A voyage into the algebras

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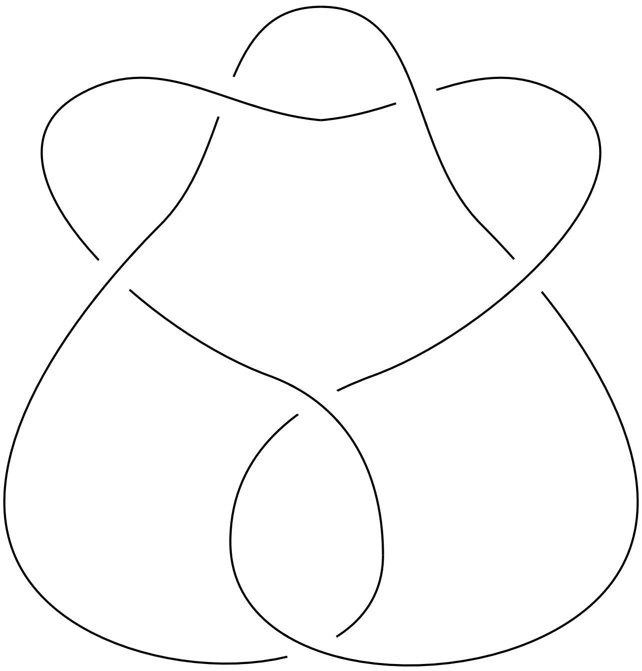


1 Knot coloring

Let R be any commutative ring with identity, let M be a module with one generator and $\phi : M^3 \to M$ be a homomorphism such that for every $m \in M$

$$\phi(m, m, m) = 0. \tag{1}$$

Notice that if $\phi(u, i, o) = au + bi + co$, then aforementioned equality demands that $(a + b + c) \in \mathrm{Ann}(M)$.

Take K to be any knot with diagram D with s arches and x crossings.

Lemma 1.1. *For diagrams of knots other than 0_1, the number of segments s is equal to the number of crossings x.*

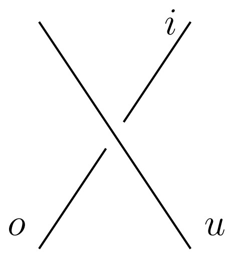
Proof. Every crossing has 2 arcs that go below it and every arc has two bottom ends that are created when this segment disappears below another segment. Thus

$$2 \cdot \#\text{arches} = \#\text{bottom ends} = 2 \cdot \#\text{crossings}.$$

$\square$

Definition 1.1. *We say that $C \subseteq M^s$ is a* coloring module *of the diagram D with elements from M if it*

1. *has s generators, each corresponding to one arc of the diagram,*
2. *and for every $u, i, o \in C$ that correspond to arcs meeting in one crossing, $\phi(u, i, o) = 0$.*



Notice that condition stated in eq. (1) makes it possible to color every diagram trivially, that is by assigning the same element of M to every arc of D.

Approach to coloring taken in definition 1.1 gives a lot of information about coloring with elements of one specific module M and if one was to change the module to some $M' \neq M$, almost all information gathered for previous coloring would be now ineffectual. Consider the following example.

Example 1.1. *Take $R = \mathbb{Z}$ with $\phi(x, y, z) = 2x - y - z$ and consider the trefoil knot 3_1. If we take $M = \mathbb{Z}$ then K admits only the trivial coloring module:*

$$C_M = \{(x, x, x) \ : \ x \in \mathbb{Z}\}.$$

However, if we take $M' = \mathbb{Z}_3$ then there exists a non-trivial coloring like the one presented in fig. 1.

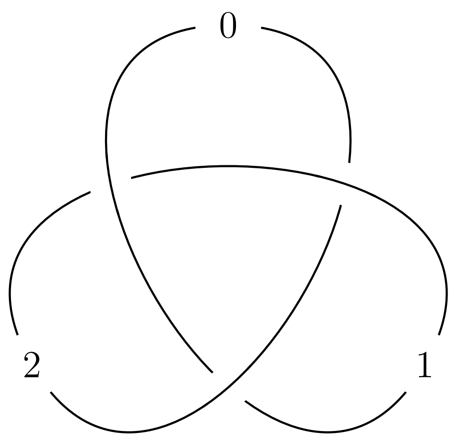


Figure 1: The trefoil knot 3_1 does not allow for nontrivial coloring over $M = \mathbb{Z}$ but it is possible to color it with $M = \mathbb{Z}_3$.

