Finite groups

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1 Introduction

Groups are a common way to formalize the notion of symmetries in mathematics. Finite groups in turn are simply groups with finitely many elements. Unsurprisingly, one of the most important invariants of a finite group G is its cardinality $|G|$. For example, one has

Theorem 1.1. *Let G be a group whose cardinality $|G|$ is a square of a prime number p. Then G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.*

More specifically, one also has

Theorem 1.2. *Let G be a group whose cardinality $|G|$ is a prime number p. Then G is isomorphic to $\mathbb{Z}_p$.*

Both of those are consequences of

Theorem 1.3. *[1, Proposition 2.2] Let G be a finite group and H a subgroup of G. Then $|H|$ divides $|G|$.*

2 Sylow Theorems

A common tool to study finite groups is the use of so-called Sylow-subgroups.

Theorem 2.1. *[1, Thereom 6.4] Let G be a finite group, p a prime number and $p^{M_p(G)}$ the maximal power of p dividing G. A subgroup S of G with $S = p^{M_p(G)}$ is called a p-Sylow subgroup of G. Let $n_p(G)$ be the number of p-Sylow-subgroup of G. Then*

$$n_p(G) \equiv 1 \mod p^{M_p(G)} \text{ and } n_p(G) \text{ divides } \frac{|G|}{p^{M_7(G)}}$$

holds. Further, all p-Sylow-subgroups of G are conjugates in G.

This can often be used to determine groups explicitly only with the information of $|G|$. For example, one has the following application. Let$G$ be a finite group with $|G| = 35$. Then

$$n_7(G) \equiv 1 \mod 7^{M_6(G)} = 7 \text{ and } n_7(G) \text{ divides } \frac{|G|}{7^{M_7(G)}} = 5.$$

But this implies that $n_7(G) = 1$ and so there is a unique 7-Sylow-sbgrou S_7 of G and so S_7 is a normal subgroup of G. But then $S_7 = \mathbb{Z}_7$ and $G/S_7 = \mathbb{Z}_5$ and hence G is solvable. Of course, one could also use

$$n_5(G) \equiv 1 \mod 5^{M_5(G)} = 5 \text{ and } n_5(G) \text{ divides } \frac{|G|}{5^{M_5(G)}} = 7$$

instead. In fact, one has

Theorem 2.2. *Let G be a subgroup whose cardinality $|G|$ is smaller than 60. Then G is solvable..*

Book on Algebra

[1] Serge Lang. *Algebra*. Vol. 211. Springer Science & Business Media, 2012.