Chapter 1
Smooth Manifolds

This book is about *smooth manifolds*. In the simplest terms, these are spaces that locally look like some Euclidean space $\mathbb{R}^n$, and on which one can do calculus. The most familiar examples, aside from Euclidean spaces themselves, are smooth plane curves such as circles and parabolas, and smooth surfaces such as spheres, tori, paraboloids, ellipsoids, and hyperboloids. Higher-dimensional examples include the set of points in $\mathbb{R}^{n+1}$ at a constant distance from the origin (an n*-sphere*) and graphs of smooth maps between Euclidean spaces.

The simplest manifolds are the topological manifolds, which are topological spaces with certain properties that encode what we mean when we say that they "locally look like" $\mathbb{R}^n$. Such spaces are studied intensively by topologists.

However, many (perhaps most) important applications of manifolds involve calculus. For example, applications of manifold theory to geometry involve such properties as volume and curvature. Typically, volumes are computed by integration, and curvatures are computed by differentiation, so to extend these ideas to manifolds would require some means of making sense of integration and differentiation on a manifold. Applications to classical mechanics involve solving systems of ordinary differential equations on manifolds, and the applications to general relativity (the theory of gravitation) involve solving a system of partial differential equations.

The first requirement for transferring the ideas of calculus to manifolds is some notion of "smoothness." For the simple examples of manifolds we described above, all of which are subsets of Euclidean spaces, it is fairly easy to describe the meaning of smoothness on an intuitive level. For example, we might want to call a curve "smooth" if it has a tangent line that varies continuously from point to point, and similarly a "smooth surface" should be one that has a tangent plane that varies continuously. But for more sophisticated applications it is an undue restriction to require smooth manifolds to be subsets of some ambient Euclidean space. The ambient coordinates and the vector space structure of $\mathbb{R}^n$ are superfluous data that often have nothing to do with the problem at hand. It is a tremendous advantage to be able to work with manifolds as abstract topological spaces, without the excess baggage of such an ambient space. For example, in general relativity, spacetime is modeled as a 4-dimensional smooth manifold that carries a certain geometric structure, called a

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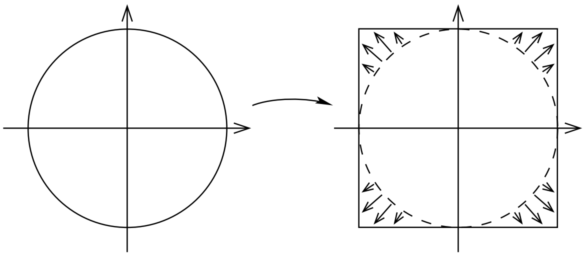

Fig. 1.1 A homeomorphism from a circle to a square

Lorentz metric, whose curvature results in gravitational phenomena. In such a model there is no physical meaning that can be assigned to any higher-dimensional ambient space in which the manifold lives, and including such a space in the model would complicate it needlessly. For such reasons, we need to think of smooth manifolds as abstract topological spaces, not necessarily as subsets of larger spaces.

It is not hard to see that there is no way to define a purely topological property that would serve as a criterion for "smoothness," because it cannot be invariant under homeomorphisms. For example, a circle and a square in the plane are homeomorphic topological spaces (Fig. 1.1), but we would probably all agree that the circle is "smooth," while the square is not. Thus, topological manifolds will not suffice for our purposes. Instead, we will think of a smooth manifold as a set with two layers of structure: first a topology, then a smooth structure.

In the first section of this chapter we describe the first of these structures. A *topological manifold* is a topological space with three special properties that express the notion of being locally like Euclidean space. These properties are shared by Euclidean spaces and by all of the familiar geometric objects that look locally like Euclidean spaces, such as curves and surfaces. We then prove some important topological properties of manifolds that we use throughout the book.

In the next section we introduce an additional structure, called a *smooth structure*, that can be added to a topological manifold to enable us to make sense of derivatives.

Following the basic definitions, we introduce a number of examples of manifolds, so you can have something concrete in mind as you read the general theory. At the end of the chapter we introduce the concept of a *smooth manifold with boundary*, an important generalization of smooth manifolds that will have numerous applications throughout the book, especially in our study of integration in Chapter 16.

Topological Manifolds

In this section we introduce topological manifolds, the most basic type of manifolds. We assume that the reader is familiar with the definition and basic properties of topological spaces, as summarized in Appendix A.

Suppose M is a topological space. We say that M is a ***topological manifold of dimension n*** or a ***topological n-manifold*** if it has the following properties:

- M is a ***Hausdorff space***: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
- M is ***second-countable***: there exists a countable basis for the topology of M.
- M is ***locally Euclidean of dimension n***: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

The third property means, more specifically, that for each $p \in M$ we can find

- an open subset $U \subseteq M$ containing p,
- an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and
- a homeomorphism $\varphi\colon U \to \widehat{U}$.

► **Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

If M is a topological manifold, we often abbreviate the dimension of M as $\dim M$. Informally, one sometimes writes "Let M^n be a manifold" as shorthand for "Let M be a manifold of dimension n." The superscript n is not part of the name of the manifold, and is usually not included in the notation after the first occurrence.

It is important to note that every topological manifold has, by definition, a specific, well-defined dimension. Thus, we do not consider spaces of mixed dimension, such as the disjoint union of a plane and a line, to be manifolds at all. In Chapter 17, we will use the theory of de Rham cohomology to prove the following theorem, which shows that the dimension of a (nonempty) topological manifold is in fact a topological invariant.

Theorem 1.2 (Topological Invariance of Dimension). *A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

For the proof, see Theorem 17.26. In Chapter 2, we will also prove a related but weaker theorem (diffeomorphism invariance of dimension, Theorem 2.17). See also [LeeTM, Chap. 13] for a different proof of Theorem 1.2 using singular homology theory.

The empty set satisfies the definition of a topological n-manifold for every n. For the most part, we will ignore this special case (sometimes without remembering to say so). But because it is useful in certain contexts to allow the empty manifold, we choose not to exclude it from the definition.

The basic example of a topological n-manifold is $\mathbb{R}^n$ itself. It is Hausdorff because it is a metric space, and it is second-countable because the set of all open balls with rational centers and rational radii is a countable basis for its topology.

Requiring that manifolds share these properties helps to ensure that manifolds behave in the ways we expect from our experience with Euclidean spaces. For example, it is easy to verify that in a Hausdorff space, finite subsets are closed and limits of convergent sequences are unique (see Exercise A.11 in Appendix A). The motivation for second-countability is a bit less evident, but it will have important

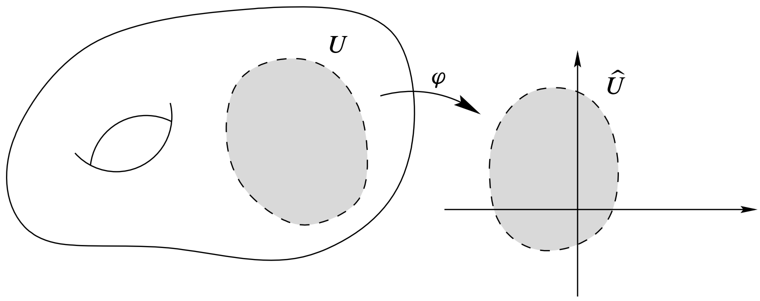

Fig. 1.2 A coordinate chart

consequences throughout the book, mostly based on the existence of partitions of unity (see Chapter 2).

In practice, both the Hausdorff and second-countability properties are usually easy to check, especially for spaces that are built out of other manifolds, because both properties are inherited by subspaces and finite products (Propositions A.17 and A.23). In particular, it follows that every open subset of a topological n-manifold is itself a topological n-manifold (with the subspace topology, of course).

We should note that some authors choose to omit the Hausdorff property or second-countability or both from the definition of manifolds. However, most of the interesting results about manifolds do in fact require these properties, and it is exceedingly rare to encounter a space "in nature" that would be a manifold except for the failure of one or the other of these hypotheses. For a couple of simple examples, see Problems 1-1 and 1-2; for a more involved example (a connected, locally Euclidean, Hausdorff space that is not second-countable), see [LeeTM, Problem 4-6].

Coordinate Charts

Let M be a topological n-manifold. A ***coordinate chart*** (or just a ***chart***) on M is a pair (U, φ), where U is an open subset of M and $\varphi\colon U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (Fig. 1.2). By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, φ). If $\varphi(p) = 0$, we say that the chart is ***centered at p***. If (U, φ) is any chart whose domain contains p, it is easy to obtain a new chart centered at p by subtracting the constant vector $\varphi(p)$.

Given a chart (U, φ), we call the set U a ***coordinate domain***, or a ***coordinate neighborhood*** of each of its points. If, in addition, $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a ***coordinate ball***; if $\varphi(U)$ is an open cube, U is a ***coordinate cube***. The map φ is called a ***(local) coordinate map***, and the component functions $\left(x^1, \dots, x^n\right)$ of φ, defined by $\varphi(p) = \left(x^1(p), \dots, x^n(p)\right)$, are called ***local coordinates*** on U. We sometimes write things such as "(U, φ) is a chart containing p" as shorthand for "(U, φ) is a chart whose domain U contains p." If we wish to emphasize the coordinate functions $(x^1, \dots, x^n)$ instead of the coordinate map φ, we sometimes denote the chart by $(U, (x^1, \dots, x^n))$ or $(U, (x^i))$.

Examples of Topological Manifolds

Here are some simple examples.

Example 1.3 (Graphs of Continuous Functions). Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $f: U \to \mathbb{R}^k$ be a continuous function. The ***graph of*** f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = f(x)\},$$

with the subspace topology. Let $\pi_1: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, and let $\varphi: \Gamma(f) \to U$ be the restriction of π_1 to $\Gamma(f)$:

$$\varphi(x, y) = x, \quad (x, y) \in \Gamma(f).$$

Because φ is the restriction of a continuous map, it is continuous; and it is a homeomorphism because it has a continuous inverse given by $\varphi^{-1}(x) = (x, f(x))$. Thus $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $(\Gamma(f), \varphi)$ is a global coordinate chart, called ***graph coordinates***. The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$. //

Example 1.4 (Spheres). For each integer $n \geq 0$, the unit n-sphere $\mathbb{S}^n$ is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each index $i = 1, \dots, n+1$ let U_i^+ denote the subset of $\mathbb{R}^{n+1}$ where the ith coordinate is positive:

$$U_i^+ = \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i > 0\}.$$

(See Fig. 1.3.) Similarly, U_i^- is the set where $x^i < 0$.

Let $f: \mathbb{B}^n \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then for each $i = 1, \dots, n+1$, it is easy to check that $U_i^+ \cap \mathbb{S}^n$ is the graph of the function

$$x^i = f\left(x^1, \dots, \widehat{x^i}, \dots, x^{n+1}\right),$$

where the hat indicates that x^i is omitted. Similarly, $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f\left(x^1, \dots, \widehat{x^i}, \dots, x^{n+1}\right).$$

Thus, each subset $U_i^\pm \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\varphi_i^\pm: U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\varphi_i^\pm(x^1, \dots, x^{n+1}) = \left(x^1, \dots, \widehat{x^i}, \dots, x^{n+1}\right)$$

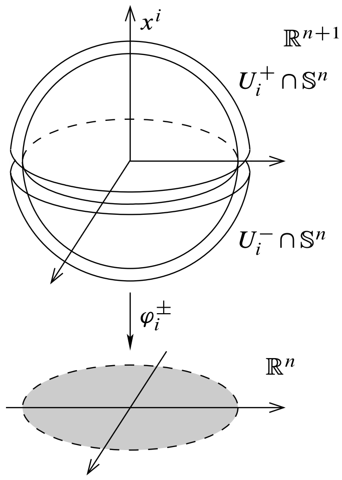

Fig. 1.3 Charts for $\mathbb{S}^n$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n+2$ charts, $\mathbb{S}^n$ is a topological n-manifold. //

Example 1.5 (Projective Spaces). The ***n-dimensional real projective space***, denoted by $\mathbb{RP}^n$ (or sometimes just $\mathbb{P}^n$), is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, with the quotient topology determined by the natural map $\pi: \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{RP}^n$ sending each point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$ to the subspace spanned by x. The 2-dimensional projective space $\mathbb{RP}^2$ is called the ***projective plane***. For any point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, let $[x] = \pi(x) \in \mathbb{RP}^n$ denote the line spanned by x.

For each $i = 1, \dots, n+1$, let $\widetilde{U}_i \subseteq \mathbb{R}^{n+1} \smallsetminus \{0\}$ be the set where $x^i \neq 0$, and let $U_i = \pi\big(\widetilde{U}_i\big) \subseteq \mathbb{RP}^n$. Since $\widetilde{U}_i$ is a saturated open subset, U_i is open and $\pi|_{\widetilde{U}_i}: \widetilde{U}_i \to U_i$ is a quotient map (see Theorem A.27). Define a map $\varphi_i: U_i \to \mathbb{R}^n$ by

$$\varphi_i\left[x^1, \dots, x^{n+1}\right] = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right).$$

This map is well defined because its value is unchanged by multiplying x by a nonzero constant. Because $\varphi_i \circ \pi$ is continuous, φ_i is continuous by the characteristic property of quotient maps (Theorem A.27). In fact, φ_i is a homeomorphism, because it has a continuous inverse given by

$$\varphi_i^{-1}\left(u^1, \dots, u^n\right) = \left[u^1, \dots, u^{i-1}, 1, u^i, \dots, u^n\right],$$

as you can check. Geometrically, $\varphi([x]) = u$ means $(u, 1)$ is the point in $\mathbb{R}^{n+1}$ where the line $[x]$ intersects the affine hyperplane where $x^i = 1$ (Fig. 1.4). Because the sets $U_1, \dots, U_{n+1}$ cover $\mathbb{RP}^n$, this shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n. The Hausdorff and second-countability properties are left as exercises. //

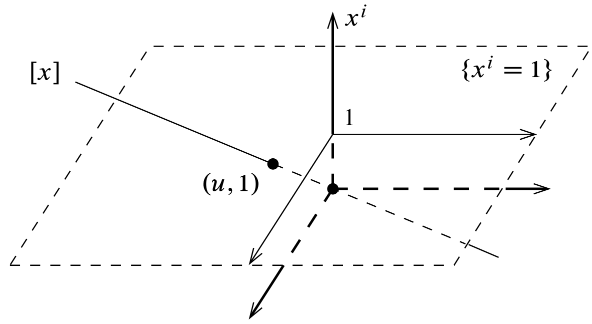

Fig. 1.4 A chart for $\mathbb{RP}^n$

► **Exercise 1.6.** Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

► **Exercise 1.7.** Show that $\mathbb{RP}^n$ is compact. [Hint: show that the restriction of π to $\mathbb{S}^n$ is surjective.]

Example 1.8 (Product Manifolds). Suppose $M_1, \dots, M_k$ are topological manifolds of dimensions $n_1, \dots, n_k$, respectively. The product space $M_1 \times \cdots \times M_k$ is shown to be a topological manifold of dimension $n_1 + \cdots + n_k$ as follows. It is Hausdorff and second-countable by Propositions A.17 and A.23, so only the locally Euclidean property needs to be checked. Given any point $(p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. The product map

$$\varphi_1 \times \cdots \times \varphi_k \colon U_1 \times \cdots \times U_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1 + \cdots + n_k}$. Thus, $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$. //

Example 1.9 (Tori). For a positive integer n, the ***n-torus*** (plural: ***tori***) is the product space $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. By the discussion above, it is a topological n-manifold. (The 2-torus is usually called simply ***the torus***.) //

Topological Properties of Manifolds

As topological spaces go, manifolds are quite special, because they share so many important properties with Euclidean spaces. Here we discuss a few such properties that will be of use to us throughout the book.

Most of the properties we discuss in this section depend on the fact that every manifold possesses a particularly well-behaved basis for its topology.

Lemma 1.10. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proof. Let M be a topological n-manifold. First we consider the special case in which M can be covered by a single chart. Suppose $\varphi\colon M \to \widehat{U} \subseteq \mathbb{R}^n$ is a global coordinate map, and let $\mathcal{B}$ be the collection of all open balls $B_r(x) \subseteq \mathbb{R}^n$ such that r is rational, x has rational coordinates, and $B_{r'}(x) \subseteq \widehat{U}$ for some $r' > r$. Each such ball is precompact in $\widehat{U}$, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of $\widehat{U}$. Because φ is a homeomorphism, it follows that the collection of sets of the form $\varphi^{-1}(B)$ for $B \in \mathcal{B}$ is a countable basis for the topology of M, consisting of precompact coordinate balls, with the restrictions of φ as coordinate maps.

Now let M be an arbitrary n-manifold. By definition, each point of M is in the domain of a chart. Because every open cover of a second-countable space has a countable subcover (Proposition A.16), M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the argument in the preceding paragraph, each coordinate domain U_i has a countable basis of coordinate balls that are precompact in U_i, and the union of all these countable bases is a countable basis for the topology of M. If $V \subseteq U_i$ is one of these balls, then the closure of V in U_i is compact, and because M is Hausdorff, it is closed in M. It follows that the closure of V in M is the same as its closure in U_i, so V is precompact in M as well. □

Connectivity

The existence of a basis of coordinate balls has important consequences for the connectivity properties of manifolds. Recall that a topological space X is

- ***connected*** if there do not exist two disjoint, nonempty, open subsets of X whose union is X;
- ***path-connected*** if every pair of points in X can be joined by a path in X; and
- ***locally path-connected*** if X has a basis of path-connected open subsets.

(See Appendix A.) The following proposition shows that connectivity and path connectivity coincide for manifolds.

Proposition 1.11. *Let M be a topological manifold.*

(a) *M is locally path-connected.*
(b) *M is connected if and only if it is path-connected.*
(c) *The components of M are the same as its path components.*
(d) *M has countably many components, each of which is an open subset of M and a connected topological manifold.*

Proof. Since each coordinate ball is path-connected, (a) follows from the fact that M has a basis of coordinate balls. Parts (b) and (c) are immediate consequences of (a) and Proposition A.43. To prove (d), note that each component is open in M by Proposition A.43, so the collection of components is an open cover of M. Because M is second-countable, this cover must have a countable subcover. But since the components are all disjoint, the cover must have been countable to begin with, which is to say that M has only countably many components. Because the components are open, they are connected topological manifolds in the subspace topology. □

Local Compactness and Paracompactness

The next topological property of manifolds that we need is local compactness (see Appendix A for the definition).

Proposition 1.12 (Manifolds Are Locally Compact). *Every topological manifold is locally compact.*

Proof. Lemma 1.10 showed that every manifold has a basis of precompact open subsets. □

Another key topological property possessed by manifolds is called *paracompactness*. It is a consequence of local compactness and second-countability, and in fact is one of the main reasons why second-countability is included in the definition of manifolds.

Let M be a topological space. A collection $\mathcal{X}$ of subsets of M is said to be ***locally finite*** if each point of M has a neighborhood that intersects at most finitely many of the sets in $\mathcal{X}$. Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a ***refinement of $\mathcal{U}$*** if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$. We say that M is ***paracompact*** if every open cover of M admits an open, locally finite refinement.

Lemma 1.13. *Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.*

(a) *The collection $\{\bar{X} : X \in \mathcal{X}\}$ is also locally finite.*
(b) $\overline{\bigcup_{X \in \mathcal{X}} X} = \bigcup_{X \in \mathcal{X}} \bar{X}$.

► **Exercise 1.14.** Prove the preceding lemma.

Theorem 1.15 (Manifolds Are Paracompact). *Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{X}$ of M, and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.*

Proof. Given M, $\mathcal{X}$, and $\mathcal{B}$ as in the hypothesis of the theorem, let $(K_j)_{j=1}^{\infty}$ be an exhaustion of M by compact sets (Proposition A.60). For each j, let $V_j = K_{j+1} \smallsetminus \operatorname{Int} K_j$ and $W_j = \operatorname{Int} K_{j+2} \smallsetminus K_{j-1}$ (where we interpret K_j as $\varnothing$ if $j < 1$). Then V_j is a compact set contained in the open subset W_j. For each $x \in V_j$, there is some $X_x \in \mathcal{X}$ containing x, and because $\mathcal{B}$ is a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X_x \cap W_j$. The collection of all such sets B_x as x ranges over V_j is an open cover of V_j, and thus has a finite subcover. The union of all such finite subcovers as j ranges over the positive integers is a countable open cover of M that refines $\mathcal{X}$. Because the finite subcover of V_j consists of sets contained in W_j, and $W_j \cap W_{j'} = \varnothing$ except when $j - 2 \le j' \le j + 2$, the resulting cover is locally finite. □

Problem 1-5 shows that, at least for connected spaces, paracompactness can be used as a substitute for second-countability in the definition of manifolds.

Fundamental Groups of Manifolds

The following result about fundamental groups of manifolds will be important in our study of covering manifolds in Chapter 4. For a brief review of the fundamental group, see Appendix A.

Proposition 1.16. *The fundamental group of a topological manifold is countable.*

Proof. Let M be a topological manifold. By Lemma 1.10, there is a countable collection $\mathcal{B}$ of coordinate balls covering M. For any pair of coordinate balls $B, B' \in \mathcal{B}$, the intersection $B \cap B'$ has at most countably many components, each of which is path-connected. Let $\mathcal{X}$ be a countable set containing a point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$ (including $B = B'$). For each $B \in \mathcal{B}$ and each $x, x' \in \mathcal{X}$ such that $x, x' \in B$, let $h^B_{x,x'}$ be some path from x to x' in B.

Since the fundamental groups based at any two points in the same component of M are isomorphic, and $\mathcal{X}$ contains at least one point in each component of M, we may as well choose a point $p \in \mathcal{X}$ as base point. Define a ***special loop*** to be a loop based at p that is equal to a finite product of paths of the form $h^B_{x,x'}$. Clearly, the set of special loops is countable, and each special loop determines an element of $\pi_1(M, p)$. To show that $\pi_1(M, p)$ is countable, therefore, it suffices to show that each element of $\pi_1(M, p)$ is represented by a special loop.

Suppose $f\colon [0,1] \to M$ is a loop based at p. The collection of components of sets of the form $f^{-1}(B)$ as B ranges over $\mathcal{B}$ is an open cover of $[0,1]$, so by compactness it has a finite subcover. Thus, there are finitely many numbers $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $[a_{i-1}, a_i] \subseteq f^{-1}(B)$ for some $B \subseteq \mathcal{B}$. For each i, let f_i be the restriction of f to the interval $[a_{i-1}, a_i]$, reparametrized so that its domain is $[0,1]$, and let $B_i \in \mathcal{B}$ be a coordinate ball containing the image of f_i. For each i, we have $f(a_i) \in B_i \cap B_{i+1}$, and there is some $x_i \in \mathcal{X}$ that lies in the same component of $B_i \cap B_{i+1}$ as $f(a_i)$. Let g_i be a path in $B_i \cap B_{i+1}$ from x_i to $f(a_i)$ (Fig. 1.5), with the understanding that $x_0 = x_k = p$, and g_0 and g_k are both equal to the constant path c_p based at p. Then, because $\overline{g}_i \cdot g_i$ is path-homotopic to a constant path (where $\overline{g}_i(t) = g_i(1-t)$ is the reverse path of g_i),

$$\begin{aligned}
f &\sim f_1 \cdot \cdots \cdot f_k \\
&\sim g_0 \cdot f_1 \cdot \overline{g}_1 \cdot g_1 \cdot f_2 \cdot \overline{g}_2 \cdot \cdots \cdot \overline{g}_{k-1} \cdot g_{k-1} \cdot f_k \cdot \overline{g}_k \\
&\sim \widetilde{f}_1 \cdot \widetilde{f}_2 \cdot \cdots \cdot \widetilde{f}_k,
\end{aligned}$$

where $\widetilde{f}_i = g_{i-1} \cdot f_i \cdot \overline{g}_i$. For each i, $\widetilde{f}_i$ is a path in B_i from x_{i-1} to x_i. Since B_i is simply connected, $\widetilde{f}_i$ is path-homotopic to $h^{B_i}_{x_{i-1},x_i}$. It follows that f is path-homotopic to a special loop, as claimed. □

Smooth Structures

The definition of manifolds that we gave in the preceding section is sufficient for studying topological properties of manifolds, such as compactness, connectedness,

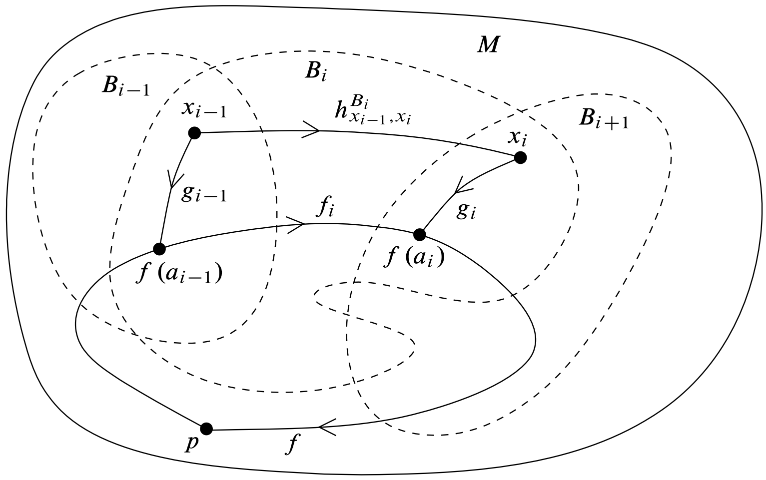

Fig. 1.5 The fundamental group of a manifold is countable

simple connectivity, and the problem of classifying manifolds up to homeomorphism. However, in the entire theory of topological manifolds there is no mention of calculus. There is a good reason for this: however we might try to make sense of derivatives of functions on a manifold, such derivatives cannot be invariant under homeomorphisms. For example, the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(u,v) = \left(u^{1/3}, v^{1/3}\right)$ is a homeomorphism, and it is easy to construct differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ \varphi$ is not differentiable at the origin. (The function $f(x,y) = x$ is one such.)

To make sense of derivatives of real-valued functions, curves, or maps between manifolds, we need to introduce a new kind of manifold called a *smooth manifold*. It will be a topological manifold with some extra structure in addition to its topology, which will allow us to decide which functions to or from the manifold are smooth.

The definition will be based on the calculus of maps between Euclidean spaces, so let us begin by reviewing some basic terminology about such maps. If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F\colon U \to V$ is said to be ***smooth*** (or $\boldsymbol{C^\infty}$, or ***infinitely differentiable***) if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a ***diffeomorphism***. A diffeomorphism is, in particular, a homeomorphism.

A review of some important properties of smooth maps is given in Appendix C. You should be aware that some authors define the word *smooth* differently—for example, to mean continuously differentiable or merely differentiable. On the other hand, some use the word *differentiable* to mean what we call *smooth*. Throughout this book, *smooth* is synonymous with C^∞.

To see what additional structure on a topological manifold might be appropriate for discerning which maps are smooth, consider an arbitrary topological n-manifold M. Each point in M is in the domain of a coordinate map $\varphi\colon U \to \widehat{U} \subseteq \mathbb{R}^n$.

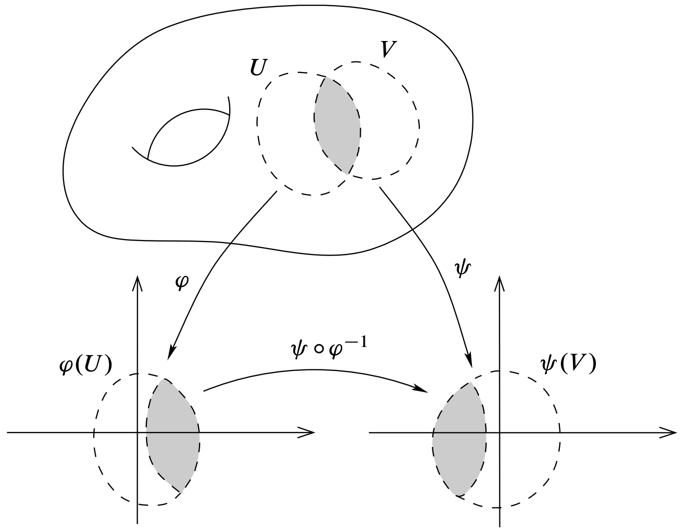

Fig. 1.6 A transition map

A plausible definition of a smooth function on M would be to say that $f\colon M \to \mathbb{R}$ is smooth if and only if the composite function $f \circ \varphi^{-1}\colon \widehat{U} \to \mathbb{R}$ is smooth in the sense of ordinary calculus. But this will make sense only if this property is independent of the choice of coordinate chart. To guarantee this independence, we will restrict our attention to "smooth charts." Since smoothness is not a homeomorphism-invariant property, the way to do this is to consider the collection of all smooth charts as a new kind of structure on M.

With this motivation in mind, we now describe the details of the construction.

Let M be a topological n-manifold. If (U,φ), (V,ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is called the ***transition map from φ to ψ*** (Fig. 1.6). It is a composition of homeomorphisms, and is therefore itself a homeomorphism. Two charts (U,φ) and (V,ψ) are said to be ***smoothly compatible*** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders.

We define an ***atlas for M*** to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a ***smooth atlas*** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

To show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U,φ) and (V,ψ) are charts in $\mathcal{A}$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $\left(\psi \circ \varphi^{-1}\right)^{-1} = \varphi \circ \psi^{-1}$ is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts (U,φ) and (V,ψ), it is often easiest to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to Corollary C.36.

Our plan is to define a "smooth structure" on M by giving a smooth atlas, and to define a function $f\colon M \to \mathbb{R}$ to be smooth if and only if $f \circ \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, φ) in the atlas. There is one minor technical problem with this approach: in general, there will be many possible atlases that give the "same" smooth structure, in that they all determine the same collection of smooth functions on M. For example, consider the following pair of atlases on $\mathbb{R}^n$:

$$\begin{aligned}\mathcal{A}_1 &= \left\{\left(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n}\right)\right\},\\ \mathcal{A}_2 &= \left\{\left(B_1(x), \mathrm{Id}_{B_1(x)}\right) : x \in \mathbb{R}^n\right\}.\end{aligned}$$

Although these are different smooth atlases, clearly a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is smooth with respect to either atlas if and only if it is smooth in the sense of ordinary calculus.

We could choose to define a smooth structure as an equivalence class of smooth atlases under an appropriate equivalence relation. However, it is more straightforward to make the following definition: a smooth atlas $\mathcal{A}$ on M is ***maximal*** if it is not properly contained in any larger smooth atlas. This just means that any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. (Such a smooth atlas is also said to be ***complete***.)

Now we can define the main concept of this chapter. If M is a topological manifold, a ***smooth structure on M*** is a maximal smooth atlas. A ***smooth manifold*** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. When the smooth structure is understood, we usually omit mention of it and just say "M is a smooth manifold." Smooth structures are also called ***differentiable structures*** or ***C^∞ structures*** by some authors. We also use the term ***smooth manifold structure*** to mean a manifold topology together with a smooth structure.

We emphasize that a smooth structure is an additional piece of data that must be added to a topological manifold before we are entitled to talk about a "smooth manifold." In fact, a given topological manifold may have many different smooth structures (see Example 1.23 and Problem 1-6). On the other hand, it is not always possible to find a smooth structure on a given topological manifold: there exist topological manifolds that admit no smooth structures at all. (The first example was a compact 10-dimensional manifold found in 1960 by Michel Kervaire [Ker60].)

It is generally not very convenient to define a smooth structure by explicitly describing a maximal smooth atlas, because such an atlas contains very many charts. Fortunately, we need only specify *some* smooth atlas, as the next proposition shows.

