object c of $\mathcal{C}$. The object A is therefore isomorphic to the quotient $l(c)/R$ of $l(c)$ by an equivalence relation R on $l(c)$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. As J_{at} is subcanonical, the equivalence relations on $l(c)$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ can be identified with the equivalence relations on $\mathcal{C}(c, -)$ in $[\mathcal{C}, \mathbf{Set}]$ which are J_{at}-closed as subobjects of $\mathcal{C}(c, -) \times \mathcal{C}(c, -)$. The condition that A should be, up to isomorphism, of the form $l(d)$ thus amounts to the existence of an object $d \in \mathcal{C}$ and an arrow $m : d \to c$ in $\mathcal{C}$ such that $l(f)$ is isomorphic to the canonical projection $l(c) \to l(c)/R$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. Notice that, since the associated sheaf functor $a_{J_{at}} : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ preserves coequalizers, this condition is equivalent to requiring the existence of an arrow $m : d \to c$ such that for any $(\xi, \chi) \in R$, $\xi \circ m = f\chi \circ m$ and the canonical arrow $- \circ m : \mathcal{C}(c, -)/R \to \mathcal{C}(d, -)$ in $[\mathcal{C}, \mathbf{Set}]$ is sent by $a_{J_{at}}$ to an isomorphism (notice that this arrow is an epimorphism since, by definition of J_{at}, $l(m)$ is an epimorphism). Now, for any elementary topos $\mathcal{E}$ and any local operator j on $\mathcal{E}$, it is possible to express the condition for an arrow $h : A \to B$ in $\mathcal{E}$ to be sent by the associated sheaf functor $a_j : \mathcal{E} \to \mathbf{sh}_j(\mathcal{E})$ to a monomorphism as the condition that the canonical monomorphism $n : A \to R$ from A to the kernel pair R of h should be j-dense; indeed, denoting by $(r_1, r_2) : R \to A$ the kernel pair of h, since a_j preserves pullbacks we have that $a_j(h)$ is a monomorphism if and only if $a_j(r_1) = a_j(r_2)$, if and only if $a_j(n)$ is an isomorphism, i.e. n is j-dense. Recall that a subobject $B \rightarrowtail E$ in $[\mathcal{C}, \mathbf{Set}]$ is J-dense for a Grothendieck topology J on $\mathcal{C}^{\mathrm{op}}$ if and only if for every object $c \in \mathcal{C}$ and any element $x \in E(c)$, the sieve $\{f : c \to d \mid E(f)(x) \in B(d)\} \in J(c)$. Applying this characterization to our particular case, we obtain that the canonical monomorphism $\mathcal{C}(c, -)/R \rightarrowtail K$ in $[\mathcal{C}, \mathbf{Set}]$, where K is the kernel pair of the arrow $- \circ m : \mathcal{C}(c, -)/R \to \mathcal{C}(d, -)$, is J_{at}-dense if and only if for every pair of arrows (a, b) such that $a \circ m = b \circ m$ there exists an arrow h such that $(h \circ a, h \circ b) \in R$.

We shall prove that (i) implies (iii) and that (iii) implies (ii).

(i) $\Rightarrow$ (iii) Let $\mathcal{D}$ be the full subcategory of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ on its atoms; if $\mathcal{C}$ is atomically complete then it is dually equivalent to $\mathcal{D}$. The condition that $\mathcal{C}$ has equalizers is thus equivalent to the condition that $\mathcal{D}$ has coequalizers, and this is obviously satisfied since $\mathcal{D}$ is closed in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ under coequalizers, as epimorphic images of atoms are atoms. Concerning the property that arbitrary intersection of subobjects exists in $\mathcal{C}$, this is clearly equivalent to the condition that for any object d of $\mathcal{D}$ and any family of arrows $\{s_i : d \to d_i \mid i \in I\}$ in $\mathcal{D}$ there exists an arrow $s : d \to e$ which factors (necessarily uniquely, as all arrows in $\mathcal{D}$ are epimorphisms) through each of the s_i and such that any arrow $s' : d \to e'$ with the same property factors through s. To prove this property of $\mathcal{D}$, we use the fact that for any $i \in I$, the arrow $s_i : d \to d_i$ is isomorphic to the quotient arrow $d \to d/R_i$, where $R_i \rightarrowtail d \times d$ is the kernel pair of s_i; if we denote by $R \rightarrowtail d \times d$ the union of the subobjects $R_i \rightarrowtail d \times d$ in the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$ then the coequalizer $s : d \to d/R$ is an arrow in $\mathcal{D}$ (since d/R is a quotient of an atom, namely d, and hence it is itself an atom) which is easily seen to satisfy the required property. Let $h, k : c \to e$ be two arrows in $\mathcal{C}$; since $\mathcal{C}$ is atomically complete then condition (ii) holds for the equivalence relation $R_{h,k}$ on yc generated by the image of the arrow $<yh, yk> : ye \to yc \times yc$, yielding an arrow $m : d \to c$ with the property that for any arrows $f, g : c \to e$ in $\mathcal{C}$, $f \circ m = g \circ m$ if and only if there exists $h : e \to e'$ such that $(h \circ f, h \circ g) \in R_{e'}$; by using the explicit description of the relation $R_{h,k}$ and the fact that all the arrows in $\mathcal{C}$ are monomorphisms, one immediately realizes that one can suppose without loss of generality that m is the equalizer of h and k in $\mathcal{C}$; condition (iii) is thus satisfied.

(iii) $\Rightarrow$ (ii) Given an equivalence relation R on yc in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$, since the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$ is atomic, R can be expressed as a (disjoint) union of relations $R_i \rightarrowtail yc \times yc$ which are atoms of the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$; every such relation R_i is of the form $R_{h,k}$ for some arrows $h, k : c \to e$ in $\mathcal{C}$, as there exists an arrow $ye \to R_i$ in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$, which is necessarily an epimorphism. Now, condition (iii) ensures that every relation R_i admits a coequalizer of the form $ym_i : yc \to yd_i$ for an arrow (in fact, a monomorphism) $m_i : d_i \to c$; but from the above discussion we know that the arrow $ym : yc \to y\,dom(m)$, where $m : dom(m) \rightarrowtail c$ is the intersection of all the subobjects $m_i : d_i \rightarrowtail c$ in $\mathcal{C}$, is a coequalizer of the relation R in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$. Condition (ii) is thus satisfied. $\square$

Remark 4.18.

