A voyage into the algebras

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1 Introduction

1.1 Order of an Ideal over PID ring

PID -> every ideal is generated by one element, every module is an image of a free module, hence it can be expressed as $M \cong R/I_1 \oplus ... \oplus R/I_n$ for some ideals I_i. This allows as to define order of a module as $\text{ord}(M) = \text{ord}(I_1...I_n)$, which is the element that generates the ideal $I_1....I_n$.

$\text{ord}(M)$ can also be described using equivalence relation $M \sim M_1 + M_2 \iff 0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence -> finitely generated abelian groups as $\mathbb{Z}$ modules and vector fields over $\mathfrak{K}$ as $\mathfrak{K}[x]$-modules.

1.2 The Problem of non-PID rings

Not every ring is a PID -> we must either find another invariant or make the ring in question a PID. E.g. for $\mathbb{Z}[x, x^{-1}]$ we can tensor it with some field, usually $\mathbb{Q}$ but we might want to try F_p for some prime p.

Maybe some example for $\mathbb{Z}[x]$?

1.3 Short Introduction to Knot Theory?

Knot - a closed curve immersed in some 3-dimensional space, or S^1 immersed in S^3

We will consider only tamed knots? That is knots that can be represented as a sum of a finite amount of straight lines?

Using Mayer-Vietoris sequence we can deduce that $H^1(S^3 \setminus K) = \mathbb{Z}$ for any knot K. Hence, if we want to find interesting invariants, we must look further.

Seifert surface of knot K is an orientable surface whose boundary is K. We can use it to create an infinite cyclic covering of $S^3 \setminus K$ by cutting copies $S^3 \setminus K$ along this surface and gluing the $+$ side of Seifert surface of one copy to the $-$ side of the next copy.

$H^1(K^*)$ is more complicated than $H^1(S^3 \setminus K)$ and things get interesting if we consider it as a $\mathbb{Z}[\mathbb{Z}]$ (or $\mathbb{Z}[x, x^{-1}]$-module. We can use the fact that $\Pi_1(K^*)^{ab} = H^1(K^*)$ and calculate this module to obtain something called Alexander ideal I: $H^1(K^*) \cong \mathbb{Z}[\mathbb{Z}]/I$. If I is a principal ideal, e.g. in the case of trefoil knot of figure eight knot, its generator is called "Alexander polynomial". If this is not the case, we must consider $H^1(K^*; \mathbb{Q})$ - kohomology module with coefficients in $\mathbb{Q}$, to obtain the Alexander polynomial. In the following paper we will consider what happens if we use F_p, a finite field, instead of $\mathbb{Q}$.

The matrix method

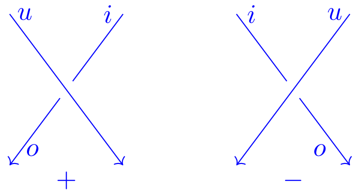
1.4 Fast notes

We might consider a module M over some ring R, usually $R = \mathbb{Z}[t, t^{-1}]$. Let K be a knot with l arches and s crossings that is oriented. We will consider a function $M^l \to M^s$ given by

$$+ : au + bi + co = 0$$

$$- : \alpha u + \beta i + \gamma o = 0,$$

where $+$ or $-$ depends on what arches u, i and o create:



The kernel of this morphism is responsible for coloring of knot K.

a, b, c (and greek) are morphisms $M \to M$ (or $M \to N$ in more general case). We can assume that c is a unit or even $c = 1 = \gamma$.

Furthermore, we can use equations above to obtain two operators $M \times M \to M \times M$ such that $(u, i) \mapsto (o, u)$ and $(i, u) \mapsto (u, o)$.

Two calculations on braids to do here, one that will give $a(a + b) = a$ and the other that states $ab = ba$!! what is the difference when $a + b = 1$ and when a is not assumed to be a unit (therefore only $a^2 + ab = a$)?

So now we can take a knot, its diagram and make it into a braid. A braid has a group (Burau representation, Markov knot theorem - moves) and we know that $\beta(w)v = v$ for the knot w and any vector v.

the braid group B_{n+1} with generators $\sigma_1, ..., \sigma_n$ can be send to S_{n+1} with relation $\sigma\eta = \eta\sigma$ for translations that are disjoint and $\sigma\eta\sigma = \eta\sigma\eta$ (i think) but we might want to do something different and add a relation that sends B_{n+1} to H_{n+1} or however this algebra was named, using $\sigma^2 + a\sigma + b = 0$.

