

# Fox knot colorings and Alexander invariants.

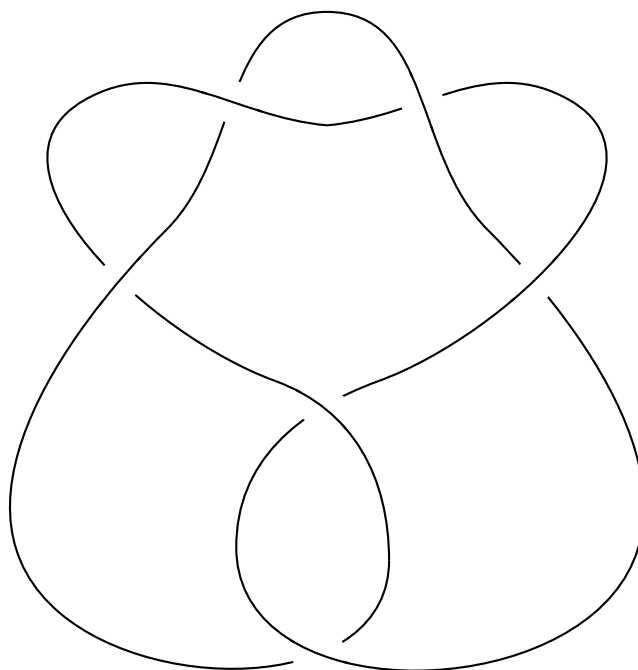
(Kolorowania Foxa i niezmienniki Alexandra)

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# 1 Preliminaries

## 1.1 Knots and diagrams

In mathematical terms, a knot is a particular embedding  $S^1 \hookrightarrow S^3$ . A knot diagram is an **immersive projection**  $D : S^1 \rightarrow \mathbb{R}^2$  along a vector such that no three points of the knot lay on this vector [5].

$S^1$  is an orientable space thus we can choose an orientation for a knot being considered. Then a diagram  $D$  is oriented if it is a projection of an oriented  $S^1$ .

Intuitively, two knots  $K_1$  and  $K_2$  are equivalent if we can deform one into the other without cutting it and only manipulating it with our hands [2]. This translates to equivalence of diagrams, which is generated by a set of moves, called the **Reidemeister moves**. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on  $D$ , 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [3].

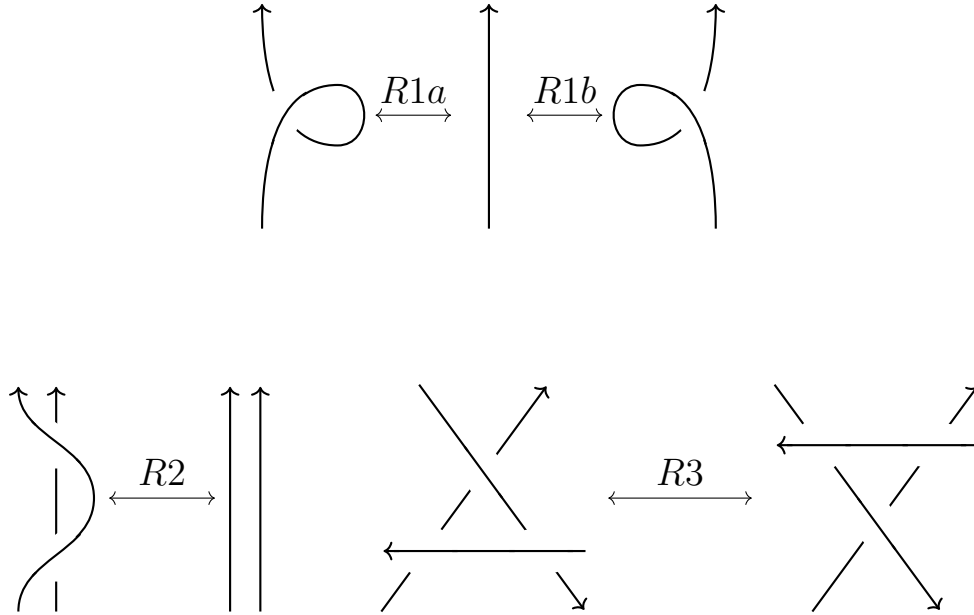


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

## 1.2 Knot group

Let  $K$  be a knot and  $D$  be its oriented diagram with  $s$  segments and  $x$  crossings.

