A voyage into the algebras

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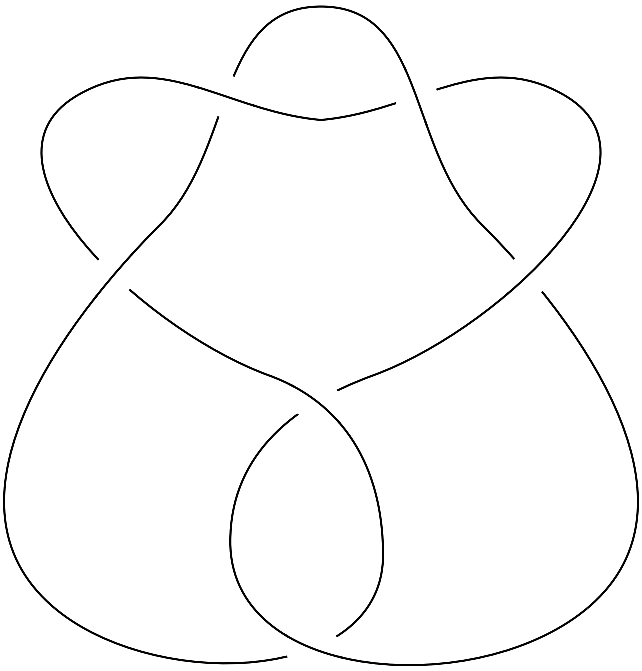

1 Introduction

1.1 What does it mean to color a knot?

What do we need?

- R - commutative ring with identity

- D - diagram of knot K with s segments and x crossings

- $\phi : M^3 \to M$ - function that dictates the rules of our coloring (and induces two operators $M^2 \to M^2$)

In order for trivial coloring to work, $\phi(m, m, m) = 0$ for all $m \in M$. This means that if we take $\phi(u, v, w) = au+bv+cw$ then $(a+b+c) \in \mathrm{Ann}(M)$.

In the most general case, $R = \mathbb{Z}[s, t, t^{-1}]/\{s(s+t-1)\}$ and $\phi(u, v, w) = su + tv - w$.

Given those we can define $f : M^s \to M^x$, which assigns values from M to segments of D according to the rules set by ϕ.

This yields an exact sequence

$$0 \longrightarrow \ker f \hookrightarrow M^s \xrightarrow{f} M^x \twoheadrightarrow \operatorname{coker} f \longrightarrow 0$$

We know that $\ker f$ always contains colorings - especially the trivial one. We expect $\operatorname{coker} f$ to contain some information about non-trivial colorings admissible.

1.2 Smith's normal form ¿and connection to Alexander polynomial?

Function f can be expressed as a $s \times x$ matrix with elements from M - we can make it into "diagonal form" where non-zero elements lower are divisible by elements at the top. This gives us information about $\ker f$ and $\operatorname{coker} f$.

Example 1.1. *Consider knot 4_1 with diagram D as seen in fig. 2 and ring $R = M = \mathbb{Z}[t, t^{-1}]$. Take function $\phi : M^3 \to M$ to be defined as*

$$\phi(u, i, o) = (1 - t)u + ti - o$$

for crossing as seen in fig. 1.

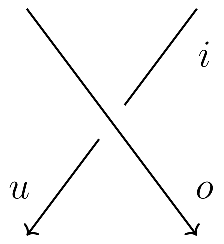

Figure 1: Crossing

Function f is then defined by matrix

$$f = \begin{pmatrix} 1-t & t & -1 & 0 \\ t^{-1} & -1 & 0 & 1-t^{-1} \\ 0 & 1-t^{-1} & t^{-1} & -1 \\ -1 & 0 & 1-t & t \end{pmatrix}$$

which has Smith's normal form:

$$f' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & t^2-3t+1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice, that $\det f' = t^2 - 3t + 1$*, which is the Alexander polynomial of* 4_1.

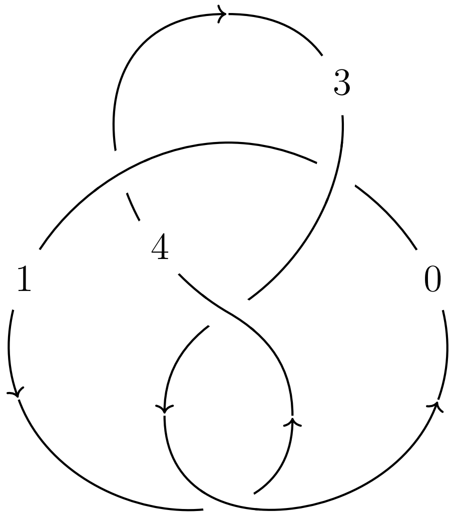

Figure 2: Coloring of knot 4_1 with elements from $\mathbb{Z}_5$.

Now, consider a homomorphism $\mathbb{Z}[t,t^{-1}] \to \mathbb{Z}$ that sends $t \mapsto -1$. This yields a new matrix for f, with Smith's normal form:

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, the $\operatorname{coker} f = \mathbb{Z} \oplus \mathbb{Z}_5$ *which hints at existence of coloring using elements from $\mathbb{Z}_5$. One of those colorings is presented in fig. 2.*

1.3 Reducing normal form of a matrix

We might want to ask the question regarding the ways to distinguish knots with the same Alexander polynomial, like 6_1 and 9_{46}. One of the answers might be to look at the function $f : M^s \to M^x$ and the equivalence class of its Smith's normal form in ring $R = \mathbb{Z}[t,t^{-1}]$.

