A voyage into the algebras

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1 What does $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4$ have in common? (problem 1)

Consider the group $\mathbb{Z}[[F]]$, where $[F]$ is the equivalence class of all finite Abelian groups isomorphic to F. Let $\heartsuit$ be the equivalence closure of relation defined as follows: if F_1, F_2, F_3 are Abelian groups the $[F_2] = [F_1] + [F_3]$ if there exists an exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

Lemma 1.1. *If F, F' are two Abelian groups of order n, then they represent the same equivalence class of relation $\heartsuit$ i.e. $[F]_\heartsuit = [F']_\heartsuit$.*

Example 1.1. *Before we prove lemma 1.1, let us examine an example. We will show that $[\mathbb{Z}_4] = [\mathbb{Z}_2 \oplus \mathbb{Z}_2]$. Consider the following exact sequence*

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \longrightarrow 0$$

which shows that $[\mathbb{Z}_4] = [\mathbb{Z}_2] + [\mathbb{Z}_2]$. On the other hand, the next sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{i_1} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0$$

which is also exact, yields $[\mathbb{Z}_2 \oplus \mathbb{Z}_2] = [\mathbb{Z}_2] + [\mathbb{Z}_2]$.

This shows that every Abelian group of order 4 is in the same equivalence class of relation given by exact sequences. We will show that all Abelian groups of the same order will belong to one equivalence class.

Proof
Every finite Abelian group is isomorphic to a direct product of its p-subgroupsDODAĆ CYTAT. Furthermore, any p-group of order p^k is isomorphic to $\mathbb{Z}_{p^k}$. We can start by examining what elements belong to equivalence class $[\mathbb{Z}_{p^k}]$.

We will start by showing that if $k = n + l,\ k, n, l \in \mathbb{N}$, then $[\mathbb{Z}_{p^k}] = [\mathbb{Z}_{p^n}] + [\mathbb{Z}_{p^l}]$. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^l} \longrightarrow 0$$

We know that $\mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n}$ is a cyclic group generated by $1 + Z_{p^n}$. Furthermore, we know that $|\,\mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n}\,| = p^l$ and thus $\mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^l}$.

Now, we will show, using induction on N, that for any $n \in \mathbb{N}$ such that $n = \prod_{i=1}^N p_i^{k_i}$, where $k_i \in \mathbb{N}$ and p_i is a prime number, we have

$$[\mathbb{Z}_n] = \sum_{i=1}^N [\mathbb{Z}_{p_i^{k_i}}] = \sum_{i=1}^N k_i \cdot [\mathbb{Z}_{p_i}] \quad (\star)$$

1. $N = 1$

   From the fact above we know that $[\mathbb{Z}_{p^{k+1}}] = [\mathbb{Z}_{p^k}] + [\mathbb{Z}_p]$ and applying the same reasoning to $\mathbb{Z}_{p^k}$ we obtain $[\mathbb{Z}_{p^k}] = k \cdot [\mathbb{Z}_p]$.

2. $N - 1 \implies N$

   We will start from the right side of the equality $(\star)$ and from inductive hypothesis we know that

   $$\sum_{i=1}^N k_i[\mathbb{Z}_{p_i}] = k_N[\mathbb{Z}_{p_N}] + \sum_{i=1}^{N-1} k_i[\mathbb{Z}_{p_i}] = [\mathbb{Z}_{p_N^{k_N}}] + [\mathbb{Z}_l]$$

   where $l = \prod_{i=1}^{N-1} p_i^{k_i}$. Consider the following sequence

   $$0 \longrightarrow \mathbb{Z}_l \longrightarrow \mathbb{Z}_n \longrightarrow \mathbb{Z}_{p_N^{k_N}} \longrightarrow 0$$

   its exactness follows from the fact that $\mathbb{Z}_n \,/\, \mathbb{Z}_l$ is a cyclic group of order $n/l = p_N^{k_N}$ and thus there exists an isomorphism

   $$\mathbb{Z}_{p_N^{k_N}} \cong \mathbb{Z}_n \,/\, \mathbb{Z}_l\,.$$

As stated before, any Abelian group of order N is isomorphic to a direct product of its p-subgroups, hence the following equality is immediate from $(\star)$:

$$\sum_{i=1}^{N} k_i[\mathbb{Z}_{p_i}] = [\mathbb{Z}_n] = \left[\bigoplus_{i=1}^{N} \mathbb{Z}_{{p_i}^{k_i}}\right]$$

♠

From this follows that every Abelian group of order n, either being a cyclic group itself or a direct sum of cyclic groups, is in one equivalence class. Hence, elements of group

$$\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\}$$

can be expressed as finite sums of equivalence classes represented by p-groups:

$$\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\} = \{\sum_{i\leq n} k_i[\mathbb{Z}_{p_i}] \ : \ p_i \text{ are prime}, n, k_i \in \mathbb{N}\}$$

2 A venture into the worlds of other rings with Euclidean algorithm (problems 2 and 3)

We will start off by showing a result very useful to showing that a sequence is exact.

