

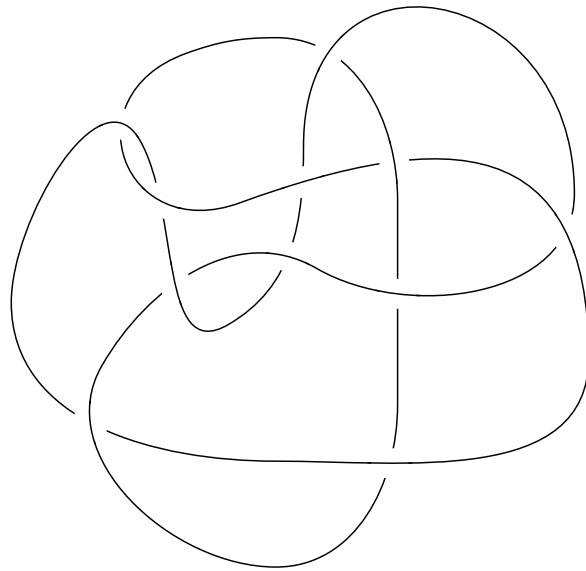
Knot colorings and homological invariants

(Kolorowania węzłów i niezmienniki homologiczne.)

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Abstract

The knot group $G = \pi_1(K)$ is a starting point for many knot invariants. Alexander matrix is a representation matrix for a subgroup of G and from its determinant, the Alexander polynomial is obtained. Another way of obtaining said polynomial is by considering a coloring matrix which assigns elements of R -module M to segments from a diagram D of knot K . This approach can be derived from the image of a resolution of Alexander module through the functor $\text{Hom}(-, M)$. Nevertheless, color checking matrices do not instantly yield a knot invariant, however it is possible to define an equivalence relation that identifies matrices stemming from the same knot. This approach is used to distinguish a pair of knots with the same Alexander polynomial. In the end, a way of generalizing the procedure of coloring diagrams is presented in terms of category theory.

Introduction

In knot theory distinguishing knots is often a difficult endeavor, usually facilitated by the notion of invariants. An interesting group of knot invariants are polynomial invariants, such as the Alexander polynomial. Another group worth mentioning are knot colorings that can also yield an element of the ring $\mathbb{Z}[\mathbb{Z}]$.

Very often, considering only one invariant is not sufficient, as there are many knots that share its value, i.e. $K11n85$ and $K11n164$ have the same Alexander polynomial. However, a more subtle application of the same method that yields the Alexander polynomial can sometimes distinguish such knots.

The following paper is a result of a year long cooperation between prof. Tadeusz Januszkiewicz, Julia Walczuk and the author of this thesis. In it, connections between the knot group, knot colorings and homology modules of infinite cyclic covering (see [5]) will be outlined. As an additional exercise, we will show a way of distinguishing already mentioned knots $K11n85$ and $K11n164$.

The first section of this paper defines the most important terms used in knot theory, as well as highlights the connection between the metabelianization of knot group and the first homology module of an infinite cyclic

covering of the complement of said knot.

Subsequently, the construction of a $\mathbb{Z}[\mathbb{Z}]$ module from the kernel of $G^m \rightarrow \mathbb{Z}$ is presented. This module is defined to be the Alexander module K_G^{ab} of knot K (definition 1.4). Then, Alexander matrix is introduced (??) as the representation matrix of the Alexander module. In this section the resolution of Alexander module is proven to be of form

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

At the end of the second section, a connection between resolution of the Alexander module and coloring of diagrams is made.

The third and last section is focused on coloring matrices (definition 3.3) and defining an equivalence relation between matrices relating to the same knot. A new knot invariant is defined (definition 3.5) and an example of its utility is presented in example 3.2.

Last section is dedicated to presenting an approach to diagram colorings from the perspective from category theory. In addition, a connection between the Alexander matrix and color checking matrix (using a particular palette, named the Alexander palette) is presented.

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1 Preliminaries

1.1 Knots and diagrams

In mathematical terms, a knot is a smooth embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \hookrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [3]. If two points are mapped to one by this projection, we say that a small neighbourhood of this point which looks locally like $-|-$ is a crossing.

S^1 is orientable, thus we can chose an orientation for any knot and, as a consequence, its diagram.

Intuitively, two knots K_1 and K_2 are equivalent if we can deform one into the other [6]. This translates to an equivalence of diagrams, which is generated by comparing diagrams that are exactly the same save for an interior of some disc in \mathbb{R}^2 . If inside of said disc the diagrams differ by one of **Reidemeister moves**, we say that they are equivalent. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D , 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [7].

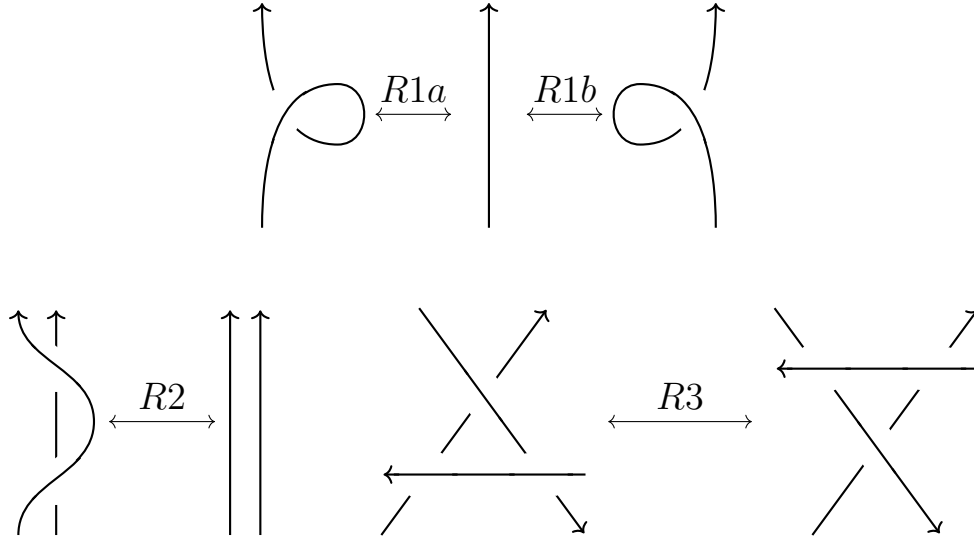


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

1.2 Knot group

Let K be a knot and D be its oriented diagram with s segments and x crossings. A segment of a diagram is a line of the diagram between two

crossings in which it disappears under another line.

