PL - knot (inclusion $S^1 \hookrightarrow S^3$) D - its diagram (find a line l such that no three points of K lay on it -> diagram is a projection of K along this line). (onto $\mathbb{R}^2$)

> tylko nie unknot, bo byłaby macierz 1x0?

ORIENTED

REIDEMEISTER 4 x

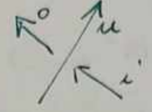
COLORING - yet another attemp at a definition.

R - ring, M ← R - module

COLORING RULES

$\mathcal{C} \subseteq M^3$ - coloring rules?

Module M^3 corresponds to the 3 arcs entering a crossing



$\mathcal{C} \subseteq M^3$ tells us what values can u, i and out take for it to be a part of a coloring.

~~Alternatively~~ We could also define a homomorphism:

$$\varphi : M^3 \longrightarrow M^3/\mathcal{C}$$

that determines what assignents to (u, i, o) are a part of a coloring.

COLORING

D has s seg. and x crossi.

A coloring is actually any assigment of values from M to arcs of the diagram. In other words: a coloring is any $(m_1, \dots, m_s) \in M^s$.

NOT ALL COLORINGS ARE ADMISSIBLE.

A coloring $(m_1, \dots, m_s)$ is admissible <=> ∀ every projection $\pi : M^s \longrightarrow M^3$ that restricts M^s to arcs forming a crossing

$\pi(m_1, \dots, m_s) \in \mathcal{C} \subseteq M^3$.

Now define a ~~This means that $(m_1, \dots, m_s)$ is an admissible coloring if a function~~ linear operator $D\varphi : M^s \to (M/\mathcal{C})^x$ given by projections. ~~a coloring~~

{ A coloring $(m_1, \dots m_s) \in M^s$ is admissible <=> $(m_1, \dots, m_s) \in \ker D\varphi$ }

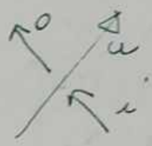
In terms of $\varphi : M^3 \to M/\mathcal{C} = N$ this means that

$$\varphi(\pi_x(m_1, \dots, m_s)) = 0, \text{ where } \pi_x : M^s \to M^3$$

is a "crossing projection"

Taking $\varphi : M^3 \to N$ to be

$$\varphi(u, i, o) = au + bi + co$$



we know that $D\varphi$ will have entries from (a, b, c) ← ideal in R. Kinda. M can have $\dim(M) \geq 1$)

Restrictions on $\mathcal{C}/\varphi$

1) trivial coloring.
2) For a choice of $\mathcal{C}$ and φ we want the following isomorphism

$$\begin{array}{ccccc} M^2 & \twoheadleftarrow & M^3 & \longleftrightarrow & \mathcal{C} \\ (u, i) & & (u, i, o) & & (u, i, \varphi'(u, i)) \end{array}$$

Where $\varphi' : M^2 \to M$ is a homomorphism taking labels of incoming rows and returning the ~~extra~~ label on the leaving arc.

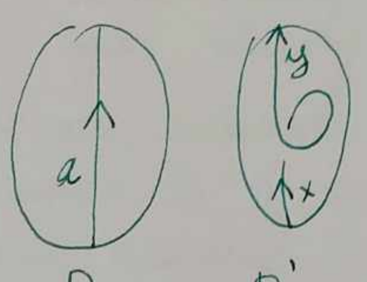
This means that c in

$$\varphi(u, i, o) = au + bi + co$$

is a unit.

REIDEMEISTER MOVES

R1.



D $\quad$ D'

$M^s \to N^x \quad M^{s+1} \to N^{x+1}$

na podmoduł $\{M^{s+1} : x = y\}$? to jakieś ilorazowe.

$$\begin{tikzcd} M^{s+1} \arrow[r, "D'\varphi"] \arrow[d, two heads] & N^{x+1} \arrow[d, two heads] \\ M^{s+1}/_{x=y} \arrow[d, "Id_{M^{s+1}} \oplus (x+y)"'] & N^x \oplus N/\varphi(M^3) \arrow[d, "Id_{N^x} + N/\varphi(M^3)"] \\ M^s \arrow[r, "D\varphi"] & N^x \end{tikzcd}$$

coker φ
czyli coker $D\varphi$ coś daje i tu?

