Fox knot colorings and Alexander invariants.

(Kolorowania Foxa i niezmienniki Alexandera)

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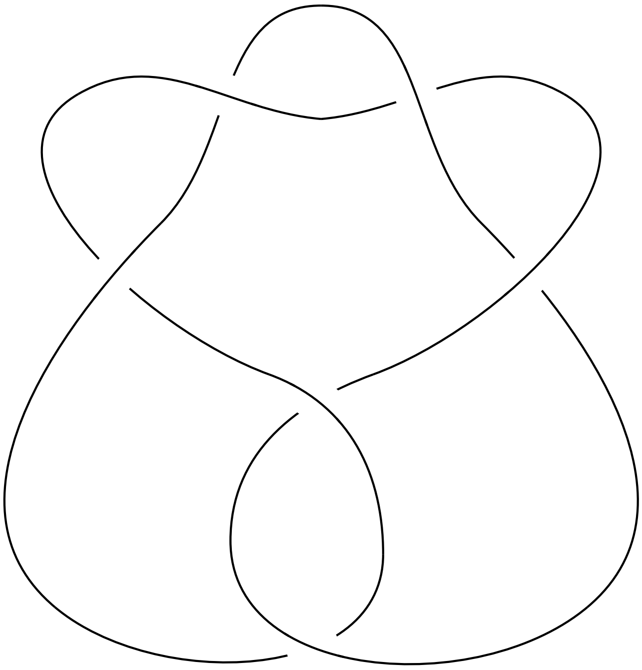

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1 Preliminaries

1.1 Knots and diagrams

In mathematical terms, a knot is a particular embedding $S^1 \hookrightarrow S^3$. A knot diagram is an immersive projection $D : S^1 \twoheadrightarrow \mathbb{R}^2$ along a vector such that no three points of the knot lay on this vector [6].

S^1 is an orientable space thus we can choose an orientation for a knot being considered. Then a diagram D is oriented if it is a projection of an oriented S^1.

Intuitively, two knots K_1 and K_2 are equivalent if we can deform one into the other without cutting it and only manipulating it with our hands [2]. This translates to equivalence of diagrams, which is generated by a set of moves, called the **Reidemeister moves**. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on D, 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [4].

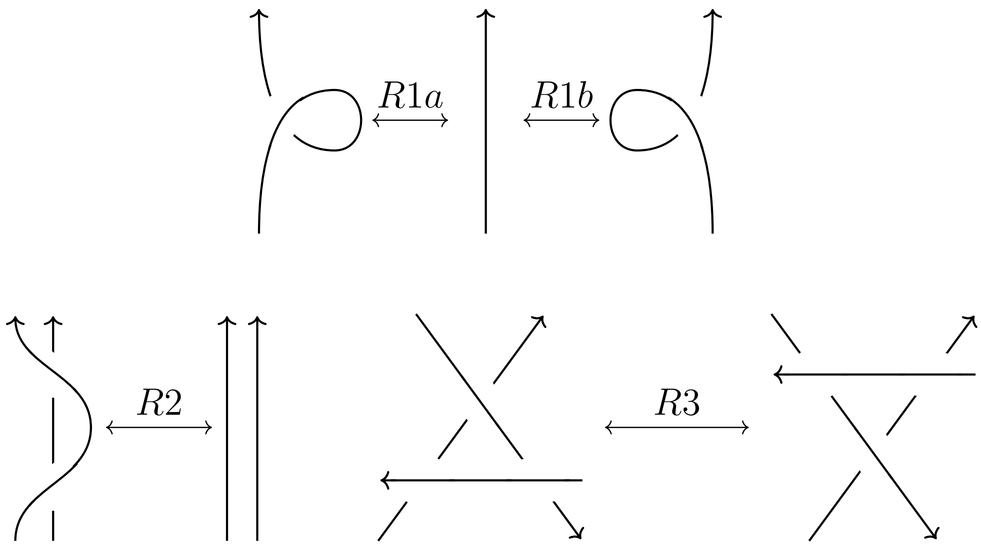


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

1.2 Knot group

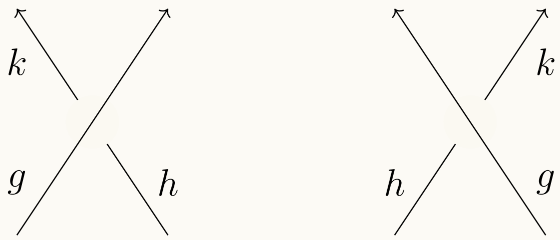
Let K be a knot and D be its oriented diagram with s segments and x crossings.

