Knot colorings and homological invariants

(Kolorowania węzłów i niezmienniki homologiczne.)

Weronika Jakimowicz
330006

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Promotor: prof. Tadeusz Januszkiewicz

Abstract

For a knot $K \subseteq S^3$ its knot group $G = \pi_1(S^3 - K)$ is a starting point for many knot invariants. One such invariant is the Alexander module K_G^{ab}, the abelianized kernel of homomorphism $G \to \mathbb{Z}$ with a $\mathbb{Z}[\mathbb{Z}]$-module structure. Associated with the Alexander module is Alexander matrix A_D with $\text{coker}(A_D) = K_G^{ab}$. The same matrix can be obtained by coloring any diagram of K with a properly chosen palette. In the following paper, the connection between the algebraical, topological and combinatorial definitions of the Alexander matrix and module are given along with connections between them.

Introduction

Let K be a knot embedded in S^3 and D its diagram. The fundamental group of the knot complement $\pi_1(S^3 - K)$, called the knot group, is a knot invariant. It is a basis for many simpler invariants, such as the Alexander polynomial [Ale28] or the Alexander module (see definition 1.4)

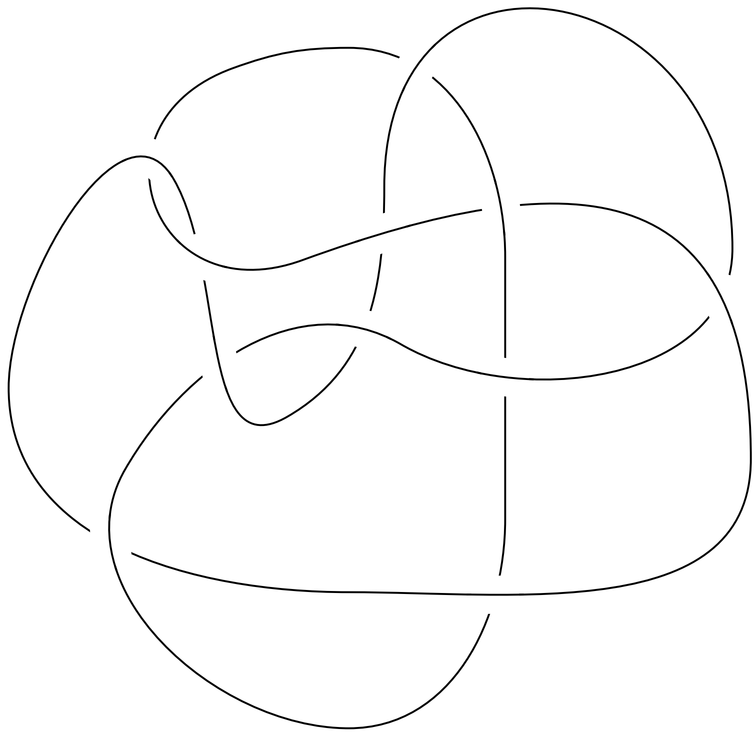
On the other hand, one can approach the search for knot invariants from the perspective of diagrams. A coloring is an assignment of elements of a module M to segments of the diagram. Assignments that satisfy specific conditions (see lemma 3.4) yield knot invariants. The following thesis is concentrated on finding a connection between the information about a knot K obtained from its group with the information obtained from a diagram coloring.

Section 1 defines the fundamental concepts of this paper, such as the knot group (see definition 1.1) its metabelianization (see definition 1.3) and the Alexander module (see definition 1.4). Two equivalent definitions for said modules are presented: an algebraic one and a topological one. In a purely algebraic sense, the Alexander module is the abelianized subgroup $[[G, G], [G, G]]$ of the knot group G, with $\mathbb{Z}$ action induced by abelianization homomorphism $G \to \mathbb{Z}$, while from a topological point of view it is the first homology module of the infinite cyclic cover (see definition 1.5). An important result finishing this section is that the Alexander module is a torsion module (see proposition 1.3).

Section 2 is dedicated to the Alexander matrix (see definition 2.1) and its properties. We start by showing an algorithm for obtaining the presentation of the Alexander module, which is then used to construct a resolution of said module. The Alexander matrix is defined as the matrix of the homomorphism $\mathbb{Z}[\mathbb{Z}]^n \, \mathbb{Z}[\mathbb{Z}]^{n-1}$ with cokernel isomorphic to the Alexander module. Over the ring of fractions F [AM69, Cahpter 2] of $\mathbb{Z}[\mathbb{Z}]$, this matrix is surjective and creates a particularly interesting short exact sequence of F-modules.

In the last section 3 diagram colorings are discussed, starting with a definition of a palette (see definition 3.1). A set of palettes of particular interest are the Alexander palette and all its images through homomorphisms induced by either a ring or a module homomorphism. From diagram coloring a matrix is obtained, called the color checking matrix (see definition 3.3). Properties of palettes that ensure the invariant nature of color checking matrices are outlined and proven. Two knots, K11n85 and Kn164, having the same Alexander polynomial, are distinguished using a reduced Smith normal form of color checking matrix (see definition 3.6). The paper ends with a proof that this reduced normal form is a knot invariant for Alexander palettes.

This thesis is a result of cooperation between Julia Walczuk and myself, and is written under supervision of prof. Tadeusz Januszkiewicz.



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1 Preliminaries

1.1 Knots and diagrams

In mathematical terms, a knot is a smooth embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \hookrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [Lik97]. If two points are mapped to one by this projection, we say that a small neighbourhood of this point which looks locally like $-|-$ is a crossing.

S^1 is orientable, thus we can chose an orientation for any knot and, as a consequence, its diagram.

Two tame knots K_1 and K_2 are equivalent if we can deform one into the other [Mur96]. This translates to an equivalence of diagrams, which is generated by comparing diagrams that are exactly the same save for an interior of some disc in R^2. If inside of said disc the diagrams differ by one of **Reidemeister moves**, we say that they are equivalent. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D, 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [Pol10].

