Knot colorings and homological invariants

(Kolorowania węzłów i niezmienniki homologiczne.)

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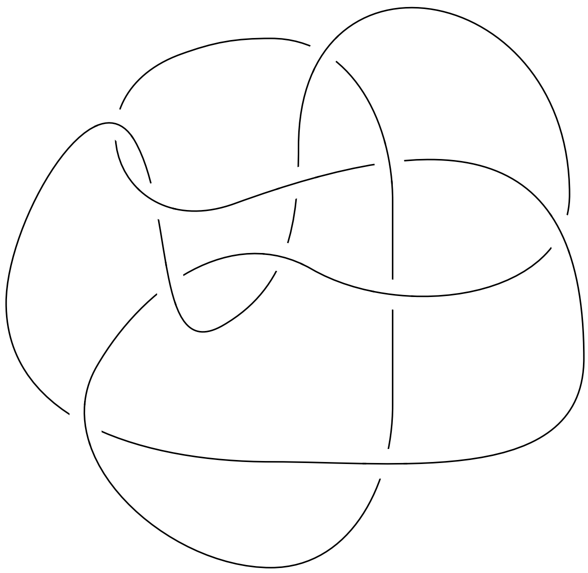

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1 Preliminaries

1.1 Knots and diagrams

In mathematical terms, a knot is a smooth embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \hookrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [3]. If two points are mapped to one by this projection, we say that a small neighbourhood of this point which looks locally like $-|-$ is a crossing.

S^1 is orientable, thus we can chose an orientation for any knot and, as a consequence, its diagram.

Intuitively, two knots K_1 and K_2 are equivalent if we can deform one into the other [6]. This translates to an equivalence of diagrams, which is generated by comparing diagrams that are exactly the same save for an interior of some disc in R^2. If inside of said disc the diagrams differ by one of **Reidemeister moves**, we say that they are equivalent. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D, 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [7].

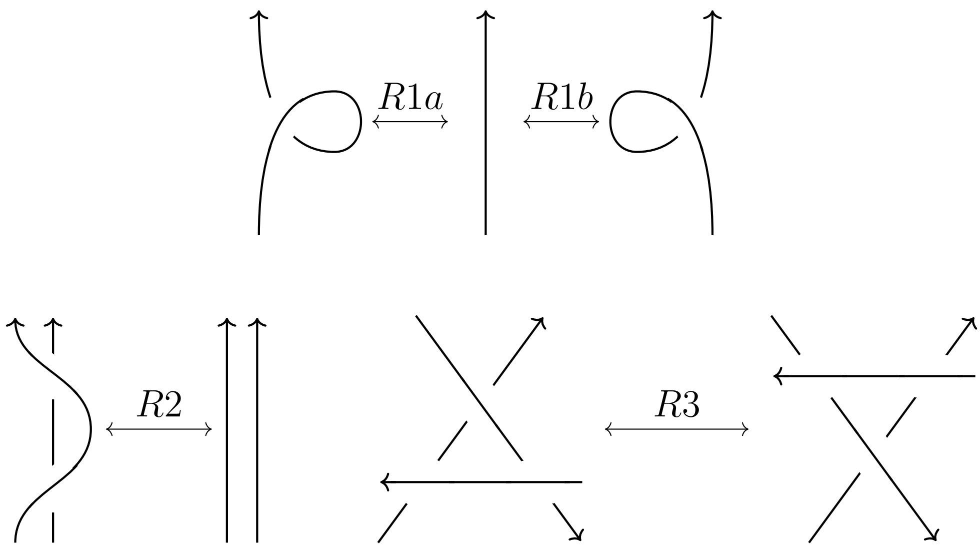


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

1.2 Knot group

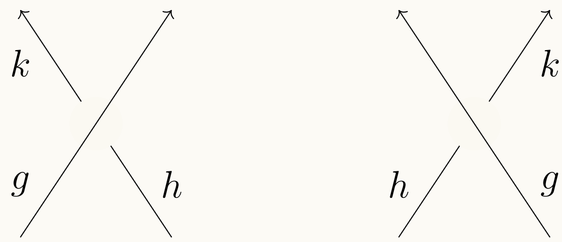
Let K be a knot and D be its oriented diagram with s segments and x crossings. A segment of a diagram is a line of the diagram between two crossings in which it is disappears under another line.