Proposition 1.17. *Let M be a topological manifold.*

(a) *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the* ***smooth structure determined by $\mathcal{A}$***.
(b) *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

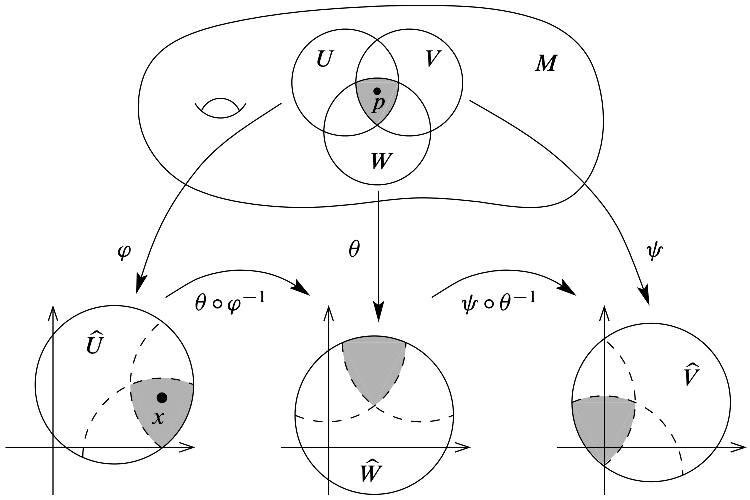

Fig. 1.7 Proof of Proposition 1.17(a)

Proof. Let $\mathcal{A}$ be a smooth atlas for M, and let $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that $\overline{\mathcal{A}}$ is a smooth atlas, we need to show that any two charts of $\overline{\mathcal{A}}$ are smoothly compatible with each other, which is to say that for any (U,φ), $(V,\psi) \in \overline{\mathcal{A}}$, the map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is smooth.

Let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary. Because the domains of the charts in $\mathcal{A}$ cover M, there is some chart $(W,\theta) \in \mathcal{A}$ such that $p \in W$ (Fig. 1.7). Since every chart in $\overline{\mathcal{A}}$ is smoothly compatible with (W,θ), both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that $\psi \circ \varphi^{-1} = \left(\psi \circ \theta^{-1}\right) \circ \left(\theta \circ \varphi^{-1}\right)$ is smooth on a neighborhood of x. Thus, $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. Therefore, $\overline{\mathcal{A}}$ is a smooth atlas. To check that it is maximal, just note that any chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ must in particular be smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. This proves the existence of a maximal smooth atlas containing $\mathcal{A}$. If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subseteq \overline{\mathcal{A}}$. By maximality of $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{A}}$.

The proof of (b) is left as an exercise. □

► **Exercise 1.18.** Prove Proposition 1.17(b).

For example, if a topological manifold M can be covered by a single chart, the smooth compatibility condition is trivially satisfied, so any such chart automatically determines a smooth structure on M.

It is worth mentioning that the notion of smooth structure can be generalized in several different ways by changing the compatibility requirement for charts. For example, if we replace the requirement that charts be smoothly compatible by the weaker requirement that each transition map $\psi \circ \varphi^{-1}$ (and its inverse) be of class C^k, we obtain the definition of a ***C^k structure***. Similarly, if we require that each transition map be real-analytic (i.e., expressible as a convergent power series in a neighborhood of each point), we obtain the definition of a ***real-analytic structure***, also called a ***C^ω structure***. If M has even dimension $n = 2m$, we can identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ and require that the transition maps be complex-analytic; this determines a ***complex-analytic structure***. A manifold endowed with one of these structures is called a ***C^k manifold***, ***real-analytic manifold***, or ***complex manifold***, respectively. (Note that a C^0 manifold is just a topological manifold.) We do not treat any of these other kinds of manifolds in this book, but they play important roles in analysis, so it is useful to know the definitions.

Local Coordinate Representations

If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a ***smooth chart***, and the corresponding coordinate map φ is called a ***smooth coordinate map***. It is useful also to introduce the terms ***smooth coordinate domain*** or ***smooth coordinate neighborhood*** for the domain of a smooth coordinate chart. A ***smooth coordinate ball*** means a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space. A ***smooth coordinate cube*** is defined similarly.

It is often useful to restrict attention to coordinate balls whose closures sit nicely inside larger coordinate balls. We say a set $B \subseteq M$ is a ***regular coordinate ball*** if there is a smooth coordinate ball $B' \supseteq \overline{B}$ and a smooth coordinate map $\varphi\colon B' \to \mathbb{R}^n$ such that for some positive real numbers $r < r'$,

$$\varphi(B) = B_r(0), \qquad \varphi\big(\overline{B}\big) = \overline{B}_r(0), \quad \text{and} \quad \varphi\left(B'\right) = B_{r'}(0).$$

Because $\overline{B}$ is homeomorphic to $\overline{B}_r(0)$, it is compact, and thus every regular coordinate ball is precompact in M. The next proposition gives a slight improvement on Lemma 1.10 for smooth manifolds. Its proof is a straightforward adaptation of the proof of that lemma.

Proposition 1.19. *Every smooth manifold has a countable basis of regular coordinate balls.*

► **Exercise 1.20.** Prove Proposition 1.19.

Here is how one usually thinks about coordinate charts on a smooth manifold. Once we choose a smooth chart (U, φ) on M, the coordinate map $\varphi\colon U \to \widehat{U} \subseteq \mathbb{R}^n$ can be thought of as giving a temporary *identification* between U and $\widehat{U}$. Using this identification, while we work in this chart, we can think of U simultaneously as an open subset of M and as an open subset of $\mathbb{R}^n$. You can visualize this identification by thinking of a "grid" drawn on U representing the preimages of the coordinate lines under φ (Fig. 1.8). Under this identification, we can represent a point $p \in U$ by its coordinates $\left(x^1, \dots, x^n\right) = \varphi(p)$, and think of this n-tuple as *being* the

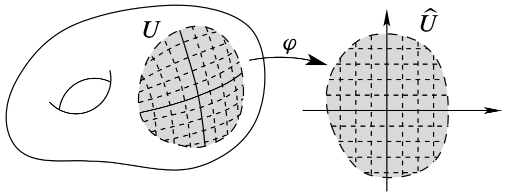

Fig. 1.8 A coordinate grid

point p. We typically express this by saying "$(x^1, \dots, x^n)$ is the (local) coordinate representation for p" or "$p = (x^1, \dots, x^n)$ in local coordinates."

Another way to look at it is that by means of our identification $U \leftrightarrow \widehat{U}$, we can think of φ as the identity map and suppress it from the notation. This takes a bit of getting used to, but the payoff is a huge simplification of the notation in many situations. You just need to remember that the identification is in general only local, and depends heavily on the choice of coordinate chart.

You are probably already used to such identifications from your study of multivariable calculus. The most common example is ***polar coordinates*** (r, θ) in the plane, defined implicitly by the relation $(x, y) = (r\cos\theta, r\sin\theta)$ (see Example C.37). On an appropriate open subset such as $U = \{(x, y) : x > 0\} \subseteq \mathbb{R}^2$, (r, θ) can be expressed as smooth functions of (x, y), and the map that sends (x, y) to the corresponding (r, θ) is a smooth coordinate map with respect to the standard smooth structure on $\mathbb{R}^2$. Using this map, we can write a given point $p \in U$ either as $p = (x, y)$ in standard coordinates or as $p = (r, \theta)$ in polar coordinates, where the two coordinate representations are related by $(r, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1} y/x\right)$ and $(x, y) = (r\cos\theta, r\sin\theta)$. Other polar coordinate charts can be obtained by restricting (r, θ) to other open subsets of $\mathbb{R}^2 \smallsetminus \{0\}$.

The fact that manifolds do not come with any predetermined choice of coordinates is both a blessing and a curse. The flexibility to choose coordinates more or less arbitrarily can be a big advantage in approaching problems in manifold theory, because the coordinates can often be chosen to simplify some aspect of the problem at hand. But we pay for this flexibility by being obliged to ensure that any objects we wish to define globally on a manifold are not dependent on a particular choice of coordinates. There are generally two ways of doing this: either by writing down a coordinate-dependent definition and then proving that the definition gives the same results in any coordinate chart, or by writing down a definition that is manifestly coordinate-independent (often called an *invariant definition*). We will use the coordinate-dependent approach in a few circumstances where it is notably simpler, but for the most part we will give coordinate-free definitions whenever possible. The need for such definitions accounts for much of the abstraction of modern manifold theory. One of the most important skills you will need to acquire in order to use manifold theory effectively is an ability to switch back and forth easily between invariant descriptions and their coordinate counterparts.

Examples of Smooth Manifolds

Before proceeding further with the general theory, let us survey some examples of smooth manifolds.

Example 1.21 (0-Dimensional Manifolds). A topological manifold M of dimension 0 is just a countable discrete space. For each point $p \in M$, the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and there is exactly one coordinate map $\varphi\colon \{p\} \to \mathbb{R}^0$. Thus, the set of all charts on M trivially satisfies the smooth compatibility condition, and each 0-dimensional manifold has a unique smooth structure. //

Example 1.22 (Euclidean Spaces). For each nonnegative integer n, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$. We call this the ***standard smooth structure on $\mathbb{R}^n$*** and the resulting coordinate map ***standard coordinates***. Unless we explicitly specify otherwise, we always use this smooth structure on $\mathbb{R}^n$. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is a diffeomorphism (in the sense of ordinary calculus) from U to another open subset $\widehat{U} \subseteq \mathbb{R}^n$. //

Example 1.23 (Another Smooth Structure on $\mathbb{R}$). Consider the homeomorphism $\psi\colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = x^3. \tag{1.1}$$

The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$. This chart is not smoothly compatible with the standard smooth structure, because the transition map $\mathrm{Id}_{\mathbb{R}} \circ \psi^{-1}(y) = y^{1/3}$ is not smooth at the origin. Therefore, the smooth structure defined on $\mathbb{R}$ by ψ is not the same as the standard one. Using similar ideas, it is not hard to construct many distinct smooth structures on any given positive-dimensional topological manifold, as long as it has one smooth structure to begin with (see Problem 1-6). //

Example 1.24 (Finite-Dimensional Vector Spaces). Let V be a finite-dimensional real vector space. Any norm on V determines a topology, which is independent of the choice of norm (Exercise B.49). With this topology, V is a topological n-manifold, and has a natural smooth structure defined as follows. Each (ordered) basis $(E_1, \dots, E_n)$ for V defines a basis isomorphism $E\colon \mathbb{R}^n \to V$ by

$$E(x) = \sum_{i=1}^{n} x^i E_i.$$

This map is a homeomorphism, so $\left(V, E^{-1}\right)$ is a chart. If $\left(\widetilde{E}_1, \dots, \widetilde{E}_n\right)$ is any other basis and $\widetilde{E}(x) = \sum_j x^j \widetilde{E}_j$ is the corresponding isomorphism, then there is some invertible matrix $\left(A_i^j\right)$ such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. The transition map between the two charts is then given by $\widetilde{E}^{-1} \circ E(x) = \widetilde{x}$, where $\widetilde{x} = \left(\widetilde{x}^1, \dots, \widetilde{x}^n\right)$

is determined by

$$\sum_{j=1}^{n} \widetilde{x}^j \widetilde{E}_j = \sum_{i=1}^{n} x^i E_i = \sum_{i,j=1}^{n} x^i A_i^j \widetilde{E}_j.$$

It follows that $\widetilde{x}^j = \sum_i A_i^j x^i$. Thus, the map sending x to $\widetilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the ***standard smooth structure on V***. //

The Einstein Summation Convention

This is a good place to pause and introduce an important notational convention that is commonly used in the study of smooth manifolds. Because of the proliferation of summations such as $\sum_i x^i E_i$ in this subject, we often abbreviate such a sum by omitting the summation sign, as in

$$E(x) = x^i E_i, \quad \text{an abbreviation for } E(x) = \sum_{i=1}^{n} x^i E_i.$$

We interpret any such expression according to the following rule, called the ***Einstein summation convention***: if the same index name (such as i in the expression above) appears exactly twice in any monomial term, once as an upper index and once as a lower index, that term is understood to be summed over all possible values of that index, generally from 1 to the dimension of the space in question. This simple idea was introduced by Einstein to reduce the complexity of expressions arising in the study of smooth manifolds by eliminating the necessity of explicitly writing summation signs. We use the summation convention systematically throughout the book (except in the appendices, which many readers will look at before the rest of the book).

Another important aspect of the summation convention is the positions of the indices. We always write basis vectors (such as E_i) with lower indices, and components of a vector with respect to a basis (such as x^i) with upper indices. These index conventions help to ensure that, in summations that make mathematical sense, each index to be summed over typically appears twice in any given term, once as a lower index and once as an upper index. Any index that is implicitly summed over is a "dummy index," meaning that the value of such an expression is unchanged if a different name is substituted for each dummy index. For example, $x^i E_i$ and $x^j E_j$ mean exactly the same thing.

Since the coordinates of a point $(x^1, \dots, x^n) \in \mathbb{R}^n$ are also its components with respect to the standard basis, in order to be consistent with our convention of writing components of vectors with upper indices, we need to use upper indices for these coordinates, and we do so throughout this book. Although this may seem awkward at first, in combination with the summation convention it offers enormous advantages when we work with complicated indexed sums, not the least of which is that expressions that are not mathematically meaningful often betray themselves quickly by violating the index convention. (The main exceptions are expressions involving the Euclidean dot product $x \cdot y = \sum_i x^i y^i$, in which the same index appears twice in the upper position, and the standard symplectic form on $\mathbb{R}^{2n}$, which we will define in Chapter 22. We always explicitly write summation signs in such expressions.)

More Examples

Now we continue with our examples of smooth manifolds.

Example 1.25 (Spaces of Matrices). Let $\mathrm{M}(m \times n, \mathbb{R})$ denote the set of $m \times n$ matrices with real entries. Because it is a real vector space of dimension mn under matrix addition and scalar multiplication, $\mathrm{M}(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold. (In fact, it is often useful to *identify* $\mathrm{M}(m \times n, \mathbb{R})$ with $\mathbb{R}^{mn}$, just by stringing all the matrix entries out in a single row.) Similarly, the space $\mathrm{M}(m \times n, \mathbb{C})$ of $m \times n$ complex matrices is a vector space of dimension $2mn$ over $\mathbb{R}$, and thus a smooth manifold of dimension $2mn$. In the special case in which $m = n$ (square matrices), we abbreviate $\mathrm{M}(n \times n, \mathbb{R})$ and $\mathrm{M}(n \times n, \mathbb{C})$ by $\mathrm{M}(n, \mathbb{R})$ and $\mathrm{M}(n, \mathbb{C})$, respectively. //

Example 1.26 (Open Submanifolds). Let U be any open subset of $\mathbb{R}^n$. Then U is a topological n-manifold, and the single chart (U, Id_U) defines a smooth structure on U.

More generally, let M be a smooth n-manifold and let $U \subseteq M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U\}.$$

Every point $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore, U is covered by the domains of charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U. Thus any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an ***open submanifold of*** $\boldsymbol{M}$. (We will define a more general class of submanifolds in Chapter 5.) //

Example 1.27 (The General Linear Group). The ***general linear group*** $\mathrm{GL}(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the n^2-dimensional vector space $\mathrm{M}(n, \mathbb{R})$, namely the set where the (continuous) determinant function is nonzero. //

Example 1.28 (Matrices of Full Rank). The previous example has a natural generalization to rectangular matrices of full rank. Suppose $m < n$, and let $\mathrm{M}_m(m \times n, \mathbb{R})$ denote the subset of $\mathrm{M}(m \times n, \mathbb{R})$ consisting of matrices of rank m. If A is an arbitrary such matrix, the fact that $\operatorname{rank} A = m$ means that A has some nonsingular $m \times m$ submatrix. By continuity of the determinant function, this same submatrix has nonzero determinant on a neighborhood of A in $\mathrm{M}(m \times n, \mathbb{R})$, which implies that A has a neighborhood contained in $\mathrm{M}_m(m \times n, \mathbb{R})$. Thus, $\mathrm{M}_m(m \times n, \mathbb{R})$ is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$, and therefore is itself a smooth mn-dimensional manifold. A similar argument shows that $\mathrm{M}_n(m \times n, \mathbb{R})$ is a smooth mn-manifold when $n < m$. //

Example 1.29 (Spaces of Linear Maps). Suppose V and W are finite-dimensional real vector spaces, and let $\mathrm{L}(V; W)$ denote the set of linear maps from V to W. Then because $\mathrm{L}(V; W)$ is itself a finite-dimensional vector space (whose dimension is the product of the dimensions of V and W), it has a natural smooth manifold structure as in Example 1.24. One way to put global coordinates on it is to choose bases for V and W, and represent each $T \in \mathrm{L}(V; W)$ by its matrix, which yields an isomorphism of $\mathrm{L}(V; W)$ with $\mathrm{M}(m \times n, \mathbb{R})$ for $m = \dim W$ and $n = \dim V$. //

Example 1.30 (Graphs of Smooth Functions). If $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is a smooth function, we have already observed above (Example 1.3) that the graph of f is a topological n-manifold in the subspace topology. Since $\Gamma(f)$ is covered by the single graph coordinate chart $\varphi : \Gamma(f) \to U$ (the restriction of π_1), we can put a canonical smooth structure on $\Gamma(f)$ by declaring the graph coordinate chart $(\Gamma(f), \varphi)$ to be a smooth chart. //

Example 1.31 (Spheres). We showed in Example 1.4 that the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold. We put a smooth structure on $\mathbb{S}^n$ as follows. For each $i = 1, \dots, n+1$, let $\left(U_i^{\pm}, \varphi_i^{\pm}\right)$ denote the graph coordinate charts we constructed in Example 1.4. For any distinct indices i and j, the transition map $\varphi_i^{\pm} \circ \left(\varphi_j^{\pm}\right)^{-1}$ is easily computed. In the case $i < j$, we get

$$\varphi_i^{\pm} \circ \left(\varphi_j^{\pm}\right)^{-1}\left(u^1, \dots, u^n\right) = \left(u^1, \dots, \widehat{u^i}, \dots, \pm\sqrt{1 - |u|^2}, \dots, u^n\right)$$

(with the square root in the jth position), and a similar formula holds when $i > j$. When $i = j$, an even simpler computation gives $\varphi_i^+ \circ \left(\varphi_i^-\right)^{-1} = \varphi_i^- \circ \left(\varphi_i^+\right)^{-1} = \mathrm{Id}_{\mathbb{B}^n}$. Thus, the collection of charts $\left\{\left(U_i^{\pm}, \varphi_i^{\pm}\right)\right\}$ is a smooth atlas, and so defines a smooth structure on $\mathbb{S}^n$. We call this its ***standard smooth structure***. //

Example 1.32 (Level Sets). The preceding example can be generalized as follows. Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $\Phi : U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set $\Phi^{-1}(c)$ is called a ***level set of Φ***. Choose some $c \in \mathbb{R}$, let $M = \Phi^{-1}(c)$, and suppose it happens that the total derivative $D\Phi(a)$ is nonzero for each $a \in \Phi^{-1}(c)$. Because $D\Phi(a)$ is a row matrix whose entries are the partial derivatives $(\partial\Phi/\partial x^1(a), \dots, \partial\Phi/\partial x^n(a))$, for each $a \in M$ there is some i such that $\partial\Phi/\partial x^i(a) \neq 0$. It follows from the implicit function theorem (Theorem C.40, with x^i playing the role of y) that there is a neighborhood U_0 of a such that $M \cap U_0$ can be expressed as a graph of an equation of the form

$$x^i = f\left(x^1, \dots, \widehat{x^i}, \dots, x^n\right),$$

for some smooth real-valued function f defined on an open subset of $\mathbb{R}^{n-1}$. Therefore, arguing just as in the case of the n-sphere, we see that M is a topological manifold of dimension $(n-1)$, and has a smooth structure such that each of the graph coordinate charts associated with a choice of f as above is a smooth chart. In Chapter 5, we will develop the theory of smooth submanifolds, which is a far-reaching generalization of this construction. //

Example 1.33 (Projective Spaces). The n-dimensional real projective space $\mathbb{RP}^n$ is a topological n-manifold by Example 1.5. Let us check that the coordinate charts (U_i, φ_i) constructed in that example are all smoothly compatible. Assuming for convenience that $i > j$, it is straightforward to compute that

$$\varphi_j \circ \varphi_i^{-1}(u^1, \dots, u^n) = \left(\frac{u^1}{u^j}, \dots, \frac{u^{j-1}}{u^j}, \frac{u^{j+1}}{u^j}, \dots, \frac{u^{i-1}}{u^j}, \frac{1}{u^j}, \frac{u^i}{u^j}, \dots, \frac{u^n}{u^j}\right),$$

which is a diffeomorphism from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$. //

Example 1.34 (Smooth Product Manifolds). If $M_1, \dots, M_k$ are smooth manifolds of dimensions $n_1, \dots, n_k$, respectively, we showed in Example 1.8 that the product space $M_1 \times \dots \times M_k$ is a topological manifold of dimension $n_1 + \dots + n_k$, with charts of the form $(U_1 \times \dots \times U_k, \varphi_1 \times \dots \times \varphi_k)$. Any two such charts are smoothly compatible because, as is easily verified,

$$(\psi_1 \times \dots \times \psi_k) \circ (\varphi_1 \times \dots \times \varphi_k)^{-1} = \left(\psi_1 \circ \varphi_1^{-1}\right) \times \dots \times \left(\psi_k \circ \varphi_k^{-1}\right),$$

which is a smooth map. This defines a natural smooth manifold structure on the product, called the ***product smooth manifold structure***. For example, this yields a smooth manifold structure on the n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$. //

In each of the examples we have seen so far, we constructed a smooth manifold structure in two stages: we started with a topological space and checked that it was a topological manifold, and then we specified a smooth structure. It is often more convenient to combine these two steps into a single construction, especially if we start with a set that is not already equipped with a topology. The following lemma provides a shortcut—it shows how, given a set with suitable "charts" that overlap smoothly, we can use the charts to define both a topology and a smooth structure on the set.

Lemma 1.35 (Smooth Manifold Chart Lemma). *Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied*:

(i) *For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.*
(ii) *For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*
(iii) *Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.*
(iv) *Countably many of the sets U_α cover M.*
(v) *Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.*

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

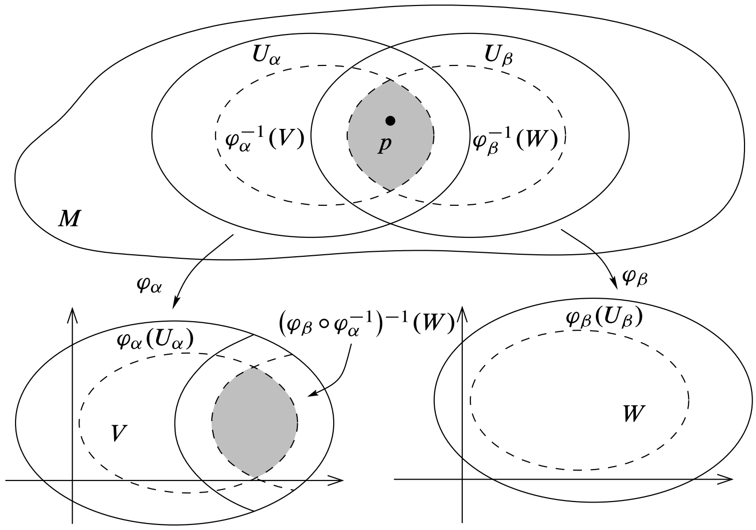

Fig. 1.9 The smooth manifold chart lemma

Proof. We define the topology by taking all sets of the form $\varphi_\alpha^{-1}(V)$, with V an open subset of $\mathbb{R}^n$, as a basis. To prove that this is a basis for a topology, we need to show that for any point p in the intersection of two basis sets $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$, there is a third basis set containing p and contained in the intersection. It suffices to show that $\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$ is itself a basis set (Fig. 1.9). To see this, observe that (iii) implies that $(\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W)$ is an open subset of $\varphi_\alpha(U_\alpha \cap U_\beta)$, and (ii) implies that this set is also open in $\mathbb{R}^n$. It follows that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}\left(V \cap \left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^{-1}(W)\right)$$

is also a basis set, as claimed.

Each map φ_α is then a homeomorphism onto its image (essentially by definition), so M is locally Euclidean of dimension n. The Hausdorff property follows easily from (v), and second-countability follows from (iv) and the result of Exercise A.22, because each U_α is second-countable. Finally, (iii) guarantees that the collection $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas. It is clear that this topology and smooth structure are the unique ones satisfying the conclusions of the lemma. □

Example 1.36 (Grassmann Manifolds). Let V be an n-dimensional real vector space. For any integer $0 \le k \le n$, we let $\mathrm{G}_k(V)$ denote the set of all k-dimensional linear subspaces of V. We will show that $\mathrm{G}_k(V)$ can be naturally given the structure of a smooth manifold of dimension $k(n-k)$. With this structure, it is called a ***Grassmann manifold***, or simply a ***Grassmannian***. In the special case $V = \mathbb{R}^n$, the Grassmannian $\mathrm{G}_k(\mathbb{R}^n)$ is often denoted by some simpler notation such as $\mathrm{G}_{k,n}$ or $\mathrm{G}(k,n)$. Note that $\mathrm{G}_1(\mathbb{R}^{n+1})$ is exactly the n-dimensional projective space $\mathbb{RP}^n$.

The construction of a smooth structure on $\mathrm{G}_k(V)$ is somewhat more involved than the ones we have done so far, but the basic idea is just to use linear algebra to construct charts for $\mathrm{G}_k(V)$, and then apply the smooth manifold chart lemma. We will give a shorter proof that $\mathrm{G}_k(V)$ is a smooth manifold in Chapter 21 (see Example 21.21).

Let P and Q be any complementary subspaces of V of dimensions k and $n-k$, respectively, so that V decomposes as a direct sum: $V = P \oplus Q$. The graph of any linear map $X\colon P \to Q$ can be identified with a k-dimensional subspace $\Gamma(X) \subseteq V$, defined by

$$\Gamma(X) = \{v + Xv : v \in P\}.$$

Any such subspace has the property that its intersection with Q is the zero subspace. Conversely, any subspace $S \subseteq V$ that intersects Q trivially is the graph of a unique linear map $X\colon P \to Q$, which can be constructed as follows: let $\pi_P\colon V \to P$ and $\pi_Q\colon V \to Q$ be the projections determined by the direct sum decomposition; then the hypothesis implies that $\pi_P|_S$ is an isomorphism from S to P. Therefore, $X = (\pi_Q|_S) \circ (\pi_P|_S)^{-1}$ is a well-defined linear map from P to Q, and it is straightforward to check that S is its graph.

Let $\mathrm{L}(P;Q)$ denote the vector space of linear maps from P to Q, and let U_Q denote the subset of $\mathrm{G}_k(V)$ consisting of k-dimensional subspaces whose intersections with Q are trivial. The assignment $X \mapsto \Gamma(X)$ defines a map $\Gamma\colon \mathrm{L}(P;Q) \to U_Q$, and the discussion above shows that Γ is a bijection. Let $\varphi = \Gamma^{-1}\colon U_Q \to \mathrm{L}(P;Q)$. By choosing bases for P and Q, we can identify $\mathrm{L}(P;Q)$ with $\mathrm{M}((n-k)\times k, \mathbb{R})$ and hence with $\mathbb{R}^{k(n-k)}$, and thus we can think of (U_Q, φ) as a coordinate chart. Since the image of each such chart is all of $\mathrm{L}(P;Q)$, condition (i) of Lemma 1.35 is clearly satisfied.

Now let (P', Q') be any other such pair of subspaces, and let $\pi_{P'}, \pi_{Q'}$ be the corresponding projections and $\varphi'\colon U_{Q'} \to \mathrm{L}(P';Q')$ the corresponding map. The set $\varphi(U_Q \cap U_{Q'}) \subseteq \mathrm{L}(P;Q)$ consists of all linear maps $X\colon P \to Q$ whose graphs intersect Q' trivially. To see that this set is open in $\mathrm{L}(P;Q)$, for each $X \in \mathrm{L}(P;Q)$, let $I_X\colon P \to V$ be the map $I_X(v) = v + Xv$, which is a bijection from P to the graph of X. Because $\Gamma(X) = \operatorname{Im} I_X$ and $Q' = \operatorname{Ker} \pi_{P'}$, it follows from Exercise B.22(d) that the graph of X intersects Q' trivially if and only if $\pi_{P'} \circ I_X$ has full rank. Because the matrix entries of $\pi_{P'} \circ I_X$ (with respect to any bases) depend continuously on X, the result of Example 1.28 shows that the set of all such X is open in $\mathrm{L}(P;Q)$. Thus property (ii) in the smooth manifold chart lemma holds.

We need to show that the transition map $\varphi' \circ \varphi^{-1}$ is smooth on $\varphi(U_Q \cap U_{Q'})$. Suppose $X \in \varphi(U_Q \cap U_{Q'}) \subseteq \mathrm{L}(P;Q)$ is arbitrary, and let S denote the subspace $\Gamma(X) \subseteq V$. If we put $X' = \varphi' \circ \varphi^{-1}(X)$, then as above, $X' = (\pi_{Q'}|_S) \circ (\pi_{P'}|_S)^{-1}$ (see Fig. 1.10). To relate this map to X, note that $I_X\colon P \to S$ is an isomorphism, so we can write

$$X' = (\pi_{Q'}|_S) \circ I_X \circ (I_X)^{-1} \circ (\pi_{P'}|_S)^{-1} = (\pi_{Q'} \circ I_X) \circ (\pi_{P'} \circ I_X)^{-1}.$$

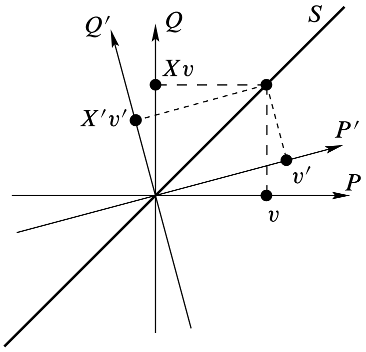

Fig. 1.10 Smooth compatibility of coordinates on $G_k(V)$

To show that this depends smoothly on X, define linear maps $A\colon P \to P'$, $B\colon P \to Q'$, $C\colon Q \to P'$, and $D\colon Q \to Q'$ as follows:

$$A = \pi_{P'}\big|_P, \qquad B = \pi_{Q'}\big|_P, \qquad C = \pi_{P'}\big|_Q, \qquad D = \pi_{Q'}\big|_Q.$$

Then for $v \in P$, we have

$$(\pi_{P'} \circ I_X)v = (A + CX)v, \qquad (\pi_{Q'} \circ I_X)v = (B + DX)v,$$

from which it follows that $X' = (B + DX)(A + CX)^{-1}$. Once we choose bases for P, Q, P', and Q', all of these linear maps are represented by matrices. Because the matrix entries of $(A + CX)^{-1}$ are rational functions of those of $A + CX$ by Cramer's rule, it follows that the matrix entries of X' depend smoothly on those of X. This proves that $\varphi' \circ \varphi^{-1}$ is a smooth map, so the charts we have constructed satisfy condition (iii) of Lemma 1.35.

To check condition (iv), we just note that $G_k(V)$ can in fact be covered by *finitely* many of the sets U_Q: for example, if $(E_1, \dots, E_n)$ is any fixed basis for V, any partition of the basis elements into two subsets containing k and $n - k$ elements determines appropriate subspaces P and Q, and any subspace S must have trivial intersection with Q for at least one of these partitions (see Exercise B.9). Thus, $G_k(V)$ is covered by the finitely many charts determined by all possible partitions of a fixed basis.

