

# Knot colorings and homological invariants

(Kolorowania węzłów i niezmienniki homologiczne.)

Weronika Jakimowicz  
330006

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**Promotor:** prof. Tadeusz Januszkiewicz

## Abstract

For a knot  $K \subseteq S^3$  its knot group  $G = \pi_1(S^3 - K)$  is a starting point for many knot invariants. One such invariant is the Alexander module  $K_G^{ab}$ , the abelianized kernel of homomorphism  $G \rightarrow \mathbb{Z}$  with a  $\mathbb{Z}[\mathbb{Z}]$ -module structure. Associated with the Alexander module is Alexander matrix  $A_D$  with  $\text{coker}(A_D) = K_G^{ab}$ . The same matrix can be obtained by coloring any diagram of  $K$  with a properly chosen palette. In the following paper, the connection between the algebraical, topological and combinatorial definitions of the Alexander matrix and module are given along with connections between them.

# Introduction

Let  $K$  be a knot embedded in  $S^3$  and  $D$  its diagram. The fundamental group of the knot complement  $\pi_1(S^3 - K)$ , called the knot group, is a knot invariant. The diagram  $D$  gives presentation of  $G$ . The knot group is a basis for many simpler invariants, such as the Alexander polynomial [Ale28] or the Alexander module (see definition 1.6) and the knot diagram helps in their computation.

One can look for other ways of defining invariants from diagrams. A coloring is an assignment of elements of a module  $M$  to segments of the diagram. Assignments that satisfy specific conditions (see lemma 3.8) yield knot invariants. The following thesis is concentrated on finding a connection between the information about a knot  $K$  obtained from its group with the information obtained from a diagram coloring.

Section 1 defines the fundamental concepts of this paper, such as the knot group (see definition 1.1) its metabelianization (see definition 1.4) and the Alexander module (see definition 1.6). Two equivalent definitions for said modules are presented: an algebraic one and a topological one. In a purely algebraic sense, the Alexander module is the abelianized subgroup  $[[G, G], [G, G]]$  of the knot group  $G$ , with  $\mathbb{Z}$  action induced by abelianization homomorphism  $G \rightarrow \mathbb{Z}$ , while from a topological point of view it is the first homology module of the infinite cyclic cover (see definition 1.7). An important result finishing this section is that the Alexander module is a torsion module (see proposition 1.8).

Section 2 is dedicated to the Alexander matrix (see definition 2.3) and its properties. We start by showing an algorithm for obtaining the presentation of the Alexander module, which is then used to construct a resolution of said module. The Alexander matrix is defined as the matrix of the homomorphism  $\mathbb{Z}[\mathbb{Z}]^n \rightarrow \mathbb{Z}[\mathbb{Z}]^{n-1}$  with cokernel isomorphic to the Alexander module. Over the ring of fractions  $F$  [AM69, Chapter 2] of  $\mathbb{Z}[\mathbb{Z}]$ , this matrix is surjective and creates a particularly interesting short exact sequence of  $F$ -modules.

In the last section 3 diagram colorings are discussed, starting with a definition of a palette (see definition 3.1). A set of palettes of particular interest are the Alexander palette and all its images through homomorphisms induced by either a ring or a module homomorphism. From diagram coloring a matrix is obtained, called the color checking matrix

(see definition 3.4). Properties of palettes that ensure the invariant nature of color checking matrices are outlined and proven. Two knots, K11n85 and Kn164, having the same Alexander polynomial, are distinguished using a reduced Smith normal form of color checking matrix (see definition 3.11). The paper ends with a proof that this reduced normal form is a knot invariant for Alexander palettes.

This thesis is a result of cooperation between Julia Walczuk and myself, and is written under supervision of prof. Tadeusz Januszkiewicz.



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# 1 Preliminaries

## 1.1 Knots and diagrams

In mathematical terms, a knot is a smooth embedding  $S^1 \hookrightarrow S^3$ . A knot diagram is an immersive projection  $D : S^1 \hookrightarrow \mathbb{R}^2$  along a vector such that no three points of the knot lay on this vector [Lik97, pages 2-3]. If two points are mapped to one by this projection, we say that a small neighbourhood of this point which looks locally like  $-|-$  is a crossing.

$S^1$  is orientable, thus we can chose an orientation for any knot and, as a consequence, its diagram.

Two tame knots  $K_1$  and  $K_2$  are equivalent if we can deform one into the other [Mur96, pages 7-8]. This translates to an equivalence of diagrams, which is generated by comparing diagrams that are exactly the same save for an interior of some disc in  $\mathbb{R}^2$ . If inside of said disc the diagrams differ by one of **Reidemeister moves**, we say that they are equivalent. In the case of a diagram without an orientation, three moves are sufficient. When an orientation is imposed on  $D$ , 4 diagram moves (pictured in fig. 1) generate the whole equivalence relation [Pol10].

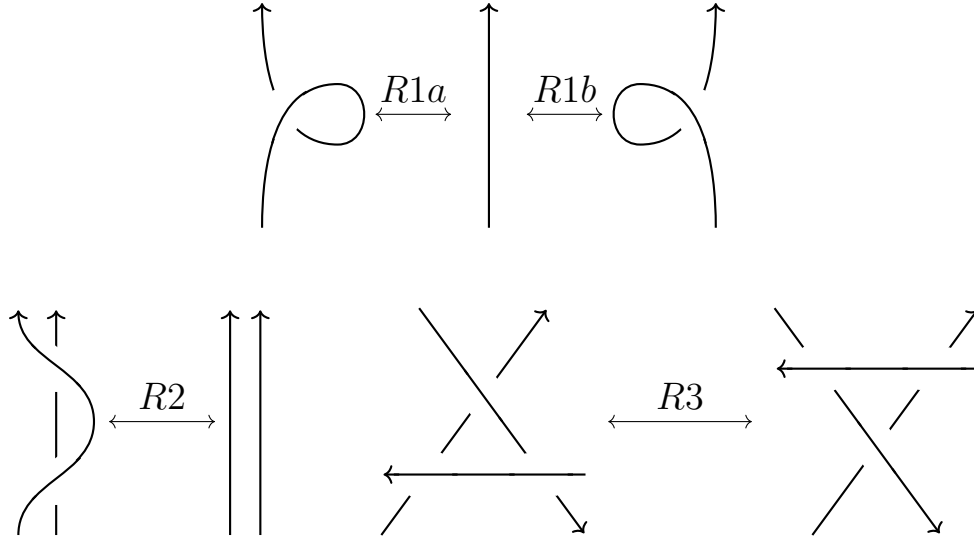


Figure 1: Generating set of Reidemeister moves in oriented diagrams.

## 1.2 Knot group

Let  $K$  be a knot and  $D$  be its oriented diagram with  $s$  segments and  $x$  crossings. A segment of a diagram is a line of the diagram between two

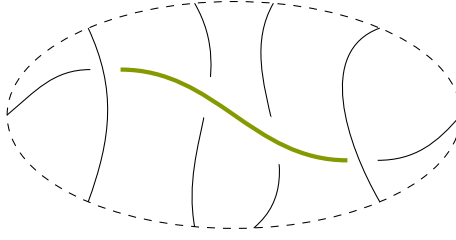


Figure 2: The green line is a segment of the diagram.

crossings in which it disappears under another line, see fig. 2.

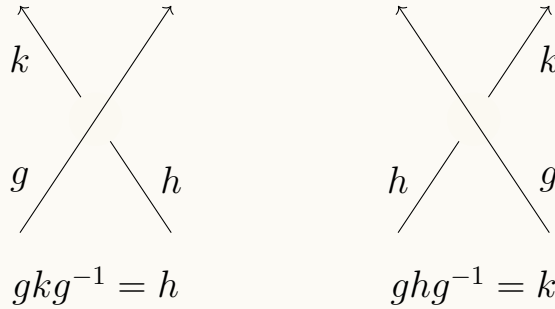
**Definition 1.1 : knot group.**

Let  $K \subseteq S^3$  be a knot. The fundamental group of knot complement  $X = S^3 - K$ ,  $\pi_1(X)$ , is called the **knot group** of  $K$ .

Although the knot itself is always  $S^1$ , the knot group has usually an interesting yet difficult structure. The most commonly used presentation of the knot group is called **the Wirtinger presentation**.

