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CONSTRUCTION OF UNIVERSAL BUNDLES, I

BY JOHN MILNOR

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1. Introduction

By an n -universal bundle is meant (see Steenrod [3] p. 102) a principal fibre bundle such that the bundle space is $(n - 1)$ -connected. We will particularly consider bundles for which the bundle space is contractible.

Serre has shown ([2] p. 481) that any arc-connected base space X is covered by a contractible fibre space, the fibre being the space of closed loops on X with a fixed base point. The present paper shows that for a suitable base space X (namely a connected, countable simplicial complex in the weak topology) there exists a contractible fibre bundle over X . In particular Serre's space of closed loops on X , is replaced by a topological group \tilde{G} .

One consequence of this is the construction of a wide variety of topological groups. (For example any Eilenberg-MacLane complex $K(\Pi, n)$ can be considered as a topological group, assuming that the group Π is countable.)

In Section 4 this result is applied to give an axiomatic characterization of homotopy groups for the category of countable CW -complexes. In Section 5 it is shown that any principal bundle over X with any group G is induced by a continuous homomorphism $\tilde{G} \rightarrow G$.

2. Lemmas concerning CW -complexes

LEMMA 2.1. *The product of two countable CW -complexes is a CW -complex.*¹

If W is a subset of $A \times B$ such that $(\bar{e} \times \bar{e}') \cap W$ is open, relative to $\bar{e} \times \bar{e}'$, for each cell e of A and e' of B , then we must prove that W is open in the product topology.

Let (a, b) be any point of W , and let A_i (resp. B_i) be the closure of the union of the first i cells of A (resp. B) numbered so that $a \in A_1$ (resp. $b \in B_1$). Suppose by induction that relative neighborhoods U_i of a in A_i and V_i of b in B_i have been constructed so that $\bar{U}_i \times \bar{V}_i \subset W$. Choose open neighborhoods $U_{i+1} \supset \bar{U}_i$ in A_{i+1} and $V_{i+1} \supset \bar{V}_i$ in B_{i+1} so that $\bar{U}_{i+1} \times \bar{V}_{i+1} \subset W$. This is certainly possible since A_{i+1} and B_{i+1} are compact spaces. Continue by induction. Then $U = U_1 \cup U_2 \cup \dots$ and $V = V_1 \cup V_2 \cup \dots$ are open neighborhoods of a and b satisfying $U \times V \subset W$. Hence W is open. This completes the proof.

PROPOSITION 2.2. *Let A be a simplicial complex and let Δ_{ij} be the subset of $A^n = A \times \dots \times A$ consisting of all (a_1, \dots, a_n) such that $a_i = a_j$. Then Δ_{ij} is a subcomplex of the first derived complex of A^n .*

(The proof is straightforward.)

Given a (not necessarily Hausdorff) space A and a collection of maps $f: \sigma^n \rightarrow$

¹ For definitions see J. H. C. Whitehead [4]. An example due to C. H. Dowker shows that this lemma would be false without the assumption of countability.

A , where each σ^n is a closed n -cell, let e^n denote the image of the interior of σ^n and let A^n denote the union of all e^i with $i \leq n$.

LEMMA 2.3. *If the following three conditions are satisfied then A , together with the collection $\{e^n\}$ of cells, forms a CW-complex.*

(a) *The interior of each σ^n is mapped one-one onto the corresponding e^n . Every point of A belongs to exactly one set e^n .*

(b) *The boundary of each σ^n is mapped into A^{n-1} .*

(c) *A subset of A^n , $0 \leq n \leq \infty$, is closed whenever its inverse image in each cell σ^i of dimension $i \leq n$ is closed.*

It is first necessary to prove that A is a Hausdorff space. It is clear that A is a T_1 -space. But now Whitehead's proof ([4] p. 225) that every CW-complex is normal can be applied, without essential change, to the space A . Since A is normal, it is certainly Hausdorff.

Together with conditions (a) and (b), this implies that the interior of each σ^n is mapped homeomorphically onto e^n . Thus $\{e^n\}$ is a complex in Whitehead's sense. Now proposition E (p. 225 of [4]) implies that this complex is a CW-complex.

PROPOSITION 2.4. *If A is a CW-complex and B is a subcomplex, then every map of B into a contractible space can be extended to A .*

(The extension is easily constructed cell by cell.)

PROPOSITION 2.5. *If A is a contractible CW-complex and B is a contractible subcomplex, then B is a strong deformation retract of A .*

(Proposition 2.4 is applied, first to construct a retraction, and then to deform this retraction into the identity map. This uses the fact that $A \times [0, 1]$ is a CW-complex ([4] p. 227).)

PROPOSITION 2.6. *Let $A \cup B$ be a CW-complex with contractible subcomplexes $A, B, A \cap B$. Then $A \cup B$ is contractible.*

(This follows from 2.5.)

PROPOSITION 2.7. *Let $A_1 \cup \dots \cup A_k$ be a CW-complex with contractible subcomplexes A_1, \dots, A_k . Suppose that each possible intersection $A_{i_1} \cap \dots \cap A_{i_r}$, $1 \leq r \leq k$, is also contractible. Then $A_1 \cup \dots \cup A_k$ is contractible.*

(This follows by induction from 2.6.)

PROPOSITION 2.8. *Let $A_0 \subset A_1 \subset A_2 \subset \dots$ be subcomplexes of the CW-complex $A = \bigcup_i A_i$. Suppose that A_0 is a point and that each A_i is a strong deformation retract of A_{i+1} . Then A is contractible.*

(Clearly each A_i is contractible. By 2.4 a contraction of A_i can be extended to a contraction of A_{i+1} . Continuing step by step, we thus construct a contraction of A .)

