A voyage into the algebras

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1 Introduction

1.1 Order of an Ideal over PID ring

PID -> every ideal is generated by one element, every module is an image of a free module, hence it can be expressed as $M \cong R/I_1 \oplus ... \oplus R/I_n$ for some ideals I_i. This allows as to define order of a module as $\text{ord}(M) = \text{ord}(I_1...I_n)$, which is the element that generates the ideal $I_1....I_n$.

$\text{ord}(M)$ can also be described using equivalence relation $M \sim M_1 + M_2 \iff 0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence -> finitely generated abelian groups as $\mathbb{Z}$ modules and vector fields over $\mathfrak{K}$ as $\mathfrak{K}[x]$-modules.

1.2 The Problem of non-PID rings

Not every ring is a PID -> we must either find another invariant or make the ring in question a PID. E.g. for $\mathbb{Z}[x, x^{-1}]$ we can tensor it with some field, usually $\mathbb{Q}$ but we might want to try F_p for some prime p.

Maybe some example for $\mathbb{Z}[x]$?

1.3 Short Introduction to Knot Theory?

Knot - a closed curve immersed in some 3-dimensional space, or S^1 immersed in S^3

We will consider only tamed knots? That is knots that can be represented as a sum of a finite amount of straight lines?

Using Mayer-Vietoris sequence we can deduce that $H^1(S^3 \setminus K) = \mathbb{Z}$ for any knot K. Hence, if we want to find interesting invariants, we must look further.

Seifert surface of knot K is an orientable surface whose boundary is K. We can use it to create an infinite cyclic covering of $S^3 \setminus K$ by cutting copies $S^3 \setminus K$ along this surface and gluing the $+$ side of Seifert surface of one copy to the $-$ side of the next copy.

$H^1(K^*)$ is more complicated than $H^1(S^3 \setminus K)$ and things get interesting if we consider it as a $\mathbb{Z}[\mathbb{Z}]$ (or $\mathbb{Z}[x, x^{-1}]$-module. We can use the fact that $\Pi_1(K^*)^{ab} = H^1(K^*)$ and calculate this module to obtain something called Alexander ideal I: $H^1(K^*) \cong \mathbb{Z}[\mathbb{Z}]/I$. If I is a principal ideal, e.g. in the case of trefoil knot of figure eight knot, its generator is called "Alexander polynomial". If this is not the case, we must consider $H^1(K^*; \mathbb{Q})$ - kohomology module with coefficients in $\mathbb{Q}$, to obtain the Alexander polynomial. In the following paper we will consider what happens if we use F_p, a finite field, instead of $\mathbb{Q}$.

2 Calculating the Alexander Module

Kinoshita-Tarasaki - does not look too promising

Conway Knot - to be examined

References