A voyage into the algebras

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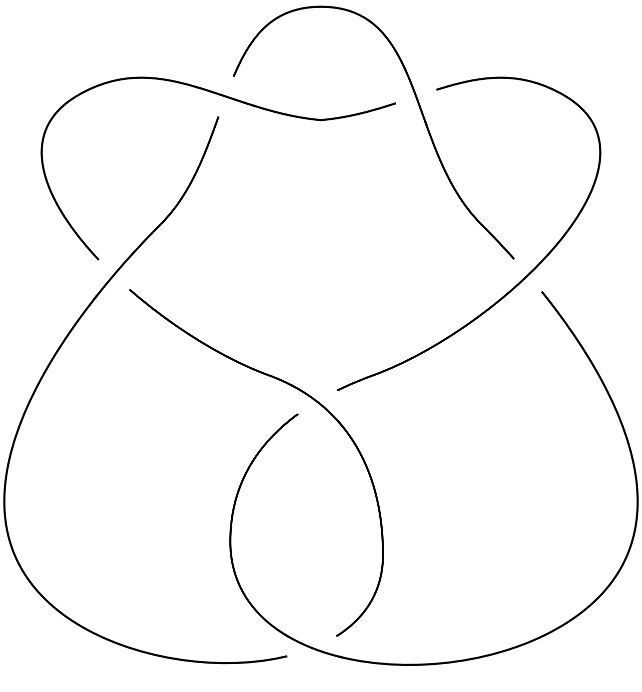


1 Knot coloring

Let R be any commutative ring with unity and let M, N be two R-modules with a module homomorphism $\omega : M^3 \to N$.

Now, consider a link diagram D with s arcs and x crossings. To each arc assign a number from 1 to s and to each crossing - from 1 to x.

Definition 1.1 (color checking matrix). *Consider modules M^s and N^x. Assign a generator of the i-th component of M^s to the arc labeled with i and a generator of the j-th component of N^x to the crossing labeled j.*

Let $(0, ..., 0, a_i, 0..., 0)$, $a_i \in M$ for $i = 1, 2, 3$ be elements taken from components of M^s corresponding to arcs creating a crossing labeled c. Let $p_c : N^x \twoheadrightarrow N$ be a projection onto component assigned to said crossing.

A module homomorphism, understood as a matrix with entries in R,

$$D\omega : M^s \to N^x$$

such that

$$p_c(D\omega(a_1 + a_2 + a_3)) = \omega(a_1, a_2, a_3)$$

will be called a ***color checking matrix****.*

We aim to find an equivalence relation that changes $D\omega$, which in definition 1.1 is dependent on diagram D of a knot.

2 Coloring oriented diagrams

In the previous section we defined coloring of a diagram without an orientation. Such a diagram has only one type of crossing, while in a diagram for which an orientation was chosen, two types of crossings are distinguishable in any knot diagram (see fig. 1).

In the case of a diagram with orientation, we must chose which type of crossing is considered by ϕ. If not explicitly stated otherwise, we will choose ϕ to determine the rules of coloring for crossing of type + as seen in fig. 1.

If u, i, o are labels assigned to arches creating a type + crossing that constitute a coloring, then we might write

$$0 = \phi(u, i, o) = au + bi + co.$$

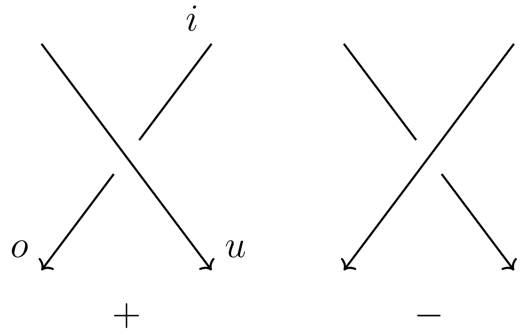


Figure 1: Two types of crossings in oriented knot diagram.