PROPOSITION 2.9. *A mapping of one CW-complex onto another which maps cells onto cells is an identification map.*

(That a set is closed if and only if its inverse image is closed. The proof is clear.)

PROPOSITION 2.10. *Let $\eta: (A, B) \rightarrow (A', B')$ be a mapping of one CW-pair onto another which maps cells onto cells, and which maps $A - B$ one-one onto*

$A' - B'$. If B is a strong deformation retract of A , then B' is a strong deformation retract of A' .

(Let $r: A \times I \rightarrow A$ be the strong deformation retraction, and define $\bar{\eta}: A \times I \rightarrow A' \times I$ by $\bar{\eta}(a, t) = (\eta(a), t)$. Then

$$\eta r \bar{\eta}^{-1}: A' \times I \rightarrow A'$$

is a strong deformation retraction of A' onto B' . Continuity follows from the fact that $\bar{\eta}$ is an identification map.)

3. Construction from a given base space

THEOREM 3.1. *For any countable, connected simplicial complex X in the weak topology, there exists an ∞ -universal bundle with base space X , the bundle space and group being countable CW-complexes.*

(“Countable” means that the collection of cells is to be countable. It would probably be possible to show that the bundle space and group are actually simplicial complexes, but the author has not succeeded in proving this.)

It is well known that every reasonable base space X is covered by a contractible fibre space. The standard construction, due to J.-P. Serre, is based on the space $X^{[0,1]}$ of paths in X . The present construction will be based on a similar space \tilde{S} of simplicial paths in X .

Let S_n be the space of all sequences $(x_n, x_{n-1}, \dots, x_0)$ of points in X such that each pair x_i, x_{i-1} lie in a common simplex of X . This is to be topologized as a subset of $X^{n+1} = X \times \dots \times X$. Let S be the topological sum of the S_n , $n \geq 1$. An equivalence relation in S is generated by the relations

$$(x_n, \dots, x_i, \dots, x_0) \sim (x_n, \dots, \hat{x}_i, \dots, x_0)$$

whenever either $x_i = x_{i-1}$ or $x_{i+1} = x_{i-1}$. (The symbol “ \hat{x} ” denotes deletion.) Let \tilde{S} denote the identification space $S/(\sim)$ with the identification topology. For each point (x_n, \dots, x_0) of S let $[x_n, \dots, x_0]$ denote the corresponding point of \tilde{S} . Let v_0 be a fixed vertex of X .

As bundle space \tilde{E} we take the subset of \tilde{S} consisting of all $[x_n, \dots, x_0]$ with $x_0 = v_0$. The projection $p: \tilde{E} \rightarrow X$ is defined by

$$p([x_n, \dots, x_1, v_0]) = x_n.$$

Thus the fibre $\tilde{G} = p^{-1}(v_0)$ consists of all $[x_n, \dots, x_0]$ with $x_n = x_0 = v_0$.

LEMMA 3.2. *The space \tilde{S} can be given the structure of a CW-complex, with subcomplexes \tilde{E} and \tilde{G} .*

LEMMA 3.3. *The projection $p: \tilde{E} \rightarrow X$ is continuous.*

(Proofs will be given later.)

A product between certain elements of \tilde{S} is defined as follows. If $[x_n, \dots, x_0]$ and $[y_m, \dots, y_0]$ satisfy $x_0 = y_m$, then define

$$[x_n, \dots, x_0] \cdot [y_m, \dots, y_0] = [x_n, \dots, x_0, y_m, \dots, y_0].$$

It is clear that this multiplication is well defined and associative (whenever it is defined).

LEMMA 3.4. *This product operation in \tilde{S} is continuous.*

Every element $[x_n, \dots, x_0]$ has an inverse $[x_n, \dots, x_0]^{-1} = [x_0, \dots, x_n]$ which satisfies

$$[x_n, \dots, x_0]^{-1} \cdot [x_n, \dots, x_0] = [x_0, x_0].$$

It is clear that the function $[x_n, \dots, x_0] \rightarrow [x_n, \dots, x_0]^{-1}$ is a homeomorphism of \tilde{S} onto itself.

Since the product of any two elements of \tilde{G} is defined, Lemma 3.4 implies

PROPOSITION 3.5. *\tilde{G} is a topological group with identity element $[v_0, v_0]$ under the above multiplication.*

The bundle structure is defined as follows. Let the coordinate neighborhood V_j be the star neighborhood of the j^{th} vertex v_j of X . For each V_j choose a fixed element

$$e_j = [v_j, x_{n-1}, \dots, x_1, v_0]$$

of $p^{-1}(v_j)$. Define the coordinate mapping

$$\phi_j: V_j \times \tilde{G} \rightarrow p^{-1}(V_j)$$

by $\phi_j(x, g) = [x, v_j] \cdot e_j \cdot g$. Lemma 3.4 implies that ϕ_j is continuous.

Define the function $p_j: p^{-1}(V_j) \rightarrow \tilde{G}$ by $p_j(e) = e_j^{-1} \cdot [v_j, p(e)] \cdot e$. Then Lemmas 3.3, 3.4 imply that p_j is continuous.

Define $g_{ij}: V_i \cap V_j \rightarrow \tilde{G}$ by $g_{ij}(x) = e_i^{-1} \cdot [v_i, x, v_j] \cdot e_j$. Lemma 3.4 implies that g_{ij} is continuous. The necessary identities

$$\begin{aligned} p\phi_j(x, g) &= x, & p_j\phi_j(x, g) &= g, \\ \phi_j(p(e), p_j(e)) &= e, & p_j\phi_j(x, g) &= g_{ij}(x) \cdot g \end{aligned}$$

are all easily verified.

