Gaußian elimination

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Abstract

We will present some variant of Gaußsian elimination. In particular, the method we present is useful in fining the number of solutions to a given system of linear equations. We note that our method does not cover finding the solution set.

0 Preliminaries

All matrices will be over a fixed field $\mathbb{F}$

Definition 0.0.1. A matrix if of *reduced echelon form* (*REF*) if:

- All rows consisting of only zeroes are at the bottom.
- The leading entry (that is the left-most nonzero entry) of every nonzero row is to the right of the leading entry of every row above.

Note that this variant doesn't require leading entries to be 1.

Definition 0.0.2. Two matrices A, B are *row equivalent* denoted $A \sim B$ if A is a product of elementary row operations on B.

Lemma 0.0.3. *$\sim$ is an equivalence relation.*

Definition 0.0.4. Let A be a matrix. Then $\text{colCount}(()A)$ is the number of columns in A.

Definition 0.0.5. Let A be an REF matrix. Then $\text{rank}(A)$ is the number of non-zero rows in A, equivalantly, the number of leading entries.

Remark 0.0.6. *Let A be an REF matrix. Then $\text{rank}(A) \leq \text{colCount}(A)$.*

1 Gaußian elimination

1.1 The results

Theorem 1.1.1. *For every matrix A, there is some matrix B such that B is REF and $A \sim B$.*

Lemma 1.1.2. *Let $(A|b)$ be an REF matrix. Then rank$(A|b) \geq$ rank(A).*

Propozycja 1.1.3. *Let $(A|b)$ be an REF matrix representing a system of linear equations, such that rank$(A) =$ rank$((A|b))$. Then the system of linear equations has at least one solution.*

Propozycja 1.1.4. *Let $(A|b)$ be an REF matrix representing a system of linear equations, such that rank$(A) <$ rank$((A|b))$. Then the system of linear equations has no solutions.*

Propozycja 1.1.5. *Let$(A|b)$ be an REF matrix representing a system of linear equations, such that rank$(A) <$ colCount(A). If the system of equations has at least one solution, then it has at least $|\mathbb{F}|$ many solutions.*

Propozycja 1.1.6. *Let $(A|b)$ be an REF matrix representing a system of linear equations, such that rank$(A) =$ colCount(A). Then the system of equations has at most one solution.*

1.2 summary

We summarize the results above in Table 1 below.

$\text{rank}(A) = \text{rank}((A\|b))$		$\text{rank}(A) < \text{rank}((A\|b))$	
$\text{rank}(A) = \text{colCount}(A)$	$\text{rank}(A) > \text{colCount}(A)$	$\text{rank}(A) = \text{colCount}(A)$	$\text{rank}(A) > \text{colCount}(A)$
$\left(\begin{array}{cc\|c} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & 2 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{array}\right)$
One solution	$\geq \|\mathbb{F}\|$ solutions	No solutions	

Table 1: summary

By Remark 0.0.6 and lemma 1.1.2 the table covers all possible cases. By Propositions 1.1.3 to 1.1.6, we obtain the number of solutions depicted in the summary.