Taking c to be a unit, we get the following equation for the label of the arch leaving the crossing:

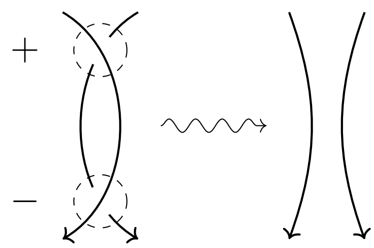
$$o = -c^{-1}au - c^{-1}bi.$$

Those assumption allow us to write a 2×2 matrix A_+ with terms in R such that multiplying an element $(u, i) \in M^2$ by A_+ will return $(o, u) \in M^2$. This means that $A_+ : M^2 \to M^2$ is the operator taking labels of incoming arches as input and returning labels of segments which leave the crossing.

$$A_+ = \begin{pmatrix} -c^{-1}a & -c^{-1}b \\ 1 & 0 \end{pmatrix}$$

It is convenient to take $c = -1$.

Allowing the following Reidemeister's move



gives equality

$$A_-A_+ = Id_2,$$

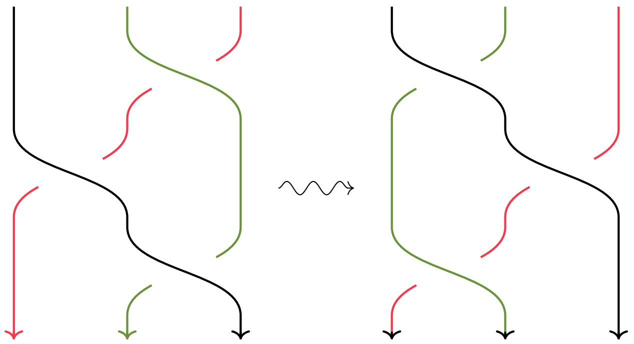
where A_+ is the matrix of operator for $+$ type crossing and $-$ - for the $-$ type crossing. Take $\alpha u + \beta i + \gamma o = 0$ to be the coloring rule for crossings of type $-$. Once again, for the sake of convenience $\gamma = -1$ and the matrix A_- must be of form

$$A_- = \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix},$$

meaning that

$$\begin{cases} b\beta = 1 \\ b\alpha - a = 0. \end{cases}$$

Consider another Reidemeister's move



Applying $A_\pm$ to each crossing separately yields the following relations

$$\begin{cases} ba = ab \\ a(a+b) = a. \end{cases}$$

We must assume that both b and β are units. In the most general situation, we are considering coloring modules as modules over the ring

$$R = \mathbb{Z}[s, t, t^{-1}]/\{s^2 + st - s\},$$

with a being send to s and b being send to t. However, it can be beneficial to at first assume yet another relation:

$$a + b = 1,$$

meaning that we are considering coloring as $\mathbb{Z}[t, t^{-1}]$ module.

Coloring a knot with $\mathbb{Z}[t, t^{-1}]$ module allows us to obtain information about coloring over $\mathbb{Z}$ or many other commutative rings by sending t to a unit in the ring in question.

Example 2.1. *Consider knot 4_1 with diagram D as seen in fig. 2 and ring $R = \mathbb{Z}[t, t^{-1}]$. Take function $\phi : M^3 \to M$ to be defined as*

$$\phi(u, i, o) = (1 - t)u + ti - o$$

The coloring homomorphism f is then defined by the matrix

$$f = \begin{pmatrix} 1-t & t & -1 & 0 \\ t^{-1} & -1 & 0 & 1-t^{-1} \\ 0 & 1-t^{-1} & t^{-1} & -1 \\ -1 & 0 & 1-t & t \end{pmatrix}$$

Changing the coefficients in R to $\mathbb{Q}$ yields the following Smith's normal form for f:

$$S = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & t^2-3t+1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice, that $\det S = t^2 - 3t + 1$, *which is the Alexander polynomial of* 4_1.

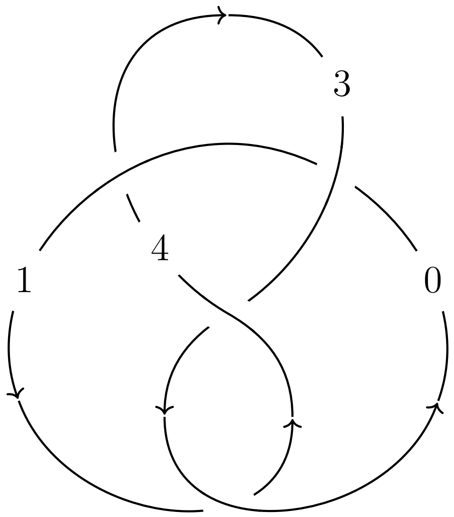


Figure 2: Coloring of knot 4_1 with elements from $\mathbb{Z}_5$.