Still assuming the proofs of the lemmas, this shows that $\{p, \tilde{E}, X, \tilde{G}, \{V_j\}, \{\phi_j\}\}$ is a principal fibre bundle. In order to complete the proof of Theorem 3.1, it is only necessary to prove the following.

LEMMA 3.6. *\tilde{E} is contractible.*

PROOF OF LEMMA 3.2. Let D be the subset of S consisting of all sequences (x_n, \dots, x_0) which are *degenerate* in the sense that $x_i = x_{i-1}$ or $x_{i+1} = x_{i-1}$ for some i . Thus every sequence in D is equivalent to a sequence of shorter length.

Making use of 2.1 it is easy to show that S is a cell complex in the weak topology. Passing to the first derived complex, we will consider S as a simplicial complex in the weak topology.

The following three facts will be needed.

(A) *Every element of S is equivalent to a unique non-degenerate element (i.e. to a unique element of $S - D$).*

(B) *D is a subcomplex of S . (This follows immediately from 2.2. It can be restated as follows:) If a simplex of S contains one non-degenerate point then every interior point of the simplex is non-degenerate.*

Such simplexes of S will be called *non-degenerate*.

(C) *If σ is any simplex of S , then there is a unique simplicial map of σ onto a non-degenerate simplex σ' which maps points onto equivalent points.*

Assuming these assertions for the moment, the CW -structure of \tilde{S} is defined as follows. As cells of \tilde{S} take the images of the interiors of the non-degenerate simplexes of S . It is now necessary to verify conditions a, b, c of 2.3.

(a) This is an immediate consequence of (A) and (B).

(b) Let s be a boundary point of the non-degenerate simplex σ^n of S . We must prove that the image \tilde{s} of s lies in the $(n-1)$ -skeleton of \tilde{S} . Let σ^i , $i \leq n-1$, be the simplex of S which contains s as interior point; and let σ^j , $j \leq i$, be the corresponding non-degenerate simplex, which is given by assertion (C). Then \tilde{s} belongs to the corresponding cell e_j of \tilde{S} .

(c) Let C be a subset of the n -skeleton of \tilde{S} such that the inverse image of C in each non-degenerate simplex of dimension $\leq n$ is closed. We must prove that C is a closed set. The following assertion will first be proved, by induction on i : *The inverse image of C in any simplex σ^i of S is closed.* Let σ^j be the non-degenerate simplex which corresponds to σ^i by assertion (C). It is sufficient to show that the inverse image of C in σ^j is closed. If $j \leq n$ this is true by hypothesis. If $j > n$ then the inverse image of C in σ^j can contain no interior points. Hence it is only necessary to consider the boundary simplexes of σ^j . But these have dimension $< i$, so that we can apply the induction hypothesis.

Since S has the weak topology, it follows that the full inverse image of C in S is a closed set. Since \tilde{S} is an identification space it follows that C is closed.

This shows that \tilde{S} is a CW -complex. It is clear that \tilde{E} and \tilde{G} are subcomplexes of \tilde{S} . Thus (still assuming (A) and (C)) this proves Lemma 3.2.

PROOF OF (A). Define the (discontinuous) function $\mu: S \rightarrow S$ by

$$\mu(x_n, \dots, x_0) = \begin{cases} (x_n, \dots, x_0) & \text{if this sequence is non-degenerate} \\ (x_n, \dots, \hat{x}_i, \dots, x_0) & \text{if } i \text{ is the largest integer with} \end{cases}$$

$$x_i = x_{i-1} \text{ or } x_{i+1} = x_{i-1}.$$

Let $\nu: S \rightarrow S - D$ be the function obtained by iterating μ until a non-degenerate point is obtained. We will prove that $\nu(s) = \nu(s')$ whenever $s \sim s'$. This clearly implies assertion (A).

It is sufficient to prove that

$$(*) \quad \nu(x_n, \dots, x_0) = \nu(x_n, \dots, \hat{x}_j, \dots, x_0)$$

where j is any index such that $x_j = x_{j-1}$ or $x_{j+1} = x_{j-1}$. The proof will be by induction on n .

Note that $i \geq j$. If $i = j$ we have

$$\nu(x_n, \dots, x_0) = \nu(\mu(x_n, \dots, x_0)) = \nu(x_n, \dots, \hat{x}_j, \dots, x_0)$$

which proves (*). A similar proof applies in case $i = j+1$ and $x_i = x_{i-1}$.

If $i = j+1$ and $x_{i+1} = x_{i-1}$ then

$$\begin{aligned} \mu\mu(x_n, \dots, x_0) &= \mu(x_n, \dots, \hat{x}_i, \dots, x_0) \\ &= (x_n, \dots, \hat{x}_{i+1}, \hat{x}_i, \dots, x_0) = (x_n, \dots, \hat{x}_i, \hat{x}_j, \dots, x_0) \\ &= \mu(x_n, \dots, \hat{x}_j, \dots, x_0), \end{aligned}$$

which proves (*). If $i > j + 1$ then

$$\begin{aligned}\nu(x_n, \dots, x_0) &= \nu(x_n, \dots, \hat{x}_i, \dots, x_0) \\ &= \nu(x_n, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_0) = \nu(x_n, \dots, \hat{x}_j, \dots, x_0),\end{aligned}$$

where the middle equality follows from the induction hypothesis. This completes the proof of (A).

