Fox knot colorings and Alexander invariants.

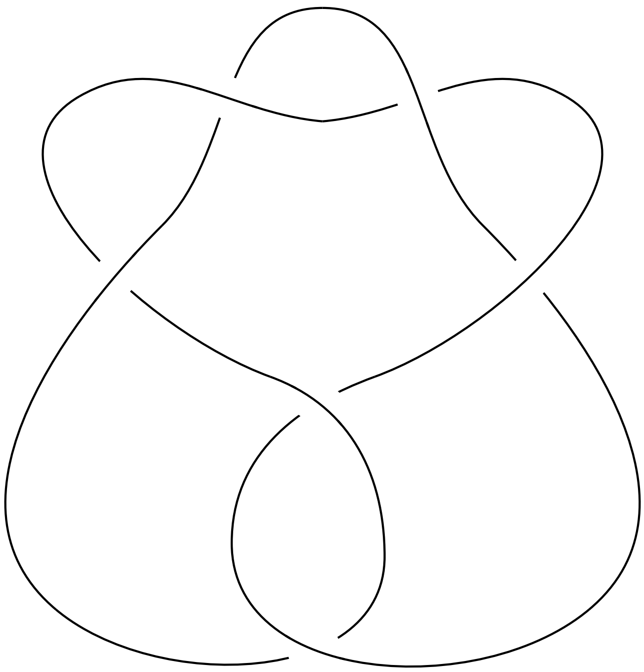
(Kolorowania Foxa i niezmienniki Alexandera)

Weronika Jakimowicz
330006

Julia Walczuk
332742

Praca licencjacka 2023-2024

Promotor: Tadeusz Januszkiewicz



Contents

1 Introduction

KTO CO ROBIŁ

In mathematical terms, a knot is a particular embedding $S^1 \hookrightarrow S^3$. A knot diagram is a projection $D : S^1 \twoheadrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [4].

S^1 is an orientable space thus we can choose an orientation for a knot being considered. Then a diagram D is oriented if it is a projection of an oriented S^1.

Intuitively, two knots K_1 and K_2 are equivalent if we can deform one into the other without cutting it and only manipulating it with our hands [2]. This translates to equivalence of diagrams, which is generated by a set of moves, called the **Reidemeister moves**. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D, at least 4 moves are required [3].

2 What is a knot coloring

Let K be a knot and D be its oriented diagram with s segments and x crossings. In such diagrams we can see two different crossing types as seen in fig. 1.

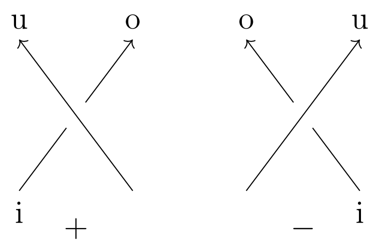


Figure 1: Two types of crossing in oriented diagram.

Take a commutative ring with unity R and an R-module M.

> **Definition 2.1 : coloring rule.**
> Take $\mathcal{C} \subseteq M^3$ to be a finitely generated submodule of M^3. We will call $\mathcal{C}$ a **coloring rule**. There are two submodules $\mathcal{C}_\pm \subseteq \mathcal{C}$, each corresponding to a type of crossing in diagram D.

We can now construct three homomorphisms

$$\phi : M^3 \to M/\mathcal{C} = N$$

$$\phi_\pm : M^3 \to M/\mathcal{C}_\pm = N_\pm.$$

We will call ϕ and $\mathcal{C}$ **coloring rule** interchangeably.

For each crossing x_j in diagram D we can construct a projection

$$\pi_{x_j} : M^s \twoheadrightarrow M^3$$

which restricts M^s to the three arcs that constitute x_j.

Definition 2.2 : diagram coloring.
A **coloring of diagram** D is any element $(m_1, ..., m_s) \in M^s$ that assigns elements of M to each arc. We will call this coloring **admissible** if for every crossing x_j of type $\pm$ we have

$$\pi_{x_j}(m_1, ..., m_s) \in \mathcal{C}_\pm \subseteq \mathcal{C}.$$

It will be beneficial to express admissibility of a coloring in terms of homomorphism ϕ.