Another approach to defining coloring module of a knot diagram D would be by starting with identifying arches with generators $(0, ..., 1, ..., 0)$ of M^s. Then, we might define a homomorphism

$$f : M^s \to M^x$$

such that arches building one crossing follow rules set by ϕ.

Definition 1.2. *Module* $\ker f$ *is a coloring module of diagram D with elements of M.*

Henceforth, we will call f described above a **coloring homomorphism** for chosen R, M and ϕ.

Corollary 1.2. *Definition 1.1 and definition 1.2 are equivalent for one dimensional modules.*

Proof. TO DO $\square$

Despite the fact that it is the kernel of f that contains colorings, examining the coloring homomorphism itself gives more information about diagram D. We might consider f as a $s \times s$ matrix and if R is a PID module, then we can represent this matrix in Smith's normal form.

Proposition 1.3. *Let A be the Smith's normal form of f. Columns of A comprised only of zeros and zero divisors contribute to the coloring module.*

Proof. An immediate result of corollary 1.2. The second part $\square$

If R is a Noetherian ring, then every finitely generated module is a quotient of a free module with the same number of generators. Thus, we might want to take M to be a finitely generated free R-module rather than any one dimensional R-module. This allows us to send M to any other R-module with at most $\dim(M)$ generators to obtain a different coloring.

The nonzero elements that appear on the diagonal of the normal form of the coloring homomorphism hint at what colorings over the ring R are admissible. Consider the following example.

Example 1.2. *As before, take $R = \mathbb{Z}$ and $\phi(x, y, z) = 2x - y - z$. Taking $M = \mathbb{Z}$ we have $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ for trefoil knot to be a matrix*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

with Smith's normal form

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The normal form of the coloring homomorphism contains a 3, hinting that $\mathbb{Z}/(3)$ allows for a non-trivial coloring. Send $M = \mathbb{Z}$ to $M' = \mathbb{Z}_3$

by taking all coefficient modulo 3 *to obtain the new Smith's normal form of f to be*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which informs about the nontrivial coloring presented in fig. 1, that was not allowed over $\mathbb{Z}$.

2 Coloring oriented diagrams

In the previous section we defined coloring of a diagram without an orientation. Such a diagram has only one type of crossing, while in a diagram for which an orientation was chosen, two types of crossings are distinguishable in any knot diagram (see fig. 2).

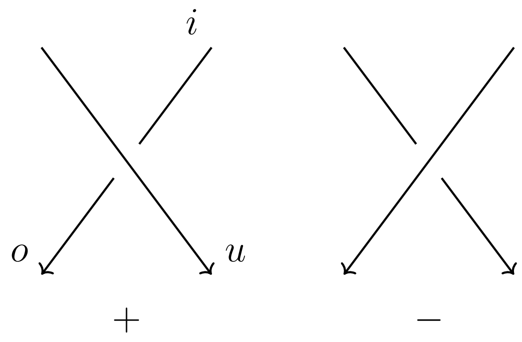


Figure 2: Two types of crossings in oriented knot diagram.

In the case of a diagram with orientation, we must chose which type of crossing is considered by ϕ. If not explicitly stated otherwise, we will choose ϕ to determine the rules of coloring for crossing of type + as seen in fig. 2.

If u, i, o are labels assigned to arches creating a type + crossing that constitute a coloring, then we might write

$$0 = \phi(u, i, o) = au + bi + co.$$

Taking c to be a unit, we get the following equation for the label of the arch leaving the crossing:

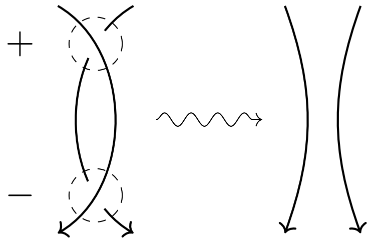
$$o = -c^{-1}au - c^{-1}bi.$$

Those assumption allow us to write a 2×2 matrix A_+ with terms in R such that multiplying an element $(u, i) \in M^2$ by A_+ will return $(o, u) \in M^2$. This means that $A_+ : M^2 \to M^2$ is the operator taking labels of incoming arches as input and returning labels of segments which leave the crossing.

$$A_+ = \begin{pmatrix} -c^{-1}a & -c^{-1}b \\ 1 & 0 \end{pmatrix}$$

It is convenient to take $c = -1$.

