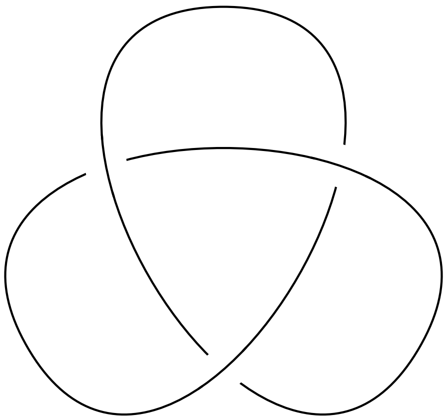
A voyage into the algebras

Weronika Jakimowicz
330006

Julia Walczuk
332742

2023-2024



1 Introduction

1.1 What does it mean to color a knot?

What do we need?

- R - commutative ring with identity

- D - diagram of knot K with s segments and x crossings

- $\phi : M^3 \to M$ - function that dictates the rules of our coloring (and induces two operators $M^2 \to M^2$)

In order for trivial coloring to work, $\phi(m, m, m) = 0$ for all $m \in M$. This means that if we take $\phi(u, v, w) = au+bv+cw$ then $(a+b+c) \in \mathrm{Ann}(M)$.

In the most general case, $R = \mathbb{Z}[s, t, t^{-1}]/\{s(s+t-1)\}$ and $\phi(u, v, w) = su + tv - w$.

Given those we can define $f : M^s \to M^x$, which assigns values from M to segments of D according to the rules set by ϕ.

This yields an exact sequence

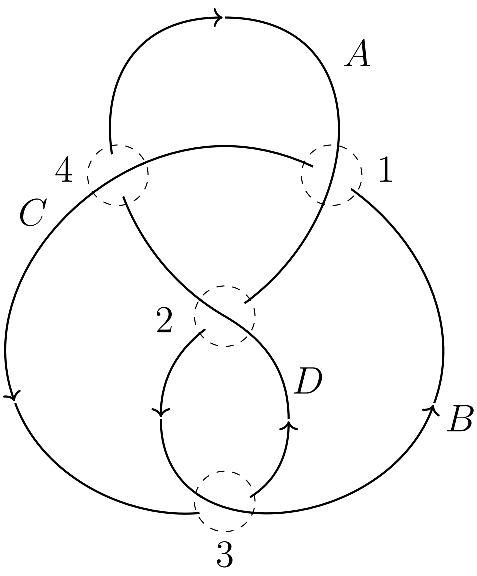
$$0 \longrightarrow \ker f \hookrightarrow M^s \xrightarrow{f} M^x \twoheadrightarrow \operatorname{coker} f \longrightarrow 0$$

We know that $\ker f$ always contains colorings - especially the trivial one. We expect $\operatorname{coker} f$ to contain some information about non-trivial colorings admissible.

1.2 Smith's normal form

Function f can be expressed as a $s \times x$ matrix with elements from M - we can make it into "diagonal form" where non-zero elements lower are divisible by elements at the top. This gives us information about $\ker f$ and $\operatorname{coker} f$.

Example 1.1. *Take $K = 4_1$ and $R = \mathbb{Z}$, which takes $t = 1$ and $s = 2$. At the beginning, $M = \mathbb{Z}$.*



$$f = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \end{pmatrix}$$

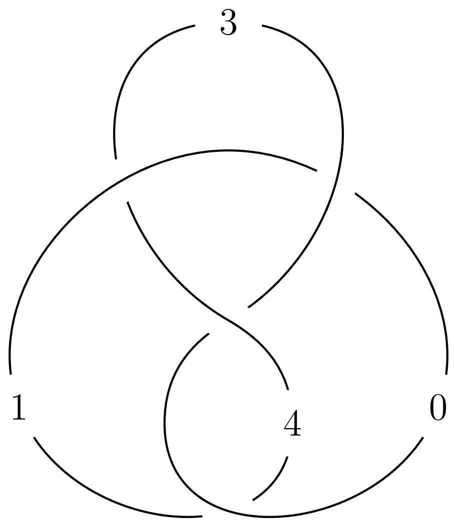
in normal form:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we know that $\ker f = \mathbb{Z}$ *and* $\operatorname{coker} f = \mathbb{Z}_5 \oplus \mathbb{Z}$*. Thus there is only a trivial coloring over* $\mathbb{Z}$ *but if we change* $\mathbb{Z}$ *to* $\mathbb{Z}_5$ *we get*