Definition 1.1 : knot group.
The fundamental group of knot complement $X = S^3 - K$ is called a **knot group**:

$$\boldsymbol{\pi_1(\mathrm{K}) := \pi_1(\mathrm{X}).}$$

Although the knot itself is always a circle S^1, the knot group has usually an interesting yet difficult structure. The most commonly used presentation of the knot group is called **the Wirtinger presentation**.

Definition 1.2 : Wirtinger presentation.
Given a diagram D of knot K with segments a_1, a_2, ..., a_s and crossings c_1, ..., c_x the knot group $\pi_1(K)$ can be represented as $\pi_1(K) = \langle G \mid R \rangle$, where G is the set of segments of D and relations R correspond to crossings in the manner described in the diagram below



$gkg^{-1} = h$ $\qquad\qquad$ $ghg^{-1} = k$

Representation $\langle G \mid R \rangle$ described above is called the **Wirtinger presentation** [4, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to $S^3 = K \oplus S^3 - K$ or noticing that every two generators are conjugate, is that the abelianization of the knot group is always $\mathbb{Z}$. This leads to a short exact sequence

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0.$$

The group $K_G = \ker(ab : G \to \mathbb{Z}) = [G, G]$ in general is not abelian nor finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization $K_G^{ab} = K_G/[K_G, K_G]$ allows a $\mathbb{Z}[\mathbb{Z}]$ module structure and thus contains obtainable information about the knot K.

Lemma 1.1.
For any group G, the commutator of its commutator K_G is a normal subgroup: $[K_G, K_G] = [[G,G],[G,G] \lhd G$.

Proof. The commutator subgroup is a characteristic subgroup, since for any automorphism $\phi : G \to G$

$$\phi(hgh^{-1}g^{-1}) = \phi(h)\phi(g)\phi(h)^{-1}\phi(g)^{-1} \in K_G = [G,G].$$

Conjugation by any element $g \in G$ is an automorphism of the commutator K_G. Thus it preserves its commutator subgroup $[K_G, K_G]$. $\square$

As a consequence, in the group $G/[K_G, K_G]$ left and right multiplication is the same. Thus, the following is an exact sequence:

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives foundation for the following definition.

Definition 1.3 : metabelianization.
The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of G.

We will return to the concept of metabelianization in section 2. For the time being, let us assign a name to K_G:

Definition 1.4 : Alexander module.
Given a group G, the abelianization of the commutator of a group G, K_G^{ab}, with $\mathbb{Z}[\mathbb{Z}]$-module structure is called the **Alexander module** of G. If G is a knot group, then it is the Alexander module of the knot K.

How the $\mathbb{Z}[\mathbb{Z}]$ module structure is obtained is described in detail in section 2.1.

1.3 Infinite cyclic covering

Let X be the complement of a knot K ($X = S^3 - K$). Take $\widetilde{X}$ to be its universal covering, meaning that it is simply connected. The fundamental group G of X acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G,G]$ is normal in G and so $\pi_1(X)/K_G$ acts on $\widetilde{X}$. Thus we might take the quotient space $\overline{X} = \widetilde{X}/[G,G]$ and call it the **infinite cyclic covering** of X. Due to this construction, the fundamental group of $\overline{X}$ is exactly

$$\pi_1(\overline{X}) = [G,G] = K_G$$

and from the perspective of homology modules, we have

$$H_1(\overline{X},\mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

Working with homology modules of an infinite cyclic cover of X instead of K_G^{ab} directly is beneficial when proving some properties of K_G^{ab}, i.e. that it is a torsion module in proposition 1.2.