PROOF OF (C). Let s_0 be some interior point of the simplex σ of S , and let $\nu(s_0)$ be the corresponding non-degenerate point. Then $\nu(s_0)$ is obtained by crossing out certain elements of the sequence $s_0 = (x_n, \dots, x_0)$. For any s in the simplex σ , let $\bar{\nu}(s)$ denote the point of S obtained by crossing out the corresponding elements of the sequence s . It follows from 2.2 that $s \sim \bar{\nu}(s)$ for every s in σ . But $\bar{\nu}$ maps σ linearly onto another simplex σ' of S . Since σ' contains the non-degenerate point $\nu(s_0)$ it is non-degenerate. This completes the proof of (C), and hence of Lemma 3.2.

PROOF OF LEMMA 3.3. Let $\eta: S \rightarrow \tilde{S}$ be the identification map. Define $\bar{p}: \tilde{S} \rightarrow X$ by $\bar{p}([x_n, \dots, x_0]) = x_n$. Since the composition $\bar{p}\eta: S \rightarrow X$ is clearly continuous, it follows from the definition of the identification topology that \bar{p} is continuous. Therefore the restriction p of \bar{p} to \tilde{E} is also continuous.

PROOF OF LEMMA 3.4. Let Δ be the subcomplex of $\tilde{S} \times \tilde{S}$ consisting of all pairs $([x_n, \dots, x_0], [y_m, \dots, y_0])$ such that x_0 and y_m lie in a common simplex of X . (Thus Δ contains all pairs with $x_0 = y_m$.) We will prove that the product map $\Delta \rightarrow \tilde{S}$ is continuous. Since $\tilde{S} \times \tilde{S}$ is a CW -complex (by 2.1) it is sufficient to verify that this map is continuous on each closed cell of Δ .

A cell of $\Delta \subset \tilde{S} \times \tilde{S}$ has the form $\eta(\sigma) \times \eta(\tau)$ where σ and τ are non-degenerate simplexes of S . It is clearly sufficient to prove that the composition $\sigma \times \tau \xrightarrow{\eta \times \eta} \Delta \rightarrow \tilde{S}$ is continuous. But this composition can be broken up into a continuous product $\sigma \times \tau \rightarrow S$ followed by $\eta: S \rightarrow \tilde{S}$. This completes the proof.

PROOF OF LEMMA 3.6. Let $E_n \subset S_n$ be the set of sequences (x_n, \dots, x_1, v_0) which end at v_0 . Let \tilde{E}_n be the image of E_n in \tilde{E} . Let $R_n = E_n \cap D$ be the set of degenerate sequences in E_n . (See proof of 3.2.)

We will prove that E_n and R_n are both contractible. By 2.5 this will imply that R_n is a strong deformation retract of E_n .

Now note that the map $E_n \rightarrow \tilde{E}_n$ carries the subcomplex R_n onto \tilde{E}_{n-1} , carries $E_n - R_n$ one-one onto $\tilde{E}_n - \tilde{E}_{n-1}$, and maps simplexes of E_n onto cells of \tilde{E}_n . Therefore, by 2.10, it will follow that \tilde{E}_{n-1} is a strong deformation retract of \tilde{E}_n .

Now since $\tilde{E}_0 \subset \tilde{E}_1 \subset \tilde{E}_2 \subset \dots$ are subcomplexes of the union \tilde{E} (where \tilde{E}_0 is defined to be the single point $[v_0, v_0]$), it will follow by 2.8 that \tilde{E} is contractible.

There is an obvious contraction of E_n which can be described as follows. Let T_n be the linear graph with vertices $[0], \dots, [n]$ and edges $[0, 1], \dots, [n-1, n]$. Then E_n can be considered as the set of all maps $(T_n, [0]) \rightarrow (X, v_0)$ which carry edges linearly into simplexes. A contraction of T_n is obtained by

first deforming the edge $[n - 1, n]$ into the vertex $[n - 1]$; then deforming $[n - 2, n - 1]$ into $[n - 2]$; etc. This induces a contraction of E_n .

To prove that R_n is contractible, first observe that $R_n = P_1 \cup \cdots \cup P_n \cup Q_1 \cup \cdots \cup Q_{n-1}$, where the subcomplex P_i is the set of all sequences in E_n for which $x_i = x_{i-1}$ and where Q_i is the set of sequences with $x_{i+1} = x_{i-1}$. By 2.7, in order to prove that R_n is contractible it is sufficient to prove that each intersection $R' = P_{i_1} \cap \cdots \cap P_{i_k} \cap Q_{j_1} \cap \cdots \cap Q_{j_k}$ is contractible.

Let R' be any such intersection, and let T' be a corresponding linear graph, obtained by identifying the two edges $[j_m + 1, j_m]$ and $[j_m - 1, j_m]$ of T_n , for $m = 1, \dots, k$; and by identifying all points of the edges $[i_m, i_m - 1]$, for $m = 1, \dots, h$. Then R' can be considered as the set of all maps $(T', [0]) \rightarrow (X, v_0)$ which carry edges linearly into simplexes.

Now observe that the graph T' is actually a tree. (This is easily proved by induction on the number of identifications made.) Hence T' can be contracted to a point by contracting one free edge after another, keeping the base point $[0]$ fixed. This induces a contraction of R' . Since R' is contractible, it follows that R_n is contractible, and therefore that \tilde{E} is contractible. This completes the proof of Theorem 3.1.

