# Knot colorings and homological invariants

(Kolorowania węzłów i niezmienniki homologiczne.)

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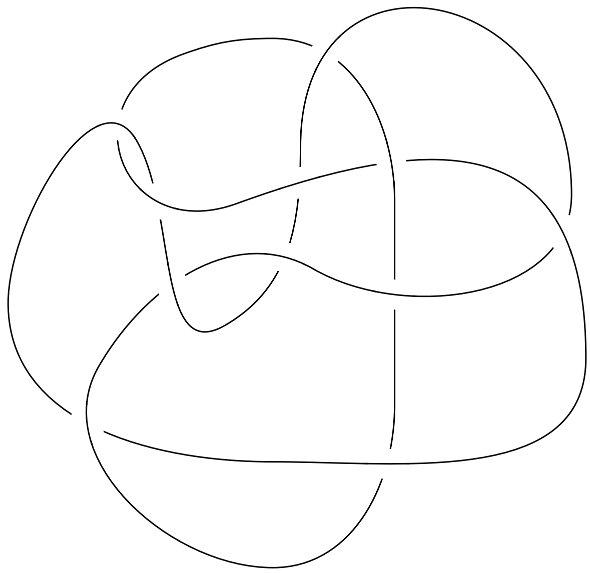


# Abstract

The knot group $G = \pi_1(K)$ is a starting point for many knot invariants. Alexander matrix is a representation matrix for a subgroup of $G$ and from its determinant, the Alexander polynomial is obtained. Another way of obtaining said polynomial is by considering a coloring matrix which assigns elements of $R$-module $M$ to segments from a diagram $D$ of knot $K$. This approach can be derived from the image of a resolution of Alexander module through the functor $\mathrm{Hom}(-, M)$. Nevertheless, color checking matrices do not instantly yield a knot invariant, however it is possible to define an equivalence relation that identifies matrices stemming from the same knot. This approach is used to distinguish a pair of knots with the same Alexander polynomial. In the end, a way of generalizing the procedure of coloring diagrams is presented in terms of category theory.

# Introduction

In knot theory distinguishing knots is often a difficult endeavor, usually facilitated by the notion of invariants. An interesting group of knot invariants are polynomial invariants, such as the Alexander polynomial. Another group worth mentioning are knot colorings that can also yield an element of the ring $\mathbb{Z}[\mathbb{Z}]$.

Very often, considering only one invariant is not sufficient, as there are many knots that share its value, i.e. $K11n85$ and $K11n164$ have the same Alexander polynomial. However, a more subtle application of the same method that yields the Alexander polynomial can sometimes distinguish such knots.

The following paper is a result of a year long cooperation between prof. Tadeusz Januszkiewicz, Julia Walczuk and the author of this thesis. In it, connections between the knot group, knot colorings and homology modules of infinite cyclic covering (see [5]) will be outlined. As an additional exercise, we will show a way of distinguishing already mentioned knots $K11n85$ and $K11n164$.

The first section of this paper defines the most important terms used in knot theory, as well as highlights the connection between the metabelianization of knot group and the first homology module of an infinite cyclic covering of the complement of said knot.

Subsequently, the construction of a $\mathbb{Z}[\mathbb{Z}]$ module from the kernel of $G^m \to \mathbb{Z}$ is presented. This module is defined to be the Alexander module $K_G^{ab}$ of knot $K$ (definition 1.4). Then, Alexander matrix is introduced (**??**) as the representation matrix of the Alexander module. In this section the resolution of Alexander module is proven to be of form

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

At the end of the second section, a connection between resolution of the Alexander module and coloring of diagrams is made.

The third and last section is focused on coloring matrices (definition 3.3) and defining an equivalence relation between matrices relating to the same knot. A new knot invariant is defined (**??**) and an example of its utility is presented in **??**.

Last section is dedicated to presenting an approach to diagram colorings from the perspective from category theory. In addition, a connection between the Alexander matrix and color checking matrix (using a particular palette, named the Alexander palette) is presented.

# Contents

# 1 Preliminaries

## 1.1 Knots and diagrams

In mathematical terms, a knot is a smooth embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \hookrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [3]. If two points are mapped to one by this projection, we say that a small neighbourhood of this point which looks locally like $^-|^-$ is a crossing.

$S^1$ is orientable, thus we can chose an orientation for any knot and, as a consequence, its diagram.

Intuitively, two tame knots $K_1$ and $K_2$ are equivalent if we can deform one into the other [6]. This translates to an equivalence of diagrams, which is generated by comparing diagrams that are exactly the same save for an interior of some disc in $R^2$. If inside of said disc the diagrams differ by one of **Reidemeister moves**, we say that they are equivalent. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on $D$, 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [7].