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{tikzcd}
\widetilde{X} \arrow[d] & \circlearrowleft & G \\
\overline{X} \arrow[d] & \circlearrowleft & G/[G,G] \\
X = S^3 - K & &
\end{tikzcd}$$

A **Seifert surface** S of knot K is an orientable surface with boundary embedded in S^3 such that $\partial S = K$. Take a countable amount of X, with S without its boundary embedded, and label each with an element from $\mathbb{Z}$. We might now cut each of the copies of X along the Seifert surface of K and identify the $+$ side of S from the i-th copy of X with the $-$ side of S from the $(i+1)$-th copy of X. Notice that the arising space with a projection to one copy of X is an infinite cyclic cover of X.

Imagine that each copy of X inside of $\overline{X}$ is a box labeled with some integer k. The ring action of $\mathbb{Z}[\mathbb{Z}]$ on $\overline{X}$ is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to $\overline{X}$.

Proposition 1.2.
The $\mathbb{Z}[\mathbb{Z}]$-module $K^{ab} = H_1(\overline{X},\mathbb{Z})$ is a torsion module.

Proof. Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \to C_*(\overline{X})$$

$$f(x) = (1-t)x.$$

It translates to removing from a cycle in the $(i+1)$-th box a corresponding cycle in the i-th box. From this it is an immediate result that $\ker f = 0$ and that $\operatorname{coker} f = C_*(X)$: after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces a long exact homology sequence

$$\begin{array}{l} \ldots \longrightarrow H_2(X,\mathbb{Z}) \longrightarrow H_1(\overline{X},\mathbb{Z}) \xrightarrow{1-t} H_1(\overline{X},\mathbb{Z}) \longrightarrow H_1(X,\mathbb{Z}) \longrightarrow \\ \longrightarrow H_0(\overline{X},\mathbb{Z}) \longrightarrow H_0(\overline{X},\mathbb{Z}) \longrightarrow H_0(X,\mathbb{Z}) \longrightarrow 0. \end{array}$$

As was mentioned previously, the following equality holds:

$$H_1(X,\mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}\,.$$

Now, because X is homology circle, then $H_2(X,\mathbb{Z}) = 0$ (one can easily check it for themselves using Alexander duality). Both X and $\overline{X}$ are connected implying that

$$H_0(X,\mathbb{Z}) = H_0(\overline{X},\mathbb{Z}) = \mathbb{Z}\,.$$

$$\begin{array}{l} \ldots \longrightarrow 0 \longrightarrow H_1(\overline{X},\mathbb{Z}) \xrightarrow{1-t} H_1(\overline{X},\mathbb{Z}) \xrightarrow{0} \mathbb{Z} \longrightarrow \\ \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \longrightarrow 0 \end{array}$$

Rewriting the sequence above we easily get that homomorphism $1-t$ is actually an isomorphism and $H_1(\overline{X},\mathbb{Z}) \cong (1-t)H_1(\overline{X},\mathbb{Z})$, which allows us to use the Nakayama's lemma [2, Proposition 2.6] to conclude that there exists $x \in \mathbb{Z}[\mathbb{Z}]$ such that

$$xH_1(\overline{X},\mathbb{Z}) = 0.$$

$\square$

2 Resolution of the Alexander module

2.1 Construction of Alexander module

Take $G = \langle G \mid R \rangle$ to be the Wirtinger presentation of G obtained from oriented diagram D. Because K is a knot and not a link, we know that the number of segments is equal to the number of crossings, thus we can take $n = s = x$ [9].

