A voyage into the algebras

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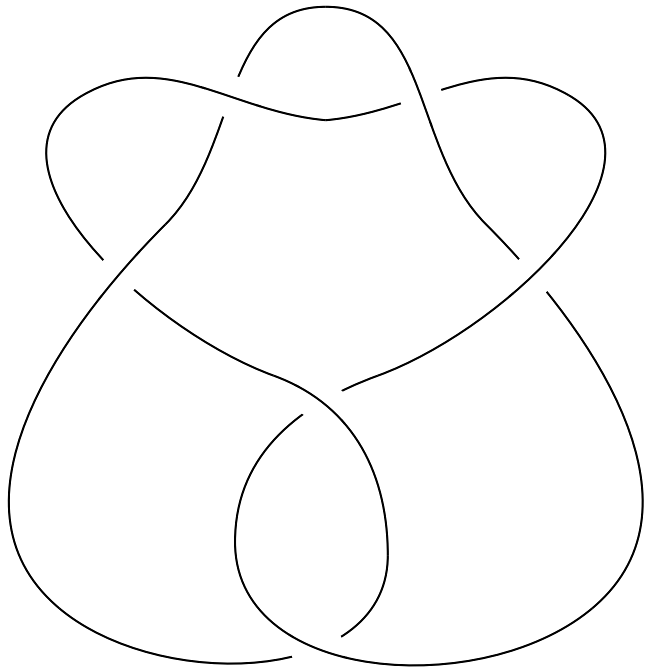


Plan działania

1. Relacje na macierzach -> Reidemeister
   (a) propagation rule - funkcja ϕ, potencjalnie dla uproszczenia będziemy pisać ϕ_+ i ϕ_- na reguły kolorowania dwóch typów skrzyżowania
   (b) Diagram, s łuczków i x skrzyżowań - macierz która bardzo nie jest niezmiennikiem węzła, a zależy od diagramu.
   (c) Wprowadzamy relację na zorientowanych diagramach (chociaż w sumie chyba nie potrzebuję orientacji, ale na takich pracuję więc elo)
2. Smith normal form
3. Skein relations
4. moduł Alexandera 6_1 i 946, czy są różne
5. rezolwenty
6. zmiana pidów

1 What is a knot coloring

Let K be a knot and D be its oriented diagram with s segments and x crossings. In such diagrams we can see two different crossing types as seen in fig. 1.

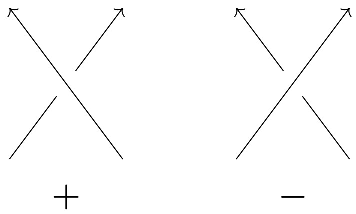


Figure 1: Two types of crossing in oriented diagram.

Let R be a commutative ring, typically $\mathbb{Z}[\mathbb{Z}]$, and take M, N to be two R-modules. Consider two module homomorphisms $\phi_+ : M^3 \to N$ and $\phi_- : M^3 \to N$ such that

$$(\forall\, x \in M)\ \phi_\pm(x, x, x) = 0.$$

This homomorphism will be used to determine whether or not a labelling of knot arcs constitutes a coloring or not.

Definition 1.1 (diagram coloring). *Let $x_1, ..., x_s \in M$ be labels of arcs in diagram D. We will say that $(x_1, ..., x_s) \in M^s$ is a **coloring** if for every crossing $\pm$ in D consisting of arcs u, i, o the following relation is satisfied*

$$\phi_\pm(u, i, o) = 0.$$

JAK WUTLUMACZYC, ZE NAPRAWDE TO WYSTARCZY JEDNA FUNKCJA, ALE TAK BEDZIE MI LATWIEJ W ZYCIU? BO JAK NARYSUJĘ DIAGRAM Z DWOMA SKRZYŻOWANIAMI JEDNO + A DRUGIE - I POŁĄCZĘ JAK PRZY WARKOCZE -> LINKI TO DOSTAJĘ LINKA Z 2 KOMPONENTAMI :V

Every crossing in the diagram D of knot K yields x relations $\phi_\pm(u, i, o) = 0$ which we might treat as linear equations of form

$$\phi_\pm(u, i, o) = au + bi + co = 0,$$

where the s arcs act as variables and $a + b + c = 0 \in \mathrm{Hom}(M, N)$ (when $M = N$ then $a + b + c \in \mathrm{Ann}(M)$).

Definition 1.2. *Matrix* $D\phi : M^s \to N^x$ *of coefficients taken from relations* $\phi_{\pm}(u,i,o) = 0$ *will be called a* ***color checking matrix****.*

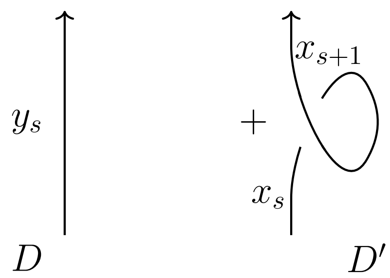
The color checking matrix in itself is obviously not a knot invariant. However, we might define a equivalence relation on the set of all matrices $M^m \to N^n$ such that all the matrices which come from the same knot fall into the same equivalence class.

