

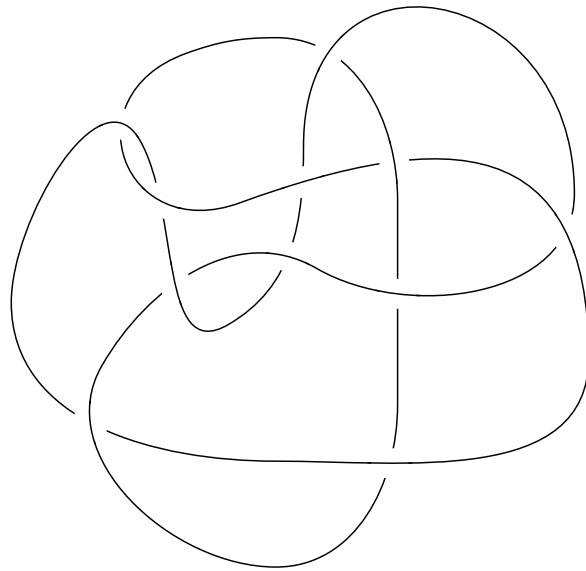
Knot colorings and homological invariants

(Kolorowania węzłów i niezmienniki homologiczne.)

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Abstract

The knot group $G = \pi_1(K)$ is a starting point for many knot invariants. Alexander matrix is a representation matrix for a subgroup of G and from its determinant, the Alexander polynomial is obtained. Another way of obtaining said polynomial is by considering a coloring matrix which assigns elements of R -module M to segments from a diagram D of knot K . This approach can be derived from the image of a resolution of Alexander module through the functor $\text{Hom}(-, M)$. Nevertheless, color checking matrices do not instantly yield a knot invariant, however it is possible to define an equivalence relation that identifies matrices stemming from the same knot. This approach is used to distinguish a pair of knots with the same Alexander polynomial. In the end, a way of generalizing the procedure of coloring diagrams is presented in terms of category theory.

Introduction

In knot theory distinguishing knots is often a difficult endeavor, usually facilitated by the notion of invariants. An interesting group of knot invariants are polynomial invariants, such as the Alexander polynomial. Another group worth mentioning are knot colorings that can also yield an element of the ring $\mathbb{Z}[\mathbb{Z}]$.

Very often, considering only one invariant is not sufficient, as there are many knots that share its value, i.e. $K11n85$ and $K11n164$ have the same Alexander polynomial. However, a more subtle application of the same method that yields the Alexander polynomial can sometimes distinguish such knots.

The following paper is a result of a year long cooperation between prof. Tadeusz Januszkiewicz, Julia Walczuk and the author of this thesis. In it, connections between the knot group, knot colorings and homology modules of infinite cyclic covering (see [5]) will be outlined. As an additional exercise, we will show a way of distinguishing already mentioned knots $K11n85$ and $K11n164$.

The first section of this paper defines the most important terms used in knot theory, as well as highlights the connection between the metabelianization of knot group and the first homology module of an infinite cyclic

covering of the complement of said knot.

Subsequently, the construction of a $\mathbb{Z}[\mathbb{Z}]$ module from the kernel of $G^m \rightarrow \mathbb{Z}$ is presented. This module is defined to be the Alexander module K_G^{ab} of knot K (definition 1.4). Then, Alexander matrix is introduced (??) as the representation matrix of the Alexander module. In this section the resolution of Alexander module is proven to be of form

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

At the end of the second section, a connection between resolution of the Alexander module and coloring of diagrams is made.

The third and last section is focused on coloring matrices (definition 3.3) and defining an equivalence relation between matrices relating to the same knot. A new knot invariant is defined (??) and an example of its utility is presented in ??.

Last section is dedicated to presenting an approach to diagram colorings from the perspective from category theory. In addition, a connection between the Alexander matrix and color checking matrix (using a particular palette, named the Alexander palette) is presented.

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1 Preliminaries

1.1 Knots and diagrams

In mathematical terms, a knot is a smooth embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \hookrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [3]. If two points are mapped to one by this projection, we say that a small neighbourhood of this point which looks locally like $-|-$ is a crossing.

S^1 is orientable, thus we can chose an orientation for any knot and, as a consequence, its diagram.

Intuitively, two tame knots K_1 and K_2 are equivalent if we can deform one into the other [6]. This translates to an equivalence of diagrams, which is generated by comparing diagrams that are exactly the same save for an interior of some disc in \mathbb{R}^2 . If inside of said disc the diagrams differ by one of **Reidemeister moves**, we say that they are equivalent. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D , 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [7].

