# Fox knot colorings and Alexander invariants.

(Kolorowania Foxa i niezmienniki Alexandera)

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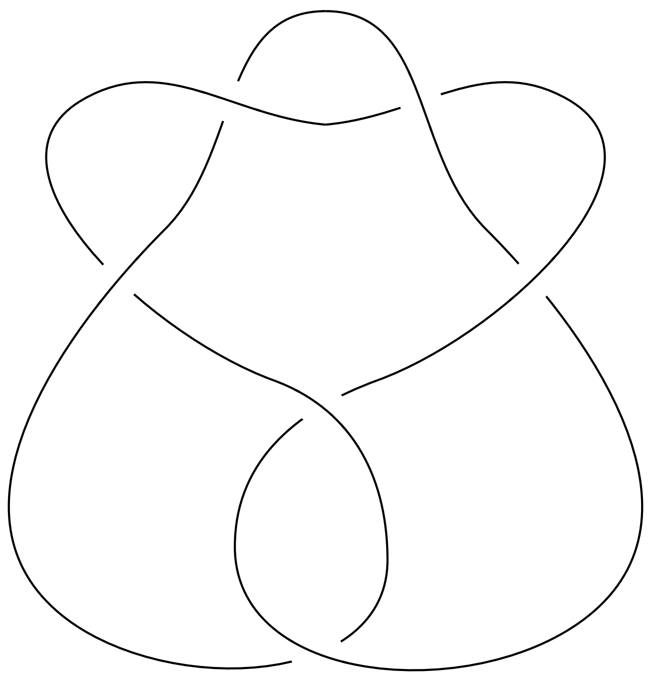


# Plan działania

1. Relacje na macierzach -> Reidemeister
   (a) propagation rule - funkcja $\phi$, potencjalnie dla uproszczenia będziemy pisać $\phi_+$ i $\phi_-$ na reguły kolorowania dwóch typów skrzyżowania
   (b) Diagram, s łuczków i x skrzyżowań - macierz która bardzo nie jest niezmiennikiem węzła, a zależy od diagramu.
   (c) Wprowadzamy relację na zorientowanych diagramach (chociaż w sumie chyba nie potrzebuję orientacji, ale na takich pracuję więc elo)
2. Smith normal form
3. Skein relations
4. moduł Alexandera $6_1$ i 946, czy są różne
5. rezolwenty
6. zmiana pidów

# 1 What is a knot coloring

Let $K$ be a knot and $D$ be its oriented diagram with $s$ segments and $x$ crossings. In such diagrams we can see two different crossing types as seen in fig. 1.

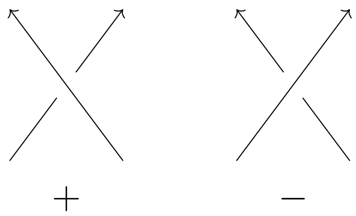



Figure 1: Two types of crossing in oriented diagram.

Take a commutative ring with unity $R$ and two $R$-modules $M$ and $N$. Take two arbitrary module homomorphisms $\phi_+ : M^3 \to N$ and $\phi_- : M^3 \to N$, one for each type of crossing.

**Definition 1.1 : diagram coloring.**
Let $x_1, ..., x_s \in M$ be labels of arcs in diagram $D$. We will say that $(x_1, ..., x_s) \in M^s$ is a **coloring** if for every crossing $\pm$ in $D$ consisting of arcs $u$, $i$, $o$ the following relation is satisfied

$$\phi_\pm(u, i, o) = 0.$$

Every crossing in the colored diagram $D$ of knot $K$ yields $x$ relations $\phi_\pm(u, i, o) = 0$ which we might treat as linear equations of form

$$\phi_+(u, i, o) = au + bi + co = 0,$$

$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o = 0,$$

where $u$, $i$ and $o$ are labels assigned to arcs entering some crossing and $a, b, c \in \mathrm{Hom}(M, N)$.

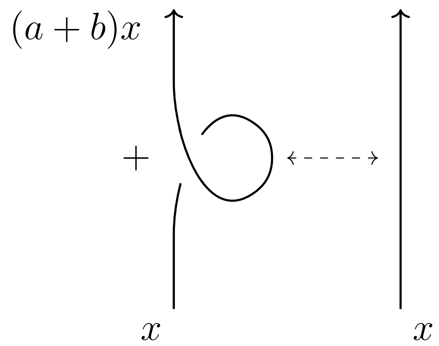
**Definition 1.2.**
Matrix $D\phi : M^s \to N^x$ of coefficients taken from relations $\phi_\pm(u, i, o)$ will be called a **color checking matrix**.