**Definition 1.1 : knot group.**

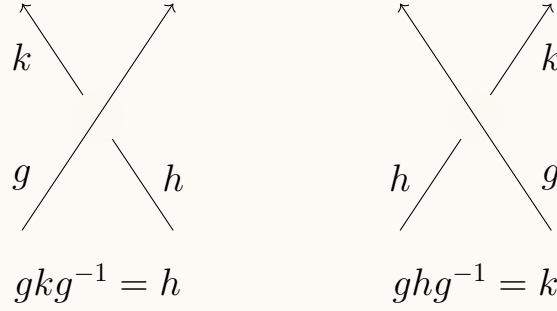
The fundamental group of a knot embedded in a three dimensional sphere  $S^3$  is called a **knot group**.

$$\pi_1(\mathbf{K}) := \pi_1(\mathbf{S}^3 - \mathbf{K}).$$

Although the knot itself is always a circle  $S^1$ , the knot group has usually an interesting yet difficult structure. The most known representation of the knot group is called **the Wirtinger presentation**.

**Definition 1.2 : Wirtinger presentation.**

Given a diagram  $D$  of knot  $K$  with segments  $a_1, a_2, \dots, a_s$  and crossings  $c_1, \dots, c_x$  the knot group  $\pi_1(K)$  can be represented as  $\pi_1(K) = \langle G \mid R \rangle$ , where  $G$  is the set of segments of  $D$  and relations  $R$  correspond to crossings in the manner described in the diagram below



Representation  $\langle G \mid R \rangle$  described above is called the **Wirtinger presentation** [1, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to  $S^3 = K \oplus S^3 - K$  or noticing that every two generators are conjugate, is that the abelianization of the knot group is always  $\mathbb{Z}$ . This leads to an acyclic complex

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0$$

The group  $K_G = \ker(ab : G \rightarrow \mathbb{Z}) = [G, G]$  is not finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization  $K_G^{ab} = K_G/[K_G, K_G]$  allows a  $\mathbb{Z}[\mathbb{Z}]$  module structure and thus contains obtainable information about the knot  $K$ .

The following is an exact sequence:

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives foundation for the following definition.

**Definition 1.3 : metabelianization.**

The quotient group  $G^{mab} = G/[K_G, K_G]$  is called the **metabelianization** of  $G$ .

We will return to the concept of metabelianization in section 2.

### 1.3 Infinite cyclic covering

Let  $X$  be the complement of a knot  $K$ , that is  $X = S^3 - K$ . Take  $\tilde{X}$  to be its universal covering, meaning that it is simply connected. The fundamental group  $G$  of  $X$  acts on its universal covering by deck transformations. The commutator subgroup  $K_G = [G, G]$  is normal in  $G$  and so the action of  $K_G$  on  $\tilde{X}$  is well defined. Thus we might take the quotient space  $\bar{X} = \tilde{X}/[G, G]$  and call it the **infinite cyclic covering** of  $X$ . The fundamental group of  $\bar{X}$  is exactly

$$\pi_1(\bar{X}) = [G, G] = K_G$$

and from the perspective of homology modules, we have

$$H_1(\bar{X}, \mathbb{Z}) = \pi_1(\bar{X})^{ab} = K_G^{ab}.$$

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{array}{ccc} \tilde{X} & \curvearrowright & G \\ \downarrow & & \\ \bar{X} & \curvearrowright & G/[G, G] \\ \downarrow & & \\ X = S^3 - K & & \end{array}$$

A **Seifert surface**  $S$  of knot  $K$  is an orientable surface with boundary embedded in  $S^3$  such that  $\partial S = K$ . Take a countable amount of  $X$ , with  $S$  without its boundary embedded, and label each with an element from  $\mathbb{Z}$ . We might now cut each of the copies of  $X$  along the Seifert surface of  $K$  and identify the  $+$  side of  $S$  from the  $i$ -th copy of  $X$  with

Imagine that each copy of  $X$  inside of  $\overline{X}$  is a box labeled with some integer  $k$ . The ring action of  $\mathbb{Z}[\mathbb{Z}]$  on  $\overline{X}$  is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of  $t \in \mathbb{Z}[\mathbb{Z}]$  in the polynomial which we apply to  $\overline{X}$ .