Lemma 2.1. *Let I, J, K be ideals in a PID ring R. If $I = JK$ and $I \subseteq K$, then $R/K \cong (R/I)/(R/J)$.*

Proof
Consider the following sequence

$$0 \longrightarrow R/J \xrightarrow{f} R/I \xrightarrow{g} R/K \longrightarrow 0$$

We will show that it is exact sequence.

Define $f : R/J \to R/I$ as

$$f(x + J) = xK + JK = xK + I$$

which gives us $\operatorname{im} f = K/I$. Of course, f is a monomorphism, because

$$f(x + J) = xK + JK = yK + JK = f(y + J) \implies (x - y)K \in JK \implies x - y \in J \implies x + J = y + J$$

Now, let $g : R/I \to R/K$ be a quotient mapping

$$g(x + I) = x + I + K = x + K$$

and so $\ker g = K/I$.

We showed that $\ker g = \operatorname{im} f$ and so the sequence above is exact and $R/K \cong (R/I)/(R/J)$.

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Theorem 2.2. *Let R be a PID ring and M be an R-module. Then there exist $p_1, ..., p_n$ elements in R such that*

$$M \cong R/(p_1) \oplus R/(p_2) \oplus ...R/(p_n).$$

We will define the ideal $(p_1....p_n)$ that stems from the decomposition above as the order of module M and denote it as $\operatorname{ord}(M)$.

Order of a module M is well defined, that is if $M' \subseteq M$ is a submodule, then we have

$$\operatorname{ord}(M) = \operatorname{ord}(M') \operatorname{ord}(M/M') \ [1].$$

2.1 When irreducible polynomials become prime numbers

Consider a field $\mathfrak{K}$ and the ring of polynomials with coefficients in $\mathfrak{K}$, $\mathfrak{K}[x]$. Obviously, the aforementioned ring is a principial ideal domain. We want to consider group $\mathbb{Z}[[M]]$, where $[M]$ is the equivalence class of all finitely generated torsion modules isomorphic to M and apply relation $\heartsuit$ to modules over the ring $\mathfrak{K}[x]$. That is, let $\heartsuit$ be the equivalence closure of relation: $[M_2] = [M_1] + [M_3] \iff$ there exists an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

Example 2.1. *As before, we will start by considering $\mathbb{Q}[x]$-modules*

$$M = \mathbb{Q}[x]/(x^3+1)$$

$$N = \mathbb{Q}[x]/(x+1) \oplus \mathbb{Q}[x]/(x^2-x+1)$$

Notice, that $x^2 - x + 1$ and $x + 1$ are both irreducible over $\mathbb{Q}$ and that

$$x^3 + 1 = (x+1)(x^2-x+1).$$

We will show that

$$\big[\mathbb{Q}[x]/(x^3+1)\big] = [\mathbb{Q}[x]/(x+1)] + \big[\mathbb{Q}[x]/(x^2-x+1)\big] = \big[\mathbb{Q}[x]/(x+1) \oplus \mathbb{Q}[x]/(x^2-x+1)\big]$$

Exactness of sequence

$$0 \longrightarrow \mathbb{Q}[x]/(x+1) \hookrightarrow \mathbb{Q}[x]/(x+1) \oplus \mathbb{Q}[x]/(x^2-x+1) \twoheadrightarrow \mathbb{Q}[x]/(x^2-x+1) \longrightarrow 0$$

is rather trivial: the left arrow is embedding of a summand to a direct sum and the right arrow is projection from direct sum.

The second sequence,

$$0 \longrightarrow \mathbb{Q}[x]/(x+1) \longrightarrow \mathbb{Q}[x]/(x^3+1) \longrightarrow \mathbb{Q}[x]/(x^2-x+1) \longrightarrow 0$$

is exact because $(x^3+1) \subseteq (x+1)$ and $(x+1)(x^2-x+1) = (x^3+1)$ allows us to use lemma 2.1 to show that there exists an isomorphism

$$\mathbb{Q}[x]/(x^2-x+1) \cong (\mathbb{Q}[x]/(x^3+1))/(\mathbb{Q}[x]/(x+1)).$$

In the previous section we showed that all Abelian groups of the same order (understood then as the number of elements) are in the same equivalence class. Now, we will show that the same is true for modules over a ring of polynomials, with order understood as a polynomial.