COROLLARY 3.7. *If Y is any connected space having the same homotopy type as a countable simplicial complex X in the weak topology, then there exists an ∞ -universal bundle with base space Y .*

Let $f: Y \rightarrow X$ be a homotopy equivalence. The universal bundle over X , together with f , induces a bundle \mathfrak{B} over Y . It is easily shown that \mathfrak{B} is a universal bundle.

In particular, this corollary applies to any connected, countable CW -complex Y . (For proof see [4] p. 239.)

4. Axiomatic characterization of homotopy groups

J.-P. Serre has pointed out that his construction of a contractible fibre space over any arc-connected base space can be used to give an axiomatic characterization of homotopy groups.² The following section contains such a characterization. The construction of Section 3 makes it possible to base our axioms on the notion of fibre bundle. [However the alternative characterization, based on fibre spaces, will be given in brackets.]

It will be convenient to ignore the group structure of the homotopy groups at first.

Consider the category consisting (1) of all triples (X, A, x) where X is a countable CW -complex, A is a subcomplex, and x is a vertex of A ; and (2) of all continuous maps between such triples. [Alternatively the category of all triples $x \in A \subset X$ of topological spaces, and all maps between such triples.]

² The possibility of such a characterization was conjectured by Steenrod and Eilenberg [1] p. 49. Added in proof: Much of the material of this section is contained in Kuranishi [5]. A proof of Lemma 4.3 is given in [6].

Let (X, x) stand for the triple (X, x, x) . Let $\pi_0(X, x)$ denote the set of all [arc]-components of X . For any map $f: (X, x) \rightarrow (Y, y)$ let $f_*: \pi_0(X, x) \rightarrow \pi_0(Y, y)$ denote the function induced by f .

THEOREM 4.1. *There exists one and, up to isomorphism, only one function which assigns*

- (a) *to each triple (X, A, x) in the category and each $i \geq 1$ a set $\pi_i(X, A, x)$,*
- (b) *to each triple and each $i \geq 1$ a function $\partial: \pi_i(X, A, x) \rightarrow \pi_{i-1}(A, x)$, and*
- (c) *to each map $f: (X, A, x) \rightarrow (Y, B, y)$ and each $i \geq 1$ a function $f_*: \pi_i(X, A, x) \rightarrow \pi_i(Y, B, y)$, such that the following seven axioms are satisfied.*

(1) *If $f: (X, A, x) \rightarrow (X, A, x)$ is the identity map, then so is $f_*: \pi_i(X, A, x) \rightarrow \pi_i(X, A, x)$.*

(2) *The identity $(gf)_* = g_*f_*$ holds for any maps $f: (X, A, x) \rightarrow (Y, B, y)$, $g: (Y, B, y) \rightarrow (Z, C, z)$.*

(3) *The identity $f_*\partial = \partial f_*$ holds in the square*

$$\begin{array}{ccc} \pi_i(X, A, x) & \xrightarrow{f_*} & \pi_i(Y, B, y) \\ \partial \downarrow & & \downarrow \partial \\ \pi_{i-1}(A, x) & \xrightarrow{f'_*} & \pi_{i-1}(B, y), \end{array}$$

where f' is the map induced by f , $i \geq 1$.

The inclusion maps $(A, x) \rightarrow (X, x) \rightarrow (X, A, x)$ induce a sequence

$$\cdots \rightarrow \pi_1(X, x) \rightarrow \pi_1(X, A, x) \xrightarrow{\partial} \pi_0(A, x) \rightarrow \pi_0(X, x).$$

(4) *The preceding sequence has the following “exactness” property. If a term and the third succeeding term both consist of a single point, then the function connecting the two intermediate sets is one-one onto.*

(5) *If f is homotopic to g then $f_* = g_*$.*

(6) *If $p: E \rightarrow X_0$ is the projection map of a fibre bundle [alternatively fibre space in the sense of Serre], where X_0 is an arc-component of X , let $q: (E, p^{-1}(A), e) \rightarrow (X, A, x)$ be induced by p . Then $q_*: \pi_i(E, p^{-1}(A), e) \rightarrow \pi_i(X, A, x)$ is one-one onto for $i \geq 1$ (assuming that both triples are in the category).*

(7) *$\pi_i(x, x)$ consists of a single point.*

PROOF. Since the existence theorem is clear, we must only prove uniqueness. Let $\{\pi_i, \partial, *\}$ and $\{\bar{\pi}_i, \bar{\partial}, \bar{*}\}$ be two such functions. Since $\pi_0(X, x)$ and $\bar{\pi}_0(X, x)$ are identical by definition we have the identity map

$$\phi_0: \pi_0(X, x) \rightarrow \bar{\pi}_0(X, x)$$

which satisfies $\bar{f}_*\phi_0 = \phi_0f_*$ for any map $f: (X, x) \rightarrow (Y, y)$. We will construct isomorphisms (i.e. one-one onto functions)

$$\phi_i: \pi_i(X, A, x) \rightarrow \bar{\pi}_i(X, A, x)$$

by induction on i , so as to satisfy

(A) the identity $\bar{\partial}\phi_i = \phi_{i-1}\partial$ holds in the square

$$\begin{array}{ccc} \pi_i(X, A, x) & \xrightarrow{\phi_i} & \bar{\pi}_i(X, A, x) \\ \partial \downarrow & & \downarrow \partial \\ \pi_{i-1}(A, x) & \xrightarrow{\phi_{i-1}} & \bar{\pi}_{i-1}(A, x) \end{array}$$

for any triple (X, A, x) ; and

(B) the identity $\bar{f}_*\phi_i = \phi_i f_*$ holds for any map $f: (X, A, x) \rightarrow (Y, B, y)$.