The  $\mathbb{Z}[\mathbb{Z}]$ -module  $K^{ab} = H_1(\overline{X}, \mathbb{Z})$  is a torsion module.

$$f : C_*(\overline{X}) \rightarrow C_*(\overline{X})$$

As a consequence, the following sequence of chain complexes is exact

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_2(X, \mathbb{Z}) & \longrightarrow & H_1(\overline{X}, \mathbb{Z}) & \xrightarrow{1-t} & H_1(\overline{X}, \mathbb{Z}) \\ & & & & & & \downarrow \\ & & & & & & H_0(\overline{X}, \mathbb{Z}) \\ & & & & & & \downarrow \\ & & & & & & H_0(\overline{X}, \mathbb{Z}) \\ & & & & & & \downarrow \\ & & & & & & H_0(X, \mathbb{Z}) \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

$$H_1(X, \mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}.$$
$$H_0(X, \mathbb{Z}) = H_0(\overline{X}, \mathbb{Z}) = \mathbb{Z}.$$

$$\begin{array}{ccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & H_1(\overline{X}, \mathbb{Z}) & \xrightarrow{1-t} & H_1(\overline{X}, \mathbb{Z}) & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \\
& & & & & & & & & \searrow & \\
& & & & & & & & & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Rewriting the sequence above we easily get that homomorphism  $1 - t$  is actually an isomorphism and  $H_1(\overline{X}, \mathbb{Z}) \cong (1 - t)H_1(\overline{X}, \mathbb{Z})$ , which allows us to use the Nakayama's lemma to conclude that there exists  $x \in \mathbb{Z}[\mathbb{Z}]$  such that

$$xH_1(\overline{X}, \mathbb{Z}) = 0.$$

□

## 2 Resolution of the Alexander module

### 2.1 Alexander module

Take  $G = \langle G \mid R \rangle$  to be the Wirtinger presentation of  $G$  obtained from diagram  $D$ . Because  $K$  is a knot and not a link, we know that the number of segments is equal to the number of crossings, thus we can take  $n = s = x$ .

Let  $a_1, \dots, a_n$  be the generators of  $G$  and  $x_1, \dots, x_n$  its relations. The homomorphism of abelianization of  $G$  is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every  $i = 1, \dots, n$ . In order to obtain a representation of  $K_G$ , the kernel of abelianization, we need to change the set of generators of  $G$  to

$$\{a_1, A_2 = a_2a_1^{-1}, \dots, A_n = a_na_1^{-1}\}.$$

It is obvious that for every  $i > 1$   $A_i \mapsto 0$  by abelianization of  $G$ . thus  $A_2, \dots, A_n$  are some of the generators of  $K_G$ . However, for each  $i = 2, \dots, n$  and  $k \in \mathbb{Z}$  the following is an element of  $K_G$ :

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the representation of  $K_G$  is infinite with generators

$$\{b_{i,k} : i = 2, \dots, n, k \in \mathbb{Z}\}.$$

Changing generators of  $G$  induced a change in relations. Suppose that the following relation was true in  $G$

$$a_k = a_i a_j a_i^{-1}.$$

If  $1 \notin \{i, k, j\}$  then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of  $a_1$  the following two relations can be obtained

$$\begin{aligned} a_1^{-1} A_k a_1 &= (a_1^{-1} A_i a_1) A_j A_i^{-1} \\ a_1^{-3} A_k a_1^3 &= (a_1^{-3} A_i a_1^3) (a_1^{-2} A_j a_1^2) (a_1^{-2} A_i^{-1} a_1^2). \end{aligned}$$

Obviously in  $G$  both of those relations are equivalent, however in  $K_G$  they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

to obtain infinitely many relations from  $K_G$ .

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Hence action of  $\mathbb{Z}$  can be defined on the group  $K_G^{ab}$ , with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}.$$

This procedure allows  $K_G^{ab}$  to be interpreted as a  $\mathbb{Z}[\mathbb{Z}]$ -module.

**Definition 2.1 : Alexander module.**

Given a group  $G$ , the abelianization of the commutator of a group  $G$ ,  $K_G^{ab}$ , with  $\mathbb{Z}[\mathbb{Z}]$ -module structure is called the **Alexander module** of  $G$ . If  $G$  is a knot group, then it is the Alexander module of the knot  $K$



**Lemma 2.1.**

The  $\mathbb{Z}[\mathbb{Z}]$  modules  $K_G^{ab}$  and  $G^{mab}$  (see definition 1.3) are isomorphic.

*Proof.* Construction presented above states that the module  $K_G^{ab}$  has  $(n - 1)$  generators.  $\square$

## 2.2 Resolution

The resolution of a module at first glance is in no way a simplification of said module. However, there are multiple ways of distilling simplifications and invariants from the resolution of the Alexander module. [In this section we want to](#)

We start writing the resolution of  $K_G^{ab}$  as follows:

$$\dots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

**Definition 2.2 : Alexander matrix.**

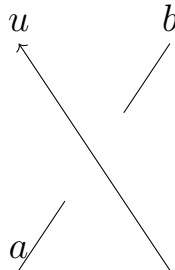
The matrix of homomorphism  $A_D$  in the diagram above is called the **Alexander matrix** of group  $G$  (knot  $K$ ).

