Algebra 2R, list 8.

All mappings/functions are R-linear (R-homomorphisms), M, N, P, Q are R-modules. R is a commutative ring with $1 \neq 0$. Homework: as usual.

1. (a)– Assume that $f : M \to N$ is an epimorphism. Prove that f splits $\iff$ $\exists g : N \to M,\ fg = id_N$.
   (b) Assume $g : M \to N$ is a monomorhism. Prove that $g(M)$ is a direct summand of module $N \iff \exists f : N \to M,\ fg = id_M$.

2. Prove that the following conditions are equivalent:
   (a) For every epimorphism $f : M \to N$ of arbitrary modules M, N and every $g : P \to N$ there is an $h : P \to M$ such that $fh = g$.
   (b) Module P is projective.
   (c) There is a module L such that $P \oplus L$ is free.
   (hint: for a proof of (c)$\Rightarrow$(a) consider projection $p : P \oplus L \to P$)

3. Prove that the following conditions are equivalent:
   (a) Module Q is injective.
   (b) For every monomorphism $f : M \to N$ of arbitrary modules M, N and homomorphism $g : M \to Q$ there is an $h : N \to Q$ such that $hf = g$.
   (hint: in a proof of (a)$\Rightarrow$(b) consider the module $M = Q \oplus N/L$, where L is a submodule of $Q \oplus N$ generated by $\{(g(m), -f(m)) : m \in M\}$).

4. (a) Prove that the module $M = \bigoplus_{i\in I} M_i$ is projective $\iff$ every M_i is projective.
   (b) Prove that the module $M = \prod_{i\in I} M_i$ is injective $\iff$ every M_i is inective.

5. Assume that $\{m_1, \ldots, m_n\}$ is a basis of a free R-module M and
$$m'_j = \sum_i r_{ij} m_i,$$
   where $r_{ij} \in R$ for $i, j = 1, \ldots, n$. Prove that the set $\{m'_1, \ldots, m'_n\}$ is a basis of $M \iff \det[r_{ij}]_{n\times n}$ is invertible in R.

6. Assume K_1, K_2 are fields and $R = K_1 \times K_2$ (ring product).
   (a) Prove that every R-module is of the form $V_1 \times V_2$, where V_i is a vector space over K_i $(i = 1, 2)$ and for $(k_1, k_2) \in R,\ (k_1, k_2) \cdot (v_1, v_2) = (k_1 v_1, k_2 v_2)$.
   (b) Prove that every R-module is projective. Which R-modules are free ?

7. (a)– Assume M is a submodule of N and $n \in N$. Prove that the set $I = \{r \in R : rn \in M\}$ is an ideal in R.
   (b) Prove that the R-module Q is injective $\iff$ for every ideal $I \subset R$ and R-homomorphism $f : I \to Q$, f extends to an R-homomorphism $R \to Q$. (hint: To prove $\Leftarrow$ use (a))
   (c) Conclude from (b) that if R is PID, then a module Q jest injektywny
$$\iff \forall r \in R \setminus \{0\} \forall m \in Q \exists m' \in Q,\ rm' = m.$$

(in paertocuar an abelian group G is an injective $\mathbb{Z}$-module $\iff$ G is divisible)

8. * Prove that every module M embeds into an injective R-module. (forsy do it for R that is PID, use the previous problem)
(hint: for an ideal $I \subset R$ and $f : I \to M$ consider the module $M \oplus R/L$, where L is generated by $(f(i), -i)$ for $i \in I$).

   One can prove that there is a smallest injective R-module containing M, it is also unique up to isomorphism. It iois called the injective hull of M.

   One can prove that a nmodule M is injective $\iff$ M is existentially closed in the class of R-modules.