Algebraic geometry

a desperate attempt to avoid failure

1. Blitzkrieg z GA

Noetherowskość, wymiary

For any $A \subseteq K[\overline{X}]$ we define

$$V(A) = \{\overline{x} \in K^n \ : \ \forall\, F \in A \ F(\overline{x}) = 0\}.$$

Let $I, J \trianglelefteq K[\overline{X}]$. Then

- $A_0 \subseteq A_1 \implies V(A_1) \subseteq V(A_0)$
- $V(\bigcup A_i) = \bigcap V(A_i)$
- $V(I \cap J) = V(IJ) = V(I) \cup V(J)$
- $V(I + J) = V(I) \cap V(J)$

For any $A \subseteq K[\overline{X}]$ there is a finite $A_0 \subseteq A$ such that $V(A) = V(A_0)$ (by the Hilbert's basis theorem).

Definicja 1.1: Noetherian ...

A topological space X is called **Noetherian** if any descending chain of closed subsets of X stabilizes. Meanwhile, a ring is Noetherian if any ascending chain of ideals stabilizes.

Definicja 1.2: irreducible space

A space X is irreducible if it is not a non-trivial union of its two closed subseteq, i.e. for any $Y_1, Y_2 \subseteq X$ closed $X = Y_1 \cup Y_2$ then $X = Y_1$ or Y_2.

If X is a Noetherian space then

- $X = X_1 \cup ... \cup X_k$ for some $X_1, ..., X_k \subseteq X$ irreducible such that $X_i \not\subseteq X_j$ for $i \neq j$
- this sequence is unique up to permutation of indices

An affinie algebraic set $V(A)$ is **affine variety** if it is irreducble as a topological space with the Zariski topology.

Definicja 1.3: dimension

$\dim(X) = n$ if there is a strictly decreasing sequence of irreducible closed subsets of X such that

$$X_n \subsetneq X_{n-1} \subsetneq \dots \subsetneq X_0 \subseteq X$$

For $V \subseteq \mathbb{A}^n$ we define the **affine coordinate ring of V** as

$$K[v] := \{f \in \mathrm{Func}(V, K) \ : \ \exists F \in K[\overline{X}] \ : F|_V = f\}$$

$K(V)$ is the field of fractions of $K[V]$, called **the field of rational functions on V**.

The **ideal of V** is defined as

$$\ker\left(K[\overline{X}] \ni F \mapsto F|_V \in K[V]\right)$$

which is the same as

$$I(V) = \{F \in K[\overline{X}] \ : \ \forall\, \overline{x} \in V \ F(\overline{x}) = 0\}.$$

The **Zariski closure** of V_0 is the set $V(I(V_0))$.

Twierdzenie 1.4: Hillbert's Nullstellensatz

weak $I \trianglelefteq K[\overline{X}] \wedge; I \neq K[\overline{X}] \implies V(I) \neq \emptyset$
strong/regular $I \trianglelefteq K[\overline{X}] \implies I(V(I)) = \sqrt{I}$

If $V \subseteq \mathbb{A}^n$ is Zariski closed, then the following are equivalent

1. V is irreducible
2. $I(V)$ is prime
3. $\exists P \trianglelefteq K[\overline{X}]$ prime such that $V = V(P)$!!the field K needs to be algebraically closed!!
4. $K[V]$ is a domain

Twierdzenie 1.5

If $F \in K[\overline{X}]$ is irreducible, then $V(F)$ is an affine variety.

Let $V \subseteq \mathbb{A}^n$ be an affine algebraic set. Then there is a bijection between $\genfrac{}{}{0pt}{}{\text{radical}}{\text{prime}}$ ideals of $K[V]$ and the set of Zariski $\genfrac{}{}{0pt}{}{\text{closed}}{\text{irreducible closed}}$ subsets of V.

Definicja 1.6: Krull dimension

For a ring R we define its Krull dimension $\dim(R)$ as the supremum $k \in \mathbb{N}$ such that there is a strictly increasing (or decreasing) sequence of prime ideals

$$P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_k$$

of R.

$$\mathbf{dim}(V) = \mathbf{dim}(K[V])$$

Twierdzenie 1.7

Let R be a finitely generated K-algebra which is a domain. Then

$$\dim(R) = \mathrm{trdeg}_K(R_0)$$

the dimension of R is equal to the transcendental degree of the field of fractions of R over K.

If $R = K[V]$ then $R_0 = K(V)$ and the above statement holds.

