A voyage into the algebras

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2023-2024

1 Introduction

1.1 Order of an Ideal over PID ring

PID -> every ideal is generated by one element, every module is an image of a free module, hence it can be expressed as $M \cong R/I_1 \oplus ... \oplus R/I_n$ for some ideals I_i. This allows as to define order of a module as $\text{ord}(M) = \text{ord}(I_1...I_n)$, which is the element that generates the ideal $I_1....I_n$.

$\text{ord}(M)$ can also be described using equivalence relation $M \sim M_1 + M_2 \iff 0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence -> finitely generated abelian groups as $\mathbb{Z}$ modules and vector fields over $\mathfrak{K}$ as $\mathfrak{K}[x]$-modules.

1.2 The Problem of non-PID rings

Not every ring is a PID -> we must either find another invariant or make the ring in question a PID. E.g. for $\mathbb{Z}[x, x^{-1}]$ we can tensor it with some field, usually $\mathbb{Q}$ but we might want to try F_p for some prime p.

Maybe some example for $\mathbb{Z}[x]$?

1.3 Short Introduction to Knot Theory?

Knot - a closed curve immersed in some 3-dimensional space, or S^1 immersed in S^3

We will consider only tamed knots? That is knots that can be represented as a sum of a finite amount of straight lines?

Using Mayer-Vietoris sequence we can deduce that $H^1(S^3 \setminus K) = \mathbb{Z}$ for any knot K. Hence, if we want to find interesting invariants, we must look further.

Seifert surface of knot K is an orientable surface whose boundary is K. We can use it to create an infinite cyclic covering of $S^3 \setminus K$ by cutting copies $S^3 \setminus K$ along this surface and gluing the $+$ side of Seifert surface of one copy to the $-$ side of the next copy.

$H^1(K^*)$ is more complicated than $H^1(S^3 \setminus K)$ and things get interesting if we consider it as a $\mathbb{Z}[\mathbb{Z}]$ (or $\mathbb{Z}[x, x^{-1}]$-module. We can use the fact that $\Pi_1(K^*)^{ab} = H^1(K^*)$ and calculate this module to obtain something called Alexander ideal I: $H^1(K^*) \cong \mathbb{Z}[\mathbb{Z}]/I$. If I is a principal ideal, e.g. in the case of trefoil knot of figure eight knot, its generator is called "Alexander polynomial". If this is not the case, we must consider $H^1(K^*; \mathbb{Q})$ - kohomology module with coefficients in $\mathbb{Q}$, to obtain the Alexander polynomial. In the following paper we will consider what happens if we use F_p, a finite field, instead of $\mathbb{Q}$.

The matrix method

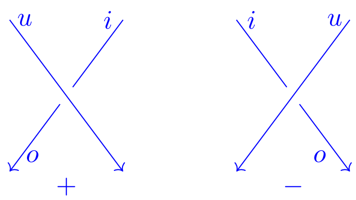
1.4 Fast notes

We might consider a module M over some ring R, usually $R = \mathbb{Z}[t, t^{-1}]$. Let K be a knot with l arches and s crossings that is oriented. We will consider a function $M^l \to M^s$ given by

$$+ : au + bi + co = 0$$

$$- : \alpha u + \beta i + \gamma o = 0,$$

where $+$ or $-$ depends on what arches u, i and o create:



The kernel of this morphism is responsible for coloring of knot K.

a, b, c (and greek) are morphisms $M \to M$ (or $M \to N$ in more general case). We can assume that c is a unit or even $c = 1 = \gamma$.

Furthermore, we can use equations above to obtain two operators $M \times M \to M \times M$ such that $(u, i) \mapsto (o, u)$ and $(i, u) \mapsto (u, o)$.

Two calculations on braids to do here, one that will give $a(a+b)=a$ and the other that states $ab=ba$!! what is the difference when $a+b=1$ and when a is not assumed to be a unit (therefore only $a^2+ab=a$)?

So now we can take a knot, its diagram and make it into a braid. A braid has a group (Burau representation, Markov knot theorem - moves) and we know that $\beta(w)v=v$ for the knot w and any vector v.

the braid group B_{n+1} with generators $\sigma_1, ..., \sigma_n$ can be send to S_{n+1} with relation $\sigma\eta = \eta\sigma$ for translations that are disjoint and $\sigma\eta\sigma = \eta\sigma\eta$ (i think) but we might want to do something different and add a relation that sends B_{n+1} to H_{n+1} or however this algebra was named, using $\sigma^2 + a\sigma + b = 0$.

