Fox knot colorings and Alexander invariants.

(Kolorowania Foxa i niezmienniki Alexandera)

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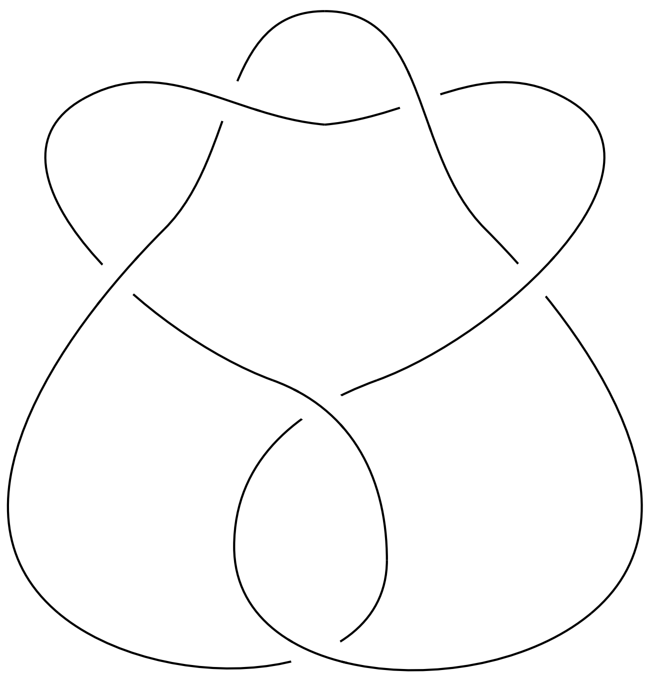

Contents

1 What is a knot coloring

Let K be a knot and D be its oriented diagram with s segments and x crossings. In such diagrams we can see two different crossing types as seen in fig. 1.

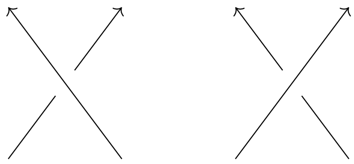

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Figure 1: Two types of crossing in oriented diagram.

Take a commutative ring with unity R and two R-modules M and N. Take two arbitrary module homomorphisms $\phi_+ : M^3 \to N$ and $\phi_- : M^3 \to N$, one for each type of crossing.

Definition 1.1 : diagram coloring.
Let $x_1, ..., x_s \in M$ be labels of arcs in diagram D. We will say that $(x_1, ..., x_s) \in M^s$ is a **coloring** if for every crossing $\pm$ in D consisting of arcs u, i, o the following relation is satisfied

$$\phi_{\pm}(u, i, o) = 0.$$

Every crossing in the colored diagram D of knot K yields x relations $\phi_{\pm}(u, i, o) = 0$ which we might treat as linear equations of form

$$\phi_+(u, i, o) = au + bi + co = 0,$$

$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o = 0,$$

where u, i and o are labels assigned to arcs entering some crossing and $a, b, c \in \mathrm{Hom}(M, N)$.

Definition 1.2.
Matrix $D\phi : M^s \to N^x$ of coefficients taken from relations $\phi_{\pm}(u, i, o)$ will be called a **color checking matrix**.

Notice that $(x_1, ..., x_s)$ is a coloring of the diagram D if and only if it is an element of $\ker D\phi$. However, we can choose ϕ to have only a trivial kernel, then only one coloring is admissible - assigning a 0 to every arc of D. Thus, to obtain valuable information about the knot K whose diagram is being colored, we must impose the following restrictions on ϕ.

1. To allow *trivial colorings*, that is colorings in which every arc is assigned the same value it is necessary that

$$(\forall\ m \in M)\ \phi_{\pm}(m, m, m) = 0.$$

2. To simplify operations of color checking matrices, if

$$\phi_{+}(u, i, o) = au + bi + co$$

$$\phi_{-}(u, i, o) = \alpha u + \beta i + \gamma o,$$

   then we take c and γ to be invertible. For the sake of simplicity, take $c = \gamma = -1$.

