Algebra 2R
Problem List 1

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EXERCISE 1.

Proof that $\mathbb{C} = \mathbb{R}[z]$ *for every complex number* $z \in \mathbb{C} \setminus \mathbb{R}$.

To begin with, let us take any $z \in \mathbb{C} \setminus \mathbb{R}$ such that $z = ai$ for some $a \in \mathbb{R}$. We have that

$$\mathbb{R}[z] = \{f(z) \ : \ f \in \mathbb{R}[X]\}.$$

Let $I = (X^2 + a^2) \triangleleft \mathbb{R}[X]$ be an ideal of $\mathbb{R}[X]$ generated by a polynomial with no real roots. We know that $\mathbb{R}[X]/I \cong \mathbb{C}$.

This is because $\mathbb{R}$ is a field and so $\mathbb{R}[X]$ is an euclidean domain: if we take any $f \in \mathbb{R}[X]$ then we can write it as $f = v(X^2 + a^2) + w$, where w is of degree 0 or 1 ($<$ def$(X^2 + a^2)$) and so f in $\mathbb{R}[X]/I$ is represented only by w. Now it is quite easy to map polynomials with real coefficients and maximal degree 1 to $\mathbb{C}$, for example $f : \mathbb{R}[X]/I \to \mathbb{C}$ such that $f(aX + b) = ai + b$. Therefore $\mathbb{R}[X]/I \cong \mathbb{C}$.

Consider the evaluation homomorphism ϕ_z which maps $\mathbb{R}[X] \ni w \mapsto w(z) \in \mathbb{R}[z]$. We can see that $\ker(\phi_z) = (X^2 + a^2) = I$. Therefore, by the fundamental theorem on ring homomorphism we have an isomorphism

$$f : Im(\phi_z) = \mathbb{R}[z] \to \mathbb{R}[X]/\ker(\phi_z) = \mathbb{R}[X]/I$$

and as mentioned above, $\mathbb{R}[X]/I \cong \mathbb{C}$. Hence, $\mathbb{R}[z] \cong \mathbb{C}$.

EXERCISE 2.

Assume that $K \subset L$ *are fields and* $a, b \in L$. *For a rational function* $f(X) \in K(X)$ *define* $f(a)$ *as* $\frac{g(a)}{h(a)}$, *where* $g, h \in K[X]$, $f = \frac{g}{h}$ *and* $h(a) \neq 0$, *provided such* g, h *exist. If not,* $f(a)$ *is undetermined. Prove that*

I know I shouldn't do this but I wanted to know if the diagram I drew in (c) is a correct solution. If not I have (a) as a more reasonable backup to get at least some points c:

(a) if $f(X) \in K(X)$ *and* $f(a)$ *is defined, then* $f(a)$ *is determined uniquely (does not depend on the choice of* g, h*)*

Suppose by contradiction that $f(a)$ depends on which g, h we choose. That means that there exist $g, h, g', h' \in K[X], h(a) \neq 0, h'(a)$ such that $f = \frac{g}{h} = \frac{g'}{h'}$ but $\frac{g(a)}{h(a)} + c = \frac{g'(a)}{h'(a)}$, where $c \in L \setminus \{0\}$.

From $f = \frac{g}{h} = \frac{g'}{h'}$ we get that $g \cdot h' = g' \cdot h$ and in particular

$$(gh')(a) = (g'h)(a)$$
$$g(a)h'(a) = g'(a)h(a)$$
$$g(a)h'(a) - g'(a)h(a) = 0$$

From the assumption that f(a) depends on the choice of polynomials we get that

$$
\begin{aligned}
\frac{g'(a)}{h'(a)} &= \frac{g(a)}{h(a)} + c \\
g'(a)h(a) &= g(a)h'(a) + ch'(a) \\
g'(a)h(a) - g(a)h'(a) &= ch'(a) \neq 0
\end{aligned}
$$

Which is a contradiction because $c \neq 0$, $h'(a) \neq 0$ and we have no zero divisors.

(c) $K(a, b) = (K(a))(b)$

Let

$$I_{ab} = I((a, b)/K[x, y])$$

$$I_a = I(a/(K[y])[x])$$

$$I_b = I(b/(K(a))(y))$$

and j_a, j_b, j_{ab} are quotient functions defined as below. We know that $\ker(j_a) = I_a$, $\ker(j_b) = I_b$ and $\ker(j_{ab}) = I_{ab}$. Let ϕ_a be an evaluation function that substitutes only one variable:

$$\phi_a : K(x, y) \to (K(a))(y)$$

$$\phi_a(f(x, y)) = f(a, y)$$

that is ϕ_a returns a rational function with changed coefficients. ϕ_b, ϕ_{ab} are defined as evaluation functions without such modifications.