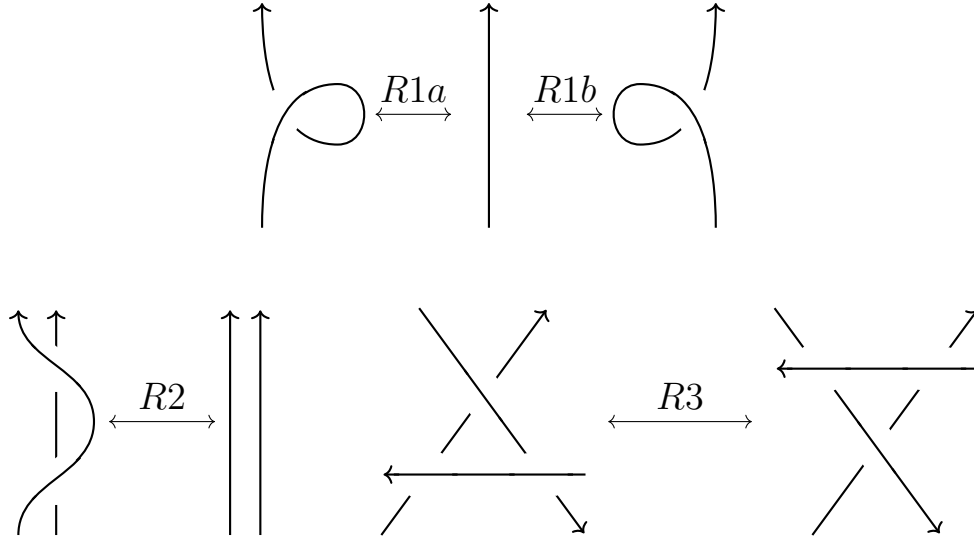


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

1.2 Knot group

Let K be a knot and D be its oriented diagram with s segments and x crossings. A segment of a diagram is a line of the diagram between two

crossings in which it disappears under another line.

Definition 1.1 : knot group.

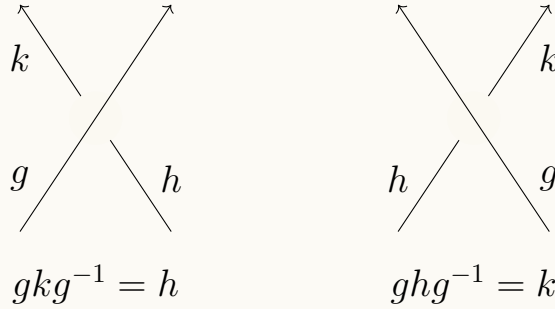
The fundamental group of knot complement $X = S^3 - K$ is called a **knot group**:

$$\pi_1(K) := \pi_1(X).$$

Although the knot itself is always a circle S^1 , the knot group has usually an interesting yet difficult structure. The most commonly used presentation of the knot group is called **the Wirtinger presentation**.

Definition 1.2 : Wirtinger presentation.

Given a diagram D of knot K with segments a_1, a_2, \dots, a_s and crossings c_1, \dots, c_x the knot group $\pi_1(K)$ can be represented as $\pi_1(K) = \langle G \mid R \rangle$, where G is the set of segments of D and relations R correspond to crossings in the manner described in the diagram below



Representation $\langle G \mid R \rangle$ described above is called the **Wirtinger presentation** [4, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to $S^3 = K \oplus S^3 - K$ or noticing that every two generators are conjugate, is that the abelianization of the knot group is always \mathbb{Z} . This leads to a short exact sequence

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0.$$

The group $K_G = \ker(ab : G \rightarrow \mathbb{Z}) = [G, G]$ in general is not abelian nor finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization $K_G^{ab} = K_G/[K_G, K_G]$ allows a $\mathbb{Z}[\mathbb{Z}]$ module structure and thus contains obtainable information about the knot K .

Lemma 1.1.

For any group G , the commutator of its commutator K_G is a normal subgroup: $[K_G, K_G] = [[G, G], [G, G]] \triangleleft G$.

Proof. The commutator subgroup is a characteristic subgroup, since for any automorphism $\phi : G \rightarrow G$

$$\phi(hgh^{-1}g^{-1}) = \phi(h)\phi(g)\phi(h)^{-1}\phi(g)^{-1} \in K_G = [G, G].$$

Conjugation by any element $g \in G$ is an automorphism of the commutator K_G . Thus it preserves its commutator subgroup $[K_G, K_G]$. \square

The exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is split, meaning that $\mathbb{Z} \rightarrow G^{mab} \rightarrow \mathbb{Z}$ is identity. As a consequence, the following is an exact sequence with $G^{mab} = K_G^{ab} \rtimes \mathbb{Z}$. This gives foundation for the following definition.

Definition 1.3 : metabelianization.

The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of G .