(a) It is clear from the proof of the theorem that in condition (ii) one can suppose without loss of generality m to be the intersection of the equalizers of the pairs $(f, g) \in R_e$ for some $e \in \mathcal{C}$.

(b) If $\mathcal{C}$ has equalizers and arbitrary intersections of subobjects then the embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{at}$ is an equivalence (i.e., $\mathcal{C}$ is atomically complete) if and only if it preserves equalizers. Indeed, by condition (ii) in the statement of the theorem, $\mathcal{C}$ is atomically complete if and only if for any arrows $f, g : a \to c$ in $\mathcal{C}$ with equalizer $m : d \to a$, the arrow ym is a coequalizer of the arrows yf and yg in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$, where $y : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \mathbf{Set}]$ is the Yoneda embedding. Now, as $\mathcal{C}_{at}$ is atomically complete, for any arrows $f, g : a \to c$ in $\mathcal{C}$ with equalizer $m : d \to a$, the arrow $y'm$ is a coequalizer of the arrows $y'f$ and $y'g$ in $\mathbf{Sh}({\mathcal{C}_{at}}^{\text{op}}, J_{at})$, where $y' : \mathcal{C}_{at} \to [{\mathcal{C}_{at}}^{\text{op}}, \mathbf{Set}]$ is the Yoneda embedding; but $\mathbf{Sh}({\mathcal{C}_{at}}^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$, and under this equivalence every representable of the form $y'e$ (for $e \in \mathcal{C}$) corresponds to the representable ye, and every arrow of the form $y'k$ for k in $\mathcal{C}$ is sent to the arrow yk. From which it follows that if the embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{at}$ preserves equalizers then ym is the equalizer of yf and yg in $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{at})$. Since this argument holds for arbitrary f, g we can conclude that the fact that the embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{at}$ preserves equalizers implies the atomic completeness of $\mathcal{C}$.

The following result provides some properties of atomically complete categories. Recall that a multilimit for a diagram $D : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ is a set T of cones on D such that for every cone x over D there exists a cone t in T such that x factors through t; the notion of multicolimit is the dual one.

Proposition 4.19. *Let $\mathcal{C}$ be an atomically complete category. Then*

(i) Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram defined on a small non-empty connected category $\mathcal{I}$. Then D has a limit in $\mathcal{C}$;
(ii) Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram defined on a small category $\mathcal{I}$. Then D has a multicolimit in $\mathcal{C}$.

Proof. Let $\mathcal{D}$ be the full subcategory of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ on its atoms. Then $\mathcal{C}$ is dually equivalent to $\mathcal{D}$.

(i) To prove condition (i) we show that every diagram $D : \mathcal{I} \to \mathcal{D}$ defined on a small non-empty connected category $\mathcal{I}$ has a colimit in $\mathcal{D}$. Let us consider the colimit $colim(D)$ of D in the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$. Since the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ is atomic, the object $colim(D)$ can be expressed as a coproduct of atoms A_k (for $k \in K$); let us denote by $j_k : A_k \to colim(D)$ the canonical coproduct arrows. For each $i \in \mathcal{I}$ we have a colimit arrow $s_i : D(i) \to colim(D)$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$; the fact that $D(i)$ is an atom and that coproducts are stable under pullback in a topos implies that there exists a unique $k_i \in K$ such that s_i factors through j_{k_i}. Notice that if there is an arrow $f : i \to i'$ in $\mathcal{I}$ then we have an arrow $D(f) : D(i) \to D(i')$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ such that $s_{i'} \circ D(f) = s_i$ and hence $k_i = k_{i'}$. Therefore, since $\mathcal{I}$ is connected, there exists a element $k \in K$ such that all the arrows s_i factor through j_k. Now, by the universal property of the colimit, the arrows s_i are jointly epimorphic, whence the coproduct arrow $j_k : A_k \to colim(D)$ is an epimorphism. But this implies that $colim(D) \cong A_k$, as j_k is also a monomorphism; in other words, the cone on D in $\mathcal{D}$ with vertex A_k given by the factorizations of the arrows s_i through j_k is a colimiting cone for D in $\mathcal{D}$.

(ii) To prove condition (ii) we show that every diagram $D : \mathcal{I} \to \mathcal{D}$ be a diagram defined on a small category $\mathcal{I}$ has a multilimit in $\mathcal{D}$. Let us consider the limit $lim(D)$ of the diagram D in the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, and decompose the object $lim(D)$ as a coproduct of atoms A_k (for $k \in K$). Then the composition of the coproduct arrows $A_k \to lim(D)$ with the limit arrows $lim(D) \to D(i)$ defines a set of cones in $\mathcal{D}$ on the diagram D with vertexes A_k; to verify that this family of cones defines a multilimit for D in $\mathcal{D}$ it suffices to observe that for any cone $(B, \{t_i : B \to D(i) \mid i \in I\})$ over D in $\mathcal{D}$ the arrow $B \to lim(D)$ induced by the universal property of the limit $lim(D)$ factors through exactly one of the coproduct arrows $A_k \rightarrowtail A$, as B is an atom and coproducts are stable under pullback in a topos. $\square$

In the following theorem, we denote by $Subgr(Aut_{\mathcal{D}}(u))$ the preorder category consisting of the subgroups of $Aut_{\mathcal{D}}(u)$.

