Algebra 2R

Problem List 1

Weronika Jakimowicz

EXERCISE 1.

Proof that $\mathbb{C} = \mathbb{R}[z]$ *for every complex number* $z \in \mathbb{C} \setminus \mathbb{R}$.

To begin with, let us take any $z \in \mathbb{C} \setminus \mathbb{R}$ such that z = ai for some $a \in \mathbb{R}$. We have that

$$\mathbb{R}[z] = \{f(z) : f \in \mathbb{R}[X]\}.$$

Let $I = (X^2 + a^2) \triangleleft \mathbb{R}[X]$ be an ideal of $\mathbb{R}[X]$ generated by a polynomial with no real roots. We know that $\mathbb{R}[X]/I \cong \mathbb{C}$.

This is because $\mathbb R$ is a field and so $\mathbb R[X]$ is an euclidean domain: if we take any $f \in \mathbb R[X]$ then we can write it as $f = v(X^2 + a^2) + w$, where w is of degree 0 or 1 (< def($X^2 + a^2$)) and so f in $\mathbb R[X]/I$ is represented only by w. Now it is quite easy to map polynomials with real coefficients and maximal degree 1 to $\mathbb C$, for example $f: \mathbb R[X]/I \to \mathbb C$ such that f(aX + b) = ai + b. Therefore $\mathbb R[X]/I \cong \mathbb C$.

Consider the evaluation homomorphism ϕ_z which maps $\mathbb{R}[X] \ni w \mapsto w(z) \in \mathbb{R}[z]$. We can see that $\ker(\phi_z) = (X^2 + a^2) = I$. Therefore, by the fundamental theorem on ring homomorphism we have an isomorphism

$$f: Im(\phi_z) = \mathbb{R}[z] \to \mathbb{R}[X]/ker(\phi_z) = \mathbb{R}[X]/I$$

and as mentioned above, $\mathbb{R}[X]/I \cong \mathbb{C}$. Hence, $\mathbb{R}[z] \cong \mathbb{C}$.

EXERCISE 2.

Assume that $K \subset L$ are fields and $a, b \in L$. For a rational function $f(X) \in K(X)$ define f(a) as $\frac{g(a)}{h(a)}$, where $g, h \in K[X]$, $f = \frac{g}{h}$ and $h(a) \neq 0$, provided such g, h exist. If not, f(a) is undetermined. Prove that

I know I shouldn't do this but I wanted to know if the diagram I drew in (c) is a correct solution. If not I have (a) as a more reasonable backup to get at least some points c:

(a) if $f(X) \in K(X)$ and f(a) is defined, then f(a) is determined uniquely (does not depend on the choice of g,h)

Suppose by contradiction that f(a) depends on which g, h we choose. That means that there exist g, h, g', $h' \in K[X]$, $h(a) \neq 0$, h'(a) such that $f = \frac{g}{h} = \frac{g'}{h'}$ but $\frac{g(a)}{h(a)} + c = \frac{g'(a)}{h'(a)}$, where $c \in L \setminus \{0\}$.

From $f = \frac{g}{h} = \frac{g'}{h'}$ we get that $g \cdot h' = g' \cdot h$ and in particular

$$(gh')(a) = (g'h)(a)$$

$$g(a)h'(a) = g'(a)h(a)$$

$$g(a)h'(a) - g'(a)h(a) = 0$$

From the assumption that f(a) depends on the choice of polynomials we get that

$$\frac{g'(a)}{h'(a)} = \frac{g(a)}{h(a)} + c$$

$$g'(a)h(a) = g(a)h'(a) + ch'(a)$$

$$g'(a)h(a) - g(a)h'(a) = ch'(a) \neq 0$$

Which is a contradiction because $c \neq 0$, $h'(a) \neq 0$ and we have no zero divisors.

(c)
$$K(a, b) = (K(a))(b)$$

Let

$$I_{ab} = I((a, b)/K[x, y])$$

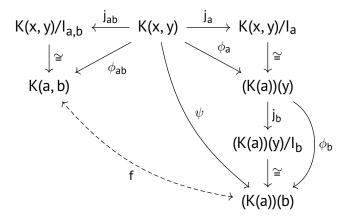
$$I_{a} = I(a/(K[y])[x])$$

$$I_{b} = I(b/(K(a))(y))$$

and j_a, j_b, j_{ab} are quotient functions defined as below. We know that $ker(j_a) = I_a$, $ker(j_b) = I_b$ and $ker(j_{ab}) = I_{ab}$. Let ϕ_a be an evaluation function that substitutes only one variable:

$$\phi_{\mathsf{a}}:\mathsf{K}(\mathsf{x},\mathsf{y})\to(\mathsf{K}(\mathsf{a}))(\mathsf{y})$$
 $\phi_{\mathsf{a}}(\mathsf{f}(\mathsf{x},\mathsf{y}))=\mathsf{f}(\mathsf{a},\mathsf{y})$

that is ϕ_a returns a rational function with changed coefficients. ϕ_b, ϕ_{ab} are defined as evaluation functions without such modifications.