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its rows will give a square matrix whose determinant is nonzero.

**Proposition 2.2.**

Let  $A'_D$  be the Alexander matrix  $A_D$  with one of its rows struck out. Then  $\det(A'_D) \neq 0$ .

*Proof.* We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator  $x$  is sent to 1 by abelianization. If said generator is  $u = x$ , then in the  $\mathbb{Z}[\mathbb{Z}]$  module  $K^{ab}$  we see the following relation

$$\pm t^n(tB - C) = 0,$$

where  $B = bx^{-1}$  and  $C = cx^{-1}$ . Otherwise, the relation is

$$\pm t^n[(1 - t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of  $A_D(1)$  in the row corresponding to the crossing is equal to 1.

The cases in which  $x$  is  $b$  or  $c$  are symmetrical and without the lose of generality assume that  $x = b$ . Then the relation is

$$\pm t^n[(t - 1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and  $\pm 1$ . Thus, the sum is not equal to zero. There are two of such rows as the segment  $b$  has to be the "out" and "in" segment of some crossing. In other words, segment  $b$  has to have a start and end in some crossings.

Let  $S_i$  be the column of the Alexander matrix corresponding to the segment labeled  $i$ . The sum  $\sum_{i \leq n-1} S_i$  is a vector with two nonzero terms. Take  $S_j$  and  $S_k$  to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both  $S_j$  and  $S_k$  by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j, k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that  $\{S_i : i \leq n - 1\}$  is a set of linearly independent vectors and thus every minor of  $A_D(1)$  has nonzero determinant. In particular,  $\det(A'_D)(1) \neq 0$ .  $\square$

The proposition 2.2 implies that image of  $A_D$  has dimension  $(n - 1)$ . We will use this knowledge later on to construct the resolution of the Alexander module.

**Theorem 2.3.**

The determinant  $\det(A'_D)$  is independent of the choice of the diagram  $D$

**Proof.** If  $D$  and  $D'$  are two diagrams of knot  $K$ , then they yield equivalent representations of  $G = \pi_1(K)$ . Thus, the chain of elementary ideals of  $A_D$  and  $A_{D'}$  are the same according to Fox [4, Chapter VII] from which immediately follows that the determinants of the maximum minors of  $A_D$  and  $A_{D'}$  are equal.  $\square$

**Definition 2.3 : Alexander polynomial.**

The **Alexander polynomial** of a knot  $K$  is the determinant of any maximal minor of the Alexander matrix  $A_D$ .

The Alexander polynomial is a knot invariant as a consequence of theorem 2.3 and proposition 2.2

**Proposition 2.4.**

Let  $G$  be a knot group of  $K$ . Then it always has a resolution

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}]^1 \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

where  $n$  is the number of crossings of the chosen diagram  $D$  of knot  $K$ .

**Proof.** Take  $R = \mathbb{Z}[\mathbb{Z}]$  and consider its ring of fractions  $R^{-1}R$ . There is an homomorphism  $R \rightarrow R^{-1}R$  which allows us to change the coefficients in the resolution of  $K_G^{ab}$  as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^a \otimes_R R^{-1}R & \longrightarrow & R^n \otimes_R R^{-1}R & \longrightarrow & R^{n-1} \otimes_R \\ & & & & & & \downarrow \\ & & & & & & K_G^{ab} \otimes_R R^{-1}R \longrightarrow 0 \end{array}$$

In proposition 1.1 it was proven that  $K_G^{ab}$  is a torsion module and so  $K_G^{ab} \otimes_R R^{-1}R = 0$ . Moreover, in proposition 2.2 we showed that  $A_D$  is a surjection onto  $(n - 1)$  dimensional module. Combining this knowledge

we obtain a short exact sequence of vector spaces, call them  $V$ , over  $R^{-1}R$ :

$$0 \longrightarrow V^a \longrightarrow V^n \longrightarrow V^{n-1} \longrightarrow 0$$

which implies that

$$n = n - 1 + a$$

and so  $a = 1$ . □

## 3 Knot colorings

### 3.1 What is a knot coloring

Let  $K$  be a knot and  $D$  be its oriented diagram with  $s$  segments and  $x$  crossings. In such diagrams we can see two different crossing types as seen in fig. 2.