Definition 1.1 : knot group.

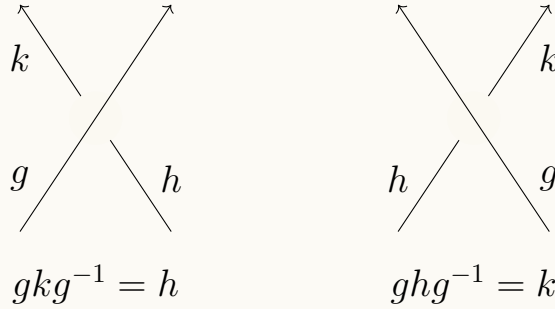
The fundamental group of knot complement $X = S^3 - K$ is called a **knot group**:

$$\pi_1(K) := \pi_1(X).$$

Although the knot itself is always a circle S^1 , the knot group has usually an interesting yet difficult structure. The most commonly used presentation of the knot group is called **the Wirtinger presentation**.

Definition 1.2 : Wirtinger presentation.

Given a diagram D of knot K with segments a_1, a_2, \dots, a_s and crossings c_1, \dots, c_x the knot group $\pi_1(K)$ can be represented as $\pi_1(K) = \langle G \mid R \rangle$, where G is the set of segments of D and relations R correspond to crossings in the manner described in the diagram below



Representation $\langle G \mid R \rangle$ described above is called the **Wirtinger presentation** [4, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to $S^3 = K \oplus S^3 - K$ or noticing that every two generators are conjugate, is that the abelianization of the knot group is always \mathbb{Z} . This leads to a short exact sequence

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0.$$

The group $K_G = \ker(ab : G \rightarrow \mathbb{Z}) = [G, G]$ in general is not abelian nor finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization $K_G^{ab} = K_G/[K_G, K_G]$ allows a $\mathbb{Z}[\mathbb{Z}]$ module structure and thus contains obtainable information about the knot K .

Lemma 1.1.

For any group G , the commutator of its commutator K_G is a normal subgroup: $[K_G, K_G] = [[G, G], [G, G]] \triangleleft G$.

Proof. The commutator subgroup is a characteristic subgroup, since for any automorphism $\phi : G \rightarrow G$

$$\phi(hgh^{-1}g^{-1}) = \phi(h)\phi(g)\phi(h)^{-1}\phi(g)^{-1} \in K_G = [G, G].$$

Conjugation by any element $g \in G$ is an automorphism of the commutator K_G . Thus it preserves its commutator subgroup $[K_G, K_G]$. \square

As a consequence, in the group $G/[K_G, K_G]$ left and right multiplication is the same. Thus, the following is an exact sequence:

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives foundation for the following definition.

Definition 1.3 : metabelianization.

The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of G .

We will return to the concept of metabelianization in section 2. For the time being, let us assign a name to K_G :

Definition 1.4 : Alexander module.

Given a group G , the abelianization of the commutator of a group G , K_G^{ab} , with $\mathbb{Z}[\mathbb{Z}]$ -module structure is called the **Alexander module** of G . If G is a knot group, then it is the Alexander module of the knot K .

How the $\mathbb{Z}[\mathbb{Z}]$ module structure is obtained is described in detail in section 2.1.

1.3 Infinite cyclic covering

Let X be the complement of a knot K ($X = S^3 - K$). Take \tilde{X} to be its universal covering, meaning that it is simply connected. The fundamental group G of X acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G, G]$ is normal in G and so $\pi_1(X)/K_G$ acts on \tilde{X} . Thus we might take the quotient space $\overline{X} = \tilde{X}/[G, G]$ and

call it the **infinite cyclic covering** of X . Due to this construction, the fundamental group of \bar{X} is exactly

$$\pi_1(\bar{X}) = [G, G] = K_G$$

and from the perspective of homology modules, we have

$$H_1(\bar{X}, \mathbb{Z}) = \pi_1(\bar{X})^{ab} = K_G^{ab}.$$

Working with homology modules of an infinite cyclic cover of X instead of K_G^{ab} directly is beneficial when proving some properties of K_G^{ab} , i.e. that it is a torsion module in proposition 1.2.

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{array}{ccc} \tilde{X} & \curvearrowright & G \\ \downarrow & & \\ \bar{X} & \curvearrowright & G/[G, G] \\ \downarrow & & \\ X = S^3 - K & & \end{array}$$

A **Seifert surface** S of knot K is an orientable surface with boundary embedded in S^3 such that $\partial S = K$. Take a countable amount of X , with S without its boundary embedded, and label each with an element from \mathbb{Z} . We might now cut each of the copies of X along the Seifert surface of K and identify the $+$ side of S from the i -th copy of X with the $-$ side of S from the $(i + 1)$ -th copy of X . Notice that the arising space with a projection to one copy of X is an infinite cyclic cover of X .

Imagine that each copy of X inside of \bar{X} is a box labeled with some integer k . The ring action of $\mathbb{Z}[\mathbb{Z}]$ on \bar{X} is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to \bar{X} .

Proposition 1.2.

The $\mathbb{Z}[\mathbb{Z}]$ -module $K^{ab} = H_1(\bar{X}, \mathbb{Z})$ is a torsion module.

Proof. Consider the following homomorphism on chain complexes:

$$f : C_*(\bar{X}) \rightarrow C_*(\bar{X})$$

2 Resolution of the Alexander module

2.1 Construction of Alexander module

Take $G = \langle G \mid R \rangle$ to be the Wirtinger presentation of G obtained from oriented diagram D . Because K is a knot and not a link, we know that the number of segments is equal to the number of crossings, thus we can take $n = s = x$ [9].