Definition 1.1 : knot group.
The fundamental group of a knot embedded in a three dimensional sphere S^3 is called a **knot group**.

$$\pi_1(\mathbf{K}) := \pi_1(\mathbf{S^3} - \mathbf{K}).$$

Although the knot itself is always a circle S^1, the knot group has usually an interesting yet difficult structure. The most known representation of the knot group is called **the Wirtinger presentation**.

Definition 1.2 : Wirtinger presentation.
Given a diagram D of knot K with segments a_1, a_2, ..., a_s and crossings c_1, ..., c_x the knot group $\pi_1(K)$ can be represented as $\pi_1(K) = \langle G \mid R \rangle$, where G is the set of segments of D and relations R correspond to crossings in the manner described in the diagram below



$gkg^{-1} = h$ $\qquad$ $ghg^{-1} = k$

Representation $\langle G \mid R \rangle$ described above is called the **Wirtinger presentation** [1, Chapter 6].

An easily obtainable result, either by applying the Mayer-Vietoris sequence to $S^3 = K \oplus S^3 - K$ or noticing that every two generators are conjugate, is that the abelianization of the knot group is always $\mathbb{Z}$. This leads to an acyclic complex

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0$$

The group $K_G = \ker(ab : G \to \mathbb{Z}) = [G, G]$ is not finitely generated, an observation that is discussed in section 2.1, and thus is a difficult group to work with. However, its abelianization $K_G^{ab} = K_G/[K_G, K_G]$ allows a $\mathbb{Z}[\mathbb{Z}]$ module structure and thus contains obtainable information about the knot K.

The following is an exact sequence:

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives foundation for the following definition.

> **Definition 1.3 : metabelianization.**
> The quotient group $G^{mab} = G/[K_G, K_G]$ is called the **metabelianization** of G.

We will return to the concept of metabelianization in section 2.

1.3 Infinite cyclic covering

Let X be the complement of a knot K, that is $X = S^3 - K$. Take $\widetilde{X}$ to be its universal covering, meaning that it is simply connected. The fundamental group G of X acts on its universal covering by deck transformations. The commutator subgroup $K_G = [G, G]$ is normal in G and so the action of K_G on $\widetilde{X}$ is well defined. Thus we might take the quotient space $\overline{X} = \widetilde{X}/[G, G]$ and call it the **infinite cyclic covering** of X. The fundamental group of $\overline{X}$ is exactly

$$\pi_1(\overline{X}) = [G, G] = K_G$$

and from the perspective of homology modules, we have

$$H_1(\overline{X}, \mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

The following diagram illustrates the construction of infinite cycle covering described above

$$\begin{tikzcd}
\widetilde{X} \arrow[d] & \circlearrowleft & G \\
\overline{X} \arrow[d] & \circlearrowleft & G/[G,G] \\
X = S^3 - K & &
\end{tikzcd}$$

A **Seifert surface** S of knot K is an orientable surface with boundary embedded in S^3 such that $\partial S = K$. Take a countable amount of X, with S without its boundary embedded, and label each with an element from $\mathbb{Z}$. We might now cut each of the copies of X along the Seifert surface of K and identify the $+$ side of S from the i-th copy of X with the $-$ side of S from the $(i+1)$-th copy of X. Notice that the arising space with a projection to one copy of X is an infinite cyclic cover of X.

Imagine that each copy of X inside of $\overline{X}$ is a box labeled with some integer k. The ring action of $\mathbb{Z}[\mathbb{Z}]$ on $\overline{X}$ is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of $t \in \mathbb{Z}[\mathbb{Z}]$ in the polynomial which we apply to $\overline{X}$.

Proposition 1.1.
The $\mathbb{Z}[\mathbb{Z}]$-module $K^{ab} = H_1(\overline{X}, \mathbb{Z})$ is a torsion module.

Proof. Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \to C_*(\overline{X})$$

$$f(x) = (1-t)x.$$

It translates to removing from a cycle in the $(i+1)$-th box a corresponding cycle in the i-th box. From this it is an immediate result that $\ker f = 0$ and that $\operatorname{coker} f = C_*(X)$: after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces an acyclic complex of homology modules

$$\begin{tikzcd}
\ldots \arrow[r] & H_2(X,\mathbb{Z}) \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r] & H_1(X,\mathbb{Z}) \arrow[dlll, rounded corners] \\
 & H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(\overline{X},\mathbb{Z}) \arrow[r] & H_0(X,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

As was mentioned previously, the following equality holds:

$$H_1(X, \mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}\,.$$

Now, because X is homotopy cycle, then $H_2(X, \mathbb{Z}) = 0$. Both X and $\overline{X}$ is connected implying that

$$H_0(X, \mathbb{Z}) = H_0(\overline{X}, \mathbb{Z}) = \mathbb{Z}\,.$$

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "1-t"] & H_1(\overline{X},\mathbb{Z}) \arrow[r, "0"] & \mathbb{Z} \arrow[dlll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Rewriting the sequence above we easily get that homomorphism $1-t$ is actually an isomorphism and $H_1(\overline{X},\mathbb{Z}) \cong (1-t)H_1(\overline{X},\mathbb{Z})$, which allows us to use the Nakayama's lemma to conclude that there exists $x \in \mathbb{Z}[\mathbb{Z}]$ such that

$$xH_1(\overline{X},\mathbb{Z}) = 0.$$

$\square$

2 Resolution of the Alexander module

2.1 Alexander module

Take $G = \langle G \mid R \rangle$ to be the Wirtinger presentation of G obtained from diagram D. Because K is a knot and not a link, we know that the number of segments is equal to the number of crossings, thus we can take $n = s = x$.