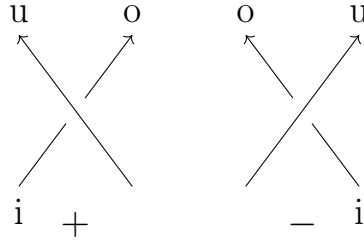


Figure 2: Two types of crossing in oriented diagram.

Take a commutative ring with unity  $R$  and an  $R$ -module  $M$ .

**Definition 3.1 : coloring rule.**

Take  $\mathcal{C} \subseteq M^3$  to be a finitely generated submodule of  $M^3$ . We will call  $\mathcal{C}$  a **coloring rule**. There are two submodules  $\mathcal{C}_{\pm} \subseteq \mathcal{C}$ , each corresponding to a type of crossing in diagram  $D$ .

We can now construct three homomorphisms

$$\phi : M^3 \rightarrow M/\mathcal{C} = N$$

$$\phi_{\pm} : M^3 \rightarrow M/\mathcal{C}_{\pm} = N_{\pm}.$$

We will call  $\phi$  and  $\mathcal{C}$  **coloring rule** interchangeably.

For each crossing  $x_j$  in diagram  $D$  we can construct a projection

$$\pi_{x_j} : M^s \twoheadrightarrow M^3$$

which restricts  $M^s$  to the three arcs that constitute  $x_j$ .

**Definition 3.2 : diagram coloring.**

A **coloring of diagram**  $D$  is any element  $(m_1, \dots, m_s) \in M^s$  that assigns elements of  $M$  to each arc. We will call this coloring **admissible** if for every crossing  $x_j$  of type  $\pm$  we have

$$\pi_{x_j}(m_1, \dots, m_s) \in \mathcal{C}_\pm \subseteq \mathcal{C}.$$

It will be beneficial to express admissibility of a coloring in terms of homomorphism  $\phi$ .

**Proposition 3.1.**

A coloring  $(m_1, \dots, m_s) \in M^s$  is a admissible  $\iff$  for each crossing  $x_j$  of type  $\pm$

$$\phi_\pm(\pi_{x_j}(m_1, \dots, m_s)) = 0.$$

**Proof.** Stems from the fact that  $\mathcal{C}_\pm = \ker \phi_\pm$ . □

**Definition 3.3 : color checking matrix.**

After assignments arcs to coordinates in  $M^s$  and crossings to coordinates in  $N^x$  it is possible to define a linear homomorphism  $D\phi : M^s \rightarrow N^x$  as

$$D\phi(m_1, \dots, m_s) = (\phi_\pm(\pi_{x_1}(m_1, \dots, m_s)), \phi_\pm(\pi_{x_2}(m_1, \dots, m_s)), \dots).$$

Matrix that is created after choosing a basis for  $M^s$  and  $N^x$  will be called a **color checking matrix**.

Taking  $\phi_\pm$  to be linear equations of form

$$\phi_+(u, i, o) = au + bi + co$$

$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o,$$

where  $u, i$  and  $o$  correspond to arcs as seen in fig. 2 and all the coefficients are linear homomorphisms  $M \rightarrow N$ , we know that all the entries for the color checking matrix will be linear combinations of  $a, b, c, \alpha, \beta, \gamma$ . If  $M$  has  $n$  generators we chose to block the matrix  $D\phi$  into  $n \times n$  blocks.

**Proposition 3.2.**

Coloring  $(m_1, \dots, m_s) \in M^s$  is admissible  $\iff (m_1, \dots, m_s) \in \ker D\phi$ .

*Proof.*  $\implies$

We know that every projection  $\pi_{x_j}(m_1, \dots, m_s)$  is in  $\ker \phi_{\pm}$ , depending on the type of  $x_j$  crossing. Thus, there is no projection  $\pi_{x_j}$  that is not being reduced by  $\phi_{\pm}$ .

$\impliedby$

□

We need to impose restrictions on the coloring rule. We want  $\mathcal{C}$  to be two dimensional (have two generators). That way we have the following diagram

$$M^2 \xhookrightarrow{\quad} M^3 \twoheadrightarrow \mathcal{C}$$

We can assume that  $M^2$  corresponds to the 'up' and 'in' segments in a crossing (compare fig. 2), then we can define  $\phi'_{\pm}$  to take  $u$  and  $i$  segments and return the out segment so that the labeling agrees with the coloring rule. Now, take the red arrow in the diagram above to be the correspondence

$$(u, i) \mapsto (u, i, \phi'_{\pm}(u, i)).$$

This demands that both  $c$  and  $\gamma$  in the definition of  $\phi_+$  and  $\phi_-$  are invertible. For the sake of simplicity, we will take  $c = \gamma = -1$ .

