Knot colorings and homological invariants

(Kolorowania węzłów i niezmienniki homologiczne.)

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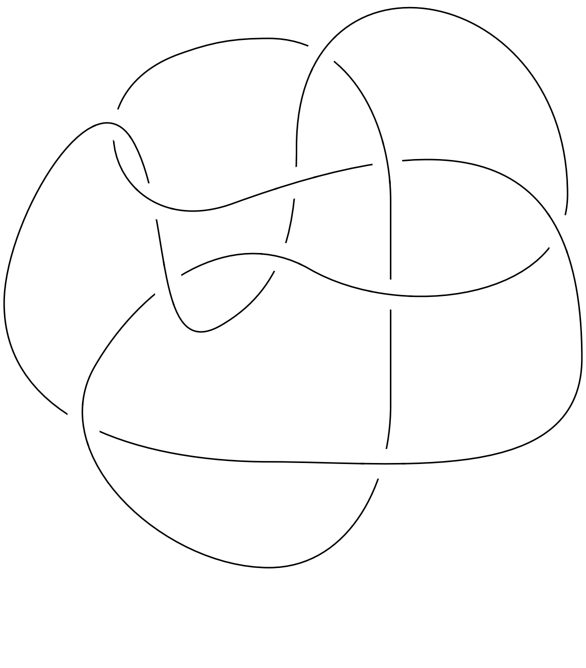


Abstract

The knot group $G = \pi_1(K)$ is a starting point for many knot invariants. Alexander matrix is a representation matrix for a subgroup of G and from its determinant, the Alexander polynomial is obtained. Another way of obtaining said polynomial is by considering a coloring matrix which assigns elements of R-module M to segments from a diagram D of knot K. This approach can be derived from the image of a resolution of Alexander module through the functor $\mathrm{Hom}(-, M)$. Nevertheless, color checking matrices do not instantly yield a knot invariant, however it is possible to define an equivalence relation that identifies matrices stemming from the same knot. This approach is used to distinguish a pair of knots with the same Alexander polynomial. In the end, a way of generalizing the procedure of coloring diagrams is presented in terms of category theory.

Introduction

In knot theory distinguishing knots is often a difficult endeavor, usually facilitated by the notion of invariants. An interesting group of knot invariants are polynomial invariants, such as the Alexander polynomial. Another group worth mentioning are knot colorings that can also yield an element of the ring $\mathbb{Z}[\mathbb{Z}]$.

Very often, considering only one invariant is not sufficient, as there are many knots that share its value, i.e. $K11n85$ and $K11n164$ have the same Alexander polynomial. However, a more subtle application of the same method that yields the Alexander polynomial can sometimes distinguish such knots.

The following paper is a result of a year long cooperation between prof. Tadeusz Januszkiewicz, Julia Walczuk and the author of this thesis. In it, connections between the knot group, knot colorings and homology modules of infinite cyclic covering (see [2]) will be outlined. As an additional exercise, we will show a way of distinguishing already mentioned knots $K11n85$ and $K11n164$.

The first section of this paper defines the most important terms used in knot theory, as well as highlights the connection between the metabelianization of knot group and the first homology module of an infinite cyclic covering of the complement of said knot.

Subsequently, the construction of a $\mathbb{Z}[\mathbb{Z}]$ module from the kernel of $G^m \to \mathbb{Z}$ is presented. This module is defined to be the Alexander module K_G^{ab} of knot K (definition 2.1). Then, Alexander matrix is introduced (definition 2.2) as the representation matrix of the Alexander module. In this section the resolution of Alexander module is proven to be of form

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

At the end of the second section, a connection between resolution of the Alexander module and coloring of diagrams is made.

The third and last section is focused on coloring matrices (definition 3.3) and defining an equivalence relation between matrices relating to the same knot. A new knot invariant is defined (definition 3.5) and an example of its utility is presented in example 3.2.

Last section is dedicated to presenting an approach to diagram colorings from the perspective from category theory. In addition, a connection between the Alexander matrix and color checking matrix (using a particular palette, named the Alexander palette) is presented.

Contents

1 Preliminaries

1.1 Knots and diagrams

In mathematical terms, a knot is a particular embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \twoheadrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [7].

S^1 is an orientable space thus we can choose an orientation for a knot being considered. Then a diagram D is oriented if it is a projection of an oriented S^1.

Intuitively, two knots K_1 and K_2 are equivalent if we can deform one into the other without cutting it and only manipulating it with our hands [3]. This translates to equivalence of diagrams, which is generated by a set of moves, called the **Reidemeister moves**. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D, 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [5].