2 Relation on color checking matrices

In order to ensure that all matrices that stem from the same knot are considered in one equivalence class we must look at how Reidemeister moves change the matrix.

In this section we will always assume that diagram D has s segments and x crossings. Furthermore, we will always put crossings and segments that are affected by the Reidemeister move as the last columns and rows of the matrix.

R1a



In the case of this Reidemeister move we have

$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+1} \to N^{x+1}.$$

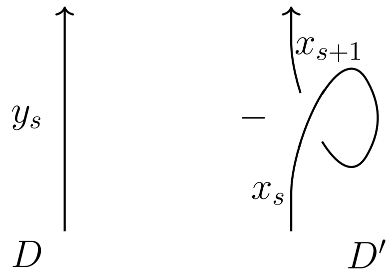
Only two arcs have changed thus

$$D\phi \restriction M^{s-1} = D'\phi \restriction M^{s-1}.$$

Furthermore, we want for any $x_s, x_{s+1} \in M$

$$\pi_{x+1}[D'\phi(0, ..., x_s, s_{s+1})] = \phi_+(x_{s+1}, x_s, x_{s+1}),$$

R1b



where π_{x+1} is projection onto the last coordinate, and

$$(D\phi(0, ..., x_s), 0) = D'\phi(0, ..., x_s, x_s).$$

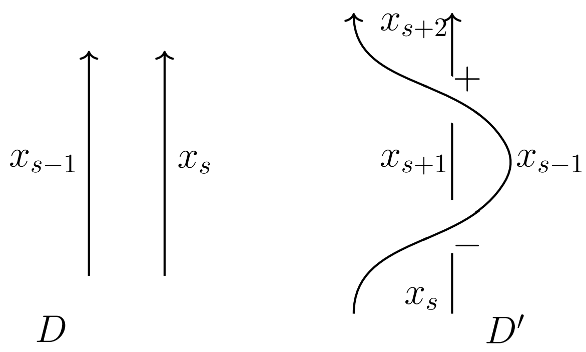
This Reidemeister move on oriented diagram is necessary in defining equivalent oriented knot diagrams but the matrix relation is the one in **R1a** with ϕ_+ changed to ϕ_-:

$$D\phi \sim D'\phi$$

if and only if

$$\begin{aligned} &D\phi \restriction M^{s-1} = D'\phi \restriction M^{s-1} \wedge \\ &\wedge \pi_{x+1}[D'\phi(0, ..., x_s, s_{s+1})] = \phi_-(x_{s+1}, x_s, x_{s+1}) \wedge \\ &\wedge (D\phi(0, ..., x_s), 0) = D'\phi(0, ..., x_s, x_s). \end{aligned}$$

R2

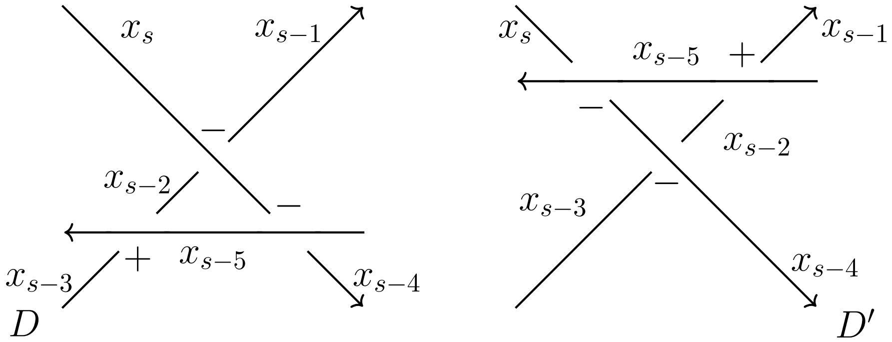


$$D\phi \sim D'\phi$$

if and only if

$$\begin{aligned}
&D\phi \restriction M^{s-2} = D'\phi \restriction M^{s-2} \wedge \\
&\wedge (\forall x \in M) D'\phi(0, ..., x, y, y, y) = \\
&\quad = (D\phi(0, ..., x, y), \phi_-(x, y, y), \phi_+(x, y, y))
\end{aligned}$$

R3



$$D\phi \sim D'\phi$$

if and only if

$$\begin{aligned}
&D\phi \restriction M^{s-5} = D'\phi \restriction M^{s-5} \wedge \\
&\wedge (\forall\, x, y, z \in M)\pi_{x-3}[D\phi(0, ..., x, y, z)] = \\
&\quad = \pi_{x-3}[D'\phi(0, ..., z, x, y, 0, 0)] \wedge \\
&\wedge D\phi(0, ..., z, y, 0, 0, 0, x) = D'\phi(0, ..., z, y, 0, 0, 0, x) \wedge \\
&\wedge D\phi(0, ..., z, 0, y, x, 0, 0) = D'\phi(0, ..., z, 0, 0, y, x, 0)
\end{aligned}$$

Theorem 2.1. *Let K be a knot and D its oriented diagram. Define*

$$K\phi := [D\phi]$$

to be the equivalence class of the matrix $D\phi$. Then, $K\phi$ is a knot invariant.

References