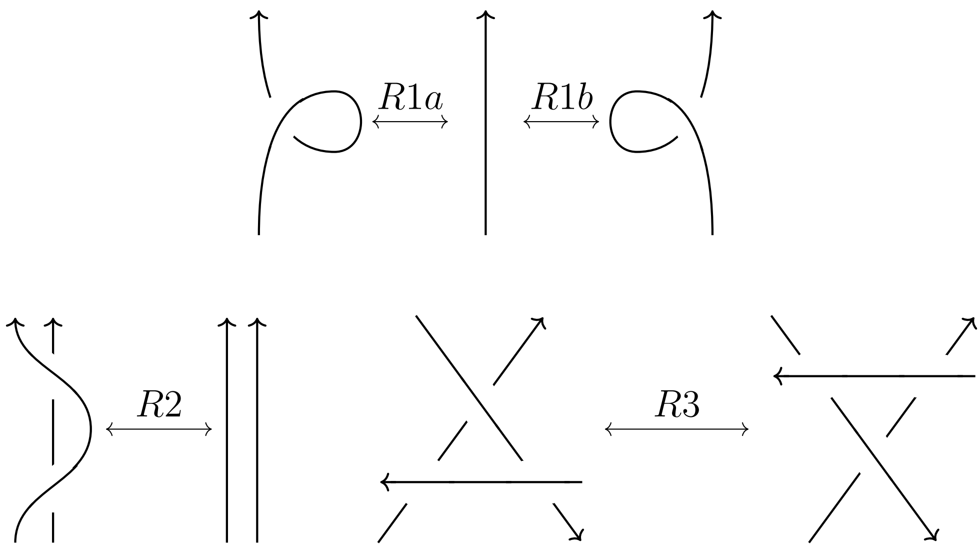



Figure 1: Generating set of Reidemeister moves in oriented diagrams.

## 1.2 Knot group

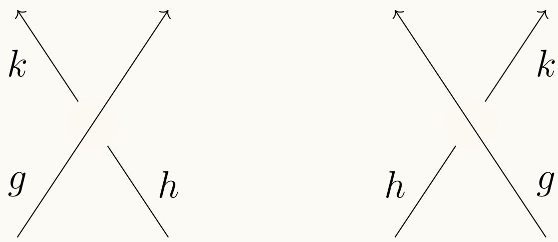
Let $K$ be a knot and $D$ be its oriented diagram with $s$ segments and $x$ crossings. A segment of a diagram is a line of the diagram between two crossings in which it is disappears under another line.

**Definition 1.1 : knot group.**
The fundamental group of knot complement $X = S^3 - K$ is called a **knot group**:

$$\boldsymbol{\pi_1(K) := \pi_1(X).}$$

Although the knot itself is always a circle $S^1$, the knot group has usually an interesting yet difficult structure. The most commonly used presentation of the knot group is called **the Wirtinger presentation**.

**Definition 1.2 : Wirtinger presentation.**
Given a diagram $D$ of knot $K$ with segments $a_1$, $a_2$, ..., $a_s$ and crossings $c_1$, ..., $c_x$ the knot group $\pi_1(K)$ can be represented as $\pi_1(K) = \langle G \mid R \rangle$, where $G$ is the set of segments of $D$ and relations $R$ correspond to crossings in the manner described in the diagram below




$gkg^{-1} = h$ $\qquad$ $ghg^{-1} = k$

Representation $\langle G \mid R \rangle$ described above is called the **Wirtinger presentation** [4, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to $S^3 = K \oplus S^3 - K$ or noticing that every two generators are conjugate, is that the abelianization of the knot group is always $\mathbb{Z}$. This leads to a short exact sequence

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0.$$

The group $K_G = \ker(ab : G \to \mathbb{Z}) = [G, G]$ in general is not abelian nor finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization $K_G^{ab} = K_G/[K_G, K_G]$ allows a $\mathbb{Z}[\mathbb{Z}]$ module structure and thus contains obtainable information about the knot $K$.

**Lemma 1.1.**
For any group $G$, the commutator of its commutator $K_G$ is a normal subgroup: $[K_G, K_G] = [[G,G],[G,G] \triangleleft G$.

***Proof.*** The commutator subgroup is a characteristic subgroup, since for any automorphism $\phi : G \to G$

$$\phi(hgh^{-1}g^{-1}) = \phi(h)\phi(g)\phi(h)^{-1}\phi(g)^{-1} \in K_G = [G,G].$$

Conjugation by any element $g \in G$ is an automorphism of the commutator $K_G$. Thus it preserves its commutator subgroup $[K_G, K_G]$. $\square$

As a consequence, in the group $G/[K_G, K_G]$ left and right multiplication is the same. Thus, the following is an exact sequence:

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives foundation for the following definition.

**Definition 1.3 : metabelianization.**
The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of $G$.

We will return to the concept of metabelianization in section 2. For the time being, let us assign a name to $K_G$:

**Definition 1.4 : Alexander module.**
Given a group $G$, the abelianization of the commutator of a group $G$, $K_G^{ab}$, with $\mathbb{Z}[\mathbb{Z}]$-module structure is called the **Alexander module** of $G$. If $G$ is a knot group, then it is the Alexander module of the knot $K$.