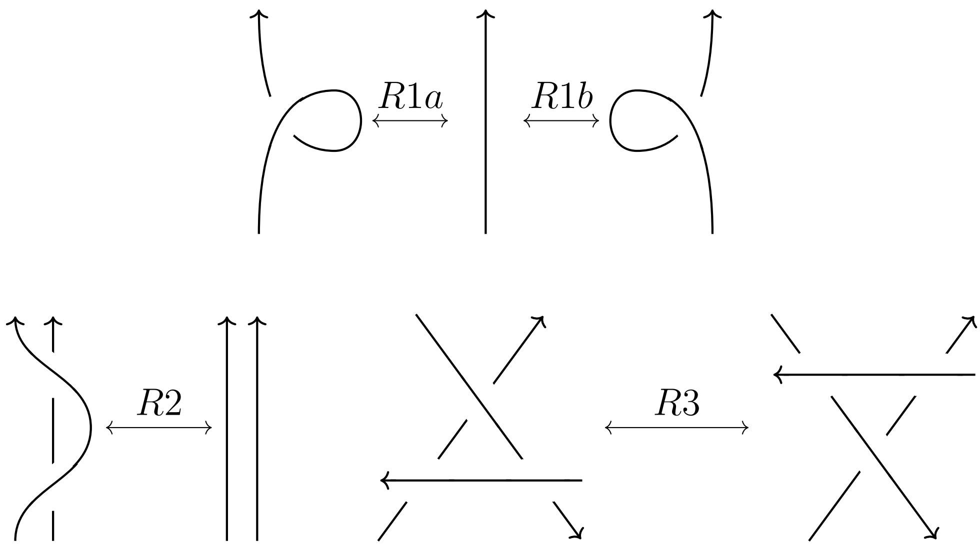


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

1.2 Knot group

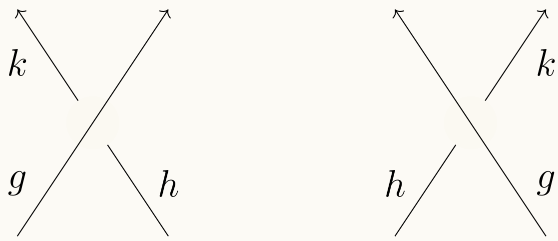
Let K be a knot and D be its oriented diagram with s segments and x crossings.

Definition 1.1 : knot group.
The fundamental group of a knot embedded in a three dimensional sphere S^3 is called a **knot group**.

$$\pi_1(\mathbf{K}) := \pi_1(\mathbf{S^3} - \mathbf{K}).$$

Although the knot itself is always a circle S^1, the knot group has usually an interesting yet difficult structure. The most known representation of the knot group is called **the Wirtinger presentation**.

Definition 1.2 : Wirtinger presentation.
Given a diagram D of knot K with segments a_1, a_2, ..., a_s and crossings c_1, ..., c_x the knot group $\pi_1(K)$ can be represented as $\pi_1(K) = \langle G \mid R \rangle$, where G is the set of segments of D and relations R correspond to crossings in the manner described in the diagram below



$gkg^{-1} = h$ $\qquad\qquad$ $ghg^{-1} = k$

Representation $\langle G \mid R \rangle$ described above is called the **Wirtinger presentation** [1, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to $S^3 = K \oplus S^3 - K$ or noticing that every two generators are conjugate, is that the abelianization of the knot group is always $\mathbb{Z}$. This leads to an acyclic complex

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0$$

The group $K_G = \ker(ab : G \to \mathbb{Z}) = [G, G]$ is not finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization $K_G^{ab} = K_G/[K_G, K_G]$ allows a $\mathbb{Z}[\mathbb{Z}]$ module structure and thus contains obtainable information about the knot K.

The following is an exact sequence:

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives foundation for the following definition.

Definition 1.3 : metabelianization.
The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of G.

We will return to the concept of metabelianization in section 2.

1.3 Infinite cyclic covering

Let X be the complement of a knot K, that is $X = S^3 - K$. Take $\widetilde{X}$ to be its universal covering, meaning that it is simply connected. The fundamental group G of X acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G, G]$ is normal in G and so the action of K_G on $\widetilde{X}$ is well defined. Thus we might take the quotient space $\overline{X} = \widetilde{X}/[G, G]$ and call it the **infinite cyclic covering** of X. The fundamental group of $\overline{X}$ is exactly

$$\pi_1(\overline{X}) = [G, G] = K_G$$

and from the perspective of homology modules, we have

$$H_1(\overline{X}, \mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{array}{ccc}
\widetilde{X} & \circlearrowleft & G \\
\downarrow & & \\
\overline{X} & \circlearrowleft & G/[G,G] \\
\downarrow & & \\
X = S^3 - K & &
\end{array}$$

A **Seifert surface** S of knot K is an orientable surface with boundary embedded in S^3 such that $\partial S = K$. Take a countable amount of X, with S without its boundary embedded, and label each with an element from $\mathbb{Z}$. We might now cut each of the copies of X along the Seifert surface of K and identify the $+$ side of S from the i-th copy of X with the $-$ side of S from the $(i+1)$-th copy of X. Notice that the arising space with a projection to one copy of X is an infinite cyclic cover of X.

Imagine that each copy of X inside of $\overline{X}$ is a box labeled with some integer k. The ring action of $\mathbb{Z}[\mathbb{Z}]$ on $\overline{X}$ is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to $\overline{X}$.

Proposition 1.1.
The $\mathbb{Z}[\mathbb{Z}]$-module $K^{ab} = H_1(\overline{X}, \mathbb{Z})$ is a torsion module.

Proof. Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \to C_*(\overline{X})$$

$$f(x) = (1-t)x.$$

It translates to removing from a cycle in the $(i+1)$-th box a corresponding cycle in the i-th box. From this it is an immediate result that $\ker f = 0$ and that $\operatorname{coker} f = C_*(X)$: after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces an acyclic complex of homology modules

$$\begin{tikzcd}
\ldots \arrow[r] & H_2(X,\mathbb{Z}) \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r] & H_1(X,\mathbb{Z}) \arrow[dlll, rounded corners] \\
& H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(X,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

As was mentioned previously, the following equality holds:

$$H_1(X, \mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}\,.$$

Now, because X is homotopy cycle, then $H_2(X, \mathbb{Z}) = 0$. Both X and $\overline{X}$ is connected implying that

$$H_0(X, \mathbb{Z}) = H_0(\overline{X}, \mathbb{Z}) = \mathbb{Z}\,.$$

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "0"] & \mathbb{Z} \arrow[dllll, rounded corners] \\
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Rewriting the sequence above we easily get that homomorphism $1-t$ is actually an isomorphism and $H_1(\overline{X},\mathbb{Z}) \cong (1-t)H_1(\overline{X},\mathbb{Z})$, which allows us to use the Nakayama's lemma to conclude that there exists $x \in \mathbb{Z}[\mathbb{Z}]$ such that

$$xH_1(\overline{X},\mathbb{Z}) = 0.$$

$\square$

2 Resolution of the Alexander module

2.1 Alexander module

Take $G = \langle G \mid R \rangle$ to be the Wirtinger presentation of G obtained from diagram D. Because K is a knot and not a link, we know that the number of segments is equal to the number of crossings, thus we can take $n = s = x$.

