Fox knot colorings and Alexander invariants.

(Kolorowania Foxa i niezmienniki Alexandera)

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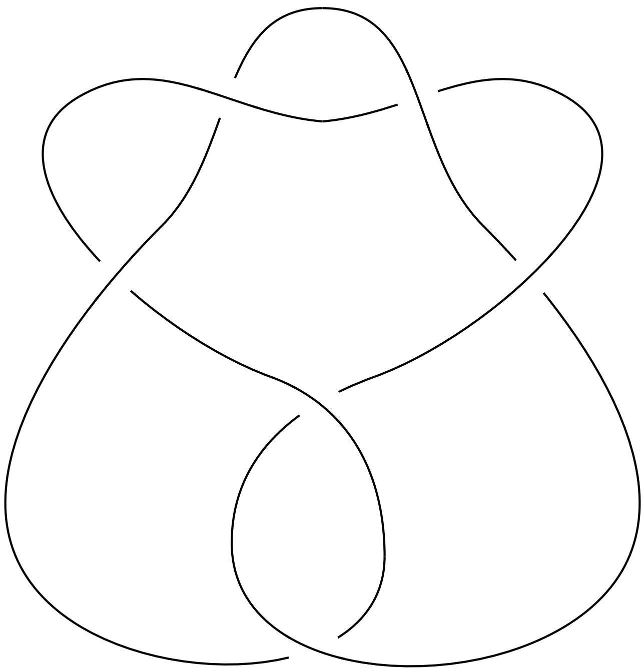

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1 Preliminaries

1.1 Knots and diagrams

In mathematical terms, a knot is a particular embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \twoheadrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [6].

S^1 is an orientable space thus we can choose an orientation for a knot being considered. Then a diagram D is oriented if it is a projection of an oriented S^1.

Intuitively, two knots K_1 and K_2 are equivalent if we can deform one into the other without cutting it and only manipulating it with our hands [2]. This translates to equivalence of diagrams, which is generated by a set of moves, called the **Reidemeister moves**. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D, 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [4].

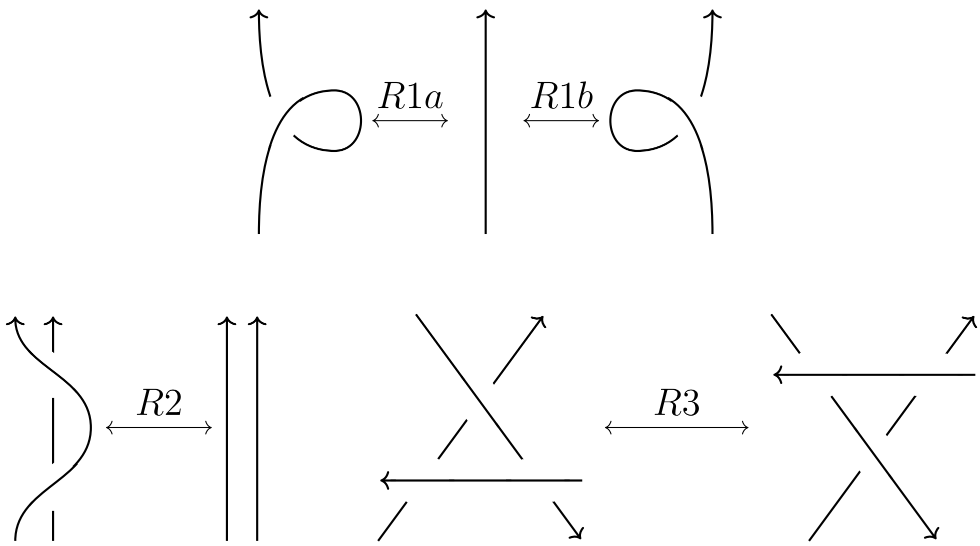


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

1.2 Knot group

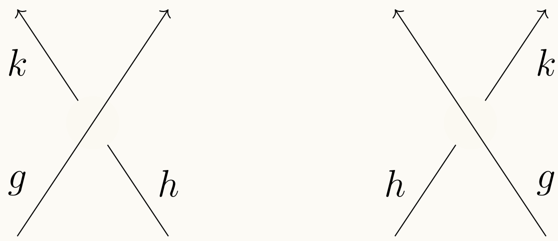
Let K be a knot and D be its oriented diagram with s segments and x crossings.