How the $\mathbb{Z}[\mathbb{Z}]$ module structure is obtained is described in detail in section 2.1.

## 1.3 Infinite cyclic covering

Let $X$ be the complement of a knot $K$ ($X = S^3 - K$). Take $\widetilde{X}$ to be its universal covering, meaning that it is simply connected. The fundamental group $G$ of $X$ acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G,G]$ is normal in $G$ and so $\pi_1(X)/K_G$ acts on $\widetilde{X}$. Thus we might take the quotient space $\overline{X} = \widetilde{X}/[G,G]$ and call it the **infinite cyclic covering** of $X$. Due to this construction, the fundamental group of $\overline{X}$ is exactly

$$\pi_1(\overline{X}) = [G, G] = K_G$$

and from the perspective of homology modules, we have

$$H_1(\overline{X}, \mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

Working with homology modules of an infinite cyclic cover of $X$ instead of $K_G^{ab}$ directly is beneficial when proving some properties of $K_G^{ab}$, i.e. that it is a torsion module in proposition 1.2.

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{array}{rcc} \widetilde{X} & \circlearrowleft & G \\ \downarrow & & \\ \overline{X} & \circlearrowleft & G/[G,G] \\ \downarrow & & \\ X = S^3 - K & & \end{array}$$

A **Seifert surface** $S$ of knot $K$ is an orientable surface with boundary embedded in $S^3$ such that $\partial S = K$. Take a countable amount of $X$, with $S$ without its boundary embedded, and label each with an element from $\mathbb{Z}$. We might now cut each of the copies of $X$ along the Seifert surface of $K$ and identify the $+$ side of $S$ from the $i$-th copy of $X$ with the $-$ side of $S$ from the $(i+1)$-th copy of $X$. Notice that the arising space with a projection to one copy of $X$ is an infinite cyclic cover of $X$.

Imagine that each copy of $X$ inside of $\overline{X}$ is a box labeled with some integer $k$. The ring action of $\mathbb{Z}[\mathbb{Z}]$ on $\overline{X}$ is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to $\overline{X}$.

**Proposition 1.2.**
The $\mathbb{Z}[\mathbb{Z}]$-module $K^{ab} = H_1(\overline{X}, \mathbb{Z})$ is a torsion module.

***Proof.*** Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \to C_*(\overline{X})$$

$$f(x) = (1-t)x.$$

It translates to removing from a cycle in the $(i+1)$-th box a corresponding cycle in the $i$-th box. From this it is an immediate result that $\ker f = 0$ and that $\operatorname{coker} f = C_*(X)$ : after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces a long exact homology sequence

$$\begin{gathered} \dots \longrightarrow H_2(X,\mathbb{Z}) \longrightarrow H_1(\overline{X},\mathbb{Z}) \xrightarrow{1-t} H_1(\overline{X},\mathbb{Z}) \longrightarrow H_1(X,\mathbb{Z}) \longrightarrow \\ H_0(\overline{X},\mathbb{Z}) \longrightarrow H_0(\overline{X},\mathbb{Z}) \longrightarrow H_0(X,\mathbb{Z}) \longrightarrow 0. \end{gathered}$$

As was mentioned previously, the following equality holds:

$$H_1(X,\mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}\,.$$

Now, because $X$ is homology circle, then $H_2(X,\mathbb{Z}) = 0$ (one can easily check it for themselves using Alexander duality). Both $X$ and $\overline{X}$ are connected implying that

$$H_0(X,\mathbb{Z}) = H_0(\overline{X},\mathbb{Z}) = \mathbb{Z}\,.$$

$$\begin{gathered} \dots \longrightarrow 0 \longrightarrow H_1(\overline{X},\mathbb{Z}) \xrightarrow{1-t} H_1(\overline{X},\mathbb{Z}) \xrightarrow{0} \mathbb{Z} \longrightarrow \\ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \longrightarrow 0 \end{gathered}$$

Rewriting the sequence above we easily get that homomorphism $1-t$ is actually an isomorphism and $H_1(\overline{X},\mathbb{Z}) \cong (1-t)H_1(\overline{X},\mathbb{Z})$, which allows us to use the Nakayama's lemma [2, Proposition 2.6] to conclude that there exists $x \in \mathbb{Z}[\mathbb{Z}]$ such that

$$xH_1(\overline{X},\mathbb{Z}) = 0.$$

$\square$

# 2 Resolution of the Alexander module

## 2.1 Construction of Alexander module

Take $G = \langle G \mid R \rangle$ to be the Wirtinger presentation of $G$ obtained from oriented diagram $D$. Because $K$ is a knot and not a link, we know that the number of segments is equal to the number of crossings, thus we can take $n = s = x$ [9].

