Algebra 2R

Problem list 9

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Exercise 1.

(a) Prove that $(\mathbb{Z}_n, +_n) \otimes_{\mathbb{Z}} (\mathbb{Z}_m, +_m) \cong (\mathbb{Z}_d, +_d)$ (tensor product of $\mathbb{Z}$-modules), where $d = GCD(m, n)$

(b) More generally, let $I, J \triangleright R$. Prove that $R/I \otimes_R R/J \cong R/(I + J)$

(a) Let $L = (\mathbb{Z}_n, +_n) \otimes_{\mathbb{Z}} (\mathbb{Z}_m, +_m)$. Take any $a \otimes b \in L$, then

$$a \otimes b = ab \otimes 1 = 1 \otimes ab$$

so $a \otimes b \neq 0$ means that ab is not divisible by n nor by m. So it also must not be divisible my d. And we have that L is created by adding $1 \otimes 1$ (like in a cyclic group) and so to get 0 we have to add an amount divisible by n and m - so at most $\gcd(n, m)$ times. This means that $d \otimes 1 = 1 \otimes d$ is actually a zero element (something like $d = \mathrm{ord}(L)$ but I am not sure if this is a group or if this has a different name).

Hence the kernel of $a \otimes b \mapsto ab \mod d$ (let us call this homomorphism) is just $0 \otimes 0$. By isomorphism theorems that I still remember from Algebra 1R we get that

$$(L{=}L/0{=})\ L/\ker \cong \mathrm{Im}$$

and it is obvious that Im $= \mathbb{Z}_d$ because if I keep $a = 1$ and move b from 0 to $d - 1$ then I get every element from $\mathbb{Z}_d$.

Why is a homomorphism? Because

$$ab + a'b' \leftarrow (a \otimes b) + (a' \otimes b') = (ab \otimes 1) + (a'b' \otimes 1) = (ab + a'b' \otimes 1) \mapsto 1 \cdot (ab + a'b') = ab + a'b' \checkmark$$

Exercise 3. Assume M is a simple R-module. Prove that $\mathrm{End}_R(M) \cong R/I$ for some maximal ideal $I \triangleright R$.

From Schur's lemma I know that every endomorphism of a simple module is actually a bijection. Hence, for every $\in \mathrm{End}_R(M)$ we have some $^{-1} \in \mathrm{End}_R(M)$. What is left is to show that this is commutative and $\mathrm{End}_R(M)$ is a field.

Take $f, g \in \mathrm{End}_R(M)$ and any $m \in \mathrm{End}_R(M)$. $f(m) = rm$ and $g(m) = sm$ for some $r, s \in R$ because Rm is a submodule of M, it is not zero hence it must be the whole thing. So now since R is commutative I have:

$$f \circ g(m) = f(g(m)) = f(sm) = sf(m) = srm = rsm = rg(m) = g(rm) = g(f(m)) = g \circ f(m)$$

Now it is simple to show that $f(x) = rx$ is the only way an endomorphism must look like because if $f(m) = rm, f(n) = sn$ (once again, Rm, Rn are submodules) then $f(m + n) = f(m) + f(n) = rm + sn$ but on the other hand there is some p such that $f(m + n) = p(m + n)$ and $p(m + n) = rm + sn \implies p - r = s - p \implies p + p = r + s \implies p = r = s$.

So given $f \in \mathrm{End}_R(m), f(x) = rx$ we can do $f \mapsto r$ and this is a unique mapping plus r must a unit from the Schur's lemma in the first paragraph.