Definition 1.1 : knot group.
The fundamental group of a knot embedded in a three dimensional sphere S^3 is called a **knot group**.

$$\pi_\mathbf{1}(\mathbf{K}) := \pi_\mathbf{1}(\mathbf{S^3} - \mathbf{K}).$$

Although the knot itself is always a circle S^1, the knot group has usually an interesting yet difficult structure. The most known representation of the knot group is called **the Wirtinger presentation**.

Definition 1.2 : Wirtinger presentation.
Given a diagram D of knot K with segments a_1, a_2, ..., a_s and crossings c_1, ..., c_x the knot group $\pi_1(K)$ can be represented as $\pi_1(K) = \langle G \mid R \rangle$, where G is the set of segments of D and relations R correspond to crossings in the manner described in the diagram below



$gkg^{-1} = h$ $\qquad$ $ghg^{-1} = k$

Representation $\langle G \mid R \rangle$ described above is called the **Wirtinger presentation** [1, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to $S^3 = K \oplus S^3 - K$ or noticing that every two generators are conjugate, is that the abelianization of the knot group is always $\mathbb{Z}$. This leads to an acyclic complex

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0$$

The group $K_G = \ker(ab : G \to \mathbb{Z}) = [G, G]$ is not finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization $K_G^{ab} = K_G/[K_G, K_G]$ allows a $\mathbb{Z}[\mathbb{Z}]$ module structure and thus contains obtainable information about the knot K.

The following is an exact sequence:

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives foundation for the following definition.

> **Definition 1.3 : metabelianization.**
> The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of G.

We will return to the concept of metabelianization in section 2.

1.3 Infinite cyclic covering

Let X be the complement of a knot K, that is $X = S^3 - K$. Take $\widetilde{X}$ to be its universal covering, meaning that it is simply connected. The fundamental group G of X acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G, G]$ is normal in G and so the action of K_G on $\widetilde{X}$ is well defined. Thus we might take the quotient space $\overline{X} = \widetilde{X}/[G, G]$ and call it the **infinite cyclic covering** of X. The fundamental group of $\overline{X}$ is exactly

$$\pi_1(\overline{X}) = [G, G] = K_G$$

and from the perspective of homology modules, we have

$$H_1(\overline{X}, \mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{array}{ccc}
\widetilde{X} & \circlearrowleft & G \\
\big\downarrow & & \\
\overline{X} & \circlearrowleft & G/[G, G] \\
\big\downarrow & & \\
X = S^3 - K & &
\end{array}$$

A **Seifert surface** S of knot K is an orientable surface with boundary embedded in S^3 such that $\partial S = K$. Take a countable amount of X, with S without its boundary embedded, and label each with an element from $\mathbb{Z}$. We might now cut each of the copies of X along the Seifert surface of K and identify the $+$ side of S from the i-th copy of X with the $-$ side of S from the $(i+1)$-th copy of X. Notice that the arising space with a projection to one copy of X is an infinite cyclic cover of X.

Imagine that each copy of X inside of $\overline{X}$ is a box labeled with some integer k. The ring action of $\mathbb{Z}[\mathbb{Z}]$ on $\overline{X}$ is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to $\overline{X}$.

Proposition 1.1.
The $\mathbb{Z}[\mathbb{Z}]$-module $K^{ab} = H_1(\overline{X}, \mathbb{Z})$ is a torsion module.