Let $a_1, ..., a_n$ be the generators of $G$ and $x_1, ..., x_n$ its relations. The homomorphism of abelianization of $G$ is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, ..., n$. In order to obtain a presentation of $K_G$, the kernel of abelianization, we need to change the set of generators of $G$ to

$$\{a_1, A_2 = a_2 a_1^{-1}, ..., A_n = a_n a_1^{-1}\}.$$

It is obvious that for every $i > 1$ $A_i \mapsto 0$ by abelianization of $G$. Thus $A_2, ..., A_n$ are some of the generators of $K_G$. However, for each $i = 2, ..., n$ and $k \in \mathbb{Z}$ the following is an element of $K_G$:

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the presentation of $K_G$ as an abelian group is infinite with (possibly redundant) generators

$$\{b_{i,k} \ : \ i = 2, ..., n, \ k \in \mathbb{Z}\}.$$

Changing generators of $G$ induced a change in relations. Suppose that the following relation was true in $G$

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of $a_1$ the following two relations can be obtained

$$a_1^{-1} A_k a_1 = (a_1^{-1} A_i a_1) A_j A_i^{-1}$$

$$a_1^{-3}A_k a_1^3 = (a_1^{-3}A_i a_1^3)(a_1^{-2}A_j a_1^2)(a_1^{-2}A_i^{-1}a_1^2).$$

Obviously in $G$ both of those relations are equivalent, however in $K_G$ they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x}b_{j,x-1}b_{i,x-1}^{-1}$$

to obtain infinitely many relations from $K_G$.

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

For any group $H$ with $H^{ab} = \mathbb{Z}$ we can write a homomorphism $\mathbb{Z} \to H$ such that composition $\mathbb{Z} \to H \to \mathbb{Z}$ is identity on $\mathbb{Z}$. Thus, this sequence splits and we can write

$$G^{mab} = K_G^{ab} \rtimes \mathbb{Z}\,.$$

Hence action of $\mathbb{Z}$ can be defined on the group $K_G^{ab}$, with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}.$$

This procedure allows $K_G^{ab}$ to be interpreted as a $\mathbb{Z}[\mathbb{Z}]$-module.

Moreover, the group $G^{mab}$ and $\mathbb{Z}[\mathbb{Z}]$ $K_G^{ab}$ can be used interchangeably as knowing the action of $\mathbb{Z}$ on $K_G^{ab}$ allows us to write the semidirect product of $\mathbb{Z}$ and $K_G^{ab}$.

## 2.2 Basic properties

Knowing the resolution of a module allows one to change said module into a matrix or even a sequence of matrices, each containing at least a portion of information about its structure.

**Definition 2.1 : Alexander matrix.**
The presentation matrix $A_D$ of $K_G^{ab}$ with Wirtinger presentation is called the **Alexander matrix** of the Alexander module $K_G^{ab}$.

We start writing the beginning $K_G^{ab}$ resolution as follows:

$$\ldots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0 \tag{1}$$

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its columns will give a square matrix whose determinant is nonzero. We will prove this statement promptly after consider the Alexander module as a vector space over the field of fractions of $R = \mathbb{Z}[\mathbb{Z}]$ [2, Chapter 3].

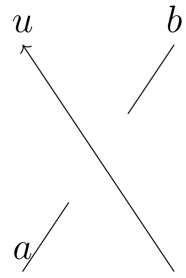
In proposition 1.2 it was shown that the Alexander module is torsion. Thus, as a vector space $K_G^{ab} \otimes_R R^{-1}R = 0$ it is trivial. Hence, the sequence in (1) translates to the following sequence of $R^{-1}R$ modules

$$\ldots \longrightarrow R^n \otimes_R R^{-1}R \xrightarrow{A_D^V} R^{n-1} \otimes_R R^{-1}R \longrightarrow 0 \tag{2}$$

As there exists an inclusion $R \hookrightarrow R^{-1}R$, every matrix with terms in $R$ can be treated as a matrix with terms in $R^{-1}R$. Naturally, $A_D^V = A_D \otimes Id_{R^{-1}R}$ is just matrix $A_D$ (with terms in $R$) with adjoined $1 \times 1$ matrix with just identity of $R^{-1}R$. Thus, we can easily translate most properties of $A_D^V$ to properties of $A_D$, i.e. its determinant and surjectivity.

**Proposition 2.1.**
Let $A'_D$ be the Alexander matrix $A_D$ with one of its rows struck out. Then $\det(A'_D) \neq 0$.

***Proof.*** We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing




is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator $x$ is send to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module $K^{ab}$ we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1-t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which $x$ is $b$ or $c$ are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t-1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and $\pm 1$. Thus, the sum is not equal to zero. There are two of such rows as the segment $b$ has to be the "out" and "in" segment of some crossing. In other words, segment $b$ has to have a start and end in some crossings.

The reasoning above is true for matrix $A_D^V$ from (2). We make the switch to vector space to use the connection between the rank of matrix and the dimension of its image.

