A voyage into the algebras

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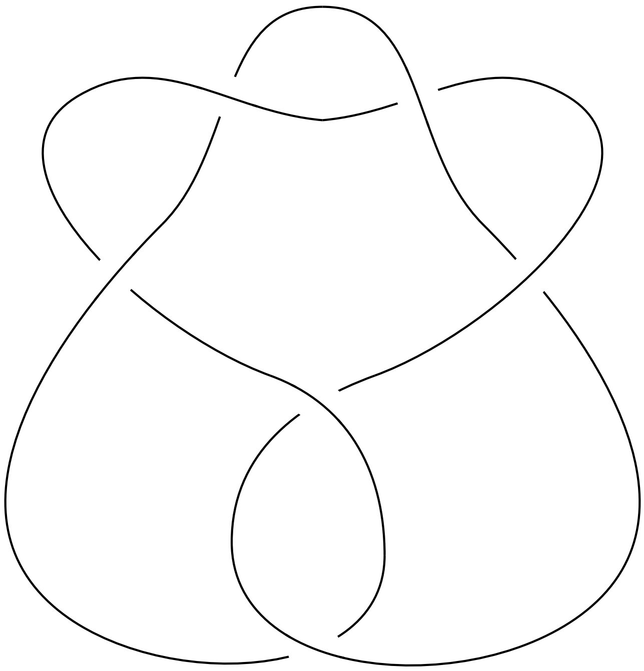


Plan działania

1. Relacje na macierzach -> Reidemeister
   (a) propagation rule - funkcja ϕ, potencjalnie dla uproszczenia będziemy pisać ϕ_+ i ϕ_- na reguły kolorowania dwóch typów skrzyżowania
   (b) Diagram, s łuczków i x skrzyżowań - macierz która bardzo nie jest niezmiennikiem węzła, a zależy od diagramu.
   (c) Wprowadzamy relację na zorientowanych diagramach (chociaż w sumie chyba nie potrzebuję orientacji, ale na takich pracuję więc elo)
2. Smith normal form
3. Skein relations
4. moduł Alexandera 6_1 i 946, czy są różne
5. rezolwenty
6. zmiana pidów

1 What is a knot coloring

Let K be a knot and D be its oriented diagram with s segments and x crossings. In such diagrams we can see two different crossing types as seen in fig. 1.

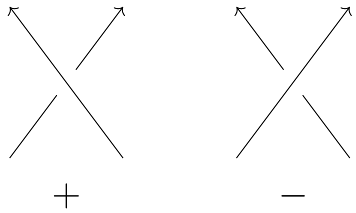


Figure 1: Two types of crossing in oriented diagram.

Let R be a commutative ring, typically $\mathbb{Z}[\mathbb{Z}]$, and take M, N to be two R-modules. Consider two module homomorphisms $\phi_+ : M^3 \to N$ and $\phi_- : M^3 \to N$ such that

$$(\forall\, x \in M)\ \phi_\pm(x, x, x) = 0.$$

This homomorphism will be used to determine whether or not a labelling of knot arcs constitutes a coloring or not.

Definition 1.1 (diagram coloring). *Let $x_1, ..., x_s \in M$ be labels of arcs in diagram D. We will say that $(x_1, ..., x_s) \in M^s$ is a **coloring** if for every crossing $\pm$ in D consisting of arcs u, i, o the following relation is satisfied*

$$\phi_\pm(u, i, o) = 0.$$

tak naprawdę wystarczy jeden homomorfizm ϕ, ale nie wiem jeszcze jak to wyjaśnić beż zahaczania o linki lub warkocze.

Every crossing in the diagram D of knot K yields x relations $\phi_\pm(u, i, o) = 0$ which we might treat as linear equations of form

$$\phi_\pm(u, i, o) = au + bi + co = 0,$$

where the s arcs act as variables and $a + b + c = 0 \in \mathrm{Hom}(M, N)$ (when $M = N$ then $a + b + c \in \mathrm{Ann}(M)$).