3. The two variations of orientation of the first Reidemeister move, pictured in fig. 2, put the following constrictions on a, b and α, β:

$$\begin{cases} a + b = 1 \\ \alpha + \beta = 1 \end{cases}$$

4. Lastly, from the second Reidemeister move, pictured in fig. 3, one can gather that

$$\begin{cases} a + b\alpha = 0 \\ b\beta = 1 \end{cases}$$

   meaning that both b and β must be units.

It is worth mentioning that examining the second Reidemeister move (fig. 3) with $\phi_{\pm}$ changed to 2×2 matrices $A_{\pm}$, which take arcs entering a crossing as input and output the arcs leaving it, we can see that

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$$A_{+}A_{-} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix} = Id.$$

This means that from homomorphism ϕ_{+} we are able to calculate ϕ_{-} and vice versa.

The color checking matrix is not a knot invariant, despite the restrictions laid on ϕ. Changing the number of crossings in a

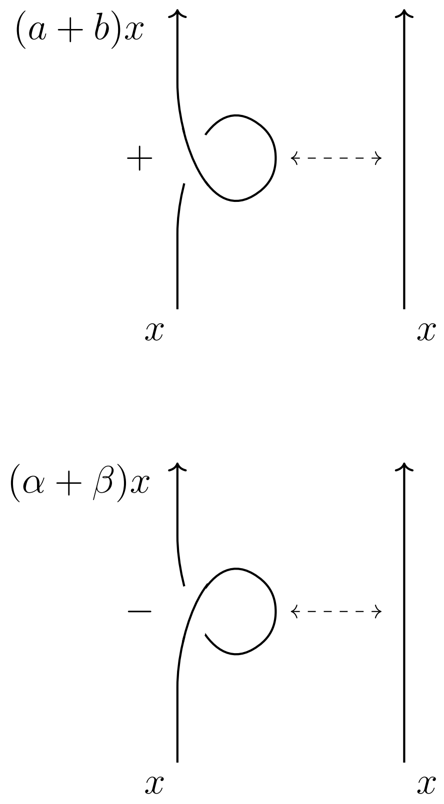


Figure 2: The two variations of the first Reidemeister move in oriented diagrams. They suggest that $(a+b)=1$ and $(\alpha+\beta)=1$.

diagram will obviously create a different matrix for the same knot. We will thus proceed to define an equivalence relation on the set of all color checking matrices of a knot K.

2 Relation on color checking matrices

In order to ensure that all matrices that stem from the same knot are considered in one equivalence class we will look at how Reidemeister moves change the matrix.

In this section we will always assume that the diagram labeled as D has s segments and x crossings. This means that always

$$D\phi : M^s \to N^x.$$

Furthermore, we will always put rows and columns corresponding to crossings and segments that are affected by the Reidemeister move as the last columns and rows of the matrix.

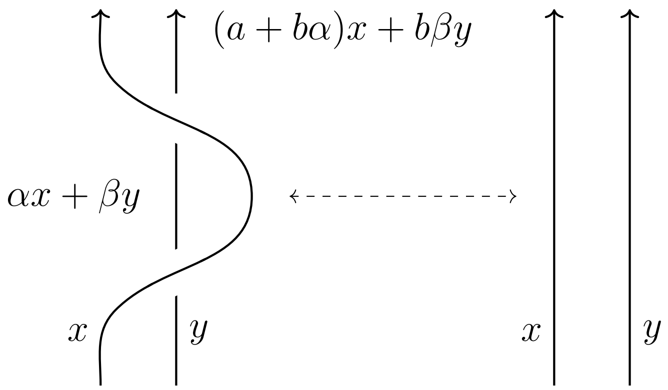
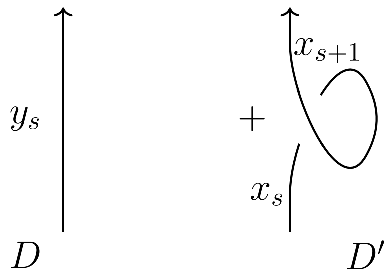


Figure 3: Second Reidemeister move. It suggests that $(a+b\alpha)=0$ and $b\beta=1$.