Finally, the Hausdorff condition (v) is easily verified by noting that for any two k-dimensional subspaces $P, P' \subseteq V$, it is possible to find a subspace Q of dimension $n - k$ whose intersections with both P and P' are trivial, and then P and P' are both contained in the domain of the chart determined by, say, (P, Q). //

Manifolds with Boundary

In many important applications of manifolds, most notably those involving integration, we will encounter spaces that would be smooth manifolds except that they

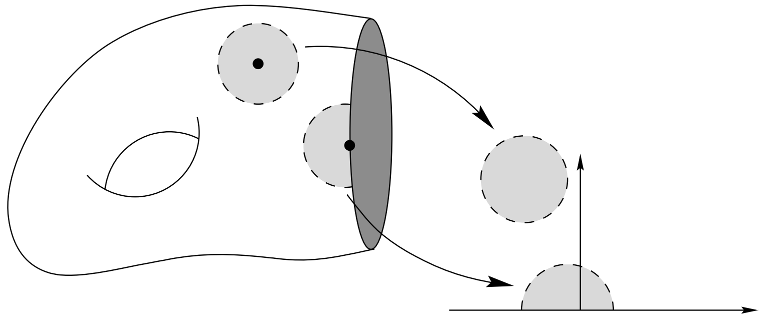

Fig. 1.11 A manifold with boundary

have a "boundary" of some sort. Simple examples of such spaces include closed intervals in $\mathbb{R}$, closed balls in $\mathbb{R}^n$, and closed hemispheres in $\mathbb{S}^n$. To accommodate such spaces, we need to extend our definition of manifolds.

Points in these spaces will have neighborhoods modeled either on open subsets of $\mathbb{R}^n$ or on open subsets of the ***closed n-dimensional upper half-space*** $\mathbb{H}^n \subseteq \mathbb{R}^n$, defined as

$$\mathbb{H}^n = \left\{ \left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^n \geq 0 \right\}.$$

We will use the notations $\operatorname{Int} \mathbb{H}^n$ and $\partial \mathbb{H}^n$ to denote the interior and boundary of $\mathbb{H}^n$, respectively, as a subset of $\mathbb{R}^n$. When $n > 0$, this means

$$\operatorname{Int} \mathbb{H}^n = \left\{ \left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^n > 0 \right\},$$
$$\partial \mathbb{H}^n = \left\{ \left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^n = 0 \right\}.$$

In the $n = 0$ case, $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so $\operatorname{Int} \mathbb{H}^0 = \mathbb{R}^0$ and $\partial \mathbb{H}^0 = \varnothing$.

An ***n-dimensional topological manifold with boundary*** is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$ (Fig. 1.11). An open subset $U \subseteq M$ together with a map $\varphi\colon U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ will be called a ***chart for M***, just as in the case of manifolds. When it is necessary to make the distinction, we will call (U, φ) an ***interior chart*** if $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case of an open subset of $\mathbb{H}^n$ that does not intersect $\partial \mathbb{H}^n$), and a ***boundary chart*** if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ such that $\varphi(U) \cap \partial \mathbb{H}^n \neq \varnothing$. A boundary chart whose image is a set of the form $B_r(x) \cap \mathbb{H}^n$ for some $x \in \partial \mathbb{H}^n$ and $r > 0$ is called a ***coordinate half-ball***.

A point $p \in M$ is called an ***interior point of M*** if it is in the domain of some interior chart. It is a ***boundary point of M*** if it is in the domain of a boundary chart that sends p to $\partial \mathbb{H}^n$. The ***boundary of M*** (the set of all its boundary points) is denoted by ∂M; similarly, its ***interior***, the set of all its interior points, is denoted by $\operatorname{Int} M$.

It follows from the definition that each point $p \in M$ is either an interior point or a boundary point: if p is not a boundary point, then either it is in the domain of an interior chart or it is in the domain of a boundary chart (U, φ) such that $\varphi(p) \notin \partial \mathbb{H}^n$, in which case the restriction of φ to $U \cap \varphi^{-1}\big(\operatorname{Int}\mathbb{H}^n\big)$ is an interior chart whose domain contains p. However, it is not obvious that a given point cannot be simultaneously an interior point with respect to one chart and a boundary point with respect to another. In fact, this cannot happen, but the proof requires more machinery than we have available at this point. For convenience, we state the theorem here.

Theorem 1.37 (Topological Invariance of the Boundary). *If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus ∂M and* Int M *are disjoint sets whose union is M.*

For the proof, see Problem 17-9. Later in this chapter, we will prove a weaker version of this result for smooth manifolds with boundary (Theorem 1.46), which will be sufficient for most of our purposes.

Be careful to observe the distinction between these new definitions of the terms *boundary* and *interior* and their usage to refer to the boundary and interior of a subset of a topological space. A manifold with boundary may have nonempty boundary in this new sense, irrespective of whether it has a boundary as a subset of some other topological space. If we need to emphasize the difference between the two notions of boundary, we will use the terms ***topological boundary*** and ***manifold boundary*** as appropriate. For example, the closed unit ball $\overline{\mathbb{B}}^n$ is a manifold with boundary (see Problem 1-11), whose manifold boundary is $\mathbb{S}^{n-1}$. Its topological boundary as a subset of $\mathbb{R}^n$ happens to be the sphere as well. However, if we think of $\overline{\mathbb{B}}^n$ as a topological space in its own right, then as a subset of itself, it has empty topological boundary. And if we think of it as a subset of $\mathbb{R}^{n+1}$ (considering $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$ in the obvious way), its topological boundary is all of $\overline{\mathbb{B}}^n$. Note that $\mathbb{H}^n$ is itself a manifold with boundary, and its manifold boundary is the same as its topological boundary as a subset of $\mathbb{R}^n$. Every interval in $\mathbb{R}$ is a 1-manifold with boundary, whose manifold boundary consists of its endpoints (if any).

The nomenclature for manifolds with boundary is traditional and well established, but it must be used with care. Despite their name, manifolds with boundary are *not* in general manifolds, because boundary points do not have locally Euclidean neighborhoods. (This is a consequence of the theorem on invariance of the boundary.) Moreover, a manifold with boundary might have empty boundary—there is nothing in the definition that requires the boundary to be a nonempty set. On the other hand, a manifold is also a manifold with boundary, whose boundary is empty. Thus, every manifold is a manifold with boundary, but a manifold with boundary is a manifold if and only if its boundary is empty (see Proposition 1.38 below).

Even though the term *manifold with boundary* encompasses manifolds as well, we will often use redundant phrases such as ***manifold without boundary*** if we wish to emphasize that we are talking about a manifold in the original sense, and ***manifold with or without boundary*** to refer to a manifold with boundary if we wish emphasize that the boundary might be empty. (The latter phrase will often appear when our primary interest is in manifolds, but the results being discussed are just as easy to state and prove in the more general case of manifolds with boundary.) Note that the word "manifold" without further qualification always means a manifold without boundary. In the literature, you will also encounter the terms ***closed manifold*** to mean a compact manifold without boundary, and ***open manifold*** to mean a noncompact manifold without boundary.

Proposition 1.38. *Let M be a topological n-manifold with boundary.*

(a) $\operatorname{Int} M$ *is an open subset of M and a topological n-manifold without boundary.*
(b) ∂M *is a closed subset of M and a topological $(n-1)$-manifold without boundary.*
(c) *M is a topological manifold if and only if $\partial M = \varnothing$.*
(d) *If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.*

► **Exercise 1.39.** Prove the preceding proposition. For this proof, you may use the theorem on topological invariance of the boundary when necessary. Which parts require it?

The topological properties of manifolds that we proved earlier in the chapter have natural extensions to manifolds with boundary, with essentially the same proofs as in the manifold case. For the record, we state them here.

Proposition 1.40. *Let M be a topological manifold with boundary.*

(a) *M has a countable basis of precompact coordinate balls and half-balls.*
(b) *M is locally compact.*
(c) *M is paracompact.*
(d) *M is locally path-connected.*
(e) *M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.*
(f) *The fundamental group of M is countable.*

► **Exercise 1.41.** Prove the preceding proposition.

Smooth Structures on Manifolds with Boundary

To see how to define a smooth structure on a manifold with boundary, recall that a map from an arbitrary subset $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$ is said to be smooth if in a neighborhood of each point of A it admits an extension to a smooth map defined on an open subset of $\mathbb{R}^n$ (see Appendix C, p. 645). Thus, if U is an open subset of $\mathbb{H}^n$, a map $F: U \to \mathbb{R}^k$ is smooth if for each $x \in U$, there exists an open subset $\widetilde{U} \subseteq \mathbb{R}^n$ containing x and a smooth map $\widetilde{F}: \widetilde{U} \to \mathbb{R}^k$ that agrees with F on $\widetilde{U} \cap \mathbb{H}^n$ (Fig. 1.12). If F is such a map, the restriction of F to $U \cap \operatorname{Int} \mathbb{H}^n$ is smooth in the usual sense. By continuity, all partial derivatives of F at points of $U \cap \partial \mathbb{H}^n$ are determined by their values in $\operatorname{Int} \mathbb{H}^n$, and therefore in particular are independent of the choice of extension. It is a fact (which we will neither prove nor use) that $F: U \to \mathbb{R}^k$ is smooth in this sense if and only if F is continuous, $F|_{U \cap \operatorname{Int} \mathbb{H}^n}$ is smooth, and the partial derivatives of $F|_{U \cap \operatorname{Int} \mathbb{H}^n}$ of all orders have continuous extensions to all of U. (One direction is obvious; the other direction depends on a lemma of Émile Borel, which shows that there is a smooth function defined in the lower half-space whose derivatives all match those of F on $U \cap \partial \mathbb{H}^n$. See, e.g., [Hör90, Thm. 1.2.6].)

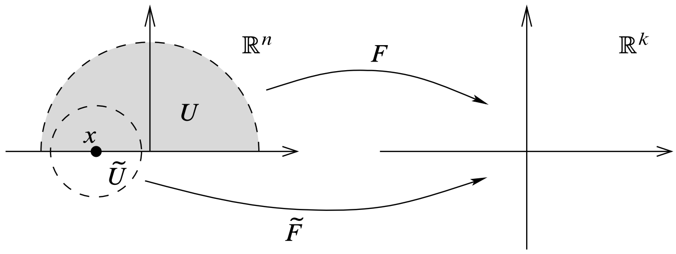

Fig. 1.12 Smoothness of maps on open subsets of $\mathbb{H}^n$

For example, let $\mathbb{B}^2 \subseteq \mathbb{R}^2$ be the open unit disk, let $U = \mathbb{B}^2 \cap \mathbb{H}^2$, and define $f: U \to \mathbb{R}$ by $f(x,y) = \sqrt{1-x^2-y^2}$. Because f extends smoothly to all of $\mathbb{B}^2$ (by the same formula), f is a smooth function on U. On the other hand, although $g(x,y) = \sqrt{y}$ is continuous on U and smooth in $U \cap \operatorname{Int}\mathbb{H}^2$, it has no smooth extension to any neighborhood of the origin in $\mathbb{R}^2$ because $\partial g/\partial y \to \infty$ as $y \to 0$. Thus g is not smooth on U.

Now let M be a topological manifold with boundary. As in the manifold case, a ***smooth structure for M*** is defined to be a maximal smooth atlas—a collection of charts whose domains cover M and whose transition maps (and their inverses) are smooth in the sense just described. With such a structure, M is called a ***smooth manifold with boundary***. Every smooth manifold is automatically a smooth manifold with boundary (whose boundary is empty).

Just as for smooth manifolds, if M is a smooth manifold with boundary, any chart in the given smooth atlas is called a ***smooth chart for M***. ***Smooth coordinate balls***, ***smooth coordinate half-balls***, and ***regular coordinate balls in M*** are defined in the obvious ways. In addition, a subset $B \subseteq M$ is called a ***regular coordinate half-ball*** if there is a smooth coordinate half-ball $B' \supseteq \overline{B}$ and a smooth coordinate map $\varphi\colon B' \to \mathbb{H}^n$ such that for some $r' > r > 0$ we have

$$\varphi(B) = B_r(0) \cap \mathbb{H}^n, \qquad \varphi(\overline{B}) = \overline{B}_r(0) \cap \mathbb{H}^n, \quad \text{and} \quad \varphi\left(B'\right) = B_{r'}(0) \cap \mathbb{H}^n.$$

► **Exercise 1.42.** Show that every smooth manifold with boundary has a countable basis consisting of regular coordinate balls and half-balls.

► **Exercise 1.43.** Show that the smooth manifold chart lemma (Lemma 1.35) holds with "$\mathbb{R}^n$" replaced by "$\mathbb{R}^n$ or $\mathbb{H}^n$" and "smooth manifold" replaced by "smooth manifold with boundary."

► **Exercise 1.44.** Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:

(a) U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V,φ) for M such that $V \subseteq U$ defines a smooth structure on U. With this topology and smooth structure, U is called an ***open submanifold with boundary***.
(b) If $U \subseteq \operatorname{Int} M$, then U is actually a smooth manifold (without boundary); in this case we call it an ***open submanifold of M***.
(c) $\operatorname{Int} M$ is an open submanifold of M (without boundary).

One important result about smooth manifolds that does *not* extend directly to smooth manifolds with boundary is the construction of smooth structures on finite products (see Example 1.8). Because a product of half-spaces $\mathbb{H}^n \times \mathbb{H}^m$ is not itself a half-space, a finite product of smooth manifolds with boundary cannot generally be considered as a smooth manifold with boundary. (Instead, it is an example of a *smooth manifold with corners*, which we will study in Chapter 16.) However, we do have the following result.

Proposition 1.45. *Suppose $M_1, \dots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then $M_1 \times \dots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \dots \times M_k \times N) = M_1 \times \dots \times M_k \times \partial N$.*

Proof. Problem 1-12. □

For smooth manifolds with boundary, the following result is often an adequate substitute for the theorem on invariance of the boundary.

Theorem 1.46 (Smooth Invariance of the Boundary). *Suppose M is a smooth manifold with boundary and $p \in M$. If there is some smooth chart (U, φ) for M such that $\varphi(U) \subseteq \mathbb{H}^n$ and $\varphi(p) \in \partial\mathbb{H}^n$, then the same is true for every smooth chart whose domain contains p.*

Proof. Suppose on the contrary that p is in the domain of a smooth interior chart (U, ψ) and also in the domain of a smooth boundary chart (V, φ) such that $\varphi(p) \in \partial\mathbb{H}^n$. Let $\tau = \varphi \circ \psi^{-1}$ denote the transition map; it is a homeomorphism from $\psi(U \cap V)$ to $\varphi(U \cap V)$. The smooth compatibility of the charts ensures that both τ and τ^{-1} are smooth, in the sense that locally they can be extended, if necessary, to smooth maps defined on open subsets of $\mathbb{R}^n$.

Write $x_0 = \psi(p)$ and $y_0 = \varphi(p) = \tau(x_0)$. There is some neighborhood W of y_0 in $\mathbb{R}^n$ and a smooth function $\eta\colon W \to \mathbb{R}^n$ that agrees with τ^{-1} on $W \cap \varphi(U \cap V)$. On the other hand, because we are assuming that ψ is an interior chart, there is an open Euclidean ball B that is centered at x_0 and contained in $\varphi(U \cap V)$, so τ itself is smooth on B in the ordinary sense. After shrinking B if necessary, we may assume that $B \subseteq \tau^{-1}(W)$. Then $\eta \circ \tau|_B = \tau^{-1} \circ \tau|_B = \mathrm{Id}_B$, so it follows from the chain rule that $D\eta(\tau(x)) \circ D\tau(x)$ is the identity map for each $x \in B$. Since $D\tau(x)$ is a square matrix, this implies that it is nonsingular. It follows from Corollary C.36 that τ (considered as a map from B to $\mathbb{R}^n$) is an open map, so $\tau(B)$ is an open subset of $\mathbb{R}^n$ that contains $y_0 = \varphi(p)$ and is contained in $\varphi(V)$. This contradicts the assumption that $\varphi(V) \subseteq \mathbb{H}^n$ and $\varphi(p) \in \partial\mathbb{H}^n$. □

Problems

1-1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the ***line with two origins***.)

1-2. Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

1-3. A topological space is said to be ***σ-compact*** if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

1-4. Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.
   (a) Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.
   (b) Give an example to show that the converse to (a) may be false.
   (c) Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

1-5. Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components. [Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]

1-6. Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

1-7. Let N denote the ***north pole*** $(0, \dots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the ***south pole*** $(0, \dots, 0, -1)$. Define the ***stereographic projection*** $\sigma: \mathbb{S}^n \smallsetminus \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1, \dots, x^{n+1}\right) = \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}.$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \smallsetminus \{S\}$.
   (a) For any $x \in \mathbb{S}^n \smallsetminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13). Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\widetilde{\sigma}$ is called ***stereographic projection from the south pole***.)
   (b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1, \dots, u^n\right) = \frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

   (c) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \smallsetminus \{N\}, \sigma)$ and $(\mathbb{S}^n \smallsetminus \{S\}, \widetilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called ***stereographic coordinates***.)
   (d) Show that this smooth structure is the same as the one defined in Example 1.31.

(*Used on pp. 201, 269, 301, 345, 347, 450.*)

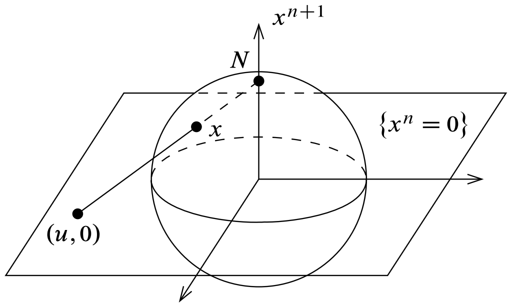

Fig. 1.13 Stereographic projection

1-8. By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An ***angle function*** on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure. (*Used on pp. 37, 152, 176.*)

1-9. ***Complex projective n-space***, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi\colon \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$\left(x^1 + iy^1, \dots, x^{n+1} + iy^{n+1}\right) \leftrightarrow \left(x^1, y^1, \dots, x^{n+1}, y^{n+1}\right)$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.) (*Used on pp. 48, 96, 172, 560, 561.*)

1-10. Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \dots, e_k)$ and $(e_{k+1}, \dots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\binom{I_k}{B}$, where I_k denotes the $k \times k$ identity matrix.

1-11. Let $M = \overline{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma\colon \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

1-12. Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Chapter 2
Smooth Maps

The main reason for introducing smooth structures was to enable us to define smooth functions on manifolds and smooth maps between manifolds. In this chapter we carry out that project.

We begin by defining smooth real-valued and vector-valued functions, and then generalize this to smooth maps between manifolds. We then focus our attention for a while on the special case of *diffeomorphisms*, which are bijective smooth maps with smooth inverses. If there is a diffeomorphism between two smooth manifolds, we say that they are *diffeomorphic*. The main objects of study in smooth manifold theory are properties that are invariant under diffeomorphisms.

At the end of the chapter, we introduce a powerful tool for blending together locally defined smooth objects, called *partitions of unity*. They are used throughout smooth manifold theory for building global smooth objects out of local ones.

Smooth Functions and Smooth Maps

Although the terms *function* and *map* are technically synonymous, in studying smooth manifolds it is often convenient to make a slight distinction between them. Throughout this book we generally reserve the term ***function*** for a map whose codomain is $\mathbb{R}$ (a ***real-valued function***) or $\mathbb{R}^k$ for some $k > 1$ (a ***vector-valued function***). Either of the words ***map*** or ***mapping*** can mean any type of map, such as a map between arbitrary manifolds.

Smooth Functions on Manifolds

Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f: M \to \mathbb{R}^k$ is any function. We say that f is a ***smooth function*** if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (Fig. 2.1). If M is a smooth manifold with boundary, the definition is exactly the same, except that

J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics 218,
DOI 10.1007/978-1-4419-9982-5_2,

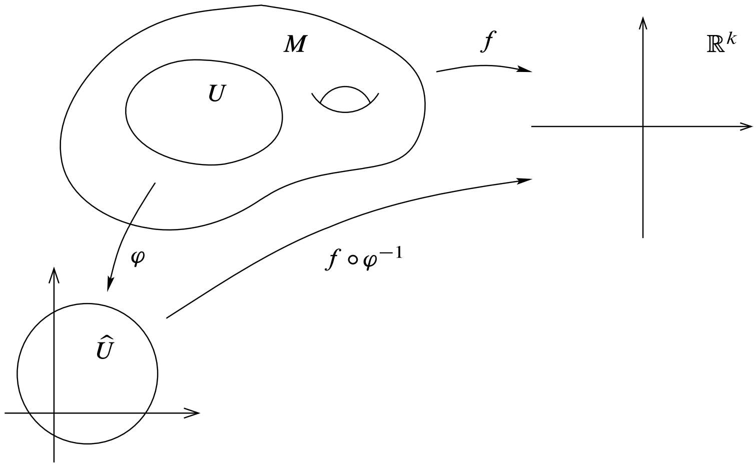

Fig. 2.1 Definition of smooth functions

$\varphi(U)$ is now an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$, and in the latter case we interpret smoothness of $f \circ \varphi^{-1}$ to mean that each point of $\varphi(U)$ has a neighborhood (in $\mathbb{R}^n$) on which $f \circ \varphi^{-1}$ extends to a smooth function in the ordinary sense.

The most important special case is that of smooth real-valued functions $f: M \to \mathbb{R}$; the set of all such functions is denoted by $C^\infty(M)$. Because sums and constant multiples of smooth functions are smooth, $C^\infty(M)$ is a vector space over $\mathbb{R}$.

► **Exercise 2.1.** Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$. (See Appendix B, p. 624, for the definition of an algebra.)

► **Exercise 2.2.** Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f: U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$ (see Exercise 1.44).

► **Exercise 2.3.** Let M be a smooth manifold with or without boundary, and suppose $f: M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}^k$ is smooth for *every* smooth chart (U, φ) for M.

Given a function $f: M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\widehat{f}: \varphi(U) \to \mathbb{R}^k$ defined by $\widehat{f}(x) = f \circ \varphi^{-1}(x)$ is called the ***coordinate representation of f***. By definition, f is smooth if and only if its coordinate representation is smooth in some smooth chart around each point. By the preceding exercise, smooth functions have smooth coordinate representations in every smooth chart.

For example, consider the real-valued function $f(x, y) = x^2 + y^2$ defined on the plane. In polar coordinates on, say, the set $U = \{(x, y) : x > 0\}$, it has the coordinate representation $\widehat{f}(r, \theta) = r^2$. In keeping with our practice of using local coordinates

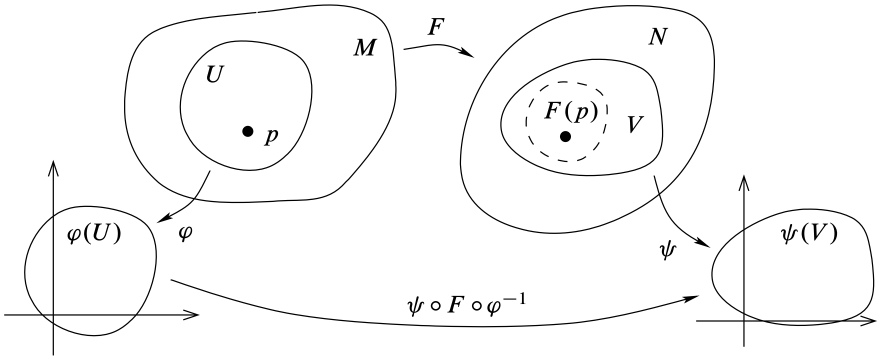

Fig. 2.2 Definition of smooth maps

to identify an open subset of a manifold with an open subset of Euclidean space, in cases where it causes no confusion we often do not even observe the distinction between $\widehat{f}$ and f itself, and instead say something like "f is smooth on U because its coordinate representation $f(r,\theta) = r^2$ is smooth."

Smooth Maps Between Manifolds

The definition of smooth functions generalizes easily to maps between manifolds. Let M, N be smooth manifolds, and let $F\colon M \to N$ be any map. We say that F is a ***smooth map*** if for every $p \in M$, there exist smooth charts (U,φ) containing p and (V,ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$ (Fig. 2.2). If M and N are smooth manifolds with boundary, smoothness of F is defined in exactly the same way, with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$. Note that our previous definition of smoothness of real-valued or vector-valued functions can be viewed as a special case of this one, by taking $N = V = \mathbb{R}^k$ and $\psi = \mathrm{Id}\colon \mathbb{R}^k \to \mathbb{R}^k$.

The first important observation about our definition of smooth maps is that, as one might expect, smoothness implies continuity.

Proposition 2.4. *Every smooth map is continuous.*

Proof. Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is smooth. Given $p \in M$, smoothness of F means there are smooth charts (U,φ) containing p and (V,ψ) containing $F(p)$, such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth, hence continuous. Since $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are homeomorphisms, this implies in turn that

$$F|_U = \psi^{-1} \circ \left(\psi \circ F \circ \varphi^{-1}\right) \circ \varphi\colon U \to V,$$

which is a composition of continuous maps. Since F is continuous in a neighborhood of each point, it is continuous on M. □

To prove that a map $F: M \to N$ is smooth directly from the definition requires, in part, that for each $p \in M$ we prove the existence of coordinate domains U containing p and V containing $F(p)$ such that $F(U) \subseteq V$. This requirement is included in the definition precisely so that smoothness automatically implies continuity. (Problem 2-1 illustrates what can go wrong if this requirement is omitted.) There are other ways of characterizing smoothness of maps between manifolds that accomplish the same thing. Here are two of them.

Proposition 2.5 (Equivalent Characterizations of Smoothness). *Suppose M and N are smooth manifolds with or without boundary, and $F: M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:*

(a) *For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.*
(b) *F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big)$ to $\psi_\beta(V_\beta)$.*

Proposition 2.6 (Smoothness Is Local). *Let M and N be smooth manifolds with or without boundary, and let $F: M \to N$ be a map.*

(a) *If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.*
(b) *Conversely, if F is smooth, then its restriction to every open subset is smooth.*

► **Exercise 2.7.** Prove the preceding two propositions.

The next corollary is essentially just a restatement of the previous proposition, but it gives a highly useful way of constructing smooth maps.

Corollary 2.8 (Gluing Lemma for Smooth Maps). *Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha: U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Then there exists a unique smooth map $F: M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.* □

If $F: M \to N$ is a smooth map, and (U, φ) and (V, ψ) are any smooth charts for M and N, respectively, we call $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ the ***coordinate representation of F*** with respect to the given coordinates. It maps the set $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.

► **Exercise 2.9.** Suppose $F: M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

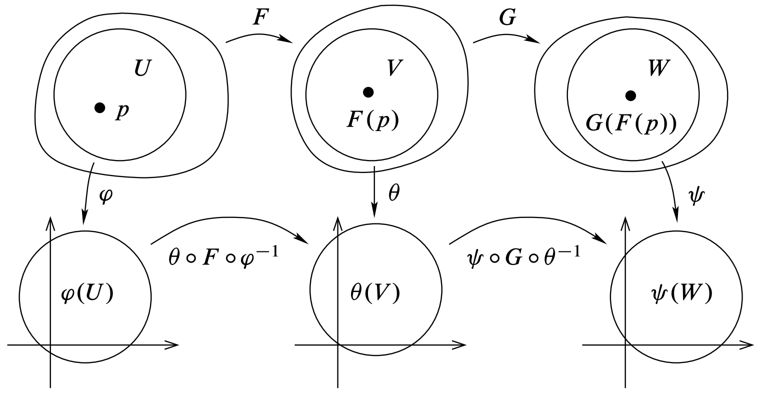

Fig. 2.3 A composition of smooth maps is smooth

As with real-valued or vector-valued functions, once we have chosen specific local coordinates in both the domain and codomain, we can often ignore the distinction between F and $\widehat{F}$.

Next we examine some simple classes of maps that are automatically smooth.

Proposition 2.10. *Let M, N, and P be smooth manifolds with or without boundary.*

(a) *Every constant map $c: M \to N$ is smooth.*
(b) *The identity map of M is smooth.*
(c) *If $U \subseteq M$ is an open submanifold with or without boundary, then the inclusion map $U \hookrightarrow M$ is smooth.*
(d) *If $F: M \to N$ and $G: N \to P$ are smooth, then so is $G \circ F: M \to P$.*

Proof. We prove (d) and leave the rest as exercises. Let $F: M \to N$ and $G: N \to P$ be smooth maps, and let $p \in M$. By definition of smoothness of G, there exist smooth charts (V,θ) containing $F(p)$ and (W,ψ) containing $G(F(p))$ such that $G(V) \subseteq W$ and $\psi \circ G \circ \theta^{-1}: \theta(V) \to \psi(W)$ is smooth. Since F is continuous, $F^{-1}(V)$ is a neighborhood of p in M, so there is a smooth chart (U,φ) for M such that $p \in U \subseteq F^{-1}(V)$ (Fig. 2.3). By Exercise 2.9, $\theta \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\theta(V)$. Then we have $G \circ F(U) \subseteq G(V) \subseteq W$, and $\psi \circ (G \circ F) \circ \varphi^{-1} = \left(\psi \circ G \circ \theta^{-1}\right) \circ \left(\theta \circ F \circ \varphi^{-1}\right): \varphi(U) \to \psi(W)$ is smooth because it is a composition of smooth maps between subsets of Euclidean spaces. □

► **Exercise 2.11.** Prove parts (a)–(c) of the preceding proposition.

Proposition 2.12. *Suppose $M_1, \dots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \dots, M_k$ has nonempty boundary. For each i, let $\pi_i: M_1 \times \dots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F: N \to M_1 \times \dots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F: N \to M_i$ is smooth.*

Proof. Problem 2-2. □

Although most of our efforts in this book are devoted to the study of smooth manifolds and smooth maps, we also need to work with topological manifolds and continuous maps on occasion. For the sake of consistency, we adopt the following conventions: without further qualification, the words "function" and "map" are to be understood purely in the set-theoretic sense, and carry no assumptions of continuity or smoothness. Most other objects we study, however, will be understood to carry some minimal topological structure by default. Unless otherwise specified, a "manifold" or "manifold with boundary" is always to be understood as a topological one, and a "coordinate chart" is to be understood in the topological sense, as a homeomorphism from an open subset of the manifold to an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. If we wish to restrict attention to smooth manifolds or smooth coordinate charts, we will say so. Similarly, our default assumptions for many other specific types of geometric objects and the maps between them will be continuity at most; smoothness will not be assumed unless explicitly specified. The only exceptions will be a few concepts that require smoothness for their very definitions.

This convention requires a certain discipline, in that we have to remember to state the smoothness hypothesis whenever it is needed; but its advantage is that it frees us (for the most part) from having to remember which types of maps are assumed to be smooth and which are not.

On the other hand, because the definition of a smooth map requires smooth structures in the domain and codomain, if we say "$F: M \to N$ is a smooth map" without specifying what M and N are, it should always be understood that they are smooth manifolds with or without boundaries.

We now have enough information to produce a number of interesting examples of smooth maps. In spite of the apparent complexity of the definition, it is usually not hard to prove that a particular map is smooth. There are basically only three common ways to do so:

- Write the map in smooth local coordinates and recognize its component functions as compositions of smooth elementary functions.
- Exhibit the map as a composition of maps that are known to be smooth.
- Use some special-purpose theorem that applies to the particular case under consideration.

Example 2.13 (Smooth Maps).

(a) Any map from a zero-dimensional manifold into a smooth manifold with or without boundary is automatically smooth, because each coordinate representation is constant.

