Algebra 2R, list 2.

Homework: Any 3 problems with numbers $\geq \boldsymbol{3}$. Each item in a problem counts as a separate problem. Out of each problem you may turn in a solution of $\leq \boldsymbol{1}$ item. You are not allowed to turn in the solution of item (b) in Problem 4. In solutions you may rely on earlier problems and items, also on the facts from the lecture. Problems marked with – are not discussed in class (unless students ask for it). These problems are not subject to students declarations.

1. – Assume that $f : K \to L$ is a non-zero field homomorphism. Prove that f is $1 - 1$.

2. – (a) Assume that $char(K) = 0,\ f : \mathbb{Q} \to K$,
$$f\left(\frac{n}{m}\right) = \frac{n \cdot 1}{m \cdot 1}.$$
Prove that this definition of f is correct (does not depend on the presentation of a rational number as a fraction $\frac{m}{n}$) and that f is a field homomorphism.
(b) Assume that $char(K) = p,\ f : Z_p \to K,\ f(n) = n \cdot 1$. Prove that f is a homomorphism.

3. Assume that $f : K \to K$ is a non-zero endomorphism (e.g. the Frobenius function). Prove that $Fix(f) = \{x \in K : f(x) = x\}$ is a subfield of the field K.

4. Assume K is a finite field, of characteristic p.
(a) Prove that every irreducible polynomial $f \in K[X]$ divides the polynomial $W_n(X) = X^n - 1$ for some n not divisible by p. (hint: prove that the splitting field of f is finite).
(b) Deduce from (a) (and Remark 3.3) that f has no multiple roots in any field $L \supseteq K$.

5. (a) Prove that if $K \subseteq L$ are finite fields, $|K| = p^m,\ |L| = p^n$, then $m|n$.
(b) Prove that every field with p^n elements contains a unique subfield with p^m elements, where $m|n$.

6. Let $F(p^n)$ be a field with p^n elements. From Problem 5 it follows from that
$$F(p) \subset F(p^2) \subset F(p^{3!}) \subset \cdots \subset F(p^{n!}) \subset \cdots$$
(after suitable identifications of isomorphic fields). Let
$$F = \bigcup_{n>0} F(p^{n!}).$$
Prove that the field F is algebraically closed. (hint: use Problem 4)

7. Let $\boldsymbol{F}$ be as in Problem 6. Let $\boldsymbol{f : F \to F}$ be the Frobenius function. Prove that $\boldsymbol{f}$ is "onto" (it is called the Frobenius automorphism then). Prove that $\boldsymbol{F(p^n) = Fix(f^n)}$.

8. * In the field $\boldsymbol{F}$ from Problem 6 find $\mathbf{2}^{\aleph_0}$ distinct subfields such that the intersection of any two of them is finite.

9. * Prove that distinct subfields of the field $\boldsymbol{F}$ from Problem 6 are not isomorphic. Does there exist a non-trivial automorphism of $\boldsymbol{F}$?

10. – Draw the addition and multiplication tables of
(a) a $\mathbf{4}$-element field,
(b) an $\mathbf{8}$-element field.