For each triple (X, A, x) in the category we may choose a triple (X', A', x') of the same homotopy type, where X' is a countable simplicial complex in the weak topology with subcomplex A' and vertex x' . (For a single CW -complex this is proved in [4] p. 239. A proof for triples can be given along the same lines.) Construct an ∞ -universal bundle $p: \tilde{E} \rightarrow X'$ as in Section 3. It is clear that $p^{-1}(A')$ will be a subcomplex of \tilde{E} . Consider the diagram

$$\begin{array}{ccccccc} \cdots & \rightarrow & \pi_i(\tilde{E}, e) & \rightarrow & \pi_i(\tilde{E}, p^{-1}(A'), e) & \xrightarrow{\partial} & \pi_{i-1}(p^{-1}(A'), e) \rightarrow \pi_{i-1}(\tilde{E}, e) \rightarrow \cdots \\ & & \downarrow p_* & & \downarrow & & \\ & & \pi_i(X', A', x') & \longrightarrow & \pi_{i-1}(A', x') & & \\ & & \downarrow g_* & & \downarrow & & \\ & & \pi_i(X, A, x) & \longrightarrow & \pi_{i-1}(A, x) & & \end{array}$$

together with the corresponding diagram for the sets $\bar{\pi}$, where the top line is part of the “exact” sequence of the triple $(\tilde{E}, p^{-1}(A'), e)$, and where g is a homotopy equivalence. Define $\phi_i: \pi_i(X, A, x) \rightarrow \bar{\pi}_i(X, A, x)$ by

$$\phi_i = \bar{g}_* \bar{p}_* \bar{\partial}^{-1} \phi_{i-1} \partial p_*^{-1} g_*^{-1}.$$

It is easily verified that each of the functions $\bar{g}_*, \bar{p}_*, \dots, g_*^{-1}$ is an isomorphism, and therefore that ϕ_i is an isomorphism.

The identity (A) can be verified by considering commutativity relations in the above diagram. The identity (B) is somewhat harder. It is clearly sufficient to consider the case where X and Y are simplicial complexes. Let $f': (X', A', x) \rightarrow (Y, B, y)$ be a simplicial approximation to f , where X' is a subdivision of X . Let $p: \tilde{E} \rightarrow X, p': \tilde{E}' \rightarrow X', q: \tilde{F} \rightarrow Y$ be the ∞ -universal bundles which were constructed in Section 3. Then f' induces a map $\bar{f}': \tilde{E}' \rightarrow \tilde{F}$ which satisfies $q\bar{f}' = f'p'$. In fact \bar{f}' is defined by $\bar{f}'([x_n, \dots, x_0]) = [f'(x_n), \dots, f'(x_0)]$. Similarly the identity map $j: X' \rightarrow X$ induces $\bar{j}: \tilde{E}' \rightarrow \tilde{E}$. The identity (B) can now be verified by considering commutativity relations in the following diagram:

$$\begin{array}{ccccc}
 & \pi_i(X, A, x) & \xleftarrow{p_*} & \pi_i(\tilde{E}, p^{-1}(A), e) & \xrightarrow{\mathfrak{d}} & \pi_{i-1}(p^{-1}(A), e) \\
 & \uparrow j_* & & \uparrow \tilde{j}_* & & \uparrow \\
 f_* & \pi_i(X', A', x) & \xleftarrow{p'_*} & \pi_i(\tilde{E}', p'^{-1}(A'), e') & \xrightarrow{\mathfrak{d}} & \pi_{i-1}(p'^{-1}(A'), e') \\
 & \downarrow f'_* & & \downarrow \tilde{f}'_* & & \downarrow \\
 & \pi_i(Y, B, y) & \xleftarrow{q_*} & \pi_i(\tilde{F}, q^{-1}(B), e'') & \xrightarrow{\mathfrak{d}} & \pi_{i-1}(q^{-1}(B), e'')
 \end{array}$$

together with the corresponding diagram for the sets $\tilde{\pi}$. This completes the proof. [The alternative theorem, stated in brackets can be proved in much the same way, except that the simplicial approximation techniques are no longer needed.]

THEOREM 4.2. *There are exactly two ways of introducing a group structure into the sets $\pi_i(X, A, x)$, $i \geq 2$, and $\pi_1(X, x)$, in such a way that the functions \mathfrak{d} and f_* are homomorphisms. These two group structures are related by the identity $(a \cdot b)_1 = (b \cdot a)_2$.*

Since the existence theorem is clear we need only prove uniqueness. During the proof of Theorem 4.1, every group π_i , $i \geq 2$, was put into one-one correspondence with some group $\pi_1(X, x)$ by a sequence of isomorphisms. Therefore it is sufficient to carry out the proof for the fundamental group $\pi_1(X, x)$.

Let ab denote the customary product of a and b in $\pi_1(X, x)$ and suppose that $a \circ b$ is some new product such that

(1) for any space X , $\pi_1(X, x)$ is a group under this product, and

(2) the induced function $f_*: \pi_1(X, x) \rightarrow \pi_1(Y, y)$ is a homomorphism with respect to this product for any map f .