**Definition 1.2 : Wirtinger presentation.**

Given an oriented diagram  $D$  of knot  $K$  with segments  $a_1, a_2, \dots, a_s$ , which follow the orientation, and crossings  $c_1, \dots, c_x$  the knot group  $\pi_1(X)$  can be represented as  $\pi_1(X) = \langle S \mid R \rangle$ , where  $S$  is the set of segments  $a_1, \dots, a_s$  of  $D$  and relations  $R$  correspond to crossings in the manner described in the diagram below



Presentation  $\langle S \mid R \rangle$  described above is called the **Wirtinger presentation** [Liv93, Chapter 6].

By applying the Mayer-Vietoris sequence to  $S^3 = K \cup_{T^2} (S^3 - K)$  or noticing that every two generators are conjugate, one can conclude that the abelianization of the knot group is always  $\mathbb{Z}$ . This leads to a short exact sequence

$$0 \longrightarrow K_G \longrightarrow G = \pi_1(K) \xrightarrow{ab} \mathbb{Z} = G^{ab} \longrightarrow 0.$$

The presentation of the group  $K_G = \ker(ab : G \rightarrow \mathbb{Z}) = [G, G]$  obtained from a Wirtinger presentation of  $G$  is discussed in section 2.1.

**Lemma 1.3.**

For any group  $G$ , the commutator of its commutator  $K_G$  is a normal subgroup:  $[K_G, K_G] = [[G, G], [G, G]] \triangleleft G$ .

**Proof.** The commutator subgroup is a characteristic subgroup, since for any automorphism  $\phi : G \rightarrow G$

$$\phi(hgh^{-1}g^{-1}) = \phi(h)\phi(g)\phi(h)^{-1}\phi(g)^{-1} \in K_G = [G, G].$$

Conjugation by any element  $g \in G$  is an automorphism of the commutator  $K_G$ . Thus it preserves its commutator subgroup  $[K_G, K_G]$ .  $\square$

The exact sequence

$$0 \longrightarrow K_G^{ab} \longrightarrow G^{mab} = G/[K_G, K_G] \longrightarrow \mathbb{Z} \longrightarrow 0 \quad (1)$$

is exact.

**Definition 1.4 : metabelianization.**

The quotient group  $G^{mab} = G/[K_G, K_G]$  is called the **metabelianization** of  $G$ .

**Proposition 1.5.**

$$G^{mab} \cong K_G^{ab} \rtimes \mathbb{Z}$$

**Proof.** By lemma 1.3 we know that  $[K_G, K_G] \triangleleft K_G$  are normal subgroups of  $G$ , then by the third isomorphism theorem,  $K_G^{ab} = K_G/[K_G, K_G]$  is also a normal subgroup in  $G/[K_G, K_G]$ . The sequence (1) is split because  $\mathbb{Z} \rightarrow G^{mab} \rightarrow \mathbb{Z}$  defined by  $1 \mapsto a \mapsto 1$ , where  $a$  is any generator of  $G^{mab}$  is the identity map on  $\mathbb{Z}$ . This completes the proof.  $\square$

Proposition 1.5 implies that there exists an action of  $\mathbb{Z}$  on the  $\mathbb{Z}$ -module  $K_G^{ab}$ . Thus, a  $\mathbb{Z}[\mathbb{Z}]$ -module structure is admissible on  $K_G^{ab}$ . We will return to the concept of metabelianization and  $K_G^{ab}$  in section 2, specifically in section 2.1. For the time being, let us assign a name to  $K_G^{ab}$ :

**Definition 1.6 : Alexander module.**

Given a group  $G$  with a homomorphism  $g : G \rightarrow \mathbb{Z}$ , the abelianization of the kernel  $\ker(g)$ ,  $K_G^{ab}$ , with  $\mathbb{Z}[\mathbb{Z}]$ -module structure is called the **Alexander module** of  $G$ . If  $G$  is a knot group, then  $\ker(g) = [G, G] = K_G$  and it is the Alexander module of the knot  $K$ .

### 1.3 Infinite cyclic covering

Let  $X$  be the complement of a knot  $K$  ( $X = S^3 - K$ ). Take  $\tilde{X}$  to be its universal covering, meaning that it is simply connected. The knot group  $G = \pi_1(X)$  of  $X$  acts on its universal covering by deck transformations. The commutator subgroup  $K_G = [G, G] \triangleleft G$  also acts on  $\tilde{X}$  and the quotient

$$\overline{X} = \tilde{X}/K_G$$

has a fundamental group

$$\pi_1(\overline{X}) = K_G$$

no matter the choice of basepoint in  $\tilde{X}$ .

Since  $K_G$  is a normal subgroup, then  $G/K_G = G^{ab} = \mathbb{Z}$  also acts on the quotient space  $\overline{X}$ . This extends to the homology groups of  $\overline{X}$  and allows a  $\mathbb{Z}[\mathbb{Z}]$ -module structure on them.

**Definition 1.7 : infinite cyclic covering.**

The quotient space  $\overline{X} = \tilde{X}/K_G$  is called the **infinite cyclic covering** of  $X$ .

Looking at homology groups of  $\overline{X}$ , we have the following equality

$$H_1(\overline{X}, \mathbb{Z}) = \pi_1(\overline{X})^{ab} = K_G^{ab}.$$

Working with homology modules of an infinite cyclic cover of  $X$  instead of  $K_G^{ab}$  directly is beneficial when proving some properties of  $K_G^{ab}$ , i.e. that it is a torsion module in proposition 1.8.

The following diagram illustrates the construction of infinite cycle covering described above



$$\begin{array}{ccc}
\tilde{X} & \curvearrowright & G \\
\downarrow & & \\
\overline{X} & \curvearrowright & G/[G, G] \\
\downarrow & & \\
X = S^3 - K & & 
\end{array}$$

A **Seifert surface**  $S$  of knot  $K$  is an orientable surface with boundary embedded in  $S^3$  such that  $\partial S = K$ . Take a countable amount of  $X$ , with  $S$  without its boundary embedded, and label each with an element from  $\mathbb{Z}$ . We might now cut each of the copies of  $X$  along the Seifert surface of  $K$  and identify the  $+$  side of  $S$  from the  $i$ -th copy of  $X$  with the  $-$  side of  $S$  from the  $(i+1)$ -th copy of  $X$ . The arising space with a projection to one copy of  $X$  is an infinite cyclic cover of  $X$ .

Imagine that each copy of  $X$  inside of  $\overline{X}$  is a box labeled with some integer  $k$  such that copies of  $X$  sharing the Seifert surface are labeled with consecutive integers. The ring action of  $\mathbb{Z}[\mathbb{Z}]$  on  $\overline{X}$  is increasing or decreasing the label on the box from which a cycle is taken, depending on the power of  $t \in \mathbb{Z}[\mathbb{Z}]$  in the polynomial which we apply to  $\overline{X}$ .

**Proposition 1.8.**

The  $\mathbb{Z}[\mathbb{Z}]$ -module  $K^{ab} = H_1(\overline{X}, \mathbb{Z})$  is a torsion module.

**Proof.** Consider the following homomorphism on chain complexes:

$$f : C_*(\overline{X}) \rightarrow C_*(\overline{X})$$

$$f(x) = (1 - t)x.$$

It translates to removing from a cycle in the  $(i+1)$ -th box a corresponding cycle in the  $i$ -th box. From this it is an immediate result that  $\ker f = 0$  and that  $\operatorname{coker} f = C_*(X)$ : after gluing all pairs of cycles from two consecutive boxes, the result is easily identified with cycles from just one box.