Allowing the following Reidemeister's move



gives equality

$$A_- A_+ = Id_2,$$

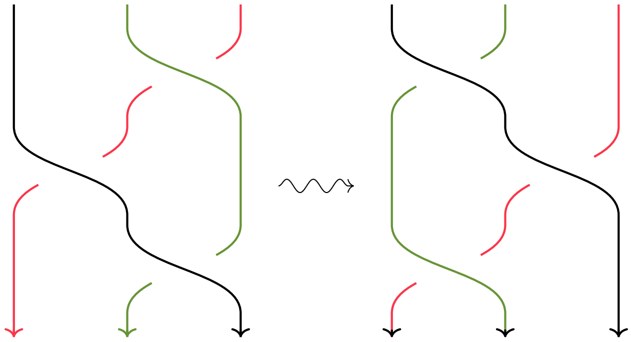
where A_+ is the matrix of operator for $+$ type crossing and $-$ - for the $-$ type crossing. Take $\alpha u + \beta i + \gamma o = 0$ to be the coloring rule for crossings of type $-$. Once again, for the sake of convenience $\gamma = -1$ and the matrix A_- must be of form

$$A_- = \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix},$$

meaning that

$$\begin{cases} b\beta = 1 \\ b\alpha - a = 0. \end{cases}$$

Consider another Reidemeister's move



Applying $A_\pm$ to each crossing separately yields the following relations

$$\begin{cases} ba = ab \\ a(a+b) = a. \end{cases}$$

We must assume that both b and β are units. In the most general situation, we are considering coloring modules as modules over the ring

$$R = \mathbb{Z}[s, t, t^{-1}]/\{s^2 + st - s\},$$

with a being send to s and b being send to t. However, it can be beneficial to at first assume yet another relation:

$$a + b = 1,$$

meaning that we are considering coloring as $\mathbb{Z}[t, t^{-1}]$ module.

Coloring a knot with $\mathbb{Z}[t, t^{-1}]$ module allows us to obtain information about coloring over $\mathbb{Z}$ or many other commutative rings by sending t to a unit in the ring in question.

Example 2.1. *Consider knot 4_1 with diagram D as seen in fig. 3 and ring $R = \mathbb{Z}[t, t^{-1}]$. Take function $\phi : M^3 \to M$ to be defined as*

$$\phi(u, i, o) = (1-t)u + ti - o$$

The coloring homomorphism f is then defined by the matrix

$$f = \begin{pmatrix} 1-t & t & -1 & 0 \\ t^{-1} & -1 & 0 & 1-t^{-1} \\ 0 & 1-t^{-1} & t^{-1} & -1 \\ -1 & 0 & 1-t & t \end{pmatrix}$$

Changing the coefficients in R to $\mathbb{Q}$ yields the following Smith's normal form for f:

$$S = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & t^2 - 3t + 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice, that $\det S = t^2 - 3t + 1$*, which is the Alexander polynomial of* 4_1.

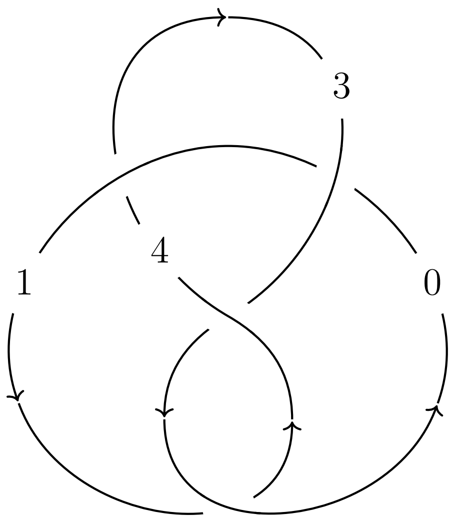


Figure 3: Coloring of knot 4_1 with elements from $\mathbb{Z}_5$.