Let T be the space which consists of two circles, intersecting at a single point t . Then $\pi_1(T, t)$ is a free group on two generators α and β , under the usual product. Given any two elements $a, b \in \pi_1(X, x)$ we can clearly construct a mapping $f: (T, t) \rightarrow (X, x)$ so that $f_*(\alpha) = a$, $f_*(\beta) = b$. Then $a \circ b = f_*(\alpha \circ \beta)$. In terms of the usual group structure, $\alpha \circ \beta$ is equal to some word $w(\alpha, \beta)$ of the free group. Hence

$$f_*(\alpha \circ \beta) = f_*(w(\alpha, \beta)) = w(f_*(\alpha), f_*(\beta)) = w(a, b)$$

and therefore

$$a \circ b = w(a, b).$$

Thus to describe the new product completely, it is sufficient to determine this word $w(\alpha, \beta)$ in the free group on two generators.

The word $w(\alpha, \beta)$ has the following two properties

- (a) $w(1, \beta) = \beta, \quad w(\alpha, 1) = \alpha$
- (b) $w(\alpha, w(\beta, \gamma)) = w(w(\alpha, \beta), \gamma)$

(where (b) is an identity in the free group on three generators). To prove (a) note that the identity element $1 \in \pi_1(T, t)$ can be defined as the image of the injection homomorphism $i_*: \pi_1(t, t) \rightarrow \pi_1(T, t)$. It follows that the new product operation must have this same identity element.

To prove (b) choose a space X for which $\pi_1(X, x)$ is a free group on three generators α, β, γ . Then formula (b) is merely the associative law $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$. To complete the proof of Theorem 4.2 it is only necessary to prove the following lemma.

LEMMA 4.3. *If a reduced word $w(\alpha, \beta)$ in the free group on two generators satisfies conditions (a) and (b), then either $w(\alpha, \beta) = \alpha\beta$ or $w(\alpha, \beta) = \beta\alpha$.*

The proof is a long but easy exercise in the manipulation of reduced words. Details will not be given.

5. A property of the universal bundle

The following property of an n -universal bundle with group G is well known ([3] p. 101). If Y is an n -dimensional complex, then any bundle over Y with group G is induced by a map of Y into the base space of the universal bundle. The following theorem describes a dual property.

If a bundle over X with group \tilde{G} is given, then every continuous homomorphism $h: \tilde{G} \rightarrow G$ induces a principal bundle over X with group G . In fact if $g_{ij}: V_i \cap V_j \rightarrow \tilde{G}$ are the coordinate transformations of the given bundle, then $hg_{ij}: V_i \cap V_j \rightarrow G$ are those of the induced bundle.

THEOREM 5.1. *Let $p: \tilde{E} \rightarrow X$ be the universal bundle of Section 3, with group \tilde{G} . Then any principal bundle \mathcal{B} over X with any group G is induced by a continuous homomorphism $h: \tilde{G} \rightarrow G$.*

The proof will be based on the construction of a certain slicing function for the bundle \mathcal{B} . Let $q: E \rightarrow X$ be any bundle with group G and coordinate functions $\phi_i: V_i \times F \rightarrow E$. Let $F_x = q^{-1}(x)$ be the fibre over x , and define $\phi_{ix}: F \rightarrow F_x$ by $\phi_{ix}(f) = \phi_i(x, f)$. A *slicing function* for this bundle is a function

$$\omega_{x,y}: F_y \rightarrow F_x$$

defined for all pairs (x, y) in some symmetric subset U of $X \times X$, which is continuous as a function of three variables, and such that $\omega_{x,x}: F_x \rightarrow F_x$ is the identity map for each $x \in X$.

A slicing function will be called *symmetric* if $\omega_{y,x} = \omega_{x,y}^{-1}$ for each $(x, y) \in U$. It will be called a *bundle slicing function* if

(1) The map $\omega_{x,y}$ is admissible for each $(x, y) \in U$. That is, if $x \in V_i, y \in V_j$, then the map

$$\phi_{ix}^{-1} \omega_{x,y} \phi_{jy}: F \rightarrow F$$

coincides with the operation of an element $g = \tilde{g}_{ij}(x, y)$ of G . And

(2) the associated map $\tilde{g}_{ij}: (V_i \times V_j) \cap U \rightarrow G$ is continuous.

THEOREM 5.2. *If X is a countable simplicial complex in the weak topology and $U \subset X \times X$ is the union over all simplexes $\tau \subset X$ of $\tau \times \tau$, then any bundle over X possesses a symmetric bundle slicing function with respect to U .*

Theorem 5.1 follows from 5.2 as follows. Let $\omega_{x,y}$ be a symmetric bundle slicing function for the principle bundle \mathfrak{B} having projection $q: E \rightarrow X$ and group G . Identify the groups G, \tilde{G} with the fibres $q^{-1}(v_0), p^{-1}(v_0)$ respectively. Let $e_0 \in q^{-1}(v_0)$ be the identity element of G . Define the function $h: \tilde{E} \rightarrow E$ by

$$h([x_n, x_{n-1}, \dots, x_1, v_0]) = \omega_{x_n, x_{n-1}} \cdots \omega_{x_2, x_1} \omega_{x_1, v_0}(e_0).$$

Then it is easily verified that h is a well defined mapping which carries \tilde{G} homomorphically into G , and that the bundle \mathfrak{B} is induced by this homomorphism.

The proof of 5.2 follows. Let $A_{x,y}$ denote the set of all admissible maps $F_y \rightarrow F_x$. Let A be the union of $A_{x,y}$ over all $(x, y) \in U$. There is an obvious projection $a: A \rightarrow U$. It will be proved that A is a fibre bundle over U with fibre (but not group) G .