As a consequence, the following sequence of chain complexes is exact

$$0 \longrightarrow C_*(\overline{X}) \xrightarrow{f} C_*(\overline{X}) \longrightarrow C_*(X) \longrightarrow 0$$

and induces a long exact homology sequence

$$\begin{array}{ccccccc}
\dots & \longrightarrow & H_2(X, \mathbb{Z}) & \longrightarrow & H_1(\overline{X}, \mathbb{Z}) & \xrightarrow{1-t} & H_1(\overline{X}, \mathbb{Z}) \longrightarrow H_1(X, \mathbb{Z}) \\
& & & & & & \downarrow \\
& & & & & & H_0(\overline{X}, \mathbb{Z}) \longrightarrow H_0(\overline{X}, \mathbb{Z}) \longrightarrow H_0(X, \mathbb{Z}) \longrightarrow 0.
\end{array}$$

As was mentioned previously, the following equality holds:

$$H_1(X, \mathbb{Z}) = \pi_1(X)^{ab} = \mathbb{Z}.$$

Now, because  $X$  is homology circle, then  $H_2(X, \mathbb{Z}) = 0$  (one can easily check it for themselves using Alexander duality). Both  $X$  and  $\overline{X}$  are connected implying that

$$H_0(X, \mathbb{Z}) = H_0(\overline{X}, \mathbb{Z}) = \mathbb{Z}.$$

$$\begin{array}{ccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & H_1(\overline{X}, \mathbb{Z}) & \xrightarrow{1-t} & H_1(\overline{X}, \mathbb{Z}) \xrightarrow{0} \mathbb{Z} \\
& & & & & & \downarrow \\
& & & & & & \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \longrightarrow 0
\end{array}$$

Rewriting the sequence above we easily get that homomorphism  $1 - t$  is actually an isomorphism and  $H_1(\overline{X}, \mathbb{Z}) \cong (1 - t)H_1(\overline{X}, \mathbb{Z})$ , which allows us to use the Nakayama's lemma [AM69, Proposition 2.6] to conclude that there exists  $x \in \mathbb{Z}[\mathbb{Z}]$  such that

$$xH_1(\overline{X}, \mathbb{Z}) = 0.$$

□

## 2 Resolution of the Alexander module

### 2.1 Presentation of the Alexander module

Take  $G = \langle S \mid R \rangle$  to be the Wirtinger presentation of  $G$  obtained from oriented diagram  $D$  (see definition 1.2). Because  $K$  is a knot and not a link, we know that the number of segments is equal to the number of crossings.

**Lemma 2.1.**

If  $K$  is a knot and  $D$  its diagram with  $x > 0$  crossings and  $s > 1$  segments, then  $s = x$ .

**Proof.** Every crossing has two segments that disappear under it. Let  $d$  be the number of segment ends that go below a crossing then

$$2 \cdot x = d.$$

Every segment starts and ends in some crossing, thus

$$2 \cdot s = d.$$

From those two equalities we have that  $d = 2x = 2s \implies x = s$ .  $\square$

Let  $a_1, \dots, a_n$  be the generators of  $G$  agreeing with the orientation of the diagram  $D$  and  $x_1, \dots, x_n$  its relations. The homomorphism of abelianization of  $G$  is defined by

$$a_i \mapsto 1 \in \mathbb{Z}$$

for every  $i = 1, \dots, n$ . In order to obtain a presentation of  $K_G$ , the kernel of abelianization, we need to change the set of generators of  $G$  to

$$\{a_1, A_2 = a_2 a_1^{-1}, \dots, A_n = a_n a_1^{-1}\}.$$

**Proposition 2.2 : presentation of  $K_G$ .**

The group  $K_G$  has an infinite presentation with generators

$$\{b_{i,k} = a_1^k A_i a_1^{-k} : i = 2, \dots, n, k \in \mathbb{Z}\}$$

and relations

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

for  $k \in \mathbb{Z}$ , where  $a_k = a_i a_j a_i^{-1}$  was a relation in  $G$ .

**Proof.** To begin with,  $(\forall i > 1) A_i \mapsto 0$  by abelianization of  $G$ . Thus  $A_2, \dots, A_n$  are some of the generators of  $K_G$ . However, for each  $i = 2, \dots, n$  and  $k \in \mathbb{Z}$  the following is an element of  $K_G$ :

$$b_{i,k} := a_1^k A_i a_1^{-k}.$$

Thus, the presentation of  $K_G$  as an abelian group is infinite with (possibly redundant) generators

$$\{b_{i,k} : i = 2, \dots, n, k \in \mathbb{Z}\}.$$

Changing generators of  $G$  induced a change in relations. Suppose that the following relation was true in  $G$

$$a_k = a_i a_j a_i^{-1}.$$

If  $1 \notin \{i, k, j\}$  then after the change of generators the relation is

$$A_k a_1 = A_i a_1 A_j a_1 (A_i a_1)^{-1}$$

forcing each element to be a conjugate of  $a_1$  the following two relations can be obtained

$$\begin{aligned} a_1^{-1} A_k a_1 &= (a_1^{-1} A_i a_1) A_j A_i^{-1} \\ a_1^{-3} A_k a_1^3 &= (a_1^{-3} A_i a_1^3) (a_1^{-2} A_j a_1^2) (a_1^{-2} A_i^{-1} a_1^2). \end{aligned}$$

In  $G$  both of those relations are equivalent, however in  $K_G$  they are distinct. Moreover, we can write

$$b_{k,x} = b_{i,x} b_{j,x-1} b_{i,x-1}^{-1}$$

to obtain infinitely many relations from  $K_G$ . □

The action of  $\mathbb{Z}$  on the group  $K_G^{ab}$  exists by proposition 1.5 and can be defined as follows

$$t(b_{i,k}) = b_{i,k+1}.$$

In particular,

$$t(A_i) = a_1 A_i a_1^{-1}$$

and so the  $\mathbb{Z}[\mathbb{Z}]$ -module  $K_G^{ab}$  is generated by  $(n-1)$  elements.

Moreover, the group  $G^{mab}$  and  $\mathbb{Z}[\mathbb{Z}]$ -module  $K_G^{ab}$  can be used interchangeably thanks to the split exact sequence (1) and proposition 1.5.

## 2.2 Basic properties

Knowing the resolution of a module allows one to change said module into a matrix or even a sequence of matrices, each containing a portion of information about its structure. We write the following resolution of the Alexander module:

$$0 \rightarrow \ker(A_D) \rightarrow \mathbb{Z}[\mathbb{Z}]^n \xrightarrow{A_D} \mathbb{Z}[\mathbb{Z}]^{n-1} \xrightarrow{f} K_G^{ab} \rightarrow 0 \quad (2)$$

where the  $f$  arrow sends every generator  $(0, \dots, 1, \dots, 0) \in \mathbb{Z}[\mathbb{Z}]^{n-1}$  to each of the  $(n-1)$  generators  $A_i$  of  $K_G^{ab}$  (compare with discussion in section 2.1). In the kernel of this homomorphism are exactly all  $n$  relations in  $K_G^{ab}$  and they are the image of  $A_D$ .

### Definition 2.3 : Alexander matrix.

The matrix  $A_D$  in the diagram above is called the **Alexander matrix** of the Alexander module  $K_G^{ab}$  using the Wirtinger presentation with the diagram  $D$ .

The Alexander matrix in the case of a knot group is not a square matrix. However, striking out any of its columns will give a square matrix whose determinant is nonzero (see lemma 2.4). Take  $R = \mathbb{Z}[\mathbb{Z}]$ .

In proposition 1.8 it was shown that the Alexander module is torsion. Let  $F = R^{-1}R$  be the field of fractions of  $R$  [AM69, Chapter 2]. Then, as a vector space  $K_G^{ab} \otimes_R F = 0$  is trivial. The field of fractions is flat  $R$ -module [AM69, Corollary 3.6] and so the sequence in (2) translates to the following exact sequence of  $F$  modules

$$0 \rightarrow \ker(A_D) \otimes_R F \rightarrow R^n \otimes_R F \xrightarrow{A_D^V} R^{n-1} \otimes_R F \rightarrow 0 \quad (3)$$

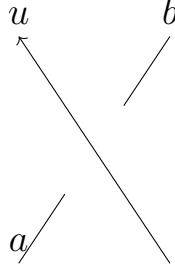
As there exists an inclusion  $R \hookrightarrow F = R^{-1}R$ , every matrix with terms in  $R$  can be treated as a matrix with terms in  $F$ . Naturally,  $A_D^V := A_D \otimes Id_F$  is just matrix  $A_D$  (with terms in  $R$ ) with adjoined  $1 \times 1$  matrix with just identity of  $F$ . Thus, if  $A_D^V$  has nonzero determinant, then so does  $A_D$ .