Proof. Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \to C_*(\overline{X})$$

$$f(x) = (1-t)x.$$

It translates to removing from a cycle in the $(i+1)$-th box a corresponding cycle in the i-th box. From this it is an immediate result that $\ker f = 0$ and that $\operatorname{coker} f = C_*(X)$: after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces an acyclic complex of homology modules

$$\begin{tikzcd}
\dots \arrow[r] & H_2(X,\mathbb{Z}) \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r] & H_1(X,\mathbb{Z}) \arrow[dll, rounded corners] \\
& H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(X,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

As was mentioned previously, the following equality holds:

$$H_1(X, \mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}\,.$$

Now, because X is homotopy cycle, then $H_2(X, \mathbb{Z}) = 0$. Both X and $\overline{X}$ is connected implying that

$$H_0(X, \mathbb{Z}) = H_0(\overline{X}, \mathbb{Z}) = \mathbb{Z}\,.$$

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "0"] & \mathbb{Z} \arrow[dlll, rounded corners] \\
& \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Rewriting the sequence above we easily get that homomorphism $1-t$ is actually an isomorphism and $H_1(\overline{X},\mathbb{Z}) \cong (1-t)H_1(\overline{X},\mathbb{Z})$, which allows us to use the Nakayama's lemma to conclude that there exists $x \in \mathbb{Z}[\mathbb{Z}]$ such that

$$xH_1(\overline{X},\mathbb{Z}) = 0.$$

$\square$

2 Resolution of the Alexander module

2.1 Alexander module

Take $G = \langle G \mid R \rangle$ to be the Wirtinger presentation of G obtained from diagram D. Because K is a knot and not a link, we know that the number of segments is equal to the number of crossings, thus we can take $n = s = x$.

Let $a_1, ..., a_n$ be the generators of G and $x_1, ..., x_n$ its relations. The homomorphism of abelianization of G is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, ..., n$. In order to obtain a representation of K_G, the kernel of abelianization, we need to change the set of generators of G to

$$\{a_1, A_2 = a_2a_1^{-1}, ..., A_n = a_na_1^{-1}\}.$$

It is obvious that for every $i > 1$ $A_i \mapsto 0$ by abelianization of G.thus $A_2, ..., A_n$ are some of the generators of K_G. However, for each $i = 2, ..., n$ and $k \in \mathbb{Z}$ the following is an element of K_G:

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the representation of K_G is infinite with generators

$$\{b_{i,k} \ : \ i = 2, ..., n, \ k \in \mathbb{Z}\}.$$

Changing generators of G induced a change in relations. Suppose that the following relation was true in G

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of a_1 the following two relations can be obtained

$$a_1^{-1} A_k a_1 = (a_1^{-1} A_i a_1) A_j A_i^{-1}$$
$$a_1^{-3} A_k a_1^3 = (a_1^{-3} A_i a_1^3)(a_1^{-2} A_j a_1^2)(a_1^{-2} A_i^{-1} a_1^2).$$

Obviously in G both of those relations are equivalent, however in K_G they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

to obtain infinitely many relations from K_G.

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Hence action of $\mathbb{Z}$ can be defined on the group K_G^{ab}, with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}.$$

This procedure allows K_G^{ab} to be interpreted as a $\mathbb{Z}[\mathbb{Z}]$-module.

Definition 2.1 : Alexander module.
Given a group G, the abelianization of the commutator of a group G, K_G^{ab}, with $\mathbb{Z}[\mathbb{Z}]$-module structure is called the **Alexander module** of G. If G is a knot group, then it is the Alexander module of the knot K

Lemma 2.1.
The $\mathbb{Z}[\mathbb{Z}]$ modules K_G^{ab} and G^{mab} (see definition 1.3) are isomorphic.

Proof. Construction presented above states that the module K_G^{ab} has $(n-1)$ generators. $\square$

2.2 Basic properties

The resolution of a module at first glance is in no way a simplification of said module. However, there are multiple ways of distilling simplifications and invariants from the resolution of the Alexander module. In this section we want to

We start writing the beginning K_G^{ab} resolution as follows:

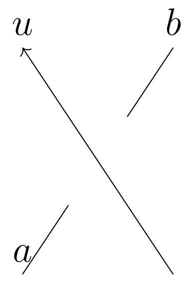
$$\ldots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

Definition 2.2 : Alexander matrix.
The matrix of homomorphism A_D in the diagram above is called the **Alexander matrix** of group G (knot K).