Let $a_1, ..., a_n$ be the generators of G and $x_1, ..., x_n$ its relations. The homomorphism of abelianization of G is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every $i = 1, ..., n$. In order to obtain a representation of K_G, the kernel of abelianization, we need to change the set of generators of G to

$$\{a_1, A_2 = a_2 a_1^{-1}, ..., A_n = a_n a_1^{-1}\}.$$

It is obvious that for every $i > 1$ $A_i \mapsto 0$ by abelianization of G.thus $A_2, ..., A_n$ are some of the generators of K_G. However, for each $i = 2, ..., n$ and $k \in \mathbb{Z}$ the following is an element of K_G:

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the representation of K_G is infinite with generators

$$\{b_{i,k} \ : \ i = 2, ..., n, \ k \in \mathbb{Z}\}.$$

Changing generators of G induced a change in relations. Suppose that the following relation was true in G

$$a_k = a_i a_j a_i^{-1}.$$

If $1 \notin \{i, k, j\}$ then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of a_1 the following two relations can be obtained

$$a_1^{-1} A_k a_1 = (a_1^{-1} A_i a_1) A_j A_i^{-1}$$
$$a_1^{-3} A_k a_1^3 = (a_1^{-3} A_i a_1^3)(a_1^{-2} A_j a_1^2)(a_1^{-2} A_i^{-1} a_1^2).$$

Obviously in G both of those relations are equivalent, however in K_G they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

to obtain infinitely many relations from K_G.

It was mentioned in the previous chapter, that following is an exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} \longrightarrow \mathbb{Z} \longrightarrow 0$$

Hence action of $\mathbb{Z}$ can be defined on the group K_G^{ab}, with presentation described above, as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}.$$

This procedure allows K_G^{ab} to be interpreted as a $\mathbb{Z}[\mathbb{Z}]$-module.

Definition 2.1 : Alexander module.
Given a group G, the abelianization of the commutator of a group G, K_G^{ab}, with $\mathbb{Z}[\mathbb{Z}]$-module structure is called the **Alexander module** of G. If G is a knot group, then it is the Alexander module of the knot K

Lemma 2.1.
The $\mathbb{Z}[\mathbb{Z}]$ modules K_G^{ab} and G^{mab} (see definition 1.3) are isomorphic.

Proof. Construction presented above states that the module K_G^{ab} has $(n-1)$ generators. $\square$

2.2 Basic properties

The resolution of a module at first glance is in no way a simplification of said module. However, there are multiple ways of distilling simplifications and invariants from the resolution of the Alexander module. In this section we want to

We start writing the beginning K_G^{ab} resolution as follows:

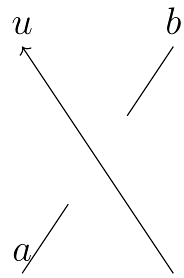
$$\dots \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

Definition 2.2 : Alexander matrix.
The matrix of homomorphism A_D in the diagram above is called the **Alexander matrix** of group G (knot K).

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its rows will give a square matrix whose determinant is nonzero.

Proposition 2.2.
Let A'_D be the Alexander matrix A_D with one of its rows struck out. Then $\det(A'_D) \neq 0$.

Proof. We start by noticing that every crossing contains three segments and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

As described in the previous section, we change the Wirtinger presentation so that only one generator x is send to 1 by abelianization. If said generator is $u = x$, then in the $\mathbb{Z}[\mathbb{Z}]$ module K^{ab} we see the following relation

$$\pm t^n(tB - C) = 0,$$

where $B = bx^{-1}$ and $C = cx^{-1}$. Otherwise, the relation is

$$\pm t^n[(1-t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of $A_D(1)$ in the row corresponding to the crossing is equal to 1.

The cases in which x is b or c are symmetrical and without the lose of generality assume that $x = b$. Then the relation is

$$\pm t^n[(t-1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and ± 1. Thus, the sum is not equal to zero. There are two of such rows as the segment b has to be the "out" and "in" segment of some crossing. In other words, segment b has to have a start and end in some crossings.