Theorem 4.20. *Let $\mathcal{C}$ be an atomically complete category and $\mathcal{C} \hookrightarrow \mathcal{D}$ be an embedding of $\mathcal{C}$ into a category $\mathcal{D}$ containing a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous u such that there is an equivalence*

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{D}}(u))$$

induced by a functor $F : \mathcal{C}^{\mathrm{op}}/u \to Subgr(Aut_{\mathcal{C}}(u))$ sending to any object $\chi : d \to u$ of $\mathcal{C}/u$ the subgroup $\mathcal{I}_\chi := \{f : u \cong u \mid f \circ \chi = \chi\}$ of $Aut_{\mathcal{C}}(u)$ (as provided for instance by Theorem 3.5 or by Theorem 4.3). Then F induces a bijection between the isomorphism classes of objects of $\mathcal{C}/u$ and the open subgroups of $Aut_{\mathcal{D}}(u)$; for any open subgroup U of $Aut_{\mathcal{D}}(u)$ the arrow $\chi_U : c_U \to u$ in $\mathcal{D}$ corresponding to it in this bijection satisfies the following universal property: it is fixed by all the automorphisms in U and any other arrow $a \to u$ in $\mathcal{D}$ which is fixed by all the automorphisms in U factors uniquely through it.

Proposition 4.21. *In Theorem 4.20, if the subgroup U corresponds to a pair $(\xi : c \to u, R)$, where R is an equivalence relation on yc in $[\mathcal{C}, \mathbf{Set}]$ as in Proposition 4.15 (i.e., $U = \{z \in Aut_{\mathcal{C}}(u) \mid \xi = \chi \circ f$ and $z \circ \xi = \chi \circ g$ for some $(f, g) \in R$ and $\chi : cod(f) \to u\}$) then the arrow $\chi_U : c_U \to u$ can be identified with the composite $\xi \circ m$ of ξ with the monomorphism $m : d \rightarrowtail c$ given by the intersection of the equalizers of the pairs $(f, g) \in R_e$ for some $e \in \mathcal{C}$.*

Proof. First, we notice that the existence of the equalizers and intersection of subobjects in $\mathcal{C}$ is guaranteed by Theorem 4.17.

To prove that $\xi \circ m$ is isomorphic to χ_U in $\mathcal{C}/u$, we exploit the universal property of χ_U given by Theorem 4.20; that is, we verify that (1) $\xi \circ m$ is fixed by all the automorphisms in U; (2) for any arrow $\beta : b \to u$ which is fixed by all the automorphisms in U, β factors uniquely through $\xi \circ m$.

To show (1), we have to verify that for every $z \in U$, $z \circ \xi \circ m = \xi \circ m$. By definition of U, there exists a pair $(f, g) \in R_e$ and an arrow $\chi : e \to u$ such that $(\xi, z \circ \xi) = (\chi \circ f, \chi \circ g)$. Therefore, as $f \circ m = g \circ m$ by definition of m, we have that $z \circ \xi \circ m = \xi \circ m$, as required.

To show (2), we remark that the uniqueness of the factorization of β through ξ is guaranteed by condition (b) in Theorem 4.3 (cf. Remark 4.4(a)). To prove its existence, we argue as follows. First, we note that β factorizes (uniquely) through ξ, say $\beta = \xi \circ w$; indeed, $\mathcal{I}_\xi \subseteq U \subseteq \mathcal{I}_\beta$. Next, we observe that w factors (uniquely) through m. To prove this, we verify that for any $(f, g) \in R_e$, $w \circ f = w \circ g$. Now, the fact that u is $\mathcal{C}$-universal, $\mathcal{C}$-homogeneous and $\mathcal{C}$-ultrahomogeneous implies that there exists an arrow $\gamma : e \to u$ and an automorphism z of u such that $\gamma \circ f = \xi$ and $z \circ \gamma \circ g = \xi$. From these identities it follows at once that $z^{-1} \in U$, which in turn implies that $z^{-1} \circ \beta = \beta$; but $\beta = \xi \circ w$ and $\xi = \gamma \circ f = z \circ \gamma \circ g$, whence $z^{-1} \circ (z \circ \gamma \circ g) \circ w = \gamma \circ f \circ w$. As γ satisfies condition (b) in Theorem 4.3, we conclude that $w \circ f = w \circ g$, as required. $\square$

For any atomically complete category $\mathcal{C}$, the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ admits a simple characterization.

Proposition 4.22. *Let $\mathcal{C}$ be an atomically complete category. Then the canonical embedding $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ realizes $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ as the free small-coproduct completion of $\mathcal{C}^{\mathrm{op}}$.*

Proof. It suffices to verify that the embedding $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ satisfies the universal property of the free small-coproduct completion of the category $\mathcal{C}^{\mathrm{op}}$. We note that $\mathcal{C}^{\mathrm{op}}$ is equivalent to the full subcategory of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ on its atoms. The thesis follows from the fact that, as $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ is an atomic topos, every object A of it can be written, in a unique way, as a disjoint union (=coproduct) of atoms A_k; also, since coproducts in a topos are stable under pullback, for any arrow $f : A \to B$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, where $A = \coprod_{k \in K} A_k$ and $B = \coprod_{j \in J} B_j$ with the A_k and the B_j atoms, there exists a unique function $s_f : K \to J$ and for any $k \in K$ a unique arrow $f_k : A_k \to B_{s_f(k)}$ such that $f \circ \alpha_k = \beta_{s_f(k)} \circ f_k$ for any k, where $\alpha_k : A_k \to A$ and $\beta_j : B_j \to B$ are the canonical coproduct arrows. Any functor $F : \mathcal{D} \to \mathcal{B}$ from $\mathcal{D}$ to a category with small coproducts $\mathcal{B}$ can thus be extended, in a unique way, to a small coproduct-preserving functor $\tilde{F} : \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \to \mathcal{B}$. Indeed, for any object A of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$, we set $\tilde{F}(A)$ equal to the coproduct $\coprod_{k \in K} F(A_k)$ and for any arrow $f : A \to B$ of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ we set $\tilde{F}(f)$ equal to the unique arrow $h : \coprod_{k \in K} F(A_k) \to \coprod_{j \in J} F(B_j)$ in $\mathcal{B}$ such that for any $k \in K$, $h \circ \xi_k = \chi_{s_f(k)} \circ F(f_k)$, where $\xi_k : F(A_k) \to \coprod_{k \in K} F(A_k)$ and $\xi_j : F(B_j) \to \coprod_{j \in J} F(B_j)$ are the canonical coproduct arrows. $\square$

5. Examples

In this section we present some applications of the theory developed above in different mathematical contexts.