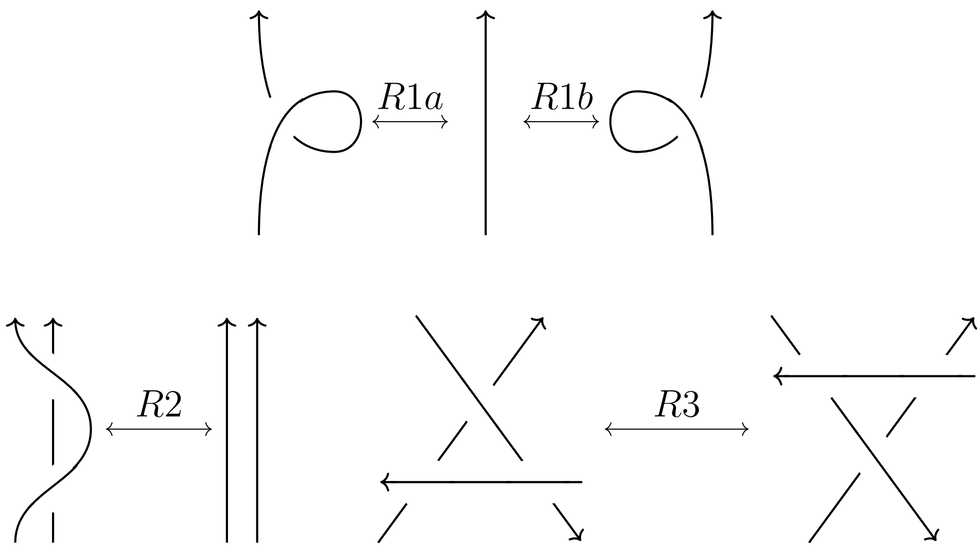


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

1.2 Knot group

Let K be a knot and D be its oriented diagram with s segments and x crossings. A segment of a diagram is a line of the diagram between two

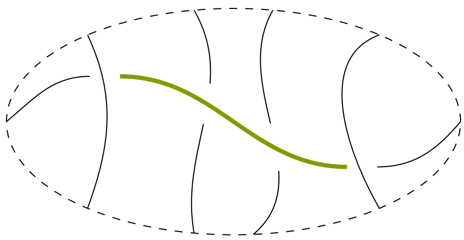


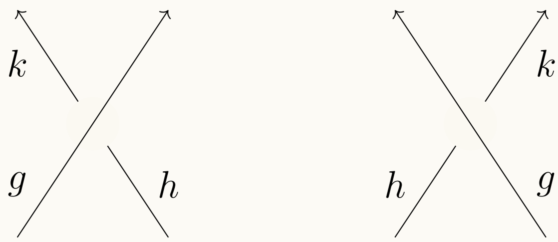
Figure 2: The green line is a segment of the diagram.

crossings in which it is disappears under another line, see fig. 2.

Definition 1.1 : knot group.
Let $K \subseteq S^3$ be a knot. The fundamental group of knot complement $X = S^3 - K$, $\pi_1(X)$, is called the **knot group** of K.

Although the knot itself is always S^1, the knot group has usually an interesting yet difficult structure. The most commonly used presentation of the knot group is called **the Wirtinger presentation**.

Definition 1.2 : Wirtinger presentation.
Given an oriented diagram D of knot K with segments a_1, a_2, ..., a_s, which follow the orientation, and crossings c_1, ..., c_x the knot group $\pi_1(X)$ can be represented as $\pi_1(X) = \langle S \mid R \rangle$, where S is the set of segments a_1, ..., a_s of D and relations R correspond to crossings in the manner described in the diagram below



$gkg^{-1} = h$ $\qquad$ $ghg^{-1} = k$

Presentation $\langle S \mid R \rangle$ described above is called the **Wirtinger presentation** [Liv93, Chapter 6].

By applying the Mayer-Vietoris sequence to $S^3 = K \cup_{T^2} (S^3 - K)$ or noticing that every two generators are conjugate, one can conclude that the abelianization of the knot group is always $\mathbb{Z}$. This leads to a short exact sequence

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0.$$

The presentation of the group $K_G = \ker(ab : G \to \mathbb{Z}) = [G, G]$ obtained from a Wirtinger presentation of G is discussed in section 2.1.

Lemma 1.1.
For any group G, the commutator of its commutator K_G is a normal subgroup: $[K_G, K_G] = [[G, G], [G, G]] \triangleleft G$.

Proof. The commutator subgroup is a characteristic subgroup, since for any automorphism $\phi : G \to G$

$$\phi(hgh^{-1}g^{-1}) = \phi(h)\phi(g)\phi(h)^{-1}\phi(g)^{-1} \in K_G = [G, G].$$

Conjugation by any element $g \in G$ is an automorphism of the commutator K_G. Thus it preserves its commutator subgroup $[K_G, K_G]$. $\square$

The exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{1}$$

is exact.

Definition 1.3 : metabelianization.
The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of G.

Proposition 1.2.

$$G^{mab} \cong K_G^{ab} \rtimes \mathbb{Z}$$

Proof. By lemma 1.1 we know that $[K_G, K_G] \triangleleft K_G$ are normal subgroups of G, then by the third isomorphism theorem, $K_G^{ab} = K_G/[K_G, K_G]$ is also a normal subgroup in $G/[K_G, K_G]$. The sequence (1) is split because $\mathbb{Z} \to G^{mab} \to \mathbb{Z}$ defined by $1 \mapsto a \mapsto 1$, where a is any generator of G^{mab} is the identity map on $\mathbb{Z}$. This completes the proof. $\square$

Proposition 1.2 implies that there exists an action of $\mathbb{Z}$ on the $\mathbb{Z}$-module K_G^{ab}. Thus, a $\mathbb{Z}[\mathbb{Z}]$-module structure is admissible on K_G^{ab}. We will return to the concept of metabelianization and K_G^{ab} in section 2, specifically in section 2.1. For the time being, let us assign a name to K_G^{ab}:

Definition 1.4 : Alexander module.
Given a group G with a homomorphism $g : G \to \mathbb{Z}$, the abelianization of the kernel $\ker(g)$, K_G^{ab}, with $\mathbb{Z}[\mathbb{Z}]$-module structure is called the **Alexander module** of G. If G is a knot group, then $\ker(g) = [G, G] = K_G$ and it is the Alexander module of the knot K.

1.3 Infinite cyclic covering

Let X be the complement of a knot K ($X = S^3 - K$). Take $\widetilde{X}$ to be its universal covering, meaning that it is simply connected. The knot group $G = \pi_1(X)$ of X acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G, G] \triangleleft G$ also acts on $\widetilde{X}$ and the quotient

$$\overline{X} = \widetilde{X}/K_G$$

has a fundamental group

$$\pi_1(\overline{X}) = K_G$$

no matter the choice of basepoint in $\widetilde{X}$.