Let $a_1, ..., a_n$ be the generators of G and $x_1, ..., x_n$ its relations. The homomorphism of abelianization of G is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, ..., n$. In order to obtain a representation of K_G, the kernel of abelianization, we need to change the set of generators of G to

$$\{a_1, A_2 = a_2 a_1^{-1}, ..., A_n = a_n a_1^{-1}\}.$$

It is obvious that for every $i > 1$ $A_i \mapsto 0$ by abelianization of G.thus $A_2, ..., A_n$ are some of the generators of K_G. However, for each $i = 2, ..., n$ and $k \in \mathbb{Z}$ the following is an element of K_G:

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the representation of K_G is infinite with generators

$$\{b_{i,k} \ : \ i = 2, ..., n, \ k \in \mathbb{Z}\}.$$

Changing generators of G induced a change in relations. Suppose that the following relation was true in G

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of a_1 the following two relations can be obtained

$$a_1^{-1} A_k a_1 = (a_1^{-1} A_i a_1) A_j A_i^{-1}$$
$$a_1^{-3} A_k a_1^3 = (a_1^{-3} A_i a_1^3)(a_1^{-2} A_j a_1^2)(a_1^{-2} A_i^{-1} a_1^2).$$

Obviously in G both of those relations are equivalent, however in K_G they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

to obtain infinitely many relations from K_G.

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Hence action of $\mathbb{Z}$ can be defined on the group K_G^{ab}, with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}.$$

This procedure allows K_G^{ab} to be interpreted as a $\mathbb{Z}[\mathbb{Z}]$-module.

Definition 2.1 : Alexander module.
Given a group G, the abelianization of the commutator of a group G, K_G^{ab}, with $\mathbb{Z}[\mathbb{Z}]$-module structure is called the **Alexander module** of G. If G is a knot group, then it is the Alexander module of the knot K

Notice that if G^m is known, one can easily reconstruct K_G^{ab} knowing that it is the $\ker(G^m \to \mathbb{Z})$. Conversely, if K_G^{ab} is known, then G^m can be found as the middle term of the exact sequence $0 \longrightarrow K_G^{ab} \longrightarrow\ ?\ \longrightarrow \mathbb{Z} \longrightarrow 0$

2.2 Basic properties

The resolution of a module at first glance is in no way a simplification of said module. However, there are multiple ways of distilling simplifications and invariants from the resolution of the Alexander module.

We start writing the beginning K_G^{ab} resolution as follows:

$$\ldots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0 \qquad (1)$$

Definition 2.2 : Alexander matrix.
The matrix of homomorphism A_D in the diagram above is called the **Alexander matrix** of group G (knot K).

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its rows will give a square matrix whose determinant is nonzero. We will prove this statement promptly after consider the Alexander module as a vector space over the field of fractions of $R = \mathbb{Z}[\mathbb{Z}]$.

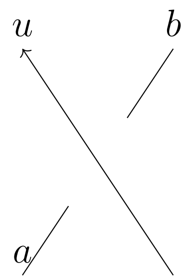
In proposition 1.1 it was shown that the Alexander module is torsion. Thus, as a vector field $K_G^{ab} \otimes_R R^{-1}R = 0$ it is trivial. Hence, the sequence in (1) translates to the following sequence of $R^{-1}R$ modules

$$\dots \longrightarrow R^n \otimes_R R^{-1}R \xrightarrow{A_D^V} R^{n-1} \otimes_R R^{-1}R \longrightarrow 0 \qquad (2)$$

Naturally, $A_D^V = A_D \otimes Id_{R^{-1}R}$ is just matrix A_D (with terms in R) with adjoined 1×1 matrix with just identity of $R^{-1}R$. Thus, we can easily translate most properties of A_D^V to properties of A_D, i.e. its determinant and surjectivity, by forgetting the $R^{-1}R$ factor.

Proposition 2.1.
Let A'_D be the Alexander matrix A_D with one of its rows struck out. Then $\det(A'_D) \neq 0$.

Proof. We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator x is send to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module K^{ab} we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1-t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which x is b or c are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t-1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and ± 1. Thus, the sum is not equal to zero. There are two of such rows as the segment b has to be the "out" and "in" segment of some crossing. In other words, segment b has to have a start and end in some crossings.

The reasoning above is true for matrix A_D^V from (2). We make the switch to vector fields to use the connection between the rank of matrix and the dimension of its image.

Let S_i be the column of the Alexander matrix corresponding to the segment labeled i. The sum $\sum_{i \leq n-1} S_i$ is a vector with two nonzero terms. Take S_j and S_k to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both S_j and S_k by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j,k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i \, : \, i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D^V(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. $\square$

The proposition 2.1 implies that image of A_D^V has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

Theorem 2.2.
The determinant $\det(A'_D)$ is independent of the choice of the diagram D

Proof. If D and D' are two diagrams of knot K, then they yield equivalent representations of $G=\pi_1(K)$. Thus, the chain of elementary ideals of A_D and $A_{D'}$ are the same according to Fox [6, Chapter VII] from which immediately follows that the determinants of the maximum minors of A_D and $A_{D'}$ are equal. $\square$

Definition 2.3 : Alexander polynomial.
The **Alexander polynomial** of a knot K is the determinant of any maximal minor of the Alexander matrix A_D.