With this assumption for any admissible coloring  $(u, i, o)$  of a crossing we have the following relation:

$$\phi_+ : o = au + bi$$

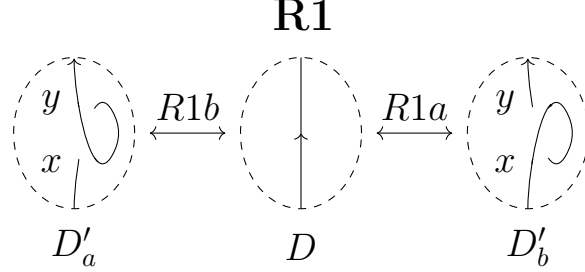
$$\phi_- : o = \alpha u + \beta i.$$

We might also demand that a trivial coloring (every arc is assigned the same element of  $M$ ) is an admissible coloring.

### 3.2 Relation on color checking matrices

The color checking matrix, defined in definition 3.3, is not a knot invariant. Its size and structure changes as Reidemeister moves are applied to the

diagram. Thus, we need to define which matrices stem from equivalent knot diagrams.



Both Reidemeister moves  $R1a$  and  $R1b$  require the following diagram to commute,

$$\begin{array}{ccc}
 M^{s+1} & \xrightarrow{D'\phi} & N^{x+1} \\
 \downarrow & & \downarrow \\
 M^{s+1}, x=y & & N^x \oplus (N/\phi_{\pm}(M^3)) \\
 f \downarrow & & \downarrow g \\
 M^s & \xrightarrow{D\phi} & N^x
 \end{array}$$

where  $\phi_{\pm}$  changes (for  $R1a$  we have  $+$  and for  $R1b$   $-$ ). We take  $f$  and  $g$  to be given by

$$\begin{aligned}
 f(m_1, \dots, m_s, m_{s+1}) &= (m_1, \dots, m_s + m_{s+1}) \\
 g(n_1, \dots, n_x, n_{x+1}) &= (n_1, \dots, n_x + n_{x+1}).
 \end{aligned}$$

The homomorphism  $f$  ensures that on the rest of diagrams  $D'$  are labeled  $x$  in figure above and  $y$  add up to the arc visible in the diagram  $D$ . Meanwhile,  $g$  ensures that the additional crossing is treated with the appropriate coloring rule.

In terms of matrices, the above diagram can be translated to

$$\begin{array}{ccc}
 D'_a & D & D'_b \\
 \begin{bmatrix} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} & \xrightarrow{R1a} & \begin{bmatrix} x_1 + y_1 & z_1 & \dots \\ \vdots & & \ddots \end{bmatrix} & \xrightarrow{R1b} & \begin{bmatrix} \beta & \alpha + \gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix}
 \end{array}$$

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**Theorem 3.3.**

The equivalence class of a color checking matrix of a diagram  $D\phi$  under relation generated by matrix relations  $R1a$ ,  $R1b$ ,  $R2$  and  $R3$  is a knot diagram. Thus we can define  $K\phi := [D\phi]$ .

**Proof.** A direct result of the definition of the equivalence relation.  $\square$

**3.3 Smith normal form**

The ring  $R$  over which we consider modules  $M$  is not necessary a principal ideal domain. However, there are plenty of PID rings and one can find at least one PID  $P$  with a homomorphism  $R \rightarrow P$  that allows to consider  $M$  as a  $P$ -module by tensoring it with  $P$ :

$$M_P = M \otimes_R P.$$

That way, we can consider a new type of equivalence relation on any color checking matrix  $D\phi$ .

**Definition 3.4 : Smith normal form.**

Take  $A \in K\phi$  and consider it as a  $s \times x$  matrix with terms in a  $P$ . Then there exist a  $s \times s$  matrix  $S$  and  $x \times x$  matrix  $T$  such that  $SAT$  is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every  $i$   $a_i | a_{i+1}$ . Such a matrix  $SAT$  is called the **Smith normal form** of matrix  $A$ .

As was mentioned in the first section,  $\bar{x} \in M^s$  is a coloring of a diagram  $D$  if and only if  $D\phi(\bar{x}) = 0$ , that is  $\bar{x} \in \ker D\phi$ . The Smith normal form hints at the structure of matrix kernel - the columns filled with zeros will contributed a free factor  $M$  to the kernel.