We will return to the concept of metabelianization in section 2. For the time being, let us assign a name to K_G :

Definition 1.4 : Alexander module.

tutaj nie rozumiem Pana uwagi " $G \rightarrow \mathbb{Z}$ nie nagie G " Given a group G , the abelianization of the commutator of a group G , K_G^{ab} , with $\mathbb{Z}[\mathbb{Z}]$ -module structure is called the **Alexander module** of G . If G is a knot group, then it is the Alexander module of the knot K .

How the $\mathbb{Z}[\mathbb{Z}]$ module structure is obtained is described in detail in section 2.1.

1.3 Infinite cyclic covering

Let X be the complement of a knot K ($X = S^3 - K$). Take \tilde{X} to be its universal covering, meaning that it is simply connected. The fundamental group $G = \pi_1(X)$ of X acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G, G] \triangleleft G$ acts

on \tilde{X} and because K_G is normal in G , the quotient $\overline{X} = \tilde{X}/K_G$ has fundamental group $\pi_1(\overline{X}) = K_G$ no matter the choice of the basepoint of \overline{X} .

Definition 1.5 : infinite cyclic covering.

The quotient space $\overline{X} = \tilde{X}/K_G$ is called the **infinite cyclic covering** of X .

Looking at homology groups of \overline{X} , we have the following equality

$$H_1(\overline{X}, \mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

Working with homology modules of an infinite cyclic cover of X instead of K_G^{ab} directly is beneficial when proving some properties of K_G^{ab} , i.e. that it is a torsion module in proposition 1.2.

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{array}{ccc} \tilde{X} & \curvearrowright & G \\ \downarrow & & \\ \overline{X} & \curvearrowright & G/[G, G] \\ \downarrow & & \\ X = S^3 - K & & \end{array}$$

A **Seifert surface** S of knot K is an orientable surface with boundary embedded in S^3 such that $\partial S = K$. Take a countable amount of X , with S without its boundary embedded, and label each with an element from \mathbb{Z} . We might now cut each of the copies of X along the Seifert surface of K and identify the $+$ side of S from the i -th copy of X with the $-$ side of S from the $(i + 1)$ -th copy of X . Notice that the arising space with a projection to one copy of X is an infinite cyclic cover of X .

Imagine that each copy of X inside of \overline{X} is a box labeled with some integer k . The ring action of $\mathbb{Z}[\mathbb{Z}]$ on \overline{X} is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to \overline{X} .

Proposition 1.2.

The $\mathbb{Z}[\mathbb{Z}]$ -module $K^{ab} = H_1(\overline{X}, \mathbb{Z})$ is a torsion module.

Proof. Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \rightarrow C_*(\overline{X})$$

$$f(x) = (1 - t)x.$$

It translates to removing from a cycle in the $(i+1)$ -th box a corresponding cycle in the i -th box. From this it is an immediate result that $\ker f = 0$ and that $\operatorname{coker} f = C_*(X)$: after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces a long exact homology sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_2(X, \mathbb{Z}) & \longrightarrow & H_1(\overline{X}, \mathbb{Z}) & \xrightarrow{1-t} & H_1(\overline{X}, \mathbb{Z}) \longrightarrow H_1(X, \mathbb{Z}) \\ & & & & & & \downarrow \\ & & & & & & \longrightarrow H_0(\overline{X}, \mathbb{Z}) \longrightarrow H_0(\overline{X}, \mathbb{Z}) \longrightarrow H_0(X, \mathbb{Z}) \longrightarrow 0. \end{array}$$

As was mentioned previously, the following equality holds:

$$H_1(X, \mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}.$$

Now, because X is homology circle, then $H_2(X, \mathbb{Z}) = 0$ (one can easily check it for themselves using Alexander duality). Both X and \overline{X} are connected implying that

$$H_0(X, \mathbb{Z}) = H_0(\overline{X}, \mathbb{Z}) = \mathbb{Z}.$$

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & H_1(\overline{X}, \mathbb{Z}) & \xrightarrow{1-t} & H_1(\overline{X}, \mathbb{Z}) \xrightarrow{0} \mathbb{Z} \longrightarrow \\ & & & & & & \downarrow \text{iso} \\ & & & & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \longrightarrow 0 \end{array}$$

Rewriting the sequence above we easily get that homomorphism $1 - t$ is actually an isomorphism and $H_1(\overline{X}, \mathbb{Z}) \cong (1 - t)H_1(\overline{X}, \mathbb{Z})$, which allows us to use the Nakayama's lemma [2, Proposition 2.6] to conclude that there exists $x \in \mathbb{Z}[\mathbb{Z}]$ such that

$$xH_1(\overline{X}, \mathbb{Z}) = 0.$$

□

2 Resolution of the Alexander module

2.1 Presentation of the Alexander module

Take $G = \langle G \mid R \rangle$ to be the Wirtinger presentation of G obtained from oriented diagram D . Because K is a knot and not a link, we know that the number of segments is equal to the number of crossings.

Lemma 2.1.