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its rows will give a square matrix whose determinant is nonzero.

Proposition 2.2.
Let A'_D be the Alexander matrix A_D with one of its rows struck out. Then $\det(A'_D) \neq 0$.

Proof. We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator x is send to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module K^{ab} we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1-t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which x is b or c are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t-1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and ± 1. Thus, the sum is not equal to zero. There are two of such rows as the segment b has to be the "out" and "in" segment of some crossing. In other words, segment b has to have a start and end in some crossings.

Let S_i be the column of the Alexander matrix corresponding to the segment labeled i. The sum $\sum_{i \leq n-1} S_i$ is a vector with two nonzero terms. Take S_j and S_k to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both S_j and S_k by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j,k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i : i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. $\square$

The proposition 2.2 implies that image of A_D has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

Theorem 2.3.
The determinant $\det(A'_D)$ is independent of the choice of the diagram D

Proof. If D and D' are two diagrams of knot K, then they yield equivalent representations of $G = \pi_1(K)$. Thus, the chain of elementary ideals of A_D and $A_{D'}$ are the same according to Fox [5, Chapter VII] from which immediately follows that the determinants of the maximum minors of A_D and $A_{D'}$ are equal. $\square$

Definition 2.3 : Alexander polynomial.
The **Alexander polynomial** of a knot K is the determinant of any maximal minor of the Alexander matrix A_D.

The Alexander polynomial is a knot invariant as a consequence of theorem 2.3 and proposition 2.2

Proposition 2.4.
Let G be a knot group of K. Then it always has a resolution

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}]^1 \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

where n is the number of crossings of the chosen diagram D of knot K.

Proof. Take $R = \mathbb{Z}[\mathbb{Z}]$ and consider its field of fractions $R^{-1}R$. There is an obvious homomorphism $R \to R^{-1}R$ which allows us to work on A_D as if it was a linear map between vector spaces

$$R \otimes_R R^{-1}R \xrightarrow{A_D \otimes_R id_{R^{-1}R}} R \otimes_R R^{-1}R$$

with $\dim(A_D \otimes_R id_{R^{-1}R}) = (n-1)$ as was proven in proposition 2.2.

Thus, the following is an exact sequence of vector spaces

$$0 \longrightarrow V \longrightarrow V^n \xrightarrow{A'_D} V^{n-1} \longrightarrow 0$$

where $V = R^{-1}R$ and $A'_D = A_D \otimes_R id_{R^{-1}R}$.

Now consider the following sequence

$$0 \longrightarrow R \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0 \tag{1}$$

The only concerning point is the leftmost arrow as it might not be an injection to $\ker A_D$.

The ring of fractions is flat [3, Chapter 3], the module K_G^{ab} is torsion proposition 1.1 and thus

$$K_G^{ab} \otimes_R R^{-1}R = 0.$$

Because of that, tensoring the sequence (1) by $R^{-1}R$ induces an isomorphism between homologies of the sequences above, wherefore it is exact. □

2.3 Hinting at colorings

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3 Knot colorings

3.1 What is a knot coloring

Let K be a knot and D be its oriented diagram with s segments and x crossings. In such diagrams we can see two different crossing types as seen in fig. 2.

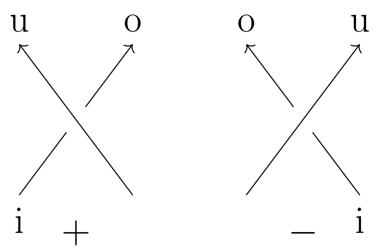

Figure 2: Two types of crossing in oriented diagram.

Take a commutative ring with unity R and an R-module M.

Definition 3.1 : coloring rule.
Take $\mathcal{C} \subseteq M^3$ to be a finitely generated submodule of M^3. We will call $\mathcal{C}$ a **coloring rule**. There are two submodules $\mathcal{C}_\pm \subseteq \mathcal{C}$, each corresponding to a type of crossing in diagram D.