### Lemma 2.4.

Let  $A'_D$  be the Alexander matrix  $A_D$  with one of its rows struck out. Then  $\det(A'_D) \neq 0$ .

**Proof.** We start by noticing that every crossing contains three segments

and so every row of the Alexander matrix has at most three non-zero terms. The relation in Wirtinger presentation generated by crossing



is of form

$$ubu^{-1} = c.$$

The particular choice of orientation on  $a$  and  $b$  does not matter as long as it agrees with the rest of the diagram.

As described in the previous section, we change the Wirtinger presentation so that only one generator  $x$  is sent to 1 by abelianization. If said generator is  $u = x$ , then in the  $\mathbb{Z}[\mathbb{Z}]$  module  $K^{ab}$  we see the following relation

$$\pm t^n(tB - C) = 0,$$

where  $B = bx^{-1}$  and  $C = cx^{-1}$ . Otherwise, the relation is

$$\pm t^n[(1 - t)U + tB - C] = 0,$$

and the row corresponding to this crossing in the Alexander matrix has exactly three terms.

In those two cases, the sum of coefficients of  $A_D(1)$  in the row corresponding to the crossing is equal to 1.

The cases in which  $x$  is  $b$  or  $c$  are symmetrical and without the lose of generality assume that  $x = b$ . Then the relation is

$$\pm t^n[(t - 1)U - tC] = 0.$$

Notice that the coefficients in row corresponding to this crossing are 0 and  $\pm 1$ . Thus, the sum is not equal to zero. There are two of such rows as the segment  $b$  has to be the "out" and "in" segment of some crossing. In other words, segment  $b$  has to have a start and end in some crossings.

The reasoning above is true for matrix  $A_D^V$  from (3). We make the switch to vector space to use the connection between the rank of matrix and its determinant.

Let  $S_i$  be the column of the Alexander matrix corresponding to the segment labeled  $i$ . The sum  $\sum_{i \leq n-1} S_i$  is a vector with two nonzero terms. Take  $S_j$  and  $S_k$  to be the vectors with those nonzero terms. The only way to cancel out those coordinates is to multiply both  $S_j$  and  $S_k$  by zero. However, doing this we eliminate two other coordinates with nonzero terms. This yields a sum

$$\sum_{\substack{i \leq n-1 \\ i \neq j, k}} S_i$$

which still has two nonzero elements. Repeat the reasoning until only one nonzero vector remains or all the vectors are multiplied by 0.

We showed that  $\{S_i : i \leq n-1\}$  is a set of linearly independent vectors and thus every minor of  $A_D^V(1)$  has nonzero determinant. In particular,  $\det(A'_D)(1) \neq 0$ .  $\square$

The lemma 2.4 implies that image of  $A_D^V$  has dimension  $(n-1)$ . We will use this knowledge later on to construct the resolution of the Alexander module.

**Theorem 2.5.**

Up to multiplication by a unit, the  $(n-1)$  minor of  $A_D$  is independent of the choice of the diagram  $D$  of a knot  $K$ .

A proof of this statement using the Dehn presentation of knot group rather than the Wirtinger presentation is presented in [Ale28].

**Definition 2.6 : Alexander polynomial.**

Let  $p(t)$  be the determinant of any maximal minor of the Alexander matrix  $A_D$ . Then, we can find  $k \in \mathbb{Z}$  such that  $t^k p(t)$  is a symmetrical polynomial, meaning that  $t^k p(t) = (t^{-1})^k p(t^{-1})$  [Ale28]. The polynomial  $t^k p(t)$  is called the **Alexander polynomial** of the knot  $K$ .

The Alexander polynomial is a knot invariant as a consequence of theorem 2.5.

Lemma 2.4 gives information about (3), implicitly stated in proposi-

tion 2.7.

**Proposition 2.7.**

Let  $G$  be a knot group of  $K$  and  $F = R^{-1}R$  the field of fractions of ring  $R$  [AM69]. Then, changing coefficients by applying functor  $- \otimes_R F$  to the resolution (2) yields the following exact sequence

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R F = 0 \longrightarrow 0$$

where  $n$  is the number of crossings of the chosen diagram  $D$  of knot  $K$ .

**Proof.** We start by saying that  $R^n \otimes_R F \cong (R \otimes_R F)^n$  and  $R \otimes_R F \cong F$  [AM69, Proposition 2.14].

Lemma 2.4 implies that the Alexander matrix is surjective thus  $\dim(\ker(A_D)) = n - (n - 1) = 1$ . Therefore, (3) is isomorphic to the following exact sequence of vector spaces:

$$0 \longrightarrow F \longrightarrow F^n \xrightarrow{A_D^V} F^{n-1} \longrightarrow K_G^{ab} \otimes_R R^{-1}R = 0 \longrightarrow 0$$

□



### 3 Knot colorings

#### 3.1 Palettes and diagram colorings

An oriented diagram  $D$  of knot  $K$  has two types of crossings, pictured in fig. 3. A diagram coloring, in essence, is an assignment of values from some mathematical object (i.e.  $R$ -module) to segments of  $D$ .

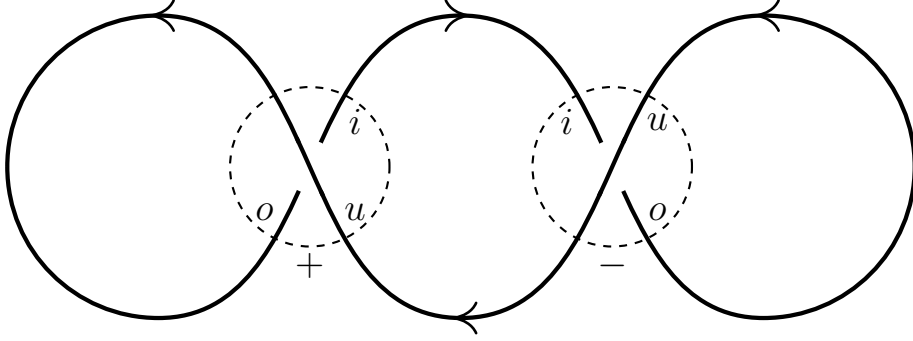


Figure 3: Two types of crossing in oriented diagram.

##### Definition 3.1 : palette.

We say that a quadruple  $(R, M, \mathcal{C}_\pm)$  is a **palette** if  $R$  is a commutative ring with unity,  $M$  an  $R$ -module and  $\mathcal{C}_\pm$  are two  $R$ -modules, corresponding to the two types of crossings (fig. 3) such that  $\mathcal{C}_\pm \subseteq M^3$ .

If a palette  $(R, M, \mathcal{C}_\pm)$  is given along with a ring homomorphism  $f : R \rightarrow S$ , then the image of this palette through induced palette homomorphism  $f_*$  is  $(S, M \otimes_R S, \mathcal{C}_\pm \otimes_R S)$ . Similarly, for an  $R$ -module homomorphism  $g : M \rightarrow N$  we write  $g_*(R, M, \mathcal{C}_\pm) = (R, N, g(\mathcal{C}_\pm))$ .

We will cumulatively call the two modules  $\mathcal{C}_\pm$  the **coloring rule** of palette  $(R, M, \mathcal{C}_\pm)$  as they determine whether a coloring is admissible.

##### Definition 3.2 : diagram coloring.

A **coloring of diagram**  $D$  with  $s$  segments and  $x$  crossings (for knots  $s = x$  lemma 2.1) is any element  $(m_1, \dots, m_s) \in M^s$  that assigns elements of  $M$  to each arc.

We will call a coloring **admissible** if for every crossing  $x_j$  of type  $\pm$  we have

$$\pi_{x_j}(m_1, \dots, m_s) \in \mathcal{C}_\pm,$$

where  $\pi_{x_j} : M^s \rightarrow M^3$  is a projection of module  $M^s$  to the  $M^3$  factor

that corresponds to segments that constitute  $x_j$ .

We can now define two module homomorphisms

$$\phi_{\pm} : M^3 \rightarrow M^3/\mathcal{C}_{\pm} = N_{\pm}$$

that take in as arguments the arcs constituting a crossing. Assuming that  $M^3/\mathcal{C}_{\pm} \cong M$  (reasoning behind this assumption will be given in section 3.2), we will take

$$\phi_+(u, i, o) = au + bi + co \quad (4)$$

$$\phi_-(u, i, o) = \alpha u + \beta i + \gamma o \quad (5)$$

for  $u, i, o$  understood like in fig. 3 and with coefficients being homomorphisms  $M \rightarrow M$ .

