Algebra 2 R, list 4.

Homework: arbitrary three, the usual rules apply. The problems marked with ”–” are excluded from homework and are not discussed during problems sessions unless students ask. $K \subset L \subset M$ denotes field extensions.

1. Calculate cyclotomic polynomials
   $F_1(X), F_2(X), F_4(X),\ F_8(X), F_{16}(X), F_{15}(X)$, and then calculate their images in te ring $\mathbb{Z}_3[X]$, under the homomorphism $\mathbb{Z}[X] \to \mathbb{Z}_3[X]$ induced by the quotient homomorphism $\mathbb{Z} \mapsto \mathbb{Z}_3$. Which of them are indecomposable over $\mathbb{Z}_3$?

2. Describe the normal closures of the following field extensions:
   (a) $\mathbb{Q}[\sqrt[n]{2}] \supset \mathbb{Q}$,
   (b) $\mathbb{Q}(\sqrt[n]{X}) \supset \mathbb{Q}(X)$,
   (c) $\mathbb{C}(\sqrt[n]{X}) \supset \mathbb{C}(X)$,
   (d) $\mathbb{Q}[\zeta] \supset \mathbb{Q}$, where ζ is a primitive root of 1 of degree $n > 1$.
   (hint: in (a)–(c) find the minimal polynomial, in (c) use the fact that C is algebraically closed. In (b) notice that X may be replaced by any transcendental number, this is not necessary, but it helps.)

3. Prove that every field extension of degree 2 is normal.

4. Assume that the field extension $K \subset L$ is algebraic and $f : L \to L$ is a monomorphism, $f|_K = id$. Prove that f is “onto”.

5. Prove that if $K \subset L \subset \hat{K}$ and the extension $K \subset L$ jest radical, then $G(\hat{K}/K) = G(\hat{K}/L)$.

6. – Assume that $char(K) = p > 0$ and $W(X) \in K[X]$ is irreducible and not separable. Prove that $W(X) \in K[X^p]$. (Hint: consider $W(X)$ the minimal polynomial of an $a \in \hat{K}$ such that a is a multiple root of W. Prove that $W'(X)$ is a zero polynomial.)

7. Assume that $char(K) = p > 0$ and $a \in \hat{K}$ is separable over K. Prove that $K(a) = K(a^p)$. (Hint: consider the minimal polynomial of a over K.)

8. (a) Prove that if $a \in L$ is radical over K, then $\deg(a/K) = \min\{p^n : a^{p^n} \in K\}$.
   (b) – Conclude that if a finite extension $K \subset L$ is radical, then its degree is a power of p (here $p = char(K)$).

9. Assume that $K \subseteq L, M \subseteq \hat{K}$ are field extensions such that $L \cap M = K$. Prove that if
   $(\forall K \subseteq_{\text{finite}} L_0 \subset L)(\forall K \subseteq_{\text{finite}} M_0 \subseteq M)[L_0(M_0) : L_0] = [M_0 : K]$,
   then $[L(M) : L] = [M : K]$.

10. Prove Remark 7.5(1) in the general case, i.e. when $[L : K]$ is infinite.

11. Assume that the numbers $m, n > 1$ are relatively prime and $\zeta_n, \zeta_m \in \mathbb{C}$ are primitive roots of 1 of degree n, m respectively. Prove that $\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$. (Hint: notice that $\mathbb{Q}(\zeta_n, \zeta_m) = \mathbb{Q}(\zeta_{mn})$. Rely on the fact $\varphi(mn) = \varphi(m)\varphi(n)$ for any co-prime m, n (without proof).)

12. * Prove that if $K \subset L \subset \hat{K}$ and every polynomial over K of degree > 0 has a root in L, then $L = \hat{K}$. (Hint: First consider the separable case, use the Abel's theorem on primitive element.)