Now, consider a homomorphism $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$ defined by $t \mapsto -1$. This yields a new matrix for f, with Smith's normal form:

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix above hints at existence of a coloring with elements from $\mathbb{Z}_5$, one of which is presented in fig. 2.

3 Reducing normal form of a matrix

In **??** we defined the coloring module of a diagram D as the kernel of coloring homomorphism. We might also want to extend this homomorphism to a short exact sequence

$$0 \longrightarrow \ker f \hookrightarrow M^s \xrightarrow{f} M^s \twoheadrightarrow \operatorname{coker} f \longrightarrow 0$$

and ask what information can be obtained from studying $\operatorname{coker} f$.

In **????** and example 2.1 nontrivial coloring was admissible only in modules $M/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal spanned by a portion of terms that appear on the diagonal of Smith's normal form of f. In the same examples, we observe also that $\operatorname{coker} f = R^k \oplus R/\mathfrak{a}$. For the knot 3_1 it was $\operatorname{coker} f = \mathbb{Z} \oplus \mathbb{Z}_3$, while in the case of knot 4_1 $\operatorname{coker} f = \mathbb{Z} \oplus \mathbb{Z}_5$.

Proposition 3.1. *Let f be a coloring homomorphism of an oriented diagram D. If $\operatorname{coker} f = R/\mathfrak{a}_1 \oplus ... \oplus R/\mathfrak{a}_k$ then D can be colored with elements from $R/\mathfrak{a}_i$ for $i = 1, ..., k$.*

Proof. To się powinno sprowadzić do rozwiązywania układu równań przy pomocy macierzy. □

The coloring homomorphism f of a diagram D carries a lot of information about the knot whose diagram it is. However, f in itself is not a knot invariant. The dimensions of its matrix will change if a new crossing is created, see the following example.

Example 3.1. *We take knot 3_1 with additional crossing, $R = \mathbb{Z}[t, t^{-1}]$ and $M = \mathbb{Z}[t, t^{-1}]$ with ϕ as in example 2.1. The coloring homomorphism has matrix*

$$\begin{pmatrix} 1-t & t & -1 & 0 \\ t & -1 & 0 & 1-t \\ -1 & 1-t & 0 & t \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

with normal form

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -t^2+t-1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which after evaluation at $t = -1$ yields

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which differs from matrix obtained in **??** *by just one trivial.*

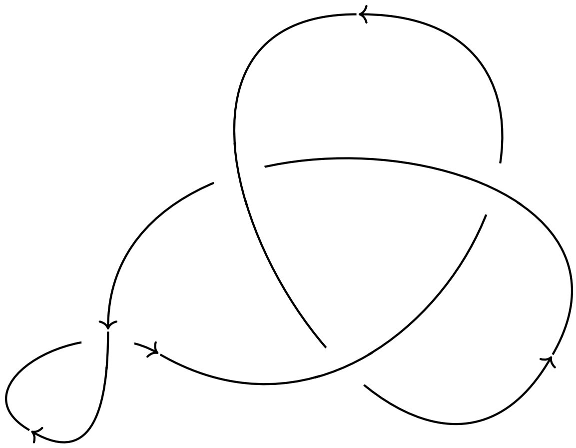


Figure 3: Diagram of knot 3_1 with additional crossing.

The nontrivial term on the diagonal in example 3.1 is the same as in **??**. The difference between matrices obtained in those two examples are their dimensions.

Definition 3.1. *Let A, B be matrices with entries from a PID ring R. We will say that they are equivalent ($A \sim B$) if and only if their Smith's normal form has the same nonzero and nonunit terms.*

Example 3.2. *Matrices of coloring homomorphisms over the ring $\mathbb{Z}$ of knot 3_1 presented in* **????** *and example 3.1 are both equivalent to a 1×1 matrix $\begin{pmatrix} 3 \end{pmatrix}$.*

Theorem 3.2. *Equivalence class of matrices under relation $\sim$ defined in definition 3.1 is a knot invariant.*

Example 3.3. *First, consider the knot* 6_1 *with diagram as seen in fig. 4, ring* $R = \mathbb{Z}[t, t^{-1}]$ *and* $M = R$. *We calculate that*