Notice that $(x_1, ..., x_s)$ is a coloring of the diagram $D$ if and only if it is an element of $\ker D\phi$. However, we can choose $\phi$ to have only a trivial kernel, then only one coloring is admissible - assigning a 0 to every arc of $D$. Thus, to obtain valuable information about the knot $K$ whose diagram is being colored, we must impose the following restrictions on $\phi$.

1. To allow *trivial colorings*, that is colorings in which every arc is assigned the same value it is necessary that
$$(\forall\, m \in M)\ \phi_{\pm}(m,m,m) = 0.$$
2. To simplify operations of color checking matrices, if
$$\phi_{+}(u,i,o) = au + bi + co$$
$$\phi_{-}(u,i,o) = \alpha u + \beta i + \gamma o,$$
then we take $c$ and $\gamma$ to be invertible. For the sake of simplicity, take $c = \gamma = -1$.
3. The two variations of orientation of the first Reidemeister move, pictured in fig. 2, put the following constrictions on $a$, $b$ and $\alpha$, $\beta$:
$$\begin{cases} a + b = 1 \\ \alpha + \beta = 1 \end{cases}$$
4. Lastly, from the second Reidemeister move, pictured in fig. 3, one can gather that
$$\begin{cases} a + b\alpha = 0 \\ b\beta = 1 \end{cases}$$
meaning that both $b$ and $\beta$ must be units.

It is worth mentioning that examining the second Reidemeister move (fig. 3) with $\phi_{\pm}$ changed to $2 \times 2$ matrices $A_{\pm}$, which take arcs entering a crossing as input and output the arcs leaving it, we can see that
$$A_{+}A_{-} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix} = Id.$$
This means that from homomorphism $\phi_{+}$ we are able to calculate $\phi_{-}$ and vice versa.

The color checking matrix is not a knot invariant, despite the restrictions laid on $\phi$. Changing the number of crossings in a diagram will obviously create a different matrix for the same knot. We will thus proceed to define an equivalence relation on the set of all color checking matrices of a knot $K$.



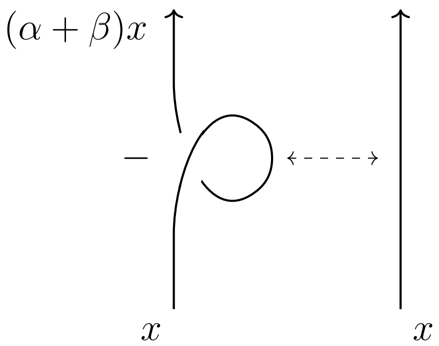




Figure 2: The two variations of the first Reidemeister move in oriented diagrams. They suggest that $(a+b)=1$ and $(\alpha+\beta)=1$.

# 2 Relation on color checking matrices

In order to ensure that all matrices that stem from the same knot are considered in one equivalence class we will look at how Reidemeister moves change the matrix.

In this section we will always assume that the diagram labeled as $D$ has $s$ segments and $x$ crossings. This means that always

$$D\phi : M^s \to N^x.$$

Furthermore, we will always put rows and columns corresponding to crossings and segments that are affected by the Reidemeister move as the last columns and rows of the matrix.

## R1

In oriented diagrams, there are four distinct Reidemeister moves [1], the first one having to account for two types of crossing. We start with **R1a**,

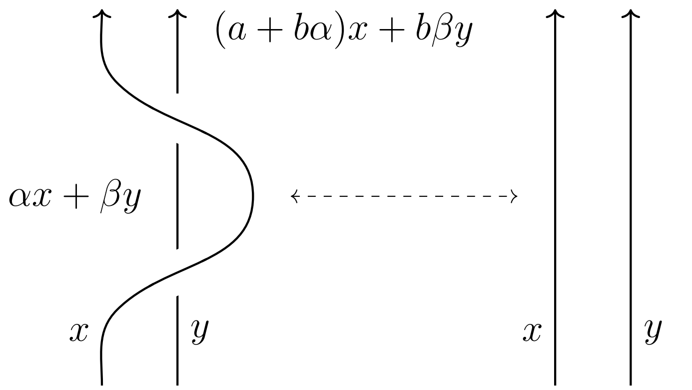
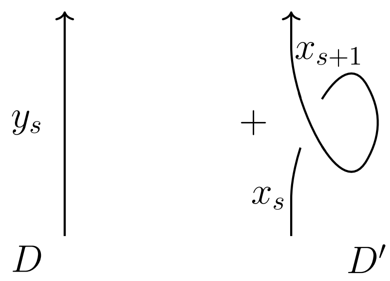



Figure 3: Second Reidemeister move. It suggests that $(a + b\alpha) = 0$ and $b\beta = 1$.