Before presenting our list of examples, we make a few remarks on theories of presheaf type that will be useful in the sequel.

Let $\mathbb{T}$ be a theory of presheaf type. It is natural to wonder if there is a theory of presheaf type $\mathbb{T}'$ whose models in **Set** can be identified with those of $\mathbb{T}$ in **Set** and whose model homomorphisms between them can be identified with the injective $\mathbb{T}$-model homomorphism. This problem is relevant for our purposes because the theories of homogeneous $\mathbb{S}$-models are interesting especially when the arrows between the finitely presentable $\mathbb{S}$-models are monic.

The most natural candidate for a theory $\mathbb{T}'$ satisfying the above-mentioned property is the theory obtained from $\mathbb{T}$ by adding to its signature a binary predicate D and the coherent sequents expressing the fact that D is a provable complement of the equality relation; we shall call this theory the *injectivization* of $\mathbb{T}$.

In general, there are various criteria that one can use for investigating whether a theory is of presheaf type (see [12]), and under some natural conditions, the injectivization of a theory of presheaf type whose models are all finite is again of presheaf type (cf. Corollary 6.57 in [12]).

5.1. Discrete Galois theory

If there exists an object c of $\mathcal{C}$ with the property that any arrow in $\mathcal{C}$ with domain $dom(\chi)$ is an isomorphism and any object of $\mathcal{C}$ admits an arrow to c then the group $Aut_{\mathcal{C}}(u)$ is discrete, c is $\mathcal{C}$-ultrahomogeneous and any $\mathcal{C}$-ultrahomogeneous and $\mathcal{C}$-universal object in Ind-$\mathcal{C}$ is isomorphic to c. In particular, Theorem 3.5 yields the following discrete 'Galois-type' theorem.

Theorem 5.1. *Let $\mathcal{C}$ be a category satisfying AP which contains an object c with the property that all arrows $c \to d$ in $\mathcal{C}$ are isomorphisms and for any $e \in \mathcal{C}$ there exists an arrow $e \to c$ in $\mathcal{C}$. Then c is $\mathcal{C}$-ultrahomogeneous and we have an equivalence $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{C}}(c))$, where $Aut_{\mathcal{C}}(c)$ is the discrete group of automorphisms of c in $\mathcal{C}$. Moreover, if all the arrows of $\mathcal{C}$ are strict monomorphisms then the functor $F : \mathcal{C}^{\mathrm{op}}/c \to Subgr(Aut_{\mathcal{C}}(c))$ sending to any object $\chi : d \to c$ of $\mathcal{C}$ the subgroup $\{f : c \cong c \mid f \circ \chi = \chi\}$ of $Aut_{\mathcal{C}}(c)$ yields a bijection between the isomorphism classes of objects of $\mathcal{C}/c$ and the subgroups of $Aut_{\mathcal{C}}(c)$ in the image of the functor F.* □

Remark 5.2. The second part of the theorem (that is, the particular case of it for subcanonical sites) is essentially equivalent to the discrete Galois theory of [16].

There are several interesting examples of discrete Galois theories, notably including the well-known ones given by the classical Galois theory of a finite Galois extension and the theory of universal coverings and the fundamental group.

5.1.1. Classical Galois theory

Let $F \subset L$ be a finite-dimensional Galois extension of a field F, and let $\mathcal{L}_F^L$ be the category of intermediate extensions and homomorphisms between them. We have an equivalence of toposes $\mathbf{Sh}({\mathcal{L}_F^L}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_F(L))$, where $Aut_F(L)$ is the discrete Galois group of L (the category $\mathcal{L}_F^L$ and the object L satisfy the hypotheses of Theorem 5.1). The fundamental theorem of classical Galois theory is exactly equivalent to the assertion that all the arrows in the category $\mathcal{L}_F^L$ are strict monomorphisms and that the category $\mathcal{L}_F^L$ is atomically complete.

5.1.2. Coverings and the fundamental group

Let B be a path connected, locally path connected and semi locally simply connected space; then there is a universal covering $\tau : \tilde{B} \to B$. Let $\mathcal{B}$ be the category whose objects are the connected coverings $p : E \to B$ of B and whose arrows $(p : E \to B) \to (q : E' \to B)$ are the continuous maps $z : E \to E'$ such that $q \circ z = p$. Let b_0 be a fixed point of B, and $\pi_1(B, b_0)$ the fundamental group of B at b_0.

The category $\mathcal{B}$ and the object $\tau : \tilde{B} \to B$ satisfy the hypotheses of Theorem 5.1, and the automorphism group of the object τ in $\mathcal{B}$ is isomorphic to the fundamental group $\pi_1(B, b_0)$; hence we have an equivalence of toposes $\mathbf{Sh}(\mathcal{B}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(\pi_1(B, b_0))$. The fundamental theorem of covering theory provides a bijective correspondence between the subgroups of $\pi_1(B, b_0)$ and the connected coverings of B; in other words, it says that this equivalence of toposes yields a standard Galois theory, or, equivalently, that every arrow in $\mathcal{B}$ is a strict monomorphism and that $\mathcal{B}$ is atomically complete.

5.1.3. *Ultrahomogeneous finite groups*

A finite group G is said to be *ultrahomogeneous* if every isomorphism between subgroups of G can be extended to an automorphism of G. Ultrahomogeneous finite groups have been completely classified, and a full list can be found in [17]. For any such group G, the category $\mathcal{C}_G$ of groups which can be embedded in G and injective homomorphisms between them, together with the object G, satisfies the hypotheses of Theorem 5.1. We thus have an equivalence of toposes $\mathbf{Sh}({\mathcal{C}_G}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(G))$, where $Aut(G)$ is the discrete automorphism group of G in $\mathcal{C}_G$.