R1

R1a



In oriented diagrams, there are four distinct Reidemeister moves [1], the first one having to account for two types of crossing. We start with **R1a**, pictured in fig. 4. Consider the two matrices

$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+1} \to N^{x+1}.$$

Only two arcs change between the two, thus

$$D\phi \restriction M^{s-1} = D'\phi \restriction M^{s-1}.$$

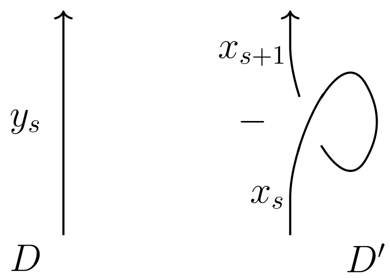
Furthermore, we want to assert that the two arcs x_s and x_{s+1}, into which y_s is split, are arranged in a $+$ type crossing. Thus, for all $x_s, x_{s+1} \in M$ we require that

$$\pi_{x+1}[D'\phi(0, ..., x_s, x_{s+1})] = \phi_+(x_{s+1}, x_s, x_{s+1}),$$

where π_{x+1} is projection onto the last coordinate. Additionally, we want the column that represented contribution of the twisted arc in diagram D to be the sum of two new arcs in $D'\phi$, meaning that for every $y_s \in M$:

$$(D\phi(0, ..., y_s), 0) = D'\phi(0, ..., y_s, y_s).$$

R1b



The second type of **R1** is seen in fig. 5. The relation for this move is almost the same as above. We start with two matrices

$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+1} \to N^{x+1}$$

that must agree on the arcs and crossings that are not changed between D and D':

$$D\phi \restriction M^{s-1} = D'\phi \restriction M^{s-1}.$$

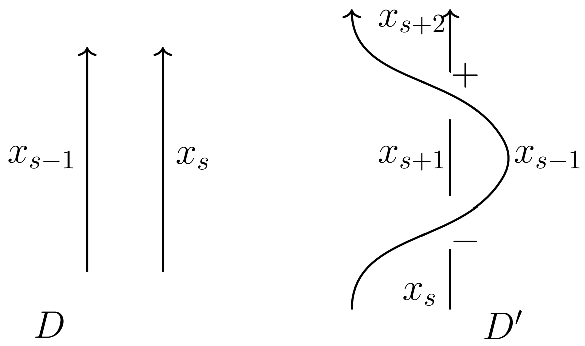
The type of crossing into which an arc of D is twisted in D' is differented than in **R1a**, thus the second equality is slightly different: for all $x_s, x_{s+1} \in M$

$$\pi_{x+1}[D'\phi(0, ..., x_s, x_{s+1})] = \phi_-(x_s, x_s, x_{s+1}).$$

The last requirement is not changed from **R1a**, meaning that for all $y_s \in M$:

$$(D\phi(0, ..., y_s), 0) = D'\phi(0, ..., y_s, y_s).$$

R2



R2

For this Reidemeinster move we start with matrices

$$D\phi : M^s \to N^x$$

$$D'\phi : M^{s+2} \to N^{x+2}.$$

Only two segments of D are manipulated, thus

$$D\phi \restriction M^{s-2} = D'\phi \restriction M^{s-2}.$$

To ensure that the poke contributes adequate crossings, we want for every $x, y \in M$

$$\begin{aligned} &D'\phi(0, ..., x, y, y, y) = \\ =&(D\phi(0, ..., x, y), \phi_-(x, y, y), \phi_+(x, y, y)) \end{aligned}$$