We can now construct three homomorphisms

$$\phi : M^3 \to M/\mathcal{C} = N$$

$$\phi_\pm : M^3 \to M/\mathcal{C}_\pm = N_\pm.$$

We will call ϕ and $\mathcal{C}$ **coloring rule** interchangeably.

For each crossing x_j in diagram D we can construct a projection

$$\pi_{x_j} : M^s \twoheadrightarrow M^3$$

which restricts M^s to the three arcs that constitute x_j.

Definition 3.2 : diagram coloring.
A **coloring of diagram** D is any element $(m_1, ..., m_s) \in M^s$ that assigns elements of M to each arc. We will call this coloring **admissible** if for every crossing x_j of type $\pm$ we have

$$\pi_{x_j}(m_1, ..., m_s) \in \mathcal{C}_\pm \subseteq \mathcal{C}.$$

It will be beneficial to express admissibility of a coloring in terms of homomorphism ϕ.

Proposition 3.1.
A coloring $(m_1, ..., m_s) \in M^s$ is a admissible $\iff$ for each crossing x_j of type $\pm$

$$\phi_\pm(\pi_{x_j}(m_1, ..., m_s)) = 0.$$

Proof. Stems from the fact that $\mathcal{C}_\pm = \ker \phi_\pm$. □

Definition 3.3 : color checking matrix.
After assignings arcs to coordinates in M^s and crossings to coordinates in N^x it is possible to define a linear homomorphism $D\phi : M^s \to N^x$ as

$$D\phi(m_1, ..., m_s) = (\phi_\pm(\pi_{x_1}(m_1, ..., m_s)), \phi_\pm(\pi_{x_2}(m_1, ..., m_s)), ...).$$

Matrix that is created after choosing a basis for M^s and N^x will be called a **color checking matrix**.

Taking $\phi_\pm$ to be linear equations of form

$$\phi_+(u,i,o) = au + bi + co$$

$$\phi_-(u,i,o) = \alpha u + \beta i + \gamma o,$$

where u, i and o correspond to arcs as seen in fig. 2 and all the coefficients are linear homomorphisms $M \to N$, we know that all the entries for the color checking matrix will be linear combinations of a, b, c, α, β, γ. If M has n generators we chose to block the matrix $D\phi$ into $n \times n$ blocks.

Proposition 3.2.
Coloring $(m_1, ..., m_s) \in M^s$ is admissible $\iff$ $(m_1, ..., m_s) \in \ker D\phi$.

Proof. $\implies$

We know that every projection $\pi_{x_j}(m_1, ..., m_s)$ is in $\ker \phi_\pm$, depending on the type of x_j crossing. Thus, there is no projection π_{x_j} that is not being reduced by $\phi_\pm$.

$\impliedby$

□

We need to impose restrictions on the coloring rule. We want $\mathcal{C}$ to be two dimensional (have two generators). That way we have the following diagram

$$\begin{tikzcd} M^2 \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & M^3 \arrow[r, two heads] & \mathcal{C} \end{tikzcd}$$

We can assume that M^2 corresponds to the 'up' and 'in' segments in a crossing (compare fig. 2), then we can define $\phi'_\pm$ to take u and i segments and return the out segment so that the labeling agrees with the coloring rule. Now, take the red arrow in the diagram above to be the correspondence

$$(u,i) \mapsto (u,i,\phi'_\pm(u,i)).$$

This demands that both c and γ in the definition of ϕ_+ and ϕ_- are invertible. For the sake of simplicity, we will take $c = \gamma = -1$.

With this assumption for any admissible coloring (u, i, o) of a crossing we have the following relation:

$$\phi_+ \ : \ o = au + bi$$

$$\phi_- \ : \ o = \alpha u + \beta i.$$

We might also demand that a trivial coloring (every arc is assigned the same element of M) is an admissible coloring.