Proposition 2.1.
A coloring $(m_1, ..., m_s) \in M^s$ is a admissible $\iff$ for each crossing x_j of type $\pm$

$$\phi_\pm(\pi_{x_j}(m_1, ..., m_s)) = 0.$$

Proof. Stems from the fact that $\mathcal{C}_\pm = \ker \phi_\pm$. □

Definition 2.3 : color checking matrix.
After assignings arcs to coordinates in M^s and crossings to coordinates in N^x it is possible to define a linear homomorphism $D\phi : M^s \to N^x$ as

$$D\phi(m_1, ..., m_s) = (\phi_\pm(\pi_{x_1}(m_1, ..., m_s)), \phi_\pm(\pi_{x_2}(m_1, ..., m_s)), ...).$$

Matrix that is created after choosing a basis for M^s and N^x will be called a **color checking matrix**.

Taking $\phi_\pm$ to be linear equations of form

$$\phi_+(u, i, o) = au + bi + co$$

$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o,$$

where u, i and o correspond to arcs as seen in fig. 1 and all the coefficients are linear homomorphisms $M \to N$, we know that all the entries for the color checking matrix will be linear combinations of a, b, c, α, β, γ. If M has n generators we chose to block the matrix $D\phi$ into $n \times n$ blocks.

Proposition 2.2.
Coloring $(m_1, ..., m_s) \in M^s$ is admissible $\iff$ $(m_1, ..., m_s) \in \ker D\phi$.

Proof. $\implies$

We know that every projection $\pi_{x_j}(m_1, ..., m_s)$ is in $\ker \phi_\pm$, depending on the type of x_j crossing. Thus, there is no projection π_{x_j} that is not being reduced by $\phi_\pm$.

$\impliedby$

□

We need to impose restrictions on the coloring rule. We want $\mathcal{C}$ to be two dimensional (have two generators). That way we have the following diagram

$$\begin{tikzcd} M^2 \arrow[r, hook] \arrow[rr, bend right, red, "\sim"'] & M^3 \arrow[r, two heads] & \mathcal{C} \end{tikzcd}$$

We can assume that M^2 corresponds to the 'up' and 'in' segments in a crossing (compare fig. 1), then we can define $\phi'_{\pm}$ to take u and i segments and return the out segment so that the labeling agrees with the coloring rule. Now, take the red arrow in the diagram above to be the correspondence

$$(u, i) \mapsto (u, i, \phi'_{\pm}(u, i)).$$

This demands that both c and γ in the definition of ϕ_+ and ϕ_- are invertible. For the sake of simplicity, we will take $c = \gamma = -1$.

With this assumption for any admissible coloring (u, i, o) of a crossing we have the following relation:

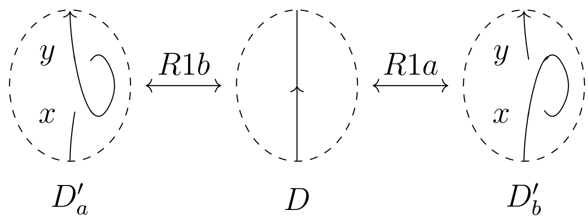
$$\phi_+ \; : \; o = au + bi$$
$$\phi_- \; : \; o = \alpha u + \beta i.$$

We might also demand that a trivial coloring (every arc is assigned the same element of M) is an admissible coloring.

The color checking matrix is not a knot invariant. Changing the diagram with accordance to the Reidemeister moves might change the dimensions of the matrix. Thus, we need to define an equivalence relation on the set of all color checking matrices.

3 Relation on color checking matrices

R1



Both Reidemeister moves $R1a$ and $R1b$ require the following diagram to commute,

$$\begin{tikzcd} M^{s+1} \arrow[r, "D'\phi"] \arrow[d, two heads] & N^{x+1} \arrow[d, two heads] \\ M^{s+1},\ x = y \arrow[d, "f"] & N^x \oplus (N/\phi_{\pm}(M^3)) \arrow[d, "g"] \\ M^s \arrow[r, "D\phi"'] & N^x \end{tikzcd}$$

where $\phi_\pm$ changes (for $R1a$ we have $+$ and for $R1b$ $-$). We take f and g to be given by

$$f(m_1, ..., m_s, m_{s+1}) = (m_1, ..., m_s + m_{s+1})$$

$$g(n_1, ..., n_x, n_{x+1}) = (n_1, ..., n_x + n_{x+1}).$$

The homomorphism f ensures that on the rest of diagrams D' arc labeled x in figure above and y add up to the arc visible in the diagram D. Meanwhile, g ensures that the additional crossing is treated with the appropriate coloring rule.