R3

$$D\phi \sim D'\phi$$

if and only if

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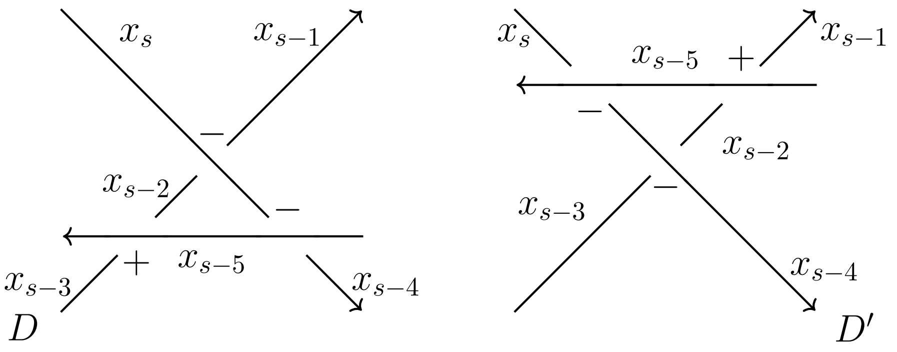
$$\begin{aligned} &D\phi \restriction M^{s-5} = D'\phi \restriction M^{s-5} \wedge \\ \wedge\ &(\forall\ x, y, z \in M)\pi_{x-3}[D\phi(0, ..., x, y, z)] = \\ &= \pi_{x-3}[D'\phi(0, ..., z, x, y, 0, 0)] \wedge \\ \wedge\ &D\phi(0, ..., z, y, 0, 0, 0, x) = D'\phi(0, ..., z, y, 0, 0, 0, x) \wedge \\ \wedge\ &D\phi(0, ..., z, 0, y, x, 0, 0) = D'\phi(0, ..., z, 0, 0, y, x, 0) \end{aligned}$$

R3



Theorem 2.1.
Let K be a knot and D its oriented diagram. Define

$$K\phi := [D\phi]$$

to be the equivalence class of the matrix $D\phi$. Then, $K\phi$ is a knot invariant.

Proof. An immediate result of construction presented above. $\square$ *brzydko brzmi*

3 Smith normal form

The ring R over which we consider modules M and N is not necessary a principal ideal domain. However, there are plenty of PID rings and one can take any unit of R and send it to any unit of a PID ring P to allow for M and N to be considered as P-modules. That way, we can consider a new type of equivalence relation on any color checking matrix $D\phi$.

Definition 3.1 : Smith normal form.
Take $A \in K\phi$ and consider it as a $s \times x$ matrix with terms in a P-module M. Then there exist a $s \times s$ matrix S and $x \times x$

matrix T such that SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every i $a_i | a_{i+1}$. Such matrix SAT is called **Smith normal form** of matrix A.

Given a matrix A one can compute its Smith normal form by the following algorithm:

dziwnie opisane?

1. Take the ideal containing all elements from A. It has a generator x.
2. If x is an element of A, then permute rows and columns so that x is in the upper left corner of A.
3. Otherwise, use the following two operations
   - adding a linear combination of remaining rows to a row
   - adding a linear combination of remaining columns to a column

   to force x to be an element of A. Go to step 2.
4. As the generator of the ideal spanned by all elements of A is in the upper left corner, we might now reduce the first row and column to only zeros with x on their intersection.
5. Strike out the first row and column to create a new matrix A' and repeat the whole process.

To obtain the most general invariants, we take R to be $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials with integer coefficients. There are multitudes of PIDs P with homomorphisms $\mathbb{Z}[t, t^{-1}] \to P$.

ten akapit to chyba niezbyt pasuje

As was mentioned in the first section, $\overline{x} \in M^s$ is a coloring of a diagram D if and only if $D\phi(\overline{x}) = 0$, that is $\overline{x} \in \ker D\phi$. The Smith normal form hints at the structure of matrix kernel - the columns filled with zeros will contributed a free factor M to the kernel.

