# A voyage into the algebras

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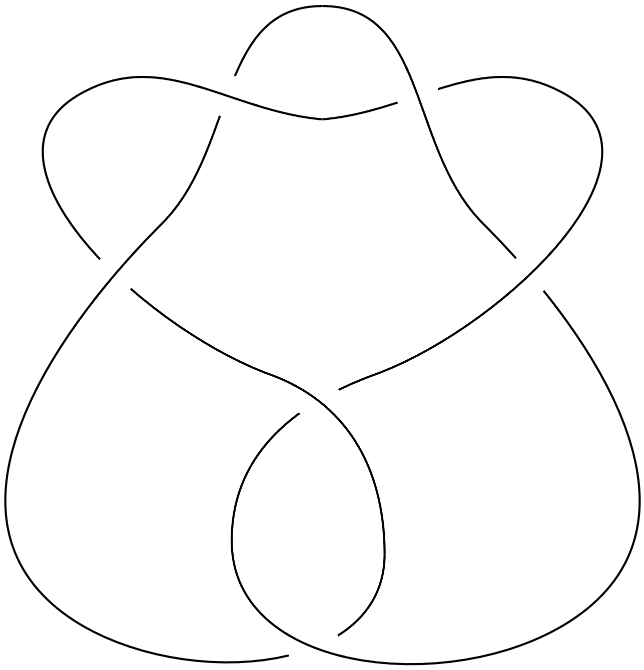


# 1 Knot coloring

Let $R$ be any commutative ring with identity, let $M$ be a module with one generator and $\phi : M^3 \to M$ be a homomorphism such that for every $m \in M$

$$\phi(m, m, m) = 0. \tag{1}$$

Notice that if $\phi(u, i, o) = au + bi + co$, then aforementioned equality demands that $(a + b + c) \in \mathrm{Ann}(M)$.

Take $K$ to be any knot with diagram $D$ with $s$ arches and $x$ crossings.

**Lemma 1.1.** *For diagrams of knots other than $0_1$, the number of segments $s$ is equal to the number of crossings $x$.*

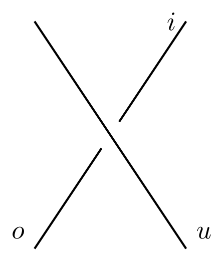
*Proof.* Every crossing has 2 arcs that go below it and every arc has two bottom ends that are created when this segment disappears below another segment. Thus

$$2 \cdot \#\text{arches} = \#\text{bottom ends} = 2 \cdot \#\text{crossings}.$$

□

**Definition 1.1.** *We say that $C \subseteq M^s$ is a* coloring module *of the diagram $D$ with elements from $M$ if it*

1. *has $s$ generators, each corresponding to one arc of the diagram,*
2. *and for every $u, i, o \in C$ that correspond to arcs meeting in one crossing, $\phi(u, i, o) = 0$.*




Notice that condition stated in eq. (1) makes it possible to color every diagram trivially, that is by assigning the same element of $M$ to every arc.

Approach to coloring taken in definition 1.1 gives a lot of information about coloring with elements of one specific module and it is rather difficult to use it for other modules. Consider the following example.

**Example 1.1.** *Take $R = \mathbb{Z}$ with $\phi(x, y, z) = 2x - y - z$ and consider the trefoil knot $3_1$. If we take $M = \mathbb{Z}$ then $K$ admits only the trivial coloring. However, if we take $M = \mathbb{Z}_3$ then there exists a non-trivial coloring like the one presented in fig. 1.*

Another approach to defining coloring of a knot diagram $D$ would be by starting with identifying arches with generators $(0, ..., 1, ..., 0)$ of $M^s$. Then, we might define a homomorphism

$$f : M^s \to M^x$$

such that arches building one crossing follow rules set by $\phi$.

**Definition 1.2.** *Module $\ker f$ is a coloring module of diagram $D$ with elements of $M$.*

**Corollary 1.2.** *Definition 1.1 and definition 1.2 are equivalent for one dimensional modules.*

*Proof.* TO DO □

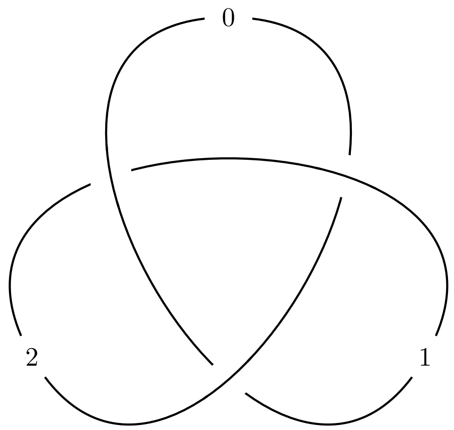



Figure 1: The trefoil knot $3_1$ does not allow for nontrivial coloring over $M = \mathbb{Z}$ but it is possible to color it with $M = \mathbb{Z}_3$.