Definition 1.2. *Matrix $D\phi : M^s \to N^x$ of coefficients taken from relations $\phi_\pm(u, i, o) = 0$ will be called a **color checking matrix**.*

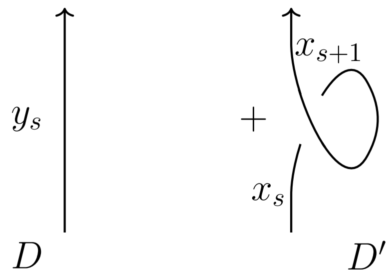
The color checking matrix in itself is obviously not a knot invariant. However, we might define an equivalence relation on the set of all matrices $M^m \to N^n$, $m, n \in \mathbb{N}$, such that all the matrices which come from the same knot fall into the same equivalence class.

2 Relation on color checking matrices

In order to ensure that all matrices that stem from the same knot are considered in one equivalence class we must look at how Reidemeister moves change the matrix.

In this section we will always assume that diagram D has s segments and x crossings. Furthermore, we will always put crossings and segments that are affected by the Reidemeister move as the last columns and rows of the matrix.

R1a



In the case of this Reidemeister move we have

$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+1} \to N^{x+1}.$$

Only two arcs have changed thus

$$D\phi \restriction M^{s-1} = D'\phi \restriction M^{s-1}.$$

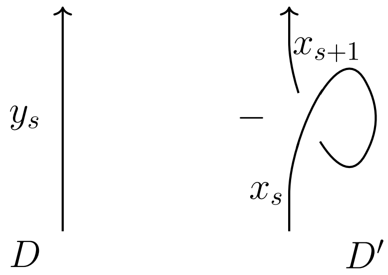
Furthermore, we want for any $x_s, x_{s+1} \in M$

$$\pi_{x+1}[D'\phi(0, ..., x_s, s_{s+1})] = \phi_+(x_{s+1}, x_s, x_{s+1}),$$

where π_{x+1} is projection onto the last coordinate, and

$$(D\phi(0, ..., x_s), 0) = D'\phi(0, ..., x_s, x_s).$$

R1b



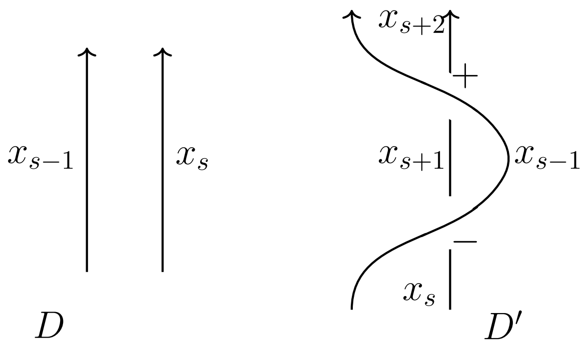
This Reidemeister move on oriented diagram is necessary in defining equivalent oriented knot diagrams but the matrix relation is the one in **R1a** with ϕ_+ changed to ϕ_-:

$$D\phi \sim D'\phi$$

if and only if

$$\begin{aligned} &D\phi \restriction M^{s-1} = D'\phi \restriction M^{s-1} \wedge \\ &\wedge\ \pi_{x+1}[D'\phi(0, ..., x_s, s_{s+1})] = \phi_-(x_{s+1}, x_s, x_{s+1}) \wedge \\ &\wedge\ (D\phi(0, ..., x_s), 0) = D'\phi(0, ..., x_s, x_s). \end{aligned}$$

