Algebra 2R, list 7.

Homework rules as usual.

1. Assume that $A, B \subseteq U$, where U is an algebraically closed field, and K is a subfield of U.
(a)– Prove that if $A \subset B$ is algebraically independent over K, then it may be extended to a transcendence basis of B, over K. (for simplicity you may assume B is finite)
(b)– Prove that if A is a transcendence basis of B over K, then it is also a transcendence basis of the set $acl_K(B)$ over K.
(c) Prove that every two transcendence bases of B over K are equinumerous (for simplicity you may assume that one of these bases is finite).
(d)– Prove in detail that if $\{a_i, i \in I\} \subset U$ is algebraically independent over K, then $K(a_i, i \in I)$ is isomorphic (over K) to the field of rational functions $K(X_i, i \in I)$.

2. (a) Prove that the field extensions $K \subset K(X, Y)$ is purely transcendental.
(b)– Prove that the field extension $K \subset K(X_i, i \in I)$ is purely transcendental.

3. Prove that the set $\{a_1, \ldots, a_n\} \subset U$ is algebraically independent over K $\iff$ there is no non-zero polynomial $W(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$ such that $W(a_1, \ldots, a_n) = 0$.

4. * Let U be the algebraic closure of the field $\mathbb{Q}(X, Y, Z)$. Inside U find algebraically closed fields K i L of transcendence degree 2 (over $\mathbb{Q}$) such that $K \cap L = \hat{\mathbb{Q}}$. (Comment: it means that the operator of algebraic closure does not satisfy the modularity law, unlike the linear closure operator in a vector space).

5. (a) Prove that $trdeg(\mathbb{C}) = 2^{\aleph_0}$ and the field $\mathbb{C}$ has $2^{2^{\aleph_0}}$ automorphisms. (hint[1])
(b)– Assume that the fields K i L are of the same characteristic, the same transcendence degree and both are algebraically closed. Prove that $K \cong L$.

6. Assume that $a_0, \ldots, a_{n-1} \in R$ are algebraically independent (over $\mathbb{Q}$). Prove that the polynomial

$$W(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$$

is irreducible over the field $\mathbb{Q}(a_0, \ldots, a_{n-1})$.

7. Assume that $a, b, c \in \mathbb{R}$ are algebraically independent. Prove that $a^2 + b + c, b^2c, ab + ac^2$ are also algebraically independent. (Hint[2])

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[1] use the finite character of acl.

[2] Think geometrically, as in a vector space. Let $u = a^2+b+c, v = b^2c, w = ab+ac^2$. It is enough to prove that $a, b, c \in acl(u, v, w)$. Present c and a as rational expressions in terms of u, v, w, b. Then write down a polynomial with coefficients being rational expressions in u, v, w such that b is its root.

8. Assume that M is an R-module. Prove that:
(a) the set $I = \{r \in R : (\forall m \in M)\ rm = 0\}$ is a two-sided ideal in R.
(b) if R is commutative, then for $r \in R$ the set $M_r = \{m \in M : rm = 0\}$ is a submodule of M.

9. Let G be an abelian group.
(a)– Prove that $End(G, +) = End_{\mathbb{Z}}(G)$.
(b) Which abelian groups G are simple $\mathbb{Z}$-modules? Find their endomorphism rings (these are fields by Wedderburne Theorem).
(c) Which abelian groups G are simple $\mathbb{Z}_n$-modules ? Find their endomorphism rings.
(d) Describe the ring $End(\mathbb{Q}, +)$. Is it a division ring? Is $\mathbb{Q}$ a simple $\mathbb{Z}$-module?

10. Prove that every non-zero vector space V over a field K is a simple $End_K(V)$-module.
(b) Find the endomorphism ring of this module.

11. Assume that M is an R-module. Then M is also R'-module, where $R' = End_R(M)$ (no proof is required for that). For $r \in R$ we define $f_r : M \to M$ by $f_r(m) = rm$. Prove that $f_r \in End_{R'}(M)$.