A voyage into the algebras

Weronika Jakimowicz
330006

Julia Walczuk
332742

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1 Problem

Consider the ring $\mathbb{Z}[[F]$, where $[F]$ is the equivalence class of all finite Abelian groups isomorphic to F. Describe the set $\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\}$, where relation $[F_2] = [F_1] + [F_3]$ means that there exists exact sequence:

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

Lemma 1.1. *If F, F' are two Abelian groups of order n, then they represent the same equivalence class of relation $\heartsuit$ i.e. $[F] = [F']$.*

Example 1.1. *Before we prove lemma 1.1, let us examine an example. We will show that $[\mathbb{Z}_4] = [\mathbb{Z}_2 \oplus \mathbb{Z}_2]$. Consider the following exact sequence*

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\mathrm{mod}\ 2} \mathbb{Z}_2 \longrightarrow 0$$

which shows that $[\mathbb{Z}_4] = [\mathbb{Z}_2] + [\mathbb{Z}_2]$. On the other hand, the next sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{i_1} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0$$

which is also exact, yields $[\mathbb{Z}_2 \oplus \mathbb{Z}_2] = [\mathbb{Z}_2] + [\mathbb{Z}_2]$.

This shows that every Abelian group of order 4 is in the same equivalence class of relation given by exact sequences. We will show that all Abelian groups of the same order will belong to one equivalence class.

Proof
Every finite Abelian group is isomorphic to a direct product of its p-subgroupsDODAĆ CYTAT. Furthermore, any p-group of order p^k is isomorphic to $\mathbb{Z}_{p^k}$. We can start by examining what elements belong to equivalence class $[\mathbb{Z}_{p^k}]$.

We will start by showing that if $k = n + l,\ k, n, l \in \mathbb{N}$, then $[\mathbb{Z}_{p^k}] = [\mathbb{Z}_{p^n}] + [\mathbb{Z}_{p^l}]$. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^l} \longrightarrow 0$$

We know that $\mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n}$ is a cyclic group generated by $1 + Z_{p^n}$. Furthermore, we know that $|\,\mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n}\,| = p^l$ and thus $\mathbb{Z}_{p^k} \,/\, \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^l}$.

Now, we will show, using induction on N, that for any $n \in \mathbb{N}$ such that $n = \prod_{i=1}^{N} p_i^{k_i}$, where $k_i \in \mathbb{N}$ and p_i is a prime number, we have

$$[\mathbb{Z}_n] = \sum_{i=1}^{N} [\mathbb{Z}_{p_i^{k_i}}] = \sum_{i=1}^{N} k_i \cdot [\mathbb{Z}_{p_i}] \quad (\star)$$

1. $N = 1$

   From the fact above we know that $[\mathbb{Z}_{p^{k+1}}] = [\mathbb{Z}_{p^k}] + [\mathbb{Z}_p]$ and applying the same reasoning to $\mathbb{Z}_{p^k}$ we obtain $[\mathbb{Z}_{p^k}] = k \cdot [\mathbb{Z}_p]$.

2. $N - 1 \implies N$

   We will start from the right side of the equality $(\star)$ and from inductive hypothesis we know that

   $$\sum_{i=1}^{N} k_i [\mathbb{Z}_{p_i}] = k_N [\mathbb{Z}_{p_N}] + \sum_{i=1}^{N-1} k_i [\mathbb{Z}_{p_i}] = [\mathbb{Z}_{p_N^{k_N}}] + [\mathbb{Z}_l]$$

   where $l = \prod_{i=1}^{N-1} p_i^{k_i}$. Consider the following sequence

   $$0 \longrightarrow \mathbb{Z}_l \longrightarrow \mathbb{Z}_n \longrightarrow \mathbb{Z}_{p_N^{k_N}} \longrightarrow 0$$

   its exactness follows from the fact that $\mathbb{Z}_n \,/\, \mathbb{Z}_l$ is a cyclic group of order $n/l = p_N^{k_N}$ and thus there exists an isomorphism

   $$\mathbb{Z}_{p_N^{k_N}} \cong \mathbb{Z}_n \,/\, \mathbb{Z}_l\,.$$

As stated before, any Abelian group of order N is isomorphic to a direct product of its p-subgroups, hence the following equality is immediate from $(\star)$:

$$\sum_{i=1}^{N} k_i[\mathbb{Z}_{p_i}] = [\mathbb{Z}_n] = \left[\bigoplus_{i=1}^{N} \mathbb{Z}_{{p_i}^{k_i}}\right]$$

