Algebra 2R
Problem List 1

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EXERCISE 1.

Proof that $\mathbb{C} = \mathbb{R}[z]$ for every complex number $z \in \mathbb{C} \setminus \mathbb{R}$.

To begin with, let us take any $z \in \mathbb{C} \setminus \mathbb{R}$ such that $z = ai$ for some $a \in \mathbb{R}$. We have that

$$\mathbb{R}[z] = \{f(z) \ : \ f \in \mathbb{R}[X]\}.$$

Let $I = (X^2 + a^2) \triangleleft \mathbb{R}[X]$ be an ideal of $\mathbb{R}[X]$ generated by a polynomial with no real roots. We know that $\mathbb{R}[X]/I \cong \mathbb{C}$.

This is because $\mathbb{R}$ is a field and so $\mathbb{R}[X]$ is an euclidean domain: if we take any $f \in \mathbb{R}[X]$ then we can write it as $f = v(X^2 + a^2) + w$, where w is of degree 0 or 1 ($< \mathrm{def}(X^2 + a^2)$) and so f in $\mathbb{R}[X]/I$ is represented only by w. Now it is quite easy to map polynomials with real coefficients and maximal degree 1 to $\mathbb{C}$, for example $f : \mathbb{R}[X]/I \to \mathbb{C}$ such that $f(aX + b) = ai + b$. Therefore $\mathbb{R}[X]/I \cong \mathbb{C}$.

Consider the evaluation homomorphism ϕ_z which maps $\mathbb{R}[X] \ni w \mapsto w(z) \in \mathbb{R}[z]$. We can see that $\ker(\phi_z) = (X^2 + a^2) = I$. Therefore, by the fundamental theorem on ring homomorphism we have an isomorphism

$$f : \mathrm{Im}(\phi_z) = \mathbb{R}[z] \to \mathbb{R}[X]/\ker(\phi_z) = \mathbb{R}[X]/I$$

and as mentioned above, $\mathbb{R}[X]/I \cong \mathbb{C}$. Hence, $\mathbb{R}[z] \cong \mathbb{C}$.

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EXERCISE 2.

Assume that $K \subset L$ are fields and $a, b \in L$. For a rational function $f(X) \in K(X)$ define $f(a)$ as $\frac{g(a)}{h(a)}$, where $g, h \in K[X]$, $f = \frac{g}{h}$ and $h(a) \neq 0$, provided such g, h exist. If not, $f(a)$ is undetermined. Prove that

(a) if $f(X) \in K(X)$ and $f(a)$ is defined, then $f(a)$ is determined uniquely (does not depend on the choice of g, h)

Suppose by contradiction that $f(a)$ depends on which g, h we choose. That means that there exist $g, h, g', h' \in K[X], h(a) \neq 0, h'(a)$ such that $f = \frac{g}{h} = \frac{g'}{h'}$ but $\frac{g(a)}{h(a)} + c = \frac{g'(a)}{h'(a)}$, where $c \in L \setminus \{0\}$.

From $f = \frac{g}{h} = \frac{g'}{h'}$ we get that $g \cdot h' = g' \cdot h$ and in particular

$$(gh')(a) = (g'h)(a)$$
$$g(a)h'(a) = g'(a)h(a)$$
$$g(a)h'(a) - g'(a)h(a) = 0$$

From the assumption that f(a) depends on the choice of polynomials we get that

$$\frac{g'(a)}{h'(a)} = \frac{g(a)}{h(a)} + c$$
$$g'(a)h(a) = g(a)h'(a) + ch'(a)$$
$$g'(a)h(a) - g(a)h'(a) = ch'(a) \neq 0$$

Which is a contradiction because $c \neq 0$, $h'(a) \neq 0$ and we have no zero divisors.

(b) $K(a) = \{f(a) \ : \ f \in K(X)$ *and* $f(a)$ *is defined*$\}$

We know that K(a) is a subfield of L that is generated by $K \cup \{a\}$. Let us label this field as L'. We will show that $L' = K(a)$.