Let $S_i$ be the column of the Alexander matrix corresponding to the segment labeled $i$. The sum $\sum_{i \leq n-1} S_i$ is a vector with two nonzero terms. Take $S_j$ and $S_k$ to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both $S_j$ and $S_k$ by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j,k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i : i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D^V(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. $\square$

The proposition 2.1 implies that image of $A_D^V$ has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

**Theorem 2.2.**
The determinant $\det(A'_D)$ up to multiplication by a unit is independent of the choice of the diagram $D$.

***Proof.*** A proof using Dehn presentation is provided in [1], while a proof of more general case is provided in [8] $\square$

**Definition 2.2 : Alexander polynomial.**
The **Alexander polynomial** of a knot $K$ is the determinant of any maximal minor of the Alexander matrix $A_D$.

The Alexander polynomial is a knot invariant as a consequence of theorem 2.2 and proposition 2.1

**Proposition 2.3.**
Let $G$ be a knot group of $K$ and $F = R^{-1}R$ the field of fraction of ring $R$. Then $K_G^{ab}$ always has a resolution

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

where $n$ is the number of crossings of the chosen diagram $D$ of knot $K$ and $M \otimes_R F \cong F$.

***Proof.*** We start by saying that $R \otimes_R F \cong F$ because $R$ is a free module over $R$ [2, Proposition 2.14].

Proposition 2.1 implies that (2) can be extended into the following exact sequence of vector spaces:

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R R^{-1}R = 0 \longrightarrow 0$$

as we proved that $\dim(\operatorname{im} A_D^V) = n-1 \implies \dim(\ker A_D^V) = 1$.

The ring of fractions is flat [2, Chapter 3] at the same time we only consider $R$-modules treated as vector spaces in this proposition. Thus, we have the following exact sequence

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

with $M \otimes_R F \cong F$. □

Notice, that sequence

$$\star:\ 0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow 0$$

is not acyclic, however it allows us to once again define the Alexander module, this time as $H_1(\star)$.

## 2.3 A homological roots of diagram colorings

Thus far a resolution of the Alexander module $K_G^{ab}$ provided a matrix and with it a polynomial invariant of knots. In this short section we will explain the connection between Alexander module and knot colorings, which will be the focus of the subsequent section.

Take $M$ to be a finitely generated $R = \mathbb{Z}[\mathbb{Z}]$-module. The functor $\mathrm{Hom}(-, M^n)$ is left exact therefore applied to the resolution of the Alexander module generates the following sequence

$$0 \longrightarrow \mathrm{Hom}(R, M) \longrightarrow \mathrm{Hom}(R^n, M) \xrightarrow{\mathrm{Hom}(A_D, M)} \mathrm{Hom}(R^{n-1}, M) \longrightarrow \mathrm{Hom}(K_G^{ab}, M^n)$$

The diagram $D$ taken as the starting point for the construction of $K_G^{ab}$ had $n = x$ crossings and $n = s$ segments. The module $K_G^{ab}$ was presented using $(n-1)$ generators, corresponding to all but one segments of the diagram. If we allow for propagation of values, then $\mathrm{Hom}(R^{n-1}, M)$ can be interpreted as assigning values from $M$ to $(n-1)$ segments in diagram $D$, with the last segment colored based on the remaining part of the diagram.

The arrow $\mathrm{Hom}(R^{n-1}, M) \to \mathrm{Hom}(K_G^{ab}, M)$ ensures that the structure of $K$ is taken into account during this assigment. Its kernel is be equal to $\mathrm{im}\,\mathrm{Hom}(A_D, M)$ and thus remembers which segments contributed to which crossings.

The above remark points at a similarity between the concept of diagram colorings, elaborated in the following section, and the more topological invariant which is the Alexander module

# 3 Knot colorings

## 3.1 Palettes and diagram colorings

For an oriented diagram $D$ of knot $K$ two types of crossings are recognized, pictured in fig. 2. A diagram coloring, in essence, is an assignment of values from some mathematical object (i.e. $R$-module) to segments of $D$.

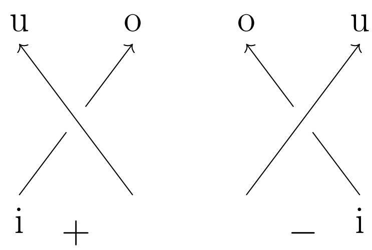



Figure 2: Two types of crossing in oriented diagram.

**Definition 3.1 : palette.**
We say that a triplet $(\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{\mathcal{C}_{\pm}})$ is a **palette** if $R$ is a commutative ring with unity, $M$ an $R$-module and $\mathcal{C}_{\pm}$ are two $R$-modules, corresponding to the two types of crossings (fig. 2) such that $\mathcal{C}_{\pm} \subseteq M^3$.

We will cumulatively call the two modules $\mathcal{C}_{\pm}$ the **coloring rule** of palette $(R, M, \mathcal{C}_{\pm})$ as they determine whether a coloring is admissible, a term that will be defined.