Despite the fact that it is the kernel of $f$ that contains colorings, examining the matrix itself gives more information about diagram $D$. We might consider $f$ as a $s \times s$ matrix and if $R$ is a PID module, then we can represent this matrix in Smith's normal form.

**Proposition 1.3.** *Let $A$ be the Smith's normal form of $f$. Columns of $A$ comprised only of zeros and zero divisors contribute to the coloring module.*

*Proof.* An immediate result of corollary 1.2. $\square$

If $R$ is a Noetherian ring, then every finitely generated module is a quotient of a free module with the same number of generators. Thus, we might want to take $M$ to be a finitely generated free $R$-module rather than one dimensional $R$-module. This allows us to send $M$ to any other $R$-module with at most $\dim(M)$ generators to obtain a different coloring.

Usually, it is the irreversible elements from the diagonal of Smith's form $f$ that hint at what colorings are admisible. Consider the following example.

**Example 1.2.** *As before, take $R = \mathbb{Z}$ and $\phi(x, y, z) = 2x - y - z$. Taking $M = \mathbb{Z}$ we have $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ for trefoil knot to be a matrix*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

*with Smith's normal form*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

*Sending $M = \mathbb{Z}$ to $M = \mathbb{Z}_3$ by taking all coefficient modulo 3 we get the new Smith's normal form of $f$ to be*

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

*which informs about the nontrivial coloring that was not allowed over $\mathbb{Z}$.*

## 2 Coloring oriented diagrams

In the previous chapter we defined coloring of a diagram without an orientation. Such a diagram has only one type of crossing, while a diagram for which an orientation was chosen, two types of crossings are distinguishable in any knot diagram (see fig. 2).

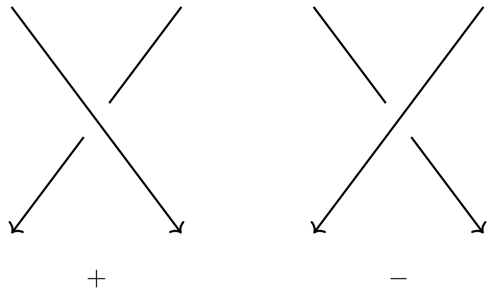



Figure 2: Two types of crossings in oriented knot diagram.

In the case of a diagram with orientation, we must chose which type of crossing is considered by $\phi$. If not explicitly mentioned, we will choose $\phi$ to determine the rules of coloring for crossing of type $+$ in fig. 2.

If $u$, $i$, $o$ are labels assigned to arches entering a $+$ type crossing that constitute a coloring, then we might write

$$0 = \phi(u, i, o) = au + bi + co.$$

Taking $c$ to be a unit, we get the following equation for the label of the arch leaving the crossing:

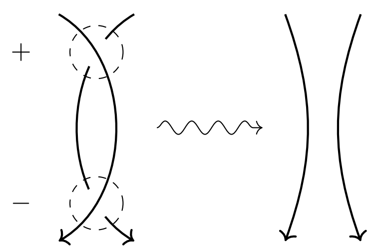
$$o = -c^{-1}au - c^{-1}bi.$$

Those assumption allow us to write an operator $M^2 \to M^2$, which takes incoming arches as input and give segments leaving the crossing as output. The matrix of the aforementioned operator takes form

$$A_+ = \begin{pmatrix} -c^{-1}a & -c^{-1}b \\ 1 & 0 \end{pmatrix}$$

It is convenient to take $c = -1$.