Take (a) to be a prime ideal with its generator a appearing in the Smith normal form of $D\phi$. Then we might consider the matrix over a new ring $R/(a)$, which is still a PID. After this change, the structure of the kernel has changed as now there are additional zero columns where a and all its multiples stood.

Definition 3.2 : reduced normal form of matrix.
Take A to be a matrix with coefficients in principal ideal domain P. Take $a_1, ..., a_k \in P$ to be all the elements of the Smith normal form of A that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix A.

The following example justifies the utility of the reduced normal form of color checking matrices in distinguishing knots.

Example 3.1. Consider the knots 6_1 with diagram as seen in fig. 8 and 9_{46} pictured in fig. 9, ring $R = \mathbb{Z}[t, t^{-1}]$, $M = R$ and

$$\begin{cases} \phi_+(u, i, o) = (1 - t)u + ti - o \\ \phi_-(u, i, o) = (1 - t^{-1})u + t^{-1}i - o. \end{cases}$$

The two rings have the same Alexander polynomial, $\Delta = -2t^{-2} + 5t^{-1} - 2$, and the same Alexander module $H^1(S^3 - K) = \mathbb{Z}[t, t^{-1}]/(\Delta)$.

For the knot 6_1 we find the matrix $D\phi$ and after changing to the PID ring $P = \mathbb{Q}[t, t^{-1}]$ we see that the Smith normal form is:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

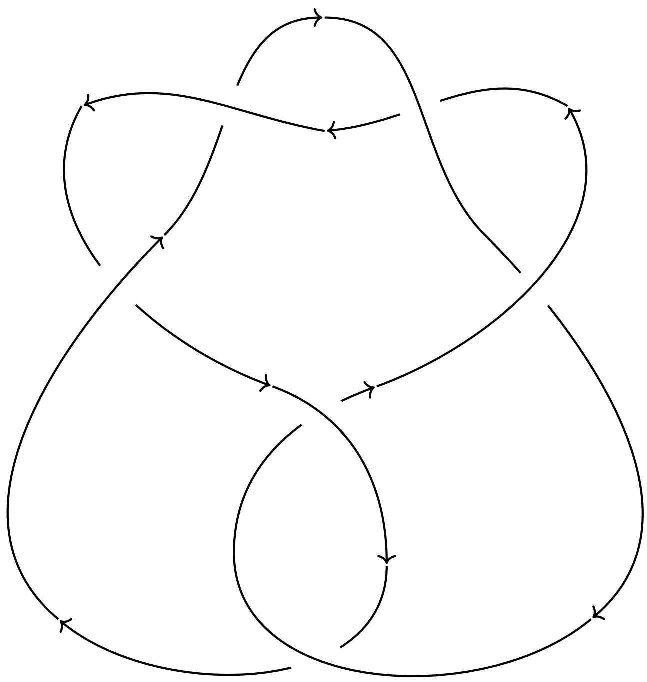


Figure 8: Diagram of knot 6_1.

which after reduction is

$$A' = \begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

a 1×1 matrix with the only term being the Alexander polynomial of 6_1.

Using diagram in fig. 9 of 9_{46} it can be calculated that the Smith normal form of $D\phi$ is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t - t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2} - 2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while reduced normal form of $D\phi$ is