Let $a_1, ..., a_n$ be the generators of G and $x_1, ..., x_n$ its relations. The homomorphism of abelianization of G is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, ..., n$. In order to obtain a presentation of K_G, the kernel of abelianization, we need to change the set of generators of G to

$$\{a_1, A_2 = a_2 a_1^{-1}, ..., A_n = a_n a_1^{-1}\}.$$

It is obvious that for every $i > 1$ $A_i \mapsto 0$ by abelianization of G. Thus $A_2, ..., A_n$ are some of the generators of K_G. However, for each $i = 2, ..., n$ and $k \in \mathbb{Z}$ the following is an element of K_G:

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the presentation of K_G as an abelian group is infinite with (possibly redundant) generators

$$\{b_{i,k} \ : \ i = 2, ..., n, \ k \in \mathbb{Z}\}.$$

Changing generators of G induced a change in relations. Suppose that the following relation was true in G

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of a_1 the following two relations can be obtained

$$a_1^{-1} A_k a_1 = (a_1^{-1} A_i a_1) A_j A_i^{-1}$$

$$a_1^{-3}A_ka_1^3 = (a_1^{-3}A_ia_1^3)(a_1^{-2}A_ja_1^2)(a_1^{-2}A_i^{-1}a_1^2).$$

Obviously in G both of those relations are equivalent, however in K_G they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x}b_{j,x-1}b_{i,x-1}^{-1}$$

to obtain infinitely many relations from K_G.

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

For any group H with $H^{ab} = \mathbb{Z}$ we can write a homomorphism $\mathbb{Z} \to H$ such that composition $\mathbb{Z} \to H \to \mathbb{Z}$ is identity on $\mathbb{Z}$. Thus, this sequence splits and we can write

$$G^{mab} = K_G^{ab} \rtimes \mathbb{Z}\,.$$

Hence action of $\mathbb{Z}$ can be defined on the group K_G^{ab}, with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1A_ia_1^{-1}.$$

This procedure allows K_G^{ab} to be interpreted as a $\mathbb{Z}[\mathbb{Z}]$-module.

Moreover, the group G^{mab} and $\mathbb{Z}[\mathbb{Z}]$ K_G^{ab} can be used interchangeably as knowing the action of $\mathbb{Z}$ on K_G^{ab} allows us to write the semidirect product of $\mathbb{Z}$ and K_G^{ab}.

2.2 Basic properties

Knowing the resolution of a module allows one to change said module into a matrix or even a sequence of matrices, each containing at least a portion of information about its structure.

Definition 2.1 : Alexander matrix.
The presentation matrix A_D of K_G^{ab} with Wirtinger presentation is called the **Alexander matrix** of the Alexander module K_G^{ab}.

We start writing the beginning K_G^{ab} resolution as follows:

$$\ldots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0 \tag{1}$$

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its columns will give a square matrix whose determinant is nonzero. We will prove this statement promptly after consider the Alexander module as a vector space over the field of fractions of $R = \mathbb{Z}[\mathbb{Z}]$ [2, Chapter 3].

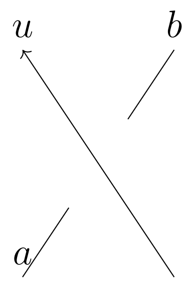
In proposition 1.2 it was shown that the Alexander module is torsion. Thus, as a vector space $K_G^{ab} \otimes_R R^{-1}R = 0$ it is trivial. Hence, the sequence in (1) translates to the following sequence of $R^{-1}R$ modules

$$\ldots \longrightarrow R^n \otimes_R R^{-1}R \xrightarrow{A_D^V} R^{n-1} \otimes_R R^{-1}R \longrightarrow 0 \tag{2}$$

As there exists an inclusion $R \hookrightarrow R^{-1}R$, every matrix with terms in R can be treated as a matrix with terms in $R^{-1}R$. Naturally, $A_D^V = A_D \otimes Id_{R^{-1}R}$ is just matrix A_D (with terms in R) with adjoined 1×1 matrix with just identity of $R^{-1}R$. Thus, we can easily translate most properties of A_D^V to properties of A_D, i.e. its determinant and surjectivity.

Proposition 2.1.
Let A'_D be the Alexander matrix A_D with one of its rows struck out. Then $\det(A'_D) \neq 0$.