Take  $(a)$  to be a prime ideal with its generator  $a$  appearing in the Smith normal form of  $D\phi$ . Then we might consider the matrix over a new ring



$P/(a)$ , which is still a PID. After this change, the structure of the kernel has changed as now there are additional zero columns where  $a$  and all its multiples stood.

**Definition 3.5 : reduced normal form of matrix.**

Take  $A$  to be a matrix with coefficients in principal ideal domain  $P$ . Take  $a_1, \dots, a_k \in P$  to be all the elements of the Smith normal form of  $A$  that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix  $A$ .

When working with knots we usually take  $R = \mathbb{Z}[t, t^{-1}]$  and  $M = \mathbb{Z}[t, t^{-1}]$ . This is not a PID ring but there are multitudes of PID rings into which  $R$  can be mapped. The following algorithm can be used to calculate the Smith normal form of a color checking matrix.

1. Let  $A = \{a_{i,j}\}_{i,j \leq n}$  be an  $n \times n$  matrix. Take the ideal  $I = (a_{i,j})$  generated by all the terms of  $A$ .
2. If we are in PID then  $I$  has one generator, call it  $a$ .
3. We can now use the following row and column operations to put  $a$  in the upper left corner of  $A$ 
  - (a) Permuting rows (columns).
  - (b) Adding a linear combination of rows (columns) to the remaining row (column).
4. With  $a$  in the upper left corner we can now use the fact that it was the generator of  $I$  to strike out the remaining terms on the first column and row, using the operations described in the previous point.
5. Repeat the same algorithm on the smaller matrix  $\{a_{i,j}\}_{1 < i,j \leq n}$ .

The following example justifies the utility of the reduced normal form of color checking matrices in distinguishing knots.

**Example 3.1.** Consider the knots  $6_1$  with diagram as seen in fig. 3 and  $9_{46}$  pictured in fig. 4, ring  $R = \mathbb{Z}[t, t^{-1}]$ ,  $M = R$  and

$$\begin{cases} \phi_+(u, i, o) = (1 - t)u + ti - o \\ \phi_-(u, i, o) = (1 - t^{-1})u + t^{-1}i - o. \end{cases}$$

The two rings have the same Alexander polynomial,  $\Delta = -2t^{-2} + 5t^{-1} - 2$ , and the same Alexander module  $H^1(S^3 - K) = \mathbb{Z}[t, t^{-1}]/(\Delta)$ .

For the knot  $6_1$  we find the matrix  $D\phi$  and after changing to the *PID* ring  $P = \mathbb{Q}[t, t^{-1}]$  we see that the Smith normal form is:

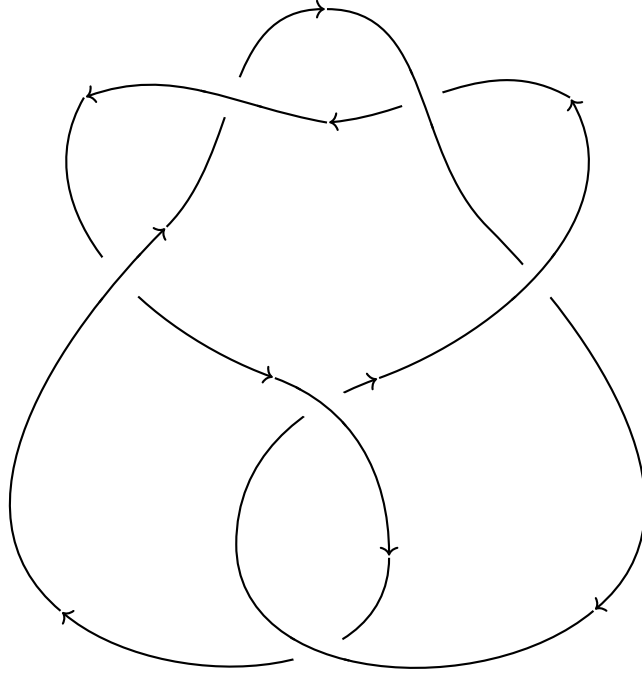


Figure 3: Diagram of knot  $6_1$ .

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which after reduction is

$$A' = (-2t^{-2} + 5t^{-1} - 2)$$

a  $1 \times 1$  matrix with the only term being the Alexander polynomial of  $6_1$ .