We notice, that the function f in itself is not a knot invariant. Its matrix changes in size with changes in the diagram of the knot that we are considering. What is an invariant is its $\ker f$ and $\operatorname{coker} f$ - information about the number of colorings and what colorings might be admissible. Furthermore, when calculated over $\mathbb{Z}[t,t^{-1}]$, the kernel always is a free module of dimension 1 and all units that appear on the diagonal will not contribute to the coker. Hence, we might *consider the normal form of f stripped of units and zeros.*

Example 1.2. *First, consider the knot* 6_1 *with diagram as seen in fig. 3, ring $R = \mathbb{Z}[t,t^{-1}]$ and $M = R$. We calculate that*

$$f = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & -2t^{-2} + 5t^{-1} - 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which agrees with the Alexander polynomial of 6_1. *Now, the reduced form of f would be*

$$\begin{pmatrix} -2t^{-2} + 5t^{-1} - 2 \end{pmatrix}$$

a 1×1 matrix.

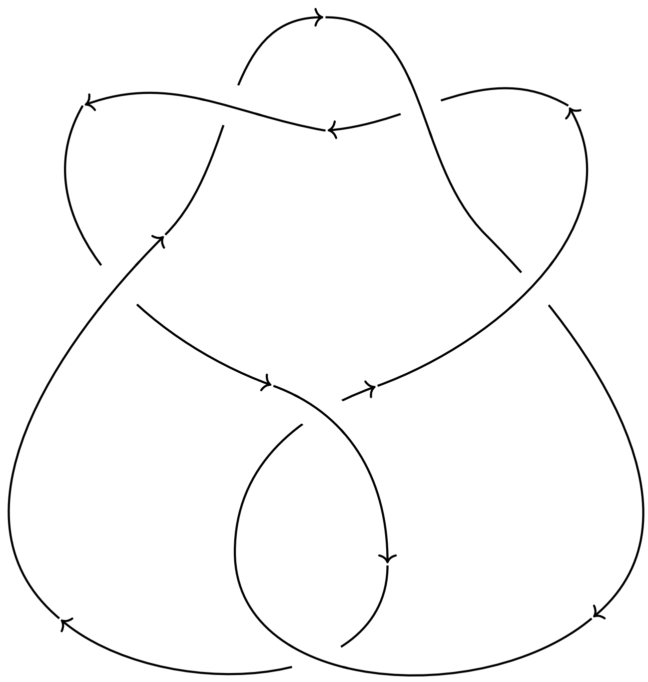

Figure 3: Diagram of knot 6_1.

There is another knot with Alexander polynomial equal $-2t^{-2}+5t^{-1}-2$*:* 9_{46}*. Using diagram in fig. 4 it can be calculated that*

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t-t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-2}-2t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\det f = (2t-t^2)(t^{-2}-2t^{-1}) = 2t^{-1} - 5 + 2t$$

is also the Alexander polynomial. The reduced form of f *is*

$$\begin{pmatrix} 2t-t^2 & 0 \\ 0 & t^{-2}-2t^{-1} \end{pmatrix}$$

which is significantly different than the one for 6_1*.*

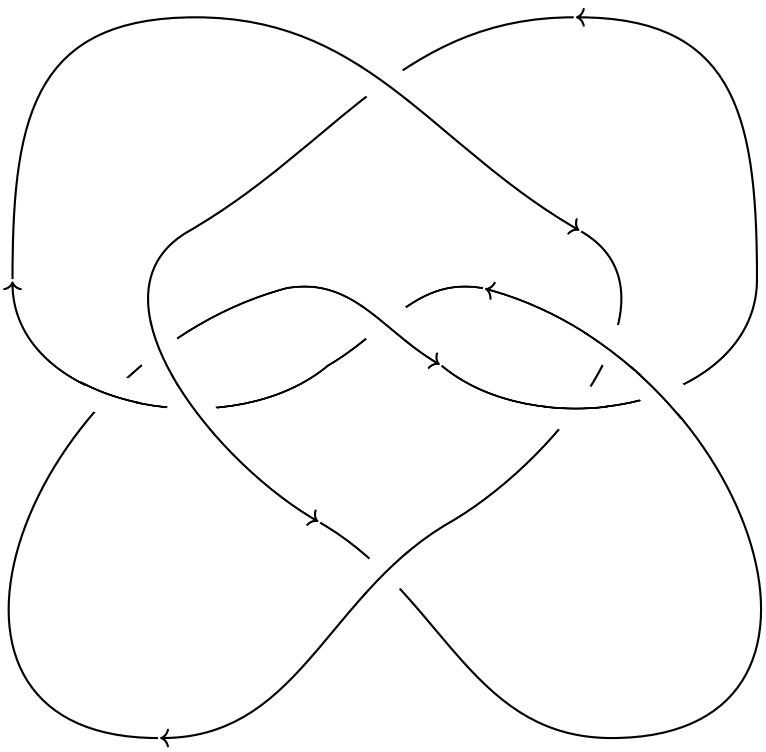

Figure 4: Diagram of knot 9_{46}.

References