Allowing the following Reidemeister's move



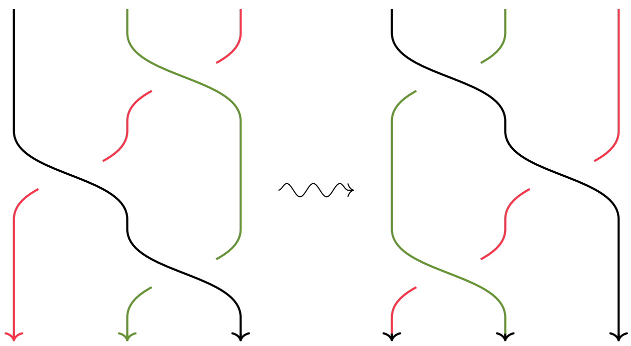

gives equality

$$A_-A_+ = Id_2,$$

where $A_+$ is the matrix of operator for $+$ type crossing and $-$ - for the $-$ type crossing. Take

$$A_- = \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix}$$

and consider another Reidemeister's move



to obtain the following relations

$$\begin{cases} b\beta = 1 \\ b\alpha - a = 0 \\ ba = ab \\ a(a+b) = a \end{cases}$$

We must assume that both $b$ and $\beta$ are units. In the most general situation, we have

$$R = \mathbb{Z}[s, t, t^{-1}]/\{s^2 + st - s\},$$

with $a$ being send to $s$ and $b$ being send to $t$. However, it can be beneficial to at first assume yet another relation:

$$a + b = 1,$$

meaning that in the ring above we have

$$s + t = 1$$

and thus $R \cong \mathbb{Z}[t, t^{-1}]$.

**Example 2.1.** *Consider knot $4_1$ with diagram $D$ as seen in fig. 4 and ring $R = M = \mathbb{Z}[t, t^{-1}]$. Take function $\phi : M^3 \to M$ to be defined as*

$$\phi(u, i, o) = (1 - t)u + ti - o$$

*for crossing as seen in fig. 3.*

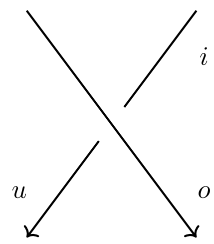



Figure 3: Crossing

*Function $f$ is then defined by matrix*

$$f = \begin{pmatrix} 1-t & t & -1 & 0 \\ t^{-1} & -1 & 0 & 1-t^{-1} \\ 0 & 1-t^{-1} & t^{-1} & -1 \\ -1 & 0 & 1-t & t \end{pmatrix}$$

*which has Smith's normal form:*

$$f' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & t^2 - 3t + 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

*Notice, that* $\det f' = t^2 - 3t + 1$*, which is the Alexander polynomial of* $4_1$*.*

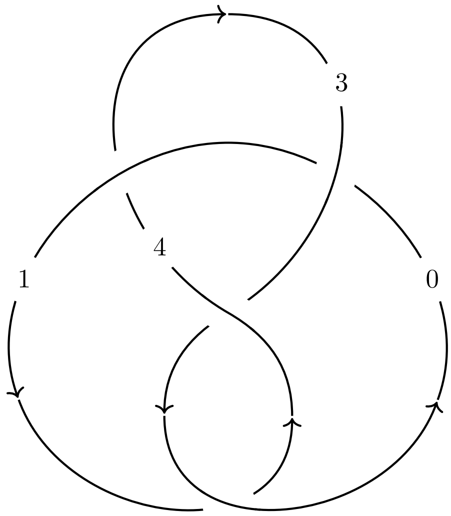



Figure 4: Coloring of knot $4_1$ with elements from $\mathbb{Z}_5$.

*Now, consider a homomorphism* $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$ *that sends* $t \mapsto -1$*. This yields a new matrix for* $f$*, with Smith's normal form:*

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

*Now, the* $\operatorname{coker} f = \mathbb{Z} \oplus \mathbb{Z}_5$ *which hints at existence of coloring using elements from* $\mathbb{Z}_5$*. One of those colorings is presented in fig. 4.*

## 2.1 Reducing normal form of a matrix

We might want to ask the question regarding the ways to distinguish knots with the same Alexander polynomial, like $6_1$ and $9_{46}$. One of the answers might be to look at the function $f : M^s \to M^x$ and the equivalence class of its Smith's normal form in ring $R = \mathbb{Z}[t, t^{-1}]$.