5.2. *Decidable objects and infinite sets*

Let $\mathbb{S}$ be the theory of decidable objects, that is the theory over a signature consisting of a unique sort and a binary relation symbol D with no axioms except for the two coherent sequents asserting that D is a (provable) complement to the equality relation, i.e. $(\top \vdash_{x,x} D(x, y) \vee (x = y))$ and $(D(x, y) \wedge x = y \vdash_{x,y} \bot)$. Notice that $\mathbb{S}$ is the injectivization of the empty theory over the signature entirely consisting of one sort. Clearly, the models of $\mathbb{S}$ in **Set** are precisely the sets, while the $\mathbb{S}$-model homomorphisms are the injective functions between them. It is easy to prove that $\mathbb{S}$ is of presheaf type (cf. the remarks above on theories of presheaf type), and that its category of finitely presentable models can be identified with the category $\mathbf{I}$ of finite sets and injections. This category satisfies both the amalgamation and joint embedding properties, and the set $\mathbb{N}$ of natural numbers is clearly a $\mathbf{I}$-ultrahomogeneous model (in fact, any *infinite* set is equally $\mathbf{I}$-ultrahomogeneous). Thus Theorem 3.5 yields an equivalence $\mathbf{Sh}(\mathbf{I}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(\mathbb{N}))$. It is easy to see that every morphism in I is a strict monomorphism, from which it follows that the given equivalence restricts to an equivalence of $\mathbf{I}^{\mathrm{op}}$ with a full subcategory of $\mathbf{Cont}_t(Aut(\mathbb{N}))$. This in particular implies the (easy) Galois-type property that for any two subsets S and T of $\mathbb{N}$, if all the automorphisms of $\mathbb{N}$ which fix T fix S then $S \subseteq T$.

5.3. *Atomless Boolean algebras*

Let **Bool** be (a skeleton of) the category of Boolean algebras and homomorphisms between them, that is the category of set-based models of the algebraic theory $\mathbb{T}$ of

Boolean algebras. This theory is clearly of presheaf type, and its finitely presentable models are precisely the finite Boolean algebras. The injectivization of $\mathbb{T}$ is also of presheaf type (cf. the remarks above on theories of presheaf type); this implies that its category of models, which coincides with the category $\mathbf{Bool}_i$ of Boolean algebras and embeddings between them, is equivalent to the ind-completion of its full subcategory $\mathbf{Bool}_i^f$ on the finite Boolean algebras. By using Stone duality between the category of finite Boolean algebras and embeddings between them and the category of finite sets and surjections, it is immediate to prove that the category $\mathbf{Bool}_i^f$ has the property that every arrow of it is a strict monomorphism. From Fraïssé's construction in Model Theory, we know that the unique countable atomless Boolean algebra B is a $\mathbf{Bool}_i^f$-universal and $\mathbf{Bool}_i^f$-ultrahomogeneous object in $\mathbf{Bool}_i$. We thus obtain a concrete Galois theory for finite Boolean algebras.

5.4. Dense linear orders without endpoints

Let **LOrd** be (a skeleton of) the category of linear orders and order-preserving injections between them. It is easy to see that **LOrd** is the ind-completion of the full subcategory $\mathbf{LOrd}^f$ of it consisting of the finite linear orders. It is immediate to verify that every arrow of $\mathbf{LOrd}^f$ is a strict monomorphism, and that the set $\mathbb{Q}$ of rational numbers, endowed with its usual ordering, is a $\mathbf{LOrd}^f$-universal and $\mathbf{LOrd}^f$-ultrahomogeneous object in **LOrd**. We thus have a concrete Galois theory for finite linear orders.

5.5. Universal locally finite groups

Let $\mathcal{C}$ be (a skeleton of) the category of finite groups and injective homomorphisms between them. The ind-completion of $\mathcal{C}$ can be identified with the category of $\mathcal{L}$ locally finite groups (i.e., the groups such that any finitely generated subgroup of them is finite) and injective homomorphisms between them.

As remarked at p. 330 of [21], the fact that $\mathcal{C}$ satisfies the amalgamation property (and hence also the joint embedding property) can be proved using the permutation products of B.H. Neumann (see section 3 of [28]). Also, it is clear that $\mathcal{C}$ satisfies all the other hypotheses of Fraïssé's construction (cf. Theorem 7.1.2 in [21]). The Fraïssé's of $\mathcal{C}$ is known as Philip Hall's universal locally finite group (cf. [20]); this group is countable, simple, and any two isomorphic finite subgroups of it are conjugate.

A locally finite group G is said to be *universal* if (a) every finite group is embeddable in G and (b) every isomorphism between finite subgroups of G can be extended to some inner automorphism of G. It follows at once that G, regarded as an object of $\mathcal{L}$, is $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous. Universal locally finite groups are known to exist in all infinite cardinalities, and Fraïssé's theorem implies that there is exactly one countable universal locally finite group (up to isomorphism), namely Philip Hall's group.

For any universal locally finite group G, Theorem 3.5 thus provides an equivalence of toposes $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(G))$, where $Aut(G)$ is endowed with the topology of pointwise convergence. Such an equivalence always yields a concrete Galois theory for the category $\mathcal{C}$ (that is, $\mathcal{C}^{\text{op}}$ embeds as a full subcategory of the topos $\mathbf{Cont}(Aut(G))$), as any arrow of $\mathcal{C}$ is a strict monomorphism (cf. Exercise 7H(a) in [1]). These Galois theories are non-standard; in other words, there are 'imaginaries' for the theory of finite groups. To prove this, we use the criterion for proving the atomic completeness of a category provided by Theorem 4.17. Clearly, the category $\mathcal{C}$ has equalizers and arbitrary intersections of subobjects; we thus have to find a pair of arrows $h, k : c \to e$ with equalizer $m : d \to c$ and a pair of arrows $k, n : c \to e'$ with $l \circ m = n \circ m$ and such that for any arrow $s : e' \to e''$, $(s \circ l, s \circ n)$ does not belong to the equivalence relation $R^{e''}_{h,k}$ on $Hom_{\mathcal{C}}(c, e'')$ generated by the relation consisting of the pairs of the form $(t \circ h, t \circ k)$ for an arrow $t : e \to e''$. Let us take $c = e$ to be the additive group $\mathbb{Z}/15\mathbb{Z}$, h to be the identity on c and k to be the function $[x] \to [2x]$; clearly, these maps are arrows in $\mathcal{C}$, and the equalizer m of h and k is the trivial group. Take e' to be the product group $\mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$, l to be the arrow $[x] \to ([x], [3x])$ and n to be the arrow $[y] \to ([5y], [y])$; obviously, $l \circ m = n \circ m$. Now, for any $a, b \in Hom_{\mathcal{C}}(c, e'')$, $(a, b) \in R^{e''}_{h,k}$ clearly implies that for any element z of c there exist natural numbers p and q such that $2^p a(z) = 2^q b(z)$. From this it follows that for any arrow $s : e' \to e''$, if $(s \circ l, s \circ n)$ belongs to $R^{e''}_{h,k}$ then there exist natural numbers p and q such that $2^p(s \circ l)([1]) = 2^q(s \circ n)([1])$. But $2^p(s \circ l)([1]) = s(2^p l([1]))$ and $2^q(s \circ n)([1]) = s(2^q n([1]))$, whence, as s is injective, $2^p l([1]) = 2^q n([1])$, which is not true since by definition of l and n we have $2^p l([1]) = ([2^p], [2^p \cdot 3])$ and $2^q l([1]) = ([2^q \cdot 5], [2^q])$.