If K is a knot and D its diagram with $x > 0$ crossings and $s > 1$ segments, then $s = x$.

Proof. Every crossing has two segments that disappear under it. Let d be the number of segment ends that go below a crossing then

$$2 \cdot x = d.$$

Every segment starts and ends in some crossing, thus

$$2 \cdot s = d.$$

From those two equalities we have that $d = 2x = 2s \implies x = s$. \square

Let a_1, \dots, a_n be the generators of G agreeing with the orientation of the diagram D and x_1, \dots, x_n its relations. The homomorphism of abelianization of G is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, \dots, n$. In order to obtain a presentation of K_G , the kernel of abelianization, we need to change the set of generators of G to

$$\{a_1, A_2 = a_2 a_1^{-1}, \dots, A_n = a_n a_1^{-1}\}.$$

It is obvious that for every $i > 1$ $A_i \mapsto 0$ by abelianization of G . Thus A_2, \dots, A_n are some of the generators of K_G . However, for each $i = 2, \dots, n$ and $k \in \mathbb{Z}$ the following is an element of K_G :

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the presentation of K_G as an abelian group is infinite with (possibly redundant) generators

$$\{b_{i,k} : i = 2, \dots, n, k \in \mathbb{Z}\}.$$

Changing generators of G induced a change in relations. Suppose that the following relation was true in G

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of a_1 the following two relations can be obtained

$$\begin{aligned} a_1^{-1} A_k a_1 &= (a_1^{-1} A_i a_1) A_j A_i^{-1} \\ a_1^{-3} A_k a_1^3 &= (a_1^{-3} A_i a_1^3) (a_1^{-2} A_j a_1^2) (a_1^{-2} A_i^{-1} a_1^2). \end{aligned}$$

Obviously in G both of those relations are equivalent, however in K_G they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

to obtain infinitely many relations from K_G .

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

For any group H with $H^{ab} = \mathbb{Z}$ we can write a homomorphism $\mathbb{Z} \rightarrow H$ such that composition $\mathbb{Z} \rightarrow H \rightarrow \mathbb{Z}$ is identity on \mathbb{Z} . Thus, this sequence splits and we can write

$$G^{mab} = K_G^{ab} \rtimes \mathbb{Z}.$$

Hence action of \mathbb{Z} can be defined on the group K_G^{ab} , with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}.$$

This procedure allows K_G^{ab} to be interpreted as a $\mathbb{Z}[\mathbb{Z}]$ -module.

Moreover, the group G^{mab} and $\mathbb{Z}[\mathbb{Z}]$ -module K_G^{ab} can be used interchangeably as knowing the action of \mathbb{Z} on K_G^{ab} allows us to write the semidirect product of \mathbb{Z} and K_G^{ab} .

2.2 Basic properties

Knowing the resolution of a module allows one to change said module into a matrix or even a sequence of matrices, each containing at least a portion of information about its structure.

Definition 2.1 : Alexander matrix.

The presentation matrix A_D of K_G^{ab} with Wirtinger presentation is the matrix $\mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1}$ such that $\ker(A_D)$ is equal to module of relations of K_G^{ab} . This matrix is called the **Alexander matrix** of the Alexander module K_G^{ab} .

We start writing the beginning K_G^{ab} resolution as follows:

$$\dots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0 \quad (1)$$

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its columns will give a square matrix whose determinant is nonzero. We will prove this statement promptly after consider the Alexander module as a vector space over the field of fractions of $R = \mathbb{Z}[\mathbb{Z}]$ [2, Chapter 3].

In proposition 1.2 it was shown that the Alexander module is torsion. Thus, as a vector space $K_G^{ab} \otimes_R R^{-1}R = 0$ it is trivial. Hence, the sequence in (1) translates to the following sequence of $R^{-1}R$ modules

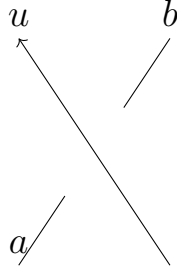
$$\dots \longrightarrow R^n \otimes_R R^{-1}R \xrightarrow{A_D^V} R^{n-1} \otimes_R R^{-1}R \longrightarrow 0 \quad (2)$$

As there exists an inclusion $R \hookrightarrow R^{-1}R$, every matrix with terms in R can be treated as a matrix with terms in $R^{-1}R$. Naturally, $A_D^V = A_D \otimes Id_{R^{-1}R}$ is just matrix A_D (with terms in R) with adjoined 1×1 matrix with just identity of $R^{-1}R$. Thus, we can easily translate most properties of A_D^V to properties of A_D , i.e. its determinant and surjectivity.

Proposition 2.2.

Let A'_D be the Alexander matrix A_D with one of its rows struck out. Then $\det(A'_D) \neq 0$.