♠

From this follows that every Abelian group of order n, either being a cyclic group itself or a direct sum of cyclic groups, is in one equivalence class. Hence, elements of group

$$\mathbb{Z}[[F]]/\{[F_2] = [F_1] + [F_3]\}$$

can be expressed as finite sums of equivalence classes represented by p-groups:

$$\mathbb{Z}[[F]/\{[F_2] = [F_1] + [F_3]\} = \{\sum_{i\leq n} k_i[\mathbb{Z}_{p_i}] \ : \ p_i \text{ are prime}, n, k_i \in \mathbb{N}\}$$

2 Problem

Consider a field $\mathfrak{K}$ and the ring of polynomials with coefficients in $\mathfrak{K}$, $\mathfrak{K}[x]$. Obviously, the aforementioned ring is a principial ideal domain. We want to consider group $\mathbb{Z}[[M]]$, where $[M]$ is the equivalence class of all finitely generated torsion modules isomorphic to M and relation $[M_2] = [M_1] + [M_3]$ defined by the existence of an exact sequence.

Example 2.1. *Let us consider $\mathbb{Q}[x]$-modules*

$$M = \mathbb{Q}[x]/(x^3 + 1)$$

$$N = \mathbb{Q}[x]/(x + 1) \oplus \mathbb{Q}[x]/(x^2 - x + 1)$$

we will show that

$$\big[\mathbb{Q}[x]/(x^3 + 1)\big] = \big[\mathbb{Q}[x]/(x + 1)\big] + \big[\mathbb{Q}[x]/(x^2 - x + 1)\big] = \big[\mathbb{Q}[x]/(x + 1) \oplus \mathbb{Q}[x]/(x^2 - x + 1)\big]$$

Exactness of sequence

$$0 \longrightarrow \mathbb{Q}[x]/(x + 1) \hookrightarrow \mathbb{Q}[x]/(x + 1) \oplus \mathbb{Q}[x]/(x^2 - x + 1) \twoheadrightarrow \mathbb{Q}[x]/(x^2 - x + 1) \longrightarrow 0$$

is rather trivial: the left arrow is embedding of a summand to a direct sum and the right arrow is projection from direct sum.

The second sequence,

$$0 \longrightarrow \mathbb{Q}[x]/(x + 1) \xrightarrow{f} \mathbb{Q}[x]/(x^3 + 1) \xrightarrow{g} \mathbb{Q}[x]/(x^2 - x + 1) \longrightarrow 0$$

We define f as

$$f(w + (x + 1)) = w(x^2 - x + 1) + (x + 1)(x^2 - x + 1) = w(x^2 - x + 1) + (x^3 + 1)$$

because $(x + 1)(x^2 - x + 1) = (x^3 + 1)$. Then $\mathrm{im}(f) = (x^2 - x + 1)/(x^3 + 1)$. Now, the second homomorphism will be

$$g(w + (x^3 + 1)) = (w + (x^3 + 1)) + (x^2 - x + 1) = w + (x^3 + 1) + (x^2 - x + 1) = w + (x^2 - x + 1)$$

because $(x^3 + 1) + (x^2 - x + 1) = (x^2 - x + 1)$ as polynomial $x^2 - x + 1$ divides $x^3 + 1$. It is clear that $\ker(g) = (x^2 - x + 1)$. Hence, the sequence is exact.

Any finitely generated module M is isomorphic to a direct sum of cyclic modules:

$$M \cong \mathfrak{K}[x]/(p_1) \oplus \mathfrak{K}[x]/(p_2) \oplus ... \oplus \mathfrak{K}[x]/(p_n)$$

Lemma 2.1. *Let $\mathfrak{K}$ be a field and consider $\mathfrak{K}[x]$-modules M, M'. Then $[M]_\heartsuit = [M']_\heartsuit$ if and only if irreducible polynomials p_i that appear in decomposition of M are the same to the ones that appear in decomposition of M'.*

Moduły które mają ten sam wielomian w rozkładzie trafiają do tego samego domku.

If p, q are two irreducible polynomials, then $(p) \oplus (q) = \mathfrak{K}[x]$ (example: $x-1, x^2+1$).

$$x^2 + 2x + 1 - (x-1)(x+3)$$