**Lemma 3.3.**

A coloring  $(m_1, \dots, m_s) \in M^s$  is a admissible  $\iff$  for each crossing  $x_j$  of type  $\pm$

$$\phi_{\pm}(\pi_{x_j}(m_1, \dots, m_s)) = 0.$$

*Proof.* Stems from the fact that  $\mathcal{C}_{\pm} = \ker \phi_{\pm}$ . □

## 3.2 Color checking matrix

Given a palette  $P = (R, M, \mathcal{C}_{\pm})$  and a diagram  $D$  with  $s$  segments and  $x$  crossings, a homomorphism  $D\phi : M^s \rightarrow N_{\pm}^x$  with  $\ker(D\phi)$  containing the admissible colorings, can be produced.

**Definition 3.4 : color checking matrix.**

Assigning segments of diagram  $D$  to coordinates in  $M^s$  and crossings to coordinates in  $N_{\pm}^x$  it is possible to define a linear homomorphism  $D\phi : M^s \rightarrow N_{\pm}^x$  as

$$D\phi(m_1, \dots, m_s) = (\phi_{\pm}(\pi_{x_1}(m_1, \dots, m_s)), \phi_{\pm}(\pi_{x_2}(m_1, \dots, m_s)), \dots).$$

Matrix that is created after choosing a basis for  $M^s$  and  $N_{\pm}^x$  will be called a **color checking matrix**.

**Proposition 3.5.**

Coloring  $(m_1, \dots, m_s) \in M^s$  is admissible  $\iff (m_1, \dots, m_s) \in \ker D\phi$ .

**Proof.** We start by saying that

$$(m_1, \dots, m_s) \in \ker D\phi \iff [(\forall x_j \text{ crossing}) \phi_{\pm}(\pi_{x_j}(m_1, \dots, m_s)) = 0].$$

meaning that every coordinate of  $D\phi(m_1, \dots, m_s)$  is zero. Lemma 3.3 says that it is equivalent with  $(m_1, \dots, m_s)$  being an admissible coloring.  $\square$

In the most basic sense, two diagrams  $D$  and  $D'$  are isomorphic if there exists an isotopy  $h_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $h_0(D) = D$  and  $h_1(D) = D'$  and  $D'$  has crossings identical to those of  $D$ .

Knot diagrams create a category with Reidemeister moves and the isotopy  $h_t$  supplying morphisms between objects. The assignment  $D \mapsto D\phi$  of a color checking matrix for a chosen palette is to be a functor from the category of diagrams to the category of color checking matrices. This means that if two diagrams were equivalent, then their color checking matrices must also be equivalent.

We say that two color checking matrices  $D\phi$  and  $D'\phi$  are equivalent if there exist isomorphisms  $\theta : M^s \rightarrow M^s$  and  $\psi : N^x \rightarrow N^x_{\pm}$  such that

$$\begin{array}{ccc} M^s & \xrightarrow{D\phi} & N^x \\ \theta \downarrow & & \downarrow \psi \\ M^s & \xrightarrow{D'\phi} & N^x \end{array}$$

is a commutative diagram. Additionally, we allow  $D\phi$  to differ from  $D'\phi$  by a identity matrix block.

**Lemma 3.6.**

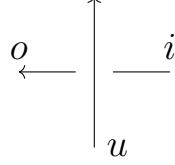
Isomorphic diagrams  $D \sim D'$  yield equivalent color checking matrices  $D\phi \sim D'\phi$ .

**Proof.** In terms of color checking matrices, an isomorphism of diagrams defined above only relabels segments (permutes columns) and crossings (permutes rows).  $\square$

### 3.3 Alexander palette

In this section we will define a palette for which the assignment  $D \mapsto D\phi$  yields a knot invariant.

Consider a crossing



and take some  $x$  to be the generator in the Wirtinger presentation of the knot group that is used to generate a representation for  $K_G^{ab}$  (see section 2.1). Then, the following is a relation in said group

$$UxCx(Ux)^{-1} = Ix$$

where  $U = ux^{-1}$ ,  $I = ix^{-1}$  and  $O = ox^{-1}$ . We can multiply both sides by  $x^{-1}$  to obtain

$$x^{-1}UxCU^{-1} = x^{-1}Ix$$

which is change in  $\mathbb{Z}[\mathbb{Z}]$  to

$$tU + C - U = tI \implies 0 = (1 - t)U + tI - C$$

The procedure for the other type of crossing is analogous and yields relation  $0 = (1 - t^{-1})U + t^{-1}I - C$ .

#### Definition 3.7 : Alexander palette.

A palette  $(\mathbf{R} = \mathbb{Z}[\mathbb{Z}], \mathbf{M} = \mathbb{Z}[\mathbb{Z}], \mathcal{C}_{\pm})$ , where  $\mathcal{C}_{\pm}$  are defined by homomorphisms

$$\phi_+(u, i, o) = (1 - t)u + ti - o$$

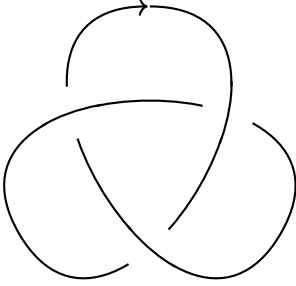
$$\phi_-(u, i, o) = (1 - t^{-1})u + t^{-1}i - o,$$

with  $u$ ,  $i$  and  $o$  defined in fig. 3, is called the **Alexander palette**.

The following example illustrates the importance of choosing a suitable palette.

**Example 3.8.** Consider a coloring of trefoil knot  $3_1$  with two palettes:  $P_1 = (\mathbb{Z}, \mathbb{Z}, \phi_{\pm}(u, i, o) = 2u - i + o)$  that is not an image of the Alexander palette and  $P_2 = (\mathbb{Z}, \mathbb{Z}, \phi_{\pm}(u, i, o) = 2u - i - o)$  which in turn is one (by

$\mathbb{Z}[\mathbb{Z}] \ni t \mapsto -1 \in \mathbb{Z}$ ). On the diagram below the color checking matrices of the diagram with 3 crossings along with example  $2 \times 2$  minor values (on the right of the matrix) of it are presented



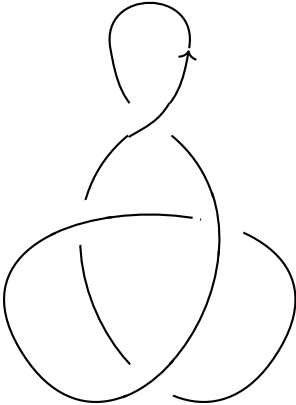
$$P_1(D) = \begin{bmatrix} \overset{1}{\boxed{\begin{matrix} 2 & -1 \\ -1 & 1 \end{matrix}}} & \begin{matrix} 1 \\ 2 \\ -1 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ -1 \end{matrix} & \boxed{\begin{matrix} 1 & 2 \\ 2 & -1 \end{matrix}} \end{bmatrix}$$

$-5$

$$P_2(D) = \begin{bmatrix} \overset{-3}{\boxed{\begin{matrix} 2 & -1 \\ -1 & -1 \end{matrix}}} & \begin{matrix} -1 \\ 2 \\ -1 \end{matrix} \\ \begin{matrix} -1 \\ 2 \\ -1 \end{matrix} & \boxed{\begin{matrix} -1 & 2 \\ 2 & -1 \end{matrix}} \end{bmatrix}$$

$-3$

while after the first Reidemeister move, the color checking matrices and their  $3 \times 3$  minors are



$$P_1(D) = \begin{bmatrix} \overset{5}{\boxed{\begin{matrix} 0 & 2 \\ 0 & -1 \end{matrix}}} & \overset{11}{\boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}}} \\ \overset{11}{\boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}}} & \boxed{\begin{matrix} 0 & 2 \\ 2 & 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 2 \\ 2 & 0 \end{matrix}} & \boxed{\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}} \end{bmatrix}$$

$5$

$$P_1(D) = \begin{bmatrix} \overset{-3}{\boxed{\begin{matrix} 0 & 2 \\ 0 & -1 \end{matrix}}} & \overset{3}{\boxed{\begin{matrix} -1 & -1 \\ 2 & 1 \end{matrix}}} \\ \overset{3}{\boxed{\begin{matrix} -1 & -1 \\ 2 & 1 \end{matrix}}} & \boxed{\begin{matrix} 0 & 2 \\ 2 & 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 2 \\ 2 & 0 \end{matrix}} & \boxed{\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}} \end{bmatrix}$$

$-3$

For the Alexander palette  $P_2$ , the minors are the same in both diagrams, while for the palette  $P_1$  there is no apparent relation between the minors.