3.2 Relation on color checking matrices

The color checking matrix, defined in definition 3.3, is not a knot invariant. Its size and structure changes as Reidemeister moves are applied to the diagram. Thus, we need to define which matrices stem from equivalent knot diagrams.

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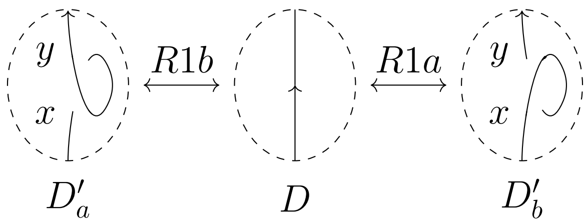

Both Reidemeister moves $R1a$ and $R1b$ require the following diagram to commute,

$$\begin{tikzcd}
M^{s+1} \arrow[rr, "D'\phi"] \arrow[d, two heads] & & N^{x+1} \arrow[d, two heads] \\
M^{s+1},\ x = y \arrow[d, "f"'] & & N^x \oplus (N/\phi_\pm(M^3)) \arrow[d, "g"] \\
M^s \arrow[rr, "D\phi"'] & & N^x
\end{tikzcd}$$

where $\phi_\pm$ changes (for $R1a$ we have $+$ and for $R1b$ $-$). We take f and g to be given by

$$f(m_1, ..., m_s, m_{s+1}) = (m_1, ..., m_s + m_{s+1})$$

$$g(n_1, ..., n_x, n_{x+1}) = (n_1, ..., n_x + n_{x+1}).$$

The homomorphism f ensures that on the rest of diagrams D' arc labeled x in figure above and y add up to the arc visible in the diagram D. Meanwhile, g ensures that the additional crossing is treated with the appropriate coloring rule.

In terms of matrices, the above diagram can be translated to

$$\begin{array}{ccccc} D'_a & & D & & D'_b \\ \begin{bmatrix} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} & \overset{R1a}{\sim} & \begin{bmatrix} x_1+y_1 & z_1 & \dots \\ \vdots & & \ddots \end{bmatrix} & \overset{R1b}{\sim} & \begin{bmatrix} \beta & \alpha+\gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} \end{array}$$

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Theorem 3.3.
The equivalence class of a color checking matrix of a diagram $D\phi$ under relation generated by matrix relations $R1a$, $R1b$, $R2$ and $R3$ is a knot diagram. Thus we can define $K\phi := [D\phi]$.

Proof. A direct result of the definition of the equivalence relation. □

3.3 Smith normal form

The ring R over which we consider modules M is not necessary a principal ideal domain. However, there are plenty of PID rings and one can find at least one PID P with a homomorphism $R \to P$ that allows to consider M as a P-module by tensoring it with P:

$$M_P = M \otimes_R P.$$

That way, we can consider a new type of equivalence relation on any color checking matrix $D\phi$.

Definition 3.4 : Smith normal form.
Take $A \in K\phi$ and consider it as a $s \times x$ matrix with terms in a P. Then there exist a $s \times s$ matrix S and $x \times x$ matrix T such that

SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every i $a_i|a_{i+1}$. Such a matrix SAT is called the **Smith normal form** of matrix A.

As was mentioned in the first section, $\overline{x} \in M^s$ is a coloring of a diagram D if and only if $D\phi(\overline{x}) = 0$, that is $\overline{x} \in \ker D\phi$. The Smith normal form hints at the structure of matrix kernel - the columns filled with zeros will contributed a free factor M to the kernel.

Take (a) to be a prime ideal with its generator a appearing in the Smith normal form of $D\phi$. Then we might consider the matrix over a new ring $P/(a)$, which is still a PID. After this change, the structure of the kernel has changed as now there are additional zero columns where a and all its multiples stood.

Definition 3.5 : reduced normal form of matrix.
Take A to be a matrix with coefficients in principal ideal domain P. Take $a_1, ..., a_k \in P$ to be all the elements of the Smith normal form of A that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix A.