The Alexander polynomial is a knot invariant as a consequence of theorem 2.2 and proposition 2.1

Proposition 2.3.
Let G be a knot group of K. Then it always has a resolution

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}]^1 \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

where n is the number of crossings of the chosen diagram D of knot K.

Proof. Proposition 2.1 implies that (2) can be extended into the following acyclic complex of vector spaces

$$0 \longrightarrow R\otimes_R R^{-1}R \longrightarrow R^n\otimes_R R^{-1}R \longrightarrow R^{n-1}\otimes_R R^{-1}R \longrightarrow K_G^{ab}\otimes_R R^{-1}R = 0 \longrightarrow 0$$

as we proved that $\dim(\operatorname{im} A_D^V) = n-1$.

The ring of fractions is flat [4, Chapter 3]. This combined with the ease of obtaining information about the sequence

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}]^1 \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

from (2) implies that the sequence above is exact and is a resolution of K_G^{ab}.

□

Notice, that sequence

$$\star : \ 0\,\mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow 0$$

is not acyclic, however it allows us to define the Alexander module as $H_1(\star)$.

2.3 A homological roots of diagram colorings

Thus far a resolution of the Alexander module K_G^{ab} provided a matrix and with it a polynomial invariant of knots. In this short section we will explain the connection between Alexander module and knot colorings, which will be the focus of the subsequent section.

Take M to be a finitely generated $R = \mathbb{Z}[\mathbb{Z}]$-module. The functor $\mathrm{Hom}(-, M^n)$ is left exact therefore applied to the resolution of the Alexander module generates the following sequence

$$0 \longrightarrow \mathrm{Hom}(R, M) \longrightarrow \mathrm{Hom}(R^n, M) \xrightarrow{\mathrm{Hom}(A_D, M)} \mathrm{Hom}(R^{n-1}, M) \longrightarrow \mathrm{Hom}(K_G^{ab}, M^n)$$

The diagram D taken as the starting point for the construction of K_G^{ab} had $n = x$ crossings and $n = s$ segments. The module K_G^{ab} was presented using $(n-1)$ generators, corresponding to all but one segments of the diagram. If we allow for propagation of values, then $\mathrm{Hom}(R^{n-1}, M)$ can be interpreted as assigning values from M to $(n-1)$ segments in diagram D, with the last segment colored based on the remaining part of the diagram.

The arrow $\mathrm{Hom}(R^{n-1}, M) \to \mathrm{Hom}(K_G^{ab}, M)$ ensures that the structure of K is taken into account during this assigment. Its kernel is be equal to $\mathrm{im}\,\mathrm{Hom}(A_D, M)$ and thus remembers which segments contributed to which crossings.

The above remark points at a similarity between the concept of diagram colorings, elaborated in the following section, and the more topological invariant which is the Alexander module

3 Knot colorings

3.1 Diagram colorings

Let K be a knot and D be its oriented diagram with s segments and x crossings. In such diagrams we can see two different crossing types as seen in fig. 2.

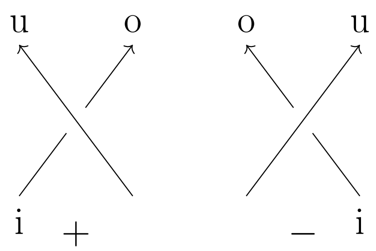


Figure 2: Two types of crossing in oriented diagram.

Take a commutative ring with unity R and an R-module M.

> **Definition 3.1 : coloring rule.**
> Take $\mathcal{C} \subseteq M^3 \oplus M^3$ to be a module such that there exists two modules $\mathcal{C}_\pm \subseteq M^3$ for which $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$. We will call $\mathcal{C}$ a **coloring rule**. The two submodules $\mathcal{C}_\pm$ each correspond to a type of crossing in diagram D.

We can now construct a pair of homomorphisms

$$\phi_\pm : M^3 \to M/\mathcal{C}_\pm = N_\pm,$$

cumulatively referred to as ϕ. We will call ϕ and $\mathcal{C}$ **coloring rule** interchangeably.

For each crossing x_j in diagram D we can construct a projection

$$\pi_{x_j} : M^s \twoheadrightarrow M^3$$

which restricts M^s to the three (or two, in which case one coordinate is zero) arcs that constitute x_j.

> **Definition 3.2 : diagram coloring.**
> A **coloring of diagram** D is any element $(m_1, ..., m_s) \in M^s$ that assigns elements of M to each arc. We will call this coloring **admissible**

if for every crossing x_j of type $\pm$ we have

$$\pi_{x_j}(m_1, ..., m_s) \in \mathcal{C}_\pm \subseteq \mathcal{C}.$$

It will be beneficial to express admissibility of a coloring in terms of homomorphisms ϕ.

Proposition 3.1.
A coloring $(m_1, ..., m_s) \in M^s$ is a admissible $\iff$ for each crossing x_j of type $\pm$

$$\phi_\pm(\pi_{x_j}(m_1, ..., m_s)) = 0.$$

Proof. Stems from the fact that $\mathcal{C}_\pm = \ker \phi_\pm$. □

3.2 Color checking matrix

Definition 3.3 : color checking matrix.
After assignings arcs to coordinates in M^s and crossings to coordinates in N^x it is possible to define a linear homomorphism $D\phi : M^s \to N^x$ as

$$D\phi(m_1, ..., m_s) = (\phi_\pm(\pi_{x_1}(m_1, ..., m_s)), \phi_\pm(\pi_{x_2}(m_1, ..., m_s)), ...).$$

Matrix that is created after choosing a basis for M^s and N^x will be called a **color checking matrix**.