Proof. We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator x is sent to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module K^{ab} we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1 - t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which x is b or c are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t - 1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and ± 1 . Thus, the sum is not equal to zero. There are two of such rows as the segment b has to be the "out" and "in" segment of some crossing. In other words, segment b has to have a start and end in some crossings.

The reasoning above is true for matrix A_D^V from (2). We make the switch to vector space to use the connection between the rank of matrix and the dimension of its image.

Let S_i be the column of the Alexander matrix corresponding to the segment labeled i . The sum $\sum_{i \leq n-1} S_i$ is a vector with two nonzero terms. Take S_j and S_k to be the vectors with those nonzero terms. The

only way to cancel out those coordinates is to multiply both S_j and S_k by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j,k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i : i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D^V(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. \square

The proposition 2.2 implies that image of A_D^V has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

Theorem 2.3.

The determinant $\det(A'_D)$ up to multiplication by a unit is independent of the choice of the diagram D .

Proof. A proof using Dehn presentation is provided in [1], while a proof of more general case is provided in [8] \square

Definition 2.2 : Alexander polynomial.

Let $p(t)$ be the determinant of any maximal minor of the Alexander matrix A_D . Then, we can find $k \in \mathbb{Z}$ such that $t^k p(t)$ is a symmetrical polynomial, meaning that the absolute values of the highest and lowest powers of variables in it are equal [1]. The polynomial $t^k p(t)$ is called the **Alexander polynomial** of a knot K .

The Alexander polynomial is a knot invariant as a consequence of theorem 2.3 and proposition 2.2

Proposition 2.4.

Let G be a knot group of K and $F = R^{-1}R$ the field of fractions of ring R [2]. Then, changing coefficients by applying functor $- \otimes_R F$ to the following exact sequence

$$0 \longrightarrow \ker(A_D) \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

yields the following exact sequence

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R F = 0 \longrightarrow 0$$

where n is the number of crossings of the chosen diagram D of knot K .

Proof. We start by saying that $R \otimes_R F \cong F$ because R is a free module over R [2, Proposition 2.14].

Proposition 2.2 implies that (2) can be extended into the following exact sequence of vector spaces:

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R R^{-1}R = 0 \longrightarrow 0$$

as we proved that $\dim(\operatorname{im} A_D^V) = n - 1 \implies \dim(\ker A_D^V) = 1$.

The ring of fractions is flat [2, Chapter 3] at the same time we only consider R -modules treated as vector spaces in this proposition. Thus, we have the following exact sequence

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

with $M \otimes_R F \cong F$. □

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Notice, that sequence

$$\star : 0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow 0$$

is not acyclic, however it allows us to once again define the Alexander module, this time as $H_1(\star)$.

3 Knot colorings

3.1 Palettes and diagram colorings

An oriented diagram D of knot K has two types of crossings, pictured in fig. 2. A diagram coloring, in essence, is an assignment of values from some mathematical object (i.e. R -module) to segments of D .

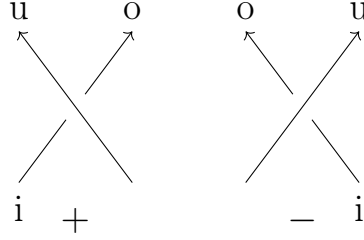


Figure 2: Two types of crossing in oriented diagram.

Definition 3.1 : palette.

We say that a triplet (R, M, \mathcal{C}_\pm) is a **palette** if R is a commutative ring with unity, M an R -module and \mathcal{C}_\pm are two R -modules, corresponding to the two types of crossings (fig. 2) such that $\mathcal{C}_\pm \subseteq M^3$.

We will cumulatively call the two modules \mathcal{C}_\pm the **coloring rule** of palette (R, M, \mathcal{C}_\pm) as they determine whether a coloring is admissible, a term that will be defined.

Definition 3.2 : diagram coloring.

A **coloring of diagram** D with s segments and x crossings (for knots $s = x$ [9]) is any element $(m_1, \dots, m_s) \in M^s$ that assigns elements of M to each arc.

We will call a coloring **admissible** if for every crossing x_j of type \pm we have

$$\pi_{x_j}(m_1, \dots, m_s) \in \mathcal{C}_\pm \subseteq \mathcal{C},$$

where $\pi_{x_j} : M^s \rightarrow M^3$ is a projection of module M^s to the M^3 factor that corresponds to segments that constitute x_j .

Usually, a crossing comprises of exactly three segments. However, as is the case for the first Reidemeister move, a crossing can be comprised of only two segments. Thus, if x_j is such a two-segment crossing, $\text{im}(\pi_{x_j})$ should be isomorphic to $\{(x, y, y) : x, y \in M\}$.