In terms of matrices, the above diagram can be translated to

$$\overset{D'_a}{\begin{bmatrix} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix}} \overset{R1a}{\sim} \overset{D}{\begin{bmatrix} x_1+y_1 & z_1 & \dots \\ \vdots & & \ddots \end{bmatrix}} \overset{R1b}{\sim} \overset{D'_b}{\begin{bmatrix} \beta & \alpha+\gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix}}$$

DOKOŃCZYĆ

Theorem 3.1.
The equivalence class of a color checking matrix of a diagram $D\phi$ under relation generated by matrix relations $R1a$, $R1b$, $R2$ and $R3$ is a knot diagram. Thus we can define $K\phi := [D\phi]$.

Proof. A direct result of the definition of the equivalence relation. □

4 Smith normal form

The ring R over which we consider modules M is not necessary a principal ideal domain. However, there are plenty of PID rings and one can find at least one PID P with a homomorphism $R \to P$ that allows to consider M as a P-module by tensoring it with P:

$$M_P = M \otimes_R P.$$

That way, we can consider a new type of equivalence relation on any color checking matrix $D\phi$.

Definition 4.1 : Smith normal form.
Take $A \in K\phi$ and consider it as a $s \times x$ matrix with terms in a P. Then there

exist a $s \times s$ matrix S and $x \times x$ matrix T such that SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every i $a_i | a_{i+1}$. Such a matrix SAT is called the **Smith normal form** of matrix A.

As was mentioned in the first section, $\overline{x} \in M^s$ is a coloring of a diagram D if and only if $D\phi(\overline{x}) = 0$, that is $\overline{x} \in \ker D\phi$. The Smith normal form hints at the structure of matrix kernel - the columns filled with zeros will contributed a free factor M to the kernel.

Take (a) to be a prime ideal with its generator a appearing in the Smith normal form of $D\phi$. Then we might consider the matrix over a new ring $P/(a)$, which is still a PID. After this change, the structure of the kernel has changed as now there are additional zero columns where a and all its multiples stood.

Definition 4.2 : reduced normal form of matrix.
Take A to be a matrix with coefficients in principal ideal domain P. Take $a_1, ..., a_k \in P$ to be all the elements of the Smith normal form of A that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix A.

When working with knots we usually take $R = \mathbb{Z}[t, t^{-1}]$ and $M = \mathbb{Z}[t, t^{-1}]$. This is not a PID ring but there are multitudes of PID rings into which R can be mapped. The following algorithm can be used to calculate the Smith normal form of a color checking matrix.

1. Let $A = \{a_{i,j}\}_{i,j \leq n}$ be an $n \times n$ matrix. Take the ideal $I = (a_{i,j})$ generated by all the terms of A.
2. If we are in PID then I has one generator, call it a.
3. We can now use the following row and column operations to put a in the upper left corner of A

(a) Permuting rows (columns).

(b) Adding a linear combination of rows (columns) to the remaining row (column).

4. With a in the upper left corner we can now use the fact that it was the generator of I to strike out the remaining terms on the first column and row, using the operations described in the previous point.
5. Repeat the same algorithm on the smaller matrix $\{a_{i,j}\}_{1<i,j\leq n}$.

The following example justifies the utility of the reduced normal form of color checking matrices in distinguishing knots.

Example 4.1. Consider the knots 6_1 with diagram as seen in fig. 2 and 9_{46} pictured in fig. 3, ring $R = \mathbb{Z}[t, t^{-1}]$, $M = R$ and

$$\begin{cases} \phi_+(u, i, o) = (1 - t)u + ti - o \\ \phi_-(u, i, o) = (1 - t^{-1})u + t^{-1}i - o. \end{cases}$$

The two rings have the same Alexander polynomial, $\Delta = -2t^{-2} + 5t^{-1} - 2$, and the same Alexander module $H^1(S^3 - K) = \mathbb{Z}[t, t^{-1}]/(\Delta)$.

For the knot 6_1 we find the matrix $D\phi$ and after changing to the PID ring $P = \mathbb{Q}[t, t^{-1}]$ we see that the Smith normal form is:

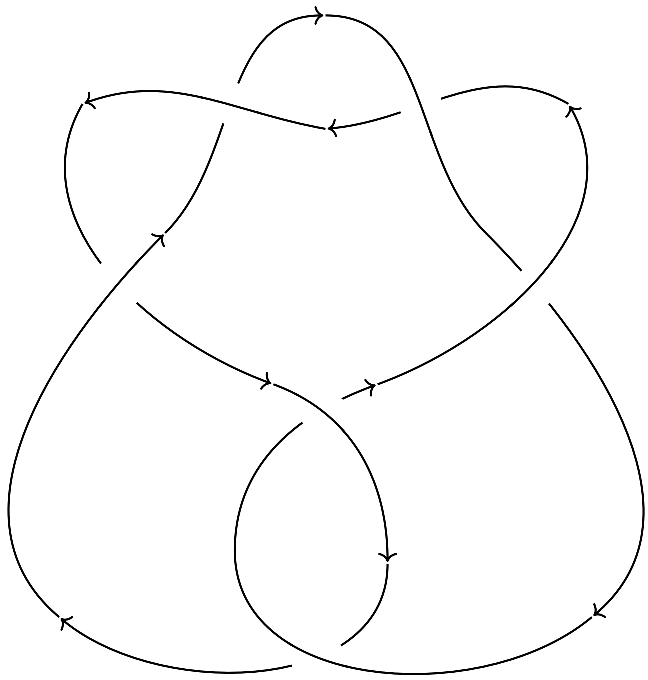


Figure 2: Diagram of knot 6_1.

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2}+5t^{-1}-2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which after reduction is

$$A' = \begin{pmatrix} -2t^{-2}+5t^{-1}-2 \end{pmatrix}$$

a 1×1 matrix with the only term being the Alexander polynomial of 6_1.

Using diagram in fig. 3 of 9_{46} it can be calculated that the Smith normal form of $D\phi$ is

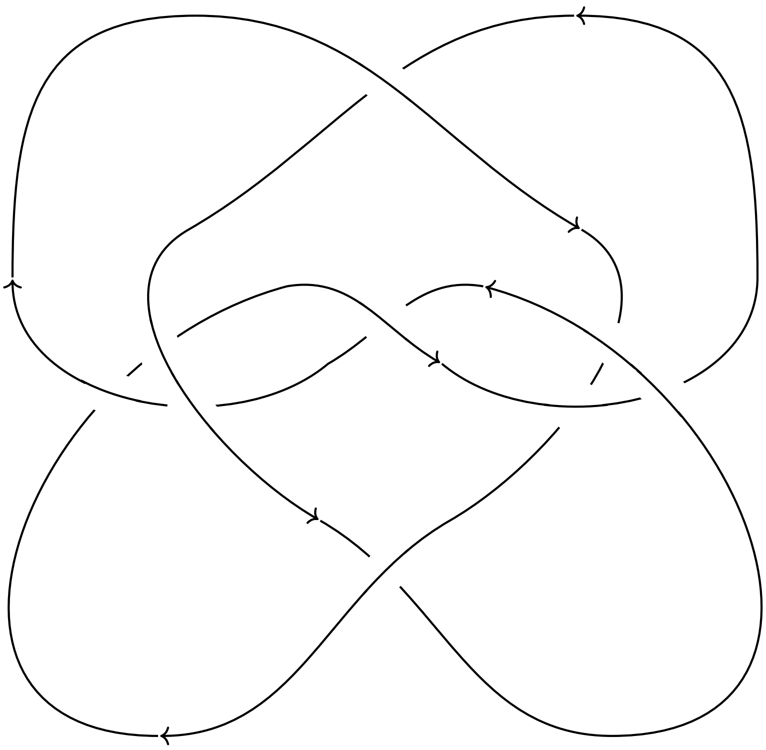


Figure 3: Diagram of knot 9_{46}.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t-t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2}-2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while reduced normal form of $D\phi$ is

$$B' = \begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

which is significantly different than the one for 6_1. Observe also that the determinant of both matrices is equal to the Alexander polynomial of corresponding knots

$$\det(A') = -2 + 5t^{-1} - 2t^{-2}$$

$$\det(B') = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t + 2 + 2t^{-1} = -t(-2 + 5t^{-1} - 2t^{-2}).$$

to wyliczenie nie wogóle jest na miejscu?

Theorem 4.1.
The reduced normal form of color checking matrix does not depend on the choice of diagram D. Thus, it is well defined for $K\phi$ and is a knot invariant.

Proof. Take a knot K and its diagram D with s segments and x crossings. We will show that applying any Reidemeister move to this knot will not change the reduced normal form of its color checking matrix.

nie wiem, czy tutaj aż tak powinno się dokładnie mówić co i jak dodaję?

R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a** (the proof of **R1b** is left as an exercise for the reader).

Take D' to be diagram D with one arc twisted into a $+$ crossing. In opposition to the assumption in previous section, we will take the arcs and crossings that differ between those two diagrams to be on first positions. Now, the matrices $D\phi$ and $D'\phi$ are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

Adding the first column of $D'\phi$ to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

because $a+b=1$. Now we know that b is a unit, thus we can easily remove the elements of the first column that are not b. This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2+y_2 & \dots & \\ 0 & x_3+y_3 & & \\ \vdots & & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like $D\phi$. The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Thus, the reduced normal form of $D'\phi$ is the same as in $D\phi$.