When working with knots we usually take $R = \mathbb{Z}[t, t^{-1}]$ and $M = \mathbb{Z}[t, t^{-1}]$. This is not a PID ring but there are multitudes of PID rings into which R can be mapped. The following algorithm can be used to calculate the Smith normal form of a color checking matrix.

1. Let $A = \{a_{i,j}\}_{i,j\leq n}$ be an $n \times n$ matrix. Take the ideal $I = (a_{i,j})$ generated by all the terms of A.
2. If we are in PID then I has one generator, call it a.
3. We can now use the following row and column operations to put a in the upper left corner of A
   (a) Permuting rows (columns).
   (b) Adding a linear combination of rows (columns) to the remaining row (column).
4. With a in the upper left corner we can now use the fact that it was the generator of I to strike out the remaining terms on the first column and row, using the operations described in the previous point.
5. Repeat the same algorithm on the smaller matrix $\{a_{i,j}\}_{1<i,j\leq n}$.

The following example justifies the utility of the reduced normal form of color checking matrices in distinguishing knots.

Example 3.1. Consider the knots 6_1 with diagram as seen in fig. 3 and 9_{46} pictured in fig. 4, ring $R = \mathbb{Z}[t, t^{-1}]$, $M = R$ and

$$\begin{cases} \phi_+(u, i, o) = (1 - t)u + ti - o \\ \phi_-(u, i, o) = (1 - t^{-1})u + t^{-1}i - o. \end{cases}$$

The two rings have the same Alexander polynomial, $\Delta = -2t^{-2}+5t^{-1}-2$, and the same Alexander module $H^1(S^3 - K) = \mathbb{Z}[t, t^{-1}]/(\Delta)$.

For the knot 6_1 we find the matrix $D\phi$ and after changing to the PID ring $P = \mathbb{Q}[t, t^{-1}]$ we see that the Smith normal form is:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which after reduction is

$$A' = \begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

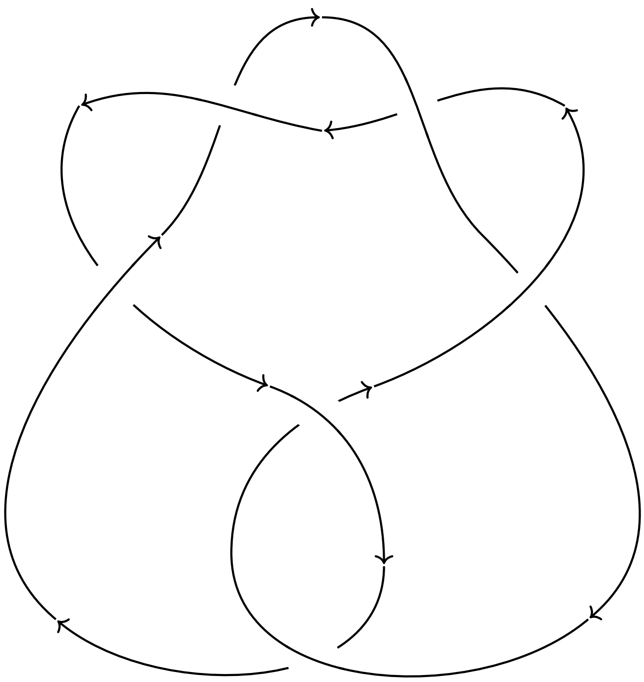

Figure 3: Diagram of knot 6_1.

a 1×1 matrix with the only term being the Alexander polynomial of 6_1. Using diagram in fig. 4 of 9_{46} it can be calculated that the Smith normal form of $D\phi$ is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t - t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2} - 2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while reduced normal form of $D\phi$ is

$$B' = \begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

which is significantly different than the one for 6_1. Observe also that the determinant of both matrices is equal to the Alexander polynomial of