Taking $\phi_\pm$ to be linear equations of form

$$\phi_+(u, i, o) = au + bi + co \tag{3}$$
$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o, \tag{4}$$

where u, i and o correspond to arcs as seen in fig. 2 and all the coefficients are linear homomorphisms $M \to N$, we know that all the entries for the color checking matrix will be linear combinations of a, b, c, α, β, γ. If M has n generators we chose to block the matrix $D\phi$ into $n \times n$ blocks.

Proposition 3.2.
Coloring $(m_1, ..., m_s) \in M^s$ is admissible $\iff$ $(m_1, ..., m_s) \in \ker D\phi$.

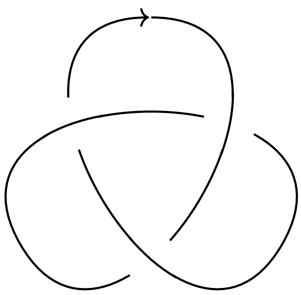
Proof. We start by saying that

$$(m_1, ..., m_s) \in \ker D\phi \iff [(\forall\ x_j \text{ crossing})\ \phi_\pm(\pi_{x_j}(m_1, ..., m_s)) = 0].$$

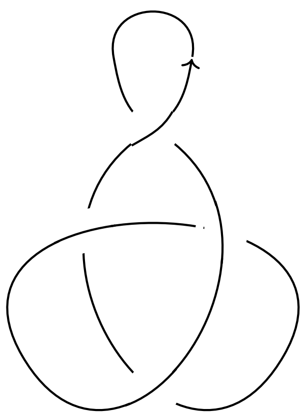
Which is to say that every coordinate of $D\phi(m_1, ..., m_s)$ is zero. Proposition proposition 3.1 says that it is equivalent with $(m_1, ..., m_s)$ being an admissible coloring. $\square$

The reasoning presented in section 2.3 points at determinant of the coloring matrix being an invariant as was the case for the Alexander matrix. However, at the very moment the color checking matrix is not a knot invariant nor is its determinant. Any module $\mathcal{C}$ and associated with it pair of homomorphisms ϕ does not necessarily yield a "nice" coloring. The following example justifies the necessity of imposing restrictions to which a coloring rule must conform in order to be considered in the latter part of this paper.

Example 3.1. Consider a coloring of trefoil knot 3_1 with $\mathbb{Z}$ over the ring $\mathbb{Z}$ with $\phi_{\pm}(u, i, o) = 2u - i + o$. The color checking matrix of its diagram with 3 crossings is



$$\det \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = -3$$



$$\det \begin{bmatrix} 0 & 2 & 1 & -1 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} = -8$$

The most important condition that $\mathcal{C}_{\pm}$ must meet is to be two dimensional.

This will allow for propagation of coloring, meaning that knowing colors of two segments creating a crossing the third one can be calculated from $\phi_\pm$.

The following diagram

$$\begin{tikzcd} M^2 \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & M^3 \arrow[r, two heads] & \mathcal{C} \end{tikzcd}$$

must commute, with the red arrow being

$$(u, i) \mapsto (u, i, \phi'_\pm(u, i))$$

where $\phi'_\pm$ calculates the "out" segment in admissible coloring of each crossing (compare fig. 2). Using (3) and (4) we can take c and γ to be any invertible elements, i.e. $c = \gamma = -1$, to have

$$\phi'_+(u, i) = au + bi$$

$$\phi'_-(u, i) = \alpha u + \beta i$$

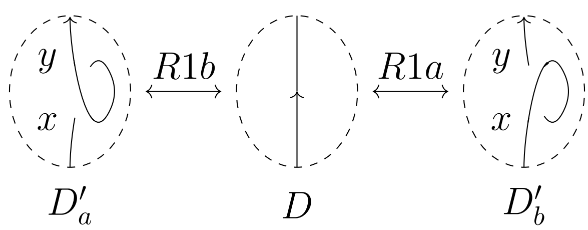
Notice that now a diagram with all but one segments colored can be easily colored in its entirety, using $\phi'_\pm$ on the crossing where the remaining segment starts.

3.3 Relation on color checking matrices

The color checking matrix, defined in definition 3.3, is not a knot invariant. Its size and structure changes as Reidemeister moves are applied to the diagram. Thus, we need to define which matrices stem from equivalent knot diagrams.

For the time being the diagram D has s segments and x crossings. Although only knots are considered in this paper, it is possible to expand definitions of color checking matrices and relations on them to links.

R1



Both Reidemeister moves $R1a$ and $R1b$ require the following diagram to commute,

$$\begin{tikzcd}
M^{s+1} \arrow[rr, "D'\phi"] \arrow[d, two heads] & & N^{x+1} \arrow[d, two heads] \\
M^{s+1},\ x=y \arrow[d, "f"'] & & N^x \oplus (N/\phi_\pm(M^3)) \arrow[d, "g"] \\
M^s \arrow[rr, "D\phi"'] & & N^x
\end{tikzcd}$$

where $\phi_\pm$ changes (for $R1a$ we have $+$ and for $R1b$ $-$). We take f and g to be given by

$$f(m_1,...,m_s,m_{s+1}) = (m_1,...,m_s+m_{s+1})$$

$$g(n_1,...,n_x,n_{x+1}) = (n_1,...,n_x+n_{x+1}).$$

The homomorphism f ensures that on the rest of diagrams D' arc labeled x in figure above and y add up to the arc visible in the diagram D. Meanwhile, g ensures that the additional crossing is treated with the appropriate coloring rule.