$L' \subseteq K(a)$

Let us take any $x \in L'$. Then x is a finite linear combination of elements from K and $\{a.a^{-1}\}$:

$$x = \sum_{0\leq k\leq n} \alpha_k a^{i_k k}, \quad i_k \in \{1, -1\},\ \alpha_k \in K.$$

We need to change this into a rational function. Take $p_k \in K[X]$ such that $p_k(X) = \alpha_k X^k$. We have that

$$x = \sum_{0\leq k\leq n} p_k(a^{i_k}).$$

It is clear that when working with rational functions we may say that $p_k(a^{-1}) = \frac{1}{p_k'(a)}$ where $p_k(X) = \alpha_k^{-1}X^k$.

$$x = \sum_{0\leq k\leq n} p_k(a^{i_k}) = \frac{\sum\limits_{0\leq k\leq n} p_k(a) \prod\limits_{\substack{0\leq l\leq n,\\ i_l=-1}} p_k'(a)}{\prod\limits_{\substack{0\leq k\leq n\\ i_k=-1}} p_k'(a)} \in K(a)$$

$K(a) \subseteq L'$

Let us take any $f \in K(X)$ such that f(a) is defined. We may write $f = \frac{g}{h}$ for $g, h \in K[X]$ and $h(a) \neq 0$. We have that $g(a) \in L'$ and $h(a) \in L'$. Therefore, $\frac{g(a)}{h(a)} = g(a) \cdot [h(a)]^{-1} \in L'$.

(c) $K(a, b) = (K(a))(b)$

Let

$$I_{ab} = I((a, b)/K[x, y])$$
$$I_a = I(a/(K[y])[x])$$
$$I_b = I(b/(K(a))(y))$$

and j_a, j_b, j_{ab} are quotient functions defined as below. We know that $\ker(j_a) = I_a$, $\ker(j_b) = I_b$ and $\ker(j_{ab}) = I_{ab}$. Let ϕ_a be an evaluation function that substitutes only one variable:

$$\phi_a : K(x, y) \to (K(a))(y)$$

$$\phi_a(f(x, y)) = f(a, y)$$

that is ϕ_a returns a rational function with changed coefficients. ϕ_b, ϕ_{ab} are defined as evaluation functions without such modifications.

$$\begin{tikzcd}
K(x,y)/I_{a,b} \arrow[d, "\cong"] & K(x,y) \arrow[l, "j_{ab}"'] \arrow[r, "j_a"] \arrow[dl, "\phi_{ab}"] \arrow[dr, "\phi_a"] \arrow[dddr, bend right, "\psi"] & K(x,y)/I_a \arrow[d, "\cong"] \\
K(a,b) \arrow[dddrr, dashed, leftrightarrow, "f"'] & & (K(a))(y) \arrow[d, "j_b"] \arrow[dd, bend left, "\phi_b"] \\
& & (K(a))(y)/I_b \arrow[d, "\cong"] \\
& & (K(a))(b)
\end{tikzcd}$$

Function ψ is a ring homomorphism defined as composition of ϕ_a and ϕ_b:

$$\psi : K(x,y) \to (K(a))(y)$$

$$\psi = \phi_b \circ \phi_a$$

For f to be an isomorphism

$$f : (K(a))(b) \to K(a,b)$$

we need to show that $\ker(\phi_{ab}) = \ker(\psi)$ because then

$$\begin{tikzcd}
& K(x,y)/\ker(\phi_{ab}) = K(x,y)/\ker(\psi) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
K(a,b) \arrow[rr, dashed, leftrightarrow, "\cong"] & & (K(a))(b)
\end{tikzcd}$$

$\ker(\phi_{ab}) = \ker(\psi)$

$\subseteq$

$f \in \ker(\phi_{ab})$ means that $f(a,b) = 0$. That is, either of the following is true for any $x, y \in K$

$f(a,b) = 0$ this directly implies that $f \in \ker(\psi)$.

$f(a,y) = 0$ the same as above.

$f(x,b) = 0$ we know that for any $x \in K$ $f(x,b) = 0$ then for $x = a$ this is also true and so $f(a,b) = 0$ and $f \in \ker(\psi)$.

$\supseteq$

$f \in \ker(\psi)$ means that $f(a,b) = 0$ or $f(a,y) = 0$. This means that $f \in \ker(\phi_{ab})$.

Therefore, there exists an isomorphism $K(a,b) \cong (K(a))(b)$.

EXERCISE 3.

Assume that $K \subseteq L$ *are fields and* $f_1, \dots, f_m \in K[X_1, \dots, X_n]$ *have degree* 1.

(a) Prove that if the system of equations $f_1 = \dots = f_m = 0$ *has a solution in* L *then it has a solution in* K. *(hint: use linear algebra).*

Let

$$f_i = \sum_{1 \le k \le n} b_{i,k} X_k$$

for $i = 1, \dots, m$.