$$
\begin{tikzcd}
K(x,y)/I_{a,b} \arrow[d, "\cong"] & K(x,y) \arrow[l, "j_{ab}"'] \arrow[r, "j_a"] \arrow[dl, "\phi_{ab}"] \arrow[dr, "\phi_a"] \arrow[dddr, bend right, "\psi"'] & K(x,y)/I_a \arrow[d, "\cong"] \\
K(a,b) & & (K(a))(y) \arrow[d, "j_b"] \arrow[dd, bend left, "\phi_b"] \\
& & (K(a))(y)/I_b \arrow[d, "\cong"] \\
& & (K(a))(b) \arrow[uull, dashed, leftrightarrow, "f"]
\end{tikzcd}
$$

Function ψ is a ring homomorphism defined as composition of ϕ_a and ϕ_b:

$$\psi : K(x, y) \to (K(a))(y)$$

$$\psi = \phi_b \circ \phi_a$$

For f to be an isomorphism

$$f : (K(a))(b) \to K(a, b)$$

we need to show that $\ker(\phi_{ab}) = \ker(\psi)$ because then

$$
\begin{tikzcd}
& K(x,y)/\ker(\phi_{ab}) = K(x,y)/\ker(\psi) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
K(a,b) \arrow[rr, dashed, leftrightarrow, "\cong"] & & (K(a))(b)
\end{tikzcd}
$$

$\ker(\phi_{ab}) = \ker(\psi)$

$\subseteq$

$f \in \ker(\phi_{ab})$ means that $f(a, b) = 0$. That is, either of the following is true for any $x, y \in K$

$f(a, b) = 0$ this directly implies that $f \in \ker(\psi)$.

$f(a, y) = 0$ the same as above.

$f(x, b) = 0$ we know that for any $x \in K$ $f(x, b) = 0$ then for $x = a$ this is also true and so $f(a, b) = 0$ and $f \in \ker(\psi)$.

$\supseteq$

$f \in \ker(\psi)$ means that $f(a, b) = 0$ or $f(a, y) = 0$. This means that $f \in \ker(\phi_{ab})$.

Therefore, there exists an isomorphism $K(a, b) \cong (K(a))(b)$.

EXERCISE 3.

Assume that $K \subseteq L$ *are fields and* $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ *have degree* 1.

(a) Prove that if the system of equations $f_1 = \dots = f_m = 0$ *has a solution in* L *then it has a solution in* K. *(hint: use linear algebra).*

We are working on linear equations, therefore we can construct a matrix that stores the same information as the system of equations $f_1 = \dots = f_m$. Let

$$f_i = \sum_{1 \leq k \leq n} b_{i,k} X_k + c_i$$

for $i = 1, \dots, m$. The matrix representation of this system of equations is:

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{m,1} & b_{m,2} & b_{m,3} & \dots & b_{m,n-1} & b_{m,n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \dots \\ c_m \end{bmatrix}.$$

Using Gaussian algorithm, we can create an upper triangular matrix with coefficients that are linear combinations of elements from K and thus are themselves in K.

If $m \leq n$, then let

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \dots & \dots & \alpha_{1,n-1} & \alpha_{1,n} \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \dots & \dots & \dots & \alpha_{2,n-1} & \alpha_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_{m,m} & \dots & \alpha_{m,n-1} & \alpha_{m,n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \dots \\ \gamma_m \end{bmatrix}$$

be the result of Gaussian elimination of the matrix above. Because Gaussian elimination returns a matrix with elements that are linear combinations of the elements of the original matrix, we have that $b_{i,k}, \gamma_i \in K$.

The solution would be found by backwards substitution. We could take $a_{m+1}, a_{m+2}, \dots, a_n = 0 \in K$ then

$$\gamma_m = \alpha_{m,m} a_m + \alpha_{m,m+1} a_{m+1} + \dots + \alpha_{m,n} a_n = \alpha_{m,m} a_m$$

$$a_m = (\alpha_{m,m})^{-1} \gamma_m \in K$$

Then

$$\gamma_{m-1} = \alpha_{m-1,m-1}a_{m-1} + \alpha_{m-1,m}a_m + \alpha_{m-1,m+1}a_{m+1} + \ldots + \alpha_{m-1,n}a_n =$$
$$= \alpha_{m-1,m}a_{m-1} + \alpha_{m-1,m}(\alpha_{m,m})^{-1}\gamma_m$$

$$a_{m-1} = (\alpha_{m-1,m})^{-1}(\gamma_{m-1} - \alpha_{m-1,m}(\alpha_{m,m})^{-1}\gamma_m) \in K$$

And so on. We know from linear algebra that this will work.

If m > n, then the upper triangular matrix would look like this:

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \ldots & \ldots & \ldots & \alpha_{1,n-1} & \alpha_{1,n} \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \ldots & \ldots & \ldots & \alpha_{2,n-1} & \alpha_{2,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \alpha_{m,m} \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \ldots \\ a_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \ldots \\ \gamma_n \end{bmatrix}$$

and such a matrix can be treated the same way as before with the condition that for i > m $\gamma_i = 0$. Otherwise no solutions exist.