Since K_G is a normal subgroup, then $G/K_G = G^{ab} = \mathbb{Z}$ also acts on the quotient space $\overline{X}$. This extends to the homology groups of $\overline{X}$ and allows a $\mathbb{Z}[\mathbb{Z}]$-module structure on them.

Definition 1.5 : infinite cyclic covering.
The quotient space $\boldsymbol{\overline{X} = \widetilde{X}/K_G}$ is called the **infinite cyclic covering** of X.

Looking at homology groups of $\overline{X}$, we have the following equality

$$H_1(\overline{X}, \mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

Working with homology modules of an infinite cyclic cover of X instead of K_G^{ab} directly is beneficial when proving some properties of K_G^{ab}, i.e. that it is a torsion module in proposition 1.3.

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{array}{ccc} \widetilde{X} & \circlearrowleft & G \\ \downarrow & & \\ \overline{X} & \circlearrowleft & G/[G,G] \\ \downarrow & & \\ X = S^3 - K & & \end{array}$$

A **Seifert surface** S of knot K is an orientable surface with boundary embedded in S^3 such that $\partial S = K$. Take a countable amount of X, with S without its boundary embedded, and label each with an element from $\mathbb{Z}$. We might now cut each of the copies of X along the Seifert surface of K and identify the $+$ side of S from the i-th copy of X with the $-$ side of S from the $(i+1)$-th copy of X. The arising space with a projection to one copy of X is an infinite cyclic cover of X.

Imagine that each copy of X inside of $\overline{X}$ is a box labeled with some integer k such that copies of X sharing the Seifert surface are labeled with consecutive integers. The ring action of $\mathbb{Z}[\mathbb{Z}]$ on $\overline{X}$ is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to $\overline{X}$.

Proposition 1.3.
The $\mathbb{Z}[\mathbb{Z}]$-module $K^{ab} = H_1(\overline{X}, \mathbb{Z})$ is a torsion module.

Proof. Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \to C_*(\overline{X})$$

$$f(x) = (1-t)x.$$

It translates to removing from a cycle in the $(i+1)$-th box a corresponding cycle in the i-th box. From this it is an immediate result that $\ker f = 0$ and that $\operatorname{coker} f = C_*(X)$: after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with cycles from just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces a long exact homology sequence

$$\begin{tikzcd}
\dots \arrow[r] & H_2(X,\mathbb{Z}) \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r] & H_1(X,\mathbb{Z}) \arrow[dlll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (Z) [near end]\tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(X,\mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

As was mentioned previously, the following equality holds:

$$H_1(X,\mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}\,.$$

Now, because X is homology circle, then $H_2(X,\mathbb{Z}) = 0$ (one can easily check it for themselves using Alexander duality). Both X and $\overline{X}$ are connected implying that

$$H_0(X,\mathbb{Z}) = H_0(\overline{X},\mathbb{Z}) = \mathbb{Z}\,.$$

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "0"] & \mathbb{Z} \arrow[dlll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (Z) [near end]\tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Rewriting the sequence above we easily get that homomorphism $1-t$ is actually an isomorphism and $H_1(\overline{X},\mathbb{Z}) \cong (1-t)H_1(\overline{X},\mathbb{Z})$, which allows us to use the Nakayama's lemma [AM69, Proposition 2.6] to conclude that there exists $x \in \mathbb{Z}[\mathbb{Z}]$ such that

$$xH_1(\overline{X},\mathbb{Z}) = 0.$$

□

2 Resolution of the Alexander module

2.1 Presentation of the Alexander module

Take $G = \langle S \mid R \rangle$ to be the Wirtinger presentation of G obtained from oriented diagram D (see definition 1.2). Because K is a knot and not a link, we know that the number of segments is equal to the number of crossings.

Lemma 2.1.
If K is a knot and D its diagram with $x > 0$ crossings and $s > 1$ segments, then $s = x$.

Proof. Every crossing has two segments that disappear under it. Let d be the number of segment ends that go below a crossing then

$$2 \cdot x = d.$$

Every segment starts and ends in some crossing, thus

$$2 \cdot s = d.$$

From those two equalities we have that $d = 2x = 2s \implies x = s$. $\square$

Let $a_1, ..., a_n$ be the generators of G agreeing with the orientation of the diagram D and $x_1, ..., x_n$ its relations. The homomorphism of abelianization of G is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, ..., n$. In order to obtain a presentation of K_G, the kernel of abelianization, we need to change the set of generators of G to

$$\{a_1, A_2 = a_2 a_1^{-1}, ..., A_n = a_n a_1^{-1}\}.$$

Proposition 2.2 : presentation of K_G.
The group K_G has an infinite presentation with generators

$$\{b_{i,k} = a_1^k A_i a_1^{-k} \ : \ i = 2, ..., n, \ k \in \mathbb{Z}\}$$

and relations

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

for $k \in \mathbb{Z}$, where $a_k = a_i a_j a_i^{-1}$ was a relation in G.

Proof. To begin with, $(\forall\ i > 1)\ A_i \mapsto 0$ by abelianization of G. Thus $A_2, ..., A_n$ are some of the generators of K_G. However, for each $i = 2, ..., n$ and $k \in \mathbb{Z}$ the following is an element of K_G:

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the presentation of K_G as an abelian group is infinite with (possibly redundant) generators

$$\{b_{i,k}\ :\ i = 2, ..., n,\ k \in \mathbb{Z}\}.$$

Changing generators of G induced a change in relations. Suppose that the following relation was true in G

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of a_1 the following two relations can be obtained

$$a_1^{-1} A_k a_1 = (a_1^{-1} A_i a_1) A_j A_i^{-1}$$
$$a_1^{-3} A_k a_1^3 = (a_1^{-3} A_i a_1^3)(a_1^{-2} A_j a_1^2)(a_1^{-2} A_i^{-1} a_1^2).$$

In G both of those relations are equivalent, however in K_G they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

to obtain infinitely many relations from K_G. □

The action of $\mathbb{Z}$ on the group K_G^{ab} exists by proposition 1.2 and can be defined as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}$$

and so the $\mathbb{Z}[\mathbb{Z}]$-module K_G^{ab} is generated by $(n-1)$ elements.