R2

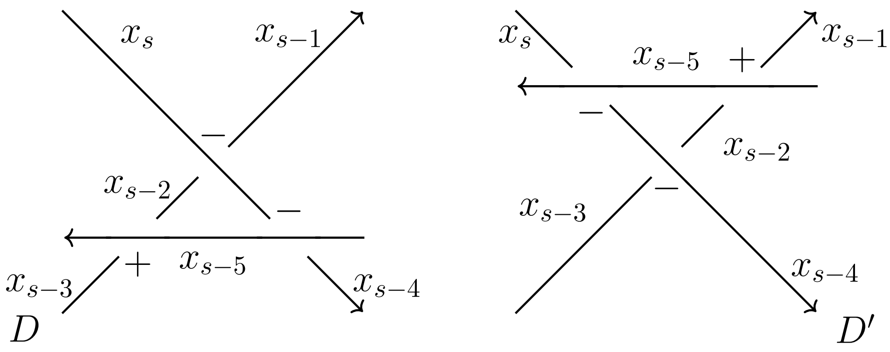


$$D\phi \sim D'\phi$$

if and only if

$$\begin{aligned} &D\phi \restriction M^{s-2} = D'\phi \restriction M^{s-2} \wedge \\ &\wedge\ (\forall x \in M) D'\phi(0, ..., x, y, y, y) = \\ &\quad = (D\phi(0, ..., x, y), \phi_-(x, y, y), \phi_+(x, y, y)) \end{aligned}$$

R3



$$D\phi \sim D'\phi$$

if and only if

$$\begin{aligned} &D\phi \restriction M^{s-5} = D'\phi \restriction M^{s-5} \wedge \\ &\wedge (\forall\, x, y, z \in M)\pi_{x-3}[D\phi(0, ..., x, y, z)] = \\ &= \pi_{x-3}[D'\phi(0, ..., z, x, y, 0, 0)] \wedge \\ &\wedge D\phi(0, ..., z, y, 0, 0, 0, x) = D'\phi(0, ..., z, y, 0, 0, 0, x) \wedge \\ &\wedge D\phi(0, ..., z, 0, y, x, 0, 0) = D'\phi(0, ..., z, 0, 0, y, x, 0) \end{aligned}$$

Theorem 2.1. Let K be a knot and D its oriented diagram. Define

$$K\phi := [D\phi]$$

to be the equivalence class of the matrix $D\phi$. Then, $K\phi$ is a knot invariant.

3 Smith normal form

Usually, when working with knots the ring R is not a PID but $\mathbb{Z}[\mathbb{Z}]$. It is possible to find invariants over this particular ring, however it is often beneficial to find a PID ring P and send $t \in \mathbb{Z}[\mathbb{Z}]$ to some unit of P.

Definition 3.1. *Take $A \in K\phi$ and consider it as a $s \times x$ matrix with terms in a P-module M. Then there exist a $s \times s$ matrix S and $x \times x$*

matrix T such that SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

*where for every i $a_i|a_{i+1}$. Such matrix SAT is called **Smith normal form** of matrix A.*

Because every true coloring $\overline{x}$ of a diagram D satisfies $D\phi(\overline{x}) = 0$, then $\overline{x} \in \ker D\phi$. The Smith normal form gives us a hint as to what is the structure of a matrix' kernel - it will be M^k, where k is the number of columns filled with zeroes.

In the case of knot coloring it might also hint at other possible colorings. For example, if a nonunit appears on the diagonal, it might generated a submodule $M' \subseteq M$ such that M/M' has new coloring that was not observed over M.

We might want to enhance the original equivalence relation on color checking matrices by stating that *if Smith normal forms of two color checking matrices contain the same nonzero, nonunit elements then those matrices are equivalent.* Consider the following example.

Example 3.1. Consider the knot 6_1 with diagram as seen in fig. 2, ring $R = \mathbb{Z}[t, t^{-1}]$, $M = R$ and

$$\begin{cases} \phi_+(u, i, o) = (1-t)u + ti - o \\ \phi_-(u, i, o) = (1-t^{-1})u + t^{-1}i - o. \end{cases}$$