Function ψ is a ring homomorphism defined as composition of ϕ_a and ϕ_b :

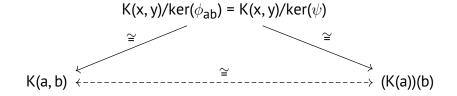
$$\psi : \mathsf{K}(\mathsf{x}, \mathsf{y}) \to (\mathsf{K}(\mathsf{a}))(\mathsf{y})$$

$$\psi = \phi_\mathsf{h} \circ \phi_\mathsf{a}$$

For f to be an isomorphism

$$f: (K(a))(b) \rightarrow K(a, b)$$

we need to show that $ker(\phi_{ab}) = ker(\psi)$ because then



 $ker(\phi_{ab}) = ker(\psi)$

 \subset

 $f \in \ker(\phi_{ab})$ means that f(a, b) = 0. That is, either of the following is true for any $x, y \in K$

f(a, b) = 0 this directly implies that $f \in ker(\psi)$.

f(a, y) = 0 the same as above.

f(x, b) = 0 we know that for any $x \in K$ f(x, b) = 0 then for x = a this is also true and so f(a, b) = 0 and $f \in ker(\psi)$.

 \supseteq

 $f \in \ker(\psi)$ means that f(a, b) = 0 or f(a, y) = 0. This means that $f \in \ker(\phi_{ab})$.

Therefore, there exists an isomorphism $K(a, b) \cong (K(a))(b)$.

EXERCISE 3.

Assume that $K \subseteq L$ are fields and $f_1, ..., f_m \in K[X_1, ..., X_n]$ have degree 1.

(a) Prove that if the system of equations $f_1 = ... = f_m = 0$ has a solution in L then it has a solution in K. (hint: use linear algebra).

We are working on linear equations, therefore we can construct a matrix that stores the same information as the system of equations $f_1 = ... = f_m$. Let

$$f_i = \sum_{1 \le k \le n} b_{i,k} X_k + c_i$$

for i = 1, ..., m. The matrix representation of this system of equations is:

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{m,1} & b_{m,2} & b_{m,3} & \dots & b_{m,n-1} & b_{m,n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \dots \\ c_m \end{bmatrix}.$$

Using Gaussian algorithm, we can create an upper triangular matrix with coefficients that are linear combinations of elements from K and thus are themselves in K.

If $m \le n$, then let

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \dots & \dots & \alpha_{1,n-1} & \alpha_{1,n} \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \dots & \dots & \dots & \alpha_{2,n-1} & \alpha_{2,n} \\ \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_{m,m} & \dots & \alpha_{m,n-1} & \alpha_{m,n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \alpha_3 \\ \dots \\ \gamma_m \end{bmatrix}$$

be the result of Gaussian elimination of the matrix above. Because Gaussian elimination returns a matrix with elements that are linear combinations of the elements of the original matrix, we have that $b_{i,k}, \gamma_i \in K$.

The solution would be found by backwards substitution. We could take $a_{m+1}, a_{m+2}, ..., a_n = 0 \in K$ then

$$\gamma_{\text{m}}$$
 = $\alpha_{\text{m,m}}$ a_m + $\alpha_{\text{m,m+1}}$ a_{m+1} + ... + $\alpha_{\text{m,n}}$ a_n = $\alpha_{\text{m,m}}$ a_m

$$a_{\text{m}} = (\alpha_{\text{m,m}})^{-1} \gamma_{\text{m}} \in K$$

Then

$$\begin{split} \gamma_{\mathsf{m}-1} &= \alpha_{\mathsf{m}-1,\mathsf{m}-1} \mathsf{a}_{\mathsf{m}-1} + \alpha_{\mathsf{m}-1,\mathsf{m}} \mathsf{a}_{\mathsf{m}} + \alpha_{\mathsf{m}-1,\mathsf{m}+1} \mathsf{a}_{\mathsf{m}+1} + \ldots + \alpha_{\mathsf{m}-1,\mathsf{n}} \mathsf{a}_{\mathsf{n}} = \\ &= \alpha_{\mathsf{m}-1,\mathsf{m}} \mathsf{a}_{\mathsf{m}-1} + \alpha_{\mathsf{m}-1,\mathsf{m}} (\alpha_{\mathsf{m},\mathsf{m}})^{-1} \gamma_{\mathsf{m}} \\ \\ \mathsf{a}_{\mathsf{m}-1} &= (\alpha_{\mathsf{m}-1,\mathsf{m}})^{-1} (\gamma_{\mathsf{m}-1} - \alpha_{\mathsf{m}-1,\mathsf{m}} (\alpha_{\mathsf{m},\mathsf{m}})^{-1} \gamma_{\mathsf{m}}) \in \mathsf{K} \end{split}$$

And so on. We know from linear algebra that this will work.

If m > n, then the upper triangular matrix would look like this:

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \dots & \alpha_{1,n-1} & \alpha_{1,n} \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \dots & \dots & \alpha_{2,n-1} & \alpha_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & \alpha_{m,m} \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \dots \\ \gamma_n \end{bmatrix}$$

and such a matrix can be treated the same way as before with the condition that for i > m γ_i = 0. Otherwise no solutions exist.