Lemma 2.3. *Let $\mathfrak{K}$ be a field and consider $\mathfrak{K}[x]$-module M with $\operatorname{ord}(M) = (p_1 \dots p_n)$, p_i are irreducible polynomials. Then*

$$[M]_\heartsuit = \sum_{i=1}^{n} [R/(p_i)]_\heartsuit$$

Proof

We will use induction on the number of polynomials n in decomposition of $\operatorname{ord}(M)$.

1. $n = 1$

   This of course means that $M \cong R/(p_1)$ and so $[M]_\heartsuit = [R/(p_1)]_\heartsuit$.

2. $n \implies n+1$

   First, notice that $R/(p_{n+1}) \subseteq M$ and so

   $$(p_1 \dots p_n p_{n+1}) = \operatorname{ord}(M) = (p_{n+1}) \operatorname{ord}(M/(p_{n+1})) = (p_{n+1}) \operatorname{ord}(M/(p_{n+1}))$$

   implies that $\operatorname{ord}(M/(p_{n+1})) = (p_1 \dots p_n)$. From inductive hypothesis we know that

   $$[M/(p_{n+1})]_\heartsuit = \sum_{i=1}^{n} [R/(p_i)]_\heartsuit.$$

   We know that sequence

$$0 \longrightarrow R/(p_{n+1}) \longrightarrow M \longrightarrow M/(p_{n+1}) \longrightarrow 0$$

is exact, hence

$$[M]_\heartsuit = [R/(p_{n+1})]_\heartsuit + [M/(p_{n+1})]_\heartsuit = \sum_{i=1}^{n+1}[R/(p_i)]_\heartsuit.$$

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2.2 Vector space as a $\mathfrak{K}[x]$-module

DO DOPRACOWANIA, BO NAPISAŁAM JAKOŚ KRZYWO, może coś o Jordan basis?

Let V be a vector space over a field $\mathfrak{K}$. Take $f : V \to V$ to be any endomorphism of V. We can now consider V as a $\mathfrak{K}[x]$-module, with multiplication of $v \in V$ by $w(x) = \sum \alpha_i x^i \in \mathfrak{K}[x]$ defined as:

$$w \cdot v = (w(f))(v) = (\sum \alpha_i f^i)(v) = \sum \alpha_i \cdot f^i(v),$$

where $f^i = \underbrace{f \circ ... \circ f}_{\text{i times}}$.

For a fixed vector space V (for simplicity let $V = \mathfrak{K}^n$) consider all the $\mathfrak{K}[x]$-modules created by choosing different endomorphisms $f : V \to V$, where endomorphisms that are represented by similar matrices will be treated as one object.

If an endomorphism $f : \mathfrak{K}^n \to \mathfrak{K}^n$ is represented by matrix A, then we know that there exists a polynomial $w \in \mathfrak{K}[x]$ such that for every $v \in \mathfrak{K}^n$ $w \cdot v = 0$, because any collection of at least $n+1$ vectors from $\mathfrak{K}^n$ cannot be linearly independent. Thus we know that $\mathfrak{K}^n$ is a torsion module over $\mathfrak{K}$.

The characteristic polynomial of A, $\chi_A(x) = \sum \alpha_i x^i$ is defined as $\det(Ix - A)$, where I is the identity matrix. We know that $\chi_A(x) = \prod(x - \lambda_i)$, where λ_i are all the eigenvalues of A. There are at most n eigenvectors of A and they are all killed by $\chi_A(x)$. This means, that

$$(\mathfrak{K}^n; f) \cong R/(\chi_A)$$

Hence, $[(\mathfrak{K}^n; f)]_\heartsuit = [(\mathfrak{K}^n; g)]_\heartsuit \iff \chi_f(x) = \chi_g(x)$ and every vector space can be expressed as

$$[(\mathfrak{K}^n; f)]_\heartsuit = \sum_{i=1}^{n}[R/(x - \lambda_i)]_\heartsuit$$

where λ_i are the eigenvalues of f.

3 Ta ambitna część dyskursu

Theorem 3.1. *Any torsion $\mathbb{Z}[x]$-module M with n generators $e_1, ..., e_n$ is isomorphic to direct product*

$$M \cong \mathbb{Z}[x]/(\mathrm{Ann}(e_1)) \oplus \mathbb{Z}[x]/(\mathrm{Ann}(e_2)) \oplus ... \oplus \mathbb{Z}[x]/(\mathrm{Ann}(e_n))$$

Proof
We will use induction on the number of generators n.