**R1a**




pictured in fig. 4. Consider the two matrices

$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+1} \to N^{x+1}.$$

Only two arcs change between the two, thus

$$D\phi \upharpoonright M^{s-1} = D'\phi \upharpoonright M^{s-1}.$$

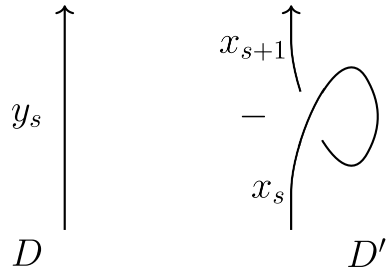
Furthermore, we want to assert that the two arcs $x_s$ and $x_{s+1}$, into which $y_s$ is split, are arranged in a + type crossing. Thus, for all $x_s, x_{s+1} \in M$ we require that

$$\pi_{x+1}[D'\phi(0, ..., x_s, x_{s+1})] = \phi_+(x_{s+1}, x_s, x_{s+1}),$$

where $\pi_{x+1}$ is projection onto the last coordinate. Additionally, we want the column that represented contribution of the twisted arc in diagram $D$ to be the sum of two new arcs in $D'\phi$, meaning that for every $y_s \in M$:

$$(D\phi(0, ..., y_s), 0) = D'\phi(0, ..., y_s, y_s).$$

### R1b




The second type of **R1** is seen in fig. 5. The relation for this move is almost the same as above. We start with two matrices

$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+1} \to N^{x+1}$$

that must agree on the arcs and crossings that are not changed between $D$ and $D'$:

$$D\phi \restriction M^{s-1} = D'\phi \restriction M^{s-1}.$$

The type of crossing into which an arc of $D$ is twisted in $D'$ is differented than in **R1a**, thus the second equality is slightly different: for all $x_s, x_{s+1} \in M$

$$\pi_{x+1}[D'\phi(0, ..., x_s, x_{s+1})] = \phi_-(x_s, x_s, x_{s+1}).$$

The last requirement is not changed from **R1a**, meaning that for all $y_s \in M$:

$$(D\phi(0, ..., y_s), 0) = D'\phi(0, ..., y_s, y_s).$$

### R2

For this Reidemeinster move we start with matrices

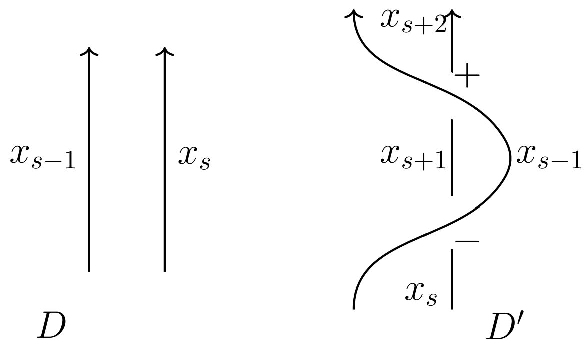
$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+2} \to N^{x+2}.$$

Beyond the two segments

$$D\phi \sim D'\phi$$

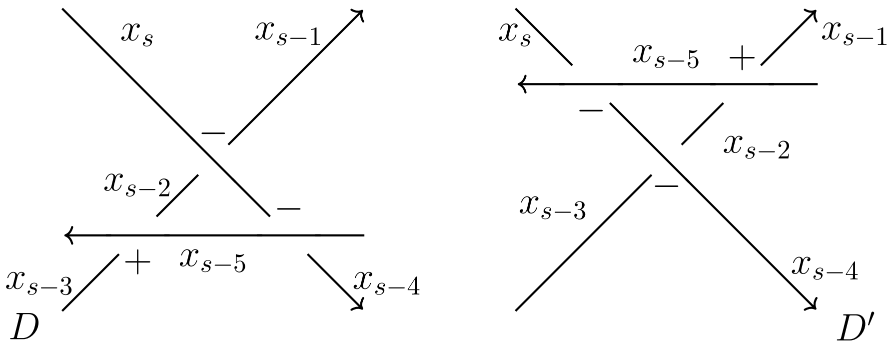
**R2**




if and only if

$$\begin{aligned} &D\phi \restriction M^{s-2} = D'\phi \restriction M^{s-2} \wedge \\ &\wedge (\forall x \in M) D'\phi(0, ..., x, y, y, y) = \\ &\quad = (D\phi(0, ..., x, y), \phi_-(x, y, y), \phi_+(x, y, y)) \end{aligned}$$