We notice, that the function $f$ in itself is not a knot invariant. Its matrix changes in size with changes in the diagram of the knot that we are considering. What is an invariant is its $\ker f$ and $\operatorname{coker} f$ - information about the number of colorings and what colorings might be admissible. Furthermore, when calculated over $\mathbb{Z}[t, t^{-1}]$, the kernel always is a free module of dimension 1 and all units that appear on the diagonal will not contribute to the coker. Hence, we might *consider the normal form of* $f$ *stripped of units and zeros.*

**Example 2.2.** *First, consider the knot* $6_1$ *with diagram as seen in fig. 5, ring* $R = \mathbb{Z}[t, t^{-1}]$ *and* $M = R$*. We calculate that*

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

*which agrees with the Alexander polynomial of* $6_1$*. Now, the reduced form of* $f$ *would be*

$$\begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

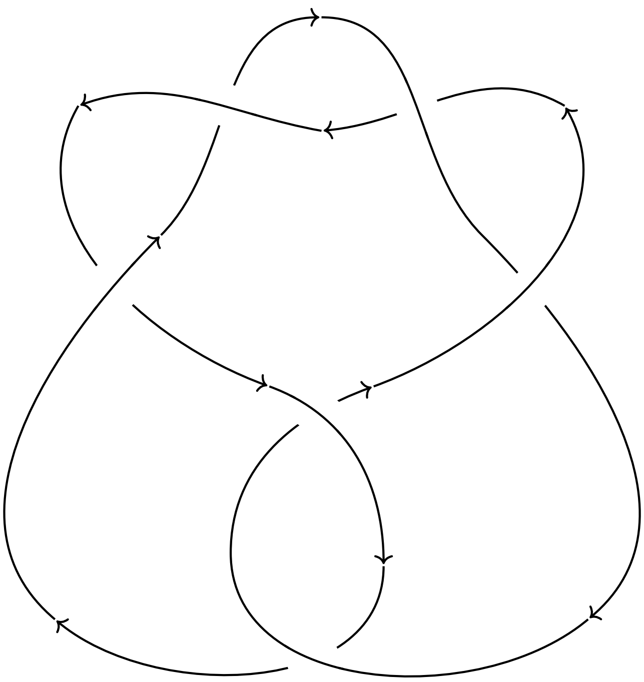


Figure 5: Diagram of knot $6_1$.

*a $1 \times 1$ matrix.*

*There is another knot with Alexander polynomial equal $-2t^{-2} + 5t^{-1} - 2$: $9_{46}$. Using diagram in fig. 6 it can be calculated that*

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t-t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2}-2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

*where*

$$\det f = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t^{-1} - 5 + 2t$$

*is also the Alexander polynomial. The reduced form of $f$ is*

$$\begin{pmatrix} 2t-t^2 & 0 \\ 0 & t^{-2}-2t^{-1} \end{pmatrix}$$

*which is significantly different than the one for $6_1$.*

# References

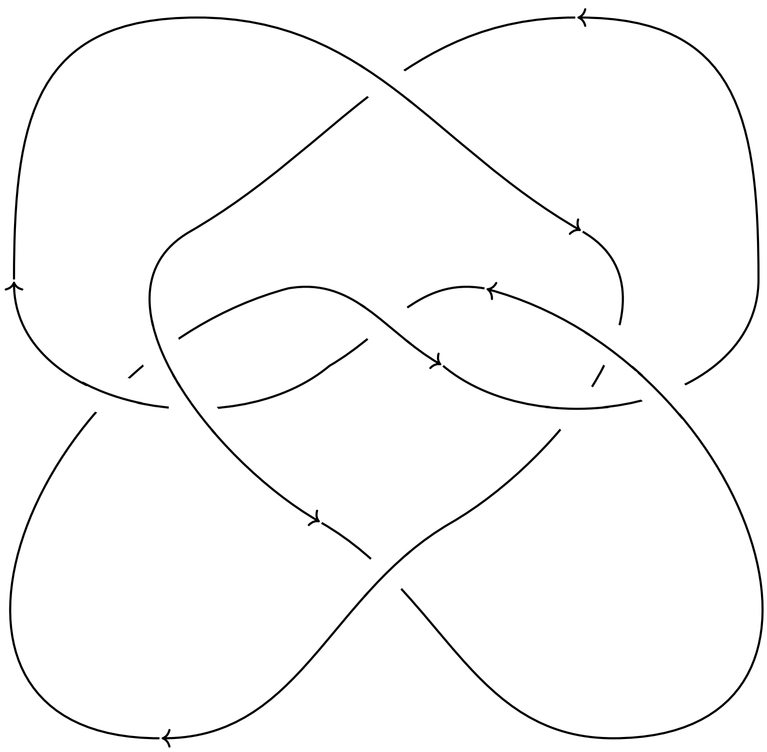


Figure 6: Diagram of knot $9_{46}$.