In terms of matrices, the above diagram can be translated to

$$\overset{D'_a}{\begin{bmatrix} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix}} \overset{R1a}{\sim} \overset{D}{\begin{bmatrix} x_1+y_1 & z_1 & \dots \\ \vdots & & \ddots \end{bmatrix}} \overset{R1b}{\sim} \overset{D'_b}{\begin{bmatrix} \beta & \alpha+\gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix}}$$

where $(\forall\ i=1,...,x)\ x_i = 0\ \vee y_i = 0$.

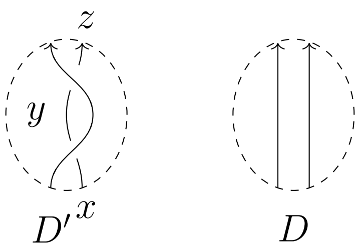
Notice, that if the propagation rule that was outlined at the beginning of this section is to be true, one must have

$$0 = a+b+c = a+b-1 \implies a = 1-b$$

$$0 = \alpha+\beta+\gamma = \alpha+\beta-1 \implies \alpha = 1-\beta,$$

as the two segments in both D' must admit coloring with one element from M. This puts further restrictions on coloring rules.

R2



For the second Reidemeister move we will say that $D\phi$ and $D'\phi$ are in relation if the following diagram commutes:

$$\begin{tikzcd}
M^{s+2} \arrow[rr, "D'\phi"] \arrow[d, two heads] & & N^{x+2} \arrow[d, two heads] \\
M^{s+2}, x = z \arrow[d] & & N^x \oplus (N/\phi_{\pm}(M^3)) \oplus (N/\phi_{\mp}(M^3)) \arrow[d] \\
M^s \arrow[rr, "D\phi"] & & N^x
\end{tikzcd}$$

In terms of matrices, the following move is admitted:

$$\begin{array}{cc}
D' & D \\
\begin{bmatrix} b & c & 0 & a & \dots \\ 0 & \beta & \gamma & \alpha & \\ x_1 & 0 & z_1 & w_1 & \\ \vdots & & & & \ddots \end{bmatrix} &
\begin{bmatrix} x_1 + z_1 & w_1 & \dots \\ \vdots & & \ddots \end{bmatrix}
\end{array}$$

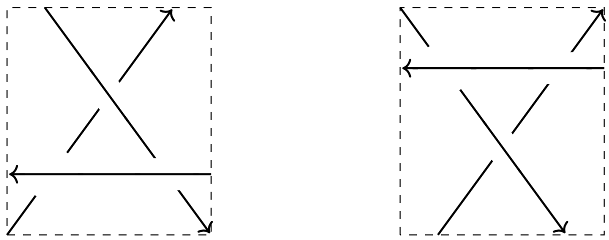
where $(\forall\ i = 1, ..., x)\ x_i = 0\ \vee z_i = 0$.

Once more, if the propagation property of coloring is to be upheld, then

$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1 \end{cases}$$

meaning that x and z can be colored with one element from M.

R3



D D'

In terms of matrices, the following move is admitted:

$$\begin{array}{c} D' \\ \begin{bmatrix} \alpha & \gamma & \beta & 0 & 0 & 0 & \dots \\ 0 & 0 & c & b & 0 & a & \\ \beta & 0 & 0 & 0 & \gamma & \alpha & \\ u_1 & 0 & v_1 & w_1 & x_4 & y_4 & \\ \vdots & & & & & & \ddots \end{bmatrix} \end{array} \quad \begin{array}{c} D \\ \begin{bmatrix} 0 & 0 & \gamma & \beta & \alpha & 0 & \dots \\ \beta & 0 & 0 & 0 & \gamma & \alpha & \\ 0 & c & b & 0 & 0 & a & \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 & \\ \vdots & & & & & \ddots & \end{bmatrix} \end{array}$$

Theorem 3.3.
The equivalence class of a color checking matrix of a diagram $D\phi$ under relation generated by matrix relations $R1a$, $R1b$, $R2$ and $R3$ is a knot diagram. Thus we can define $K\phi := [D\phi]$.

Proof. A direct result of the definition of the equivalence relation. □

3.4 Reduced Smith normal form

The ring R over which we consider modules M is not necessary a principal ideal domain. However, there are plenty of PID rings and more often than not, one can find at least one PID P with a homomorphism $R \to P$ that allows to consider M as a P-module by tensoring it with P:

$$M_P = M \otimes_R P.$$

We will use this idea to define a new type of equivalence relation on any color checking matrices.

Definition 3.4 : Smith normal form.
Take $A \in K\phi$ and consider it as an $n \times n$ matrix with terms in a P by the procedure outlined above. Then there exist a $n \times n$ matrix S

and $n \times n$ matrix T such that SAT is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every i $a_i | a_{i+1}$. Such a matrix SAT is called the **Smith normal form** of matrix A.

As was mentioned in the first section, $\overline{x} \in M^n$ is a coloring of a diagram D if and only if $D\phi(\overline{x}) = 0$, that is $\overline{x} \in \ker D\phi$. The Smith normal form hints at the structure of matrix kernel - the columns filled with zeros will contributed a "free" factor M to the kernel.

Take (a) to be a prime ideal with its generator a appearing in the Smith normal form of $D\phi$. Then we might consider the matrix over a new ring $P/(a)$, which is still a PID. After this change, the structure of the kernel has changed as now there are additional zero columns where a and all its multiples stood. Meaning that kernel became bigger and more colorings are admissible over $P/(a)$.

Definition 3.5 : reduced normal form of matrix.
Take A to be a matrix with coefficients in principal ideal domain P. Take $a_1, ..., a_k \in P$ to be all the elements of the Smith normal form of A that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix A.