**Definition 3.2 : diagram coloring.**
A **coloring of diagram** $D$ with $s$ segments and $x$ crossings (for knots $s = x$ [9]) is any element $(m_1, ..., m_s) \in M^s$ that assigns elements of $M$ to each arc.

We will call a coloring **admissible** if for every crossing $x_j$ of type $\pm$ we have

$$\pi_{x_j}(m_1, ..., m_s) \in \mathcal{C}_{\pm} \subseteq \mathcal{C},$$

where $\pi_{x_j} : M^s \to M^3$ is a projection of the diagram to the segments that constitute $x_j$. TUTAJ NIE MOGĘ NAPISAĆ ŻE TO JEST RZUTOWANIE DIAGRAMU, TYLKO TRZEBA FORMALNIE

Usually, a crossing comprises of exactly three segments. However, as is the case for the first Reidemeister move, a crossing can be comprised of only two segments. Thus, if $x_j$ is such a two-segment crossing, $\mathrm{im}(\pi_{x_j})$

should be isomorphic to $\{(x, y, y) : x, y \in M\}$.

We can now define two linear homomorphisms

$$\phi_{\pm} : M^3 \to M^3/\mathcal{C}_{\pm} = N_{\pm}$$

that take in as arguments the arcs constituting a crossing. Assuming that $M^3/\mathcal{C}_{\pm} \cong M$ (reasoning behind this assumption will be given in section 3.2), we will take

$$\phi_{+}(u, i, o) = au + bi + co \tag{3}$$
$$\phi_{-}(u, i, o) = \alpha u + \beta i + \gamma o \tag{4}$$

for $u$, $i$, $o$ understood like in fig. 2 and with coefficients being homomorphisms $M \to M$.

**Lemma 3.1.**
A coloring $(m_1, ..., m_s) \in M^s$ is a admissible $\iff$ for each crossing $x_j$ of type $\pm$

$$\phi_{\pm}(\pi_{x_j}(m_1, ..., m_s)) = 0.$$

***Proof.*** Stems from the fact that $\mathcal{C}_{\pm} = \ker \phi_{\pm}$. □

## 3.2 Color checking matrix

We can think of coloring diagrams with a chosen palette $(R, M, \mathcal{C}_{\pm})$ as being a functor from a set of diagrams to a set of matrices.

**Definition 3.3 : color checking matrix.**
Assigning segments of diagram $D$ to coordinates in $M^s$ and crossings to coordinates in $N_{\pm}^x$ it is possible to define a linear homomorphism $D\phi : M^s \to N_{\pm}^x$ as

$$D\phi(m_1, ..., m_s) = (\phi_{\pm}(\pi_{x_1}(m_1, ..., m_s)), \phi_{\pm}(\pi_{x_2}(m_1, ..., m_s)), ...).$$

Matrix that is created after choosing a basis for $M^s$ and $N_{\pm}^x$ will be called a **color checking matrix**.

**Proposition 3.2.**
Coloring $(m_1, ..., m_s) \in M^s$ is admissible $\iff$ $(m_1, ..., m_s) \in \ker D\phi$.

***Proof.*** We start by saying that

$$(m_1, ..., m_s) \in \ker D\phi \iff [(\forall\ x_j \text{ crossing})\ \phi_{\pm}(\pi_{x_j}(m_1, ..., m_s)) = 0].$$

Which is to say that every coordinate of $D\phi(m_1, ..., m_s)$ is zero. Proposition lemma 3.1 says that it is equivalent with $(m_1, ..., m_s)$ being an admissible coloring. □

We want to define which color checking matrices are equivalent. We will say that $D\phi$ and $D'\phi$ are equivalent if

1. they differ by a permutation of rows or columns,
2. one can be obtained from the other by adding a linear combination of rows or columns to another row or columns
3. one can be obtained from the other by adding a new row and a new column with only 0 save for the term on their intersection, which is a unit.

The first two points mean that two color checking matrices $D\phi, D'\phi : M^s \to N^x$ are equivalent if there exist an isomorphisms $\theta : M^s \to M^s$ and $\psi : N^x \to N^x$ such that

$$\begin{tikzcd} M^s \arrow[r, "D\phi"] \arrow[d, "\theta"'] & N^x \arrow[d, "\psi"] \\ M^s \arrow[r, "D'\phi"'] & N^x \end{tikzcd}$$

is a commutative diagram.

In the most basic sense, two diagrams $D$ and $D'$ are isomorphic if there exists an isotopy $h_t : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h_0(D) = D$ and $h_1(D) = D'$ and $D'$ has crossings identical to those of $D$.