Let a_1, \dots, a_n be the generators of G and x_1, \dots, x_n its relations. The homomorphism of abelianization of G is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, \dots, n$. In order to obtain a presentation of K_G , the kernel of abelianization, we need to change the set of generators of G to

$$\{a_1, A_2 = a_2 a_1^{-1}, \dots, A_n = a_n a_1^{-1}\}.$$

It is obvious that for every $i > 1$ $A_i \mapsto 0$ by abelianization of G . Thus A_2, \dots, A_n are some of the generators of K_G . However, for each $i = 2, \dots, n$ and $k \in \mathbb{Z}$ the following is an element of K_G :

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the presentation of K_G as an abelian group is infinite with (possibly redundant) generators

$$\{b_{i,k} : i = 2, \dots, n, k \in \mathbb{Z}\}.$$

Changing generators of G induced a change in relations. Suppose that the following relation was true in G

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of a_1 the following two relations can be obtained

$$a_1^{-1} A_k a_1 = (a_1^{-1} A_i a_1) A_j A_i^{-1}$$

$$a_1^{-3}A_ka_1^3 = (a_1^{-3}A_ia_1^3)(a_1^{-2}A_ja_1^2)(a_1^{-2}A_i^{-1}a_1^2).$$

Obviously in G both of those relations are equivalent, however in K_G they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x}b_{j,x-1}b_{i,x-1}^{-1}$$

to obtain infinitely many relations from K_G .

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

For any group H with $H^{ab} = \mathbb{Z}$ we can write a homomorphism $\mathbb{Z} \rightarrow H$ such that composition $\mathbb{Z} \rightarrow H \rightarrow \mathbb{Z}$ is identity on \mathbb{Z} . Thus, this sequence splits and we can write

$$G^{mab} = K_G^{ab} \rtimes \mathbb{Z}.$$

Hence action of \mathbb{Z} can be defined on the group K_G^{ab} , with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1A_ia_1^{-1}.$$

This procedure allows K_G^{ab} to be interpreted as a $\mathbb{Z}[\mathbb{Z}]$ -module.

Moreover, the group G^{mab} and $\mathbb{Z}[\mathbb{Z}] K_G^{ab}$ can be used interchangeably as knowing the action of \mathbb{Z} on K_G^{ab} allows us to write the semidirect product of \mathbb{Z} and K_G^{ab} .

2.2 Basic properties

Knowing the resolution of a module allows one to change said module into a matrix or even a sequence of matrices, each containing at least a portion of information about its structure.

Definition 2.1 : Alexander matrix.

The presentation matrix A_D of K_G^{ab} with Wirtinger presentation is called the **Alexander matrix** of the Alexander module K_G^{ab} .

We start writing the beginning K_G^{ab} resolution as follows:

$$\dots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0 \quad (1)$$

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its columns will give a square matrix whose determinant is nonzero. We will prove this statement promptly after consider the Alexander module as a vector space over the field of fractions of $R = \mathbb{Z}[\mathbb{Z}]$ [2, Chapter 3].

In proposition 1.2 it was shown that the Alexander module is torsion. Thus, as a vector space $K_G^{ab} \otimes_R R^{-1}R = 0$ it is trivial. Hence, the sequence in (1) translates to the following sequence of $R^{-1}R$ modules

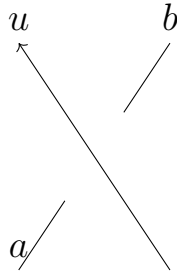
$$\dots \longrightarrow R^n \otimes_R R^{-1}R \xrightarrow{A_D^V} R^{n-1} \otimes_R R^{-1}R \longrightarrow 0 \quad (2)$$

As there exists an inclusion $R \hookrightarrow R^{-1}R$, every matrix with terms in R can be treated as a matrix with terms in $R^{-1}R$. Naturally, $A_D^V = A_D \otimes Id_{R^{-1}R}$ is just matrix A_D (with terms in R) with adjoined 1×1 matrix with just identity of $R^{-1}R$. Thus, we can easily translate most properties of A_D^V to properties of A_D , i.e. its determinant and surjectivity.

Proposition 2.1.

Let A'_D be the Alexander matrix A_D with one of its rows struck out. Then $\det(A'_D) \neq 0$.

Proof. We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator x is sent to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module K^{ab} we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1 - t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which x is b or c are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t - 1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and ± 1 . Thus, the sum is not equal to zero. There are two of such rows as the segment b has to be the "out" and "in" segment of some crossing. In other words, segment b has to have a start and end in some crossings.

The reasoning above is true for matrix A_D^V from (2). We make the switch to vector space to use the connection between the rank of matrix and the dimension of its image.

Let S_i be the column of the Alexander matrix corresponding to the segment labeled i . The sum $\sum_{i \leq n-1} S_i$ is a vector with two nonzero terms. Take S_j and S_k to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both S_j and S_k by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j, k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i : i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D^V(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. \square

The proposition 2.1 implies that image of A_D^V has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

Theorem 2.2.

The determinant $\det(A'_D)$ up to multiplication by a unit is independent of the choice of the diagram D .

Proof. A proof using Dehn presentation is provided in [1], while a proof of more general case is provided in [8] \square

Definition 2.2 : Alexander polynomial.

The **Alexander polynomial** of a knot K is the determinant of any maximal minor of the Alexander matrix A_D .

The Alexander polynomial is a knot invariant as a consequence of theorem 2.2 and proposition 2.1

Proposition 2.3.

Let G be a knot group of K and $F = R^{-1}R$ the field of fraction of ring R . Then K_G^{ab} always has a resolution

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

where n is the number of crossings of the chosen diagram D of knot K and $M \otimes_R F \cong F$.

Proof. We start by saying that $R \otimes_R F \cong F$ because R is a free module over R [2, Proposition 2.14].

