Algebra 2R, list 9.

$R$ denotes a commutative ring with $1 \neq 0$.

1. (a) Prove that $(\mathbb{Z}_n, +_n) \otimes_{\mathbb{Z}} (\mathbb{Z}_m, +_m) \cong (\mathbb{Z}_d, +_d)$ (tensor product of $\mathbb{Z}$-modules), where $d = GCD(m, n)$.
   (b) More generally, let $I, J \triangleleft R$. Prove that $R/I \otimes_R R/J \cong R/(I + J)$.
2. (a) Let $G$ be an abelian group. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} G$ is a divisbke torsion-free group, hence a vector space over $\mathbb{Q}$.
   (b) What is the dimension of this vector space ?
   (c) More generally let $R$ be an integral domain, $K$ its field of fractions (hence an $R$-module) and $M$ an $R$-module. Prove that $M$ is a torsion $R$-module $\iff K \otimes_R M = \{0\}$.
3. Assume $M$ is a simle $R$-module. Prove that $End_R(M) \cong R/I$ for some maximal ideal $I \triangleleft R$.
4. From now on in the next problems $R$ is a PID. Assume $M$ is a torsion $R$-module.
   (a)- Let $p \in R$ be prime. Check that $M_p$ jest is a submodule of $M$.
   (b) Let $\{p_i, i \in I\} \subset R$ be a se of represenatives of the assosiation classes of the prime elements of $R$. Prove that
$$M = \bigoplus_{i \in I} M_{p_i}.$$
5. (a) Assume $M$ is a $p$-primary cyclic $R$-module. Prove that $M \cong R/(p^k)$ dla pewnego $k \leqslant 0$.
   (b) Assume that $M$ is cyclic. Prove that $M$ is indecomposable $\iff$ $M$ is torsion free or $p$-primary for some prime $p \in R$.
6. Which finitely generated indecomposable $R$-modules are simple ?
7. – Using the Jordan Theorem prove that if $V$ is an $n$-dimensional vector space over a field $K$ and $f \in End_K(V)$, then $\varphi_f(f) = 0$, where $\varphi_f(X)$ is the characteristic polynomial of $f$ (Cayley-Hamilton Theorem).