In both palettes from the example 3.7.1 coloring rule modules  $\mathcal{C}_{\pm}$  are

isomorphic to  $M^2$  like in the diagram:

$$\begin{array}{ccccc}
 M^3 & \xrightleftharpoons{\quad} & M^2 & \xleftarrow[\cong]{\quad} & \mathcal{C}_{\pm} \\
 (u, i, o) & & (u, i) & & (u, i, \phi'_{\pm}(u, i))
 \end{array}$$

The red isomorphism  $(u, i) \mapsto (u, i, \phi'_{\pm}(u, i)) \in \mathcal{C}_{\pm}$ , with segments labeled like in fig. 3, means that  $c$  and  $\gamma$  in (4) and (5) respectively are units. For the sake of convenience, take  $c = \gamma = -1$ .

This property of palettes will be called a **propagation rule** as knowing colors of two of the three segments allows one to calculate the color assigned to the remaining segment.

**Lemma 3.9.**

If a palette has the propagation rule and the following two properties:

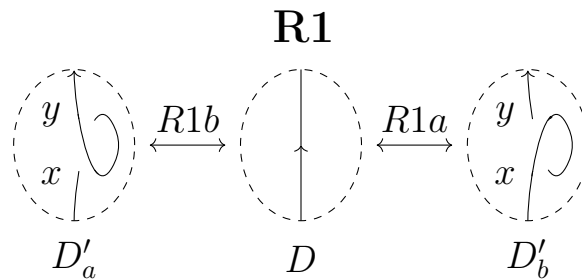
1.  $a = 1 - b$  and  $\alpha = 1 - \beta$
2.  $b$  and  $\beta$  are mutual inverses ( $b\beta = 1$ ) and  $\alpha\beta + \alpha = 0$

then equivalent diagrams yield equivalent matrices under the relation presented in the next section.

Proof of the lemma is divided into two parts, each given in the next session after defining the relevant relation induced by the Reidemeister moves on color checking matrices.

### 3.4 Reidemeister relation on color checking matrices

The diagram  $D$  has  $s$  segments and  $x$  crossings and  $D\phi : M^s \rightarrow N_{\pm}^x$ .



The first Reidemeister move allows the following two moves on color

checking matrices

$$\begin{matrix} D'_a & & D & & D'_b \\ \begin{bmatrix} b & a+c & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} & \xrightarrow{D(R1a)} & \begin{bmatrix} x_1+y_1 & z_1 & \dots \\ \vdots & & \ddots \end{bmatrix} & \xrightarrow{D(R1b)} & \begin{bmatrix} \beta & \alpha+\gamma & 0 & \dots \\ x_1 & y_1 & z_1 & \\ \vdots & & & \ddots \end{bmatrix} \end{matrix}$$

where  $(\forall i = 1, \dots, x) x_i = 0 \vee y_i = 0$ .

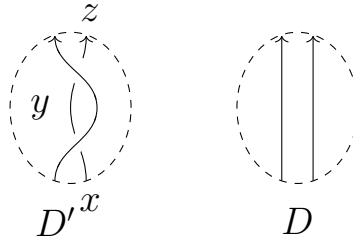
**Proof of lemma 3.8.1.** The propagation rule mandates the following equalities

$$0 = a + b + c = a + b - 1 \implies a = 1 - b$$

$$0 = \alpha + \beta + \gamma = \alpha + \beta - 1 \implies \alpha = 1 - \beta,$$

as the up and out segments in  $D'_a$  and the up and in segments in  $D'_b$  must admit coloring with the same element from  $M$ .  $\square$

## R2



For the second Reidemeister move we will say that  $D\phi$  and  $D'\phi$  are in relation if they differ by the following matrix move

$$\begin{matrix} D' & & D \\ \begin{bmatrix} b & c & 0 & a & \dots \\ 0 & \beta & \gamma & \alpha & \\ x_1 & 0 & z_1 & w_1 & \\ \vdots & & & & \ddots \end{bmatrix} & \xrightarrow{D(R2)} & \begin{bmatrix} x_1+z_1 & w_1 & \dots \\ \vdots & & \ddots \end{bmatrix} \end{matrix}$$

where  $(\forall i = 1, \dots, x) x_i = 0 \vee z_i = 0$ , in addition to permuting rows and columns and adding linear combination of rows or columns to another row or column.

**Proof of lemma 3.8.2.** In the case of this Reidemeister move, we would like to be able to color the diagram  $D'$  exactly like the diagram  $D$

save for the segments contributing to the two additional crossings. This means that segments labeled  $z$  and  $x$  on the diagram above must admit a coloring with the same element from  $M$ .

The restrictions stemming from this observation are more easily calculated if homomorphisms  $\phi_+$  and  $\phi_-$  are made into two matrices,  $A_+$  and  $A_-$ , that take the incoming segments (up and in segments) and return the output segments (out and up segments). This is possible because of the propagation rule.

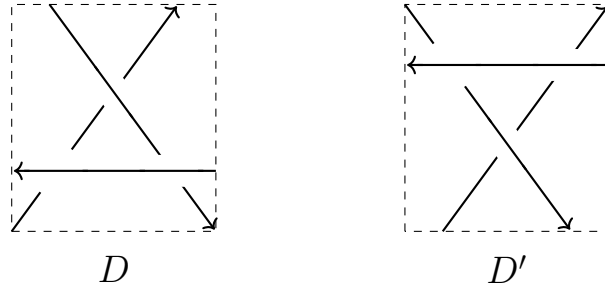
$$A_+ A_- \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} u \\ i \end{bmatrix}$$

Comparing the terms of the matrix  $A_+ A_-$  with terms of the identity matrix yields:

$$\begin{cases} a\beta + \alpha = 0 \\ \beta b = 1 \end{cases}$$

□

### R3



The last Reidemeister move does not change the size of matrices but only permutes the terms appearing in columns and rows corresponding to the three crossing that are manipulated in the diagram.

$$\begin{matrix} & D' \\ \begin{bmatrix} \alpha & \gamma & \beta & 0 & 0 & 0 & \dots \\ 0 & 0 & c & b & 0 & a \\ \beta & 0 & 0 & 0 & \gamma & \alpha \\ u_1 & 0 & v_1 & w_1 & x_4 & y_4 \\ \vdots & & & & & \ddots \end{bmatrix} & \xrightarrow{D(R3)} & \begin{matrix} D \\ \begin{bmatrix} 0 & 0 & \gamma & \beta & \alpha & 0 & \dots \\ \beta & 0 & 0 & 0 & \gamma & \alpha \\ 0 & c & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \\ \vdots & & & & & \ddots \end{bmatrix} \end{matrix} \end{matrix}$$



Let  $D(R)$  be the equivalence relation generated by moves  $D(R1a)$ ,  $D(R1b)$ ,  $D(R2)$  and  $D(R3)$ .

**Theorem 3.10.**

For a diagram  $D$  colored with an Alexander palette (or its image) the equivalence class of the color checking matrix  $D\phi$  under the equivalence relation  $D(R)$  is a knot invariant.

*Proof.* A direct result of the definition of the equivalence relation.  $\square$

Theorem 3.9 justifies defining  $K\phi := [D\phi]$ .

### 3.5 Smith normal form

The ring  $R$  of palette  $(R, M, \mathcal{C}_\pm)$  is not necessarily a PID ring, e.g.  $\mathbb{Z}[\mathbb{Z}]$  ring of the Alexander palette has ideal  $(2, t + 1)$  which is not principal. However, usually one can find a PID ring  $P$  with homomorphism  $R \rightarrow P$  which creates a new palette  $(P, M \otimes_R P, \mathcal{C}_\pm \otimes_R P)$  derived from  $(R, M, \mathcal{C}_\pm)$ . Matrices over PID rings have many interesting properties, like having a Smith normal form.