5.6. The random graph

Let $\mathcal{C}$ be (a skeleton of) the category of finite (irreflexive) graphs and injective homomorphisms between them. It is easy to verify that $\mathcal{C}$ satisfies the amalgamation and joint embedding properties, and that its ind-completion can be identified with the category of (irreflexive) graphs and injective homomorphisms between them. Also, one can easily prove that every arrow in $\mathcal{C}$ is a strict monomorphism. The *random graph* (also known as the Rado graph) is a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous; indeed, it can be identified with the Fraïssé's limit of the class of finite graphs (cf. Theorems 7.1.2 and 7.4.4 in [21]). We thus have a concrete Galois theory for finite graphs.

5.7. Infinite Galois theory

Let $F \subset L$ be a Galois extension (not necessarily finite-dimensional), and let $\mathcal{L}^L_F$ be the category of finite intermediate extensions and field homomorphisms between them. The ind-completion Ind-$\mathcal{L}^L_F$ of $\mathcal{L}^L_F$ can be identified with the category of intermediate (separable) extensions of F, and L is a $\mathcal{L}^L_F$-universal and $\mathcal{L}^L_F$-ultrahomogeneous object in Ind-$\mathcal{L}^L_F$. The fundamental theorem of classical Galois theory ensures that all the arrows in $\mathcal{L}^L_F$ are strict monomorphisms and that the category $\mathcal{L}^L_F$ is atomically complete (in fact, the theorem is equivalent to the conjunction of these two assertions). Notice that if L is finite-dimensional then the resulting Galois theory is discrete. Of course, infinite Galois theory also falls into the framework of Grothendieck's Galois theory (see below).

5.8. Grothendieck's Galois theory

Let $\mathcal{C}$ be a Galois category (in the sense of [19]), with fibre functor $F : \mathcal{C} \to \mathbf{Set}$, and let $\mathcal{C}_t$ be the full subcategory of $\mathcal{C}$ formed by the objects c of $\mathcal{C}$ such that $F(c)$ is non-empty and the action of $Aut(F)$ on $F(c)$ is transitive (equivalently, by the atomic objects of $\mathcal{C}$, i.e. the objects of $\mathcal{C}$ which are not isomorphic to 0 and which do not admit any proper subobjects). Grothendieck's theory provides an equivalence $\mathcal{C} \simeq \mathbf{Cont}_f(Aut(F))$, where $\mathbf{Cont}_f(Aut(F))$ is the category of continuous actions of the profinite group $Aut(F)$ on a finite set. The open subgroups of $Aut(F)$ are exactly those of the form $U_{c,x} := \{\alpha : F \cong F \mid \alpha_c(x) = x\}$ for $c \in \mathcal{C}$ and $x \in F(c)$; from this it follows in particular that every transitive continuous action of $Aut(F)$ lies in $\mathbf{Cont}_f(Aut(F))$ (that is, its underlying set is finite).

The equivalence $\mathcal{C} \simeq \mathbf{Cont}_f(Aut(F))$ thus restricts to an equivalence $\mathcal{C}_t \simeq \mathbf{Cont}_t(Aut(F))$, where $\mathbf{Cont}_t(Aut(F))$ is the category of continuous transitive actions of $Aut(F)$ over discrete sets. On the other hand, the equivalence $\mathcal{C} \simeq \mathbf{Cont}_f(Aut(F))$ can be extended to an equivalence of toposes by equipping each of the two categories with the coherent topology. By the Comparison Lemma, the category of sheaves on $\mathbf{Cont}_f(Aut(F))$ with respect to the atomic topology is equivalent to the topos $\mathbf{Cont}(Aut(F))$ of continuous actions of $Aut(F)$; indeed, as every continuous action can be written as a coproduct of transitive ones, and every transitive action lies in $\mathbf{Cont}_f(Aut(F))$, the subcategory $\mathbf{Cont}_f(Aut(F))$ of $\mathbf{Cont}(Aut(F))$ is dense with respect to the canonical topology on $\mathbf{Cont}(Aut(F))$, and the induced Grothendieck topology on $\mathbf{Cont}_f(Aut(F))$ coincides with the coherent one. Thus we have an equivalence of toposes

$$\mathbf{Sh}(\mathcal{C}, J_{coh}) \simeq \mathbf{Cont}(Aut(F)),$$

where J_{coh} is the coherent topology on $\mathcal{C}$.