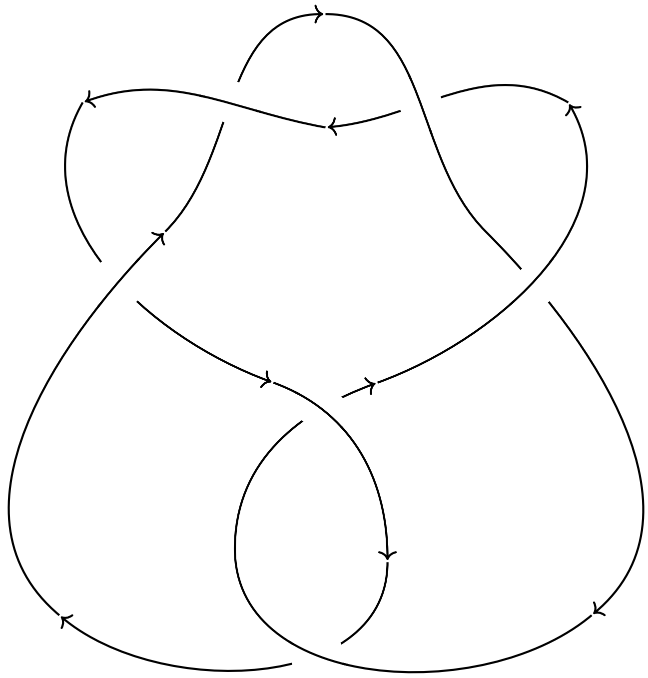


Figure 4: Diagram of knot 6_1.

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which agrees with the Alexander polynomial of 6_1. *Now, the reduced form of* f *would be*

$$\begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

a 1×1 *matrix.*

There is another knot with Alexander polynomial equal $-2t^{-2} + 5t^{-1} - 2$: 9_{46}. *Using diagram in fig. 5 it can be calculated that*

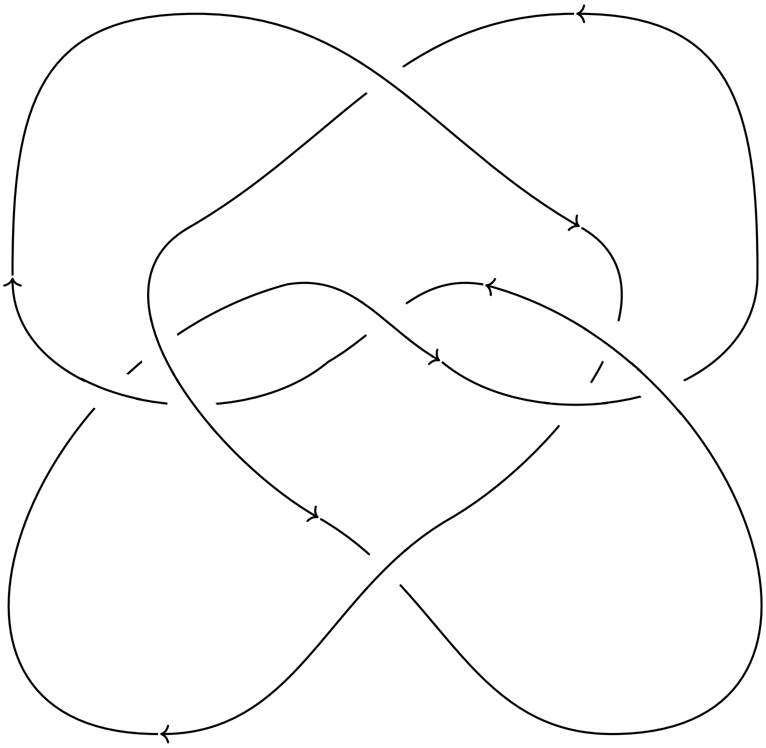


Figure 5: Diagram of knot 9_{46}.

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t-t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2}-2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\det f = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t^{-1} - 5 + 2t$$

is also the Alexander polynomial. The reduced form of f is

$$\begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

which is significantly different than the one for 6_1.

TO DO: sprawdzić te węzły wyżej za pomocą pow. Seiferta, czy mają różne moduły Alexandera

References