Going back to the $M \times M$ stuff -> we can have a matrix $\begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}$ and we can assosiate it with translation σ_i from B_{n+1} and it acts on the braid. This gives us a coloring of the braid.

1.5 Coloring an unoriented knot diagram

Let R be a ring with identity and let M be an R-module. If we consider a diagram of a knot K without any orientation, the only type of crossing we will encounter is pictured in fig. 1

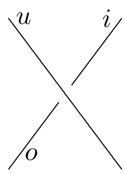


Figure 1: Crossing in an unoriented knot diagram.

Notice, that rotating it by 180 degrees changes i and o position (see fig. 2). Thus, segments passing under a crossing are indistinguishable.

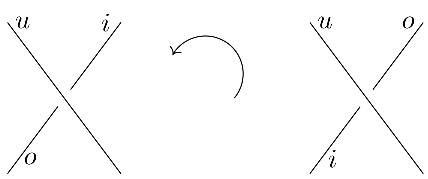


Figure 2: Segments going under a crossing in an unoriented knot diagram are indistinguishable.

When K has s segments and x crossings, we can write a labeling homomorphism

$$\phi : M^s \to M^x$$

which for segments that form a crossing pictured in fig. 1 takes value

$$\phi(u, i, o) = au + bi + co$$

for fixed $a, b, c \in \mathrm{End}(M)$. However, as we noted before, i and o are indistinguishable in fig. 1 and thus $b = c$, which yields a simpler definition:

$$\phi(u, i, o) = au + b(i + o).$$

tutaj trzeba sie dokladnie zastanowic jak to idzie bardzo formalnie w zapisie

$$\phi(u + i + o) = au + bi + co = 0$$

for $a, b, c \in \mathrm{End}(M)$ that are fixed for the entirety of K. However, because i and o are impossible to tell apart, we must take $b = c$ and thus arrive at a very simple equation:

$$au + b(i + o) = 0.$$

A coloring of a knot diagram without orientation is a labeling of its segments with elements from some module that agrees on crossings. That is, if a segment started in one crossing with label x then it must be labeled with x in every other crossing until another segment passes over it. Every diagram has a trivial coloring, in which every segment is labeled with the same element.

In other words, a coloring is an element from M^s that agrees with a and b on every crossing and thus it belongs to $\ker\phi$. For $(m_1, ..., m_s) \in \ker\phi$ we have a coloring such that segment i is labeled with m_i.

If we extend the morphism $M^s \to M^x$ to an exact sequence, we obtain

$$0 \to \ker\phi \to M^s \xrightarrow{\phi} M^x \to \operatorname{coker}\phi \to 0.$$

Module $\ker\phi$ can be viewed as a coloring of the diagram of K with elements of module M.

Example 1.1. *Let $M = \mathbb{Z}_n$, $R = \mathbb{Z}$, and consider the trefoil knot with 3 segments and 3 crossings.*

TO DO: function such that $2x - y - z = 0$ always when x is the upper strand, using Smith's normal form show that only $\mathbb{Z}_3$ can be used to make a non-trivial coloring

MAYHAPSE A DIFFERENT KNOT? like 4_1 - figure 8 knot?

2 The case of oriented knot diagrams

2.1 Labeling homomorphism with orientation

Every knot diagram can be endowed with orientation in two different ways. Given an oriented knot diagram there are two distinct types of crossings as seen in fig. 4. In such crossings segments i and o are easily distinguishable, allowing us to write two different elements to which labeling homomorphism ϕ will map segments that contribute to one crossing:

$$+ : \phi(u, i, o) = au + bi + co$$

$$- : \phi(u, i, o) = \alpha u + \beta i + \gamma o.$$

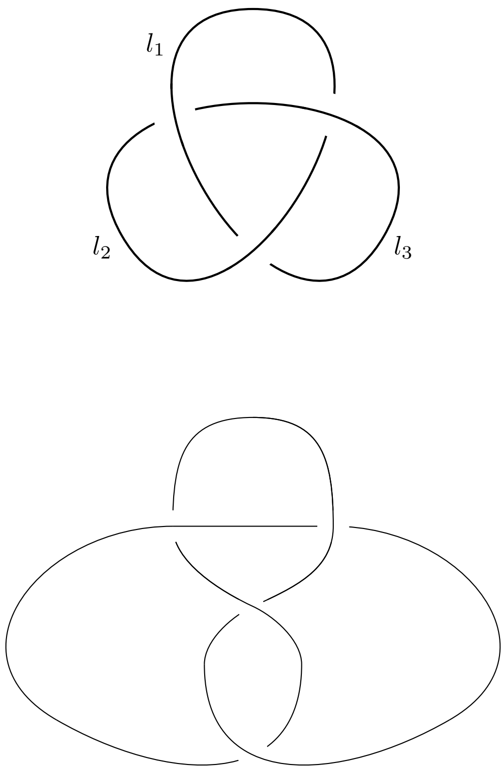


Figure 3: An alternating diagram of trefoil knot 3_1.