We can now define two linear homomorphisms

$$\phi_{\pm} : M^3 \rightarrow M^3/\mathcal{C}_{\pm} = N_{\pm}$$

that take in as arguments the arcs constituting a crossing. Assuming that $M^3/\mathcal{C}_{\pm} \cong M$ (reasoning behind this assumption will be given in section 3.2), we will take

$$\phi_+(u, i, o) = au + bi + co \quad (3)$$

$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o \quad (4)$$

for u, i, o understood like in fig. 2 and with coefficients being homomorphisms $M \rightarrow M$.

Lemma 3.1.

A coloring $(m_1, \dots, m_s) \in M^s$ is a admissible \iff for each crossing x_j of type \pm

$$\phi_{\pm}(\pi_{x_j}(m_1, \dots, m_s)) = 0.$$

Proof. Stems from the fact that $\mathcal{C}_{\pm} = \ker \phi_{\pm}$. □

3.2 Color checking matrix

Coloring diagrams with a chosen palette $(R, M, \mathcal{C}_{\pm})$ is a functor from the category of knot diagrams, whose objects are diagrams and morphisms represent Reidemeister moves, to a category of matrices, with morphisms corresponding to matrix equivalence explained in this section.

Moreover, if a palette $P = (R, M, \mathcal{C}_{\pm})$ is given alongside a ring homomorphism $f : R \rightarrow S$, then an image of this palette P through induced palette homomorphism is $(S, M \otimes_R S, \mathcal{C}_{\pm} \otimes_R S)$. Similarly, if an R -module homomorphism $g : M \rightarrow N$ is given, then the image of palette P is $(R, N, g(\mathcal{C}_{\pm}))$.

Definition 3.3 : color checking matrix.

Assigning segments of diagram D to coordinates in M^s and crossings to coordinates in N_{\pm}^x it is possible to define a linear homomorphism $D\phi : M^s \rightarrow N_{\pm}^x$ as

$$D\phi(m_1, \dots, m_s) = (\phi_{\pm}(\pi_{x_1}(m_1, \dots, m_s)), \phi_{\pm}(\pi_{x_2}(m_1, \dots, m_s)), \dots).$$

Matrix that is created after choosing a basis for M^s and N_{\pm}^x will be called a **color checking matrix**.

Proposition 3.2.

Coloring $(m_1, \dots, m_s) \in M^s$ is admissible $\iff (m_1, \dots, m_s) \in \ker D\phi$.

Proof. We start by saying that

$$(m_1, \dots, m_s) \in \ker D\phi \iff [(\forall x_j \text{ crossing}) \phi_{\pm}(\pi_{x_j}(m_1, \dots, m_s)) = 0].$$

Which is to say that every coordinate of $D\phi(m_1, \dots, m_s)$ is zero. Proposition lemma 3.1 says that it is equivalent with (m_1, \dots, m_s) being an admissible coloring. \square

We want to define which color checking matrices are equivalent. We will say that $D\phi$ and $D'\phi$ are equivalent if

1. they differ by a permutation of rows or columns,
2. one can be obtained from the other by adding a linear combination of rows or columns to another row or columns
3. one can be obtained from the other by adding a new row and a new column with only 0 save for the term on their intersection, which is a unit.

The first two points mean that two color checking matrices $D\phi, D'\phi : M^s \rightarrow N^x$ are equivalent if there exist an isomorphisms $\theta : M^s \rightarrow M^s$ and $\psi : N^x \rightarrow N^x$ such that

$$\begin{array}{ccc} M^s & \xrightarrow{D\phi} & N^x \\ \theta \downarrow & & \downarrow \psi \\ M^s & \xrightarrow{D'\phi} & N^x \end{array}$$

is a commutative diagram.

In the most basic sense, two diagrams D and D' are isomorphic if there exists an isotopy $h_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $h_0(D) = D$ and $h_1(D) = D'$ and D' has crossings identical to those of D .

Lemma 3.3.

Isomorphic diagrams $D \sim D'$ yield equivalent color checking matrices $D\phi \sim D'\phi$.

Proof. In terms of color checking matrices, an isomorphism of diagrams defined above only relabels segments (permutes columns) and crossings (permutes rows). \square

In reality, the isomorphism between diagrams which relates diagrams of the same knot is much more subtle. This relation R is generated by the Reidemeister moves (see fig. 1). There is an induced relation on the set of color checking matrices, called $D(R)$, that relates color checking matrices of the same knot. For suitably chosen palettes, an invariant of the equivalent class of matrices of the same knot is known.

Definition 3.4 : Alexander palette.