Moreover, the group G^{mab} and $\mathbb{Z}[\mathbb{Z}]$-module K_G^{ab} can be used interchangeably thanks to the split exact sequence (1) and proposition 1.2.

2.2 Basic properties

Knowing the resolution of a module allows one to change said module into a matrix or even a sequence of matrices, each containing a portion of information about its structure. We write the following resolution of the Alexander module:

$$0 \longrightarrow \ker(A_D) \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \xrightarrow{f} K_G^{ab} \longrightarrow 0 \tag{2}$$

where the f arrow sends every generator $(0, ..., 1, ..., 0) \in \mathbb{Z}[\mathbb{Z}]^{n-1}$ to each of the $(n-1)$ generators A_i of K_G^{ab} (compare with discussion in section 2.1). In the kernel of this homomorphism are exactly all n relations in K_G^{ab} and they are the image of A_D.

Definition 2.1 : Alexander matrix.
The matrix A_D in the diagram above is called the **Alexander matrix** of the Alexander module K_G^{ab} using the Wirtinger presentation with the diagram D.

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its columns will give a square matrix whose determinant is nonzero (see lemma 2.3). Take $R = \mathbb{Z}[\mathbb{Z}]$.

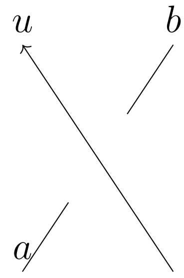
In proposition 1.3 it was shown that the Alexander module is torsion. Let $F = R^{-1}R$ be the field of fractions of R [AM69, Chapter 2]. Then, as a vector space $K_G^{ab} \otimes_R F = 0$ is trivial. Hence, the sequence in (2) translates to the following sequence of F modules

$$0 \longrightarrow \ker(A_D) \otimes_R F \longrightarrow R^n \otimes_R F \xrightarrow{A_D^V} R^{n-1} \otimes_R F \longrightarrow 0 \tag{3}$$

As there exists an inclusion $R \hookrightarrow F = R^{-1}R$, every matrix with terms in R can be treated as a matrix with terms in F. Naturally, $A_D^V := A_D \otimes Id_F$ is just matrix A_D (with terms in R) with adjoined 1×1 matrix with just identity of F. Thus, if A_D^V has nonzero determinant, then so does A_D.

Lemma 2.3.
Let A'_D be the Alexander matrix A_D with one of its rows struck out. Then $\det(A'_D) \neq 0$.

Proof. We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

The particular choice of orientation on a and b does not matter as long as it agrees with the rest of the diagram.

As described in the previous section, we change the Wirtinger presentation so that only one generator x is send to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module K^{ab} we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1-t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which x is b or c are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t-1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and ± 1. Thus, the sum is not equal to zero. There are two of such rows as the segment b has to be the "out" and "in" segment of some crossing. In other words, segment b has to have a start and end in some crossings.

The reasoning above is true for matrix A_D^V from (3). We make the switch to vector space to use the connection between the rank of matrix and its determinant.

Let S_i be the column of the Alexander matrix corresponding to the segment labeled i. The sum $\sum_{i\leq n-1} S_i$ is a vector with two nonzero terms. Take S_j and S_k to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both S_j and S_k by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i\leq n-1\\ i\neq j,k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i \ : \ i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D^V(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. □

The lemma 2.3 implies that image of A_D^V has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

Theorem 2.4.
Up to multiplication by a unit, the $(n-1)$ minor of A_D is independent of the choice of the diagram D of a knot K.

A proof of this statement using the Dehn presentation of knot group rather than the Wirtinger presentation is presented in [Ale28].

Definition 2.2 : Alexander polynomial.
Let $p(t)$ be the determinant of any maximal minor of the Alexander matrix A_D. Then, we can find $k \in \mathbb{Z}$ such that $t^k p(t)$ is a symmetrical polynomial, meaning that $t^k p(t) = (t^{-1})^k p(t^{-1})$ [Ale28]. The polynomial $t^k p(t)$ is called the **Alexander polynomial** of the knot K.

The Alexander polynomial is a knot invariant as a consequence of theorem 2.4.

Proposition 2.5.
Let G be a knot group of K and $F = R^{-1}R$ the field of fractions of ring R [AM69]. Then, changing coefficients by applying functor $- \otimes_R F$ to the resolution (2) yields the following exact sequence

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R F = 0 \longrightarrow 0$$

where n is the number of crossings of the chosen diagram D of knot K.

Proof. We start by saying that $R^n \otimes_R F \cong (R \otimes_R F)^n$ and $R \otimes_R F \cong F$ [AM69, Proposition 2.14].

Lemma 2.3 implies that the Alexander matrix is surjective thus $\dim(\ker(A_D)) = n - (n-1) = 1$. Therefore, (3) is isomorphic to the following exact sequence of vector spaces:

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R R^{-1}R = 0 \longrightarrow 0$$

$\square$

3 Knot colorings

3.1 Palettes and diagram colorings

An oriented diagram D of knot K has two types of crossings, pictured in fig. 3. A diagram coloring, in essence, is an assignment of values from some mathematical object (i.e. R-module) to segments of D.

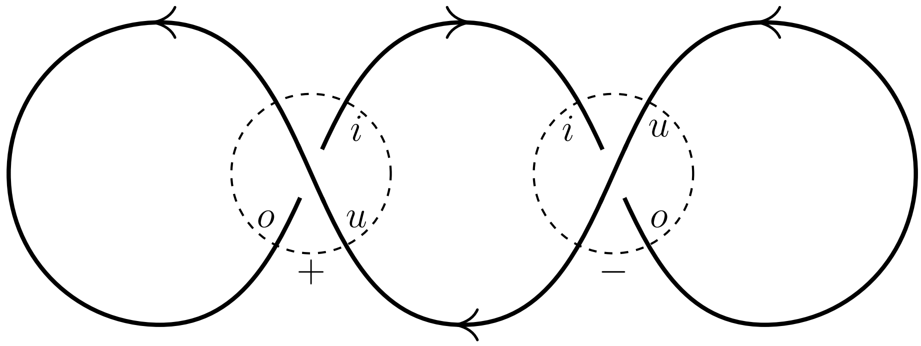


Figure 3: Two types of crossing in oriented diagram.