Let S_i be the column of the Alexander matrix corresponding to the segment labeled i. The sum $\sum_{i \leq n-1} S_i$ is a vector with two nonzero terms. Take S_j and S_k to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both S_j and S_k by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j,k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that $\{S_i \,:\, i \leq n-1\}$ is a set of linearly independent vectors and thus every minor of $A_D(1)$ has nonzero determinant. In particular, $\det(A'_D)(1) \neq 0$. $\square$

The proposition 2.2 implies that image of A_D has dimension $(n-1)$. We will use this knowledge later on to construct the resolution of the Alexander module.

Theorem 2.3.
The determinant $\det(A'_D)$ is independent of the choice of the diagram D

Proof. If D and D' are two diagrams of knot K, then they yield equivalent representations of $G = \pi_1(K)$. Thus, the chain of elementary ideals of A_D and $A_{D'}$ are the same according to Fox [5, Chapter VII] from which immediately follows that the determinants of the maximum minors of A_D and $A_{D'}$ are equal. □

Definition 2.3 : Alexander polynomial.
The **Alexander polynomial** of a knot K is the determinant of any maximal minor of the Alexander matrix A_D.

The Alexander polynomial is a knot invariant as a consequence of theorem 2.3 and proposition 2.2

Proposition 2.4.
Let G be a knot group of K. Then it always has a resolution

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}]^1 \longrightarrow \mathbb{Z}[\mathbb{Z}]^n \longrightarrow \mathbb{Z}[\mathbb{Z}]^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0$$

where n is the number of crossings of the chosen diagram D of knot K.

Proof. Take $R = \mathbb{Z}[\mathbb{Z}]$ and consider its field of fractions $R^{-1}R$. There is an obvious homomorphism $R \to R^{-1}R$ which allows us to work on A_D as if it was a linear map between vector spaces

$$R \otimes_R R^{-1}R \xrightarrow{A_D \otimes_R id_{R^{-1}R}} R \otimes_R R^{-1}R$$

with $\dim(A_D \otimes_R id_{R^{-1}R}) = (n-1)$ as was proven in proposition 2.2.

Thus, the following is an exact sequence of vector spaces

$$0 \longrightarrow V \longrightarrow V^n \xrightarrow{A'_D} V^{n-1} \longrightarrow 0$$

where $V = R^{-1}R$ and $A'_D = A_D \otimes_R id_{R^{-1}R}$.

Now consider the following sequence

$$0 \longrightarrow R \longrightarrow R^n \xrightarrow{A_D} R^{n-1} \longrightarrow K_G^{ab} \longrightarrow 0 \tag{1}$$

The only concerning point is the leftmost arrow as it might not be an injection to $\ker A_D$.

The ring of fractions is flat [3, Chapter 3], the module K_G^{ab} is torsion proposition 1.1 and thus

$$K_G^{ab} \otimes_R R^{-1}R = 0.$$

Because of that, tensoring the sequence (1) by $R^{-1}R$ induces an isomorphism between homologies of the sequences above, wherefore it is exact. □

2.3 A homological roots of diagram colorings

Thus far a resolution of the Alexander module K_G^{ab} provided a matrix and with it a polynomial invariant of knots. In this short section we will explain the connection between Alexander module and knot colorings, which will be the focus of the subsequent section.

Take M to be a finitely generated $R = \mathbb{Z}[\mathbb{Z}]$-module. The functor $\mathrm{Hom}(-, M^n)$ is left exact therefore applied to the resolution of the Alexander module generates the following sequence

$$0 \longrightarrow \mathrm{Hom}(R, M) \longrightarrow \mathrm{Hom}(R^n, M) \xrightarrow{\mathrm{Hom}(A_D, M)} \mathrm{Hom}(R^{n-1}, M) \longrightarrow \mathrm{Hom}(K_G^{ab}, M^n)$$

The diagram D taken as the starting point for the construction of K_G^{ab} had $n = x$ crossings and $n = s$ segments. The module K_G^{ab} was presented using $(n-1)$ generators, corresponding to all but one segments of the diagram. If we allow for propagation of values, then $\mathrm{Hom}(R^{n-1}, M)$ can be interpreted as assigning values from M to $(n-1)$ segments in diagram D, with the last segment colored based on the remaining part of the diagram.

The arrow $\mathrm{Hom}(R^{n-1}, M) \to \mathrm{Hom}(K_G^{ab}, M)$ ensures that the structure of K is taken into account during this assigment. Its kernel is be equal to $\operatorname{im} \mathrm{Hom}(A_D, M)$ and thus remembers which segments contributed to which crossings.

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$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4+v_4 & w_4+v_4 & x_4-v_4 & y_4+u_4+x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4+v_4 & w_4+v_4 & x_4-v_4 & y_4+u_4+x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as b and β were taken to be units.

□

4 A look at category theory

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