One of such palettes is the **Alexander palette** ($\mathbf{R} = \mathbb{Z}[\mathbb{Z}], \mathbf{M} = \mathbb{Z}[\mathbb{Z}], \mathbf{C}_{\pm}$), where \mathbf{C}_{\pm} are defined by homomorphisms

$$\phi_+(u, i, o) = (1 - t)u + ti - o$$

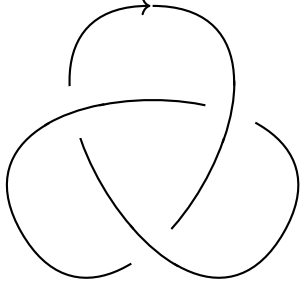
$$\phi_-(u, i, o) = (1 - t^{-1})u + t^{-1}i - o,$$

with u, i and o defined in fig. 2.

For the Alexander palette said invariant is e.g. the determinant of a $n - 1$ minor of the color checking matrix up to multiplication by a unit.

The following example illustrates the importance of choosing a suitable palette.

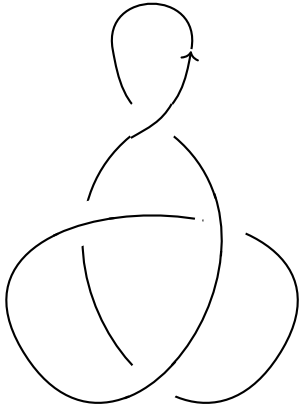
Example 3.1. Consider a coloring of trefoil knot 3_1 with two palettes: $P_1 = (\mathbb{Z}, \mathbb{Z}, \phi_{\pm}(u, i, o) = 2u - i + o)$ that is not an image of the Alexander palette and $P_2 = (\mathbb{Z}, \mathbb{Z}, \phi_{\pm}(u, i, o) = 2u - i - o)$ which in turn is one. The color checking matrix of its diagram with 3 crossings is



$$P_1(D) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \mapsto \{1, -3, -5\}$$

$$P_2(D) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \mapsto \{-3\}$$

while after the first Reidemeister move, the color checking matrix is



$$P_1(D) = \begin{bmatrix} 0 & 2 & 1 & -1 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} \mapsto \{5, 11, 3, \dots\}$$

$$P_2(D) = \begin{bmatrix} 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ -1 & -1 & 0 & 0 \end{bmatrix} \mapsto \{-3, 3\}$$

In order to ensure that the equivalence classes of color checking matrices under the relation induced by Reidemeister moves on diagrams have a known invariant, rules must be imposed on palettes. A particular set of palettes that will satisfy conditions to be mention are the Alexander palette and all palettes derived from it by either a ring homomorphism or a module homomorphism.

The coloring rule modules \mathcal{C}_{\pm} are to be isomorphic to M^2 , with the red arrow being an isomorphism

$$M^2 \hookrightarrow M^3 \twoheadrightarrow \mathcal{C}_{\pm}.$$

\cong

We want the red isomorphism to be $(u, i) \mapsto (u, i, \phi'_\pm(u, i)) \in \mathcal{C}_\pm$, with segments labeled like in fig. 2, meaning that c and γ in (3) and (4) respectively are units. For the sake of convenience, take $c = \gamma = -1$.

This property of palettes will be called a **propagation rule** as knowing colors of two of the three segments allows one to calculate the color assigned to the remaining segment.

Lemma 3.4.

A palette that meets the propagation rule has the following two properties determined by Reidemeister moves.

1. The first Reidemeister move requires that

$$a = 1 - b$$

$$\alpha = 1 - \beta,$$

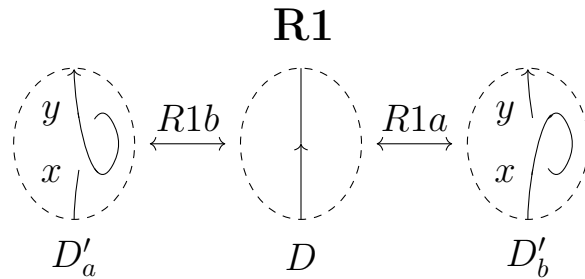
where the variables are coefficients from (3) and (4).

2. Similarly, the second Reidemeister move requires

$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1. \end{cases}$$

3.3 Reidemeister relation on color checking matrices

The diagram D has s segments and x crossings.



The first Reidemeister move allows the following two moves on color

checking matrices

$$\begin{array}{c} D'_a \\ \left[\begin{array}{cccc} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{array} \right] \end{array} \xrightarrow{D(R1a)} \begin{array}{c} D \\ \left[\begin{array}{cccc} x_1 + y_1 & z_1 & \dots & \\ \vdots & & \ddots & \end{array} \right] \end{array} \xrightarrow{D(R1b)} \begin{array}{c} D'_b \\ \left[\begin{array}{cccc} \beta & \alpha + \gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{array} \right] \end{array}$$

where $(\forall i = 1, \dots, x) x_i = 0 \vee y_i = 0$.