We are working on linear equations, therefore we can construct a matrix that stores the same information as the system of equations $f_1 = \ldots = f_m$. Let

$$f_i = \sum_{1 \leq k \leq n} b_{i,k} X_k$$

for $i = 1, \ldots, m$. The matrix representation of this system of equations is:

$$\begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \ldots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \ldots & b_{2,n-1} & b_{2,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{m,1} & b_{m,2} & b_{m,3} & \ldots & b_{m,n-1} & b_{1,n} \end{bmatrix} X = 0.$$

Using Gaussian algorithm, we can create an upper triangular matrix with coefficients from K. The solution would be found by backwards substitution. That is, a_n would be in the bottom right corner of the matrix and it is an element of K because such are the coefficients within my matrix. Then a_{n-1} would be a combination of a_n with two elements of K, hence it would still be in K and so on. Each a_i would be a linear combination of elements from K and a_k, $k < i$, which we know are in K.

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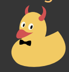

(b) Does K contain a generic solution of this system (over K)?

From Remark 1.4. we know that $\bar{a}$ is a generic solution $\iff$ for any other solution $\bar{a}' \in K^n$ we have only one homomorphism $h : K[\bar{a}] \to K[\bar{a}]$ such that $h(\bar{a}) = h(\bar{a}')$ and $h \restriction K = id_K$. It is suffice to notice that because $K[\bar{a}]$ and $K[\bar{a}']$ are evaluations of polynomials with coefficients from K, then they are finite combinations of elements from K and therefore $K[\bar{a}] \subseteq K$ and $K[\bar{a}'] \subseteq K$. Therefore $h \subseteq id_K$ and thus is unique.

ZADANIE 5.

Which of the following solutions of the equation $X_1^2 - X_2^3 = 0$ in the field of rational functions $\mathbb{C}(X)$ are generic over the field $\mathbb{Q}$?

(a) $(1, 1)$

ZADANIE 6.

Assume that $f \in K[X]$ is irreducible, $\deg(f) = n > 0$, $\mathrm{char}(K) = 0$ and L is the splitting field of polynomial f over K. Prove that the field L has at least n distinct automorphisms.

First of all, I need f to have n distinct roots in L.

If a is at least a double root of f then $f'(a) = 0$. Let

$$f(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_1 x + \alpha_0$$

where $\alpha_n \neq 0$. Then, the derivative is

$$f'(x) = n\alpha_n x^{n-1} + (n-1)\alpha_{n-1} x^{n-1} + \ldots + \alpha_1$$

and because we $\mathrm{char}(K) = 0$, then $n\alpha_n = \alpha_n + \ldots + \alpha_n \neq 0$. Thus, $f'(x) \not\equiv 0$.

We know that $f \in K[X]$ is irreducible and f' has lower degree, hence f' does not divide f. From Bezout's identity I get that there exist $p, q \in K[X] \setminus \{0\}$ such that

$$fp + f'q = 1.$$

If $f'(a) = 0$, then

$$0 = f(a)p(a) + f'(a)q(a) = 1$$

which is a contradiction, hence $f'(a) \neq 0$ and f has only simple roots.

Let $\phi \in \mathrm{Aut}(L)$ such that $\phi_{\restriction K} = id_K$. Let $a_1, \ldots, a_n \in L$ be roots of f. Then for $i = 1, \ldots, n$ we have

$$\begin{aligned} 0 = \phi(f(a_i)) = \phi\Big(\sum_{k=0}^{n} \alpha_k a_i^k\Big) = \sum_{k=0}^{n} \phi(\alpha_k a_i^k) = \\ = \sum_{k=0}^{n} \phi(\alpha_k)\phi(a_i^k) = \sum_{k=0}^{n} \alpha_k \phi(a_i)^k = f(\phi(a_i)) \end{aligned}$$

which implies that we can define an automorphism on L by simply mapping a_i to any of the roots of f and keeping the coefficients from K in place. This gives us with at least n such permutations of roots.

EXERCISE 8.

Prove that the set $\{\sqrt{p} \ : \ p$ is a prime number$\}$ is linearly independent over the field $\mathbb{Q}$.

Consider a polynomial $a_1x_1 + a_2x_2 + \ldots + a_nx_n \in \mathbb{Q}[x_1, \ldots, x_n]$