Kategoryje

Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine algebraic sets. Then $\varphi : V \to W$ is a **morphism** if there are $f_1, \ldots, f_m \in K[V]$ such that

$$\varphi(v) = (f_1(v), \ldots, f_m(v)).$$

For such a morphism we define

$$\varphi^* : K[W] \to K[V]$$
$$\varphi^*(f) = f \circ \varphi.$$

1. The mapping $\varphi \mapsto \varphi^*$ is an isomorphism between the set of morphisms $V \to W$ and the set of morphisms $K[W]$ to $K[V]$.
2. Any finitely generated K-algebra R which is reduced (no nilpotent elements) is isomorphic over V to $K[V]$ for some affine algebraic V.

$$\boldsymbol{V \cong W \iff K[V] \cong_K K[W]}$$

For $f \in K(V)$, the **domain** of f, $\operatorname{dom} f$, is the set of points $v \in V$ such that there are $f_1, f_2 \in K[V]$, $f = \frac{f_1}{f_2}$ and $f_2(v) \neq 0$.

Definicja 1.8

Let $f \in K(V)$ and $v \in V$.

1. f is regular at v if $v \in \operatorname{dom} f$
2. $\mathcal{O}_{V,v} := \{f \in K(V) \ : \ v \in \operatorname{dom} f\}$
3. f is regular if f is regular at each $v \in V$, i.e. $\operatorname{dom} f = V$.

Fakt 1.9

For any $v \in V$

$$\mathcal{O}_{V,v} = K[V]_{I_V(v)} = \{\frac{a}{b} \ : \ a \in K[V], b \in K[V] - I_V(v)\}$$

Denote by

$$\mathfrak{m}_{V,v} \trianglelefteq \mathcal{O}_{V,v}$$

the maximal ideal of $\mathcal{O}_{V,v}$.

f is regular $\iff$ $f \in K[V]$

Lemat 1.10

For a morphism $\varphi : V \to W$ its dual φ^* is a monomorphism $\iff$ φ is **dominant**, i.e. $\varphi(V)$ is Zariski dense in W (is an epimorphism in its category).

A function $\varphi : U \subseteq V \to W$ is a **rational function** between V and W if there are $f_1, ..., f_m \in K(V)$ such that

$$U = \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_m$$

and for all $v \in U$ there is $\varphi(v) = (f_1(v), ..., f_m(v))$.

$\varphi : V \dashrightarrow W$ denotes a **dominant rational function** from V to W.

For any field extension $K \subseteq L$ such that L is finitely generated over K there is an affine variety V such that $L \cong_K K(V)$.

The category of affine varieties and dominant rational maps is *entiequivalent* or *dually equivalent* to the category of finitely generated field extensions of K.

Smooooth *like the fur of a newborn goat*

Let R be a ring. The map $\partial R \to R$ is called a **derivation** on R if for all $a, b \in R$

$$\partial(a+b) = \partial(a) + \partial(b)$$

$$\partial(ab) = \partial(a)b + a\partial(b).$$

The **Jacobian matrix** of $\overline{F} = (F_1, \ldots, F_m)$, $F_i \in K[\overline{X}]$ is

$$J_{\overline{F}} := \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial X_1} & \cdots & \frac{\partial F_m}{\partial X_n} \end{pmatrix}$$

Fakt 1.11

If $(G_1, \ldots, G_k) = I = (F_1, \ldots, F_m) \trianglelefteq K[\overline{X}]$ and $v \in V(I)$, then

$$\operatorname{rank}(J_{\overline{G}}(v)) = \operatorname{rank}(J_{\overline{F}}(v)).$$

Definicja 1.12: non-singular variety

Let $V \subseteq \mathbb{A}^n$ and $F_1, \ldots, F_m \in I(V) = (F_1, \ldots, F_m)$. We say that $a \in V$ is a **non-singular** or smooth point of V if

$$\operatorname{rank}(J_{\overline{F}}(a)) = n - \dim(V).$$

We say that V is a non-singular variety or a smooth variety if V is irreducible and all points of V are smooth.