(b) If the circle $\mathbb{S}^1$ is given its standard smooth structure, the map $\varepsilon: \mathbb{R} \to \mathbb{S}^1$ defined by $\varepsilon(t) = e^{2\pi i t}$ is smooth, because with respect to any angle coordinate θ for $\mathbb{S}^1$ (see Problem 1-8) it has a coordinate representation of the form $\widehat{\varepsilon}(t) = 2\pi t + c$ for some constant c, as you can check.

(c) The map $\varepsilon^n: \mathbb{R}^n \to \mathbb{T}^n$ defined by $\varepsilon^n\left(x^1, \dots, x^n\right) = \left(e^{2\pi i x^1}, \dots, e^{2\pi i x^n}\right)$ is smooth by Proposition 2.12.

(d) Now consider the n-sphere $\mathbb{S}^n$ with its standard smooth structure. The inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is certainly continuous, because it is the inclusion map of a topological subspace. It is a smooth map because its coordinate representation with respect to any of the graph coordinates of Example 1.31 is

$$\begin{aligned}\widehat{\iota}\left(u^1,\dots,u^n\right) &= \iota \circ \left(\varphi_i^{\pm}\right)^{-1}\left(u^1,\dots,u^n\right)\\ &= \left(u^1,\dots,u^{i-1},\pm\sqrt{1-|u|^2},u^i,\dots,u^n\right),\end{aligned}$$

which is smooth on its domain (the set where $|u|^2<1$).

(e) The quotient map $\pi\colon \mathbb{R}^{n+1}\smallsetminus\{0\}\to\mathbb{RP}^n$ used to define $\mathbb{RP}^n$ is smooth, because its coordinate representation in terms of any of the coordinates for $\mathbb{RP}^n$ constructed in Example 1.33 and standard coordinates on $\mathbb{R}^{n+1}\smallsetminus\{0\}$ is

$$\begin{aligned}\widehat{\pi}\left(x^1,\dots,x^{n+1}\right) &= \varphi_i\circ\pi\left(x^1,\dots,x^{n+1}\right) = \varphi_i\left[x^1,\dots,x^{n+1}\right]\\ &= \left(\frac{x^1}{x^i},\dots,\frac{x^{i-1}}{x^i},\frac{x^{i+1}}{x^i},\dots,\frac{x^{n+1}}{x^i}\right).\end{aligned}$$

(f) Define $q\colon \mathbb{S}^n\to\mathbb{RP}^n$ as the restriction of $\pi\colon \mathbb{R}^{n+1}\smallsetminus\{0\}\to\mathbb{RP}^n$ to $\mathbb{S}^n\subseteq\mathbb{R}^{n+1}\smallsetminus\{0\}$. It is a smooth map, because it is the composition $q=\pi\circ\iota$ of the maps in the preceding two examples.

(g) If $M_1,\dots,M_k$ are smooth manifolds, then each projection map $\pi_i\colon M_1\times\dots\times M_k\to M_i$ is smooth, because its coordinate representation with respect to any of the product charts of Example 1.8 is just a coordinate projection. //

Diffeomorphisms

If M and N are smooth manifolds with or without boundary, a ***diffeomorphism from M to N*** is a smooth bijective map $F\colon M\to N$ that has a smooth inverse. We say that ***M and N are diffeomorphic*** if there exists a diffeomorphism between them. Sometimes this is symbolized by $M\approx N$.

Example 2.14 (Diffeomorphisms).

(a) Consider the maps $F\colon \mathbb{B}^n\to\mathbb{R}^n$ and $G\colon \mathbb{R}^n\to\mathbb{B}^n$ given by

$$F(x)=\frac{x}{\sqrt{1-|x|^2}},\qquad G(y)=\frac{y}{\sqrt{1+|y|^2}}. \tag{2.1}$$

These maps are smooth, and it is straightforward to compute that they are inverses of each other. Thus they are both diffeomorphisms, and therefore $\mathbb{B}^n$ is diffeomorphic to $\mathbb{R}^n$.

(b) If M is any smooth manifold and (U,φ) is a smooth coordinate chart on M, then $\varphi\colon U\to\varphi(U)\subseteq\mathbb{R}^n$ is a diffeomorphism. (In fact, it has an identity map as a coordinate representation.) //

Proposition 2.15 (Properties of Diffeomorphisms).

(a) *Every composition of diffeomorphisms is a diffeomorphism.*
(b) *Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.*
(c) *Every diffeomorphism is a homeomorphism and an open map.*
(d) *The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.*
(e) "*Diffeomorphic*" *is an equivalence relation on the class of all smooth manifolds with or without boundary.*

► **Exercise 2.16.** Prove the preceding proposition.

The following theorem is a weak version of invariance of dimension, which suffices for many purposes.

Theorem 2.17 (Diffeomorphism Invariance of Dimension). *A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.*

Proof. Suppose M is a nonempty smooth m-manifold, N is a nonempty smooth n-manifold, and $F: M \to N$ is a diffeomorphism. Choose any point $p \in M$, and let (U, φ) and (V, ψ) be smooth coordinate charts containing p and $F(p)$, respectively. Then (the restriction of) $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is a diffeomorphism from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$, so it follows from Proposition C.4 that $m = n$. $\square$

There is a similar invariance statement for boundaries.

Theorem 2.18 (Diffeomorphism Invariance of the Boundary). *Suppose M and N are smooth manifolds with boundary and $F: M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from* $\operatorname{Int} M$ *to* $\operatorname{Int} N$.

► **Exercise 2.19.** Use Theorem 1.46 to prove the preceding theorem.

Just as two topological spaces are considered to be "the same" if they are homeomorphic, two smooth manifolds with or without boundary are essentially indistinguishable if they are diffeomorphic. The central concern of smooth manifold theory is the study of properties of smooth manifolds that are preserved by diffeomorphisms. Theorem 2.17 shows that dimension is one such property.

It is natural to wonder whether the smooth structure on a given topological manifold is unique. This straightforward version of the question is easy to answer: we observed in Example 1.21 that every zero-dimensional manifold has a unique smooth structure, but as Problem 1-6 showed, each positive-dimensional manifold admits many distinct smooth structures as soon as it admits one.

A more subtle and interesting question is whether a given topological manifold admits smooth structures that are not diffeomorphic to each other. For example, let $\widetilde{\mathbb{R}}$ denote the topological manifold $\mathbb{R}$, but endowed with the smooth structure described in Example 1.23 (defined by the global chart $\psi(x) = x^3$). It turns out that $\widetilde{\mathbb{R}}$ is diffeomorphic to $\mathbb{R}$ with its standard smooth structure. Define a map $F\colon \mathbb{R} \to \widetilde{\mathbb{R}}$ by $F(x) = x^{1/3}$. The coordinate representation of this map is $\widehat{F}(t) = \psi \circ F \circ \mathrm{Id}_{\mathbb{R}}^{-1}(t) = t$, which is clearly smooth. Moreover, the coordinate representation of its inverse is

$$\widehat{F^{-1}}(y) = \mathrm{Id}_{\mathbb{R}} \circ F^{-1} \circ \psi^{-1}(y) = y,$$

which is also smooth, so F is a diffeomorphism. (This is a case in which it *is* important to maintain the distinction between a map and its coordinate representation!)

In fact, as you will see later, there is only one smooth structure on $\mathbb{R}$ up to diffeomorphism (see Problem 15-13). More precisely, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are any two smooth structures on $\mathbb{R}$, there exists a diffeomorphism $F\colon (\mathbb{R}, \mathcal{A}_1) \to (\mathbb{R}, \mathcal{A}_2)$. In fact, it follows from work of James Munkres [Mun60] and Edwin Moise [Moi77] that every topological manifold of dimension less than or equal to 3 has a smooth structure that is unique up to diffeomorphism. The analogous question in higher dimensions turns out to be quite deep, and is still largely unanswered. Even for Euclidean spaces, the question of uniqueness of smooth structures was not completely settled until late in the twentieth century. The answer is surprising: as long as $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure (up to diffeomorphism); but $\mathbb{R}^4$ has uncountably many distinct smooth structures, no two of which are diffeomorphic to each other! The existence of nonstandard smooth structures on $\mathbb{R}^4$ (called ***fake*** $\boldsymbol{\mathbb{R}^4}$***'s***) was first proved by Simon Donaldson and Michael Freedman in 1984 as a consequence of their work on the geometry and topology of compact 4-manifolds; the results are described in [DK90] and [FQ90].

For compact manifolds, the situation is even more fascinating. In 1956, John Milnor [Mil56] showed that there are smooth structures on $\mathbb{S}^7$ that are not diffeomorphic to the standard one. Later, he and Michel Kervaire [KM63] showed (using a deep theorem of Steve Smale [Sma62]) that there are exactly 15 diffeomorphism classes of such structures (or 28 classes if you restrict to diffeomorphisms that preserve a property called *orientation*, which will be discussed in Chapter 15).

On the other hand, in all dimensions greater than 3 there are compact topological manifolds that have no smooth structures at all. The problem of identifying the number of smooth structures (if any) on topological 4-manifolds is an active subject of current research.

Partitions of Unity

A frequently used tool in topology is the gluing lemma (Lemma A.20), which shows how to construct continuous maps by "gluing together" maps defined on open or closed subsets. We have a version of the gluing lemma for smooth maps defined on *open* subsets (Corollary 2.8), but we cannot expect to glue together smooth maps defined on *closed* subsets and obtain a smooth result. For example, the two functions $f_+\colon [0,\infty) \to \mathbb{R}$ and $f_-\colon (-\infty, 0] \to \mathbb{R}$ defined by

$$f_+(x) = +x, \quad x \in [0,\infty),$$
$$f_-(x) = -x, \quad x \in (-\infty, 0],$$

are both smooth and agree at the point 0 where they overlap, but the continuous function $f : \mathbb{R} \to \mathbb{R}$ that they define, namely $f(x) = |x|$, is not smooth at the origin.

A disadvantage of Corollary 2.8 is that in order to use it, we must construct maps that agree exactly on relatively large subsets of the manifold, which is too restrictive for some purposes. In this section we introduce *partitions of unity*, which are tools for "blending together" local smooth objects into global ones without necessarily assuming that they agree on overlaps. They are indispensable in smooth manifold theory and will reappear throughout the book.

All of our constructions in this section are based on the existence of smooth functions that are positive in a specified part of a manifold and identically zero in some other part. We begin by defining a smooth function on the real line that is zero for $t \le 0$ and positive for $t > 0$.

Lemma 2.20. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

is smooth.

Proof. The function in question is pictured in Fig. 2.4. It is smooth on $\mathbb{R} \smallsetminus \{0\}$ by composition, so we need only show f has continuous derivatives of all orders at the origin. Because existence of the $(k+1)$st derivative implies continuity of the kth, it suffices to show that each such derivative exists. We begin by noting that f is continuous at 0 because $\lim_{t \searrow 0} e^{-1/t} = 0$. In fact, a standard application of l'Hôpital's rule and induction shows that for any integer $k \ge 0$,

$$\lim_{t \searrow 0} \frac{e^{-1/t}}{t^k} = \lim_{t \searrow 0} \frac{t^{-k}}{e^{1/t}} = 0. \tag{2.2}$$

We show by induction that for $t > 0$, the kth derivative of f is of the form

$$f^{(k)}(t) = p_k(t) \frac{e^{-1/t}}{t^{2k}} \tag{2.3}$$

for some polynomial p_k of degree at most k. This is clearly true (with $p_0(t) = 1$) for $k = 0$, so suppose it is true for some $k \ge 0$. By the product rule,

$$\begin{aligned} f^{(k+1)}(t) &= p_k'(t) \frac{e^{-1/t}}{t^{2k}} + p_k(t) \frac{t^{-2} e^{-1/t}}{t^{2k}} - 2k p_k(t) \frac{e^{-1/t}}{t^{2k+1}} \\ &= \left(t^2 p_k'(t) + p_k(t) - 2k t p_k(t) \right) \frac{e^{-1/t}}{t^{2(k+1)}}, \end{aligned}$$

which is of the required form.

Finally, we prove by induction that $f^{(k)}(0) = 0$ for each integer $k \ge 0$. For $k = 0$ this is true by definition, so assume that it is true for some $k \ge 0$. To prove that $f^{(k+1)}(0)$ exists, it suffices to show that $f^{(k)}$ has one-sided derivatives from both sides at $t = 0$ and that they are equal. Clearly, the derivative from the left is zero.

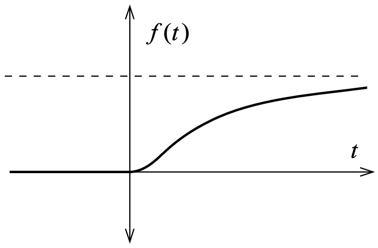

Fig. 2.4 $f(t) = e^{-1/t}$

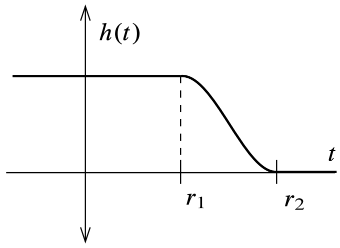

Fig. 2.5 A cutoff function

Using (2.3) and (2.2) again, we find that the derivative of $f^{(k)}$ from the right at $t = 0$ is equal to

$$\lim_{t \searrow 0} \frac{p_k(t)\dfrac{e^{-1/t}}{t^{2k}} - 0}{t} = \lim_{t \searrow 0} p_k(t)\frac{e^{-1/t}}{t^{2k+1}} = p_k(0) \lim_{t \searrow 0} \frac{e^{-1/t}}{t^{2k+1}} = 0.$$

Thus $f^{(k+1)}(0) = 0$. □

Lemma 2.21. *Given any real numbers r_1 and r_2 such that $r_1 < r_2$, there exists a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(t) \equiv 1$ for $t \le r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$, and $h(t) \equiv 0$ for $t \ge r_2$.*

Proof. Let f be the function of the previous lemma, and set

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

(See Fig. 2.5.) Note that the denominator is positive for all t, because at least one of the expressions $r_2 - t$ and $t - r_1$ is always positive. The desired properties of h follow easily from those of f. □

A function with the properties of h in the preceding lemma is usually called a ***cutoff function***.

Lemma 2.22. *Given any positive real numbers $r_1 < r_2$, there is a smooth function $H\colon \mathbb{R}^n \to \mathbb{R}$ such that $H \equiv 1$ on $\overline{B}_{r_1}(0)$, $0 < H(x) < 1$ for all $x \in B_{r_2}(0) \smallsetminus \overline{B}_{r_1}(0)$, and $H \equiv 0$ on $\mathbb{R}^n \smallsetminus B_{r_2}(0)$.*

Proof. Just set $H(x) = h(|x|)$, where h is the function of the preceding lemma. Clearly, H is smooth on $\mathbb{R}^n \smallsetminus \{0\}$, because it is a composition of smooth functions there. Since it is identically equal to 1 on $B_{r_1}(0)$, it is smooth there too. □

The function H constructed in this lemma is an example of a ***smooth bump function***, a smooth real-valued function that is equal to 1 on a specified set and is zero outside a specified neighborhood of that set. Later in this chapter, we will generalize this notion to manifolds.

If f is any real-valued or vector-valued function on a topological space M, the ***support of f***, denoted by $\operatorname{supp} f$, is the closure of the set of points where f is nonzero:

$$\operatorname{supp} f = \overline{\{p \in M : f(p) \neq 0\}}.$$

(For example, if H is the function constructed in the preceding lemma, then $\operatorname{supp} H = \overline{B}_{r_2}(0)$.) If $\operatorname{supp} f$ is contained in some set $U \subseteq M$, we say that f is ***supported in U***. A function f is said to be ***compactly supported*** if $\operatorname{supp} f$ is a compact set. Clearly, every function on a compact space is compactly supported.

The next construction is the most important application of paracompactness. Suppose M is a topological space, and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an arbitrary open cover of M, indexed by a set A. A ***partition of unity subordinate to $\mathcal{X}$*** is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha \colon M \to \mathbb{R}$ with the following properties:

(i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.
(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.
(iii) The family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is locally finite, meaning that every point has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many values of α.
(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Because of the local finiteness condition (iii), the sum in (iv) actually has only finitely many nonzero terms in a neighborhood of each point, so there is no issue of convergence. If M is a smooth manifold with or without boundary, a ***smooth partition of unity*** is one for which each of the functions ψ_α is smooth.

Theorem 2.23 (Existence of Partitions of Unity). *Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Proof. For simplicity, suppose for this proof that M is a smooth manifold without boundary; the general case is left as an exercise. Each set X_α is a smooth manifold in its own right, and thus has a basis $\mathcal{B}_\alpha$ of regular coordinate balls by Proposition 1.19, and it is easy to check that $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$ is a basis for the topology of M. It follows from Theorem 1.15 that $\mathcal{X}$ has a countable, locally finite refinement $\{B_i\}$ consisting of elements of $\mathcal{B}$. By Lemma 1.13(a), the cover $\{\overline{B}_i\}$ is also locally finite.

For each i, the fact that B_i is a regular coordinate ball in some X_α guarantees that there is a coordinate ball $B_i' \subseteq X_\alpha$ such that $B_i' \supseteq \overline{B}_i$, and a smooth coordinate map $\varphi_i \colon B_i' \to \mathbb{R}^n$ such that $\varphi_i(\overline{B}_i) = \overline{B}_{r_i}(0)$ and $\varphi_i(B_i') = B_{r_i'}(0)$ for some $r_i < r_i'$. For each i, define a function $f_i \colon M \to \mathbb{R}$ by

$$f_i = \begin{cases} H_i \circ \varphi_i & \text{on } B_i', \\ 0 & \text{on } M \smallsetminus \overline{B}_i, \end{cases}$$

where $H_i \colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function that is positive in $B_{r_i}(0)$ and zero elsewhere, as in Lemma 2.22. On the set $B_i' \smallsetminus \overline{B}_i$ where the two definitions overlap, both definitions yield the zero function, so f_i is well defined and smooth, and $\operatorname{supp} f_i = \overline{B}_i$.

Define $f: M \to \mathbb{R}$ by $f(x) = \sum_i f_i(x)$. Because of the local finiteness of the cover $\{\bar{B}_i\}$, this sum has only finitely many nonzero terms in a neighborhood of each point and thus defines a smooth function. Because each f_i is nonnegative everywhere and positive on B_i, and every point of M is in some B_i, it follows that $f(x) > 0$ everywhere on M. Thus, the functions $g_i: M \to \mathbb{R}$ defined by $g_i(x) = f_i(x)/f(x)$ are also smooth. It is immediate from the definition that $0 \le g_i \le 1$ and $\sum_i g_i \equiv 1$.

Finally, we need to reindex our functions so that they are indexed by the same set A as our open cover. Because the cover $\{B_i'\}$ is a refinement of $\mathcal{X}$, for each i we can choose some index $a(i) \in A$ such that $B_i' \subseteq X_{a(i)}$. For each $\alpha \in A$, define $\psi_\alpha: M \to \mathbb{R}$ by

$$\psi_\alpha = \sum_{i:a(i)=\alpha} g_i.$$

If there are no indices i for which $a(i) = \alpha$, then this sum should be interpreted as the zero function. It follows from Lemma 1.13(b) that

$$\operatorname{supp} \psi_\alpha = \overline{\bigcup_{i:a(i)=\alpha} B_i} = \bigcup_{i:a(i)=\alpha} \bar{B}_i \subseteq X_\alpha.$$

Each ψ_α is a smooth function that satisfies $0 \le \psi_\alpha \le 1$. Moreover, the family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is still locally finite, and $\sum_\alpha \psi_\alpha \equiv \sum_i g_i \equiv 1$, so this is the desired partition of unity. □

► **Exercise 2.24.** Show how the preceding proof needs to be modified for the case in which M has nonempty boundary.

There are basically two different strategies for patching together locally defined smooth maps to obtain a global one. If you can define a map in a neighborhood of each point in such a way that the locally defined maps all agree where they overlap, then the local definitions piece together to yield a global smooth map by Corollary 2.8. (This usually requires some sort of uniqueness result.) But if the local definitions are not guaranteed to agree, then you usually have to resort to a partition of unity. The trick then is showing that the patched-together objects still have the required properties. We use both strategies repeatedly throughout the book.

Applications of Partitions of Unity

As our first application of partitions of unity, we extend the notion of bump functions to arbitrary closed subsets of manifolds. If M is a topological space, $A \subseteq M$ is a closed subset, and $U \subseteq M$ is an open subset containing A, a continuous function $\psi: M \to \mathbb{R}$ is called a ***bump function for A supported in U*** if $0 \le \psi \le 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp} \psi \subseteq U$.

Proposition 2.25 (Existence of Smooth Bump Functions). *Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.*

Proof. Let $U_0 = U$ and $U_1 = M \smallsetminus A$, and let $\{\psi_0, \psi_1\}$ be a smooth partition of unity subordinate to the open cover $\{U_0, U_1\}$. Because $\psi_1 \equiv 0$ on A and thus $\psi_0 = \sum_i \psi_i = 1$ there, the function ψ_0 has the required properties. □

Our second application is an important result concerning the possibility of extending smooth functions from closed subsets. Suppose M and N are smooth manifolds with or without boundary, and $A \subseteq M$ is an arbitrary subset. We say that a map $F: A \to N$ is ***smooth on A*** if it has a smooth extension in a neighborhood of each point: that is, if for every $p \in A$ there is an open subset $W \subseteq M$ containing p and a smooth map $\widetilde{F}: W \to N$ whose restriction to $W \cap A$ agrees with F.

Lemma 2.26 (Extension Lemma for Smooth Functions). *Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and $f: A \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\widetilde{f}: M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and* $\operatorname{supp} \widetilde{f} \subseteq U$.

Proof. For each $p \in A$, choose a neighborhood W_p of p and a smooth function $\widetilde{f}_p: W_p \to \mathbb{R}^k$ that agrees with f on $W_p \cap A$. Replacing W_p by $W_p \cap U$, we may assume that $W_p \subseteq U$. The family of sets $\{W_p : p \in A\} \cup \{M \smallsetminus A\}$ is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \smallsetminus A$.

For each $p \in A$, the product $\psi_p \widetilde{f}_p$ is smooth on W_p, and has a smooth extension to all of M if we interpret it to be zero on $M \smallsetminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open subset $W_p \smallsetminus \operatorname{supp} \psi_p$ where they overlap.) Thus we can define $\widetilde{f}: M \to \mathbb{R}^k$ by

$$\widetilde{f}(x) = \sum_{p \in A} \psi_p(x) \widetilde{f}_p(x).$$

Because the collection of supports $\{\operatorname{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of M, and therefore defines a smooth function. If $x \in A$, then $\psi_0(x) = 0$ and $\widetilde{f}_p(x) = f(x)$ for each p such that $\psi_p(x) \neq 0$, so

$$\widetilde{f}(x) = \sum_{p \in A} \psi_p(x) f(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) f(x) = f(x),$$

so $\widetilde{f}$ is indeed an extension of f. It follows from Lemma 1.13(b) that

$$\operatorname{supp} \widetilde{f} = \overline{\bigcup_{p \in A} \operatorname{supp} \psi_p} = \bigcup_{p \in A} \operatorname{supp} \psi_p \subseteq U.$$

□

► **Exercise 2.27.** Give a counterexample to show that the conclusion of the extension lemma can be false if A is not closed.

The assumption in the extension lemma that the codomain of f is $\mathbb{R}^k$, and not some other smooth manifold, is needed: for other codomains, extensions can fail to exist for topological reasons. (For example, the identity map $\mathbb{S}^1 \to \mathbb{S}^1$ is smooth, but does not have even a *continuous* extension to a map from $\mathbb{R}^2$ to $\mathbb{S}^1$.) Later we will show that a smooth map from a closed subset of a smooth manifold into a smooth manifold has a smooth extension if and only if it has a continuous one (see Corollary 6.27).

This extension lemma, by the way, illustrates an essential difference between smooth manifolds and real-analytic manifolds. The analogue of the extension lemma for real-analytic functions on real-analytic manifolds is decidedly false, because a real-analytic function that is defined on a connected domain and vanishes on an open subset must be identically zero.

Next, we use partitions of unity to construct a special kind of smooth function. If M is a topological space, an ***exhaustion function for*** M is a continuous function $f: M \to \mathbb{R}$ with the property that the set $f^{-1}\big((-\infty, c]\big)$ (called a ***sublevel set of*** f) is compact for each $c \in \mathbb{R}$. The name comes from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}\big((-\infty, n]\big)$ form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{B}^n \to \mathbb{R}$ given by

$$f(x) = |x|^2, \qquad g(x) = \frac{1}{1 - |x|^2}$$

are smooth exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.28 (Existence of Smooth Exhaustion Functions). *Every smooth manifold with or without boundary admits a smooth positive exhaustion function.*

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^{\infty}$ be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j\,\psi_j(p).$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \geq \sum_j \psi_j(p) = 1$.

To see that f is an exhaustion function, let $c \in \mathbb{R}$ be arbitrary, and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^{N} \overline{V}_j$, then $\psi_j(p) = 0$ for $1 \leq j \leq N$, so

$$f(p) = \sum_{j=N+1}^{\infty} j\,\psi_j(p) \geq \sum_{j=N+1}^{\infty} N\psi_j(p) = N\sum_{j=1}^{\infty}\psi_j(p) = N > c.$$

Equivalently, if $f(p) \leq c$, then $p \in \bigcup_{j=1}^{N} \overline{V}_j$. Thus $f^{-1}\big((-\infty, c]\big)$ is a closed subset of the compact set $\bigcup_{j=1}^{N} \overline{V}_j$ and is therefore compact. □

As our final application of partitions of unity, we will prove the remarkable fact that every closed subset of a manifold can be expressed as a level set of some smooth real-valued function. We will not use this result in this book (except in a few of the problems), but it provides an interesting contrast with the result of Example 1.32.

Theorem 2.29 (Level Sets of Smooth Functions). *Let M be a smooth manifold. If K is any closed subset of M, there is a smooth nonnegative function $f: M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

Proof. We begin with the special case in which $M = \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is a closed subset. For each $x \in M \smallsetminus K$, there is a positive number $r \leq 1$ such that $B_r(x) \subseteq M \smallsetminus K$. By Proposition A.16, $M \smallsetminus K$ is the union of countably many such balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$.

Let $h: \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function that is equal to 1 on $\overline{B}_{1/2}(0)$ and supported in $B_1(0)$. For each positive integer i, let $C_i \geq 1$ be a constant that bounds the absolute values of h and all of its partial derivatives up through order i. Define $f: \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{(r_i)^i}{2^i C_i} h\left(\frac{x - x_i}{r_i}\right).$$

The terms of the series are bounded in absolute value by those of the convergent series $\sum_i 1/2^i$, so the entire series converges uniformly to a continuous function by the Weierstrass M-test. Because the ith term is positive exactly when $x \in B_{r_i}(x_i)$, it follows that f is zero in K and positive elsewhere.

It remains only to show that f is smooth. We have already shown that it is continuous, so suppose $k \geq 1$ and assume by induction that all partial derivatives of f of order less than k exist and are continuous. By the chain rule and induction, every kth partial derivative of the ith term in the series can be written in the form

$$\frac{(r_i)^{i-k}}{2^i C_i} D_k h\left(\frac{x - x_i}{r_i}\right),$$

where $D_k h$ is some kth partial derivative of h. By our choices of r_i and C_i, as soon as $i \geq k$, each of these terms is bounded in absolute value by $1/2^i$, so the differentiated series also converges uniformly to a continuous function. It then follows from Theorem C.31 that the kth partial derivatives of f exist and are continuous. This completes the induction, and shows that f is smooth.

Now let M be an arbitrary smooth manifold, and $K \subseteq M$ be any closed subset. Let $\{B_\alpha\}$ be an open cover of M by smooth coordinate balls, and let $\{\psi_\alpha\}$ be a subordinate partition of unity. Since each B_α is diffeomorphic to $\mathbb{R}^n$, the preceding argument shows that for each α there is a smooth nonnegative function $f_\alpha: B_\alpha \to \mathbb{R}$ such that $f_\alpha^{-1}(0) = B_\alpha \cap K$. The function $f = \sum_\alpha \psi_\alpha f_\alpha$ does the trick. □

Problems

2-1. Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\big(U \cap f^{-1}(V)\big)$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

2-2. Prove Proposition 2.12 (smoothness of maps into product manifolds).

2-3. For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n: \mathbb{S}^1 \to \mathbb{S}^1$ is the ***nth power map*** for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.

(b) $\alpha: \mathbb{S}^n \to \mathbb{S}^n$ is the ***antipodal map*** $\alpha(x) = -x$.

(c) $F: \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$, where we think of $\mathbb{S}^3$ as the subset $\big\{(w, z) : |w|^2 + |z|^2 = 1\big\}$ of $\mathbb{C}^2$.

2-4. Show that the inclusion map $\overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.

2-5. Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f: \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.

(b) Show that f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

2-6. Let $P: \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{R}^{k+1} \smallsetminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \smallsetminus \{0\}$ and $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$. (Such a function is said to be ***homogeneous of degree d***.) Show that the map $\widetilde{P}: \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\widetilde{P}([x]) = [P(x)]$ is well defined and smooth.

2-7. Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional. [Hint: show that if $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, then they are linearly independent.]

2-8. Define $F: \mathbb{R}^n \to \mathbb{RP}^n$ by $F\big(x^1, \ldots, x^n\big) = \big[x^1, \ldots, x^n, 1\big]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G: \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G\big(z^1, \ldots, z^n\big) = \big[z^1, \ldots, z^n, 1\big]$ (see Problem 1-9).

2-9. Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\widetilde{p}: \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G\colon \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2-8:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "G"] \arrow[d, "p"'] & \mathbb{CP}^1 \arrow[d, "\widetilde{p}"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{CP}^1. \end{tikzcd}$$

(*Used on p. 465.*)

2-10. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f\colon M \to \mathbb{R}$. Given a continuous map $F\colon M \to N$, define $F^*\colon C(N) \to C(M)$ by $F^*(f) = f \circ F$.
(a) Show that F^* is a linear map.
(b) Suppose M and N are smooth manifolds. Show that $F\colon M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.
(c) Suppose $F\colon M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.
[Remark: this result shows that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subseteq C(M)$. In fact, some authors *define* a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties; see, e.g., [Nes03].] (*Used on p. 75.*)

2-11. Suppose V is a real vector space of dimension $n \geq 1$. Define the ***projectivization of V***, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi\colon V \smallsetminus \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \dots, E_n)$ for V, the map $E\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E[v^1, \dots, v^n] = [v^i E_i]$ (where brackets denote equivalence classes) is a diffeomorphism. (*Used on p. 561.*)

2-12. State and prove an analogue of Problem 2-11 for complex vector spaces.

2-13. Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.

2-14. Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Chapter 3
Tangent Vectors

The central idea of calculus is *linear approximation*. This arises repeatedly in the study of calculus in Euclidean spaces, where, for example, a function of one variable can be approximated by its tangent line, a parametrized curve in $\mathbb{R}^n$ by its velocity vector, a surface in $\mathbb{R}^3$ by its tangent plane, or a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by its total derivative (see Appendix C).

In order to make sense of calculus on manifolds, we need to introduce the *tangent space to a manifold at a point*, which we can think of as a sort of "linear model" for the manifold near the point. Because of the abstractness of the definition of a smooth manifold, this takes some work, which we carry out in this chapter.

We begin by studying much more concrete objects: *geometric tangent vectors* in $\mathbb{R}^n$, which can be visualized as "arrows" attached to points. Because the definition of smooth manifolds is built around the idea of identifying which functions are smooth, the property of a geometric tangent vector that is amenable to generalization is its action on smooth functions as a "directional derivative." The key observation, which we prove in the first section of this chapter, is that the process of taking directional derivatives gives a natural one-to-one correspondence between geometric tangent vectors and linear maps from $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}$ satisfying the product rule. (Such maps are called *derivations*.) With this as motivation, we then *define* a tangent vector on a smooth manifold as a derivation of $C^\infty(M)$ at a point.