We find the matrix $D\phi$ and after changing to the PID ring $P = \mathbb{Q}[t, t^{-1}]$ we see that the Smith normal form is:

$$N = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

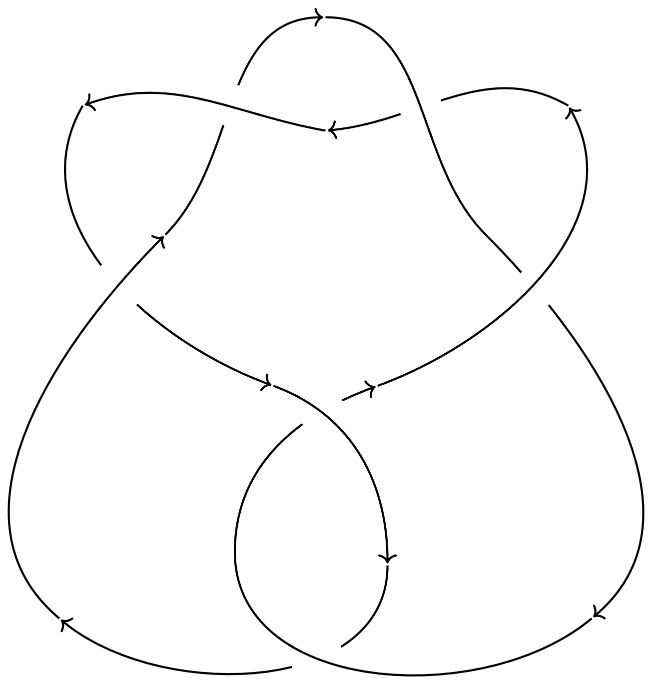


Figure 2: Diagram of knot 6_1.

which after reduction is

$$N' = \begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

a 1×1 matrix with the only term being the Alexander polynomial of 6_1.

There is another knot with Alexander polynomial equal $-2t^{-2} + 5t^{-1} - 2$: 9_{46}. Using diagram in fig. 3 it can be calculated that the Smith normal form of $D\phi$ is

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t - t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2} - 2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced form of N is

$$N' = \begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

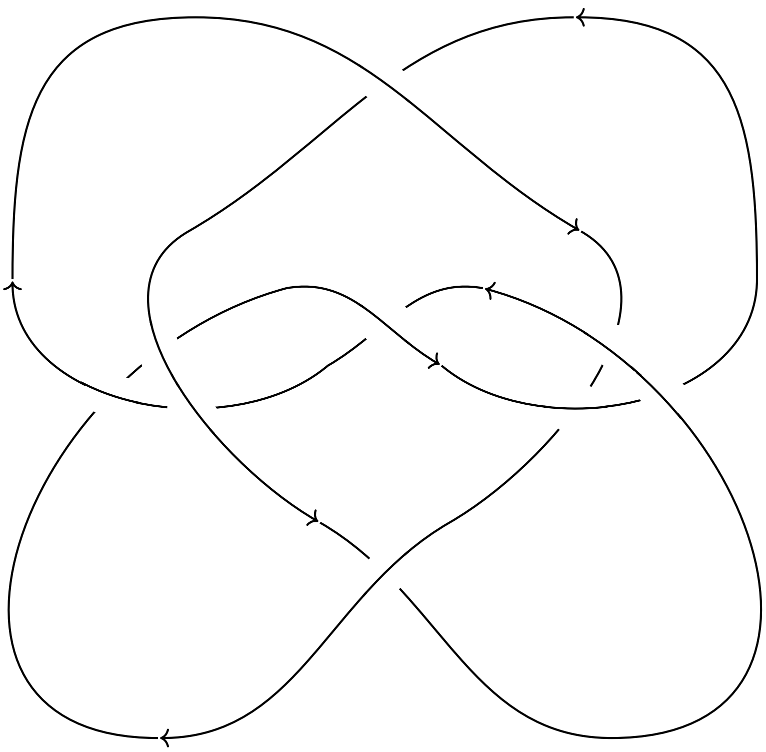


Figure 3: Diagram of knot 9_{46}.

which is significantly different than the one for 6_1 yet

$$\det(N') = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t + 2 + 2t^{-1} = -t(-2 + 5t^{-1} - 2t^{-2})$$

is the Alexander polynomial of 9_{46}.

Proposition 3.1. The reduced Smith normal form of color checking matrix is a knot invariant. Thus, it is well defined for $K\phi$.