Algebra 2R
Problem List 1

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EXERCISE 1.

Proof that $\mathbb{C} = \mathbb{R}[z]$ *for every complex number* $z \in \mathbb{C} \setminus \mathbb{R}$.

To begin with, let us take any $z \in \mathbb{C} \setminus \mathbb{R}$ such that $z = ai$ for some $a \in \mathbb{R}$. We have that

$$\mathbb{R}[z] = \{f(z) \ : \ f \in \mathbb{R}[X]\}.$$

Let $I = (X^2 + a^2) \lhd \mathbb{R}[X]$ be an ideal of $\mathbb{R}[X]$ generated by a polynomial with no real roots. We know that $\mathbb{R}[X]/I \cong \mathbb{C}$.

This is because $\mathbb{R}$ is a field and so $\mathbb{R}[X]$ is an euclidean domain: if we take any $f \in \mathbb{R}[X]$ then we can write it as $f = v(X^2 + a^2) + w$, where w is of degree 0 or 1 ($< \mathrm{def}(X^2 + a^2)$) and so f in $\mathbb{R}[X]/I$ is represented only by w. Now it is quite easy to map polynomials with real coefficients and maximal degree 1 to $\mathbb{C}$, for example $f : \mathbb{R}[X]/I \to \mathbb{C}$ such that $f(aX + b) = ai + b$. Therefore $\mathbb{R}[X]/I \cong \mathbb{C}$.

Consider the evaluation homomorphism ϕ_z which maps $\mathbb{R}[X] \ni w \mapsto w(z) \in \mathbb{R}[z]$. We can see that $\ker(\phi_z) = (X^2 + a^2) = I$. Therefore, by the fundamental theorem on ring homomorphism we have an isomorphism

$$f : \mathrm{Im}(\phi_z) = \mathbb{R}[z] \to \mathbb{R}[X]/\ker(\phi_z) = \mathbb{R}[X]/I$$

and as mentioned above, $\mathbb{R}[X]/I \cong \mathbb{C}$. Hence, $\mathbb{R}[z] \cong \mathbb{C}$.

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EXERCISE 2.

Assume that $K \subset L$ *are fields and* $a, b \in L$. *For a rational function* $f(X) \in K(X)$ *define* $f(a)$ *as* $\frac{g(a)}{h(a)}$, *where* $g, h \in K[X]$, $f = \frac{g}{h}$ *and* $h(a) \neq 0$, *provided such* g, h *exist. If not,* $f(a)$ *is undetermined. Prove that*

(a) if $f(X) \in K(X)$ *and* $f(a)$ *is defined, then* $f(a)$ *is determined uniquely (does not depend on the choice of* g, h*)*

Suppose by contradiction that $f(a)$ depends on which g, h we choose. That means that there exist $g, h, g', h' \in K[X], h(a) \neq 0, h'(a)$ such that $f = \frac{g}{h} = \frac{g'}{h'}$ but $\frac{g(a)}{h(a)} + c = \frac{g'(a)}{h'(a)}$, where $c \in L \setminus \{0\}$.

From $f = \frac{g}{h} = \frac{g'}{h'}$ we get that $g \cdot h' = g' \cdot h$ and in particular

$$(gh')(a) = (g'h)(a)$$
$$g(a)h'(a) = g'(a)h(a)$$
$$g(a)h'(a) - g'(a)h(a) = 0$$

From the assumption that f(a) depends on the choice of polynomials we get that

$$\frac{g'(a)}{h'(a)} = \frac{g(a)}{h(a)} + c$$
$$g'(a)h(a) = g(a)h'(a) + ch'(a)$$
$$g'(a)h(a) - g(a)h'(a) = ch'(a) \neq 0$$

Which is a contradiction because $c \neq 0$, $h'(a) \neq 0$ and we are working in a field that is a ring without zero divisors.