**Definition 3.11 : Smith normal form.**

Take  $A \in K\phi$  and consider it as an  $s \times x$  matrix with terms in a  $P$  by the procedure outlined above. Then there exist a  $s \times s$  matrix  $S$  and  $x \times x$  matrix  $T$  such that  $SAT$  is of form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & a_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & a_r & & & \\ 0 & & \dots & & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \dots & & 0 & \dots & 0 \end{bmatrix}$$

where for every  $i$   $a_i | a_{i+1}$ . Such a matrix  $SAT$  is called the **Smith normal form** of matrix  $A$ .

The following is an algorithm for computing the Smith normal form of a matrix  $A$ :

1. Let  $A = \{a_{i,j}\}_{i,j \leq n}$  be an  $n \times n$  matrix. Take the ideal  $I =$

$(a_{i,j})_{0 < i,j \leq n}$  generated by all the terms of  $A$ .

2. If we are in PID then  $I$  has one generator, call it  $a$ .
3. We can now use the following row and column operations to put  $a$  in the upper left corner of  $A$ 
  - (a) Permuting rows (columns).
  - (b) Adding a linear combination of rows (columns) to the remaining row (column).
4. With  $a$  in the upper left corner we can now use the fact that it was the generator of  $I$  to strike out the remaining terms on the first column and row, using the operations described in the previous point.
5. Repeat the same algorithm on the smaller matrix  $\{a_{i,j}\}_{1 < i,j \leq n}$ .

The terms  $a_1, \dots, a_k$  of the Smith normal form of a color checking matrix  $D\phi$  that are not units give  $\text{coker}(D\phi)$  as a module  $P/(a_1) \oplus P/(a_2) \oplus \dots \oplus P/(a_k)$ . In definition 2.3 the  $\text{coker}(A_D)$  of the Alexander matrix was equal to the Alexander module. The same invariant information can be obtained from the color checking matrix.

**Definition 3.12 : reduced normal form of matrix.**

Take  $A$  to be a matrix with coefficients in principal ideal domain  $P$ . Take  $a_1, \dots, a_k \in P$  to be all the elements of the Smith normal form of  $A$  that are neither zero nor invertible. Consider a new square matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & a_k \end{bmatrix}$$

which will be called the **reduced normal form** of matrix  $A$ .

Consider the following as a motivation behind definition 3.11.

**Example 3.13.** Consider the knots  $K11n85$  and  $K11n164$  pictured in figs. 4 and 5 respectively. They both have the Alexander polynomial equal

$$\Delta(t) = -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3}.$$

Coloring them with the Alexander palette yields two  $11 \times 11$  matrices

whose any  $10 \times 10$  minor is equal to the Alexander polynomial (up to multiplication by a unit). However, the reduced Smith normal forms are distinguishable

$$D_{11n85}\phi = [-t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3}]$$

$$D_{11n164}\phi = \begin{bmatrix} 1 - t + t^2 & 0 \\ 0 & -t^{-1} + 4 - 5t + 4t^2 - t^3 \end{bmatrix}$$

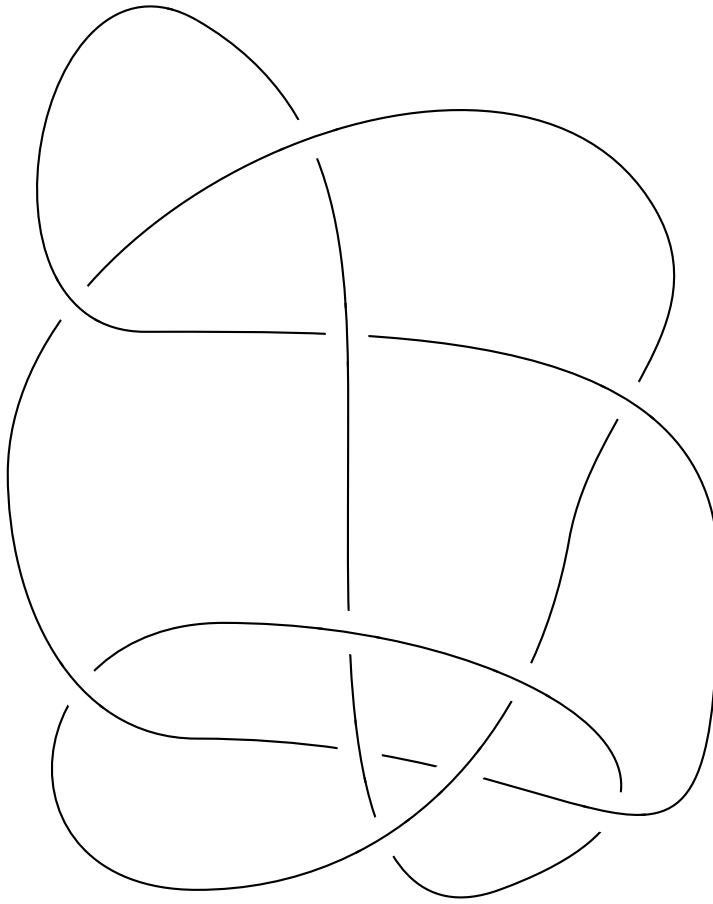


Figure 4: A diagram for knot  $K11n85$ .

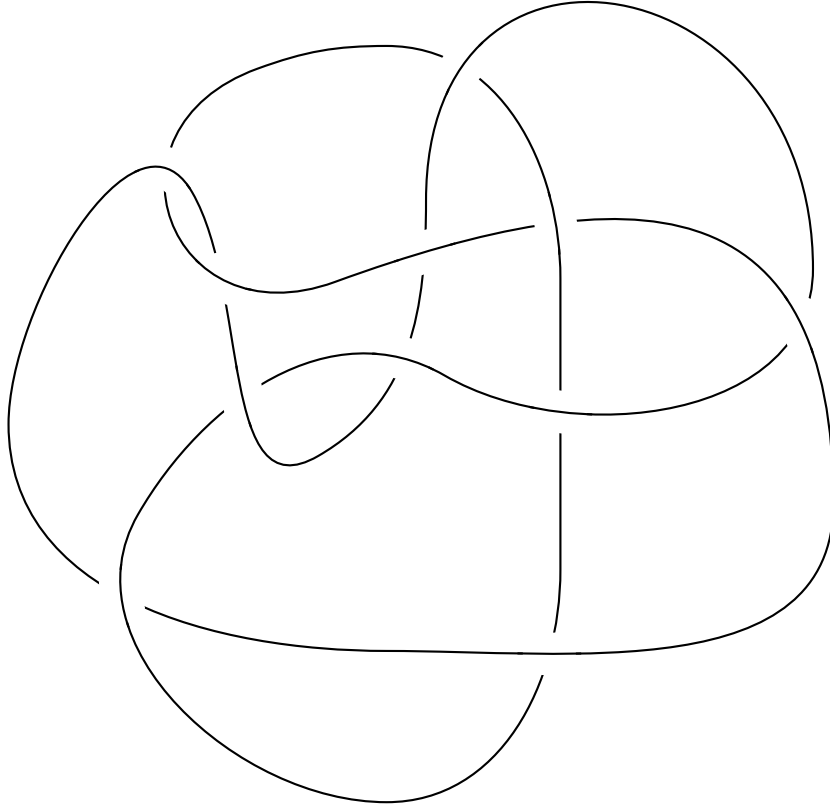


Figure 5: A diagram for knot  $K11n164$ .

**Theorem 3.14.**

The reduced normal form of color checking matrix using the Alexander palette does not depend on the choice of diagram  $D$  of a knot  $K$ . Thus, it is well defined for  $K\phi$  and is a knot invariant.

**Proof.** Take a knot  $K$  and its diagram  $D$  with  $s$  segments and  $x$  crossings ( $s = x$  by lemma 2.1). We will start by showing that applying any Reidemeister move to obtain a new diagram  $D'$  will not change the reduced normal form of its color checking matrix.

## R1

The first Reidemeister move is split into **R1a** and **R1b**. Due to those two cases being analogous, we will focus on the move **R1a**.