**R3**




$$D\phi \sim D'\phi$$

if and only if

$$\begin{aligned} &D\phi \restriction M^{s-5} = D'\phi \restriction M^{s-5} \wedge \\ &\wedge (\forall\ x, y, z \in M) \pi_{x-3}[D\phi(0, ..., x, y, z)] = \\ &\quad = \pi_{x-3}[D'\phi(0, ..., z, x, y, 0, 0)] \wedge \\ &\wedge D\phi(0, ...., z, y, 0, 0, 0, x) = D'\phi(0, ..., z, y, 0, 0, 0, x) \wedge \\ &\wedge D\phi(0, ..., z, 0, y, x, 0, 0) = D'\phi(0, ..., z, 0, 0, y, x, 0) \end{aligned}$$

**Theorem 2.1.**
Let $K$ be a knot and $D$ its oriented diagram. Define

$$K\phi := [D\phi]$$

to be the equivalence class of the matrix $D\phi$. Then, $K\phi$ is a knot invariant.

# 3 Smith normal form

The ring $R$ over which we consider modules $M$ and $N$ is not necessary a principal ideal domain. However, there are plenty of PID rings and one can take any unit of $R$ and send it to any unit of a PID ring $P$ to allow for $M$ and $N$ to be considered as $P$-modules. That way, we can consider a new type of equivalence relation on any color checking matrix $D\phi$.

**Definition 3.1.**
Take $A \in K\phi$ and consider it as a $s \times x$ matrix with terms in a $P$-module $M$. Then there exist a $s \times s$ matrix $S$ and $x \times x$ matrix $T$ such that $SAT$ is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every $i$ $a_i | a_{i+1}$. Such matrix $SAT$ is called **Smith normal form** of matrix $A$.

To obtain the most general invariants, we take $R$ to be $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials with integer coefficients. There are multitudes of PIDs $P$ with homomorphisms $\mathbb{Z}[t, t^{-1}] \to P$.

As was mentioned in the first section, $\overline{x} \in M^s$ is a coloring of a diagram $D$ if and only if $D\phi(\overline{x}) = 0$, that is $\overline{x} \in \ker D\phi$. The Smith normal form hints at the structure of matrix kernel - the columns filled with zeros will contributed a free factor $M$ to the kernel.

Take $(a)$ to be a prime ideal with its generator $a$ appearing in the Smith normal form of $D\phi$. Then we might consider the matrix over a new ring $R/(a)$, which is still a PID. After this change, the structure of the kernel has changed as now there are additional zero columns where $a$ and all its multiples stood.

**Definition 3.2 : reduced normal form of matrix.**
Take $A$ to be a matrix with coefficients in principal ideal domain $P$. Take $a_1, ..., a_k \in P$ to be elements of the Smith normal form of $A$ that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix $A$.

The following example justifies the utility of the reduced normal form of color checking matrices in distinguishing knots.

**Example 3.1.** Consider the knots $6_1$ with diagram as seen in fig. 7 and $9_{46}$ pictured in fig. 8, ring $R = \mathbb{Z}[t, t^{-1}]$, $M = R$ and

$$\begin{cases} \phi_+(u, i, o) = (1 - t)u + ti - o \\ \phi_-(u, i, o) = (1 - t^{-1})u + t^{-1}i - o. \end{cases}$$

The two rings have the same Alexander polynomial, $\Delta = -2t^{-2}+5t^{-1}-2$, and the same Alexander module $H^1(S^3 - K) = \mathbb{Z}[t, t^{-1}]/(\Delta)$.