Proof. We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator x is send to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module K^{ab} we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1-t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which x is b or c are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t-1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and ± 1. Thus, the sum is not equal to zero. There are two of such rows as the segment b has to be the "out" and "in" segment of some crossing. In other words, segment b has to have a start and end in some crossings.

The reasoning above is true for matrix A_D^V from (2). We make the switch to vector space to use the connection between the rank of matrix and the dimension of its image.

Let S_i be the column of the Alexander matrix corresponding to the segment labeled i. The sum $\sum_{i \leq n-1} S_i$ is a vector with two nonzero terms. Take S_j and S_k to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both S_j and S_k by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j,k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i : i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D^V(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. $\square$

The proposition 2.1 implies that image of A_D^V has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

Theorem 2.2.
The determinant $\det(A'_D)$ up to multiplication by a unit is independent of the choice of the diagram D.

Proof. A proof using Dehn presentation is provided in [1], while a proof of more general case is provided in [8] $\square$

Definition 2.2 : Alexander polynomial.
The **Alexander polynomial** of a knot K is the determinant of any maximal minor of the Alexander matrix A_D.

The Alexander polynomial is a knot invariant as a consequence of theorem 2.2 and proposition 2.1

Proposition 2.3.
Let G be a knot group of K and $F = R^{-1}R$ the field of fraction of ring R. Then K_G^{ab} always has a resolution

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

where n is the number of crossings of the chosen diagram D of knot K and $M \otimes_R F \cong F$.

Proof. We start by saying that $R \otimes_R F \cong F$ because R is a free module over R [2, Proposition 2.14].

Proposition 2.1 implies that (2) can be extended into the following exact sequence of vector spaces:

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R R^{-1}R = 0 \longrightarrow 0$$

as we proved that $\dim(\operatorname{im} A_D^V) = n-1 \implies \dim(\ker A_D^V) = 1$.

The ring of fractions is flat [2, Chapter 3] at the same time we only consider R-modules treated as vector spaces in this proposition. Thus, we have the following exact sequence

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

with $M \otimes_R F \cong F$. □

Notice, that sequence

$$\star:\ 0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow 0$$

is not acyclic, however it allows us to once again define the Alexander module, this time as $H_1(\star)$.

2.3 A homological roots of diagram colorings

Thus far a resolution of the Alexander module K_G^{ab} provided a matrix and with it a polynomial invariant of knots. In this short section we will explain the connection between Alexander module and knot colorings, which will be the focus of the subsequent section.

Take M to be a finitely generated $R = \mathbb{Z}[\mathbb{Z}]$-module. The functor $\mathrm{Hom}(-, M^n)$ is left exact therefore applied to the resolution of the Alexander module generates the following sequence

$$0 \to \mathrm{Hom}(R, M) \to \mathrm{Hom}(R^n, M) \xrightarrow{\mathrm{Hom}(A_D, M)} \mathrm{Hom}(R^{n-1}, M) \to \mathrm{Hom}(K_G^{ab}, M^n)$$

The diagram D taken as the starting point for the construction of K_G^{ab} had $n = x$ crossings and $n = s$ segments. The module K_G^{ab} was presented using $(n-1)$ generators, corresponding to all but one segments of the diagram. If we allow for propagation of values, then $\mathrm{Hom}(R^{n-1}, M)$ can be interpreted as assigning values from M to $(n-1)$ segments in diagram D, with the last segment colored based on the remaining part of the diagram.

The arrow $\mathrm{Hom}(R^{n-1}, M) \to \mathrm{Hom}(K_G^{ab}, M)$ ensures that the structure of K is taken into account during this assigment. Its kernel is be equal to $\mathrm{im}\,\mathrm{Hom}(A_D, M)$ and thus remembers which segments contributed to which crossings.

The above remark points at a similarity between the concept of diagram colorings, elaborated in the following section, and the more topological invariant which is the Alexander module

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