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and now $\ker f' = \mathbb{Z}_5 \oplus \mathbb{Z}_5$ *and* $\operatorname{coker} = \mathbb{Z}_5 \oplus \mathbb{Z}_5$*. Thus there is a coloring using elements of* $\mathbb{Z}_5$*, for example:*



In a more general case, we would orient the diagram and use $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ *as the ring:*

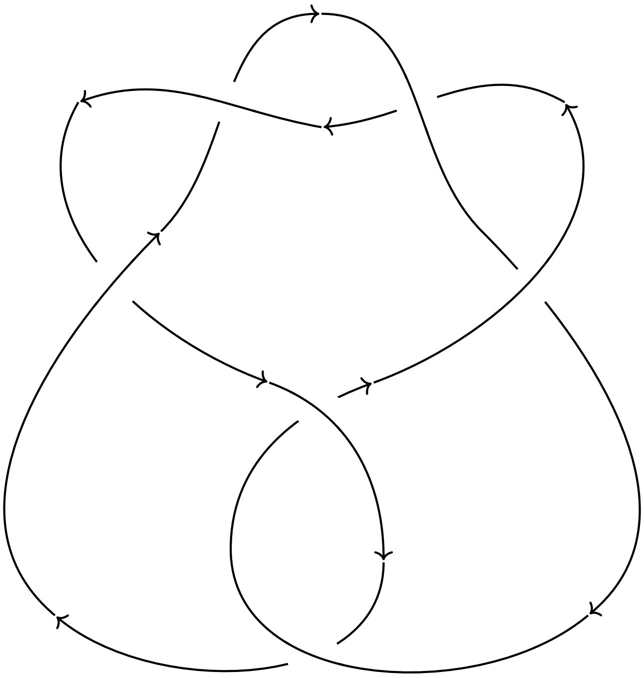
$$f = \begin{pmatrix} 1-t & t & -1 & 0 \\ t^{-1} & -1 & 0 & 1-t^{-1} \\ 0 & 1-t^{-1} & t^{-1} & -1 \\ -1 & 0 & 1-t & t \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & t^2-3t+1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the Alexander polynomial of 4_1 *is equal to* $t - 3 + t^{-1}$*, which is the same up to multiplication by a unit to the last term of the Smith's normal form.*

1.3 Misc

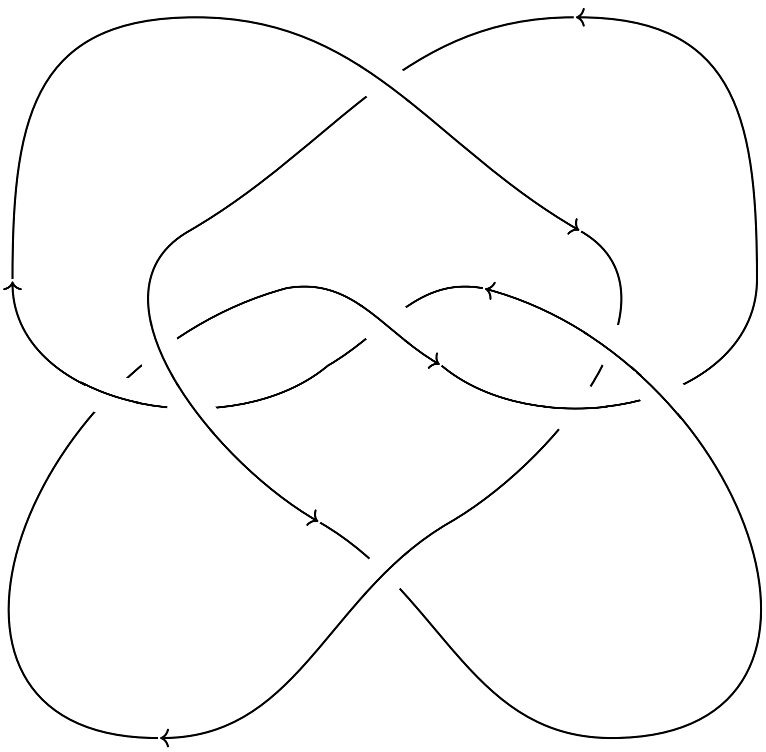
$K = 6_1$



Tutaj mamy

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2}+5t^{-1}-2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$K = 9_{46}$



$$f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t-t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2}-2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

References