Another application of the Comparison Lemma yields an equivalence

$$\mathbf{Sh}(\mathcal{C}, J_{coh}) \simeq \mathbf{Sh}(\mathcal{C}_t, J_{at}),$$

where J_{at} is the atomic topology on $\mathcal{C}_t$. We thus have an equivalence

$$\mathbf{Sh}(\mathcal{C}_t, J_{at}) \simeq \mathbf{Cont}(Aut(F)).$$

These equivalences show that F corresponds to a ${\mathcal{C}_t}^{\mathrm{op}}$-universal and ${\mathcal{C}_t}^{\mathrm{op}}$-ultrahomogeneous object of Ind-${\mathcal{C}_t}^{\mathrm{op}}$, that all the arrows of ${\mathcal{C}_t}^{\mathrm{op}}$ are strict monomorphisms and that the category ${\mathcal{C}_t}^{\mathrm{op}}$ is atomically complete. In particular, the canonical embedding of $\mathcal{C}_t$ into the topos $\mathbf{Sh}(\mathcal{C}_t, J_{at}) \simeq \mathbf{Sh}(\mathcal{C}, J_{coh})$ can be identified with the free small coproduct-completion of $\mathcal{C}_t$ (cf. Proposition 4.22), while the canonical embedding $\mathcal{C}_t \hookrightarrow \mathcal{C}$ realizes $\mathcal{C}$ as the free finite-coproduct completion of the category $\mathcal{C}_t$.

Notice that, as $Aut(F)$ is a profinite group, the topos $\mathbf{Cont}(Aut(F))$ is coherent (see section D3.4 [23]). In fact, the category $\mathcal{C}$ is a pretopos, as it is equivalent to the full subcategory of $\mathbf{Cont}(Aut(F))$ on its coherent objects; indeed, from the proof of Lemma D3.4.2 in [23] we know that the coherent objects of the topos $\mathbf{Cont}(Aut(F))$ are exactly the compact objects, and it is clear that an object of $\mathbf{Cont}(Aut(F))$ is compact if and only if it lies in $\mathbf{Cont}_f(Aut(F))$.

The discussion above shows in particular that the Galois categories can be characterized as the pretoposes $\mathcal{D}$ in which every object can be written as a coproduct of atomic objects and with the property that the opposite $\mathcal{D}_a$ of their subcategory of atomic objects satisfies AP and JEP and there is a $\mathcal{D}_a$-universal and $\mathcal{D}_a$-ultrahomogeneous object in Ind-$\mathcal{D}_a$. Indeed, we observed above that any Galois category has this property, while the converse follows from the fact that if a category $\mathcal{D}$ satisfies this condition then, denoting by u the $\mathcal{D}_a$-universal and $\mathcal{D}_a$-ultrahomogeneous object in Ind-$\mathcal{D}_a$, we have, by the Comparison Lemma and Theorem 3.5, equivalences

$$\mathbf{Sh}(\mathcal{D}, J_{coh}) \simeq \mathbf{Sh}({\mathcal{D}_a}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut_{\mathcal{D}_a}(u)),$$

and hence $\mathcal{D}$ is equivalent to $\mathbf{Cont}_f(Aut_{\mathcal{D}_a}(u))$ (recall that any pretopos can be recovered, up to equivalence, as the category of coherent objects of the topos of coherent sheaves on it).

The infinitary version of Grothendieck's Galois theory obtained in [29] admits a similar interpretation in the context of our framework; Noohi's Galois categories $\mathcal{C}$ are themselves toposes which can be represented as categories of atomic sheaves on the subcategory $\mathcal{C}_t$ of atomic objects of $\mathcal{C}$.

6. The four ways to topological Galois representations

Our abstract approach to Galois theory is based on the intuition that the generating 'kernel' of a Galois theory lies at the topos-theoretic level; more precisely, it is expressed by an equivalence between a topos of sheaves on an atomic site $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at})$ and a topos $\mathbf{Cont}(G)$ of continuous actions of a topological group G. Starting from such an equivalence

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(G),$$

one can establish relationships between the category $\mathcal{C}$ and the group G by considering appropriate topos-theoretic invariants in terms of the two different representations of the classifying topos of the given 'Galois-type' theory; in other words, the given topos acts as a 'bridge' (in the sense of [11]) for transferring information between the two sides.

In particular, it is natural to investigate under which conditions the given equivalence restricts to a categorical equivalence at the level of sites, leading to a 'concrete'

Galois theory (we did this in section 4). The fact that every concrete Galois theory can be obtained as a restriction of a topos-theoretic 'Galois-type' equivalence and that, on the other hand, there are many equivalences of the latter kind which do not specialize to equivalences of sites (but which are nonetheless relevant for other purposes, notably including the computation of cohomological invariants), shows that this 'top-down', topos-theoretic viewpoint carries with itself a higher generality and a greater technical flexibility with respect to the categorical, site-level, analysis.

Another illustration of the naturality of our topos-theoretic viewpoint is provided by the fact that, as a given topos can admit several different representations, one can arrive at establishing such Galois-type topos-theoretic equivalences in a variety of different ways. We shall briefly discuss these different ways below, organizing them in four main groups.

6.1. Representation theory of Grothendieck toposes

The representation theory of Grothendieck toposes provides natural ways for generating Galois-type equivalences of toposes. For instance, as we saw in section 5.1, discrete Galois theories can all be obtained as instances of Grothendieck's Comparison Lemma. Another natural source of Galois-type equivalences of the above kind is provided by the representation theory of atomic connected toposes in terms of localic groups established by Dubuc in [16]; indeed, these representations yield, in the case of spatial localic groups, topological Galois representations fitting into our framework.

6.2. Ultrahomogeneous structures

Theorems 3.5 and 4.8 establish a central connection between the theory of ultrahomogeneous structures and the subject of Galois representations. Ultrahomogeneous structures naturally arise in a great variety of different mathematical contexts (cf. section 5 for a list of notable examples). Their existence can be proved either directly through an explicit construction or through abstract logical arguments. A general method for building countable ultrahomogeneous structures is provided by Fraïssé's construction in Model Theory (cf. Chapter 7 of [21]), while the categorical generalization established in [6] allows to construct ultrahomogeneous structures of arbitrary cardinality.

6.3. Special models

Recall from section 3 that a model M of an atomic complete theory $\mathbb{T}$ is said to be *special* if every $\mathbb{T}$-complete formula $\phi(\vec{x})$ is realized in M and for any tuples $\vec{a}$ and $\vec{b}$ of elements of M which satisfy the same $\mathbb{T}$-complete formulae there is an automorphism $f : M \to M$ of M which sends $\vec{a}$ to $\vec{b}$. Theorem 3.1 shows that, in presence of any special model for an atomic complete theory, we have a Galois-type representation for its classifying topos.