Using diagram in fig. 4 of  $9_{46}$  it can be calculated that the Smith normal form of  $D\phi$  is

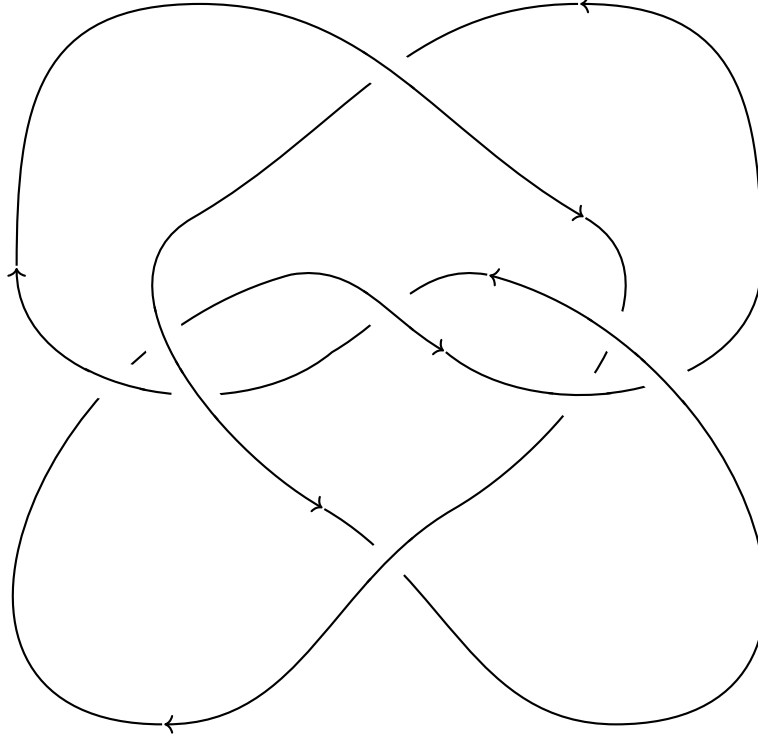


Figure 4: Diagram of knot  $9_{46}$ .

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t - t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2} - 2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while reduced normal form of  $D\phi$  is

$$B' = \begin{pmatrix} 2t - t^2 & 0 \\ 0 & t^{-2} - 2t^{-1} \end{pmatrix}$$

which is significantly different than the one for  $6_1$ . Observe also that the determinant of both matrices is equal to the Alexander polynomial of

corresponding knots

$$\det(A') = -2 + 5t^{-1} - 2t^{-2}$$

$$\det(B') = (2t - t^2)(t^{-2} - 2t^{-1}) = 2t + 2 + 2t^{-1} = -t(-2 + 5t^{-1} - 2t^{-2}).$$

### Theorem 3.4.

The reduced normal form of color checking matrix does not depend on the choice of diagram  $D$ . Thus, it is well defined for  $K\phi$  and is a knot invariant.

**Proof.** Take a knot  $K$  and its diagram  $D$  with  $s$  segments and  $x$  crossings. We will show that applying any Reidemeister move to this knot will not change the reduced normal form of its color checking matrix.

## R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a** (the proof of **R1b** is left as an exercise for the reader).

Take  $D'$  to be diagram  $D$  with one arc twisted into a  $+$  crossing. In opposition to the assumption in previous section, we will take the arcs and crossings that differ between those two diagrams to be on first positions. Now, the matrices  $D\phi$  and  $D'\phi$  are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

Adding the first column of  $D'\phi$  to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

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because  $a + b = 1$ . Now we know that  $b$  is a unit, thus we can easily remove the elements of the first column that are not  $b$ . This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2 + y_2 & \dots \\ 0 & x_3 + y_3 & \\ \vdots & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like  $D\phi$ . The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Thus, the reduced normal form of  $D'\phi$  is the same as in  $D\phi$ .

## R2

Now the diagram  $D'$  is a diagram  $D$  with one arc poked onto another. Once again we will put those changed arcs at the beginning of the color checking matrix to obtain following matrices:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3 + v_3 & \dots \\ x_4 & u_4 + v_4 & \\ \vdots & & \end{bmatrix}$$

Adding the third column of  $D'\phi$  multiplied by  $\alpha$  and  $\beta$  to first and second column respectively we are able to reduce the first row to only zeros and  $-1$ . Now, adding this row to the second one creates a column with only  $-1$  and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a + b\alpha & 0 & -1 & \\ 0 & x_3 & u_3 & v_3 & \\ 0 & x_4 & u_4 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that  $a + b\alpha = 0$  and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 & \\ 0 & v_3 + u_3 & v_3 + u_3 & x_3 & \\ 0 & v_4 + u_4 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & v_3 + u_3 & x_3 & \\ 0 & 0 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as  $D\phi$ .

### R3

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as  $b$  and  $\beta$  were taken to be units.

□

## 4 A look at category theory

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