R2

Now the diagram D' is a diagram D with one arc poked onto another. Once again we will put those changed arcs at the beggining of the color checking matrix to obtain following matrices:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3+v_3 & \dots \\ x_4 & u_4+v_4 & \\ \vdots & & \end{bmatrix}$$

Adding the third column of $D'\phi$ multiplied by α and β to first and second column respectively we are able to reduce the first row to only zeros and -1. Now, adding this row to the second one creates a column with only -1 and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a+b\alpha & 0 & -1 & \\ 0 & x_3 & u_3 & v_3 & \\ 0 & x_4 & u_4 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that $a+b\alpha=0$ and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 & \\ 0 & v_3+u_3 & v_3+u_3 & x_3 & \\ 0 & v_4+u_4 & v_4+u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & v_3+u_3 & x_3 & \\ 0 & 0 & v_4+u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as $D\phi$.

R3

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4+v_4 & w_4+v_4 & x_4-v_4 & y_4+u_4+x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4+v_4 & w_4+v_4 & x_4-v_4 & y_4+u_4+x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as b and β were taken to be units.

□

5 Skein relations

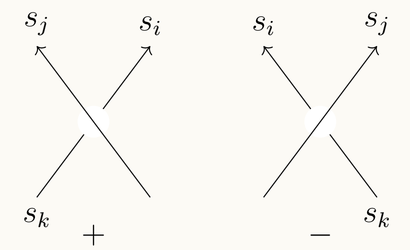
6 Another way to look at knot colorings

Alexander in his work colored the regions of a knot diagram as he related the resulting knot invariant to the Dehn presentation of the knot group [1]. Today we know about the Wirtinger presentation to which coloring of arcs of a diagram can be related.

Definition 6.1 : Wirtinger presentation.
Take D to be an oriented diagram of knot K with n segments $s_1, ..., s_n$ and n crossings $x_1, ..., x_n$. The fundamental group of K has the following presentation

$$\pi_1(S^3 - K) = \langle s_1, ..., s_n \mid x_1, ..., x_n \rangle,$$

where each crossing x_i represents one of the following relations, depending on its type



$s_k = s_j s_i s_j^{-1}$ $\qquad$ $s_k = s_j^{-1} s_i s_j$

Take $G = \pi_1(S^3 - K)$. Using the Mayer-Vietoris sequence we can show that $G^{ab} = \mathbb{Z}$ regardless of the structure of K. Thus we need to work more carefully with G to obtain a reasonable invariant. Consider the following two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K = [G,G] \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & K^{ab} \arrow[r] & G^{mab} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where both arrows $K \to K^{ab}$ and $G \to G^{mab}$ are quotient maps of $[K, K]$.

Lemma 6.1.
Let $s_1, ..., s_n$ be the generating set of G and take $a_i = s_i s_1^{-1}$ for $i = 2, ..., n$ and $a_1 = s_1$ to be the new generating set. Then K is generated by $s_1^k a_i s_1^{-k}$ for $k \in \mathbb{Z}$ and $i = 2, ..., n$.

Proof. TODO ☐

Thanks to lemma 6.1 we can now define the structure of $\mathbb{Z}[\mathbb{Z}]$ module on K^{ab} by

$$t(a_i) = s_1 a_i s_1^{-1}$$

for $i = 2, ..., n$. From now on we will think of K^{ab} as a $\mathbb{Z}[\mathbb{Z}]$-module.

Definition 6.2 : Alexander matrix.
Let G be the fundamental group of a knot K. Consider the following resolution of $\mathbb{Z}[\mathbb{Z}]$-module K^{ab}

$$\dots \longrightarrow \mathbb{Z}[\mathbb{Z}]^a \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^b \longrightarrow K^{ab} \longrightarrow 0$$

The matrix associated with homomorphism A_D is called the **Alexander matrix** of G.

When G has the Wirtinger presentation with n generators, then K^{ab} has resolution

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n} \longrightarrow K^{ab} \longrightarrow 0$$

References

[1] J.W. Alexander. Topological invariants of knots and links. *Transactions of the American Mathematical Society*, 1928.

[2] Kunio Murasagi. *Knot theory and its applications*, pages 7–8. Springer, 1996.

[3] Michael Polyak. Minimal generating set of reidemeister moves. *Quantum Topology vol 1*, 2010.

[4] W.B. Raymond Likorish. *An Introduction to Knot Theory*, pages 2–3. Springer, 1997.