When working with knots we usually take $R = \mathbb{Z}[t, t^{-1}]$ and $M = \mathbb{Z}[t, t^{-1}]$. This is not a PID ring but there are multitudes of PID rings into which R can be mapped. The following algorithm can be used to calculate the Smith normal form of a color checking matrix over a PID ring.

1. Let $A = \{a_{i,j}\}_{i,j \leq n}$ be an $n \times n$ matrix. Take the ideal $I = (a_{i,j})$ generated by all the terms of A.
2. If we are in PID then I has one generator, call it a.
3. We can now use the following row and column operations to put a in the upper left corner of A
   (a) Permuting rows (columns).
   (b) Adding a linear combination of rows (columns) to the remaining row (column).
4. With a in the upper left corner we can now use the fact that it was the generator of I to strike out the remaining terms on the first column and row, using the operations described in the previous point.
5. Repeat the same algorithm on the smaller matrix $\{a_{i,j}\}_{1<i,j\leq n}$.

The following example justifies the utility of the reduced normal form of color checking matrices in distinguishing knots.

Example 3.2. Consider the knots $K11n85$ and $K11n164$ pictured in figs. 3 and 4. They both have the Alexander polynomial equal

$$\Delta(t) = -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3}.$$

Coloring them over ring $\mathbb{Z}[\mathbb{Z}]$ with module $M = \mathbb{Z}[\mathbb{Z}]$ and coloring rules

$$\phi_+(u, i, o) = (1 - t)u + tb - o$$
$$\phi_-(u, i, o) = (1 - t^{-1})y + t^{-1}b - o$$

yields two 11×11 matrices whose any 10×10 minor is equal to the Alexander polynomial (up to multiplication by a unit). However, the reduced Smith normal form are

$$D_{11n85}\phi = \begin{bmatrix} -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3} \end{bmatrix}$$
$$D_{11n164}\phi = \begin{bmatrix} 1 - t + t^2 & 0 \\ 0 & -t^{-1} + 4 - 5t + 4t^2 - t^3 \end{bmatrix}$$

Having witnessed the utility of the reduced normal form of a coloring matrix we proceed to show that it is in fact a knot invariant.

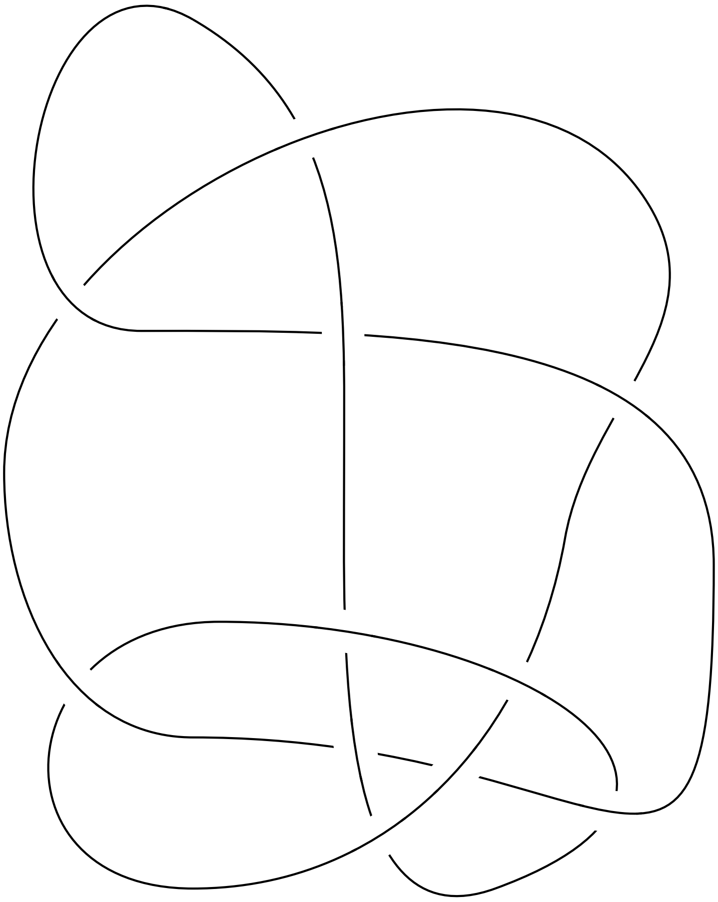


Figure 3: A diagram for knot $K11n85$.

Theorem 3.4.
The reduced normal form of color checking matrix does not depend on the choice of diagram D. Thus, it is well defined for $K\phi$ and is a knot invariant.

Proof. Take a knot K and its diagram D with n segments and n crossings. We will start by showing that applying any Reidemeister move to obtain a new diagram D' will not change the reduced normal form of its color checking matrix.