In the second section of the chapter, we show how a smooth map between manifolds yields a linear map between tangent spaces, called the *differential* of the map, which generalizes the total derivative of a map between Euclidean spaces. This allows us to connect the abstract definition of tangent vectors to our concrete geometric picture by showing that any smooth coordinate chart (U, φ) gives a natural isomorphism from the space of tangent vectors to M at p to the space of tangent vectors to $\mathbb{R}^n$ at $\varphi(p)$, which in turn is isomorphic to the space of geometric tangent vectors at $\varphi(p)$. Thus, any smooth coordinate chart yields a basis for each tangent space. Using this isomorphism, we describe how to do concrete computations in such a basis. Based on these coordinate computations, we show how the union of all the tangent spaces at all points of a smooth manifold can be "glued together" to form a new manifold, called the *tangent bundle* of the original manifold.

J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics 218,
DOI 10.1007/978-1-4419-9982-5_3,

Next we show how a smooth curve determines a tangent vector at each point, called its *velocity*, which can be regarded as the derivation of $C^\infty(M)$ that takes the derivative of each function along the curve.

In the final two sections we discuss and compare several other approaches to defining tangent spaces, and give a brief overview of the terminology of *category theory*, which puts the tangent space and differentials in a larger context.

Tangent Vectors

Imagine a manifold in Euclidean space—for example, the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. What do we mean by a "tangent vector" at a point of $\mathbb{S}^{n-1}$? Before we can answer this question, we have to come to terms with a dichotomy in the way we think about elements of $\mathbb{R}^n$. On the one hand, we usually think of them as *points* in space, whose only property is location, expressed by the coordinates $\left(x^1, \dots, x^n\right)$. On the other hand, when doing calculus we sometimes think of them instead as *vectors*, which are objects that have magnitude and direction, but whose location is irrelevant. A vector $v = v^i e_i$ (where e_i denotes the ith standard basis vector) can be visualized as an arrow with its initial point anywhere in $\mathbb{R}^n$; what is relevant from the vector point of view is only which direction it points and how long it is.

What we really have in mind here is a separate copy of $\mathbb{R}^n$ at each point. When we talk about vectors tangent to the sphere at a point a, for example, we imagine them as living in a copy of $\mathbb{R}^n$ with its origin translated to a.

Geometric Tangent Vectors

Here is a preliminary definition of tangent vectors in Euclidean space. Given a point $a \in \mathbb{R}^n$, let us define the ***geometric tangent space to $\mathbb{R}^n$ at a***, denoted by $\mathbb{R}^n_a$, to be the set $\{a\} \times \mathbb{R}^n = \{(a, v) : v \in \mathbb{R}^n\}$. A ***geometric tangent vector*** in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$. As a matter of notation, we abbreviate (a, v) as v_a (or sometimes $v|_a$ if it is clearer, for example if v itself has a subscript). We think of v_a as the vector v with its initial point at a (Fig. 3.1).The set $\mathbb{R}^n_a$ is a real vector space under the natural operations

$$v_a + w_a = (v + w)_a, \qquad c(v_a) = (cv)_a.$$

The vectors $e_i|_a$, $i = 1, \dots, n$, are a basis for $\mathbb{R}^n_a$. In fact, as a vector space, $\mathbb{R}^n_a$ is essentially the same as $\mathbb{R}^n$ itself; the only reason we add the index a is so that the geometric tangent spaces $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ at distinct points a and b will be disjoint sets.

With this definition we could think of the tangent space to $\mathbb{S}^{n-1}$ at a point $a \in \mathbb{S}^{n-1}$ as a certain subspace of $\mathbb{R}^n_a$ (Fig. 3.2), namely the space of vectors that are orthogonal to the radial unit vector through a, using the inner product that $\mathbb{R}^n_a$ inherits from $\mathbb{R}^n$ via the natural isomorphism $\mathbb{R}^n \cong \mathbb{R}^n_a$. The problem with this definition, however, is that it gives us no clue as to how we might define tangent vectors on an arbitrary smooth manifold, where there is no ambient Euclidean space. So we

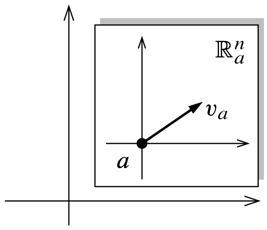

Fig. 3.1 Geometric tangent space

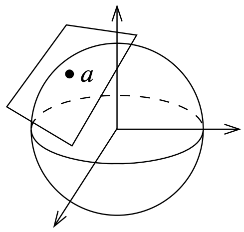

Fig. 3.2 Tangent space to $\mathbb{S}^{n-1}$

need to look for another characterization of tangent vectors that might make sense on a manifold.

The only things we have to work with on smooth manifolds so far are smooth functions, smooth maps, and smooth coordinate charts. One thing that a geometric tangent vector provides is a means of taking directional derivatives of functions. For example, any geometric tangent vector $v_a \in \mathbb{R}^n_a$ yields a map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, which takes the directional derivative in the direction v at a:

$$D_v|_a f = D_v f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a+tv). \tag{3.1}$$

This operation is linear over $\mathbb{R}$ and satisfies the product rule:

$$D_v|_a(fg) = f(a)D_v|_a g + g(a)D_v|_a f. \tag{3.2}$$

If $v_a = v^i e_i|_a$ in terms of the standard basis, then by the chain rule $D_v|_a f$ can be written more concretely as

$$D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a).$$

(Here we are using the summation convention as usual, so the expression on the right-hand side is understood to be summed over $i = 1, \ldots, n$. This sum is consistent with our index convention if we stipulate that an upper index "in the denominator" is to be regarded as a lower index.) For example, if $v_a = e_j|_a$, then

$$D_v|_a f = \frac{\partial f}{\partial x^j}(a).$$

With this construction in mind, we make the following definition. If a is a point of $\mathbb{R}^n$, a map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a ***derivation at a*** if it is linear over $\mathbb{R}$ and satisfies the following product rule:

$$w(fg) = f(a)wg + g(a)wf. \tag{3.3}$$

Let $T_a\mathbb{R}^n$ denote the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. Clearly, $T_a\mathbb{R}^n$ is a vector space under the operations

$$(w_1 + w_2)f = w_1 f + w_2 f, \qquad (cw)f = c(wf).$$

The most important (and perhaps somewhat surprising) fact about $T_a\mathbb{R}^n$ is that it is finite-dimensional, and in fact is naturally isomorphic to the geometric tangent space $\mathbb{R}^n_a$ that we defined above. The proof will be based on the following lemma.

Lemma 3.1 (Properties of Derivations). *Suppose $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$.*

(a) *If f is a constant function, then $wf = 0$.*
(b) *If $f(a) = g(a) = 0$, then $w(fg) = 0$.*

Proof. It suffices to prove (a) for the constant function $f_1(x) \equiv 1$, for then $f(x) \equiv c$ implies $wf = w(cf_1) = cwf_1 = 0$ by linearity. For f_1, the product rule gives

$$wf_1 = w(f_1 f_1) = f_1(a)wf_1 + f_1(a)wf_1 = 2wf_1,$$

which implies that $wf_1 = 0$. Similarly, (b) also follows from the product rule:

$$w(fg) = f(a)wg + g(a)wf = 0 + 0 = 0. \qquad \square$$

The next proposition shows that derivations at a are in one-to-one correspondence with geometric tangent vectors.

Proposition 3.2. *Let $a \in \mathbb{R}^n$.*

(a) *For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by (3.1) is a derivation at a.*
(b) *The map $v_a \mapsto D_v|_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.*

Proof. The fact that $D_v|_a$ is a derivation at a is an immediate consequence of the product rule (3.2).

To prove that the map $v_a \mapsto D_v|_a$ is an isomorphism, we note first that it is linear, as is easily checked. To see that it is injective, suppose $v_a \in \mathbb{R}^n_a$ has the property that $D_v|_a$ is the zero derivation. Writing $v_a = v^i e_i|_a$ in terms of the standard basis, and taking f to be the jth coordinate function $x^j \colon \mathbb{R}^n \to \mathbb{R}$, thought of as a smooth function on $\mathbb{R}^n$, we obtain

$$0 = D_v|_a\left(x^j\right) = v^i \frac{\partial}{\partial x^i}\left(x^j\right)\bigg|_{x=a} = v^j,$$

where the last equality follows because $\partial x^j/\partial x^i = 0$ except when $i = j$, in which case it is equal to 1. Since this is true for each j, it follows that v_a is the zero vector.

To prove surjectivity, let $w \in T_a\mathbb{R}^n$ be arbitrary. Motivated by the computation in the preceding paragraph, we define $v = v^i e_i$, where the real numbers $v^1, \dots, v^n$ are given by $v^i = w\left(x^i\right)$. We will show that $w = D_v|_a$.

To see this, let f be any smooth real-valued function on $\mathbb{R}^n$. By Taylor's theorem (Theorem C.15), we can write

$$\begin{aligned} f(x) = f(a) &+ \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)\left(x^i - a^i\right) \\ &+ \sum_{i,j=1}^n \left(x^i - a^i\right)\left(x^j - a^j\right)\int_0^1 (1-t)\frac{\partial^2 f}{\partial x^i \partial x^j}\big(a + t(x-a)\big)\,dt. \end{aligned}$$

Note that each term in the last sum above is a product of two smooth functions of x that vanish at $x = a$: one is $\left(x^i - a^i\right)$, and the other is $\left(x^j - a^j\right)$ times the integral. The derivation w annihilates this entire sum by Lemma 3.1(b). Thus

$$\begin{aligned} wf &= w\big(f(a)\big) + \sum_{i=1}^{n} w\left(\frac{\partial f}{\partial x^i}(a)\left(x^i - a^i\right)\right) \\ &= 0 + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\left(w\left(x^i\right) - w\left(a^i\right)\right) \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) v^i = D_v|_a f. \end{aligned}$$

□

Corollary 3.3. *For any $a \in \mathbb{R}^n$, the n derivations*

$$\left.\frac{\partial}{\partial x^1}\right|_a, \ldots, \left.\frac{\partial}{\partial x^n}\right|_a \quad \text{defined by} \quad \left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a basis for $T_a\mathbb{R}^n$, which therefore has dimension n.

Proof. Apply the previous proposition and note that $\partial/\partial x^i|_a = D_{e_i}|_a$. □

Tangent Vectors on Manifolds

Now we are in a position to define tangent vectors on manifolds and manifolds with boundary. The definition is the same in both cases. Let M be a smooth manifold with or without boundary, and let p be a point of M. A linear map $v \colon C^\infty(M) \to \mathbb{R}$ is called a ***derivation at p*** if it satisfies

$$v(fg) = f(p)vg + g(p)vf \quad \text{for all } f, g \in C^\infty(M). \tag{3.4}$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the ***tangent space to M at p***. An element of T_pM is called a ***tangent vector at p***.

The following lemma is the analogue of Lemma 3.1 for manifolds.

Lemma 3.4 (Properties of Tangent Vectors on Manifolds). *Suppose M is a smooth manifold with or without boundary, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$.*

(a) *If f is a constant function, then $vf = 0$.*
(b) *If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

► **Exercise 3.5.** Prove Lemma 3.4.

With the motivation of geometric tangent vectors in $\mathbb{R}^n$ in mind, you should visualize tangent vectors to M as "arrows" that are tangent to M and whose base points are attached to M at the given point. Proofs of theorems about tangent vectors must, of course, be based on the abstract definition in terms of derivations, but your intuition should be guided as much as possible by the geometric picture.

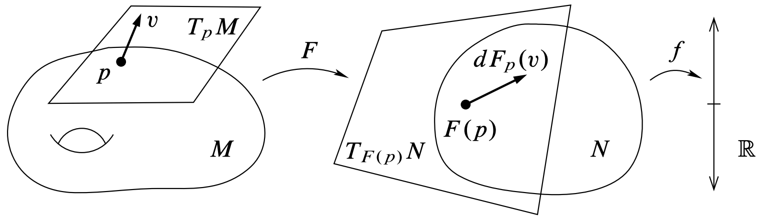

Fig. 3.3 The differential

The Differential of a Smooth Map

To relate the abstract tangent spaces we have defined on manifolds to geometric tangent spaces in $\mathbb{R}^n$, we have to explore the way smooth maps affect tangent vectors. In the case of a smooth map between Euclidean spaces, the total derivative of the map at a point (represented by its Jacobian matrix) is a linear map that represents the "best linear approximation" to the map near the given point. In the manifold case there is a similar linear map, but it makes no sense to talk about a linear map between manifolds. Instead, it will be a linear map between tangent spaces.

If M and N are smooth manifolds with or without boundary and $F: M \to N$ is a smooth map, for each $p \in M$ we define a map

$$dF_p: T_pM \to T_{F(p)}N,$$

called the ***differential of F at p*** (Fig. 3.3), as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

Note that if $f \in C^\infty(N)$, then $f \circ F \in C^\infty(M)$, so $v(f \circ F)$ makes sense. The operator $dF_p(v): C^\infty(N) \to \mathbb{R}$ is linear because v is, and is a derivation at $F(p)$ because for any $f, g \in C^\infty(N)$ we have

$$\begin{aligned} dF_p(v)(fg) &= v\big((fg) \circ F\big) = v\big((f \circ F)(g \circ F)\big) \\ &= f \circ F(p) v(g \circ F) + g \circ F(p) v(f \circ F) \\ &= f\big(F(p)\big) dF_p(v)(g) + g\big(F(p)\big) dF_p(v)(f). \end{aligned}$$

Proposition 3.6 (Properties of Differentials). *Let M, N, and P be smooth manifolds with or without boundary, let $F: M \to N$ and $G: N \to P$ be smooth maps, and let $p \in M$.*

(a) *$dF_p: T_pM \to T_{F(p)}N$ is linear.*
(b) *$d(G \circ F)_p = dG_{F(p)} \circ dF_p: T_pM \to T_{G\circ F(p)}P$.*
(c) *$d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}: T_pM \to T_pM$.*
(d) *If F is a diffeomorphism, then $dF_p: T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d\big(F^{-1}\big)_{F(p)}$.*

► **Exercise 3.7.** Prove Proposition 3.6.

Our first important application of the differential will be to use coordinate charts to relate the tangent space to a point on a manifold with the Euclidean tangent space. But there is an important technical issue that we must address first: while the tangent space is defined in terms of smooth functions on the whole manifold, coordinate charts are in general defined only on open subsets. The key point, expressed in the next proposition, is that tangent vectors act locally.

Proposition 3.8. *Let M be a smooth manifold with or without boundary, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.*

Proof. Let $h = f - g$, so that h is a smooth function that vanishes in a neighborhood of p. Let $\psi \in C^\infty(M)$ be a smooth bump function that is identically equal to 1 on the support of h and is supported in $M \smallsetminus \{p\}$. Because $\psi \equiv 1$ where h is nonzero, the product ψh is identically equal to h. Since $h(p) = \psi(p) = 0$, Lemma 3.4 implies that $vh = v(\psi h) = 0$. By linearity, this implies $vf = vg$. □

Using this proposition, we can identify the tangent space to an open submanifold with the tangent space to the whole manifold.

Proposition 3.9 (The Tangent Space to an Open Submanifold). *Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset, and let $\iota\colon U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p\colon T_pU \to T_pM$ is an isomorphism.*

Proof. To prove injectivity, suppose $v \in T_pU$ and $d\iota_p(v) = 0 \in T_pM$. Let B be a neighborhood of p such that $\overline{B} \subseteq U$. If $f \in C^\infty(U)$ is arbitrary, the extension lemma for smooth functions guarantees that there exists $\widetilde{f} \in C^\infty(M)$ such that $\widetilde{f} \equiv f$ on $\overline{B}$. Then since f and $\widetilde{f}|_U$ are smooth functions on U that agree in a neighborhood of p, Proposition 3.8 implies

$$vf = v\big(\widetilde{f}|_U\big) = v\big(\widetilde{f} \circ \iota\big) = d\iota(v)_p\widetilde{f} = 0.$$

Since this holds for every $f \in C^\infty(U)$, it follows that $v = 0$, so $d\iota_p$ is injective.

On the other hand, to prove surjectivity, suppose $w \in T_pM$ is arbitrary. Define an operator $v\colon C^\infty(U) \to \mathbb{R}$ by setting $vf = w\widetilde{f}$, where $\widetilde{f}$ is any smooth function on all of M that agrees with f on $\overline{B}$. By Proposition 3.8, vf is independent of the choice of $\widetilde{f}$, so v is well defined, and it is easy to check that it is a derivation of $C^\infty(U)$ at p. For any $g \in C^\infty(M)$,

$$d\iota_p(v)g = v(g \circ \iota) = w\big(\widetilde{g \circ \iota}\big) = wg,$$

where the last two equalities follow from the facts that $g \circ \iota$, $\widetilde{g \circ \iota}$, and g all agree on B. Therefore, $d\iota_p$ is also surjective. □

Given an open subset $U \subseteq M$, the isomorphism $d\iota_p$ between T_pU and T_pM is canonically defined, independently of any choices. From now on we *identify* T_pU with T_pM for any point $p \in U$. This identification just amounts to the observation

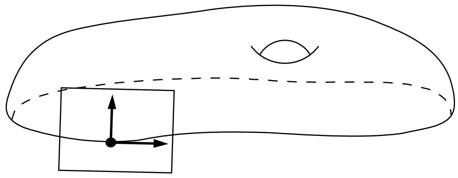

Fig. 3.4 The tangent space to a manifold with boundary

that $d\iota_p(v)$ is the *same derivation as* v, thought of as acting on functions on the bigger manifold M instead of functions on U. Since the action of a derivation on a function depends only on the values of the function in an arbitrarily small neighborhood, this is a harmless identification. In particular, this means that any tangent vector $v \in T_pM$ can be unambiguously applied to functions defined only in a neighborhood of p, not necessarily on all of M.

Proposition 3.10 (Dimension of the Tangent Space). *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

Proof. Given $p \in M$, let (U, φ) be a smooth coordinate chart containing p. Because φ is a diffeomorphism from U onto an open subset $\widehat{U} \subseteq \mathbb{R}^n$, it follows from Proposition 3.6(d) that $d\varphi_p$ is an isomorphism from T_pU to $T_{\varphi(p)}\widehat{U}$. Since Proposition 3.9 guarantees that $T_pM \cong T_pU$ and $T_{\varphi(p)}\widehat{U} \cong T_{\varphi(p)}\mathbb{R}^n$, it follows that $\dim T_pM = \dim T_{\varphi(p)}\mathbb{R}^n = n$. □

Next we need to prove an analogous result for manifolds with boundary. In fact, if M is an n-manifold with boundary, it might not be immediately clear what one should expect the tangent space at a boundary point of M to look like. Should it be an n-dimensional vector space, like the tangent space at an interior point? Or should it be $(n-1)$-dimensional, like the boundary? Or should it be an n-dimensional half-space, like the space $\mathbb{H}^n$ on which M is modeled locally?

As we will show below, our definition implies that the tangent space at a boundary point is an n-dimensional vector space (Fig. 3.4), just like the tangent spaces at interior points. This may or may not seem like the most geometrically intuitive choice, but it has the advantage of making most of the definitions of geometric objects on a manifold with boundary look exactly the same as those on a manifold.

First, we need to relate the tangent spaces $T_a\mathbb{H}^n$ and $T_a\mathbb{R}^n$ for points $a \in \partial\mathbb{H}^n$. Since $\mathbb{H}^n$ is not an open subset of $\mathbb{R}^n$, Proposition 3.9 does not apply. As a substitute, we have the following lemma.

Lemma 3.11. *Let $\iota\colon \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ denote the inclusion map. For any $a \in \partial\mathbb{H}^n$, the differential $d\iota_a\colon T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ is an isomorphism.*

Proof. Suppose $a \in \partial\mathbb{H}^n$. To show that $d\iota_a$ is injective, assume $d\iota_a(v) = 0$. Suppose $f\colon \mathbb{H}^n \to \mathbb{R}$ is smooth, and let $\widetilde{f}$ be any extension of f to a smooth function defined on all of $\mathbb{R}^n$. (Such an extension exists by the extension lemma for smooth functions, Lemma 2.26.) Then $\widetilde{f} \circ \iota = f$, so

$$v f = v(\widetilde{f} \circ \iota) = d\iota_a(v)\widetilde{f} = 0,$$

which implies that $d\iota_a$ is injective.

To show surjectivity, let $w \in T_a\mathbb{R}^n$ be arbitrary. Define $v \in T_a\mathbb{H}^n$ by

$$v f = w\widetilde{f},$$

where $\widetilde{f}$ is any smooth extension of f. Writing $w = w^i \partial/\partial x^i|_a$ in terms of the standard basis for $T_a\mathbb{R}^n$, this means that

$$v f = w^i \frac{\partial \widetilde{f}}{\partial x^i}(a).$$

This is independent of the choice of $\widetilde{f}$, because by continuity the derivatives of $\widetilde{f}$ at a are determined by those of f in $\mathbb{H}^n$. It is easy to check that v is a derivation at a and that $w = d\iota_a(v)$, so $d\iota_a$ is surjective. □

Just as we use Proposition 3.9 to identify T_pU with T_pM when U is an open subset of M, we use this lemma to identify $T_a\mathbb{H}^n$ with $T_a\mathbb{R}^n$ when $a \in \partial\mathbb{H}^n$, and we do not distinguish notationally between an element of $T_a\mathbb{H}^n$ and its image in $T_a\mathbb{R}^n$.

Proposition 3.12 (Dimension of Tangent Spaces on a Manifold with Boundary). *Suppose M is an n-dimensional smooth manifold with boundary. For each $p \in M$, T_pM is an n-dimensional vector space.*

Proof. Let $p \in M$ be arbitrary. If p is an interior point, then because $\operatorname{Int} M$ is an open submanifold of M, Proposition 3.9 implies that $T_p(\operatorname{Int} M) \cong T_pM$. Since $\operatorname{Int} M$ is a smooth n-manifold without boundary, its tangent spaces all have dimension n.

On the other hand, if $p \in \partial M$, let (U, φ) be a smooth boundary chart containing p, and let $\widehat{U} = \varphi(U) \subseteq \mathbb{H}^n$. There are isomorphisms $T_pM \cong T_pU$ (by Proposition 3.9); $T_pU \cong T_{\varphi(p)}\widehat{U}$ (by Proposition 3.6(d), because φ is a diffeomorphism); $T_{\varphi(p)}\widehat{U} \cong T_{\varphi(p)}\mathbb{H}^n$ (by Proposition 3.9 again); and $T_{\varphi(p)}\mathbb{H}^n \cong T_{\varphi(p)}\mathbb{R}^n$ (by Lemma 3.11). The result follows. □

Recall from Example 1.24 that every finite-dimensional vector space has a natural smooth manifold structure that is independent of any choice of basis or norm. The following proposition shows that the tangent space to a vector space can be naturally identified with the vector space itself.

Suppose V is a finite-dimensional vector space and $a \in V$. Just as we did earlier in the case of $\mathbb{R}^n$, for any vector $v \in V$, we define a map $D_v|_a \colon C^\infty(V) \to \mathbb{R}$ by

$$D_v|_a f = \left.\frac{d}{dt}\right|_{t=0} f(a + tv). \tag{3.5}$$

Proposition 3.13 (The Tangent Space to a Vector Space). *Suppose V is a finite-dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the map $v \mapsto D_v|_a$ defined by (3.5) is a canonical isomorphism from V to T_aV, such that for any linear map $L\colon V \to W$, the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[d, "L"'] & T_aV \arrow[d, "dL_a"] \\ W \arrow[r, "\cong"'] & T_{La}W. \end{tikzcd} \tag{3.6}$$

Proof. Once we choose a basis for V, we can use the same argument as in the proof of Proposition 3.2 to show that $D_v|_a$ is indeed a derivation at a, and that the map $v \mapsto D_v|_a$ is an isomorphism.

Now suppose $L\colon V \to W$ is a linear map. Because its components with respect to any choices of bases for V and W are linear functions of the coordinates, L is smooth. Unwinding the definitions and using the linearity of L, we compute

$$\begin{aligned} dL_a\big(D_v|_a\big)f &= D_v|_a(f \circ L) \\ &= \frac{d}{dt}\bigg|_{t=0} f\big(L(a+tv)\big) = \frac{d}{dt}\bigg|_{t=0} f(La + tLv) \\ &= D_{Lv}|_{La}f. \end{aligned}$$ □

It is important to understand that each isomorphism $V \cong T_aV$ is canonically defined, independently of any choice of basis (notwithstanding the fact that we used a choice of basis to prove that it is an isomorphism). Because of this result, we can routinely *identify* tangent vectors to a finite-dimensional vector space with elements of the space itself. More generally, if M is an open submanifold of a vector space V, we can combine our identifications $T_pM \leftrightarrow T_pV \leftrightarrow V$ to obtain a canonical identification of each tangent space to M with V. For example, since $\mathrm{GL}(n,\mathbb{R})$ is an open submanifold of the vector space $\mathrm{M}(n,\mathbb{R})$, we can identify its tangent space at each point $X \in \mathrm{GL}(n,\mathbb{R})$ with the full space of matrices $\mathrm{M}(n,\mathbb{R})$.

There is another natural identification for tangent spaces to a product manifold.

Proposition 3.14 (The Tangent Space to a Product Manifold). *Let $M_1,\dots,M_k$ be smooth manifolds, and for each j, let $\pi_j\colon M_1 \times \dots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1,\dots,p_k) \in M_1 \times \dots \times M_k$, the map*

$$\alpha\colon T_p(M_1 \times \dots \times M_k) \to T_{p_1}M_1 \oplus \dots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = \big(d(\pi_1)_p(v),\dots,d(\pi_k)_p(v)\big) \tag{3.7}$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Proof. See Problem 3-2. □

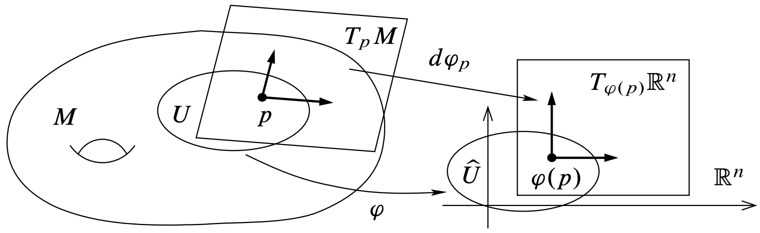

Fig. 3.5 Tangent vectors in coordinates

Once again, because the isomorphism (3.7) is canonically defined, independently of any choice of coordinates, we can consider it as a canonical identification, and we will always do so. Thus, for example, we identify $T_{(p,q)}(M \times N)$ with $T_pM \oplus T_qN$, and treat T_pM and T_qN as subspaces of $T_{(p,q)}(M \times N)$.

Computations in Coordinates

Our treatment of the tangent space to a manifold so far might seem hopelessly abstract. To bring it down to earth, we will show how to do computations with tangent vectors and differentials in local coordinates.

First, suppose M is a smooth manifold (without boundary), and let (U, φ) be a smooth coordinate chart on M. Then φ is, in particular, a diffeomorphism from U to an open subset $\widehat{U} \subseteq \mathbb{R}^n$. Combining Propositions 3.9 and 3.6(d), we see that $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism.

By Corollary 3.3, the derivations $\partial/\partial x^1|_{\varphi(p)}, \dots, \partial/\partial x^n|_{\varphi(p)}$ form a basis for $T_{\varphi(p)}\mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $d\varphi_p$ form a basis for T_pM (Fig. 3.5). In keeping with our standard practice of treating coordinate maps as identifications whenever possible, we use the notation $\partial/\partial x^i|_p$ for these vectors, characterized by either of the following expressions:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = d\left(\varphi^{-1}\right)_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right). \tag{3.8}$$

Unwinding the definitions, we see that $\partial/\partial x^i|_p$ acts on a function $f \in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \left(f \circ \varphi^{-1}\right) = \frac{\partial \widehat{f}}{\partial x^i}(\widehat{p}),$$

where $\widehat{f} = f \circ \varphi^{-1}$ is the coordinate representation of f, and $\widehat{p} = \left(p^1, \dots, p^n\right) = \varphi(p)$ is the coordinate representation of p. In other words, $\partial/\partial x^i|_p$ is just the derivation that takes the ith partial derivative of (the coordinate representation of) f at (the coordinate representation of) p. The vectors $\partial/\partial x^i|_p$ are called the ***coordinate vectors at p*** associated with the given coordinate system. In the special case of standard coordinates on $\mathbb{R}^n$, the vectors $\partial/\partial x^i|_p$ are literally the partial derivative operators.

When M is a smooth manifold with boundary and p is an interior point, the discussion above applies verbatim. For $p \in \partial M$, the only change that needs to be made is to substitute $\mathbb{H}^n$ for $\mathbb{R}^n$, with the understanding that the notation $\partial/\partial x^i|_{\varphi(p)}$ can be used interchangeably to denote either an element of $T_{\varphi(p)}\mathbb{R}^n$ or an element of $T_{\varphi(p)}\mathbb{H}^n$, in keeping with our convention of considering the isomorphism $d\iota_{\varphi(p)}\colon T_{\varphi(p)}\mathbb{H}^n \to T_{\varphi(p)}\mathbb{R}^n$ as an identification. The nth coordinate vector $\partial/\partial x^n|_p$ should be interpreted as a one-sided derivative in this case.

The following proposition summarizes the discussion so far.

Proposition 3.15. *Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $\left(U,\left(x^i\right)\right)$ containing p, the coordinate vectors $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$ form a basis for T_pM.* □

Thus, a tangent vector $v \in T_pM$ can be written uniquely as a linear combination

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

where we use the summation convention as usual, with an upper index in the denominator being considered as a lower index, as explained on p. 52. The ordered basis $\left(\partial/\partial x^i|_p\right)$ is called a ***coordinate basis for T_pM***, and the numbers $\left(v^1,\dots,v^n\right)$ are called the ***components of v*** with respect to the coordinate basis. If v is known, its components can be computed easily from its action on the coordinate functions. For each j, the components of v are given by $v^j = v\left(x^j\right)$ (where we think of x^j as a smooth real-valued function on U), because

$$v\left(x^j\right) = \left(v^i \left.\frac{\partial}{\partial x^i}\right|_p \right)\left(x^j\right) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j.$$

The Differential in Coordinates

Next we explore how differentials look in coordinates. We begin by considering the special case of a smooth map $F\colon U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets of Euclidean spaces. For any $p \in U$, we will determine the matrix of $dF_p\colon T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Using $\left(x^1,\dots,x^n\right)$ to denote the coordinates in the domain and $\left(y^1,\dots,y^m\right)$ to denote those in the codomain, we use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p \right) f &= \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial f}{\partial y^j}\left(F(p)\right)\frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\frac{\partial F^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_{F(p)} \right) f. \end{aligned}$$

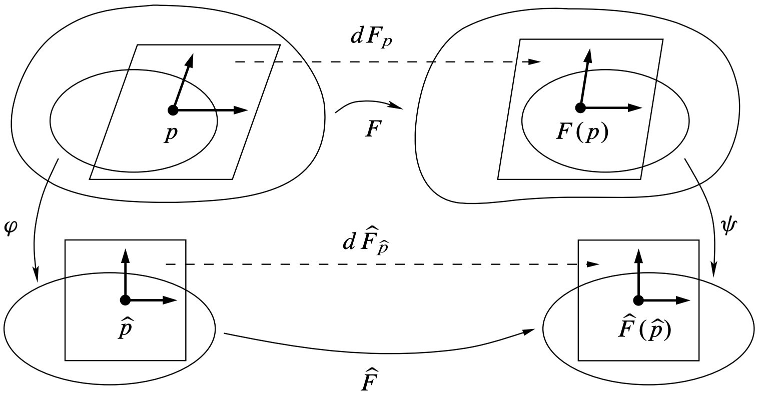

Fig. 3.6 The differential in coordinates

Thus

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \tag{3.9}$$

In other words, the matrix of dF_p in terms of the coordinate bases is

$$\begin{pmatrix} \dfrac{\partial F^1}{\partial x^1}(p) & \cdots & \dfrac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F^m}{\partial x^1}(p) & \cdots & \dfrac{\partial F^m}{\partial x^n}(p) \end{pmatrix}.$$

(Recall that the columns of the matrix are the components of the images of the basis vectors.) This matrix is none other than the Jacobian matrix of F at p, which is the matrix representation of the total derivative $DF(p)\colon \mathbb{R}^n \to \mathbb{R}^m$. Therefore, in this case, $dF_p\colon T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ corresponds to the total derivative $DF(p)\colon \mathbb{R}^n \to \mathbb{R}^m$, under our usual identification of Euclidean spaces with their tangent spaces. The same calculation applies if U is an open subset of $\mathbb{H}^n$ and V is an open subset of $\mathbb{H}^m$.

