Functions

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The modern concept of functions $f : \mathbb{R} \to \mathbb{R}$ has a surprisingly complicated history and grew mostly out of an attempt to solve various problems arising from earlier concepts.

One such problem is the following related to differentiability:

Question 1. Are all functions $f : \mathbb{R} \to \mathbb{R}$ differentiable?

The answer here is obviously no, as seen by the following example:

$$f(x) := |x| := \begin{cases} x+1, \text{ if } x \leq 0 \\ x, \text{ if } x < 0 \end{cases}$$

Obviously, the only problem here is the point $x = 0$. So one might ask the following question:

Question 2. Are all functions f differentiable at all but finitely many points?

Again, the answer is no, as seen by the following example:

$$f(x) := \chi_{\mathbb{Q}}(x) := \begin{cases} 1, \text{ if } x \in \mathbb{Q} \\ 0, \text{ if } x \notin \mathbb{Q} \end{cases}$$

Clearly, this function is discountinuous at all points though and hence one can ask the following even more precise question:

Question 3. *Are all continuous functions differentiable everywhere but a countable number of points?*

This however is also wrong:

Theorem 4. *There are functions $f : \mathbb{R} \to \mathbb{R}$ which are everywhere continuous but nowhere differentiable.*

Examples like the one in Theorem 4 were found by Karl Weierstraß pictured in Figure 1 contemplating how to destroy 19th century real Analysis.

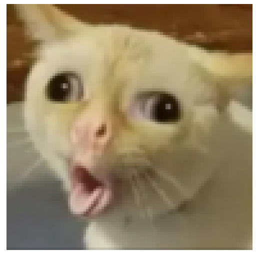

Figure 1: An utter madman!

$$
\begin{array}{||c|c|c||}
\hline\hline
f(x) := & f'(x) & \int f(t)dt \\
\hline
x & 1 & \frac{x^2}{2} \\
\hline
e^x & e^x & e^x \\
\hline
\sin(x) & \cos(x) & -\cos(x) \\
\hline
x^{-1} & -x^{-2} & \log(x) \\
\hline\hline
\end{array}
\tag{1}
$$

We want to note differentials and integral of some well-known functions in (1). Summarising some of the examples above we obtain in Table 1 below. One can reasonably ask what the issue with the integrability of $\chi_{\mathbb{Q}}$ is. We shall return to this later.

$f(x) :=$	continuous	differentiable	integrable
x	YES	YES	YES
$\lvert x\rvert$	YES	Everywhere but 0	YES
$\chi_{\mathbb{Q}}(x)$	Nowhere continuous or differentiable		Ehhh...
Weierstraß functions	YES	NO	YES

Table 1: This whole mess!

(Remark: Question 3 must not be italicised by using a font-environment like *emph*.)