Problem List 3

Algebra 2r

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Exercise 1. Let K be a field.

(a) Prove that the field extension $L \supseteq K$ is transcendental, where $L = K(X)$ is the field of rational functions in varriable X over K.

(b) Let $M = L[\sqrt{X}]$ be an algebraic extension of the field L by an element $Y = \sqrt{X}$ such that $Y^2 - X = 0$ in the field M. Prove that M and L are isomorphic over K.

(b)

$$M = L[\sqrt{X}] \qquad\qquad L = K[X]$$

$$K$$

Exercise 3. Let $v_1, ..., v_n$ be vertices of a regular n-gon inscribed in a circle on the plane $\mathbb{R}^2$ with equation $x^2 + y^2 = 1$. What is the linear dimension over $\mathbb{Q}$ of the system of vectors $v_1, ..., v_n$.

Without the loss of generality, I will consider polygons with one vertex in $(1, 0)$. Then, the remaining vertices are in $(\cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n})$, for $k = 1, ..., n-1$. Now, let me switch where I live and let us consider roots of

$$x^n - 1.$$

We have n roots $z_1, ..., z_n$ in $\mathbb{C}$. Notice, that $z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ and adding complex numbers works almost like adding vectors in 2D. The minimal polynomial over $\mathbb{Q}$ of each of z_k is $F_n(x)$. Therefore, $\dim(v_1, ..., v_n) = \dim(z_1, ..., z_n) = \phi(n)$, where ϕ is Euler's function.

Exercise 4. Assume that $K \supseteq F(p)$ is a finite field extension of $F(p)$. Assume that $a \in K$ is a primitive root of 1 of degree m. Let n be the smallest natural number > 0 such that $m | p^n - 1$.

(a) Prove that n equals the degree of a over $F(p)$

(b) Prove that $n | \phi(m)$. Give an example where $n < \phi(m)$.

Well, I think I kinda showed it before XD

Exercise 6. Find the minimal polynomials over $\mathbb{Q}$ fot the following numbers:

(a) $\sqrt{2} + \sqrt{3}$

$$x - (\sqrt{2} + \sqrt{3}) = 0$$
$$x - \sqrt{2} = \sqrt{3}$$
$$(x - \sqrt{2})^2 = 3$$
$$x^2 - x2\sqrt{2} + 2 = 3$$
$$x^2 - 1 = x2\sqrt{2}$$
$$(x^2 - 1)^2 = 8x^2$$
$$x^4 - 2x^2 + 1 = 8x^2$$
$$x^4 - 10x^2 + 1 = 0$$