Algebra IIR, list 1.

Students choose 3 homework problems from each list. Each item from a problem from list counts as a separate homework problem. Students can not choose as their homework problem more than one item from each problem from a list. Exceptionally in the first list two homework problems are already assigned and marked with D. Students choose additionally the third problem.

1. D Prove that $\mathbb{C} = \mathbb{R}[z]$ for every complex number $z \in \mathbb{C} \setminus \mathbb{R}$.

2. Assume that $K \subset L$ are fields and $a, b \in L$. For a rational function $f(X) \in K(X)$ define $f(a)$ as $g(a)/h(a)$, where $g, h \in K[X]$, $f = g/h$ and $h(a) \neq 0$, provided such g, h exist. If not, $f(a)$ is undetermined. Prove that
   (a) if $f(X) \in K(X)$ and $f(a)$ is defined, then $f(a)$ is determined uniquely (does not depend on the choice of g, h),
   (b) $K(a) = \{f(a) : f \in K(X) \text{ and } f(a) \text{ is defined}\}$,
   (c) $K(a, b) = (K(a))(b)$.

3. Assume that $K \subset L$ are fields and $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ have degree 1.
   (a)D Prove that if the system of equations $f_1 = \cdots = f_m = 0$ has a solution in L, then it has a solution in K. (hint: use linear algebra).
   (b) Does K contain a generic solution of this system (over K) ?

4. Prove in detail that the definition of h in Remark 1.2 is correct.

5. Which of the following solutions of the equation $X_1^2 - X_2^3 = 0$ in the field of rational functions $\mathbb{C}(X)$ are generic over the field $\mathbb{Q}$?
   (a) $(1, 1)$, (b) $(\sqrt[6]{8}, \sqrt[6]{4})$, (c) $(1, \cos\frac{2}{3}\pi + i \sin\frac{2}{3}\pi)$,
   (d) (X^3, X^2), (e) $(\frac{X^3}{(X-1)^3}, \frac{X^2}{(X-1)^2})$.
   hint: in (d) you need to work a bit with polynomials, in (e) consider some automorphisms.

6. Assume that $f \in K[X]$ is irreducible, of degree $n > 0$, $\operatorname{char} K = 0$, L is the splitting field of polynomial f over K. Prove that the field L has at least n distinct automorphisms.

7. Assume that $K_1 \subset K_2 \subset K_3 \subset \ldots$ is an increasing sequence of fields. Verify in detail that $\bigcup_n K_n$ is also a field, containing $K_1, K_2, K_3, \ldots$ as subfields.

8. Prove that the set $\{\sqrt{p} : p \text{ is a prime number}\}$ is linearly independent over the field $\mathbb{Q}$.