Now consider the more general case of a smooth map $F\colon M \to N$ between smooth manifolds with or without boundary. Choosing smooth coordinate charts (U,φ) for M containing p and (V,ψ) for N containing $F(p)$, we obtain the coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}\colon \varphi(U \cap F^{-1}(V)) \to \psi(V)$ (Fig. 3.6). Let $\widehat{p} = \varphi(p)$ denote the coordinate representation of p. By the computation above, $d\widehat{F}_{\widehat{p}}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\widehat{F}$ at $\widehat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we compute

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = dF_p\left(d\left(\varphi^{-1}\right)_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right) = d\left(\psi^{-1}\right)_{\widehat{F}(\widehat{p})}\left(d\widehat{F}_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right)$$

$$= d\left(\psi^{-1}\right)_{\widehat{F}(\widehat{p})}\left(\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\, \frac{\partial}{\partial y^j}\bigg|_{\widehat{F}(\widehat{p})}\right)$$

$$= \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\, \frac{\partial}{\partial y^j}\bigg|_{F(p)}. \tag{3.10}$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representative of) F. In fact, the definition of the differential was cooked up precisely to give a coordinate-independent meaning to the Jacobian matrix.

In the differential geometry literature, the differential is sometimes called the ***tangent map***, the ***total derivative***, or simply the ***derivative of F***. Because it "pushes" tangent vectors forward from the domain manifold to the codomain, it is also called the **(*pointwise*) *pushforward***. Different authors denote it by symbols such as

$$F'(p), \quad DF, \quad DF(p), \quad F_*, \quad TF, \quad T_pF.$$

We will stick with the notation dF_p for the differential of a smooth map between manifolds, and reserve $DF(p)$ for the total derivative of a map between finite-dimensional vector spaces, which in the case of Euclidean spaces we identify with the Jacobian matrix of F.

Change of Coordinates

Suppose (U,φ) and (V,ψ) are two smooth charts on M, and $p \in U \cap V$. Let us denote the coordinate functions of φ by $\left(x^i\right)$ and those of ψ by $\left(\widetilde{x}^i\right)$. Any tangent vector at p can be represented with respect to either basis $\left(\partial/\partial x^i|_p\right)$ or $\left(\partial/\partial \widetilde{x}^i|_p\right)$. How are the two representations related?

In this situation, it is customary to write the transition map $\psi\circ\varphi^{-1}\colon \varphi(U\cap V) \to \psi(U\cap V)$ in the following shorthand notation:

$$\psi\circ\varphi^{-1}(x) = \left(\widetilde{x}^1(x),\dots,\widetilde{x}^n(x)\right).$$

Here we are indulging in a typical abuse of notation: in the expression $\widetilde{x}^i(x)$, we think of $\widetilde{x}^i$ as a coordinate *function* (whose domain is an open subset of M, identified with an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$); but we think of x as representing a *point* (in this case, in $\varphi(U\cap V)$). By (3.9), the differential $d\left(\psi\circ\varphi^{-1}\right)_{\varphi(p)}$ can be written

$$d\left(\psi\circ\varphi^{-1}\right)_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(\varphi(p))\, \frac{\partial}{\partial \widetilde{x}^j}\bigg|_{\psi(p)}.$$

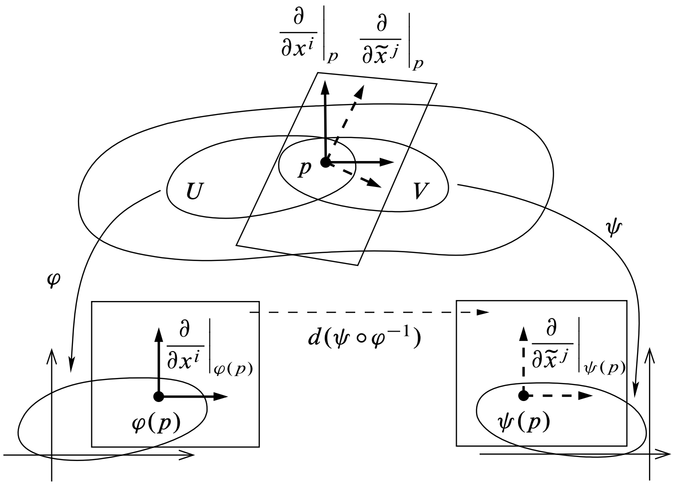

Fig. 3.7 Change of coordinates

(See Fig. 3.7.) Using the definition of coordinate vectors, we obtain

$$\begin{aligned}\left.\frac{\partial}{\partial x^i}\right|_p &= d\left(\varphi^{-1}\right)_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right)\\ &= d\left(\psi^{-1}\right)_{\psi(p)} \circ d\left(\psi\circ\varphi^{-1}\right)_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right)\\ &= d\left(\psi^{-1}\right)_{\psi(p)}\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\left(\varphi(p)\right)\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_{\psi(p)}\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p, \end{aligned} \tag{3.11}$$

where again we have written $\widehat{p} = \varphi(p)$. (This formula is easy to remember, because it looks exactly the same as the chain rule for partial derivatives in $\mathbb{R}^n$.) Applying this to the components of a vector $v = v^i \partial/\partial x^i|_p = \widetilde{v}^j \partial/\partial \widetilde{x}^j|_p$, we find that the components of v transform by the rule

$$\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(\widehat{p})v^i. \tag{3.12}$$

Example 3.16. The transition map between polar coordinates and standard coordinates in suitable open subsets of the plane is given by $(x, y) = (r\cos\theta, r\sin\theta)$. Let p be the point in $\mathbb{R}^2$ whose polar coordinate representation is $(r,\theta) = (2, \pi/2)$, and let $v \in T_p\mathbb{R}^2$ be the tangent vector whose polar coordinate representation is

$$v = 3\left.\frac{\partial}{\partial r}\right|_p - \left.\frac{\partial}{\partial \theta}\right|_p.$$

Applying (3.11) to the coordinate vectors, we find

$$\begin{aligned}
\left.\frac{\partial}{\partial r}\right|_p &= \cos\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial x}\right|_p + \sin\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial y}\right|_p = \left.\frac{\partial}{\partial y}\right|_p, \\
\left.\frac{\partial}{\partial \theta}\right|_p &= -2\sin\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial x}\right|_p + 2\cos\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial y}\right|_p = -2\left.\frac{\partial}{\partial x}\right|_p,
\end{aligned}$$

and thus v has the following coordinate representation in standard coordinates:

$$v = 3\left.\frac{\partial}{\partial y}\right|_p + 2\left.\frac{\partial}{\partial x}\right|_p. \qquad //$$

One important fact to bear in mind is that each coordinate vector $\partial/\partial x^i|_p$ depends on the entire *coordinate system*, not just on the single coordinate function x^i. Geometrically, this reflects the fact that $\partial/\partial x^i|_p$ is the derivation obtained by differentiating with respect to x^i while *all the other coordinates are held constant*. If the coordinate functions other than x^i are changed, then the direction of this coordinate derivative can change. The next exercise illustrates how this can happen.

► **Exercise 3.17.** Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\tilde{x} = x, \qquad \tilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p,$$

even though the coordinate functions x and $\tilde{x}$ are identically equal.

The Tangent Bundle

Often it is useful to consider the set of all tangent vectors at all points of a manifold. Given a smooth manifold M with or without boundary, we define the ***tangent bundle of M***, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:

$$TM = \coprod_{p \in M} T_pM.$$

We usually write an element of this disjoint union as an ordered pair (p, v), with $p \in M$ and $v \in T_pM$ (instead of putting the point p in the second position, as elements of a disjoint union are more commonly written). The tangent bundle comes equipped with a natural ***projection map*** $\pi: TM \to M$, which sends each vector in T_pM to the point p at which it is tangent: $\pi(p, v) = p$. We will often commit the usual mild sin of identifying T_pM with its image under the canonical injection $v \mapsto (p, v)$, and will use any of the notations (p, v), v_p, and v for a tangent vector in T_pM, depending on how much emphasis we wish to give to the point p.

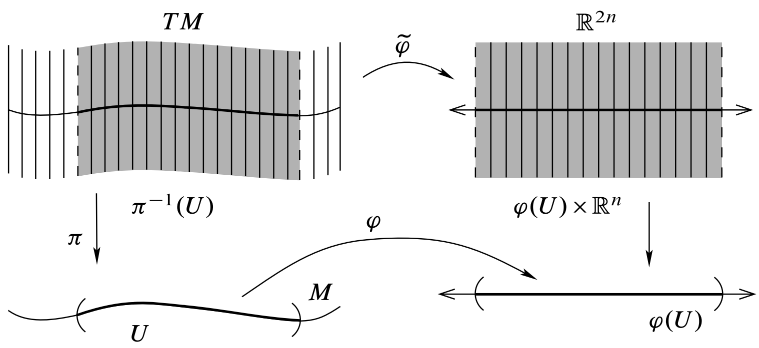

Fig. 3.8 Coordinates for the tangent bundle

For example, in the special case $M = \mathbb{R}^n$, using Proposition 3.2, we see that the tangent bundle of $\mathbb{R}^n$ can be canonically identified with the union of its geometric tangent spaces, which in turn is just the Cartesian product of $\mathbb{R}^n$ with itself:

$$T\mathbb{R}^n = \coprod_{a \in \mathbb{R}^n} T_a\mathbb{R}^n \cong \coprod_{a \in \mathbb{R}^n} \mathbb{R}^n_a = \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

An element (a, v) of this Cartesian product can be thought of as representing either the geometric tangent vector v_a or the derivation $D_v|_a$ defined by (3.1). Be warned, however, that in general the tangent bundle of a smooth manifold cannot be identified in any natural way with a Cartesian product, because there is no canonical way to identify tangent spaces at different points with each other. We will have more to say about this below.

If M is a smooth manifold, the tangent bundle TM can be thought of simply as a disjoint union of vector spaces; but it is much more than that. The next proposition shows that TM can be considered as a smooth manifold in its own right.

Proposition 3.18. *For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi: TM \to M$ is smooth.*

Proof. We begin by defining the maps that will become our smooth charts. Given any smooth chart (U, φ) for M, note that $\pi^{-1}(U) \subseteq TM$ is the set of all tangent vectors to M at all points of U. Let $(x^1, \dots, x^n)$ denote the coordinate functions of φ, and define a map $\widetilde{\varphi}: \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right). \tag{3.13}$$

(See Fig. 3.8.) Its image set is $\varphi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. It is a bijection onto its image, because its inverse can be written explicitly as

$$\widetilde{\varphi}^{-1}\left(x^1, \dots, x^n, v^1, \dots, v^n\right) = v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)}.$$

Now suppose we are given two smooth charts (U,φ) and (V,ψ) for M, and let $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$, $\left(\pi^{-1}(V),\widetilde{\psi}\right)$ be the corresponding charts on TM. The sets

$$\widetilde{\varphi}\left(\pi^{-1}(U)\cap\pi^{-1}(V)\right)=\varphi(U\cap V)\times\mathbb{R}^n \quad \text{and}$$
$$\widetilde{\psi}\left(\pi^{-1}(U)\cap\pi^{-1}(V)\right)=\psi(U\cap V)\times\mathbb{R}^n$$

are open in $\mathbb{R}^{2n}$, and the transition map $\widetilde{\psi}\circ\widetilde{\varphi}^{-1}\colon \varphi(U\cap V)\times\mathbb{R}^n\to\psi(U\cap V)\times\mathbb{R}^n$ can be written explicitly using (3.12) as

$$\begin{aligned}&\widetilde{\psi}\circ\widetilde{\varphi}^{-1}\left(x^1,\dots,x^n,v^1,\dots,v^n\right)\\&\quad=\left(\widetilde{x}^1(x),\dots,\widetilde{x}^n(x),\frac{\partial\widetilde{x}^1}{\partial x^j}(x)v^j,\dots,\frac{\partial\widetilde{x}^n}{\partial x^j}(x)v^j\right).\end{aligned}$$

This is clearly smooth.

Choosing a countable cover $\{U_i\}$ of M by smooth coordinate domains, we obtain a countable cover of TM by coordinate domains $\left\{\pi^{-1}(U_i)\right\}$ satisfying conditions (i)–(iv) of the smooth manifold chart lemma (Lemma 1.35). To check the Hausdorff condition (v), just note that any two points in the same fiber of π lie in one chart, while if (p,v) and (q,w) lie in different fibers, there exist disjoint smooth coordinate domains U, V for M such that $p\in U$ and $q\in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing (p,v) and (q,w), respectively.

To see that π is smooth, note that with respect to charts (U,φ) for M and $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$ for TM, its coordinate representation is $\pi(x,v)=x$. □

The coordinates $\left(x^i,v^i\right)$ given by (3.13) are called ***natural coordinates on TM***.

► **Exercise 3.19.** Suppose M is a smooth manifold with boundary. Show that TM has a natural topology and smooth structure making it into a smooth manifold with boundary, such that if $\left(U,\left(x^i\right)\right)$ is any smooth boundary chart for M, then rearranging the coordinates in the natural chart $\left(\pi^{-1}(U),\left(x^i,v^i\right)\right)$ for TM yields a boundary chart $\left(\pi^{-1}(U),\left(v^i,x^i\right)\right)$.

Proposition 3.20. *If M is a smooth n-manifold with or without boundary, and M can be covered by a single smooth chart, then TM is diffeomorphic to $M\times\mathbb{R}^n$.*

Proof. If (U,φ) is a global smooth chart for M, then φ is, in particular, a diffeomorphism from $U=M$ to an open subset $\widehat{U}\subseteq\mathbb{R}^n$ or $\mathbb{H}^n$. The proof of the previous proposition showed that the natural coordinate chart $\widetilde{\varphi}$ is a bijection from TM to $\widehat{U}\times\mathbb{R}^n$, and the smooth structure on TM is defined essentially by declaring $\widetilde{\varphi}$ to be a diffeomorphism. □

Although the picture of a product $U\times\mathbb{R}^n$ is a useful way to visualize the smooth structure on a tangent bundle locally as in Fig. 3.8, do not be misled into imagining that every tangent bundle is *globally* diffeomorphic (or even homeomorphic) to a product of the manifold with $\mathbb{R}^n$. This is not the case for most smooth manifolds. We will revisit this question in Chapters 8, 10, and 16.

By putting together the differentials of F at all points of M, we obtain a globally defined map between tangent bundles, called the ***global differential*** or ***global tangent map*** and denoted by $dF: TM \to TN$. This is just the map whose restriction to each tangent space $T_pM \subseteq TM$ is dF_p. When we apply the differential of F to a specific vector $v \in T_pM$, we can write either $dF_p(v)$ or $dF(v)$, depending on how much emphasis we wish to give to the point p. The former notation is more informative, while the second is more concise.

One important feature of the smooth structure we have defined on TM is that it makes the differential of a smooth map into a smooth map between tangent bundles.

Proposition 3.21. *If $F: M \to N$ is a smooth map, then its global differential $dF: TM \to TN$ is a smooth map.*

Proof. From the local expression (3.9) for dF_p in coordinates, it follows that dF has the following coordinate representation in terms of natural coordinates for TM and TN:

$$dF\left(x^1,\dots,x^n,v^1,\dots,v^n\right) = \left(F^1(x),\dots,F^n(x),\frac{\partial F^1}{\partial x^i}(x)v^i,\dots,\frac{\partial F^n}{\partial x^i}(x)v^i\right).$$

This is smooth because F is. □

The following properties of the global differential follow immediately from Proposition 3.6.

Corollary 3.22 (Properties of the Global Differential). *Suppose $F: M \to N$ and $G: N \to P$ are smooth maps.*

(a) $d(G \circ F) = dG \circ dF$.
(b) $d(\mathrm{Id}_M) = \mathrm{Id}_{TM}$.
(c) *If F is a diffeomorphism, then $dF: TM \to TN$ is also a diffeomorphism, and* $(dF)^{-1} = d\left(F^{-1}\right)$. □

Because of part (c) of this corollary, when F is a diffeomorphism we can use the notation dF^{-1} unambiguously to mean either $(dF)^{-1}$ or $d\left(F^{-1}\right)$.

Velocity Vectors of Curves

The *velocity* of a smooth parametrized curve in $\mathbb{R}^n$ is familiar from elementary calculus. It is just the vector whose components are the derivatives of the component functions of the curve. In this section we extend this notion to curves in manifolds.

If M is a manifold with or without boundary, we define a ***curve in M*** to be a continuous map $\gamma: J \to M$, where $J \subseteq \mathbb{R}$ is an interval. (Most of the time, we will be interested in curves whose domains are open intervals, but for some purposes it is useful to allow J to have one or two endpoints; the definitions all make sense with minor modifications in that case, either by considering J as a manifold with boundary or by interpreting derivatives as one-sided derivatives.) Note that in this

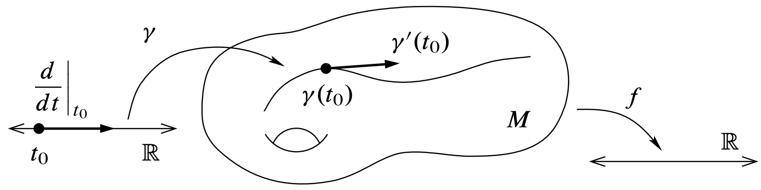

Fig. 3.9 The velocity of a curve

book the term *curve* always refers to a map from an interval into M (a parametrized curve), not just a set of points in M.

Now let M be a smooth manifold, still with or without boundary. Our definition of tangent spaces leads to a natural interpretation of velocity vectors: given a smooth curve $\gamma\colon J \to M$ and $t_0 \in J$, we define the ***velocity of γ at t_0*** (Fig. 3.9), denoted by $\gamma'(t_0)$, to be the vector

$$\gamma'(t_0) = d\gamma\left(\frac{d}{dt}\bigg|_{t_0}\right) \in T_{\gamma(t_0)}M,$$

where $d/dt|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$. (As in ordinary calculus, it is customary to use d/dt instead of $\partial/\partial t$ when the manifold is 1-dimensional.) Other common notations for the velocity are

$$\dot{\gamma}(t_0), \quad \frac{d\gamma}{dt}(t_0), \quad \text{and} \quad \frac{d\gamma}{dt}\bigg|_{t=t_0}.$$

This tangent vector acts on functions by

$$\gamma'(t_0)f = d\gamma\left(\frac{d}{dt}\bigg|_{t_0}\right)f = \frac{d}{dt}\bigg|_{t_0}(f \circ \gamma) = (f \circ \gamma)'(t_0).$$

In other words, $\gamma'(t_0)$ is the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ. (If t_0 is an endpoint of J, this still holds, provided that we interpret the derivative with respect to t as a one-sided derivative, or equivalently as the derivative of any smooth extension of $f \circ \gamma$ to an open subset of $\mathbb{R}$.)

Now let (U, φ) be a smooth chart with coordinate functions $\left(x^i\right)$. If $\gamma(t_0) \in U$, we can write the coordinate representation of γ as $\gamma(t) = \left(\gamma^1(t), \dots, \gamma^n(t)\right)$, at least for t sufficiently close to t_0, and then the coordinate formula for the differential yields

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0)\,\frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}.$$

This means that $\gamma'(t_0)$ is given by essentially the same formula as it would be in Euclidean space: it is the tangent vector whose components in a coordinate basis are the derivatives of the component functions of γ.

The next proposition shows that every tangent vector on a manifold is the velocity vector of some curve. This gives a different and somewhat more geometric way to think about the tangent bundle: it is just the set of all velocity vectors of smooth curves in M.

Proposition 3.23. *Suppose M is a smooth manifold with or without boundary and $p \in M$. Every $v \in T_pM$ is the velocity of some smooth curve in M.*

Proof. First suppose that $p \in \operatorname{Int} M$ (which includes the case $\partial M = \varnothing$). Let (U, φ) be a smooth coordinate chart centered at p, and write $v = v^i \partial/\partial x^i|_p$ in terms of the coordinate basis. For sufficiently small $\varepsilon > 0$, let $\gamma\colon (-\varepsilon, \varepsilon) \to U$ be the curve whose coordinate representation is

$$\gamma(t) = \left(tv^1, \dots, tv^n\right). \tag{3.14}$$

(Remember, this really means $\gamma(t) = \varphi^{-1}\left(tv^1, \dots, tv^n\right)$.) This is a smooth curve with $\gamma(0) = p$, and the computation above shows that $\gamma'(0) = v^i \partial/\partial x^i|_{\gamma(0)} = v$.

Now suppose $p \in \partial M$. Let (U, φ) be a smooth boundary chart centered at p, and write $v = v^i \partial/\partial x^i|_p$ as before. We wish to let γ be the curve whose coordinate representation is (3.14), but this formula represents a point of M only when $tv^n \geq 0$. We can accommodate this requirement by suitably restricting the domain of γ: if $v^n = 0$, we define $\gamma\colon (-\varepsilon, \varepsilon) \to U$ as before; if $v^n > 0$, we let the domain be $[0, \varepsilon)$; and if $v^n < 0$, we let it be $(-\varepsilon, 0]$. In each case, γ is a smooth curve in M with $\gamma(0) = p$ and $\gamma'(0) = v$. □

The next proposition shows that velocity vectors behave well under composition with smooth maps.

Proposition 3.24 (The Velocity of a Composite Curve). *Let $F\colon M \to N$ be a smooth map, and let $\gamma\colon J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma\colon J \to N$ is given by*

$$(F \circ \gamma)'(t_0) = dF\left(\gamma'(t_0)\right).$$

Proof. Just go back to the definition of the velocity of a curve:

$$(F \circ \gamma)'(t_0) = d(F \circ \gamma)\left(\left.\frac{d}{dt}\right|_{t_0}\right) = dF \circ d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) = dF\left(\gamma'(t_0)\right). \qquad \square$$

On the face of it, the preceding proposition tells us how to compute the velocity of a composite curve in terms of the differential. However, it is often much more useful to turn it around the other way, and use it as a streamlined way to compute differentials. Suppose $F\colon M \to N$ is a smooth map, and we need to compute the differential dF_p at some point $p \in M$. We can compute $dF_p(v)$ for any $v \in T_pM$ by choosing a smooth curve γ whose initial tangent vector is v, and then applying Proposition 3.23 to the composite curve $F \circ \gamma$. The next corollary summarizes the result.

Corollary 3.25 (Computing the Differential Using a Velocity Vector). *Suppose $F\colon M \to N$ is a smooth map, $p \in M$, and $v \in T_pM$. Then*

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma\colon J \to M$ such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$. □

This corollary frequently yields a much more succinct computation of dF, especially if F is presented in some form other than an explicit coordinate representation. We will see many examples of this technique in later chapters.

Alternative Definitions of the Tangent Space

In the literature you will find tangent vectors to a smooth manifold defined in several different ways. Here we describe the most common ones. (Yet another definition is suggested in the remark following Problem 11-4.) It is good to be conversant with all of them. Throughout this section, M represents an arbitrary smooth manifold with or without boundary.

Tangent Vectors as Derivations of the Space of Germs

The most common alternative definition is based on the notion of "germs" of smooth functions, which we now define.

A ***smooth function element*** on M is an ordered pair (f,U), where U is an open subset of M and $f: U \to \mathbb{R}$ is a smooth function. Given a point $p \in M$, let us define an equivalence relation on the set of all smooth function elements whose domains contain p by setting $(f,U) \sim (g,V)$ if $f \equiv g$ on some neighborhood of p. The equivalence class of a function element (f,U) is called the ***germ of f at p***. The set of all germs of smooth functions at p is denoted by $C_p^\infty(M)$. It is a real vector space and an associative algebra under the operations

$$\begin{aligned}
c\big[(f,U)\big] &= \big[(cf,U)\big],\\
\big[(f,U)\big] + \big[(g,V)\big] &= \big[(f+g, U\cap V)\big],\\
\big[(f,U)\big]\big[(g,V)\big] &= \big[(fg, U\cap V)\big].
\end{aligned}$$

(The zero element of this algebra is the equivalence class of the zero function on M.) Let us denote the germ at p of the function element (f,U) simply by $[f]_p$; there is no need to include the domain U in the notation, because the same germ is represented by the restriction of f to any neighborhood of p. To say that two germs $[f]_p$ and $[g]_p$ are equal is simply to say that $f \equiv g$ on some neighborhood of p, however small.

A ***derivation of $C_p^\infty(M)$*** is a linear map $v: C_p^\infty(M) \to \mathbb{R}$ satisfying the following product rule analogous to (3.4):

$$v[fg]_p = f(p)v[g]_p + g(p)v[f]_p.$$

It is common to define the tangent space to M at p as the vector space $\mathcal{D}_p M$ of derivations of $C_p^\infty(M)$. Thanks to Proposition 3.8, it is a simple matter to prove that $\mathcal{D}_p M$ is naturally isomorphic to the tangent space as we have defined it (see Problem 3-7).

The germ definition has a number of advantages. One of the most significant is that it makes the local nature of the tangent space clearer, without requiring the use of bump functions. Because there do not exist analytic bump functions, the germ definition of tangent vectors is the only one available on real-analytic or complex-analytic manifolds. The chief disadvantage of the germ approach is simply that it adds an additional level of complication to an already highly abstract definition.

Tangent Vectors as Equivalence Classes of Curves

Another common approach to tangent vectors is to define an intrinsic equivalence relation on the set of smooth curves with the same starting point, which captures the idea of "having the same velocity," and to define a tangent vector as an equivalence class of curves. Here we describe one such equivalence relation.

Suppose p is a point of M. We wish to define an equivalence relation on the set of all smooth curves of the form $\gamma\colon J \to M$, where J is an interval containing 0 and $\gamma(0) = p$. Given two such curves $\gamma_1\colon J_1 \to M$ and $\gamma_2\colon J_2 \to M$, let us say that $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Let $\mathcal{V}_p M$ denote the set of equivalence classes. The tangent space to M at p is often defined to be the set $\mathcal{V}_p M$.

Using this definition, it is very easy to define the differential of a smooth map $F\colon M \to N$ as the map that sends $[\gamma] \in \mathcal{V}_p M$ to $[F \circ \gamma] \in \mathcal{V}_{F(p)} N$. Velocity vectors of smooth curves are almost as easy to define. Suppose $\gamma\colon J \to M$ is any smooth curve. If $0 \in J$, then the velocity of γ at 0 is just the equivalence class of γ in $\mathcal{V}_{\gamma(0)} M$. The velocity at any other point $t_0 \in J$ can be defined as the equivalence class in $\mathcal{V}_{\gamma(t_0)} M$ of the curve γ_{t_0} defined by $\gamma_{t_0}(t) = \gamma(t_0 + t)$.

Problem 3-8 shows that there is a natural one-to-one correspondence between $\mathcal{V}_p M$ and $T_p M$. This definition has the advantage of being geometrically more intuitive, but it has the serious drawback that the existence of a vector space structure on $\mathcal{V}_p M$ is not at all obvious.

Tangent Vectors as Equivalence Classes of n-Tuples

Yet another approach to defining the tangent space is based on the transformation rule (3.12) for the components of tangent vectors in coordinates. One defines a tangent vector at a point $p \in M$ to be a rule that assigns an ordered n-tuple $\left(v^1, \dots, v^n\right) \in \mathbb{R}^n$ to each smooth coordinate chart containing p, with the property that the n-tuples assigned to overlapping charts transform according to (3.12). (This is, in fact, the oldest definition of all, and many physicists are still apt to think of tangent vectors this way.)

In this approach, the velocity of a curve is defined by the usual Euclidean formula in coordinates, and the differential of $F\colon M \to N$ is defined as the linear map determined by the Jacobian matrix of F in coordinates. One then has to show, by means of tedious computations involving the chain rule, that these operations are well defined, independently of the choices of coordinates.

It is a matter of individual taste which of the various characterizations of T_pM one chooses to take as the definition. The definition we have chosen, however abstract it may seem at first, has several advantages: it is relatively concrete (tangent vectors are actual derivations of $C^\infty(M)$, with no equivalence classes involved); it makes the vector space structure on T_pM obvious; and it leads to straightforward coordinate-independent definitions of differentials, velocities, and many of the other geometric objects we will be studying.

Categories and Functors

Another useful perspective on tangent spaces and differentials is provided by the theory of categories. In this section we summarize the basic definitions of category theory. We do not do much with the theory in this book, but we mention it because it provides a convenient and powerful language for talking about many of the mathematical structures we will meet.

A ***category*** C consists of the following things:

- a class $\mathrm{Ob}(\mathsf{C})$, whose elements are called ***objects of*** C,
- a class $\mathrm{Hom}(\mathsf{C})$, whose elements are called ***morphisms of*** C,
- for each morphism $f \in \mathrm{Hom}(\mathsf{C})$, two objects $X, Y \in \mathrm{Ob}(\mathsf{C})$ called the ***source*** and ***target of*** f, respectively,
- for each triple $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$, a mapping called ***composition***:

$$\mathrm{Hom}_{\mathsf{C}}(X,Y) \times \mathrm{Hom}_{\mathsf{C}}(Y,Z) \to \mathrm{Hom}_{\mathsf{C}}(X,Z),$$

  written $(f,g) \mapsto g \circ f$, where $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ denotes the class of all morphisms with source X and target Y.

The morphisms are required to satisfy the following axioms:

(i) ASSOCIATIVITY: $(f \circ g) \circ h = f \circ (g \circ h)$.
(ii) EXISTENCE OF IDENTITIES: For each object $X \in \mathrm{Ob}(\mathsf{C})$, there exists an ***identity morphism*** $\mathrm{Id}_X \in \mathrm{Hom}_{\mathsf{C}}(X,X)$, such that $\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X$ for all $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$.

A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$ is called an ***isomorphism in*** C if there exists a morphism $g \in \mathrm{Hom}_{\mathsf{C}}(Y,X)$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

Example 3.26 (Categories). In most of the categories that one meets "in nature," the objects are sets with some extra structure, the morphisms are maps that preserve that structure, and the composition laws and identity morphisms are the obvious ones. Some of the categories of this type that appear in this book (implicitly or explicitly) are listed below. In each case, we describe the category by giving its objects and its morphisms.