Proposition 2.1 implies that (2) can be extended into the following exact sequence of vector spaces:

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R R^{-1}R = 0 \longrightarrow 0$$

as we proved that $\dim(\text{im } A_D^V) = n-1 \implies \dim(\ker A_D^V) = 1$.

The ring of fractions is flat [2, Chapter 3] at the same time we only consider R -modules treated as vector spaces in this proposition. Thus, we have the following exact sequence

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

with $M \otimes_R F \cong F$. □

Notice, that sequence

$$\star : 0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow 0$$

is not acyclic, however it allows us to once again define the Alexander module, this time as $H_1(\star)$.

2.3 A homological roots of diagram colorings

Thus far a resolution of the Alexander module K_G^{ab} provided a matrix and with it a polynomial invariant of knots. In this short section we will explain the connection between Alexander module and knot colorings, which will be the focus of the subsequent section.

Take M to be a finitely generated $R = \mathbb{Z}[\mathbb{Z}]$ -module. The functor $\text{Hom}(-, M^n)$ is left exact therefore applied to the resolution of the Alexander module generates the following sequence

$$0 \longrightarrow \text{Hom}(R, M) \longrightarrow \text{Hom}(R^n, M) \xrightarrow{\text{Hom}(A_D, M)} \text{Hom}(R^{n-1}, M) \longrightarrow \text{Hom}(K_G^{ab}, M^n)$$

The diagram D taken as the starting point for the construction of K_G^{ab} had $n = x$ crossings and $n = s$ segments. The module K_G^{ab} was presented using $(n - 1)$ generators, corresponding to all but one segments of the diagram. If we allow for propagation of values, then $\text{Hom}(R^{n-1}, M)$ can be interpreted as assigning values from M to $(n - 1)$ segments in diagram D , with the last segment colored based on the remaining part of the diagram.

The arrow $\text{Hom}(R^{n-1}, M) \longrightarrow \text{Hom}(K_G^{ab}, M)$ ensures that the structure of K is taken into account during this assignment. Its kernel is be equal to $\text{im Hom}(A_D, M)$ and thus remembers which segments contributed to which crossings.

The above remark points at a similarity between the concept of diagram colorings, elaborated in the following section, and the more topological invariant which is the Alexander module

3 Knot colorings

3.1 Diagram colorings

Let K be a knot and D be its oriented diagram with s segments and x crossings. In such diagrams we can see two different crossing types as seen in fig. 2.

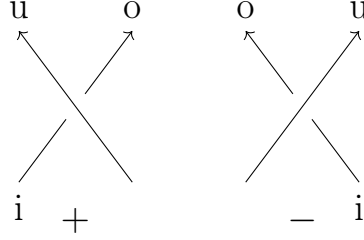


Figure 2: Two types of crossing in oriented diagram.

Take a commutative ring with unity R and an R -module M .

Definition 3.1 : coloring rule.

Take $\mathcal{C} \subseteq M^3 \oplus M^3$ to be a module such that there exists two modules $\mathcal{C}_{\pm} \subseteq M^3$ for which $\mathcal{C} = \mathcal{C}_{+} \oplus \mathcal{C}_{-}$. We will call \mathcal{C} a **coloring rule**. The two submodules \mathcal{C}_{\pm} each correspond to a type of crossing in diagram D .

We can now construct a pair of homomorphisms

$$\phi_{\pm} : M^3 \rightarrow M/\mathcal{C}_{\pm} = N_{\pm},$$

cumulatively referred to as ϕ . We will call ϕ and \mathcal{C} **coloring rule** interchangeably.

For each crossing x_j in diagram D we can construct a projection

$$\pi_{x_j} : M^s \twoheadrightarrow M^3$$

which restricts M^s to the three (or two, in which case one coordinate is zero) arcs that constitute x_j .

Definition 3.2 : diagram coloring.

A **coloring of diagram** D is any element $(m_1, \dots, m_s) \in M^s$ that assigns elements of M to each arc. We will call this coloring **admissible**

if for every crossing x_j of type \pm we have

$$\pi_{x_j}(m_1, \dots, m_s) \in \mathcal{C}_\pm \subseteq \mathcal{C}.$$

It will be beneficial to express admissibility of a coloring in terms of homomorphisms ϕ .

Proposition 3.1.

A coloring $(m_1, \dots, m_s) \in M^s$ is a admissible \iff for each crossing x_j of type \pm

$$\phi_\pm(\pi_{x_j}(m_1, \dots, m_s)) = 0.$$

Proof. Stems from the fact that $\mathcal{C}_\pm = \ker \phi_\pm$. □

3.2 Color checking matrix

tutaj dam definicje color checking matrix z innego pliku

Definition 3.3 : color checking matrix.

After assignings arcs to coordinates in M^s and crossings to coordinates in N^x it is possible to define a linear homomorphism $D\phi : M^s \rightarrow N^x$ as

$$D\phi(m_1, \dots, m_s) = (\phi_\pm(\pi_{x_1}(m_1, \dots, m_s)), \phi_\pm(\pi_{x_2}(m_1, \dots, m_s)), \dots).$$

Matrix that is created after choosing a basis for M^s and N^x will be called a **color checking matrix**.

Taking ϕ_\pm to be linear equations of form

$$\phi_+(u, i, o) = au + bi + co \tag{3}$$

$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o, \tag{4}$$

where u, i and o correspond to arcs as seen in fig. 2 and all the coefficients are linear homomorphisms $M \rightarrow N$, we know that all the entries for the color checking matrix will be linear combinations of $a, b, c, \alpha, \beta, \gamma$. If M has n generators we chose to block the matrix $D\phi$ into $n \times n$ blocks.

Proposition 3.2.

Coloring $(m_1, \dots, m_s) \in M^s$ is admissible $\iff (m_1, \dots, m_s) \in \ker D\phi$.