**Lemma 3.3.**
Isomorphic diagrams $D \sim D'$ yield isomorphic color checking matrices $D\phi \sim D'\phi$.

***Proof.*** In terms of color checking matrices, an isomorphism of diagrams defined above only relabels segments (permutes columns) and crossings (permutes rows). □

In reality, the isomorphism between diagrams which relates diagrams of the same knot is much more subtle. This relation $R$ is generated by the Reidemeister moves (see fig. 1). There is an induced relation on the set of color checking matrices, called $D(R)$, that relates color checking matrices of the same knot. For suitably chosen palettes, an invariant of the equivalent class of matrices of the same knot is easily obtainable. One of such palettes is the **Alexander palette** $(\boldsymbol{R} = \mathbb{Z}[\mathbb{Z}], \boldsymbol{M} = \mathbb{Z}[\mathbb{Z}], \boldsymbol{\mathcal{C}}_{\pm})$, where $\mathcal{C}_{\pm}$ are defined by homomorphisms

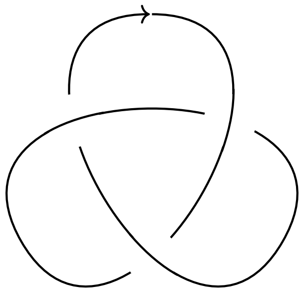
$$\phi_{+}(u, i, o) = (1 - t)u + ti - o$$

$$\phi_{-}(u, i, o) = (1 - t^{-1})u + t^{-1}i - o,$$

with $u$, $i$ and $o$ defined in fig. 2. For the Alexander palette said invariant is e.g. the determinant of a color checking matrix up to multiplication by a unit.

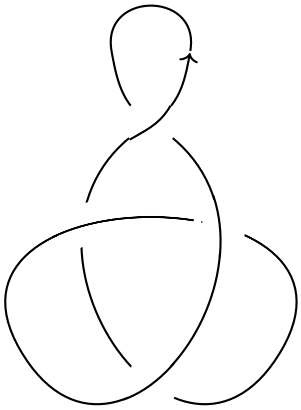
The following example illustrates the importance of choosing a suitable palette.

**Example 3.1.** Consider a coloring of trefoil knot $3_1$ with palette podmienic na bardziej brzydka $(\mathbb{Z}, \mathbb{Z}, \phi_{\pm}(u, i, o) = 2u - i + o)$. The color checking matrix of its diagram with 3 crossings is



$$\det \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = -3$$

while after the first Reidemeister move, the color checking matrix is



$$\det \begin{bmatrix} 0 & 2 & 1 & -1 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} = -8$$

In order to ensure that the equivalence classes of color checking matrices under the relation induced by Reidemeister moves on diagrams have a known invariant, rules must be imposed on palettes. A particular set of palettes that will satisfy conditions to be mention are the Alexander palette and all palettes derived from it by either a ring homomorphism or a module homomorphism.

1. To begin with, the coloring rule modules $\mathcal{C}_\pm$ are to be isomorphic to $M^2$, with the red arrow being an isomorphism

$$\begin{tikzcd} M^2 \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & M^3 \arrow[r, two heads] & \mathcal{C}_\pm. \end{tikzcd}$$

   We want the red isomorphism to be $(u, i) \mapsto (u, i, \phi'_\pm(u, i)) \in \mathcal{C}_\pm$, with segments labeled like in fig. 2, meaning that $c$ and $\gamma$ in (3) and (4) respectively are units. For the sake of convenience, take $c = \gamma = -1$.

   This property of palettes will be called a **propagation rule** as knowing colors of two of the three segments allows one to calculate the color assigned to the remaining segment.

2. The first Reidemeister move requires that

$$a = 1 - b$$

$$\alpha = 1 - \beta,$$

   where the variables are coefficients from (3) and (4). This will be explained in greater detail in the next section.

3. Similarly, the second Reidemeister move requires

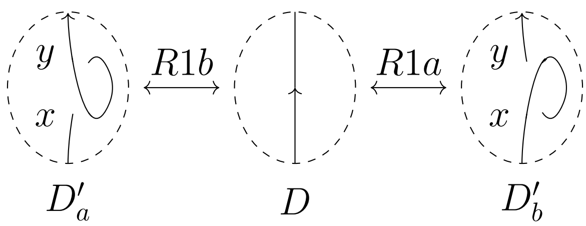
$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1. \end{cases}$$

The reasoning behind this restriction, once again, will be elaborated on in the next section.

## 3.3 Reidemeister relation on color checking matrices

The diagram $D$ has $s$ segments and $x$ crossings.

**R1**




The first Reidemeister move allows the following two moves on color checking matrices

$$\begin{array}{ccccc} D'_a & & D & & D'_b \\ \begin{bmatrix} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} & \overset{D(R1a)}{\sim} & \begin{bmatrix} x_1+y_1 & z_1 & \dots \\ \vdots & & \ddots \end{bmatrix} & \overset{D(R1b)}{\sim} & \begin{bmatrix} \beta & \alpha+\gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} \end{array}$$

where $(\forall\ i = 1, ..., x)\ x_i = 0\ \vee y_i = 0$.