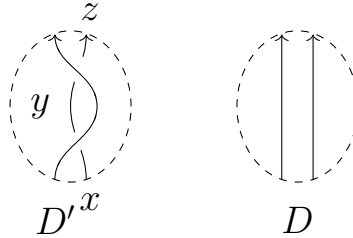
Notice, that if the propagation rule that was outlined at the beginning of this section is to be true, looking closely at the crossing in diagrams above should yield equations

$$0 = a + b + c = a + b - 1 \implies a = 1 - b$$

$$0 = \alpha + \beta + \gamma = \alpha + \beta - 1 \implies \alpha = 1 - \beta,$$

as the up and out segments in D'_a and the up and in segments in D'_b must admit coloring with the same element from M .

R2



For the second Reidemeister move we will say that $D\phi$ and $D'\phi$ are in relation if they differ by the following matrix move

$$\begin{array}{c} D' \\ \left[\begin{array}{cccc} b & c & 0 & a & \dots \\ 0 & \beta & \gamma & \alpha & \\ x_1 & 0 & z_1 & w_1 & \\ \vdots & & & & \ddots \end{array} \right] \end{array} \xrightarrow{D(R2)} \begin{array}{c} D \\ \left[\begin{array}{cccc} x_1 + z_1 & w_1 & \dots & \\ \vdots & & \ddots & \end{array} \right] \end{array}$$

where $(\forall i = 1, \dots, x) x_i = 0 \vee z_i = 0$, in addition to permuting rows and columns and adding linear combination of rows or columns to another row or column.

In the case of this Reidemeister move, we would like to be able to color the diagram D' exactly like the diagram D save for the segments contributing to the two additional crossings. This means that segments labeled z and x on the diagram above must admit a coloring with the same element from M .

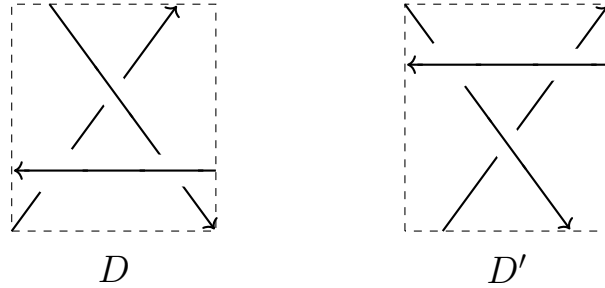
The restrictions stemming from this observation are more easily calculated if homomorphisms ϕ_+ and ϕ_- are made into two matrices, A_+ and A_- , that the incoming segments (up and in segments) and return the output segments (out and up segments). This is possible because of the propagation rule.

$$A_+ A_- \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} u \\ i \end{bmatrix}$$

Comparing the terms of the matrix $A_+ A_-$ with terms of the identity matrix yields:

$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1 \end{cases}$$

R3



The last Reidemeister move does not change the size of matrices but only permutes the terms appearing in columns and rows corresponding to the three crossing that are manipulated in the diagram.

$$\begin{matrix} & D' \\ \begin{bmatrix} \alpha & \gamma & \beta & 0 & 0 & 0 & \dots \\ 0 & 0 & c & b & 0 & a \\ \beta & 0 & 0 & 0 & \gamma & \alpha \\ u_1 & 0 & v_1 & w_1 & x_4 & y_4 \\ \vdots & & & & & \ddots \end{bmatrix} & \stackrel{D(R3)}{\sim} & \begin{matrix} D \\ \begin{bmatrix} 0 & 0 & \gamma & \beta & \alpha & 0 & \dots \\ \beta & 0 & 0 & 0 & \gamma & \alpha \\ 0 & c & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \\ \vdots & & & & & \ddots \end{bmatrix} \end{matrix} \end{matrix}$$

Theorem 3.5.

The equivalence class of a color checking matrix using the Alexander palette, or its derivative, of a diagram D under relation $D(R)$ generated by matrix relations $D(R1a)$, $D(R1b)$, $D(R2)$ and $D(R3)$ is a knot invariant. Thus we can define $K\phi := [D\phi]$.

Proof. A direct result of the definition of the equivalence relation. \square

3.4 Smith normal form

The ring R of palette (R, M, \mathcal{C}_\pm) is not necessarily a PID ring, e.g. $\mathbb{Z}[\mathbb{Z}]$ ring of the Alexander palette has ideal $(2, t + 1)$ which is not principal. However, usually one can find a PID ring P with homomorphism $R \rightarrow P$ which creates a new palette $(P, M \otimes_R P, \mathcal{C}_\pm \otimes_R P)$ derived from (R, M, \mathcal{C}_\pm) . Matrices over PID rings have many interesting properties, like having a Smith normal form.

Definition 3.5 : Smith normal form.

Take $A \in K\phi$ and consider it as an $n \times n$ matrix with terms in a P by the procedure outlined above. Then there exist a $n \times n$ matrix S and $n \times n$ matrix T such that SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every i $a_i | a_{i+1}$. Such a matrix SAT is called the **Smith normal form** of matrix A .

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