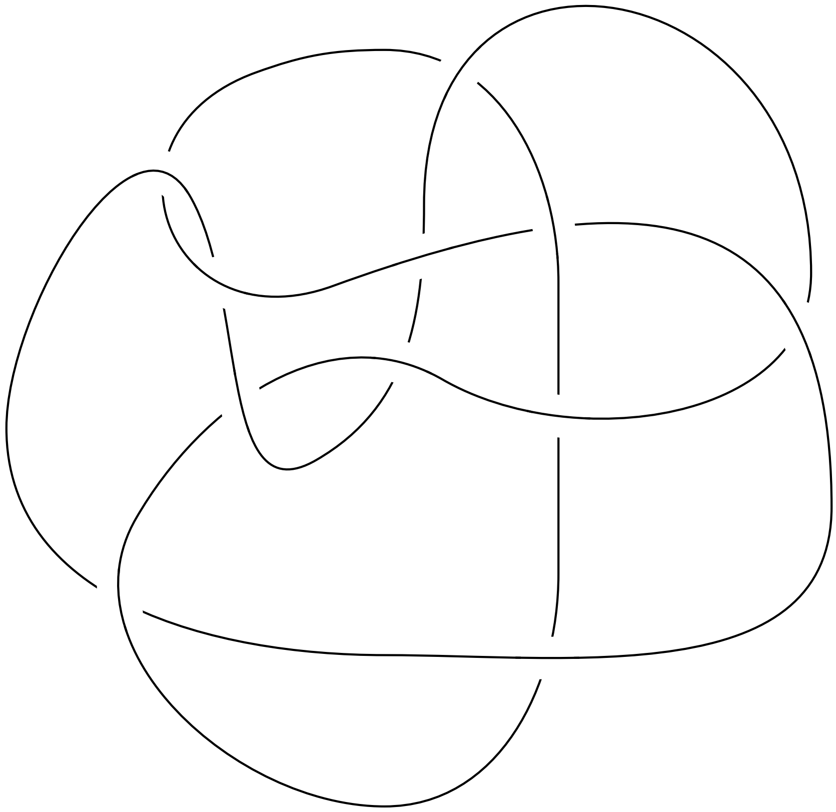


Figure 4: A diagram for knot $K11n164$.

R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a** (the proof of **R1b** is left as an exercise for the reader).

Take D' to be diagram D with one arc twisted into a $+$ crossing. In opposition to the assumption in previous section, we will take the arcs and crossings that differ between those two diagrams to be on first positions. Now, the matrices $D\phi$ and $D'\phi$ are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

Adding the first column of $D'\phi$ to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

because $a + b = 1$. Now we know that b is a unit, thus we can easily remove the elements of the first column that are not b. This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2 + y_2 & \dots & \\ 0 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like $D\phi$. The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Thus, the reduced normal form of $D'\phi$ is the same as in $D\phi$.

R2

Now the diagram D' is a diagram D with one arc poked onto another. Once again we will put those changed arcs at the beggining of the color checking matrix to obtain following matrices:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3 + v_3 & \dots \\ x_4 & u_4 + v_4 & \\ \vdots & & \end{bmatrix}$$

Adding the third column of $D'\phi$ multiplied by α and β to first and second column respectively we are able to reduce the first row to only zeros and

-1. Now, adding this row to the second one creates a column with only -1 and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a+b\alpha & 0 & -1 & \\ 0 & x_3 & u_3 & v_3 & \\ 0 & x_4 & u_4 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that $a + b\alpha = 0$ and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 & \\ 0 & v_3+u_3 & v_3+u_3 & x_3 & \\ 0 & v_4+u_4 & v_4+u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & v_3+u_3 & x_3 & \\ 0 & 0 & v_4+u_4 & x_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as $D\phi$.

R3

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as b and β were taken to be units.

Notice that if $A \sim B$ and $B \sim C$, where $\sim$ means having the same reduced Smith normal form, then $A \sim C$. Thus, if two knots differ by a finite sequence of Reidemeister moves (as is the case for different diagrams of the same knot), then their reduced Smith normal forms are equal.

$\square$

4 Category of palettes

We will work towards defining a category of palettes for a chosen knot K. This will allow us to change rules of colorings as we see fit.

Definition 4.1 : palette.
Let R be a commutative ring with unity, M a finitely generated R-module and $\mathcal{C} \subseteq M^3 \oplus M^3$ to be a coloring rule confomring to all rules outlined in the previous section. We say that a triplet $(R, M, \mathcal{C})$ is a **palette**.

Notice that if there is a ring homomorphism $f : R \to S$ then we can consider M as a S module by tensoring with S. This allows us to write a morphism between palettes

$$\overline{f} : (R, M, \mathcal{C}) \to (S, M_S, \mathcal{C}_S).$$

Similarly, if there is a module homomorphism $g : M \to M'$, then the induced morphism of palettes is

$$\overline{g} : (R, M, \mathcal{C}) \to (R, M', \mathcal{C}').$$

Definition 4.2 : category of palettes for knot K.
We define $\mathrm{Col}(K)$ to be a **category of palettes** of K with

$$\mathrm{Ob}(\mathrm{Col}(K)) = \{(R, M, \mathcal{C})\}$$

being all palettes as described above and for any two palettes

$$\mathrm{Hom}((R, M, \mathcal{C}), (S, N, \mathcal{K}))$$

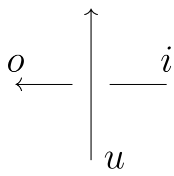
is the set of induced morphisms $\overline{g}$ between modules or $\overline{f}$ between rings, when modules are M and $N = M_S$ respectively.

It is beneficial to highlight one palette in particular:

$$(\mathbb{Z}[\mathbb{Z}], \mathbb{Z}[\mathbb{Z}], \{(u, i, (1-t)u + ti) \ : \ u, i \in \mathbb{Z}[\mathbb{Z}]\}),$$

which will be referred to as the **Alexander palette**. We can derive this palette (which was used in example 3.2) from the Wirtinger presentation of a knot as follows.

Consider a crossing



and take some x to be the generator that is used to generate a representation for K_G^{ab}. Then, the following is a relation in said group:

$$UxCx(Ux)^{-1} = Ix$$

where $U = ux^{-1}$, $I = ix^{-1}$ and $O = ox^{-1}$. We can multiply both sides by x^{-1} to obtain

$$x^{-1}UxCU^{-1} = x^{-1}Ix$$

which is change in $\mathbb{Z}[\mathbb{Z}]$ to

$$tU + C - U = tI \implies 0 = (1-t)U + tI - C$$

The procedure for the other type of crossing is analogous.

The question that presents itself here is whether using row and column operations one can obtain representation matrix for the Alexander module from coloring matrices.

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