Going back to the $M \times M$ stuff -> we can have a matrix $\begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}$ and we can assosiate it with translation σ_i from B_{n+1} and it acts on the braid. This gives us a coloring of the braid.

1.5 Coloring an unoriented knot diagram

Let R be a ring with identity and let M be an R-module. If we consider a diagram of a knot K without any orientation, the only type of crossing we will encounter is pictured in fig. 1

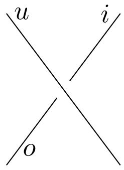


Figure 1: Crossing in an unoriented knot diagram.

Notice, that rotating it by 180 degrees changes i and o position (see fig. 2). Thus, segments passing under a crossing are indistinguishable.

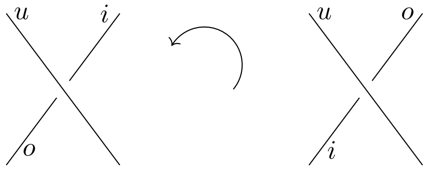


Figure 2: Segments going under a crossing in an unoriented knot diagram are indistinguishable.

When K has s segments and x crossings, we can write a labeling homomorphism

$$\phi : M^s \to M^x$$

which for segments that form a crossing pictured in fig. 1 takes value

$$\phi(u, i, o) = au + bi + co$$

for fixed $a, b, c \in \mathrm{End}(M)$. However, as we noted before, i and o are indistinguishable in fig. 1 and thus $b = c$, which yields a simpler definition:

$$\phi(u, i, o) = au + b(i + o).$$

tutaj trzeba sie dokladnie zastanowic jak to idzie bardzo formalnie w zapisie

$$\phi(u + i + o) = au + bi + co = 0$$

for $a, b, c \in \mathrm{End}(M)$ that are fixed for the entirety of K. However, because i and o are impossible to tell apart, we must take $b = c$ and thus arrive at a very simple equation:

$$au + b(i + o) = 0.$$

A coloring of a knot diagram without orientation is a labeling of its segments with elements from some module that agrees on crossings. That is, if a segment started in one crossing with label x then it must be labeled with

x in every other crossing until another segment passes over it. Every diagram has a trivial coloring, in which every segment is labeled with the same element.

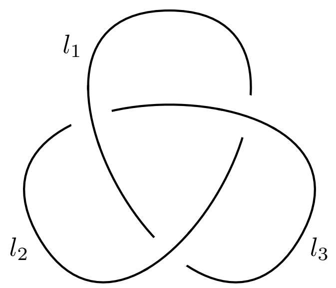
In other words, a coloring is an element from M^s that agrees with a and b on every crossing and thus it belongs to $\ker\phi$. For $(m_1, ..., m_s) \in \ker\phi$ we have a coloring such that segment i is labeled with m_i.

If we extend the morphism $M^s \to M^x$ to an exact sequence, we obtain

$$0 \to \ker\phi \to M^s \xrightarrow{\phi} M^x \to \operatorname{coker}\phi \to 0.$$

Module $\ker\phi$ can be viewed as a coloring of the diagram of K with elements of module M.

Example 1.1. *Let $M = \mathbb{Z}_n$, $R = \mathbb{Z}$, and consider the trefoil knot with 3 segments and 3 crossings.*



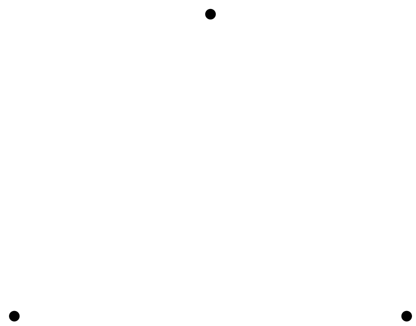
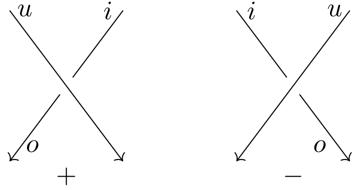


Figure 3: An alternating diagram of trefoil knot 3_1.

TO DO: function such that $2x - y - z = 0$ always when x is the upper strand, using Smith's normal form show that only $\mathbb{Z}_3$ can be used to make a non-trivial coloring

MAYHAPSE A DIFFERENT KNOT?

1.6 The case of oriented knot diagram



2 Calculating the Alexander Module

Kinoshita-Tarasaki - does not look too promising

Conway Knot - to be examined

Torus knots are useless -> 5_2 but not the $5_1 = T(5,2)$ one. Could not find a seifert surface for this bad boy.

References