Now, consider a homomorphism $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$ defined by $t \mapsto -1$. This yields a new matrix for f, with Smith's normal form:

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix above hints at existence of a coloring with elements from $\mathbb{Z}_5$, one of which is presented in fig. 3.

3 Reducing normal form of a matrix

In definition 1.2 we defined the coloring module of a diagram D as the kernel of coloring homomorphism. We might also want to extend this homomorphism to a short exact sequence

$$0 \longrightarrow \ker f \hookrightarrow M^s \xrightarrow{f} M^s \twoheadrightarrow \operatorname{coker} f \longrightarrow 0$$

and ask what information can be obtained from studying $\operatorname{coker} f$.

In examples 1.2 and 2.1 nontrivial coloring was admissible only in modules $M/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal spanned by a portion of terms that appear on the diagonal of Smith's normal form of f. In the same examples, we observe also that $\operatorname{coker} f = R^k \oplus R/\mathfrak{a}$. For the knot 3_1 it was

$\operatorname{coker} f = \mathbb{Z} \oplus \mathbb{Z}_3$, while in the case of knot 4_1 $\operatorname{coker} f = \mathbb{Z} \oplus \mathbb{Z}_5$.

Proposition 3.1. *Let f be a coloring homomorphism of an oriented diagram D. If $\operatorname{coker} f = R/\mathfrak{a}_1 \oplus ... \oplus R/\mathfrak{a}_k$ then D can be colored with elements from $R/\mathfrak{a}_i$ for $i = 1, ..., k$.*

Proof. To się powinno sprowadzić do rozwiązywania układu równań przy pomocy macierzy. □

The coloring homomorphism f of a diagram D carries a lot of information about the knot whose diagram it is. However, f in itself is not a knot invariant. The dimensions of its matrix will change if a new crossing is created, see the following example.

Example 3.1. *We take knot 3_1 with additional crossing, $R = \mathbb{Z}[t, t^{-1}]$ and $M = \mathbb{Z}[t, t^{-1}]$ with ϕ as in example 2.1. The coloring homomorphism has matrix*

$$\begin{pmatrix} 1-t & t & -1 & 0 \\ t & -1 & 0 & 1-t \\ -1 & 1-t & 0 & t \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

with normal form

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -t^2+t-1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which after evaluation at $t = -1$ yields

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which differs from matrix obtained in example 1.2 by just one trivial.

The nontrivial term on the diagonal in example 3.1 is the same as in example 1.2. The difference between matrices obtained in those two examples are their dimensions.

Definition 3.1. *Let A, B be matrices with entries from a PID ring R. We will say that they are equivalent ($A \sim B$) if and only if their Smith's normal form has the same nonzero and nonunit terms.*

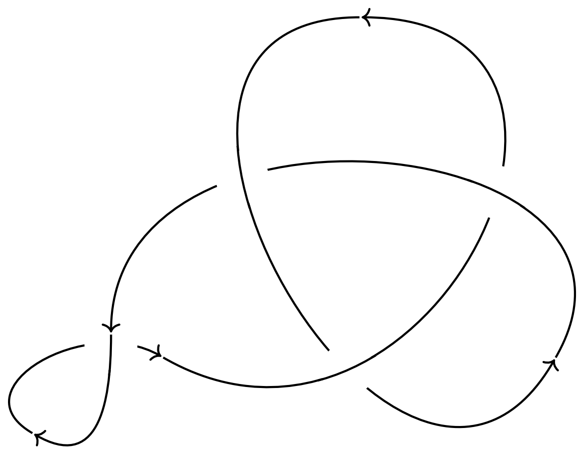


Figure 4: Diagram of knot 3_1 with additional crossing.