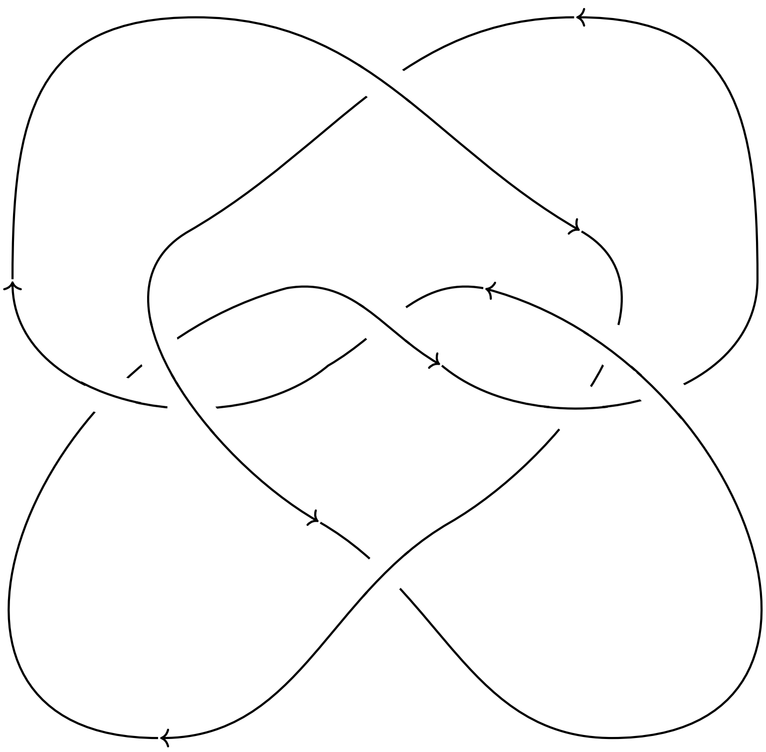

Figure 4: Diagram of knot 9_{46}.

corresponding knots

$$\det(A') = -2 + 5t^{-1} - 2t^{-2}$$

$$\det(B') = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t + 2 + 2t^{-1} = -t(-2 + 5t^{-1} - 2t^{-2}).$$

to wyliczenie wogóle jest na miejscu?

Theorem 3.4.
The reduced normal form of color checking matrix does not depend on the choice of diagram D. Thus, it is well defined for $K\phi$ and is a knot invariant.

Proof. Take a knot K and its diagram D with s segments and x crossings. We will show that applying any Reidemeister move to this knot will not change the reduced normal form of its color checking matrix.

nie wiem, czy tutaj aż tak powinno się dokładnie mówić co i jak dodaję?

R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a** (the proof of **R1b** is left as an exercise for the reader).

Take D' to be diagram D with one arc twisted into a + crossing. In opposition to the assumption in previous section, we will take the arcs and crossings that differ between those two diagrams to be on first positions. Now, the matrices $D\phi$ and $D'\phi$ are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

Adding the first column of $D'\phi$ to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

because $a + b = 1$. Now we know that b is a unit, thus we can easily remove the elements of the first column that are not b. This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2 + y_2 & \dots & \\ 0 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like $D\phi$. The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Thus, the reduced normal form of $D'\phi$ is the same as in $D\phi$.

R2

Now the diagram D' is a diagram D with one arc poked onto another. Once again we will put those changed arcs at the beggining of the color checking matrix to obtain following matrices:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3 + v_3 & \dots \\ x_4 & u_4 + v_4 & \\ \vdots & & \end{bmatrix}$$

Adding the third column of $D'\phi$ multiplied by α and β to first and second column respectively we are able to reduce the first row to only zeros and -1. Now, adding this row to the second one creates a column with only -1 and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a + b\alpha & 0 & -1 & \\ 0 & x_3 & u_3 & v_3 & \\ 0 & x_4 & u_4 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that $a + b\alpha = 0$ and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 & \\ 0 & v_3 + u_3 & v_3 + u_3 & x_3 & \\ 0 & v_4 + u_4 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & v_3 + u_3 & x_3 & \\ 0 & 0 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as $D\phi$.

R3

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as b and β were taken to be units.

$\square$

4 A look at category theory

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