Algebra 2R, list 6.

Homework: as usual. Problems/items marked with – are excluded from homework.

1. Prove that in the definition of field extension by radicals (before Thm. 9.6) we may assume that $L_k \subset L_0$ is Galois[1].

2. – Assume that G is a group and $H \triangleleft G$. Prove that if H and G/H are solvable, then G is solvable.

3. Find $a \in L$ such that $L = \mathbb{Q}(a)$, where L is the splitting field of polynomial:
   (a) $X^3 - 3$,
   (b) $(X^3 - 3)(X^2 - 2)$.

4. Find $a \in L$ such that $L = K(a)$, where
   (a) $K = \mathbb{Q}$ and L is the splitting field of polynomial $X^4 - 2$ over K
   (b) $K = \mathbb{Q}(i)$ and L is the slitting field of polynomial $X^4 - 2$ over K

5. Let $K = F_p(X^p, Y^p)$ i $L = F_p(X, Y)$. Prove that there is no $a \in L$ such that $L = K(a)$.

6. Describe $G(L/K)$ where:
   (a) $K = \mathbb{C}(X^4), L = \mathbb{C}(X)$,
   (b) $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt[4]{2}, i)$.

7. Describe $G(L/K)$ where:
   (a) $K = \mathbb{Q}$ and L is the splitting field of polynomial $(X^3 - 3)(X^2 - 2)$ over K,
   (b) $K = \mathbb{Q}$ and L is the splitting field of polynomial $(X^3 - 3)(X^3 - 2)$ over K.
   Comment: In this and the previous problem "describe $G(L/K)$" means: describe the algebraic structure of this group.

8. Find (pointing generators over $\mathbb{Q}$) all intermediate fields
   (a) between $\mathbb{Q}$ and $Q(\sqrt{5}, \sqrt{7})$
   (b) between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$.

[1] Wlog $L_0 \subset \widehat{L_k}$. Let L' be the normal closure of L_0 in $\widehat{L_k}$, over L_k. Then L' is a composition of fields $f_0[L_0], \ldots, f_m[L_0]$ for some finitely many $f_0, \ldots, f_m \in Gal(\widehat{L_k}/L_k)$, where $f_0 = id$. Using the fields $f_i[L_j]$ (in proper order) extend the sequence $L_k \subset L_{k-1} \subset \ldots \subset L_0$ so that it ends with L'.