$F \in K[X, Y]$ and $V = V(F) \subseteq \mathbb{A}^2$

1. $F \notin K \implies |V| = \infty$
2. $|V(F, \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y})| < \infty \implies \sqrt{(F)} = (F) \ \wedge \ I(V) = (F)$
3. $V(F, \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}) = \emptyset \implies V$ is smooth

Lemat 1.13

A point $a \in V$ is smooth $\iff \dim_K(I_V(a)/I_V(a)^2) = \dim(V)$.
If V is an affine variety and $a \in V$, then we have

$$I_V(a)/I_V(a)^2 \cong_K \mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^2$$

A Noetherian local ring $(R, \mathfrak{m})$ is **regular** if $\dim(R) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

The K-vector space $\mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^2$ is the **cotangent space** of V at a. The dual space is the *tangent space*.

DVR

A local ring $(R, \mathfrak{m})$ is a **discrete valuation ring** if

1. R is Noetherian domain
2. R is not a field
3. $\mathfrak{m}$ is principal (generated by a single element)

In any Noetherian domain R and any $I \triangleleft R$ we have

$$\bigcup_{n\geq 1} I^n = \{0\}.$$

Any DVR is PID (the generator of $\mathfrak{m}$ is the uniformizing parameter)

Let R be a UFD, $r \in R$ irreducible and R_0 be the field of fractions of R. Define

$$v_r : R_0^* \to \mathbb{Z}$$

$$v_r(r^n \frac{a}{b}) = n, \quad r \not| a, \ r \not| b.$$

We call v_r the **r-addic valuation** on R_0.

Fakt 1.14

R, r, R_0 and v_r as above. Then for all $\alpha, \beta \in R_0^*$

1. $\alpha + \beta \in R_0^* \implies v_r(\alpha+\beta) \geq \min\{v_r(\alpha), v_r(\beta)\}$
2. $v_r(\alpha\beta) = v_r(\alpha) + v_r(\beta)$
3. $v_r(R_0^*) = \mathbb{Z}$

For any irreducible $r, s \in R$ if $(r) = (s)$ then $v_r = v_s$.

Definicja 1.15: discrete valuation

Let L be a field. Any function $v : L^* \to \mathbb{Z}$ satisfying 1-3 from the fact above is called a **(discrete) valuation** on L. For any valuation $v : L^* \to \mathbb{Z}$

- $\mathcal{O}_v := \{\alpha \in L^* \ : \ v(\alpha) \geq 0\} \cup \{0\}$ -> *valuation ring* of v
- $\mathfrak{m}_v := \{\alpha \in L^* \ : \ v(\alpha) > 0\} \cup \{0\}$ -> *valuation ideal* of v

For a valuation $v : L^* \to \mathbb{Z}$, $(\mathcal{O}_v, \mathfrak{m}_n)$ is a DVR.

Twierdzenie 1.16

Let C be an affine curve and $a \in C$. Then a is smooth $\iff$ $(\mathcal{O}_{C,a}, \mathfrak{m}_{C,a})$ is a DVR.

Definicja 1.17

Let C be an affine curve and $a \in C$ be a smooth point

1. a uniformizing parameter $f \in \mathcal{O}_{C,a}$ is a **local parameter** for C at a
2. the unique valuation on $K(C)$ given by $(\mathcal{O}_{C,a}, \mathfrak{m}_{C,a})$ is denoted ord_a
3. for $f \in K(C) - \{0\}$ and $n \in \mathbb{N}_{>0}$
    - $\operatorname{ord}_a(f) = n \implies f$ has a zero at a of order n
    - $\operatorname{ord}_a(f) = -n \implies r$ has a pole at a of order n

$$\operatorname{ord}_a(f) = \dim_K(\mathcal{O}_{C,a}/f\mathcal{O}_{C,a})$$

$$\dim_K(K[X]/(F)) = \sum_{a \in V(F)} \operatorname{ord}_a(F)$$

Definicja 1.18: intersection number

The intersection number of F and G at $a = (x, y) \in \mathbb{A}^2$ is

$$I(a, F \cap G) := \dim_K(\mathcal{O}/(F, G)\mathcal{O}),$$

where

$$\mathcal{O} := K[X, Y]_{(X-x, Y-y)} = K[X, Y]_{I(a)} = \mathcal{O}_{\mathbb{A}^2, a}$$

For curves C_1, C_2 such that $F = I(C_1)$ and $G = I(C_2)$ we define $I(a, C_1 \cap C_2) := I(a, F \cap G)$.

$$I(a, F \cap G) > 0 \iff a \in V(F, G)$$

$$|V(F, G)| < \infty \implies I(a, F \cap G) < \infty$$

F irreducible and $a \in V(F)$ smooth then

$$I(a, F \cap G) = \operatorname{ord}_a(G|_{V(F)}).$$