Definition 3.1 : palette.
We say that a quadruple $(\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{\mathcal{C}_\pm})$ is a **palette** if R is a commutative ring with unity, M an R-module and $\mathcal{C}_\pm$ are two R-modules, corresponding to the two types of crossings (fig. 3) such that $\mathcal{C}_\pm \subseteq M^3$.

If a palette $(R, M, \mathcal{C}_\pm)$ is given along with a ring homomorphism $f : R \to S$, then the image of this palette through induced palette homomorphism f_* is $(S, M \otimes_R S, \mathcal{C}_\pm \otimes_R S)$. Similarly, for an R-module homomorphism $g : M \to N$ we write $g_*(R, M, \mathcal{C}_\pm) = (R, N, g(\mathcal{C})_\pm)$.

We will cumulatively call the two modules $\mathcal{C}_\pm$ the **coloring rule** of palette $(R, M, \mathcal{C}_\pm)$ as they determine whether a coloring is admissible.

Definition 3.2 : diagram coloring.
A **coloring of diagram** D with s segments and x crossings (for knots $s = x$ lemma 2.1) is any element $(m_1, ..., m_s) \in M^s$ that assigns elements of M to each arc.

We will call a coloring **admissible** if for every crossing x_j of type $\pm$ we have

$$\pi_{x_j}(m_1, ..., m_s) \in \mathcal{C}_\pm,$$

where $\pi_{x_j} : M^s \to M^3$ is a projection of module M^s to the M^3 factor that corresponds to segments that constitute x_j.

We can now define two module homomorphisms

$$\phi_\pm : M^3 \to M^3/\mathcal{C}_\pm = N_\pm$$

that take in as arguments the arcs constituting a crossing. Assuming that $M^3/\mathcal{C}_\pm \cong M$ (reasoning behind this assumption will be given in section 3.2), we will take

$$\phi_+(u, i, o) = au + bi + co \tag{4}$$
$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o \tag{5}$$

for u, i, o understood like in fig. 3 and with coefficients being homomorphisms $M \to M$.

Lemma 3.1.
A coloring $(m_1, ..., m_s) \in M^s$ is a admissible $\iff$ for each crossing x_j of type $\pm$

$$\phi_\pm(\pi_{x_j}(m_1, ..., m_s)) = 0.$$

Proof. Stems from the fact that $\mathcal{C}_\pm = \ker \phi_\pm$. □

3.2 Color checking matrix

Given a palette $P = (R, M, \mathcal{C}_\pm)$ and a diagram D with s segments and x crossings, a homomorphism $D\phi : M^s \to N_\pm^x$ with $\ker(D\phi)$ containing the admissible colorings, can be produced.

Definition 3.3 : color checking matrix.
Assigning segments of diagram D to coordinates in M^s and crossings to coordinates in $N_\pm^x$ it is possible to define a linear homomorphism $D\phi : M^s \to N_\pm^x$ as

$$D\phi(m_1, ..., m_s) = (\phi_\pm(\pi_{x_1}(m_1, ..., m_s)), \phi_\pm(\pi_{x_2}(m_1, ..., m_s)), ...).$$

Matrix that is created after choosing a basis for M^s and $N_\pm^x$ will be called a **color checking matrix**.

Proposition 3.2.
Coloring $(m_1, ..., m_s) \in M^s$ is admissible $\iff$ $(m_1, ..., m_s) \in \ker D\phi$.

Proof. We start by saying that

$$(m_1, ..., m_s) \in \ker D\phi \iff [(\forall\ x_j \text{ crossing})\ \phi_\pm(\pi_{x_j}(m_1, ..., m_s)) = 0].$$

meaning that every coordinate of $D\phi(m_1, ..., m_s)$ is zero. Lemma 3.1 says that it is equivalent with $(m_1, ..., m_s)$ being an admissible coloring. $\square$

In the most basic sense, two diagrams D and D' are isomorphic if there exists an isotopy $h_t : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h_0(D) = D$ and $h_1(D) = D'$ and D' has crossings identical to those of D.

Knot diagrams create a category with Reidemeister moves and the isotopy h_t supplying morphisms between objects. The assigment $D \mapsto D\phi$ of a color checking matrix for a chosen palette is to be a functor from the category of diagrams to the category of color checking matrices. This means that if two diagrams were equivalent, then their color checking matrices must also be equivalent.

We say that two color checking matrices $D\phi$ and $D'\phi$ are equivalent if there exist isomorphisms $\theta : M^s \to M^s$ and $\psi : N^x \to N^x_\pm$ such that

$$\begin{tikzcd} M^s \arrow[r, "D\phi"] \arrow[d, "\theta"'] & N^x \arrow[d, "\psi"] \\ M^s \arrow[r, "D'\phi"'] & N^x \end{tikzcd}$$

is a commutative diagram. Additionally, we allow $D\phi$ to differ from $D'\phi$ by a identity matrix block.

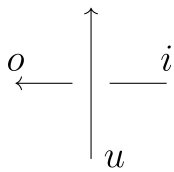
Lemma 3.3.
Isomorphic diagrams $D \sim D'$ yield equivalent color checking matrices $D\phi \sim D'\phi$.

Proof. In terms of color checking matrices, an isomorphism of diagrams defined above only relabels segments (permutes columns) and crossings (permutes rows). $\square$

3.3 Alexander palette

In this section we will define a palette for which the assigment $D \mapsto D\phi$ has all the properties of a functor.

Consider a crossing



and take some x to be the generator in the Wirtinger presentation of the knot group that is used to generate a representation for K_G^{ab} (see section 2.1). Then, the following is a relation in said group

$$UxCx(Ux)^{-1} = Ix$$

where $U = ux^{-1}$, $I = ix^{-1}$ and $O = ox^{-1}$. We can multiply both sides by x^{-1} to obtain

$$x^{-1}UxCU^{-1} = x^{-1}Ix$$

which is change in $\mathbb{Z}[\mathbb{Z}]$ to

$$tU + C - U = tI \implies 0 = (1-t)U + tI - C$$

The procedure for the other type of crossing is analogous and yields relation $0 = (1-t^{-1})U + t^{-1}I - C$.