Proof. We start by saying that

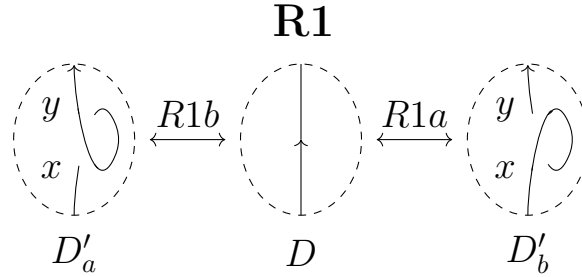
$$(m_1, \dots, m_s) \in \ker D\phi \iff [(\forall x_j \text{ crossing}) \phi_{\pm}(\pi_{x_j}(m_1, \dots, m_s)) = 0].$$

Which is to say that every coordinate of $D\phi(m_1, \dots, m_s)$ is zero. Proposition 3.1 says that it is equivalent with (m_1, \dots, m_s) being an admissible coloring. \square

3.3 Relation on color checking matrices

The color checking matrix, defined in definition 3.3, is not a knot invariant. Its size and structure changes as Reidemeister moves are applied to the diagram. Thus, we need to define which matrices stem from equivalent knot diagrams.

For the time being the diagram D has s segments and x crossings. Although only knots are considered in this paper, it is possible to expand definitions of color checking matrices and relations on them to links.



Both Reidemeister moves $R1a$ and $R1b$ require the following diagram to commute,

$$\begin{array}{ccc}
M^{s+1} & \xrightarrow{D'\phi} & N^{x+1} \\
\downarrow \cong & & \downarrow \cong \\
M^{s+1}, x=y & & N^x \oplus (N/\phi_{\pm}(M^3)) \\
f \downarrow & & \downarrow g \\
M^s & \xrightarrow{D\phi} & N^x
\end{array}$$

where ϕ_{\pm} changes (for $R1a$ we have $+$ and for $R1b$ $-$). We take f and g to be given by

$$f(m_1, \dots, m_s, m_{s+1}) = (m_1, \dots, m_s + m_{s+1})$$

$$g(n_1, \dots, n_x, n_{x+1}) = (n_1, \dots, n_x + n_{x+1}).$$

The homomorphism f ensures that on the rest of diagrams D' are labeled x in figure above and y add up to the arc visible in the diagram D . Meanwhile, g ensures that the additional crossing is treated with the appropriate coloring rule.

In terms of matrices, the above diagram can be translated to

$$\begin{array}{c} D'_a \\ \left[\begin{array}{cccc} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{array} \right] \end{array} \underset{\sim}{\overset{R1a}{\rightsquigarrow}} \begin{array}{c} D \\ \left[\begin{array}{cccc} x_1 + y_1 & z_1 & \dots & \\ \vdots & & \ddots & \end{array} \right] \end{array} \underset{\sim}{\overset{R1b}{\rightsquigarrow}} \begin{array}{c} D'_b \\ \left[\begin{array}{cccc} \beta & \alpha + \gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{array} \right] \end{array}$$

where $(\forall i = 1, \dots, x) x_i = 0 \vee y_i = 0$.

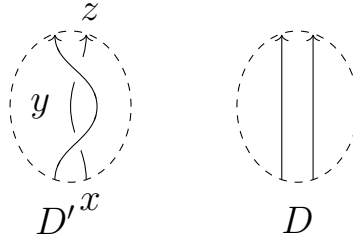
Notice, that if the propagation rule that was outlined at the beginning of this section is to be true, one must have

$$0 = a + b + c = a + b - 1 \implies a = 1 - b$$

$$0 = \alpha + \beta + \gamma = \alpha + \beta - 1 \implies \alpha = 1 - \beta,$$

as the two segments in both D' must admit coloring with one element from M . This puts further restrictions on coloring rules.

R2



For the second Reidemeister move we will say that $D\phi$ and $D'\phi$ are in relation if the following diagram commutes:

$$\begin{array}{ccc} M^{s+2} & \xrightarrow{D'\phi} & N^{x+2} \\ \downarrow & & \downarrow \\ M^{s+2}, x = z & & N^x \oplus (N/\phi_{\pm}(M^3)) \oplus (N/\phi_{\mp}(M^3)) \\ \downarrow & & \downarrow \\ M^s & \xrightarrow{D\phi} & N^x \end{array}$$

In terms of matrices, the following move is admitted:

$$\begin{array}{ccc} D' & & D \\ \begin{bmatrix} b & c & 0 & a & \dots \\ 0 & \beta & \gamma & \alpha & \\ x_1 & 0 & z_1 & w_1 & \\ \vdots & & & & \ddots \end{bmatrix} & & \begin{bmatrix} x_1 + z_1 & w_1 & \dots \\ \vdots & & \ddots \end{bmatrix} \end{array}$$

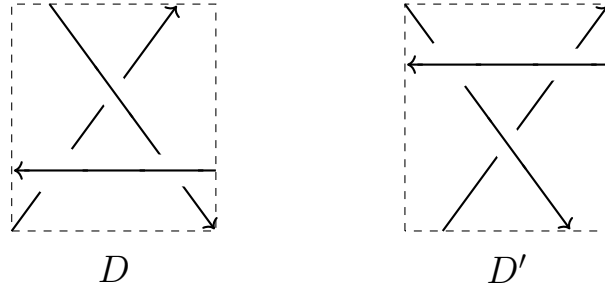
where $(\forall i = 1, \dots, x) x_i = 0 \vee z_i = 0$.

Once more, if the propagation property of coloring is to be upheld, then

$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1 \end{cases}$$

meaning that x and z can be colored with one element from M .

R3



In terms of matrices, the following move is admitted:

$$\begin{array}{ccc} D' & & D \\ \begin{bmatrix} \alpha & \gamma & \beta & 0 & 0 & 0 & \dots \\ 0 & 0 & c & b & 0 & a & \\ \beta & 0 & 0 & 0 & \gamma & \alpha & \\ u_1 & 0 & v_1 & w_1 & x_4 & y_4 & \\ \vdots & & & & & & \ddots \end{bmatrix} & & \begin{bmatrix} 0 & 0 & \gamma & \beta & \alpha & 0 & \dots \\ \beta & 0 & 0 & 0 & \gamma & \alpha & \\ 0 & c & b & 0 & 0 & a & \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 & \\ \vdots & & & & & & \ddots \end{bmatrix} \end{array}$$

Theorem 3.3.

