Algebra 2R, list 3.

Homework : any 3 problems from the list. An item counts as a separate problem. ≤ 1 item from each problem.

1. Let K be a field .
   a) Prove that the field extension $L \supset K$ is transcendental, where $L = K(X)$ is the field of rational functions in varriable X over K.
   b) Let $M = L[\sqrt{X}]$ be an algebraic extension of the field L by an element $Y = \sqrt{X}$ such that $Y^2 - X = 0$ in the field M. Prove that M i L are isomorphic over K.

2. Let K be a field.
   a) Let $g \in K(X) \setminus K$. Prove that X is algebraic over the field $K(g)$. In particular $[K(X) : K(g)] < \infty$. What is the degree of this extension ?
   b) For g as in (c) prove that $K(g)$ isi somorphic with $K(X)$, over K.

3. Let $v_1, \ldots, v_n$ be vertices of a regular n-gon inscribed in a circle on the plane $\mathbb{R}^2$ with equation $x^2 + y^2 = 1$. What is the linear dimension over $\mathbb{Q}$ of the system of vectors $v_1, \ldots, v_n$?

4. Assume that $K \supset F(p)$ is a finite field extension of $F(p)$. Assume that $a \in K$ is a primitive root of 1 of degree m. Let n be the smallest natural number > 0 such that $m \mid p^n - 1$.
   (a) Prove that n equals the degree of a over $F(p)$.
   (b) Prove that $n \mid \varphi(m)$. Give an example where $n < \varphi(m)$.

5. (a) Prove that $j(F_m(X))$ need not be irreducible over $F(p)$. ($j : \mathbb{Z} \to \mathbb{Z}_p$ is the quotient map. Hint: use the previous problem)
   (b) Prove that if $k, l \in N^+$ are co-prime, then $k \mid l^{\varphi(k)} - 1$. (hint: consider the ring $\mathbb{Z}_k$)

6. Find the minimal polynomials over $\mathbb{Q}$ for the following numbers:
   (a) $\sqrt{2} + \sqrt{3}$, (b) $1 + \sqrt{5} - \sqrt{3}$, (c) $1 + \sqrt[3]{17}$.

7. Prove (using Liouville Lemma) that the number
   $$\sum_{n=1}^{\infty} \frac{1}{2^{n!}}$$
   is transcendental. (the real numbers, whose transcendence follows from Liouville Lemma, are called Liouville numbers)

8. Assume that $M \supset K$ is an algebraic field extension and L_1, L_2 are intermediate fields (that is: $K \subset L_1, L_2 \subset M$). As usual, $L_1[L_2] = L_2[L_1]$ denotes the subring of M generated by $L_1 \cup L_2$. Prove that
   (a) $L_1[L_2]$ is a subfield of M (denoted by L_1L_2),
   (b) $[L_1L_2 : K] \leq [L_1 : K] \cdot [L_2 : K]$
   (c)* Assume $L_1 \cap L_2 = K$. Do we have equality in (b) then?