$Id_{M^{s+1}} \oplus (x+y)$ ma zapewnić, że x i y na wersji diagramu "sumują" się do a i reszta jest nienaruszona

tu chodzi o Id na prawie całości, ale mam dodawać gdzieś dodatkowe skrzyżowanie.
Jeśli to dobrze zrobiony ✗ to i tak nic nie dodam

In terms of matrices:

$D'\varphi$

X	Y	...
b	$a+c$	...
x_1	x_2	...
⋮	⋮	⋱

$D\varphi$

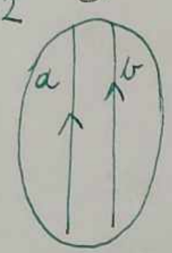
A	...
x_1+x_2	...
⋮	⋱

$\square$

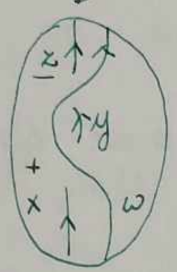
tu powinno być rozróżnianie cały czas na φ_+ i φ_-, ale mi się nie chciało pisać.

R_2

D



D'



$$\begin{array}{ccc}
M^{s+2} & \xrightarrow{D'\varphi} & N^{x+2} \\
\Big\downarrow x=2 \text{ Znowu to co wchodzi i wychodzi ma być równe} & & \Big\Downarrow d \\
M^{s+2}{}_{,\,x=2} & & N^{x} \oplus N/\varphi_+(M^3) \oplus N/\varphi_-(M^3) \quad c \\
\Big\downarrow Id_{M^{s-1}} \oplus (x+y+z) \text{ Upewniamy się, że } x \text{ i } y \text{ sumują się do } a & & \Big\downarrow Id\, N^x + d + c \\
M^s & \xrightarrow[D\varphi]{} & N^x
\end{array}$$

$\varphi_+(u, i, o) = au + bi + c\omega$

$\varphi_-(u, i, o) = \alpha u + \beta i + \gamma o$

$D'\varphi$

X	Y	Z	W	
b	c		a	...
	β	γ	α	...
x_2	0	x_1	z_1	⋱
⋮	⋮	⋮	⋮	

$D\varphi$

A	B	...
x_1+x_2	z_1	⋱
⋮	⋮	

R3

$\mathcal{G} = \pi_1(S^3 - K) = \langle S \mid R \rangle$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G^{ab} = \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underbrace{K^{ab}}_{\boxed{\text{Alexander module}}} \arrow[r] & G^{mab} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

if we interpret it as a $\mathbb{Z}[\mathbb{Z}]$-module

algorithm: $a_1, \dots, a_n$ -> generators of G.

$a_i \mapsto 1$ $\qquad a_j = a_i^{-1} a_w a_i$ ← ~~example~~ one of relations

$A_j \mapsto 0$ $\qquad A_j = a_j a_1^{-1}$

$$A_j a_1 = (A_i a_1)^{-1} (A_w a_1)(A_i a_1)$$

$$A_j a_1 = a_1^{-1} A_i^{-1} A_w a_1 A_i a_1$$

$$1 = a_1^{-1} A_j^{-1} a_1^{-1} A_i^{-1} A_w a_1 A_i a_1 =$$

$$= (a_1^{-1} A_j^{-1} a_1)(a_1^{-2} A_i^{-1} a_1^{2})(a_1^{-2} A_w a_1^{2})(a_1^{-1} A_i a_1)$$

$$0 = -tA_j - t^2 A_i + t^2 A_w + tA_i$$

(this shows that Wirtinger gives coloring. I still need the other way)

RESOLUTION OF K^{ab}

$$0 \to \mathbb{Z}[\mathbb{Z}]^a \to \mathbb{Z}[\mathbb{Z}]^b \xrightarrow[\text{matrix}]{\text{Alex}} \mathbb{Z}[\mathbb{Z}]^c \to K^{ab} \to 0$$

$b = \#$ generators $= n-1$

$c = \#$ relations $= n$

Changing to ~~rings of~~ field of fractions

$$(\mathbb{Z}[\mathbb{Z}])^{-1}(\mathbb{Z}[\mathbb{Z}])$$

the K^{ab} part disappears because K^{ab} is a torsion module. this gives acyclix complex

$$0 \to \Lambda^a \to \Lambda^b \to \Lambda^c \to 0$$

$$b = a + c$$

Hence ~~bet~~ $a = 1$ for a knot group G ALWAYS.

WE COULD COMPARE RESOLUTIONS OR SMTH.

Some knots will never have a shorter resolution, when their reduced normal form of Alex matrix or color checking matrix is not 1×1.

But the same module comes from the infinite cyclic covering.

We can calculate $H_1(\bar{X}, R)$ when R is any ring (commutative). Univeral coefficient thm but more universal than the are presented in <u>Hatcher</u>.

$$\begin{array}{ll}
\tilde{X} \text{ - universal cover} & \pi_1 = 1. \\
\big\downarrow /[G,G] & \\
\bar{X} \text{ inf. cycl.} & \pi_1 = [G,G] \\
 & H_1 = K^{ab} \\
\big\downarrow & \\
S^3 - K &
\end{array}$$

TORSION OF RESOLUTION

how does it depend on the choice of R.