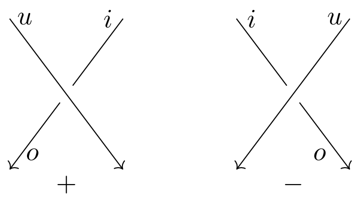


Figure 4: Two types of crossings in oriented knot diagram.

Just as before, elements from M^s that map to the trivial element in M^x are responsible for coloring of the diagram being examined.

Those two definitions of homomorphism can be used to define two operators $M \times M \to M \times M$ which take segments that enter a crossing and return segments that leave said crossing. Looking from top to bottom of fig. 4, said operators are represented by matrices:

Those two definitions of homomorphism can be used to define two operators $M \times M \to M \times M$ act on braids by taking strings that go into a crossing and returning those that leave the crossing. Looking from top to bottom of fig. 4, said operators are represented by matrices:

$$A_+ = \begin{pmatrix} -c^{-1}a & -c^{-1}b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ i \end{pmatrix} \qquad A_- = \begin{pmatrix} 0 & 1 \\ -\gamma^{-1}\beta & -\gamma^{-1}\alpha \end{pmatrix} \begin{pmatrix} i \\ u \end{pmatrix}$$

Considering the braid diagram in fig. 5 we can see that given two strings, if we first arrange them in crossing $+$ from fig. 4 and immediately after in crossing $-$ we can use Reidemeister's move to obtain identity on those two strands. Thus, we get equality

$$A_+A_- = \begin{pmatrix} -c^{-1}a & -c^{-1}b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\gamma^{-1}\beta & -\gamma^{-1}\alpha \end{pmatrix} =$$

$$= \begin{pmatrix} c^{-1}b\gamma^{-1}\beta & c^{-1}b\gamma^{-1}\alpha - c^{-1}a \\ 0 & 1 \end{pmatrix} = Id$$

assuming that $c = 1 = \gamma$ we get

$$b\beta = 1$$

$$b\alpha - a = 0$$

From the first equality we know that both b and β are units in ring $R = \mathbb{Z}[t, t^{-1}]$, therefore we can set $b = t = \beta$.

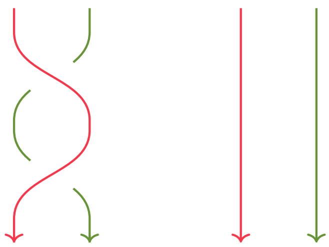


Figure 5: Braid diagram.

Using another Reidemeister move as seen in fig. 6 we get

$$(a^2 + ba + a)x = (ba - ab)y$$

and because x and y are independent strings, both brackets must equal 0. Thus we arrive at

$$ba = ab$$

$$a(a + b) = -a$$

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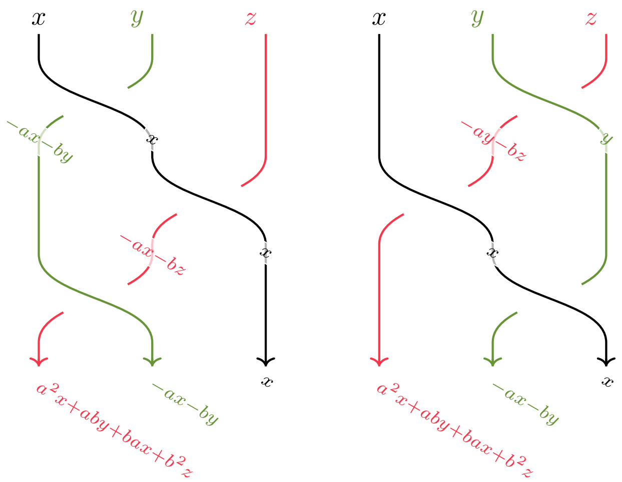

Figure 6: Braid diagram.

References