The equivalence class of a color checking matrix of a diagram $D\phi$

under relation generated by matrix relations $R1a$, $R1b$, $R2$ and $R3$ is a knot diagram. Thus we can define $K\phi := [D\phi]$.

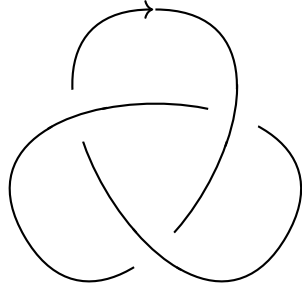
Proof. A direct result of the definition of the equivalence relation. \square

3.4 Determinant of color checking matrix as a knot invariant

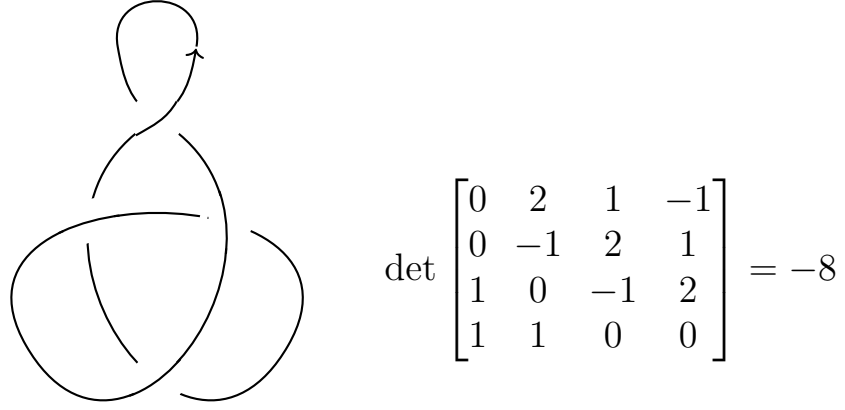
tutaj zostawie restrykcje na color checking matrix tak, zeby wyszedl knot invariant

The reasoning presented in section 2.3 points at determinant of the coloring matrix being an invariant as was the case for the Alexander matrix. However, at the very moment the color checking matrix is not a knot invariant nor is its determinant. Any module \mathcal{C} and associated with it pair of homomorphisms ϕ does not necessarily yield a "nice" coloring. The following example justifies the necessity of imposing restrictions to which a coloring rule must conform in order to be considered in the latter part of this paper.

Example 3.1. Consider a coloring of trefoil knot 3_1 with \mathbb{Z} over the ring \mathbb{Z} with $\phi_{\pm}(u, i, o) = 2u - i + o$. The color checking matrix of its diagram with 3 crossings is



$$\det \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = -3$$




$$\det \begin{bmatrix} 0 & 2 & 1 & -1 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} = -8$$

The most important condition that \mathcal{C}_{\pm} must meet is to be two dimensional. This will allow for propagation of coloring, meaning that knowing colors of two segments creating a crossing the third one can be calculated from ϕ_{\pm} .

The following diagram

$$M^2 \xrightarrow{\quad} M^3 \xrightarrow{\quad} \mathcal{C}$$



must commute, with the red arrow being

$$(u, i) \mapsto (u, i, \phi'_{\pm}(u, i))$$

where ϕ'_{\pm} calculates the "out" segment in admissible coloring of each crossing (compare fig. 2). Using (3) and (4) we can take c and γ to be any invertible elements, i.e. $c = \gamma = -1$, to have

$$\phi'_+(u, i) = au + bi$$

$$\phi'_-(u, i) = \alpha u + \beta i$$

Notice that now a diagram with all but one segments colored can be easily colored in its entirety, using ϕ'_{\pm} on the crossing where the remaining segment starts.

3.5 Reduced Smith normal form

The ring R over which we consider modules M is not necessary a principal ideal domain. However, there are plenty of PID rings and more often

than not, one can find at least one PID P with a homomorphism $R \rightarrow P$ that allows to consider M as a P -module by tensoring it with P :

$$M_P = M \otimes_R P.$$

We will use this idea to define a new type of equivalence relation on any color checking matrices.

Definition 3.4 : Smith normal form.

Take $A \in K\phi$ and consider it as an $n \times n$ matrix with terms in a P by the procedure outlined above. Then there exist a $n \times n$ matrix S and $n \times n$ matrix T such that SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every i $a_i | a_{i+1}$. Such a matrix SAT is called the **Smith normal form** of matrix A .

As was mentioned in the first section, $\bar{x} \in M^n$ is a coloring of a diagram D if and only if $D\phi(\bar{x}) = 0$, that is $\bar{x} \in \ker D\phi$. The Smith normal form hints at the structure of matrix kernel - the columns filled with zeros will contributed a "free" factor M to the kernel.

Take (a) to be a prime ideal with its generator a appearing in the Smith normal form of $D\phi$. Then we might consider the matrix over a new ring $P/(a)$, which is still a PID. After this change, the structure of the kernel has changed as now there are additional zero columns where a and all its multiples stood. Meaning that kernel became bigger and more colorings are admissible over $P/(a)$.

Definition 3.5 : reduced normal form of matrix.

Take A to be a matrix with coefficients in principal ideal domain P . Take $a_1, \dots, a_k \in P$ to be all the elements of the Smith normal form of A that are neither zero nor invertible. Consider a new square

matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix A .

When working with knots we usually take $R = \mathbb{Z}[t, t^{-1}]$ and $M = \mathbb{Z}[t, t^{-1}]$. This is not a PID ring but there are multitudes of PID rings into which R can be mapped. The following algorithm can be used to calculate the Smith normal form of a color checking matrix over a PID ring.