Definition 3.4 : Alexander palette.
A palette $\boldsymbol{(R = \mathbb{Z}[\mathbb{Z}], M = \mathbb{Z}[\mathbb{Z}], \mathcal{C}_\pm)}$, where $\mathcal{C}_\pm$ are defined by homomorphisms

$$\phi_+(u,i,o) = (1-t)u + ti - o$$

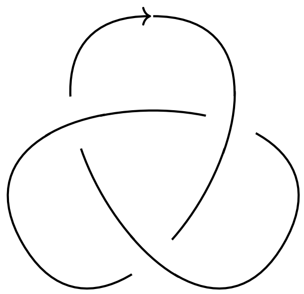
$$\phi_-(u,i,o) = (1-t^{-1})u + t^{-1}i - o,$$

with u, i and o defined in fig. 3, is called the **Alexander palette**.

The following example illustrates the importance of choosing a suitable palette.

Example 3.1. Consider a coloring of trefoil knot 3_1 with two palettes: $P_1 = (\mathbb{Z}, \mathbb{Z}, \phi_\pm(u,i,o) = 2u - i + o)$ that is not an image of the Alexander palette and $P_2 = (\mathbb{Z}, \mathbb{Z}, \phi_\pm(u,i,o) = 2u - i - o)$ which in turn is one (by

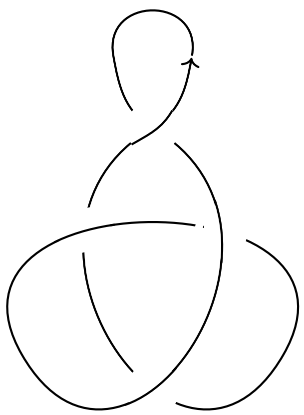
$\mathbb{Z}[\mathbb{Z}] \ni t \mapsto -1 \in \mathbb{Z}$). On the diagram below the color checking matrices of the diagram with 3 crossings along with a set of 2×2 minor values (on the right of the matrix) of it are presented



$$P_1(D) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \mapsto \{1, -3, -5\}$$

$$P_2(D) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \mapsto \{-3\}$$

while after the first Reidemeister move, the color checking matrix is



$$P_1(D) = \begin{bmatrix} 0 & 2 & 1 & -1 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} \mapsto \{5, 11, 3, ...\}$$

$$P_2(D) = \begin{bmatrix} 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & -1 & 0 & 0 \end{bmatrix} \mapsto \{-3, 3\}$$

For any palette for which $D \mapsto D\phi$ is a functor, in particular the Alexander palette, the coloring rule modules $\mathcal{C}_{\pm}$ are isomorphic to M^2

$$\begin{tikzcd} M^2 \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & M^3 \arrow[r, two heads] & \mathcal{C}_{\pm}. \end{tikzcd}$$

The red isomorphism to be $(u, i) \mapsto (u, i, \phi'_{\pm}(u, i)) \in \mathcal{C}_{\pm}$, with segments labeled like in fig. 3, meaning that c and γ in (4) and (5) respectively are units. For the sake of convenience, take $c = \gamma = -1$.

This property of palettes will be called a **propagation rule** as knowing colors of two of the three segments allows one to calculate the color assigned to the remaining segment.

Lemma 3.4.
A palette that meets the propagation rule has the following two properties determined by Reidemeister moves.

1. The first Reidemeister move requires that

$$a = 1 - b$$

$$\alpha = 1 - \beta,$$

where the variables are coefficients from (4) and (5).

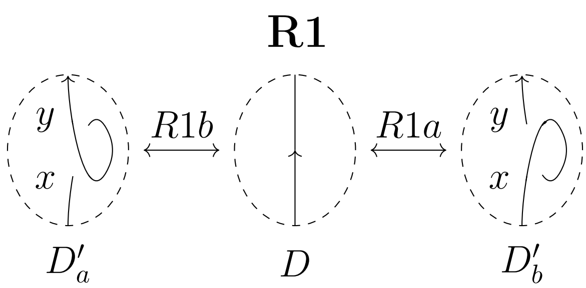
2. Similarly, the second Reidemeister move requires

$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1. \end{cases}$$

Proof of the lemma is divided into two parts, each given in the next session after defining the relation induced by the Reidemeister moves on color checking matrices.

3.4 Reidemeister relation on color checking matrices

The diagram D has s segments and x crossings and $D\phi : M^s \to N^x_{\pm}$.



The first Reidemeister move allows the following two moves on color checking matrices

$$\begin{array}{c} D'_a \\ \begin{bmatrix} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} \end{array} \overset{D(R1a)}{\sim} \begin{array}{c} D \\ \begin{bmatrix} x_1+y_1 & z_1 & \dots \\ \vdots & & \ddots \end{bmatrix} \end{array} \overset{D(R1b)}{\sim} \begin{array}{c} D'_b \\ \begin{bmatrix} \beta & \alpha+\gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} \end{array}$$

where $(\forall\ i = 1, ..., x)\ x_i = 0\ \vee y_i = 0$.

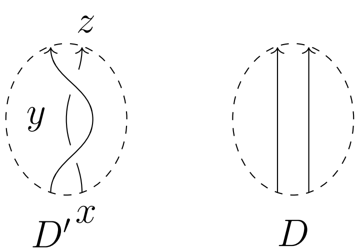
Proof of lemma 3.4.1. The propagation rule mandates the following equalities

$$0 = a + b + c = a + b - 1 \implies a = 1 - b$$

$$0 = \alpha + \beta + \gamma = \alpha + \beta - 1 \implies \alpha = 1 - \beta,$$

as the up and out segments in D'_a and the up and in segments in D'_b must admit coloring with the same element from M. $\square$

R2



For the second Reidemeister move we will say that $D\phi$ and $D'\phi$ are in relation if they differ by the following matrix move

$$\begin{array}{c} D' \\ \begin{bmatrix} b & c & 0 & a & \dots \\ 0 & \beta & \gamma & \alpha & \\ x_1 & 0 & z_1 & w_1 & \\ \vdots & & & & \ddots \end{bmatrix} \end{array} \overset{D(R2)}{\sim} \begin{array}{c} D \\ \begin{bmatrix} x_1+z_1 & w_1 & \dots \\ \vdots & & \ddots \end{bmatrix} \end{array}$$

where $(\forall\ i = 1, ..., x)\ x_i = 0\ \vee z_i = 0$, in addition to permuting rows and columns and adding linear combination of rows or columns to another row or column.

Proof of lemma 3.4.2. In the case of this Reidemeister move, we would like to be able to color the diagram D' exactly like the diagram D

save for the segments contributing to the two additional crossings. This means that segments labeled z and x on the diagram above must admit a coloring with the same element from M.

