Algebra 2R

Problem List 1

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EXERCISE 1.

Proof that $\mathbb{C} = \mathbb{R}[z]$ *for every complex number* $z \in \mathbb{C} \setminus \mathbb{R}$.

To begin with, let us take any $z \in \mathbb{C} \setminus \mathbb{R}$ such that $z = ai$ for some $a \in \mathbb{R}$. We have that

$$\mathbb{R}[z] = \{f(z) \ : \ f \in \mathbb{R}[X]\}.$$

Let $I = (X^2 + a^2) \triangleleft \mathbb{R}[X]$ be an ideal of $\mathbb{R}[X]$ generated by a polynomial with no real roots. We know that $\mathbb{R}[X]/I \cong \mathbb{C}$.

This is because $\mathbb{R}$ is a field and so $\mathbb{R}[X]$ is an euclidean domain: if we take any $f \in \mathbb{R}[X]$ then we can write it as $f = v(X^2 + a^2) + w$, where w is of degree 0 or 1 ($< def(X^2 + a^2)$) and so f in $\mathbb{R}[X]/I$ is represented only by w. Now it is quite easy to map polynomials with real coefficients and maximal degree 1 to $\mathbb{C}$, for example $f : \mathbb{R}[X]/I \to \mathbb{C}$ such that $f(aX + b) = ai + b$. Therefore $\mathbb{R}[X]/I \cong \mathbb{C}$.

Consider the evaluation homomorphism ϕ_z which maps $\mathbb{R}[X] \ni w \mapsto w(z) \in \mathbb{R}[z]$. We can see that $\ker(\phi_z) = (X^2 + a^2) = I$. Therefore, by the fundamental theorem on ring homomorphism we have an isomorphism

$$f : Im(\phi_z) = \mathbb{R}[z] \to \mathbb{R}[X]/\ker(\phi_z) = \mathbb{R}[X]/I$$

and as mentioned above, $\mathbb{R}[X]/I \cong \mathbb{C}$. Hence, $\mathbb{R}[z] \cong \mathbb{C}$.

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EXERCISE 2.

Assume that $K \subset L$ *are fields and* $a, b \in L$. *For a rational function* $f(X) \in K(X)$ *define* $f(a)$ *as* $\frac{g(a)}{h(a)}$, *where* $g, h \in K[X]$, $f = \frac{g}{h}$ *and* $h(a) \neq 0$, *provided such* g, h *exist. If not,* $f(a)$ *is undetermined. Prove that*

(a) if $f(X) \in K(X)$ *and* $f(a)$ *is defined, then* $f(a)$ *is determined uniquely (does not depend on the choice of* g, h*)*

Let $f = \frac{g}{h}$ for some $g, h \in K[X]$.

(b) $K(a) = \{f(a) \ : \ f \in K(X)$ *i* $f(a)$ *jest określone*$\}$

We know that $K(a)$ is a subset of L that is generated by $K \cup \{a\}$.

EXERCISE 3.

Assume that $K \subseteq L$ *are fields and* $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ *have degree* 1.

(a) Prove that if the system of equations $f_1 = \dots = f_m = 0$ *has a solution in* L *then it has a solution in* K. *(hint: use linear algebra).*

Let

$$f_i = \sum_{1 \le k \le n} b_{i,k} X_k$$

for $i = 1, \dots, m$.

We are working on linear equations, therefore we can construct a matrix that stores the same information as the system of equations $f_1 = \dots = f_m$. Let

$$f_i = \sum_{1 \le k \le n} b_{i,k} X_k$$

for $i = 1, \dots, m$. The matrix representation of this system of equations is:

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{m,1} & b_{m,2} & b_{m,3} & \dots & b_{m,n-1} & b_{1,n} \end{bmatrix} X = 0.$$

Using Gaussian algorithm, we can create an upper triangular matrix with coefficients from K. The solution would be found by backwards substitution. That is, a_n would be in the bottom right corner of the matrix and it is an element of K because such are the coefficients within my matrix. Then a_{n-1} would be a combination of a_n with two elements of K, hence it would still be in K and so on. Each a_i would be a linear combination of elements from K and a_k, $k < i$, which we know are in K.

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EXERCISE 8.

Prove that the set $\{\sqrt{p} \ : \ p$ *is a prime number*$\}$ *is linearly independent over the field* $\mathbb{Q}$.

Assume that the set $S = \{\sqrt{p} \ : \ p \text{ is a prime number}\}$ is not linearly independent. That means that there is a sequence $p_1, \dots, p_n$ of prime numbers and $a_1, \dots, a_n \in \mathbb{Q}$ such that

$$\sum_{1 \le k \le n} a_k \sqrt{p_j} = 0$$