1. Let $A = \{a_{i,j}\}_{i,j \leq n}$ be an $n \times n$ matrix. Take the ideal $I = (a_{i,j})$ generated by all the terms of A .
2. If we are in PID then I has one generator, call it a .
3. We can now use the following row and column operations to put a in the upper left corner of A
 - (a) Permuting rows (columns).
 - (b) Adding a linear combination of rows (columns) to the remaining row (column).
4. With a in the upper left corner we can now use the fact that it was the generator of I to strike out the remaining terms on the first column and row, using the operations described in the previous point.
5. Repeat the same algorithm on the smaller matrix $\{a_{i,j}\}_{1 < i,j \leq n}$.

The following example justifies the utility of the reduced normal form of color checking matrices in distinguishing knots.

Example 3.2. Consider the knots $K11n85$ and $K11n164$ pictured in figs. 3 and 4. They both have the Alexander polynomial equal

$$\Delta(t) = -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3}.$$

Coloring them over ring $\mathbb{Z}[\mathbb{Z}]$ with module $M = \mathbb{Z}[\mathbb{Z}]$ and coloring rules

$$\phi_+(u, i, o) = (1 - t)u + tb - o$$

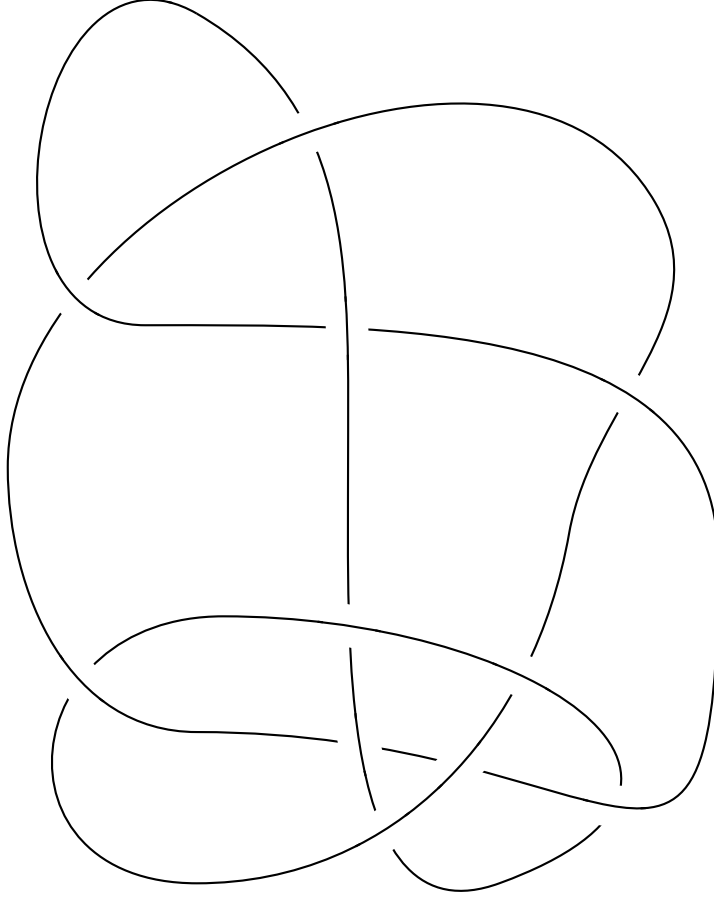


Figure 3: A diagram for knot $K11n85$.

$$\phi_-(u, i, o) = (1 - t^{-1})y + t^{-1}b - o$$

yields two 11×11 matrices whose any 10×10 minor is equal to the Alexander polynomial (up to multiplication by a unit). However, the reduced Smith normal form are

$$D_{11n85}\phi = [-t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3}]$$

$$D_{11n164}\phi = \begin{bmatrix} 1 - t + t^2 & 0 \\ 0 & -t^{-1} + 4 - 5t + 4t^2 - t^3 \end{bmatrix}$$

Having witnessed the utility of the reduced normal form of a coloring matrix we proceed to show that it is in fact a knot invariant.

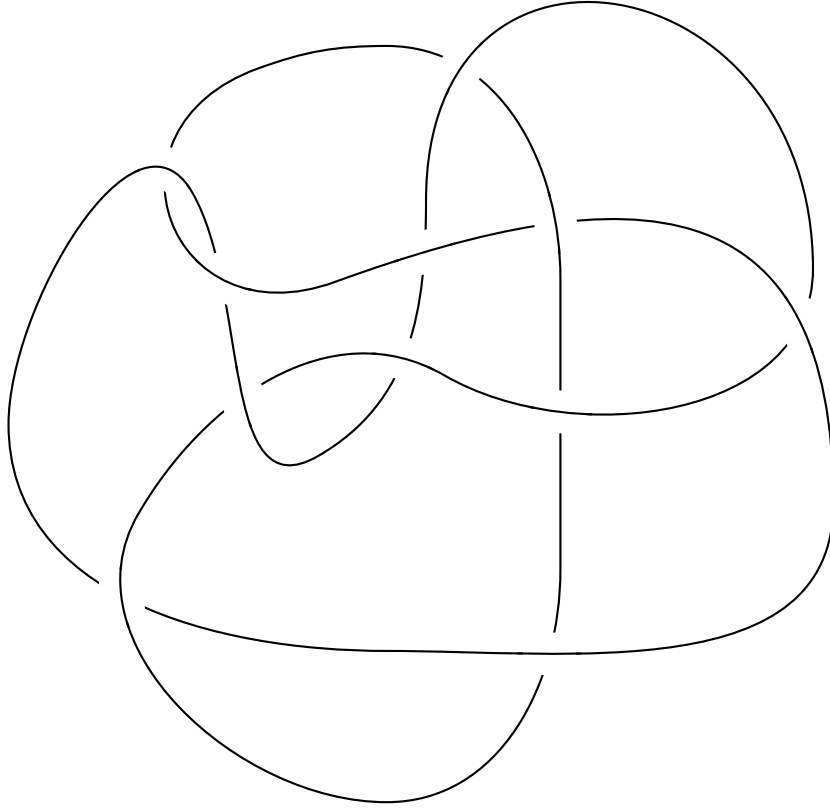


Figure 4: A diagram for knot $K11n164$.