1. $n = 1$ then M is a cyclic module and the case is rather trivial.
2. $n \implies n+1$

   Let M' be a submodule of M generated by $e_1, ..., e_n$ and let M'' be a submodule of M generated by e_{n+1}. Then $M \cong M' \oplus M''$. From inductive hypothesis we have that

   $$M' \cong \mathbb{Z}[x]/(\mathrm{Ann}(e_1)) \oplus ... \oplus \mathbb{Z}[x]/(\mathrm{Ann}(e_n))$$

   and

   $$M'' \cong \mathbb{Z}[x]/(\mathrm{Ann}(e_{n+1}))$$

hence

$$M \cong M' \oplus M'' \cong \mathbb{Z}[x]/(\mathrm{Ann}(e_1)) \oplus \ldots \mathbb{Z}[x]/(\mathrm{Ann}(e_n)) \oplus \mathbb{Z}[x]/(\mathrm{Ann}(e_{n+1})).$$

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Lemma 3.2. *Summands that appear in decomposition of a finitely generated torsion module presented in theorem 3.1 are uniquely determined.*

Proof
Firstly, notice that a torsion finitely generated module is Artinian and Noetherian and thus has a well defined composition series. From Jordan-Hölder theorem it follows that any two such composition series have the same number of elements that are isomorphic in pairs.

Let

$$0 \subset M_1 \subset M_2 \subset \ldots M_k$$

and

$$0 \subset M_1' \subset M_2' \subset \ldots \subset M_k'$$

be two composition series of finitely generated torsion module M with n coordinates. Theorem 3.1 states that both M_k and M_k' must have n summands. Furthermore, we know that both M_1 and M_1' must have just one coordinate. From this we can deduce that subsequent modules of each composition series must gradually build up to having n summands (coordinates). Hence, if $M_i \subset M_{i+1}$ is such that M_i has m summands while M_{i+1} has $m+1$ summands, then

$$M_i \cong \mathbb{Z}[x]/(\mathrm{Ann}(e_{i_1})) \oplus \ldots \mathbb{Z}[x]/(\mathrm{Ann}(e_{i_m}))$$

while

$$M_{i+1} \cong \mathbb{Z}[x]/(\mathrm{Ann}(e_{i_1})) \oplus \ldots \mathbb{Z}[x]/(\mathrm{Ann}(e_{i_m})) \oplus N$$

where $N \neq 0$. And if $M_j' \subset M_{j+1}'$ are analogous steps in the second composition series, that is if M_j' also has m summands while M_{j+1}' has $m+1$ of them, then from Jordan-Hölder theorem we can conclude that $M_j' \cong M_i$. Thus, the decomposition has summands unique up to isomorphism.

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Example 3.1. *Consider module $M = \mathbb{Z}_2[x] \oplus \mathbb{Z}$, where $\mathbb{Z}$ we treat as a $\mathbb{Z}[x]$-module. That is, multiplying by x is the same as multiplying by 0 in $\mathbb{Z}$. Then the generators are $1 + (2)$ and $1 + (x)$ and the annihilator of the whole module is $\mathrm{Ann}(M) = (2x)$. At the same time, we have*

$$\mathbb{Z}_2[x] \oplus \mathbb{Z} \cong (\mathbb{Z}[x]/(2)) \oplus (\mathbb{Z}[x]/(x))$$

where (2) is the annihilator of $\mathbb{Z}_2[x]$ and (x) annihilates $\mathbb{Z}[x]/(x)$.

$$[\mathbb{Z}_2[x] \oplus \mathbb{Z}]_\heartsuit = [\mathbb{Z}[x]/(2x)]_\heartsuit$$

$$0 \longrightarrow \mathbb{Z}[x]/(2) \xrightarrow{\cdot x} \mathbb{Z}[x]/(2x) \xrightarrow{x=0} \mathbb{Z}[x]/(x) \longrightarrow 0$$

$$\mathbb{Z}[x]/(2, x) \subseteq \mathbb{Z}[x]/(2x)$$

Warunek konieczny na bycie w tej samej klasie równoważności to równość anihilatorów.

Jeśli w theorem 3.1 ograniczymy się do cyklicznych prostych modułów torsyjnych i ich sum prostych to warunek konieczny jest też warunkiem dostatecznym.

References

[1] John. W. Milnor. Infinite cyclic coverings. 1967.