Take  $D'$  to be diagram  $D$  with one arc twisted into a  $+$  crossing. In the

previous section, due to the relation  $D(R1a)$ , the matrices  $D\phi$  and  $D'\phi$  are as follows

$$D'\phi = \begin{bmatrix} b & a-1 & 0 & \dots \\ x_2 & y_2 & \dots & \\ x_3 & y_3 & & \\ \vdots & & & \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_2 + y_2 & \dots \\ x_3 + y_3 & \\ \vdots & \end{bmatrix}$$

with  $(\forall i \geq 2) x_i = 0 \vee y_i = 0$ . Adding the first column of  $D'\phi$  to the second column will yield

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ x_2 & x_2 + y_2 & \dots & \\ x_3 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

because  $a + b = 1$ . Now we know that  $b$  is a unit, thus we can easily remove the elements of the first column that are not  $b$ . This results in

$$D'\phi = \begin{bmatrix} b & 0 & 0 & \dots \\ 0 & x_2 + y_2 & \dots & \\ 0 & x_3 + y_3 & & \\ \vdots & & & \end{bmatrix}$$

notice that the lower right portion of this matrix looks exactly like  $D\phi$ . The only difference is a column containing a singular unit element and thus it will be struck out when computing the reduced normal form. Therefore, the reduced normal form of  $D'\phi$  is the same as in  $D\phi$ .

## R2

Now the diagram  $D'$  is a diagram  $D$  with one arc poked onto another. Once again the color checking matrices are:

$$D'\phi = \begin{bmatrix} \alpha & \beta & -1 & 0 & \dots \\ a & 0 & b & -1 & \\ x_3 & u_3 & 0 & v_3 & \\ x_4 & u_4 & 0 & v_4 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$D\phi = \begin{bmatrix} x_3 & u_3 + v_3 & \dots \\ x_4 & u_4 + v_4 \\ \vdots \end{bmatrix}$$

with  $(\forall i \geq 3) u_i = 0 \vee v_i = 0$ . Adding the third column of  $D'\phi$  multiplied by  $\alpha$  and  $\beta$  to first and second column respectively we are able to reduce the first row to only zeros and  $-1$ . Now, adding this row to the second one creates a column with only  $-1$  and zeros. We can put it as the first column:

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & a + b\alpha & 0 & -1 \\ 0 & x_3 & u_3 & v_3 \\ 0 & x_4 & u_4 & v_4 \\ \vdots & & & & \ddots \end{bmatrix}$$

Notice that  $a + b\alpha = 0$  and so we can transform this matrix into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -1 & 0 \\ 0 & v_3 + u_3 & v_3 + u_3 & x_3 \\ 0 & v_4 + u_4 & v_4 + u_4 & x_4 \\ \vdots & & & & \ddots \end{bmatrix}$$

and then into

$$D'\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 \\ 0 & 0 & v_3 + u_3 & x_3 \\ 0 & 0 & v_4 + u_4 & x_4 \\ \vdots & & & & \ddots \end{bmatrix}$$

which obviously has the same reduced normal form as  $D\phi$ .

### R3

The last Reidemeister move creates the following two matrices

$$D\phi = \begin{bmatrix} \alpha & -1 & \beta & 0 & 0 & 0 \\ 0 & 0 & -1 & b & 0 & a \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} 0 & 0 & -1 & \beta & \alpha & 0 \\ \beta & 0 & 0 & 0 & -1 & \alpha \\ 0 & -1 & b & 0 & 0 & a \\ u_4 & 0 & v_4 & w_4 & x_4 & y_4 \end{bmatrix}$$

Applying row and column operations on those matrices results in

$$D\phi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

$$D'\phi = \begin{bmatrix} b & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -1 & 0 \\ 0 & 0 & u_4 + v_4 & w_4 + v_4 & x_4 - v_4 & y_4 + u_4 + x_4 \end{bmatrix}$$

which makes clear that those matrices have the same reduced normal form as  $b$  and  $\beta$  were taken to be units.

Notice that if  $A \sim B$  and  $B \sim C$ , where  $\sim$  means having the same reduced Smith normal form, then  $A \sim C$ . Thus, if two knots differ by a finite sequence of Reidemeister moves (as is the case for different diagrams of the same knot), then their reduced Smith normal forms are equal.

□

### 3.6 Color checking matrix and Alexander matrix

The Alexander matrix (definition 2.3) is derived from the presentation of the  $\mathbb{Z}[\mathbb{Z}]$ -module  $K_G^{ab}$ . We showed that the Alexander palette is also closely connected with the Wirtinger presentation of this module. In this section a connection between the Alexander matrix and the color checking matrix obtained by using the Alexander palette will be showed.

#### **Theorem 3.15.**

Let  $D\phi$  be the color checking matrix of diagram  $D$  with  $s = x = n$  crossings, colored with the Alexander palette. Then  $\ker(A_D) \cong \ker(D\phi)$ .

**Proof.** In the color checking matrix, every row has entries  $(1 - t)$ ,  $t$  and  $-1$  or  $(1 - t^{-1})$ ,  $t^{-1}$  and  $-1$ , while in the Alexander matrix every row has entries  $\pm t^k(1 - t)$ ,  $\pm t^{k+1}$  and  $\mp t^k$  or  $\pm t^k(1 - t^{-1})$ ,  $\pm t^{k-1}$  and  $\mp t^k$ . Let  $a_i$  be the  $i$ -th row of the color checking matrix, then  $\pm t^{k_i} \cdot a_i$  with one coordinate, call this coordinate  $j$ , struck out is the  $i$ -th row of the Alexander matrix.

For any  $x = (x_1, \dots, x_n) \in \mathbb{Z}[\mathbb{Z}]^n$  its image through the color checking matrix is

$$\begin{bmatrix} \langle x, a_1 \rangle \\ \vdots \\ \langle x, a_n \rangle \end{bmatrix} \in \mathbb{Z}[\mathbb{Z}]^n,$$

where  $\langle x, a_i \rangle = x^T a_i$ . The Alexander matrix will produce

$$\begin{bmatrix} \pm t^{k_i} \langle x, a_1 \rangle \\ \vdots \\ \pm t^{k_{j-1}} \langle x, a_{j-1} \rangle \\ \pm t^{k_{j+1}} \langle x, a_{j+1} \rangle \\ \vdots \\ \pm t^{k_n} \langle x, a_n \rangle \end{bmatrix} \in \mathbb{Z}[\mathbb{Z}]^{n-1}.$$

From this inclusion  $\ker(D\phi) \subseteq \ker(A_D)$  is obtained.

In section 2.1 the choice of the segment in transition from presentation of  $G$  to presentation of  $K_G^{ab}$  determined which coordinate  $j$  is struck out in  $K_G^{ab}$ . The choice of this segment does not impact kernel of  $A_D$  and only changes exponents  $k_i$  to  $k'_i$ . Hence, if  $j' \neq j$  is a new segment struck out, then  $x \in \ker(A_D) \iff \langle x, a_i \rangle = 0$  for all  $i \neq j'$ , in particular for  $i = j$ , which was not known before the change of  $j$ . Hence  $\ker(A_D) \subseteq \ker(D\phi)$ .  $\square$

Theorem 3.13 combined with the first isomorphism theorem for modules yields

$$\text{im}(A_D) \cong \mathbb{Z}[\mathbb{Z}]^n / \ker(A_D) \cong \mathbb{Z}[\mathbb{Z}]^n / \ker(D\phi) \cong \text{im}(D\phi).$$

In lemma 2.4 it was shown that  $\text{im}(A_D) \cong \mathbb{Z}[\mathbb{Z}]^{n-1}$ . Thus, the following



is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(A_D) & \longrightarrow & \mathbb{Z}[\mathbb{Z}]^n & \xrightarrow{A_D} & \mathbb{Z}[\mathbb{Z}]^{n-1} \\
& & \downarrow \cong & & \downarrow id & & \downarrow \cong \\
0 & \longrightarrow & \ker(D\phi) & \longrightarrow & \mathbb{Z}[\mathbb{Z}]^n & \xrightarrow{D\phi} & \text{im}(D\phi) \longrightarrow \mathbb{Z}[\mathbb{Z}]^n
\end{array}$$

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