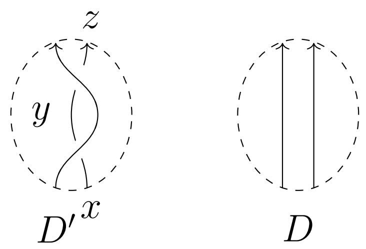
Notice, that if the propagation rule that was outlined at the beginning of this section is to be true, looking closely at the crossing in diagrams above should yield equations

$$0 = a + b + c = a + b - 1 \implies a = 1 - b$$

$$0 = \alpha + \beta + \gamma = \alpha + \beta - 1 \implies \alpha = 1 - \beta,$$

as the up and out segments in $D'_a$ and the up and in segments in $D'_b$ must admit coloring with the same element from $M$.

## R2




For the second Reidemeister move we will say that $D\phi$ and $D'\phi$ are in relation if they differ by the following matrix move

$$\begin{array}{c} D' \\ \begin{bmatrix} b & c & 0 & a & \dots \\ 0 & \beta & \gamma & \alpha & \\ x_1 & 0 & z_1 & w_1 & \\ \vdots & & & & \ddots \end{bmatrix}\end{array} \overset{D(R2)}{\sim} \begin{array}{c} D \\ \begin{bmatrix} x_1+z_1 & w_1 & \dots \\ \vdots & & \ddots \end{bmatrix}\end{array}$$

where $(\forall\ i = 1, ..., x)\ x_i = 0\ \vee z_i = 0$, in addition to permuting rows and columns and adding linear combination of rows or columns to another row or column.

In the case of this Reidiemeister move, we would like to be able to color the diagram $D'$ exactly like the diagram $D$ save for the segments contributing to the two additional crossings. This means that segments labeled $z$ and $x$ on the diagram above must admit a coloring with the same element from $M$.

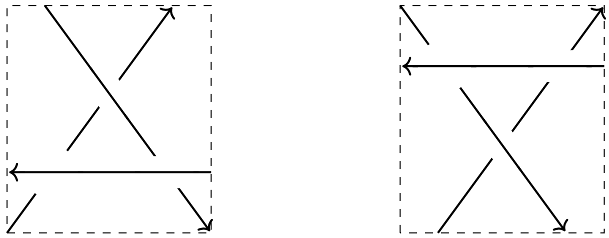
The restrictions stemming from this observation are more easily calculated if homomorphisms $\phi_+$ and $\phi_-$ are made into two matrices, $A_+$ and $A_-$, that the incoming segments (up and in segments) and return the output segments (out and up segments). This is possible because of the propagation rule.

$$A_+A_- \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} u \\ i \end{bmatrix}$$

Comparing the terms of the matrix $A_+A_-$ with terms of the identity matrix yields:

$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1 \end{cases}$$

**R3**



$D$ $D'$

The last Reidemeister move does not change the size of matrices but only permutes the terms appearing in columns and rows corresponding to the three crossing that are manipulated in the diagram.

$$\begin{array}{c} D' \\ \begin{bmatrix} \alpha & \gamma & \beta & 0 & 0 & 0 & \dots \\ 0 & 0 & c & b & 0 & a & \\ \beta & 0 & 0 & 0 & \gamma & \alpha & \\ u_1 & 0 & v_1 & w_1 & x_4 & y_4 & \\ \vdots & & & & & & \ddots \end{bmatrix} \end{array} \overset{D(R3)}{\sim} \begin{array}{c} D \\ \begin{bmatrix} 0 & 0 & \gamma & \beta & \alpha & 0 & \dots \\ \beta & 0 & 0 & 0 & \gamma & \alpha & \\ 0 & c & b & 0 & 0 & a & \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 & \\ \vdots & & & & & \ddots & \end{bmatrix} \end{array}$$

**Theorem 3.4.**
The equivalence class of a color checking matrix using the Alexander palette, or its derivative, of a diagram $D$ under relation $D(R)$ generated by matrix relations $D(R1a)$, $D(R1b)$, $D(R2)$ and $D(R3)$ is a knot invariant. Thus we can define $K\phi := [D\phi]$.

***Proof.*** A direct result of the definition of the equivalence relation. □

## 3.4 Smith normal form

The ring $R$ of palette $(R, M, \mathcal{C}_\pm)$ is not necessarily a PID ring, e.g. $\mathbb{Z}[\mathbb{Z}]$ ring of the Alexander palette has ideal $(2, t+1)$ which is not principal. However, usually one can find a PID ring $P$ with homomorphism $R \to P$ which creates a new palette $(P, M \otimes_R P, \mathcal{C}_\pm \otimes_R P)$ derived from $(R, M, \mathcal{C}_\pm)$. Matrices over PID rings have many interesting properties, like having a Smith normal form.

**Definition 3.4 : Smith normal form.**
Take $A \in K\phi$ and consider it as an $n \times n$ matrix with terms in a $P$

by the procedure outlined above. Then there exist a $n \times n$ matrix $S$ and $n \times n$ matrix $T$ such that $SAT$ is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every $i$ $a_i | a_{i+1}$. Such a matrix $SAT$ is called the **Smith normal form** of matrix $A$.

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