As coordinate neighborhoods take $U_{ij} = (V_i \times V_j) \cap U$. The coordinate function

$$\psi_{ij}: U_{ij} \times G \rightarrow A$$

is defined as the function which assigns to each $((x, y), g)$ the map

$$F_y \xrightarrow{\phi_{jy}^{-1}} F \xrightarrow{g \cdot} F \xrightarrow{\phi_{ix}} F_x.$$

These coordinate functions can be used to define a topology for A , and the rest of the bundle structure is easily defined. (The group of this bundle turns out to be $G \times G$ modulo the center of its diagonal; the action on the fibre G being given by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$.) Let $\Delta \subset U$ be the diagonal of $X \times X$.

A cross-section $c: U \rightarrow A$ of this bundle clearly gives rise to a bundle slicing function $\omega_{x,y}$ (defined by $\omega_{x,y}(e) = c(x, y)(e)$) for the original bundle, providing only that c maps each $(x, x) \in \Delta$ into the identity map

$$I(x, x): F_x \rightarrow F_x.$$

Let $T: A \rightarrow A$ be the map which carries $f: F_y \rightarrow F_x$ into $f^{-1}: F_x \rightarrow F_y$. The corresponding symmetry $\bar{T}: U \rightarrow U$ of the base space is defined by $\bar{T}(x, y) = (y, x)$. Then the cross-section $c: U \rightarrow A$ corresponds to a symmetric slicing function if and only if it satisfies $Tc = c\bar{T}$. Thus our objective is now to prove:

PROPOSITION 5.3. *There exists a cross-section $c: U \rightarrow A$ which extends the given cross-section $I: \Delta \rightarrow A$ and which satisfies $Tc = c\bar{T}$.*

Let G_0 be the arc-component of the identity element in G . Observe that the bundle space A contains a subspace A_0 which may be considered as a bundle over U with fibre G_0 . In fact for each cell $\tau \times \tau$ of U let $a_0^{-1}(\tau \times \tau)$ denote the arc-component of $a^{-1}(\tau \times \tau)$ which contains the points $I(x, x)$ for $x \in \tau$. Since $a^{-1}(\tau \times \tau)$ is homeomorphic to $(\tau \times \tau) \times G$ it follows that $a_0^{-1}(\tau \times \tau)$ is homeomorphic to $(\tau \times \tau) \times G_0$. The union of these sets $a_0^{-1}(\tau \times \tau)$ over all τ gives the required space A_0 .

Since the fibre G_0 of the bundle $a_0: A_0 \rightarrow U$ is n -simple for all n , the homotopy groups $\pi_n(a_0^{-1}(x, y))$ form a bundle \mathfrak{B}_n of coefficients over U (that is a system of local coefficients). The restriction $T_0: A_0 \rightarrow A_0$ of T to A_0 induces a homomorphism $T_*: \mathfrak{B}_n \rightarrow \mathfrak{B}_n$.

It will be necessary to consider the equivariant cohomology groups of $U \bmod \Delta$ with coefficients in \mathfrak{B}_n . First note that $X \times X$ is a cell complex with subcomplex U . Passing to the first derived complex, U will be considered as a simplicial complex with subcomplex Δ . The equivariant cochain group $C^k(U \bmod \Delta, \mathfrak{B}_n, T)$ is now defined. (An element of C^k is a function γ which assigns to each k -simplex σ of U an element $\gamma(\sigma)$ of the group of cross-sections of $\mathfrak{B}_n|_\sigma$, subject to the conditions $\gamma(\sigma) = 0$ for $\sigma \subset \Delta$, and $\gamma(\bar{T}(\sigma)) = T_*(\gamma(\sigma))$ for all σ .)

A strong deformation retraction $r: U \times [0, \frac{1}{2}] \rightarrow U$ of U into Δ is defined by $r((x, y), t) = ((1-t)x + ty, (1-t)y + tx)$. Since this commutes with the symmetry $\bar{T}: U \rightarrow U$, it follows that the cohomology groups $H^k(U \bmod \Delta, \mathfrak{B}_n, T)$ are all trivial.

Proposition 5.3 now follows by an obstruction argument. Let U^n be the n -skeleton of U . Suppose by induction that a cross section $c_{n-1}: (\Delta \cup U^{n-1}) \rightarrow A_0$ has been constructed so that

- (1) $c_{n-1}(x, x) = I(x, x)$ for all $(x, x) \in \Delta$, and
- (2) $c_{n-1}\bar{T} = T_0 c_{n-1}$.

Let $\gamma \in C^n(U \bmod \Delta, \mathfrak{B}_{n-1})$ be the obstruction to the extension of c_{n-1} to $(\Delta \cup U^n)$. Clearly γ is an equivariant cocycle. Since the equivariant cohomology group $H^n(U \bmod \Delta, \mathfrak{B}_{n-1}, T)$ is trivial, it follows that, after making a symmetric modification of c_{n-1} on the $(n-1)$ -simplexes of $U - \Delta$, we can extend it to $\Delta \cup U^n$. The extension c_n can be chosen arbitrarily on half of the n -simplexes of $U - \Delta$. It is then determined for the other half by the symmetry condition $c_n\bar{T} = T_0 c_n$.

Continuing by induction we construct the required cross-section c . The continuity of c follows from the fact that U is a complex in the weak topology (by 2.1). This completes the proof.

One consequence of theorem 5.1 is the following.

COROLLARY 5.4. *Let X and \tilde{G} be as in Section 3. If X is the base space of a second ∞ -universal bundle with group G , then there is a continuous homomorphism $\tilde{G} \rightarrow G$ which induces isomorphisms of the homotopy groups. In particular if G is a CW-complex, then it has the same homotopy type as \tilde{G} .*

The proof is clear.

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