Problem List 3

Algebra 2r

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Exercise 3. Let $v_1, \ldots, v_n$ be vertices of a regular n-gon inscribed in a circle on the plane $\mathbb{R}^2$ with equation $x^2 + y^2 = 1$. What is the linear dimension over $\mathbb{Q}$ of the system of vectors $v_1, \ldots, v_n$.

Without the loss of generality, I will consider polygons with one vertex in $(1, 0)$. Then, the remaining vertices are in $(\cos \frac{2\pi k}{n}, \sin \frac{2\pi k}{n})$, for $k = 1, \ldots, n-1$. Now, let me switch where I live and let us consider roots of

$$x^n - 1.$$

We have n roots $z_1, \ldots, z_n$ in $\mathbb{C}$. Notice, that $z_k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$ and adding complex numbers works almost like adding vectors in 2D. The minimal polynomial over $\mathbb{Q}$ of each of z_k is $F_n(x)$. Therefore, $\dim(v_1, \ldots, v_n) = \dim(z_1, \ldots, z_n) = \phi(n)$, where ϕ is Euler's function.

Well, I think I kinda showed it before XD

Exercise 6. Find the minimal polynomials over $\mathbb{Q}$ fot the following numbers:

(a) $\sqrt{2} + \sqrt{3}$

$$\begin{aligned}
x - (\sqrt{2} + \sqrt{3}) &= 0 \\
x - \sqrt{2} &= \sqrt{3} \\
(x - \sqrt{2})^2 &= 3 \\
x^2 - x2\sqrt{2} + 2 &= 3 \\
x^2 - 1 &= x2\sqrt{2} \\
(x^2 - 1)^2 &= 8x^2 \\
x^4 - 2x^2 + 1 &= 8x^2 \\
x^4 - 10x^2 + 1 &= 0
\end{aligned}$$

Exercise 7. Prove (using Liouville Lemma) that the number

$$\sum_{n=1}^{\infty} \frac{1}{2^{n!}}$$

is transcendental. (the real numbers, whose transcendence follows from Liouville Lemma are called Liouville numbers).

Liouville Lemma states that if $a \in \mathbb{R}$ is an algebraic number of degree $N > 1$, then there exists $c \in \mathbb{R}_+$ such that for all $\frac{p}{q} \in \mathbb{Q}$ the following is true:

$$\left| a - \frac{p}{q} \right| \geq \frac{c}{q^N}$$

If a number fails to meet this criterion, then it is called transcendental.

Ok, so I have no clue what the degree of my number is, but let me assume that it is some $N \in \mathbb{N}$. Now, let

$$p = \sum_{n=1}^{N+k} 2^{(N+k)! \cdot n!}.$$

Then, we have that

$$\sum_{n=1}^{\infty} \frac{1}{2^{n!}} = \frac{p}{2^{(N+k)!}} + \sum_{n=N+k}^{\infty} \frac{1}{2^{n!}}$$

with $q = 2^{(N+k)!}$. From this we get

$$\left| \sum_{n=1}^{\infty} \frac{1}{2^{n!}} - \frac{p}{2^{(N+k)!}} \right| = \left| \sum_{n=N+k+1}^{\infty} \frac{1}{2^{n!}} \right| \leq (\text{☕})$$

and notice that

$$\sum_{n=N+k+1}^{\infty} \frac{1}{2^{n!}} \leq \sum_{n=N+k+1}^{\infty} \frac{1}{2^{n}} = \frac{1}{2^{(N+k+1)!}} \frac{1}{1 - \frac{1}{2}} = \frac{2}{2^{(N+k+1)!}}$$

$$(\text{☕}) \leq \frac{2}{2^{(N+k+1)!}} = \frac{2}{q^{N+k+1}} < \frac{1}{q^{N+k}}$$

for any $k \in \mathbb{N}$ and so we cannot choose one universal c such that this inequality changes to $\geq$ for all. Thus, the number from the problem is a Liouville number.