$$B' = \begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

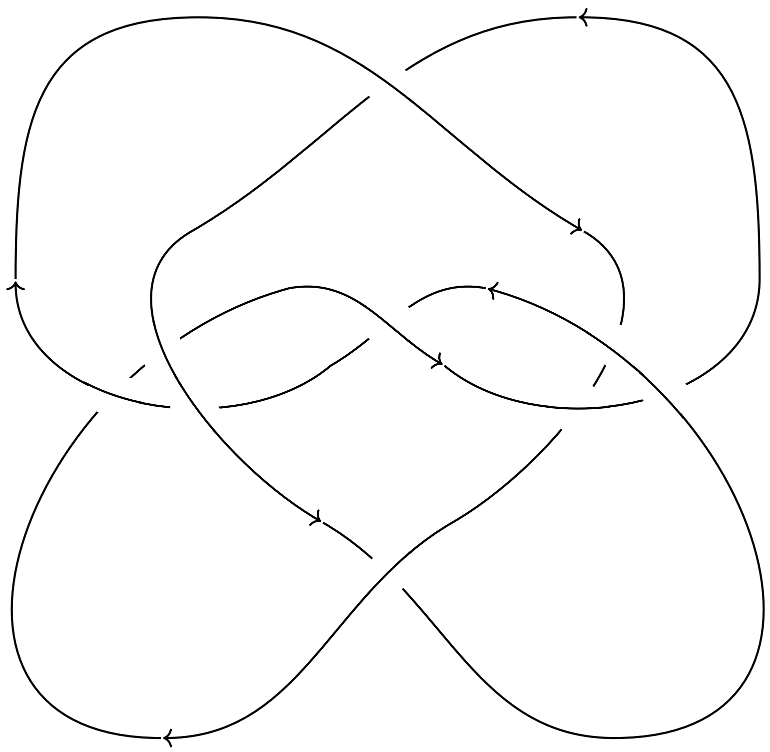


Figure 9: Diagram of knot 9_{46}.

which is significantly different than the one for 6_1. Observe also that the determinant of both matrices is equal to the Alexander polynomial of corresponding knots

$$\det(A') = -2 + 5t^{-1} - 2t^{-2}$$

$$\det(B') = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t + 2 + 2t^{-1} = -t(-2 + 5t^{-1} - 2t^{-2}).$$

to wyliczenie wogóle jest na miejscu?

Theorem 3.1.
The reduced normal form of color checking matrix does not depend on the choice of diagram D. Thus, it is well defined for $K\phi$ and is a knot invariant.

Proof. Take a knot K and its diagram D with s segments and x crossings. We will show that applying any Reidemeister move to this knot will not change the reduced normal form of its color checking matrix.

nie wiem, czy tutaj aż tak powinno się dokładnie mówić co i jak dodaję?

R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a** (the proof of **R1b** is left as an exercise for the reader).

Take D' to be diagram D with one arc twisted into a + crossing. In opposition to the assumption in previous section, we will take the arcs and crossings that differ between those two diagrams to be on first positions. Now, the matrices $D\phi$ and $D'\phi$ are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

Adding the first column of $D'\phi$ to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

because $a + b = 1$. Now we know that b is a unit, thus we can easily remove the elements of the first column that are not b. This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2 + y_2 & \dots & \\ 0 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like $D\phi$. The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Thus, the reduced normal form of $D'\phi$ is the same as in $D\phi$.

R2

Now the diagram D' is a diagram D with one arc poked onto another. Once again we will put those changed arcs at the beggining of the color checking matrix to obtain following matrices:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3 + v_3 & \dots \\ x_4 & u_4 + v_4 & \\ \vdots & & \end{bmatrix}$$

Adding the third column of $D'\phi$ multiplied by α and β to first and second column respectively we are able to reduce the first row to only zeros and -1. Now, adding this row to the second one creates a column with only -1 and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a + b\alpha & 0 & -1 & \\ 0 & x_3 & u_3 & v_3 & \\ 0 & x_4 & u_4 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that $a + b\alpha = 0$ and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 & \\ 0 & v_3 + u_3 & v_3 + u_3 & x_3 & \\ 0 & v_4 + u_4 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & v_3 + u_3 & x_3 & \\ 0 & 0 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as $D\phi$.

R3

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as b and β were taken to be units.

□

4 Skein relations

5 Investigation into resolution of fundamental group

References

[1] Michael Polyak. Minimal generating set of reidemeister moves. 2010.