Problem List 3

Algebra 2r

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Exercise 3. *Let $v_1, \ldots, v_n$ be vertices of a regular n-gon inscribed in a circle on the plane $\mathbb{R}^2$ with equation $x^2 + y^2 = 1$. What is the linear dimension over $\mathbb{Q}$ of the system of vectors $v_1, \ldots, v_n$.*

Without the loss of generality, I will consider polygons with one vertex in $(1, 0)$. Then, the remaining vertices are in $(\cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n})$, for $k = 1, \ldots, n-1$. Now, let me switch where I live and let us consider roots of

$$x^n - 1.$$

We have n roots $z_1, \ldots, z_n$ in $\mathbb{C}$. Notice, that $z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ and adding complex numbers works almost like adding vectors in 2D. The minimal polynomial over $\mathbb{Q}$ of each of z_k is $F_n(x)$. Therefore, $\dim(v_1, \ldots, v_n) = \dim(z_1, \ldots, z_n) = \phi(n)$, where ϕ is Euler's function.

Exercise 4. *Assume that $K \supseteq F(p)$ is a finite field extension of $F(p)$. Assume that $a \in K$ is a primitive root of 1 of degree m. Let n be the smallest natural number > 0 such that $m | p^n - 1$.*

(a) Prove that n equals the degree of a over $F(p)$

(b) Prove that $n | \phi(m)$. Give an example where $n < \phi(m)$.

Well, I think I kinda showed it before XD

Exercise 6. *Find the minimal polynomials over $\mathbb{Q}$ fot the following numbers:*

(a) $\sqrt{2} + \sqrt{3}$

$$\begin{aligned}
x - (\sqrt{2} + \sqrt{3}) &= 0 \\
x - \sqrt{2} &= \sqrt{3} \\
(x - \sqrt{2})^2 &= 3 \\
x^2 - x2\sqrt{2} + 2 &= 3 \\
x^2 - 1 &= x2\sqrt{2} \\
(x^2 - 1)^2 &= 8x^2 \\
x^4 - 2x^2 + 1 &= 8x^2 \\
x^4 - 10x^2 + 1 &= 0
\end{aligned}$$