Theorem 3.4.

The reduced normal form of color checking matrix does not depend on the choice of diagram D . Thus, it is well defined for $K\phi$ and is a knot invariant.

Proof. Take a knot K and its diagram D with n segments and n crossings. We will start by showing that applying any Reidemeister move to obtain a new diagram D' will not change the reduced normal form of its color checking matrix.

R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a** (the proof of **R1b** is left as an exercise for the reader).

Take D' to be diagram D with one arc twisted into a $+$ crossing. In

opposition to the assumption in previous section, we will take the arcs and crossings that differ between those two diagrams to be on first positions. Now, the matrices $D\phi$ and $D'\phi$ are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

Adding the first column of $D'\phi$ to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

because $a + b = 1$. Now we know that b is a unit, thus we can easily remove the elements of the first column that are not b . This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2 + y_2 & \dots & \\ 0 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like $D\phi$. The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Thus, the reduced normal form of $D'\phi$ is the same as in $D\phi$.

R2

Now the diagram D' is a diagram D with one arc poked onto another. Once again we will put those changed arcs at the beginning of the color

checking matrix to obtain following matrices:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3 + v_3 & \dots \\ x_4 & u_4 + v_4 & \\ \vdots & & \end{bmatrix}$$

Adding the third column of $D'\phi$ multiplied by α and β to first and second column respectively we are able to reduce the first row to only zeros and -1 . Now, adding this row to the second one creates a column with only -1 and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a + b\alpha & 0 & -1 & \\ 0 & x_3 & u_3 & v_3 & \\ 0 & x_4 & u_4 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that $a + b\alpha = 0$ and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 & \\ 0 & v_3 + u_3 & v_3 + u_3 & x_3 & \\ 0 & v_4 + u_4 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & v_3 + u_3 & x_3 & \\ 0 & 0 & v_4 + u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as $D\phi$.

R3

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as b and β were taken to be units.

Notice that if $A \sim B$ and $B \sim C$, where \sim means having the same reduced Smith normal form, then $A \sim C$. Thus, if two knots differ by a finite sequence of Reidemeister moves (as is the case for different diagrams of the same knot), then their reduced Smith normal forms are equal.

□

4 Category of palettes

We will work towards defining a category of palettes for a chosen knot K . This will allow us to change rules of colorings as we see fit.

Definition 4.1 : palette.

Let R be a commutative ring with unity, M a finitely generated R -module and $\mathcal{C} \subseteq M^3 \oplus M^3$ to be a coloring rule conforming to all rules outlined in the previous section. We say that a triplet (R, M, \mathcal{C}) is a **palette**.

Notice that if there is a ring homomorphism $f : R \rightarrow S$ then we can consider M as a S module by tensoring with S . This allows us to write a morphism between palettes

$$\bar{f} : (R, M, \mathcal{C}) \rightarrow (S, M_S, \mathcal{C}_S).$$

Similarly, if there is a module homomorphism $g : M \rightarrow M'$, then the induced morphism of palettes is

$$\bar{g} : (R, M, \mathcal{C}) \rightarrow (R, M', \mathcal{C}').$$

Definition 4.2 : category of palettes for knot K .

We define $\text{Col}(K)$ to be a **category of palettes** of K with

$$\text{Ob}(\text{Col}(K)) = \{(R, M, \mathcal{C})\}$$

being all palettes as described above and for any two palettes

$$\text{Hom}((R, M, \mathcal{C}), (S, N, \mathcal{K}))$$

is the set of induced morphisms \bar{g} between modules or \bar{f} between rings, when modules are M and $N = M_S$ respectively.

It is beneficial to highlight one palette in particular:

$$(\mathbb{Z}[\mathbb{Z}], \mathbb{Z}[\mathbb{Z}], \{(u, i, (1-t)u + ti) : u, i \in \mathbb{Z}[\mathbb{Z}]\}),$$

which will be referred to as the **Alexander palette**. We can derive this palette (which was used in example 3.2) from the Wirtinger presentation of a knot as follows.

Consider a crossing

$$\begin{array}{c}
\begin{array}{ccc}
& \uparrow & \\
O & \longleftarrow & \text{---} i \\
& \downarrow & \\
& u &
\end{array}
\end{array}$$

and take some x to be the generator that is used to generate a representation for K_G^{ab} . Then, the following is a relation in said group:

$$UxCx(Ux)^{-1} = Ix$$

where $U = ux^{-1}$, $I = ix^{-1}$ and $O = ox^{-1}$. We can multiply both sides by x^{-1} to obtain

$$x^{-1}UxCU^{-1} = x^{-1}Ix$$

which is change in $\mathbb{Z}[\mathbb{Z}]$ to

$$tU + C - U = tI \implies 0 = (1 - t)U + tI - C$$

The procedure for the other type of crossing is analogous.

The question that presents itself here is whether using row and column operations one can obtain representation matrix for the Alexander module from coloring matrices.

References

- [1] James W. Alexander. Topological invariants of knots and links. *Transactions of the American Mathematical Society*, 30:275–306, 1928.
- [2] M. F. Atiyah and I. G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Company, 1969.
- [3] W.B. Raymond Likorish. *An Introduction to Knot Theory*, pages 2–3. Springer, 1997.
- [4] Charles Livingston. *Knot Theory*. The Mathematical Assosiation of America, 1993.
- [5] John. W. Milnor. Infinite cyclic coverings. 1967.
- [6] Kunio Murasagi. *Knot theory and its applications*, pages 7–8. Springer, 1996.
- [7] Michael Polyak. Minimal generating set of reidemeister moves. *Quantum Topology vol 1*, 2010.
- [8] Ralph H. Fox Richard H. Crowell. *Introduction to Knot Theory*. Ginn and Company, 1963.
- [9] Julia Walczuk. private communication, III 2024.