

# A voyage into the algebras

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# 1 Problem

Consider the ring  $\mathbb{Z}[[F]]$ , where  $[F]$  is the equivalence class of all finite Abelian groups isomorphic to  $F$ . Describe the set  $\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\}$ , where relation  $[F_2] = [F_1] + [F_3]$  means that there exists exact sequence:

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

**Lemma 1.1.** If  $F, F'$  are two Abelian groups of order  $n$ , then they represent the same equivalence class of relation  $\heartsuit$  i.e.  $[F] = [F']$ .

**Example 1.1.** Before we prove lemma 1.1, let us examine an example. We will show that  $[\mathbb{Z}_4] = [\mathbb{Z}_2 \oplus \mathbb{Z}_2]$ . Consider the following exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \longrightarrow 0$$

which shows that  $[\mathbb{Z}_4] = [\mathbb{Z}_2] + [\mathbb{Z}_2]$ . On the other hand, the next sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{i_1} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0$$

which is also exact, yields  $[\mathbb{Z}_2 \oplus \mathbb{Z}_2] = [\mathbb{Z}_2] + [\mathbb{Z}_2]$ .

This shows that every Abelian group of order 4 is in the same equivalence class of relation given by exact sequences. We will show that all Abelian groups of the same order will belong to one equivalence class.

## Proof

Every finite Abelian group is isomorphic to a direct product of its  $p$ -subgroups **DODAC CYTAT**. Furthermore, any  $p$ -group of order  $p^k$  is isomorphic to  $\mathbb{Z}_{p^k}$ . We can start by examining what elements belong to equivalence class  $[\mathbb{Z}_{p^k}]$ .

We will start by showing that if  $k = n + l$ ,  $k, n, l \in \mathbb{N}$ , then  $[\mathbb{Z}_{p^k}] = [\mathbb{Z}_{p^n}] + [\mathbb{Z}_{p^l}]$ . Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^k} / \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^l} \longrightarrow 0$$

We know that  $\mathbb{Z}_{p^k} / \mathbb{Z}_{p^n}$  is a cyclic group generated by  $1 + \mathbb{Z}_{p^n}$ . Furthermore, we know that  $|\mathbb{Z}_{p^k} / \mathbb{Z}_{p^n}| = p^l$  and thus  $\mathbb{Z}_{p^k} / \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^l}$ .

Now, we will show, using induction on  $N$ , that for any  $n \in \mathbb{N}$  such that  $n = \prod_{i=1}^N p_i^{k_i}$ , where  $k_i \in \mathbb{N}$  and  $p_i$  is a prime number, we have

$$[\mathbb{Z}_n] = \sum_{i=1}^N [\mathbb{Z}_{p_i^{k_i}}] = \sum_{i=1}^N k_i \cdot [\mathbb{Z}_{p_i}] \quad (\star)$$

1.  $N = 1$

From the fact above we know that  $[\mathbb{Z}_{p^{k+1}}] = [\mathbb{Z}_{p^k}] + [\mathbb{Z}_p]$  and applying the same reasoning to  $\mathbb{Z}_{p^k}$  we obtain  $[\mathbb{Z}_{p^k}] = k \cdot [\mathbb{Z}_p]$ .

2.  $N - 1 \implies N$

We will start from the right side of the equality  $(\star)$  and from inductive hypothesis we know that

$$\sum_{i=1}^N k_i [\mathbb{Z}_{p_i}] = k_N [\mathbb{Z}_{p_N}] + \sum_{i=1}^{N-1} k_i [\mathbb{Z}_{p_i}] = [\mathbb{Z}_{p_N^{k_N}}] + [\mathbb{Z}_l]$$

where  $l = \prod_{i=1}^{N-1} p_i^{k_i}$ . Consider the following sequence

$$0 \longrightarrow \mathbb{Z}_l \longrightarrow \mathbb{Z}_n \longrightarrow \mathbb{Z}_{p_N^{k_N}} \longrightarrow 0$$

its exactness follows from the fact that  $\mathbb{Z}_n / \mathbb{Z}_l$  is a cyclic group of order  $n/l = p_N^{k_N}$  and thus there exists an isomorphism

$$\mathbb{Z}_{p_N^{k_N}} \cong \mathbb{Z}_n / \mathbb{Z}_l.$$

As stated before, any Abelian group of order  $N$  is isomorphic to a direct product of its  $p$ -subgroups, hence the

following equality is immediate from  $(\star)$ :

$$\sum_{i=1}^N k_i [\mathbb{Z}_{p_i}] = [\mathbb{Z}_n] = \left[ \bigoplus_{i=1}^N \mathbb{Z}_{p_i^{k_i}} \right]$$

♠

From this follows that every Abelian group of order  $n$ , either being a cyclic group itself or a direct sum of cyclic groups, is in one equivalence class. Hence, elements of group

$$\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\}$$

can be expressed as finite sums of equivalence classes represented by  $p$ -groups:

$$\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\} = \left\{ \sum_{i \leq n} k_i [\mathbb{Z}_{p_i}] : p_i \text{ are prime, } n, k_i \in \mathbb{N} \right\}$$

## 2 Problem

Consider a field  $\mathfrak{K}$  and the ring of polynomials with coefficients in  $\mathfrak{K}$ ,  $\mathfrak{K}[x]$ . Obviously, the aforementioned ring is a principal ideal domain. We want to consider group  $\mathbb{Z}[[M]]$ , where  $[M]$  is the equivalence class of all finitely generated torsion modules isomorphic to  $M$  and relation  $[M_2] = [M_1] + [M_3]$  defined by the existence of an exact sequence.

Any finitely generated module  $M$  is isomorphic to a direct sum of cyclic modules:

$$M \cong p_1 \mathfrak{K}[x] \oplus p_2 \mathfrak{K}[x] \oplus \dots \oplus p_n \mathfrak{K}[x]$$

Moduły które mają ten sam wielomian w rozkładzie trafiają do tego samego domku.

If  $p, q$  are two irreducible polynomials, then  $(p) \oplus (q) = \mathfrak{K}[x]$  (example:  $x - 1, x^2 + 1$ ).

$$x^2 + 2x + 1 - (x - 1)(x + 3)$$