The restrictions stemming from this observation are more easily calculated if homomorphisms ϕ_+ and ϕ_- are made into two matrices, A_+ and A_-, that take the incoming segments (up and in segments) and return the output segments (out and up segments). This is possible because of the propagation rule.

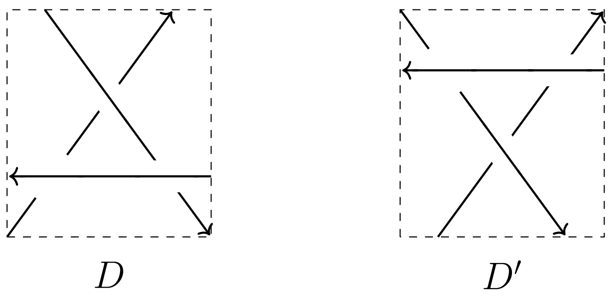
$$A_+A_-\begin{bmatrix}u\\i\end{bmatrix}=\begin{bmatrix}0&1\\b&a\end{bmatrix}\begin{bmatrix}\alpha&\beta\\1&0\end{bmatrix}\begin{bmatrix}u\\i\end{bmatrix}=\begin{bmatrix}u\\i\end{bmatrix}$$

Comparing the terms of the matrix A_+A_- with terms of the identity matrix yields:

$$\begin{cases}a\beta+\alpha=0\\ \beta b=1\end{cases}$$

□

R3



The last Reidemeister move does not change the size of matrices but only permutes the terms appearing in columns and rows corresponding to the three crossing that are manipulated in the diagram.

$$\overset{D'}{\begin{bmatrix}\alpha&\gamma&\beta&0&0&0&\dots\\0&0&c&b&0&a&\\\beta&0&0&0&\gamma&\alpha&\\u_1&0&v_1&w_1&x_4&y_4&\\\vdots&&&&&&\ddots\end{bmatrix}}\overset{D(R3)}{\sim}\overset{D}{\begin{bmatrix}0&0&\gamma&\beta&\alpha&0&\dots\\\beta&0&0&0&\gamma&\alpha&\\0&c&b&0&0&a&\\u_4&0&v_4&w_4&x_4&y_4&\\\vdots&&&&&\ddots&\end{bmatrix}}$$

Let $D(R)$ be the equivalence relation generated by moves $D(R1a)$, $D(R1b)$, $D(R2)$ and $D(R3)$.

Theorem 3.5.
For a diagram D colored with an Alexander palette (or its image) the equivalence class of the color checking matrix $D\phi$ under the equivalence relation $D(R)$ is a knot invariant.

Proof. A direct result of the definition of the equivalence relation. $\square$

Theorem 3.5 justifies defining $\boldsymbol{K\phi := [D\phi]}$.

3.5 Smith normal form

The ring R of palette $(R, M, \mathcal{C}_\pm)$ is not necessarily a PID ring, e.g. $\mathbb{Z}[\mathbb{Z}]$ ring of the Alexander palette has ideal $(2, t+1)$ which is not principal. However, usually one can find a PID ring P with homomorphism $R \to P$ which creates a new palette $(P, M \otimes_R P, \mathcal{C}_\pm \otimes_R P)$ derived from $(R, M, \mathcal{C}_\pm)$. Matrices over PID rings have many interesting properties, like having a Smith normal form.

Definition 3.5 : Smith normal form.
Take $A \in K\phi$ and consider it as an $s \times x$ matrix with terms in a P by the procedure outlined above. Then there exist a $s \times s$ matrix S and $x \times x$ matrix T such that SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every i $a_i | a_{i+1}$. Such a matrix SAT is called the **Smith normal form** of matrix A.

The following is an algorithm for computing the Smith normal form of a matrix A:

1. Let $A = \{a_{i,j}\}_{i,j \le n}$ be an $n \times n$ matrix. Take the ideal $I =$

$(a_{i,j})_{0<i,j\leq n}$ generated by all the terms of A.

2. If we are in PID then I has one generator, call it a.
3. We can now use the following row and column operations to put a in the upper left corner of A
   (a) Permuting rows (columns).
   (b) Adding a linear combination of rows (columns) to the remaining row (column).
4. With a in the upper left corner we can now use the fact that it was the generator of I to strike out the remaining terms on the first column and row, using the operations described in the previous point.
5. Repeat the same algorithm on the smaller matrix $\{a_{i,j}\}_{1<i,j\leq n}$.

The terms a_1,..., a_k of the Smith normal form of a color checking matrix $D\phi$ that are not units give $\mathrm{coker}(D\phi)$ as a module $P/(a_1) \oplus P/(a_2) \oplus \ldots \oplus P/(a_k)$. In definition 2.1 the $\mathrm{coker}(A_D)$ of the Alexander matrix was equal to the Alexanderl module. The same invariant information can be obtained from the color checking matrix.

Definition 3.6 : reduced normal form of matrix.
Take A to be a matrix with coefficients in principal ideal domain P. Take $a_1, \ldots, a_k \in P$ to be all the elements of the Smith normal form of A that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix A.

Consider the following as a motivation behind definition 3.6.

Example 3.2. Consider the knots $K11n85$ and $K11n164$ pictured in figs. 4 and 5 respectively. They both have the Alexander polynomial equal

$$\Delta(t) = -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3}.$$

Coloring them with the Alexander palette yields two 11×11 matrices whose any 10×10 minor is equal to the Alexander polynomial (up to multiplication by a unit). However, the reduced Smith normal forms are distinguishable

$$D_{11n85}\phi = \begin{bmatrix} -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3} \end{bmatrix}$$

$$D_{11n164}\phi = \begin{bmatrix} 1 - t + t^2 & 0 \\ 0 & -t^{-1} + 4 - 5t + 4t^2 - t^3 \end{bmatrix}$$

