Exercise 8. Let S, T be multiplicatively closed subsets of A, such that $S \subseteq T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $\frac{a}{s} \in S^{-1}A$ to $\frac{a}{s}$ considered as a member of $T^{-1}A$. Show that the following statements are equivalent:

1. ϕ is bijective
2. $(\forall\, t \in T)\ \frac{t}{1}$ is a unit in $S^{-1}A$
3. $(\forall\, t \in T)(\exists\, x \in A)\ xt \in S$
4. Every prime ideal which meets T also meets S

$1 \implies 2$ Obvious?

$2 \implies 3$ Take any $t \in T$. We know that $\frac{t}{1}$ is a unit in both $T^{-1}A$ and $S^{-1}A$. Hence, there exists $\frac{a}{s}$ such that $\frac{t}{1} \cdot \frac{a}{s} = \frac{at}{s} = 1$ which implies that $at \in S$

$3 \implies 4$