- Set: sets and maps
- Top: topological spaces and continuous maps
- Man: topological manifolds and continuous maps

- $\mathsf{Man_b}$: topological manifolds with boundary and continuous maps
- Diff: smooth manifolds and smooth maps
- $\mathsf{Diff_b}$: smooth manifolds with boundary and smooth maps
- $\mathsf{Vec}_{\mathbb{R}}$: real vector spaces and real-linear maps
- $\mathsf{Vec}_{\mathbb{C}}$: complex vector spaces and complex-linear maps
- Grp: groups and group homomorphisms
- Ab: abelian groups and group homomorphisms
- Rng: rings and ring homomorphisms
- CRng: commutative rings and ring homomorphisms

There are also important categories whose objects are sets with distinguished base points, in addition to (possibly) other structures. A ***pointed set*** is an ordered pair (X, p), where X is a set and p is an element of X. Other pointed objects such as ***pointed topological spaces*** or ***pointed smooth manifolds*** are defined similarly. If (X, p) and (X', p') are pointed sets (or topological spaces, etc.), a map $F\colon X \to X'$ is said to be a ***pointed map*** if $F(p) = p'$; in this case, we write $F\colon (X, p) \to (X', p')$. Here are some important examples of categories of pointed objects.

- Set_*: pointed sets and pointed maps
- Top_*: pointed topological spaces and pointed continuous maps
- Man_*: pointed topological manifolds and pointed continuous maps
- Diff_*: pointed smooth manifolds and pointed smooth maps //

We use the word *class* instead of *set* for the collections of objects and morphisms in a category because in some categories they are "too large" to be considered sets. For example, in the category Set, $\mathrm{Ob}(\mathsf{Set})$ is the class of all sets; any attempt to treat it as a set in its own right leads to the well-known Russell paradox of set theory. (See [LeeTM, Appendix A] or almost any book on set theory for more.) Even though the classes of objects and morphisms might not constitute sets, we still use notations such as $X \in \mathrm{Ob}(\mathsf{C})$ and $f \in \mathrm{Hom}(\mathsf{C})$ to indicate that X is an object and f is a morphism in C. A category in which both $\mathrm{Ob}(\mathsf{C})$ and $\mathrm{Hom}(\mathsf{C})$ are sets is called a ***small category***, and one in which each class of morphisms $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is a set is called ***locally small***. All the categories listed above are locally small but not small.

If C and D are categories, a ***covariant functor from C to D*** is a rule $\mathcal{F}$ that assigns to each object $X \in \mathrm{Ob}(\mathsf{C})$ an object $\mathcal{F}(X) \in \mathrm{Ob}(\mathsf{D})$, and to each morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$, so that identities and composition are preserved:

$$\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}; \qquad \mathcal{F}(g \circ h) = \mathcal{F}(g) \circ \mathcal{F}(h).$$

We also need to consider functors that reverse morphisms: a ***contravariant functor from C to D*** is a rule $\mathcal{F}$ that assigns to each object $X \in \mathrm{Ob}(\mathsf{C})$ an object $\mathcal{F}(X) \in \mathrm{Ob}(\mathsf{D})$, and to each morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(Y), \mathcal{F}(X))$, such that

$$\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}; \qquad \mathcal{F}(g \circ h) = \mathcal{F}(h) \circ \mathcal{F}(g).$$

► **Exercise 3.27.** Show that any (covariant or contravariant) functor from C to D takes isomorphisms in C to isomorphisms in D.

One trivial example of a covariant functor is the ***identity functor*** from any category to itself: it takes each object and each morphism to itself. Another example is the ***forgetful functor***: if C is a category whose objects are sets with some additional structure and whose morphisms are maps preserving that structure (as are all the categories listed in the first part of Example 3.26 except Set itself), the forgetful functor $\mathcal{F}: \mathsf{C} \to \mathsf{Set}$ assigns to each object its underlying set, and to each morphism the same map thought of as a map between sets.

More interesting functors arise when we associate "invariants" to classes of mathematical objects. For example, the fundamental group is a covariant functor from Top_* to Grp. The results of Problem 2-10 show that there is a contravariant functor from Diff to $\mathsf{Vec}_{\mathbb{R}}$ defined by assigning to each smooth manifold M the vector space $C^\infty(M)$, and to each smooth map $F: M \to N$ the linear map $F^*: C^\infty(N) \to C^\infty(M)$ defined by $F^*(f) = f \circ F$.

The discussion in this chapter has given us some other important examples of functors. First, the ***tangent space functor*** is a covariant functor from the category Diff_* of pointed smooth manifolds to the category $\mathsf{Vec}_{\mathbb{R}}$ of real vector spaces. To each pointed smooth manifold (M, p) it assigns the tangent space T_pM, and to each pointed smooth map $F: (M, p) \to (N, F(p))$ it assigns the differential dF_p. The fact that this is a functor is the content of parts (b) and (c) of Proposition 3.6.

Similarly, we can think of the assignments $M \mapsto TM$ and $F \mapsto dF$ (sending each smooth manifold to its tangent bundle and each smooth map to its global differential) as a covariant functor from Diff to itself, called the ***tangent functor***.

Problems

3-1. Suppose M and N are smooth manifolds with or without boundary, and $F: M \to N$ is a smooth map. Show that $dF_p: T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

3-2. Prove Proposition 3.14 (the tangent space to a product manifold).

3-3. Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

3-4. Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

3-5. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F: \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\left(\mathbb{S}^1\right) = K$, but there is no *diffeomorphism* with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $dF(\gamma'(t))$ on the coordinate functions x and y.] (*Used on p. 123.*)

3-6. Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = \left(z^1, z^2\right) \in \mathbb{S}^3$, define a curve $\gamma_z: \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = \left(e^{it}z^1, e^{it}z^2\right)$. Show that γ_z is a smooth curve whose velocity is never zero.

3-7. Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_p M$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi\colon \mathcal{D}_p M \to T_p M$ by $(\Phi v) f = v([f]_p)$. Show that Φ is an isomorphism. (*Used on p. 71.*)

3-8. Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi\colon \mathcal{V}_p M \to T_p M$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. (*Used on p. 72.*)

Chapter 4
Submersions, Immersions, and Embeddings

Because the differential of a smooth map is supposed to represent the "best linear approximation" to the map near a given point, we can learn a great deal about a map by studying linear-algebraic properties of its differential. The most essential property of the differential—in fact, just about the only property that can be defined independently of choices of bases—is its *rank* (the dimension of its image).

In this chapter we undertake a detailed study of the ways in which geometric properties of smooth maps can be detected from their differentials. The maps for which differentials give good local models turn out to be the ones whose differentials have constant rank. Three categories of such maps play special roles: *smooth submersions* (whose differentials are surjective everywhere), *smooth immersions* (whose differentials are injective everywhere), and *smooth embeddings* (injective smooth immersions that are also homeomorphisms onto their images). Smooth immersions and embeddings, as we will see in the next chapter, are essential ingredients in the theory of submanifolds, while smooth submersions play a role in smooth manifold theory closely analogous to the role played by quotient maps in topology.

The engine that powers this discussion is the *rank theorem*, a corollary of the inverse function theorem. In the first section of the chapter, we prove the rank theorem and some of its important consequences. Then we delve more deeply into smooth embeddings and smooth submersions, and apply the theory to a particularly useful class of smooth submersions, the *smooth covering maps*.

Maps of Constant Rank

The key linear-algebraic property of a linear map is its rank. In fact, as Theorem B.20 shows, the rank is the *only* property that distinguishes different linear maps if we are free to choose bases independently for the domain and codomain.

Suppose M and N are smooth manifolds with or without boundary. Given a smooth map $F: M \to N$ and a point $p \in M$, we define the ***rank of F at p*** to be the rank of the linear map $dF_p: T_pM \to T_{F(p)}N$; it is the rank of the Jacobian matrix

J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics 218,
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of F in any smooth chart, or the dimension of $\operatorname{Im} dF_p \subseteq T_{F(p)}N$. If F has the same rank r at every point, we say that it has ***constant rank***, and write $\operatorname{rank} F = r$.

Because the rank of a linear map is never higher than the dimension of either its domain or its codomain (Exercise B.22), the rank of F at each point is bounded above by the minimum of $\{\dim M, \dim N\}$. If the rank of dF_p is equal to this upper bound, we say that ***F has full rank at p***, and if F has full rank everywhere, we say ***F has full rank***.

The most important constant-rank maps are those of full rank. A smooth map $F: M \to N$ is called a ***smooth submersion*** if its differential is surjective at each point (or equivalently, if $\operatorname{rank} F = \dim N$). It is called a ***smooth immersion*** if its differential is injective at each point (equivalently, $\operatorname{rank} F = \dim M$).

Proposition 4.1. *Suppose $F: M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.*

Proof. If we choose any smooth coordinates for M near p and for N near $F(p)$, either hypothesis means that Jacobian matrix of F in coordinates has full rank at p. Example 1.28 shows that the set of $m \times n$ matrices of full rank is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$ (where $m = \dim M$ and $n = \dim N$), so by continuity, the Jacobian of F has full rank in some neighborhood of p. □

As we will see in this chapter, smooth submersions and immersions behave locally like surjective and injective linear maps, respectively. (There are also analogous notions of *topological submersions* and *topological immersions*, which apply to maps that are merely continuous. We do not have any need to use these, but for the sake of completeness, we describe them later in the chapter.)

Example 4.2 (Submersions and Immersions).

(a) Suppose $M_1, \dots, M_k$ are smooth manifolds. Then each of the projection maps $\pi_i: M_1 \times \dots \times M_k \to M_i$ is a smooth submersion. In particular, the projection $\pi: \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a smooth submersion.

(b) If $\gamma: J \to M$ is a smooth curve in a smooth manifold M with or without boundary, then γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

(c) If M is a smooth manifold and its tangent bundle TM is given the smooth manifold structure described in Proposition 3.18, the projection $\pi: TM \to M$ is a smooth submersion. To verify this, just note that with respect to any smooth local coordinates $\left(x^i\right)$ on an open subset $U \subseteq M$ and the corresponding natural coordinates $\left(x^i, v^i\right)$ on $\pi^{-1}(U) \subseteq TM$ (see Proposition 3.18), the coordinate representation of π is $\widehat{\pi}(x, v) = x$.

(d) The smooth map $X: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(u, v) = \big((2 + \cos 2\pi u)\cos 2\pi v, (2 + \cos 2\pi u)\sin 2\pi v, \sin 2\pi u\big)$$

is a smooth immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is the doughnut-shaped surface obtained by revolving the circle $(y-2)^2 + z^2 = 1$ in the (y, z)-plane about the z-axis (Fig. 4.1). //

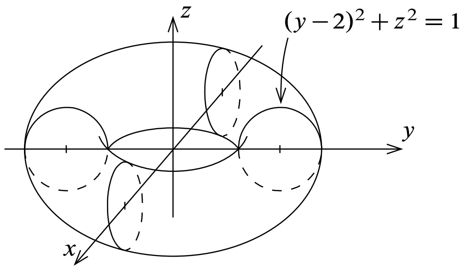

Fig. 4.1 A torus of revolution in $\mathbb{R}^3$

► **Exercise 4.3.** Verify the claims made in the preceding example.

► **Exercise 4.4.** Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.

Local Diffeomorphisms

If M and N are smooth manifolds with or without boundary, a map $F\colon M \to N$ is called a ***local diffeomorphism*** if every point $p \in M$ has a neighborhood U such that $F(U)$ is open in N and $F|_U\colon U \to F(U)$ is a diffeomorphism. The next theorem is the key to the most important properties of local diffeomorphisms.

Theorem 4.5 (Inverse Function Theorem for Manifolds). *Suppose M and N are smooth manifolds, and $F\colon M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism.*

Proof. The fact that dF_p is bijective implies that M and N have the same dimension, say n. Choose smooth charts (U,φ) centered at p and (V,ψ) centered at $F(p)$, with $F(U) \subseteq V$. Then $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is a smooth map from the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ into $\widehat{V} = \psi(V) \subseteq \mathbb{R}^n$, with $\widehat{F}(p) = 0$. Because φ and ψ are diffeomorphisms, the differential $d\widehat{F}_0 = d\psi_{F(p)} \circ dF_p \circ d\left(\varphi^{-1}\right)_0$ is nonsingular. The ordinary inverse function theorem (Theorem C.34) shows that there are connected open subsets $\widehat{U}_0 \subseteq \widehat{U}$ and $\widehat{V}_0 \subseteq \widehat{V}$ containing 0 such that $\widehat{F}$ restricts to a diffeomorphism from $\widehat{U}_0$ to $\widehat{V}_0$. Then $U_0 = \varphi^{-1}\big(\widehat{U}_0\big)$ and $V_0 = \psi^{-1}\big(\widehat{V}_0\big)$ are connected neighborhoods of p and $F(p)$, respectively, and it follows by composition that $F|_{U_0}$ is a diffeomorphism from U_0 to V_0. □

It is important to notice that we have stated Theorem 4.5 only for manifolds without boundary. In fact, it can fail for a map whose domain has nonempty boundary (see Problem 4-1). However, when the *codomain* has nonempty boundary, there is something useful that can be said: provided the map takes its values in the interior of the codomain, the same conclusion holds because the interior is a smooth manifold without boundary. Problem 4-2 shows that this is always the case at points where the differential is invertible.

Proposition 4.6 (Elementary Properties of Local Diffeomorphisms).

(a) *Every composition of local diffeomorphisms is a local diffeomorphism.*
(b) *Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.*
(c) *Every local diffeomorphism is a local homeomorphism and an open map.*
(d) *The restriction of a local diffeomorphism to an open submanifold with or without boundary is a local diffeomorphism.*
(e) *Every diffeomorphism is a local diffeomorphism.*
(f) *Every bijective local diffeomorphism is a diffeomorphism.*
(g) *A map between smooth manifolds with or without boundary is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.*

► **Exercise 4.7.** Prove the preceding proposition.

Proposition 4.8. *Suppose M and N are smooth manifolds* (*without boundary*), *and $F: M \to N$ is a map.*

(a) *F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.*
(b) *If $\dim M = \dim N$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.*

Proof. Suppose first that F is a local diffeomorphism. Given $p \in M$, there is a neighborhood U of p such that F is a diffeomorphism from U to $F(U)$. It then follows from Proposition 3.6(d) that $dF_p: T_pM \to T_{F(p)}N$ is an isomorphism. Thus $\operatorname{rank} F = \dim M = \dim N$, so F is both a smooth immersion and a smooth submersion. Conversely, if F is both a smooth immersion and a smooth submersion, then dF_p is an isomorphism at each $p \in M$, and the inverse function theorem for manifolds (Theorem 4.5) shows that p has a neighborhood on which F restricts to a diffeomorphism onto its image. This proves (a).

To prove (b), note that if M and N have the same dimension, then either injectivity or surjectivity of dF_p implies bijectivity, so F is a smooth submersion if and only if it is a smooth immersion, and thus (b) follows from (a). □

► **Exercise 4.9.** Show that the conclusions of Proposition 4.8 still hold if N is allowed to be a smooth manifold with boundary, but not if M is. (See Problems 4-1 and 4-2.)

► **Exercise 4.10.** Suppose M, N, P are smooth manifolds with or without boundary, and $F: M \to N$ is a local diffeomorphism. Prove the following:

(a) If $G: P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.
(b) If in addition F is surjective and $G: N \to P$ is any map, then G is smooth if and only if $G \circ F$ is smooth.

Example 4.11 (Local Diffeomorphisms). The map $\varepsilon\colon \mathbb{R} \to \mathbb{S}^1$ defined in Example 2.13(b) is a local diffeomorphism because in a neighborhood of each point it has a coordinate representation of the form $t \mapsto 2\pi t + c$, which is a local diffeomorphism. Similarly, the map $\varepsilon^n\colon \mathbb{R}^n \to \mathbb{T}^n$ defined in Example 2.13(c) is a local diffeomorphism because it is a product of local diffeomorphisms. //

At the end of this chapter, we will explore an important special class of local diffeomorphisms, the *smooth covering maps*.

The Rank Theorem

The most important fact about constant-rank maps is the following consequence of the inverse function theorem, which says that a constant-rank smooth map can be placed locally into a particularly simple canonical form by a change of coordinates. It is a nonlinear version of the canonical form theorem for linear maps given in Theorem B.20.

Theorem 4.12 (Rank Theorem). *Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F\colon M \to N$ is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts (U,φ) for M centered at p and (V,ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form*

$$\widehat{F}\left(x^1,\dots,x^r,x^{r+1},\dots,x^m\right) = \left(x^1,\dots,x^r,0,\dots,0\right). \tag{4.1}$$

In particular, if F is a smooth submersion, this becomes

$$\widehat{F}\left(x^1,\dots,x^n,x^{n+1},\dots,x^m\right) = \left(x^1,\dots,x^n\right), \tag{4.2}$$

and if F is a smooth immersion, it is

$$\widehat{F}\left(x^1,\dots,x^m\right) = \left(x^1,\dots,x^m,0,\dots,0\right). \tag{4.3}$$

Proof. Because the theorem is local, after choosing smooth coordinates we can replace M and N by open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. The fact that $DF(p)$ has rank r implies that its matrix has some $r \times r$ submatrix with nonzero determinant. By reordering the coordinates, we may assume that it is the upper left submatrix, $\left(\partial F^i/\partial x^j\right)$ for $i,j = 1,\dots,r$. Let us relabel the standard coordinates as $(x,y) = \left(x^1,\dots,x^r,y^1,\dots,y^{m-r}\right)$ in $\mathbb{R}^m$ and $(v,w) = \left(v^1,\dots,v^r,w^1,\dots,w^{n-r}\right)$ in $\mathbb{R}^n$. By initial translations of the coordinates, we may assume without loss of generality that $p = (0,0)$ and $F(p) = (0,0)$. If we write $F(x,y) = \big(Q(x,y),R(x,y)\big)$ for some smooth maps $Q\colon U \to \mathbb{R}^r$ and $R\colon U \to \mathbb{R}^{n-r}$, then our hypothesis is that $\left(\partial Q^i/\partial x^j\right)$ is nonsingular at $(0,0)$.

Define $\varphi\colon U \to \mathbb{R}^m$ by $\varphi(x,y) = \big(Q(x,y),y\big)$. Its total derivative at $(0,0)$ is

$$D\varphi(0,0) = \begin{pmatrix} \dfrac{\partial Q^i}{\partial x^j}(0,0) & \dfrac{\partial Q^i}{\partial y^j}(0,0) \\ 0 & \delta^i_j \end{pmatrix},$$

where we have used the following standard notation: for positive integers i and j, the symbol δ^i_j, called the ***Kronecker delta***, is defined by

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.4}$$

The matrix $D\varphi(0,0)$ is nonsingular by virtue of the hypothesis. Therefore, by the inverse function theorem, there are connected neighborhoods U_0 of $(0,0)$ and $\widetilde{U}_0$ of $\varphi(0,0) = (0,0)$ such that $\varphi\colon U_0 \to \widetilde{U}_0$ is a diffeomorphism. By shrinking U_0 and $\widetilde{U}_0$ if necessary, we may assume that $\widetilde{U}_0$ is an open cube. Writing the inverse map as $\varphi^{-1}(x,y) = \big(A(x,y), B(x,y)\big)$ for some smooth functions $A\colon \widetilde{U}_0 \to \mathbb{R}^r$ and $B\colon \widetilde{U}_0 \to \mathbb{R}^{m-r}$, we compute

$$(x,y) = \varphi\big(A(x,y), B(x,y)\big) = \Big(Q\big(A(x,y), B(x,y)\big), B(x,y)\Big). \tag{4.5}$$

Comparing y components shows that $B(x,y) = y$, and therefore φ^{-1} has the form

$$\varphi^{-1}(x,y) = \big(A(x,y), y\big).$$

On the other hand, $\varphi \circ \varphi^{-1} = \mathrm{Id}$ implies $Q\big(A(x,y), y\big) = x$, and therefore $F \circ \varphi^{-1}$ has the form

$$F \circ \varphi^{-1}(x,y) = \big(x, \widetilde{R}(x,y)\big),$$

where $\widetilde{R}\colon \widetilde{U}_0 \to \mathbb{R}^{n-r}$ is defined by $\widetilde{R}(x,y) = R\big(A(x,y), y\big)$. The Jacobian matrix of this composite map at an arbitrary point $(x,y) \in \widetilde{U}_0$ is

$$D\big(F \circ \varphi^{-1}\big)(x,y) = \begin{pmatrix} \delta^i_j & 0 \\ \dfrac{\partial \widetilde{R}^i}{\partial x^j}(x,y) & \dfrac{\partial \widetilde{R}^i}{\partial y^j}(x,y) \end{pmatrix}.$$

Since composing with a diffeomorphism does not change the rank of a map, this matrix has rank r everywhere in $\widetilde{U}_0$. The first r columns are obviously linearly independent, so the rank can be r only if the derivatives $\partial \widetilde{R}^i / \partial y^j$ vanish identically on $\widetilde{U}_0$, which implies that $\widetilde{R}$ is actually independent of $\big(y^1, \dots, y^{m-r}\big)$. (This is one reason we arranged for $\widetilde{U}_0$ to be a cube.) Thus, if we let $S(x) = \widetilde{R}(x,0)$, then we have

$$F \circ \varphi^{-1}(x,y) = \big(x, S(x)\big). \tag{4.6}$$

To complete the proof, we need to define an appropriate smooth chart in some neighborhood of $(0,0) \in V$. Let $V_0 \subseteq V$ be the open subset defined by $V_0 = \big\{(v,w) \in V : (v,0) \in \widetilde{U}_0\big\}$. Then V_0 is a neighborhood of $(0,0)$. Because $\widetilde{U}_0$ is a cube and $F \circ \varphi^{-1}$ has the form (4.6), it follows that $F \circ \varphi^{-1}\big(\widetilde{U}_0\big) \subseteq V_0$, and therefore $F(U_0) \subseteq V_0$. Define $\psi\colon V_0 \to \mathbb{R}^n$ by $\psi(v,w) = \big(v, w - S(v)\big)$. This is a diffeomorphism onto its image, because its inverse is given explicitly by $\psi^{-1}(s,t) = \big(s, t + S(s)\big)$; thus (V_0, ψ) is a smooth chart. It follows from (4.6) that

$$\psi \circ F \circ \varphi^{-1}(x,y) = \psi\big(x, S(x)\big) = \big(x, S(x) - S(x)\big) = (x,0),$$

which was to be proved. □

The next corollary can be viewed as a more invariant statement of the rank theorem. It says that constant-rank maps are precisely the ones whose local behavior is the same as that of their differentials.

Corollary 4.13. *Let M and N be smooth manifolds, let $F\colon M \to N$ be a smooth map, and suppose M is connected. Then the following are equivalent:*

(a) *For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.*
(b) *F has constant rank.*

Proof. First suppose F has a linear coordinate representation in a neighborhood of each point. Since every linear map has constant rank, it follows that the rank of F is constant in a neighborhood of each point, and thus by connectedness it is constant on all of M. Conversely, if F has constant rank, the rank theorem shows that it has the linear coordinate representation (4.1) in a neighborhood of each point. □

The rank theorem is a purely local statement. However, it has the following powerful global consequence.

Theorem 4.14 (Global Rank Theorem). *Let M and N be smooth manifolds, and suppose $F\colon M \to N$ is a smooth map of constant rank.*

(a) *If F is surjective, then it is a smooth submersion.*
(b) *If F is injective, then it is a smooth immersion.*
(c) *If F is bijective, then it is a diffeomorphism.*

Proof. Let $m = \dim M$, $n = \dim N$, and suppose F has constant rank r. To prove (a), assume that F is not a smooth submersion, which means that $r < n$. By the rank theorem, for each $p \in M$ there are smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$ and the coordinate representation of F is given by (4.1). (See Fig. 4.2.) Shrinking U if necessary, we may assume that it is a regular coordinate ball and $F\big(\bar{U}\big) \subseteq V$. This implies that $F\big(\bar{U}\big)$ is a compact subset of the set $\big\{y \in V : y^{r+1} = \cdots = y^n = 0\big\}$, so it is closed in N and contains no open subset of N; hence it is nowhere dense in N. Since every open cover of a manifold has a countable subcover, we can choose countably many such charts $\{(U_i, \varphi_i)\}$ covering M, with corresponding charts $\{(V_i, \psi_i)\}$ covering $F(M)$. Because $F(M)$ is equal to the countable union of the nowhere dense sets $F\big(\bar{U}_i\big)$, it follows from the Baire category theorem (Theorem A.58) that $F(M)$ has empty interior in N, which means F cannot be surjective.

To prove (b), assume that F is not a smooth immersion, so that $r < m$. By the rank theorem, for each $p \in M$ we can choose charts on neighborhoods of p and $F(p)$ in which F has the coordinate representation (4.1). It follows that $F(0, \dots, 0, \varepsilon) = F(0, \dots, 0, 0)$ for any sufficiently small ε, so F is not injective.

Finally, (c) follows from (a) and (b), because a bijective smooth map of constant rank is a smooth submersion by part (a) and a smooth immersion by part (b); so Proposition 4.8 implies that F is a local diffeomorphism, and because it is bijective, it is a diffeomorphism. □

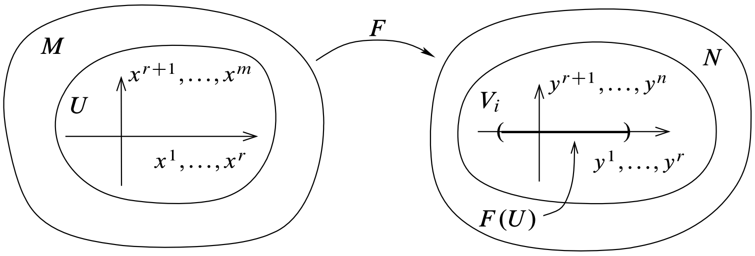

Fig. 4.2 Proof of Theorem 4.14(a)

The Rank Theorem for Manifolds with Boundary

In the context of manifolds with boundary, we need the rank theorem only in one special case: that of a smooth immersion whose domain is a smooth manifold with boundary. Of course, since the interior of a smooth manifold with boundary is a smooth manifold, near any interior point of the domain the ordinary rank theorem applies. For boundary points, we have the following substitute for the rank theorem.

Theorem 4.15 (Local Immersion Theorem for Manifolds with Boundary). *Suppose M is a smooth m-manifold with boundary, N is a smooth n-manifold, and $F: M \to N$ is a smooth immersion. For any $p \in \partial M$, there exist a smooth boundary chart (U, φ) for M centered at p and a smooth coordinate chart (V, ψ) for N centered at $F(p)$ with $F(U) \subseteq V$, in which F has the coordinate representation*

$$\widehat{F}\left(x^1, \dots, x^m\right) = \left(x^1, \dots, x^m, 0, \dots, 0\right). \tag{4.7}$$

Proof. By choosing initial smooth charts for M and N, we may assume that M and N are open subsets of $\mathbb{H}^m$ and $\mathbb{R}^n$, respectively, and also that $p = 0 \in \mathbb{H}^m$, and $F(p) = 0 \in \mathbb{R}^n$. By definition of smoothness for functions on $\mathbb{H}^m$, F extends to a smooth map $\widetilde{F}: W \to \mathbb{R}^n$, where W is some open subset of $\mathbb{R}^m$ containing 0. Because $d\widetilde{F}_0 = dF_0$ is injective, by shrinking W if necessary, we may assume that $\widetilde{F}$ is a smooth immersion. Let us write the coordinates on $\mathbb{R}^m$ as $x = \left(x^1, \dots, x^m\right)$, and those on $\mathbb{R}^n$ as $(v, w) = \left(v^1, \dots, v^m, w^1, \dots, w^{n-m}\right)$.

By the rank theorem, there exist smooth charts (U_0, φ_0) for $\mathbb{R}^m$ centered at 0 and (V_0, ψ_0) for $\mathbb{R}^n$ centered at 0 such that $\widehat{F} = \psi_0 \circ \widetilde{F} \circ \varphi_0^{-1}$ is given by (4.7). The only problem with these coordinates is that φ_0 might not restrict to a boundary chart for M. But we can correct this easily as follows. Because φ_0 is a diffeomorphism from U_0 to an open subset $\widehat{U}_0 = \varphi_0(U_0) \subseteq \mathbb{R}^m$, the map $\varphi_0^{-1} \times \mathrm{Id}_{\mathbb{R}^{n-m}}$ is a diffeomorphism from $\widehat{U}_0 \times \mathbb{R}^{n-m}$ to $U_0 \times \mathbb{R}^{n-m}$. Let $\psi = \left(\varphi_0^{-1} \times \mathrm{Id}_{\mathbb{R}^{n-m}}\right) \circ \psi_0$, which is a diffeomorphism from some open subset $V \subseteq V_0$ containing 0 to a neighborhood of 0 in $\mathbb{R}^n$. Using (4.7), we compute

$$\begin{aligned}
\psi \circ F(x) &= \left(\varphi_0^{-1} \times \mathrm{Id}_{\mathbb{R}^{n-m}}\right) \circ \psi_0 \circ F \circ \varphi_0^{-1} \circ \varphi_0(x) \\
&= \left(\varphi_0^{-1} \times \mathrm{Id}_{\mathbb{R}^{n-m}}\right) \circ \widehat{F}\left(\varphi_0(x)\right)
\end{aligned}$$

$$= \left(\varphi_0^{-1} \times \mathrm{Id}_{\mathbb{R}^{n-m}}\right)\left(\varphi_0(x), 0\right) = (x, 0).$$

Thus, the original coordinates for M (restricted to a sufficiently small neighborhood of 0) and the chart (V, ψ) for N satisfy the desired conditions. □

It is possible to prove a similar theorem for more general maps with constant rank out of manifolds with boundary, but the proof is more elaborate because an extension of F to an open subset does not automatically have constant rank. Since we have no need for this more general result, we leave it to the interested reader to pursue (Problem 4-3). On the other hand, the situation is considerably more complicated for a map whose *codomain* is a manifold with boundary: since the image of the map could intersect the boundary in unpredictable ways, there is no way to put such a map into a simple canonical form without strong restrictions on the map.

Embeddings

One special kind of immersion is particularly important. If M and N are smooth manifolds with or without boundary, a ***smooth embedding of M into N*** is a smooth immersion $F: M \to N$ that is also a topological embedding, i.e., a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology. A smooth embedding is a map that is both a topological embedding and a smooth immersion, not just a topological embedding that happens to be smooth.

► **Exercise 4.16.** Show that every composition of smooth embeddings is a smooth embedding.

Example 4.17 (Smooth Embeddings).

(a) If M is a smooth manifold with or without boundary and $U \subseteq M$ is an open submanifold, the inclusion map $U \hookrightarrow M$ is a smooth embedding.

(b) If $M_1, \dots, M_k$ are smooth manifolds and $p_i \in M_i$ are arbitrarily chosen points, each of the maps $\iota_j: M_j \to M_1 \times \cdots \times M_k$ given by

$$\iota_j(q) = (p_1, \dots, p_{j-1}, q, p_{j+1}, \dots, p_k)$$

is a smooth embedding. In particular, the inclusion map $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ given by sending $\left(x^1, \dots, x^n\right)$ to $\left(x^1, \dots, x^n, 0, \dots, 0\right)$ is a smooth embedding.

(c) Problem 4-12 shows that the map $X: \mathbb{R}^2 \to \mathbb{R}^3$ of Example 4.2(d) descends to a smooth embedding of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ into $\mathbb{R}^3$. //

To understand more fully what it means for a map to be a smooth embedding, it is useful to bear in mind some examples of injective smooth maps that are *not* smooth embeddings. The next three examples illustrate three rather different ways in which this can happen.