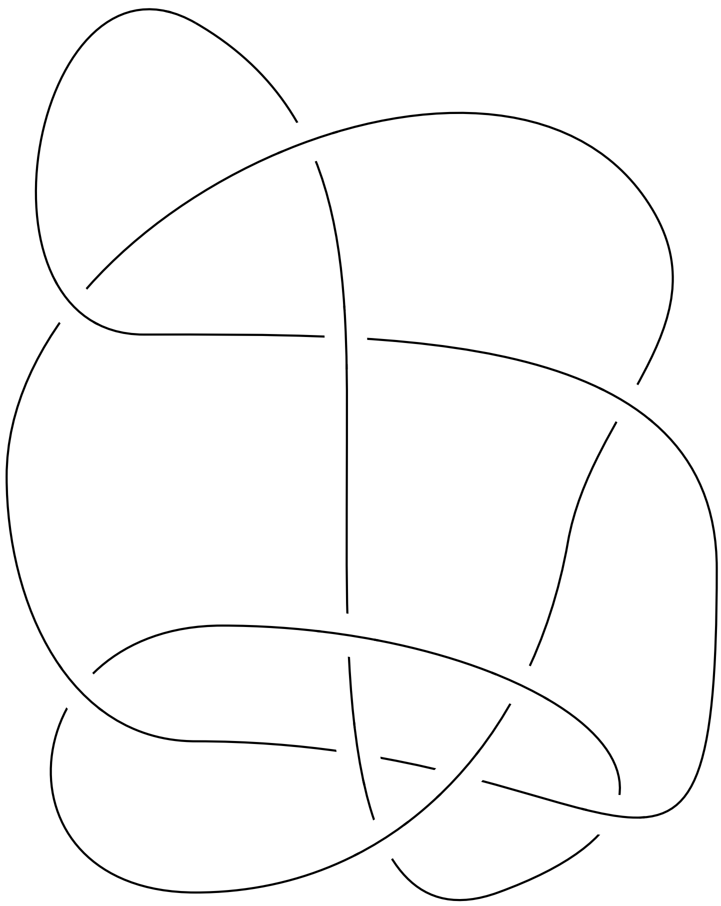


Figure 4: A diagram for knot $K11n85$.

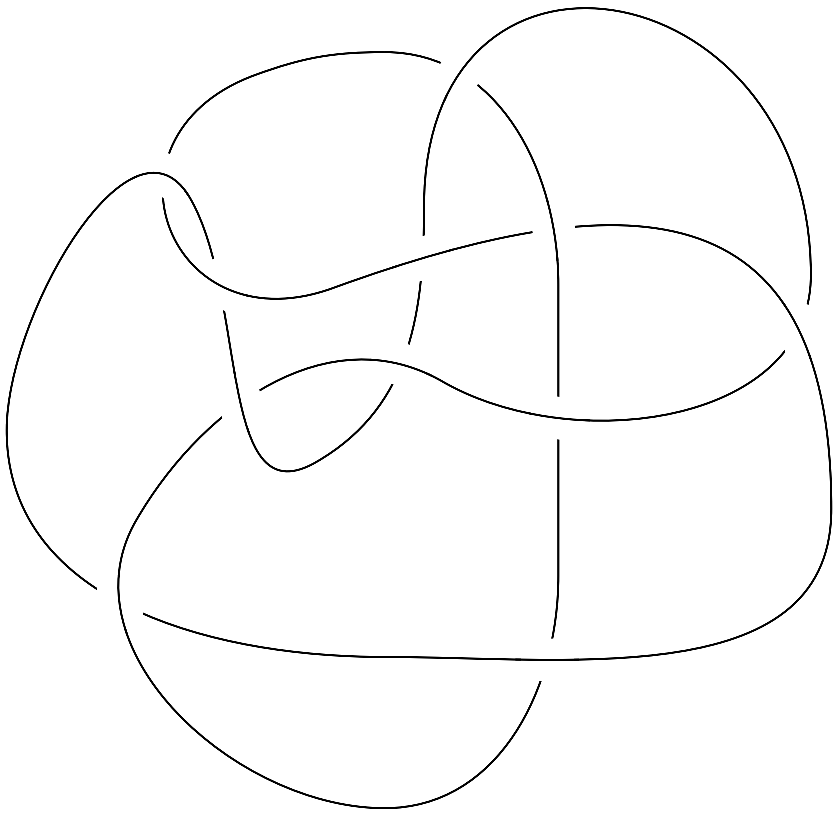


Figure 5: A diagram for knot $K11n164$.

Theorem 3.6.
The reduced normal form of color checking matrix using the Alexander palette does not depend on the choice of diagram D of a knot K. Thus, it is well defined for $K\phi$ and is a knot invariant.

Proof. Take a knot K and its diagram D with s segments and x crossings ($s = x$ by lemma 2.1). We will start by showing that applying any Reidemeister move to obtain a new diagram D' will not change the reduced normal form of its color checking matrix.

R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a**.

Take D' to be diagram D with one arc twisted into a + crossing. In the previous section, due to the relation $D(R1a)$, the matrices $D\phi$ and $D'\phi$ are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

with $(\forall\, i \geq 2)\ x_i = 0 \ \vee\ y_i = 0$. Adding the first column of $D'\phi$ to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

because $a + b = 1$. Now we know that b is a unit, thus we can easily remove the elements of the first column that are not b. This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2 + y_2 & \dots & \\ 0 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like $D\phi$. The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Therefore, the reduced normal form of $D'\phi$ is the same as in $D\phi$.

R2

Now the diagram D' is a diagram D with one arc poked onto another. Once again the color checking matrices are:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3 + v_3 & \dots \\ x_4 & u_4 + v_4 & \\ \vdots & & \end{bmatrix}$$

with $(\forall\ i \geq 3)\ u_3 = 0\ \vee\ v_3 = 0$. Adding the third column of $D'\phi$ multiplied by α and β to first and second column respectively we are able to reduce the first row to only zeros and -1. Now, adding this row to the second one creates a column with only -1 and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a + b\alpha & 0 & -1 & \\ 0 & x_3 & u_3 & v_3 & \\ 0 & x_4 & u_4 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that $a + b\alpha = 0$ and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 & \\ 0 & v_3 + u_3 & v_3 + u_3 & x_3 & \\ 0 & v_4 + u_4 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & v_3 + u_3 & x_3 & \\ 0 & 0 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as $D\phi$.

R3

The last Reidemeister move creates the following two matrices

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as b and β were taken to be units.

Notice that if $A \sim B$ and $B \sim C$, where $\sim$ means having the same reduced Smith normal form, then $A \sim C$. Thus, if two knots differ by a finite sequence of Reidemeister moves (as is the case for different diagrams of the same knot), then their reduced Smith normal forms are equal.

□

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