Example 3.2. *Matrices of coloring homomorphisms over the ring $\mathbb{Z}$ of knot 3_1 presented in examples 1.2 and 3.1 are both equivalent to a 1×1 matrix $\begin{pmatrix}3\end{pmatrix}$.*

Theorem 3.2. *Equivalence class of matrices under relation $\sim$ defined in definition 3.1 is a knot invariant.*

Example 3.3. *First, consider the knot 6_1 with diagram as seen in fig. 5, ring $R = \mathbb{Z}[t, t^{-1}]$ and $M = R$. We calculate that*

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which agrees with the Alexander polynomial of 6_1. Now, the reduced form of f would be

$$\begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

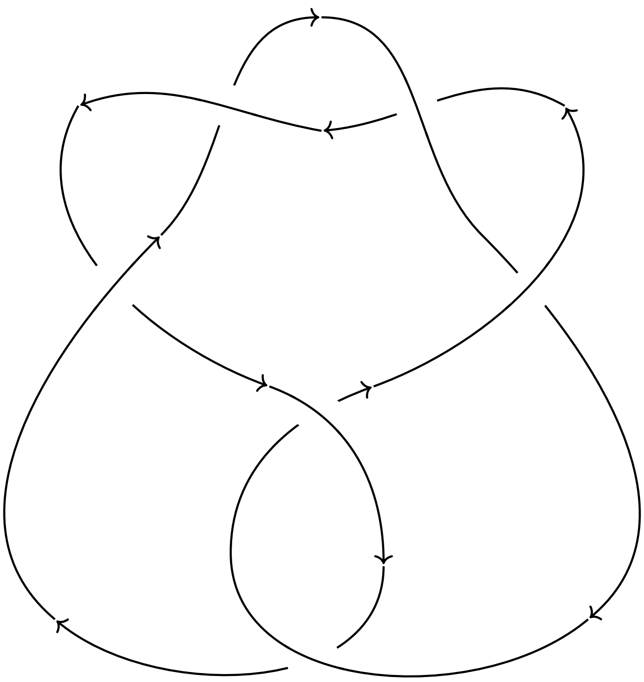


Figure 5: Diagram of knot 6_1.

a 1×1 *matrix.*

There is another knot with Alexander polynomial equal $-2t^{-2} + 5t^{-1} - 2$*:* 9_{46}*. Using diagram in fig. 6 it can be calculated that*

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t - t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2} - 2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\det f = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t^{-1} - 5 + 2t$$

is also the Alexander polynomial. The reduced form of f *is*

$$\begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

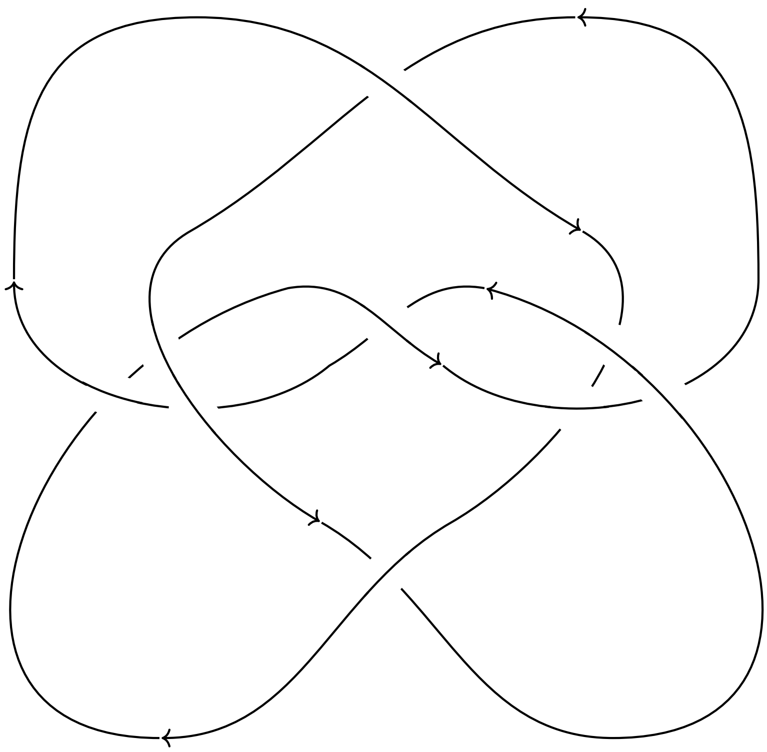


Figure 6: Diagram of knot 9_{46}.

which is significantly different than the one for 6_1.

TO DO: sprawdzić te węzły wyżej za pomocą pow. Seiferta, czy mają różne moduły Alexandera

References