An interesting aspect of special models is that their existence can be proved in many contexts by using model-theoretic techniques. For instance, the unique (up to isomorphism) countable model of a countably categorical theory is always special. Also, it follows from the fact that every consistent finitary first-order theory has a ω-big model (cf. Chapter 10 of [21]) that every consistent finitary atomic complete theory has a special model. This implies that for any consistent (i.e., with at least a set-based model) finitary theory of presheaf type $\mathbb{T}$ satisfying the hypotheses of Theorem 3.7 there exists a f.p.$\mathbb{T}$-mod($\mathbf{Set}$)-universal and f.p.$\mathbb{T}$-mod($\mathbf{Set}$)-ultrahomogeneous model of $\mathbb{T}$.

Notice also that, since the concept of special model for an atomic complete theory is expressible through a topos-theoretic invariant (cf. Remark 3.2(c)), one can obtain special models for a theory $\mathbb{T}$ starting from special models of any theory $\mathbb{T}'$ which is Morita-equivalent to it.

Another result from Chapter 10 of [21] is that any consistent finitary first-order theory has a λ-big model for any infinite cardinal λ; in particular, it has a model M such that every model of $\mathbb{T}$ of cardinality less than λ can be elementarily embedded in M and any isomorphism between any two such models can be extended to an automorphism of M. Let $\mathbb{T}\text{-mod}_e(\mathbf{Set})$ be the category of $\mathbb{T}$-models in $\mathbf{Set}$ and elementary embeddings between them. In order to obtain a Galois-type equivalence from these data by applying Theorem 4.3, we would need the embedding $\mathbb{T}\text{-mod}_\lambda(\mathbf{Set}) \hookrightarrow \mathbb{T}\text{-mod}_e(\mathbf{Set})$, where $\mathbb{T}\text{-mod}_\lambda(\mathbf{Set})$ is the full subcategory of $\mathbb{T}\text{-mod}_e(\mathbf{Set})$ on the $\mathbb{T}$-models of cardinality less than λ, together with the object M of $\mathbb{T}\text{-mod}_e(\mathbf{Set})$, to satisfy the hypotheses of the theorem. Notice that if λ is chosen in such a way that all finitely generated $\mathbb{T}$-models have cardinality less than λ (note that the cardinality of a finitely generated model of $\mathbb{T}$ is always $< \omega + \mathit{card}(L)$, where $\mathit{card}(L)$ is the cardinality of the signature of $\mathbb{T}$, cf. Theorem 1.2.3 in [21]) and every $\mathbb{T}$-model can be written as a filtered colimit in $\mathbb{T}\text{-mod}_e(\mathbf{Set})$ of finitely generated models (notice that this condition is always satisfied if $\mathbb{T}$ is the theory of homogeneous models of a theory of presheaf type, as such a theory is Boolean being atomic and hence all the $\mathbb{T}$-model homomorphisms are elementary embeddings) then the embedding $\mathbb{T}\text{-mod}_{f.g.}(\mathbf{Set}) \hookrightarrow \mathbb{T}\text{-mod}_e(\mathbf{Set})$, where $\mathbb{T}\text{-mod}_{f.g.}(\mathbf{Set})$ is the full subcategory of $\mathbb{T}\text{-mod}_e(\mathbf{Set})$ on the finitely generated $\mathbb{T}$-models satisfies, together with the object M, the hypotheses of Theorem 4.3, provided that the category $\mathbb{T}\text{-mod}_{f.g.}(\mathbf{Set})$ satisfies the amalgamation and joint embedding properties and every arrow of it is a strict monomorphism.

6.4. Galois categories

As remarked in 5.8, to every Galois category we can naturally associate an equivalence of toposes which ‘materializes’ the relationship between it and the associated Galois group. In fact, the theory of Galois categories provides a significant number of examples of standard Galois theories, as the duals of the subcategories of atomic objects of a Galois category are atomically complete.

7. Conclusions and future directions

The theory developed in this paper provides simple means for constructing 'Galois-type' equivalences in a variety of different mathematical contexts. In fact, Galois-type theories (i.e., theories whose classifying topos can be represented as a topos of continuous actions of a topological group) are ubiquitous in Mathematics; indeed, they are maximal elements in the lattice of geometric theories over their signature, and every geometric theory over a countable signature whose Booleanization is consistent can be extended to a Galois-type theory (cf. [6,7,10] and section 1 above). A natural direction of future investigation thus consists in trying to generate new meaningful applications of our general machinery across different mathematical domains; for instance, one could study Connes' topos of cyclic sets ([13] and [14]), exploring the possibility of building an associated Galois theory, or the notion of random group, introduced by M. Gromov in [18], in relation to the purpose of building Galois-type theories for various classes of discrete groups. Another specific subject worth of detailed investigation is that of imaginaries in the theory of finite groups; indeed, as shown in this paper, they define an extension of the category of finite groups and embeddings between them which is categorically much better behaved.

On the theoretical side, it would be interesting to explore possible connections between the theory developed in this paper and the categorical Galois theory of Borceux and Janelidze [5], which is based on a rather different framework.

Finally, let us comment on the relationship between our work and the model-theoretic approach to Galois theory pioneered by B. Poizat in [30]. Poizat's main result is a Galois theory for complete first-order theories which eliminate imaginaries. It should be noted that the properties of completeness and elimination of imaginaries are syntactic properties of logical theories which it is generally hard to verify in practice. Also, Poizat's theory is only applicable to finitary theories; as a result, the automorphisms groups which arise in that context are all profinite. The level of generality of Poizat's theory does therefore not go beyond that of Grothendieck's theory of Galois categories. On the other hand, our categorical treatment and interpretation of completeness as an outcome of simple properties such as AP and JEP opens the way for an easy identification of Galois-type theories in a great variety of different mathematical contexts, as well as for a geometric investigation of model-theoretic imaginaries.

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