(b) $K(a) = \{f(a) \ : \ f \in K(X)$ *i* $f(a)$ *jest określone*$\}$

We know that $K(a)$ is a subfield of L that is generated by $K \cup \{a\}$. Let us label this field as L'. We will show that $L' = K(a)$.

$L' \subseteq K(a)$

Let us take any $x \in L'$. Then x is a finite linear combination of elements from K and $\{a.a^{-1}\}$:

$$x = \sum_{0 \leq k \leq n} \alpha_k a^{i_k k}, \quad i_k \in \{1, -1\}, \ \alpha_k \in K.$$

We need to change this into a rational function. Take $p_k \in K[X]$ such that $p_k(X) = \alpha_k X^k$. We have that

$$x = \sum_{0 \leq k \leq n} p_k(a^{i_k}).$$

It is clear that when working with rational functions we may say that $p_k(a^{-1}) = \frac{1}{p_k'(a)}$ where $p_k(X) = \alpha_k^{-1} X^k$.

$$x = \sum_{0 \leq k \leq n} p_k(a^{i_k}) = \frac{\sum\limits_{0 \leq k \leq n} p_k(a) \prod\limits_{\substack{0 \leq l \leq n, \\ i_l = -1}} p_k'(a)}{\prod\limits_{\substack{0 \leq k \leq n \\ i_k = -1}} p_k'(a)} \in K(a)$$

$K(a) \subseteq L'$

Let us take any $f \in K(X)$ such that $f(a)$ is defined. We may write $f = \frac{g}{h}$ for $g, h \in K[X]$ and $h(a) \neq 0$. We have that $g(a) \in L'$ and $h(a) \in L'$. Therefore, $\frac{g(a)}{h(a)} = g(a) \cdot [h(a)]^{-1} \in L'$.

(c) $K(a, b) = (K(a))(b)$

EXERCISE 3.

Assume that $K \subseteq L$ *are fields and* $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ *have degree* 1.

(a) Prove that if the system of equations $f_1 = \ldots = f_m = 0$ *has a solution in* L *then it has a solution in* K. *(hint: use linear algebra).*

Let

$$f_i = \sum_{1 \leq k \leq n} b_{i,k} X_k$$

for $i = 1, \ldots, m$.

We are working on linear equations, therefore we can construct a matrix that stores the same information as the system of equations $f_1 = \ldots = f_m$. Let

$$f_i = \sum_{1 \leq k \leq n} b_{i,k} X_k$$

for $i = 1, \ldots, m$. The matrix representation of this system of equations is:

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \ldots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \ldots & b_{2,n-1} & b_{2,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{m,1} & b_{m,2} & b_{m,3} & \ldots & b_{m,n-1} & b_{1,n} \end{bmatrix} X = 0.$$

Using Gaussian algorithm, we can create an upper triangular matrix with coefficients from K. The solution would be found by backwards substitution. That is, a_n would be in the bottom right corner of the matrix and it is an element of K because such are the coefficients within my matrix. Then a_{n-1} would be a combination of a_n with two elements of K, hence it would still be in K and so on. Each a_i would be a linear combination of elements from K and a_k, $k < i$, which we know are in K.

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EXERCISE 8.

Prove that the set $\{\sqrt{p} \ : \ p$ is a prime number$\}$ is linearly independent over the field $\mathbb{Q}$.

Assume that the set $S = \{\sqrt{p} \ : \ p$ is a prime number$\}$ is not linearly independent. That means that there is a sequence $p_1, \ldots, p_n$ of prime numbers and $a_1, \ldots, a_n \in \mathbb{Q}$ such that

$$\sum_{1 \leq k \leq n} a_k \sqrt{p_j} = 0$$

Because we are working on a field, we can square both sides of the equation to get

$$0 = \left(a_n \sqrt{p_n} + \sum_{1 \leq k \leq n-1} a_k \sqrt{p_k} \right) \left(a_n \sqrt{p_1} - \sum_{1 \leq k \leq n-1} a_k \sqrt{p_k} \right) =$$