A voyage into the algebras

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1 Problem

Consider the ring $\mathbb{Z}[[F]$, where $[F]$ is the equivalence class of all finite abelian groups isomorphic to F. Describe the set $\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\}$, where relation $[F_2] = [F_1] + [F_3]$ means that there exists exact sequence:

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

Every finite abelian group is isomorphic to either a cyclic group or a finite product of cyclic groups. We will use this fact alongside the knowledge that every cyclic group is isomorphic with $\mathbb{Z}_n$ for some $n \in \mathbb{N}$.

We will start by showing that if $n = k \cdot l$ then $[\mathbb{Z}_n] = [\mathbb{Z}_k] + [\mathbb{Z}_l]$. Consider the sequence

$$0 \longrightarrow \mathbb{Z}_k \xrightarrow{f} \mathbb{Z}_n \xrightarrow{g} \mathbb{Z}_l \longrightarrow 0$$

Define $f(1) = l \mod n$ and $g(1) = 1 \mod l$. We now need to check if $\ker g = \operatorname{im} f$. Take any $x \in \ker g$, then $x = l \cdot m \mod n$ for some $m \in \{0, 1, 2, ..., k-1\}$. Then for $m \mod l \in \mathbb{Z}_l$ we have $f(m \mod l) = ml \mod n$ and so $x \in \ker g \iff x \in \operatorname{im} f$. This shows that the sequence above is exact and $[\mathbb{Z}_n] = [\mathbb{Z}_k] + [\mathbb{Z}_l]$.

Furthermore, if $n = \prod_{i \leq m} k_i$, then

$$[\mathbb{Z}_n] = [\mathbb{Z}_{\prod_{i \leq m-1} k_i}] + [\mathbb{Z}_{k_m}] = ... = \sum_{i \leq m} [\mathbb{Z}_{k_i}]$$

and if $n = p^k$ then by applying the above equation we have $[\mathbb{Z}_n] = k[\mathbb{Z}_p]$.

Next, we observe that for any $n, k \in \mathbb{Z}$ $[\mathbb{Z}_n \oplus \mathbb{Z}_k] = [\mathbb{Z}_n] + [\mathbb{Z}_k]$. This is because

$$0 \longrightarrow \mathbb{Z}_n \xrightarrow{f} \mathbb{Z}_n \oplus \mathbb{Z}_k \xrightarrow{g} \mathbb{Z}_k \longrightarrow 0$$

with $f(x) = x \oplus 0$ and $g(x \oplus y) = y$ is an exact sequence. For any $x \oplus y \in \ker g$ we must have $y = 0$ while x is unrestricted thus $x \oplus y \in \operatorname{im} f$.

From the latter statement, the following equality is immediately obtained:

$$\Big[\bigoplus_{i \leq n} \mathbb{Z}_{k_i}\Big] = \Big[(\bigoplus_{i \leq n-1} \mathbb{Z}_{k_i}) \oplus \mathbb{Z}_{k_n}\Big] = \Big[\bigoplus_{i \leq n-1} \mathbb{Z}_{k_i}\Big] + [\mathbb{Z}_{k_n}] = ... = \sum_{i \leq n} [\mathbb{Z}_{k_i}]$$

If F_n and F'_n are two abelian groups of order n and if $F_n = \bigoplus_{i \leq m} \mathbb{Z}_{k_i}$ then we have

$$[F_n] = \sum_{i \leq m} [\mathbb{Z}_{k_i}]$$

But because $n = |F_n| = |\oplus \mathbb{Z}_{k_i}| = \prod k_i$, then from the first two observations we obtain

$$[F'_n] = \sum_{i \leq m} [\mathbb{Z}_{k_i}]$$

and so $[F_n] = [F'_n]$. This allows us to replace every $\sum n_i [F_i] \in \mathbb{Z}[[F]$ with $\sum k_i [\mathbb{Z}_{p_i}]$ for prime p_i. Therefore,

$$\mathbb{Z}[[F]/\{[F_2] = [F_1] + [F_3]\} = \{\sum_{i \leq n} k_i [\mathbb{Z}_{p_i}] \ : \ p_i \text{ are prime}, n, k_i \in \mathbb{N}\}$$