Example 4.18 (A Smooth Topological Embedding). The map $\gamma: \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = \left(t^3, 0\right)$ is a smooth map and a topological embedding, but it is not a smooth embedding because $\gamma'(0) = 0$. //

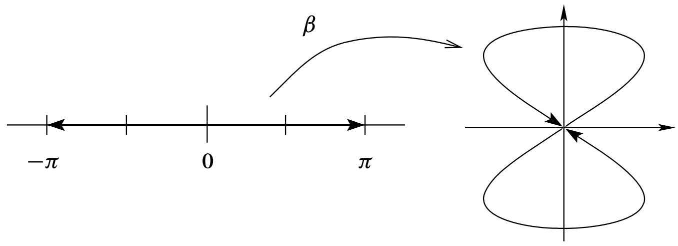

Fig. 4.3 The figure-eight curve of Example 4.19

Example 4.19 (The Figure-Eight Curve). Consider the curve $\beta\colon (-\pi,\pi) \to \mathbb{R}^2$ defined by

$$\beta(t) = (\sin 2t, \sin t).$$

Its image is a set that looks like a figure-eight in the plane (Fig. 4.3),sometimes called a ***lemniscate***. (It is the locus of points (x,y) where $x^2 = 4y^2\left(1-y^2\right)$, as you can check.) It is easy to see that β is an injective smooth immersion because $\beta'(t)$ never vanishes; but it is not a topological embedding, because its image is compact in the subspace topology, while its domain is not. //

Example 4.20 (A Dense Curve on the Torus). Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus, and let α be any irrational number. The map $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ given by

$$\gamma(t) = \left(e^{2\pi i t}, e^{2\pi i \alpha t}\right)$$

is a smooth immersion because $\gamma'(t)$ never vanishes. It is also injective, because $\gamma(t_1) = \gamma(t_2)$ implies that both $t_1 - t_2$ and $\alpha t_1 - \alpha t_2$ are integers, which is impossible unless $t_1 = t_2$.

Consider the set $\gamma(\mathbb{Z}) = \{\gamma(n) : n \in \mathbb{Z}\}$. It follows from Dirichlet's approximation theorem (see below) that for every $\varepsilon > 0$, there are integers n, m such that $|\alpha n - m| < \varepsilon$. Using the fact that $\left|e^{it_1} - e^{it_2}\right| \le |t_1 - t_2|$ for $t_1, t_2 \in \mathbb{R}$ (because the line segment from e^{it_1} to e^{it_2} is shorter than the circular arc of length $|t_1 - t_2|$), we have $\left|e^{2\pi i \alpha n} - 1\right| = \left|e^{2\pi i \alpha n} - e^{2\pi i m}\right| \le \left|2\pi(\alpha n - m)\right| < 2\pi\varepsilon$. Therefore,

$$\left|\gamma(n) - \gamma(0)\right| = \left|\left(e^{2\pi i n},\ e^{2\pi i \alpha n}\right) - (1,1)\right| = \left|\left(1,\ e^{2\pi i \alpha n}\right) - (1,1)\right| < 2\pi\varepsilon.$$

Thus, $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z})$. But this means that γ is not a homeomorphism onto its image, because $\mathbb{Z}$ has no limit point in $\mathbb{R}$. In fact, it is not hard to show that the image set $\gamma(\mathbb{R})$ is actually dense in $\mathbb{T}^2$ (see Problem 4-4). //

The preceding example and Problem 4-4 depend on the following elementary result from number theory.

Lemma 4.21 (Dirichlet's Approximation Theorem). *Given $\alpha \in \mathbb{R}$ and any positive integer N, there exist integers n, m with $1 \le n \le N$ such that $|n\alpha - m| < 1/N$.*

Proof. For any real number x, let $f(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Since the $N+1$ numbers $\{f(i\alpha) : i = 0, \ldots, N\}$ all lie in the interval $[0,1)$, by the pigeonhole principle there must exist integers i and j with $0 \le i < j \le N$ such that both $f(i\alpha)$ and $f(j\alpha)$ lie in one of the N subintervals $[0,1/N)$, $[1/N,2/N)$, $\dots$, $[(N-1)/N,1)$. This means that $|f(j\alpha) - f(i\alpha)| < 1/N$, so we can take $n = j - i$ and $m = \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor$. □

The following proposition gives a few simple sufficient criteria for an injective immersion to be an embedding.

Proposition 4.22. *Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is an injective smooth immersion. If any of the following holds, then F is a smooth embedding.*

(a) *F is an open or closed map.*
(b) *F is a proper map.*
(c) *M is compact.*
(d) *M has empty boundary and* $\dim M = \dim N$.

Proof. If F is open or closed, then it is a topological embedding by Theorem A.38, so it is a smooth embedding. Either (b) or (c) implies that F is closed: if F is proper, then it is closed by Theorem A.57, and if M is compact, then F is closed by the closed map lemma. Finally, assume M has empty boundary and $\dim M = \dim N$. Then dF_p is nonsingular everywhere, and Problem 4-2 shows that $F(M) \subseteq \operatorname{Int} N$. Proposition 4.8(b) shows that $F\colon M \to \operatorname{Int} N$ is a local diffeomorphism, so it is an open map. It follows that $F\colon M \to N$ is a composition of open maps $M \to \operatorname{Int} N \hookrightarrow N$, so it is an embedding. □

Example 4.23. Let $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map. We showed in Example 2.13(d) that ι is smooth by computing its coordinate representation with respect to graph coordinates. It is easy to verify in the same coordinates that its differential is injective at each point, so it is an injective smooth immersion. Because $\mathbb{S}^n$ is compact, ι is a smooth embedding by Proposition 4.22. //

► **Exercise 4.24.** Give an example of a smooth embedding that is neither an open map nor a closed map.

Theorem 4.25 (Local Embedding Theorem). *Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is a smooth map. Then F is a smooth immersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U\colon U \to N$ is a smooth embedding.*

Proof. One direction is immediate: if every point has a neighborhood on which F is a smooth embedding, then F has full rank everywhere, so it is a smooth immersion.

Conversely, suppose F is a smooth immersion, and let $p \in M$. We show first that p has a neighborhood on which F is injective. If $F(p) \notin \partial N$, then either the rank theorem (if $p \notin \partial M$) or Theorem 4.15 (if $p \in \partial M$) implies that there is a neighborhood U_1 of p on which F has a coordinate representation of the form (4.3). It follows from this formula that $F|_{U_1}$ is injective. On the other hand, suppose $F(p) \in \partial N$, and let (W,ψ) be any smooth boundary chart for N centered at $F(p)$. If we let $U_0 = F^{-1}(W)$, which is a neighborhood of p, and let $\iota\colon \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ be the inclusion map, then the preceding argument can be applied to the composite map $\iota \circ \psi \circ F|_{U_0} \colon U_0 \to \mathbb{R}^n$, to show that p has a neighborhood $U_1 \subseteq U_0$ such that $\iota \circ \psi \circ F|_{U_1}$ is injective, from which it follows that $F|_{U_1}$ is injective.

Now let $p \in M$ be arbitrary, and let U_1 be a neighborhood of p on which F is injective. There exists a precompact neighborhood U of p such that $\bar{U} \subseteq U_1$. The restriction of F to $\bar{U}$ is an injective continuous map with compact domain, so it is a topological embedding by the closed map lemma. Because any restriction of a topological embedding is again a topological embedding, $F|_U$ is both a topological embedding and a smooth immersion, hence a smooth embedding. □

Theorem 4.25 points the way to a notion of immersions that makes sense for arbitrary topological spaces: if X and Y are topological spaces, a continuous map $F \colon X \to Y$ is called a ***topological immersion*** if every point of X has a neighborhood U such that $F|_U$ is a topological embedding. Thus, every smooth immersion is a topological immersion; but, just as with embeddings, a topological immersion that happens to be smooth need not be a smooth immersion (cf. Example 4.18).

Submersions

One of the most important applications of the rank theorem is to vastly expand our understanding of the properties of submersions. If $\pi \colon M \to N$ is any continuous map, a ***section of π*** is a continuous right inverse for π, i.e., a continuous map $\sigma \colon N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$:

$$\begin{tikzcd} M \arrow[d, "\pi"', shift right] \\ N. \arrow[u, "\sigma"', bend right] \end{tikzcd}$$

A ***local section of π*** is a continuous map $\sigma \colon U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = \mathrm{Id}_U$. Many of the important properties of smooth submersions follow from the fact that they admit an abundance of smooth local sections.

Theorem 4.26 (Local Section Theorem). *Suppose M and N are smooth manifolds and $\pi \colon M \to N$ is a smooth map. Then π is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.*

Proof. First suppose that π is a smooth submersion. Given $p \in M$, let $q = \pi(p) \in N$. By the rank theorem, we can choose smooth coordinates $\left(x^1, \dots, x^m\right)$ centered at p and $\left(y^1, \dots, y^n\right)$ centered at q in which π has the coordinate representation $\pi\left(x^1, \dots, x^n, x^{n+1}, \dots, x^m\right) = \left(x^1, \dots, x^n\right)$. If ε is a sufficiently small positive number, the coordinate cube

$$C_\varepsilon = \left\{x : \left|x^i\right| < \varepsilon \text{ for } i = 1, \dots, m\right\}$$

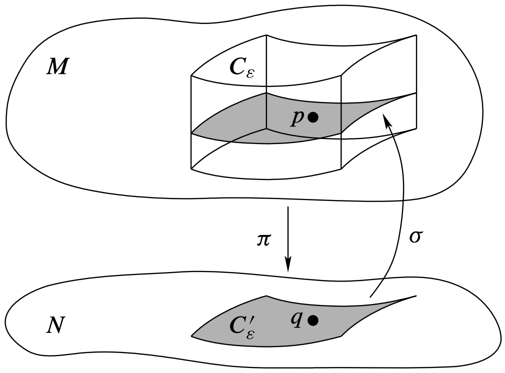

Fig. 4.4 Local section of a submersion

is a neighborhood of p whose image under π is the cube

$$C'_\varepsilon = \left\{ y : \left| y^i \right| < \varepsilon \text{ for } i = 1, \dots, n \right\}.$$

The map $\sigma\colon C'_\varepsilon \to C_\varepsilon$ whose coordinate representation is

$$\sigma\left(x^1, \dots, x^n\right) = \left(x^1, \dots, x^n, 0, \dots, 0\right)$$

is a smooth local section of π satisfying $\sigma(q) = p$ (Fig. 4.4).

Conversely, assume each point of M is in the image of a smooth local section. Given $p \in M$, let $\sigma\colon U \to M$ be a smooth local section such that $\sigma(q) = p$, where $q = \pi\big(\sigma(q)\big) = \pi(p) \in N$. The equation $\pi \circ \sigma = \mathrm{Id}_U$ implies that $d\pi_p \circ d\sigma_q = \mathrm{Id}_{T_qN}$, which in turn implies that $d\pi_p$ is surjective. □

This theorem motivates the following definition: if $\pi\colon X \to Y$ is a continuous map, we say π is a ***topological submersion*** if every point of X is in the image of a (continuous) local section of π. The preceding theorem shows that every smooth submersion is a topological submersion.

► **Exercise 4.27.** Give an example of a smooth map that is a topological submersion but not a smooth submersion.

Proposition 4.28 (Properties of Smooth Submersions). *Let M and N be smooth manifolds, and suppose $\pi\colon M \to N$ is a smooth submersion. Then π is an open map, and if it is surjective it is a quotient map.*

Proof. Suppose W is an open subset of M and q is a point of $\pi(W)$. For any $p \in W$ such that $\pi(p) = q$, there is a neighborhood U of q on which there exists a smooth local section $\sigma\colon U \to M$ of π satisfying $\sigma(q) = p$. For each $y \in \sigma^{-1}(W)$, the fact that $\sigma(y) \in W$ implies $y = \pi\big(\sigma(y)\big) \in \pi(W)$. Thus $\sigma^{-1}(W)$ is a neighborhood of q contained in $\pi(W)$, which implies that $\pi(W)$ is open. The second assertion follows from the first because every surjective open continuous map is a quotient map. □

The next three theorems provide important tools that we will use frequently when studying submersions. Notice the similarity between these results and Theorems A.27(a), A.30, and A.31. This demonstrates that surjective smooth submersions play a role in smooth manifold theory analogous to the role of quotient maps in topology. The first theorem generalizes the result of Exercise 4.10(b).

Theorem 4.29 (Characteristic Property of Surjective Smooth Submersions). *Suppose M and N are smooth manifolds, and $\pi\colon M \to N$ is a surjective smooth submersion. For any smooth manifold P with or without boundary, a map $F\colon N \to P$ is smooth if and only if $F \circ \pi$ is smooth*:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F \circ \pi"] & \\ N \arrow[r, "F"'] & P. \end{tikzcd}$$

Proof. If F is smooth, then $F \circ \pi$ is smooth by composition. Conversely, suppose that $F \circ \pi$ is smooth, and let $q \in N$ be arbitrary. Since π is surjective, there is a point $p \in \pi^{-1}(q)$, and then the local section theorem guarantees the existence of a neighborhood U of q and a smooth local section $\sigma\colon U \to M$ of π such that $\sigma(q) = p$. Then $\pi \circ \sigma = \mathrm{Id}_U$ implies

$$F|_U = F|_U \circ \mathrm{Id}_U = F|_U \circ (\pi \circ \sigma) = (F \circ \pi) \circ \sigma,$$

which is a composition of smooth maps. This shows that F is smooth in a neighborhood of each point, so it is smooth. □

Problem 4-7 explains the sense in which this property is "characteristic."

The next theorem gives a very general sufficient condition under which a smooth map can be "pushed down" by a submersion.

Theorem 4.30 (Passing Smoothly to the Quotient). *Suppose M and N are smooth manifolds and $\pi\colon M \to N$ is a surjective smooth submersion. If P is a smooth manifold with or without boundary and $F\colon M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\widetilde{F}\colon N \to P$ such that $\widetilde{F} \circ \pi = F$*:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\widetilde{F}"'] & P. \end{tikzcd}$$

Proof. Because a surjective smooth submersion is a quotient map, Theorem A.30 shows that there exists a unique continuous map $\widetilde{F}\colon N \to P$ satisfying $\widetilde{F} \circ \pi = F$. It is smooth by Theorem 4.29. □

Finally, we have the following uniqueness result.

Theorem 4.31 (Uniqueness of Smooth Quotients). *Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ are surjective smooth* *submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism* $F: N_1 \to N_2$ *such that* $F \circ \pi_1 = \pi_2$:

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "F"'] & & N_2. \end{tikzcd}$$

► **Exercise 4.32.** Prove Theorem 4.31.

Smooth Covering Maps

In this section, we introduce a class of local diffeomorphisms that play a significant role in smooth manifold theory. You are probably already familiar with the notion of a ***covering map*** between topological spaces: this is a surjective continuous map $\pi: E \to M$ between connected, locally path-connected spaces with the property that each point of M has a neighborhood U that is ***evenly covered***, meaning that each component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π. The basic properties of covering maps are summarized in Appendix A (pp. 615–616).

In the context of smooth manifolds, it is useful to introduce a slightly more restrictive type of covering map. If E and M are connected smooth manifolds with or without boundary, a map $\pi: E \to M$ is called a ***smooth covering map*** if π is smooth and surjective, and each point in M has a neighborhood U such that each component of $\pi^{-1}(U)$ is mapped *diffeomorphically* onto U by π. In this context we also say that U is evenly covered. The space M is called the ***base of the covering***, and E is called a ***covering manifold of M***. If E is simply connected, it is called the ***universal covering manifold of M***.

To distinguish this new definition from the previous one, we often call an ordinary (not necessarily smooth) covering map a ***topological covering map***. A smooth covering map is, in particular, a topological covering map. But as with other types of maps we have studied in this chapter, a smooth covering map is more than just a topological covering map that happens to be smooth: the definition requires in addition that the restriction of π to each component of the preimage of an evenly covered set be a diffeomorphism, not just a smooth homeomorphism.

Proposition 4.33 (Properties of Smooth Coverings).

(a) *Every smooth covering map is a local diffeomorphism, a smooth submersion, an open map, and a quotient map.*
(b) *An injective smooth covering map is a diffeomorphism.*
(c) *A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.*

► **Exercise 4.34.** Prove Proposition 4.33.

Example 4.35 (Smooth Covering Maps). The map $\varepsilon: \mathbb{R} \to \mathbb{S}^1$ defined in Example 2.13(b) is a topological covering map and a local diffeomorphism (see also

Example 4.11), so it is a smooth covering map. Similarly, the map $\varepsilon^n \colon \mathbb{R}^n \to \mathbb{T}^n$ of Example 2.13(c) is a smooth covering map. For each $n \geq 1$, the map $q \colon \mathbb{S}^n \to \mathbb{RP}^n$ defined in Example 2.13(f) is a two-sheeted smooth covering map (see Problem 4-10). //

Because smooth covering maps are surjective smooth submersions, all of the results in the preceding section about smooth submersions can be applied to them. For example, Theorem 4.30 is a particularly useful tool for defining a smooth map out of the base of a covering space. See Problems 4-12 and 4-13 for examples of this technique.

For smooth covering maps, the local section theorem can be strengthened.

Proposition 4.36 (Local Section Theorem for Smooth Covering Maps). *Suppose E and M are smooth manifolds with or without boundary, and $\pi \colon E \to M$ is a smooth covering map. Given any evenly covered open subset $U \subseteq M$, any $q \in U$, and any p in the fiber of π over q, there exists a unique smooth local section $\sigma \colon U \to E$ such that $\sigma(q) = p$.*

Proof. Suppose $U \subseteq M$ is evenly covered, $q \in U$, and $p \in \pi^{-1}(q)$. Let $\widetilde{U}_0$ be the component of $\pi^{-1}(U)$ containing p. Since the restriction of π to $\widetilde{U}_0$ is a diffeomorphism onto U, the map $\sigma = \left(\pi|_{\widetilde{U}_0}\right)^{-1}$ is the required smooth local section.

To prove uniqueness, suppose $\sigma' \colon U \to E$ is any other smooth local section satisfying $\sigma'(q) = p$. Since U is connected, $\sigma'(U)$ is contained in the component $\widetilde{U}_0$ containing p. Because σ' is a right inverse for the bijective map $\pi|_{\widetilde{U}_0}$, it must be equal to its inverse, and therefore equal to σ. □

► **Exercise 4.37.** Suppose $\pi \colon E \to M$ is a smooth covering map. Show that every local section of π is smooth.

► **Exercise 4.38.** Suppose $E_1, \dots, E_k$ and $M_1, \dots, M_k$ are smooth manifolds (without boundary), and $\pi_i \colon E_i \to M_i$ is a smooth covering map for each $i = 1, \dots, k$. Show that $\pi_1 \times \cdots \times \pi_k \colon E_1 \times \cdots \times E_k \to M_1 \times \cdots \times M_k$ is a smooth covering map.

► **Exercise 4.39.** Suppose $\pi \colon E \to M$ is a smooth covering map. Since π is also a topological covering map, there is a potential ambiguity about what it means for a subset $U \subseteq M$ to be evenly covered: does π map the components of $\pi^{-1}(U)$ diffeomorphically onto U, or merely homeomorphically? Show that the two concepts are equivalent: if $U \subseteq M$ is evenly covered in the topological sense, then π maps each component of $\pi^{-1}(U)$ diffeomorphically onto U.

Proposition 4.40 (Covering Spaces of Smooth Manifolds). *Suppose M is a connected smooth n-manifold, and $\pi \colon E \to M$ is a topological covering map. Then E is a topological n-manifold, and has a unique smooth structure such that π is a smooth covering map.*

Proof. Because π is a local homeomorphism, E is locally Euclidean. To show that it is Hausdorff, let p_1 and p_2 be distinct points in E. If $\pi(p_1) = \pi(p_2)$ and $U \subseteq M$ is an evenly covered open subset containing $\pi(p_1)$, then the components of $\pi^{-1}(U)$ containing p_1 and p_2 are disjoint open subsets of E that separate p_1 and p_2. On

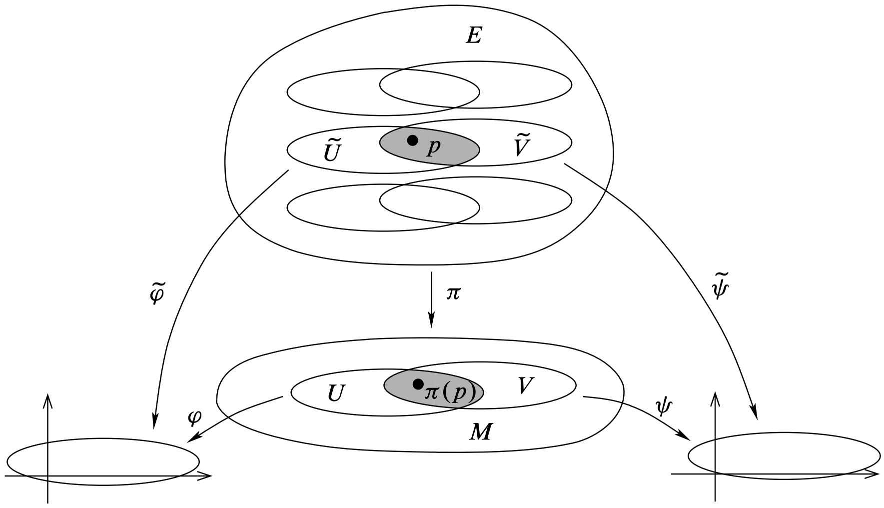

Fig. 4.5 Smooth compatibility of charts on a covering manifold

the other hand, if $\pi(p_1) \neq \pi(p_2)$, there are disjoint open subsets $U_1, U_2 \subseteq M$ containing $\pi(p_1)$ and $\pi(p_2)$, respectively, and then $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint open subsets of E containing p_1 and p_2. Thus E is Hausdorff.

To show that E is second-countable, we will show first that each fiber of π is countable. Given $q \in M$ and an arbitrary point $p_0 \in \pi^{-1}(q)$, we will construct a surjective map $\beta\colon \pi_1(M,q) \to \pi^{-1}(q)$; since $\pi_1(M,q)$ is countable by Proposition 1.16, this suffices. Let $[f] \in \pi_1(M,q)$ be the path class of an arbitrary loop $f\colon [0,1] \to M$ based at q. The path-lifting property of covering maps (Proposition A.77(b)) guarantees that there is a lift $\widetilde{f}\colon [0,1] \to E$ of f starting at p_0, and the monodromy theorem (Proposition A.77(c)) shows that the endpoint $\widetilde{f}(1) \in \pi^{-1}(q)$ depends only on the path class of f, so it makes sense to define $\beta[f] = \widetilde{f}(1)$. To see that β is surjective, just note that for any point $p \in \pi^{-1}(q)$, there is a path $\widetilde{f}$ in E from p_0 to p, and then $f = \pi \circ \widetilde{f}$ is a loop in M such that $p = \beta[f]$.

The collection of all evenly covered open subsets is an open cover of M, and therefore has a countable subcover $\{U_i\}$. For any given i, each component of $\pi^{-1}(U_i)$ contains exactly one point in each fiber over U_i, so $\pi^{-1}(U_i)$ has countably many components. The collection of all components of all sets of the form $\pi^{-1}(U_i)$ is thus a countable open cover of E; since each such component is second-countable, it follows from Exercise A.22 that E is second-countable. This completes the proof that E is a topological manifold.

To construct a smooth structure on E, suppose p is any point in E, and let U be an evenly covered neighborhood of $\pi(p)$. After shrinking U if necessary, we may assume also that it is the domain of a smooth coordinate map $\varphi\colon U \to \mathbb{R}^n$ (see Fig. 4.5).If $\widetilde{U}$ is the component of $\pi^{-1}(U)$ containing p, and $\widetilde{\varphi} = \varphi \circ \pi|_{\widetilde{U}}\colon \widetilde{U} \to \mathbb{R}^n$, then $\left(\widetilde{U}, \widetilde{\varphi}\right)$ is a chart on E. If two such charts $\left(\widetilde{U}, \widetilde{\varphi}\right)$ and

$(\widetilde{V},\widetilde{\psi})$ overlap, the transition map can be written

$$\begin{aligned}\widetilde{\psi}\circ\widetilde{\varphi}^{-1} &= \left(\psi\circ\pi|_{\widetilde{U}\cap\widetilde{V}}\right)\circ\left(\varphi\circ\pi|_{\widetilde{U}\cap\widetilde{V}}\right)^{-1}\\ &= \psi\circ\left(\pi|_{\widetilde{U}\cap\widetilde{V}}\right)\circ\left(\pi|_{\widetilde{U}\cap\widetilde{V}}\right)^{-1}\circ\varphi^{-1}\\ &= \psi\circ\varphi^{-1},\end{aligned}$$

which is smooth. Thus the collection of all such charts defines a smooth structure on E. The uniqueness of this smooth structure is left to the reader (Problem 4-9).

Finally, π is a smooth covering map because its coordinate representation in terms of any pair of charts $(\widetilde{U},\widetilde{\varphi})$ and (U,φ) constructed above is the identity. □

Here is the analogous result for manifolds with boundary.

Proposition 4.41 (Covering Spaces of Smooth Manifolds with Boundary). *Suppose M is a connected smooth n-manifold with boundary, and $\pi\colon E\to M$ is a topological covering map. Then E is a topological n-manifold with boundary such that $\partial E=\pi^{-1}(\partial M)$, and it has a unique smooth structure such that π is a smooth covering map.*

► **Exercise 4.42.** Prove the preceding proposition.

Corollary 4.43 (Existence of a Universal Covering Manifold). *If M is a connected smooth manifold, there exists a simply connected smooth manifold $\widetilde{M}$, called the **universal covering manifold of M**, and a smooth covering map $\pi\colon\widetilde{M}\to M$. The universal covering manifold is unique in the following sense: if $\widetilde{M}'$ is any other simply connected smooth manifold that admits a smooth covering map $\pi'\colon\widetilde{M}'\to M$, then there exists a diffeomorphism $\Phi\colon\widetilde{M}\to\widetilde{M}'$ such that $\pi'\circ\Phi=\pi$.*

► **Exercise 4.44.** Prove the preceding corollary.

► **Exercise 4.45.** Generalize the preceding corollary to smooth manifolds with boundary.

There are not many simple criteria for determining whether a given map is a smooth covering map, even if it is known to be a surjective local diffeomorphism. The following proposition gives one useful sufficient criterion. (It is not necessary, however; see Problem 4-11.)

Proposition 4.46. *Suppose E and M are nonempty connected smooth manifolds with or without boundary. If $\pi\colon E\to M$ is a proper local diffeomorphism, then π is a smooth covering map.*

Proof. Because π is a local diffeomorphism, it is an open map, and because it is proper, it is a closed map (Theorem A.57). Thus $\pi(E)$ is both open and closed in M. Since it is obviously nonempty, it is all of M, so π is surjective.

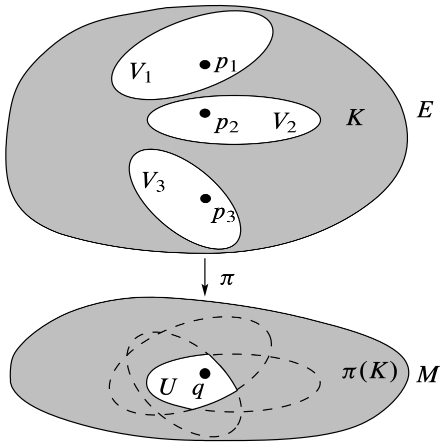

Fig. 4.6 A proper local diffeomorphism is a covering map

Let $q \in M$ be arbitrary. Since π is a local diffeomorphism, each point of $\pi^{-1}(q)$ has a neighborhood on which π is injective, so $\pi^{-1}(q)$ is a discrete subset of E. Since π is proper, $\pi^{-1}(q)$ is also compact, so it is finite. Write $\pi^{-1}(q) = \{p_1, \ldots, p_k\}$. For each i, there exists a neighborhood V_i of p_i on which π is a diffeomorphism onto an open subset $U_i \subseteq M$. Shrinking each V_i if necessary, we may assume also that $V_i \cap V_j = \varnothing$ for $i \neq j$.

Set $U = U_1 \cap \cdots \cap U_k$ (Fig. 4.6), which is a neighborhood of q. Then U satisfies

$$U \subseteq U_i \quad \text{for each } i. \tag{4.8}$$

Because $K = E \smallsetminus (V_1 \cup \cdots \cup V_k)$ is closed in E and π is a closed map, $\pi(K)$ is closed in M. Replacing U by $U \smallsetminus \pi(K)$, we can assume that U also satisfies

$$\pi^{-1}(U) \subseteq V_1 \cup \cdots \cup V_k. \tag{4.9}$$

Finally, after replacing U by the connected component of U containing q, we can assume that U is connected and still satisfies (4.8) and (4.9). We will show that U is evenly covered.

Let $\widetilde{V}_i = \pi^{-1}(U) \cap V_i$. By virtue of (4.9), $\pi^{-1}(U) = \widetilde{V}_1 \cup \cdots \cup \widetilde{V}_k$. Because $\pi\colon V_i \to U_i$ is a diffeomorphism, (4.8) implies that $\pi\colon \widetilde{V}_i \to U$ is still a diffeomorphism, and in particular $\widetilde{V}_i$ is connected. Because $\widetilde{V}_1, \ldots, \widetilde{V}_k$ are disjoint connected open subsets of $\pi^{-1}(U)$, they are exactly the components of $\pi^{-1}(U)$. □

Problems

4-1. Use the inclusion map $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to show that Theorem 4.5 does not extend to the case in which M is a manifold with boundary. (*Used on p. 80.*)

4-2. Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F: M \to N$ is smooth. Show that if $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in \operatorname{Int} N$. (*Used on pp. 80, 87.*)

4-3. Formulate and prove a version of the rank theorem for a map of constant rank whose domain is a smooth manifold with boundary. [Hint: after extending F arbitrarily as we did in the proof of Theorem 4.15, follow through the proof of the rank theorem until the point at which the constant-rank hypothesis is used, and then explain how to modify the extended map so that it has constant rank.]

4-4. Let $\gamma: \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$. (*Used on pp. 502, 542.*)

4-5. Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space, as defined in Problem 1-9.
(a) Show that the quotient map $\pi: \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion.
(b) Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.
(*Used on pp. 172, 560.*)

4-6. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F: M \to \mathbb{R}^k$ for any $k > 0$.

4-7. Suppose M and N are smooth manifolds, and $\pi: M \to N$ is a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map $F: \widetilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\widetilde{N}$. [Remark: this shows that the property described in Theorem 4.29 is "characteristic" in the same sense as that in which Theorem A.27(a) is characteristic of the quotient topology.]

4-8. This problem shows that the converse of Theorem 4.29 is false. Let $\pi: \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F: \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

4-9. Let M be a connected smooth manifold, and let $\pi: E \to M$ be a topological covering map. Complete the proof of Proposition 4.40 by showing that there is only one smooth structure on E such that π is a smooth covering map. [Hint: use the existence of smooth local sections.]

4-10. Show that the map $q: \mathbb{S}^n \to \mathbb{RP}^n$ defined in Example 2.13(f) is a smooth covering map. (*Used on p. 550.*)

4-11. Show that a topological covering map is proper if and only if its fibers are finite, and therefore the converse of Proposition 4.46 is false.

4-12. Using the covering map $\varepsilon^2 \colon \mathbb{R}^2 \to \mathbb{T}^2$ (see Example 4.35), show that the immersion $X \colon \mathbb{R}^2 \to \mathbb{R}^3$ defined in Example 4.2(d) descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\widetilde{X} \colon \mathbb{T}^2 \to \mathbb{R}^3$, and then show that $\widetilde{X}$ is a smooth embedding whose image is the given surface of revolution.

4-13. Define a map $F \colon \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x, y, z) = \left(x^2 - y^2, xy, xz, yz\right)$. Using the smooth covering map of Example 2.13(f) and Problem 4-10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.