For the knot $6_1$ we find the matrix $D\phi$ and after changing to the $PID$ ring $P = \mathbb{Q}[t, t^{-1}]$ we see that the Smith normal form is:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

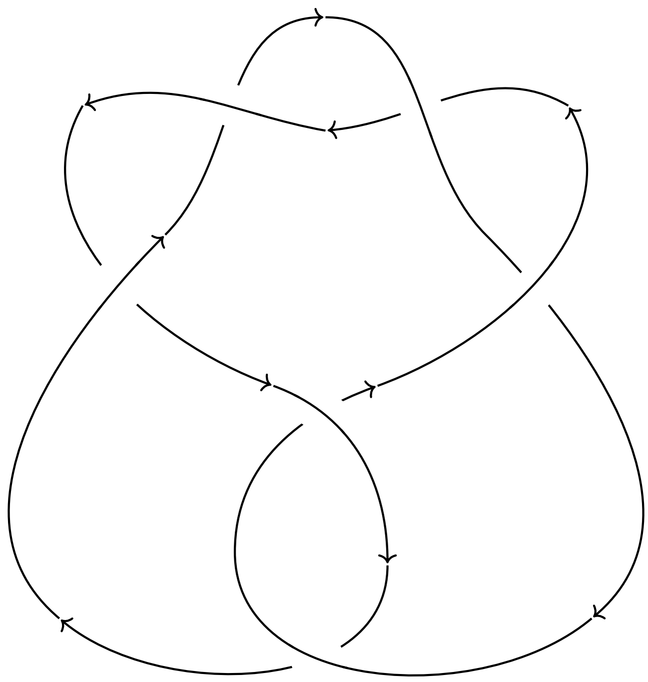


Figure 7: Diagram of knot $6_1$.

which after reduction is

$$A' = \begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

a $1 \times 1$ matrix with the only term being the Alexander polynomial of $6_1$.

Using diagram in fig. 8 of $9_{46}$ it can be calculated that the Smith normal form of $D\phi$ is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t - t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2} - 2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while reduced normal form of $D\phi$ is

$$B' = \begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

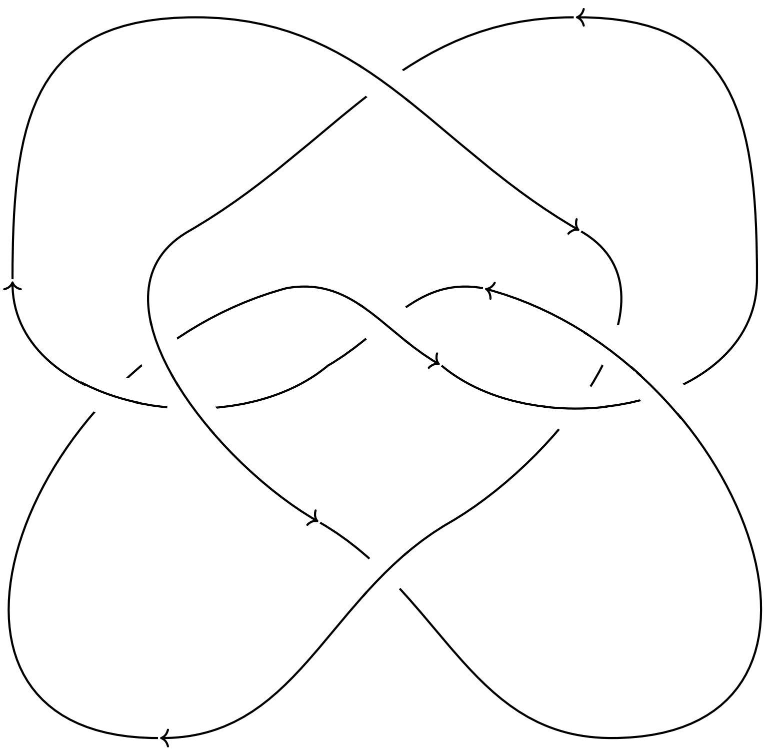


Figure 8: Diagram of knot $9_{46}$.

which is significantly different than the one for $6_1$. Observe also that the determinant of both matrices is equal to the Alexander polynomial of corresponding knots

$$\det(A') = -2 + 5t^{-1} - 2t^{-2}$$

$$\det(B') = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t + 2 + 2t^{-1} = -t(-2 + 5t^{-1} - 2t^{-2}).$$

**Theorem 3.1.**
The reduced normal form of color checking matrix does not depend on the choice of diagram $D$. Thus, it is well defined for $K\phi$ and is a knot invariant.

*Proof.* Take a knot $K$ and its diagram $D$ with $s$ segments and $x$ crossings. We will show that applying any Reidemeister move to this knot will not change the reduced normal form of its color checking matrix.

**R1**

**R2**

**R3**

□

# References


[1] Khaled Bataineh. New polynomial invariants of knotoids and the theory of polar knots. 2022.