11. $3ty+y^2+(t^2+ty)\dfrac{dy}{dt}=0, \qquad y(2)=1$

In each of Problems 12–14, determine the constant a so that the equation is exact, and then solve the resulting equation.

12. $t+ye^{2ty}+ate^{2ty}\dfrac{dy}{dt}=0$

13. $\dfrac{1}{t^2}+\dfrac{1}{y^2}+\dfrac{(at+1)}{y^3}\dfrac{dy}{dt}=0$

14. $e^{at+y}+3t^2y^2+(2yt^3+e^{at+y})\dfrac{dy}{dt}=0$

15. Show that every separable equation of the form $M(t)+N(y)dy/dt=0$ is exact.

16. Find all functions $f(t)$ such that the differential equation

$$y^2\sin t+yf(t)(dy/dt)=0$$

is exact. Solve the differential equation for these $f(t)$.

17. Show that if $((\partial N/\partial t)-(\partial M/\partial y))/M=Q(y)$, then the differential equation $M(t,y)+N(t,y)dy/dt=0$ has an integrating factor $\mu(y)=\exp\left(\int Q(y)\,dy\right)$.

18. The differential equation $f(t)(dy/dt)+t^2+y=0$ is known to have an integrating factor $\mu(t)=t$. Find all possible functions $f(t)$.

19. The differential equation $e^t\sec y-\tan y+(dy/dt)=0$ has an integrating factor of the form $e^{-at}\cos y$ for some constant a. Find a, and then solve the differential equation.

20. The Bernoulli differential equation is $(dy/dt)+a(t)y=b(t)y^n$. Multiplying through by $\mu(t)=\exp\left(\int a(t)\,dt\right)$, we can rewrite this equation in the form $d/dt(\mu(t)y)=b(t)\mu(t)y^n$. Find the general solution of this equation by finding an appropriate integrating factor. *Hint*: Divide both sides of the equation by an appropriate function of y.

1.10 The existence–uniqueness theorem; Picard iteration

Consider the initial-value problem

$$\frac{dy}{dt}=f(t,y), \qquad y(t_0)=y_0 \tag{1}$$

where f is a given function of t and y. Chances are, as the remarks in Section 1.9 indicate, that we will be unable to solve (1) explicitly. This leads us to ask the following questions.

1. How are we to know that the initial-value problem (1) actually has a solution if we can't exhibit it?

2. How do we know that there is only one solution $y(t)$ of (1)? Perhaps there are two, three, or even infinitely many solutions.
3. Why bother asking the first two questions? After all, what's the use of determining whether (1) has a unique solution if we won't be able to explicitly exhibit it?

The answer to the third question lies in the observation that it is never necessary, in applications, to find the solution $y(t)$ of (1) to more than a finite number of decimal places. Usually, it is more than sufficient to find $y(t)$ to four decimal places. As we shall see in Sections 1.13–17, this can be done quite easily with the aid of a digital computer. In fact, we will be able to compute $y(t)$ to eight, and even sixteen, decimal places. Thus, the knowledge that (1) has a unique solution $y(t)$ is our hunting license to go looking for it.

To resolve the first question, we must establish the existence of a function $y(t)$ whose value at $t=t_0$ is y_0, and whose derivative at any time t equals $f(t,y(t))$. In order to accomplish this, we must find a theorem which enables us to establish the existence of a function having certain properties, without our having to exhibit this function explicitly. If we search through the Calculus, we find that we encounter such a situation exactly once, and this is in connection with the theory of limits. As we show in Appendix B, it is often possible to prove that a sequence of functions $y_n(t)$ has a limit $y(t)$, without our having to exhibit $y(t)$. For example, we can prove that the sequence of functions

$$y_n(t)=\frac{\sin \pi t}{1^2}+\frac{\sin 2\pi t}{2^2}+\dots+\frac{\sin n\pi t}{n^2}$$

has a limit $y(t)$ even though we cannot exhibit $y(t)$ explicitly. This suggests the following algorithm for proving the existence of a solution $y(t)$ of (1).

(a) Construct a sequence of functions $y_n(t)$ which come closer and closer to solving (1).
(b) Show that the sequence of functions $y_n(t)$ has a limit $y(t)$ on a suitable interval $t_0 \leqslant t \leqslant t_0+\alpha$.
(c) Prove that $y(t)$ is a solution of (1) on this interval.

We now show how to implement this algorithm.

(a) *Construction of the approximating sequence* $y_n(t)$

The problem of finding a sequence of functions that come closer and closer to satisfying a certain equation is one that arises quite often in mathematics. Experience has shown that it is often easiest to resolve this problem when our equation can be written in the special form

$$y(t)=L(t,y(t)), \tag{2}$$

where L may depend explicitly on y, and on integrals of functions of y.

For example, we may wish to find a function $y(t)$ satisfying

$$y(t)=1+\sin\left[t+y(t)\right],$$

or

$$y(t)=1+y^2(t)+\int_0^t y^3(s)\,ds.$$

In these two cases, $L(t,y(t))$ is an abbreviation for

$$1+\sin\left[t+y(t)\right]$$

and

$$1+y^2(t)+\int_0^t y^3(s)\,ds,$$

respectively.

The key to understanding what is special about Equation (2) is to view $L(t,y(t))$ as a "machine" that takes in one function and gives back another one. For example, let

$$L(t,y(t))=1+y^2(t)+\int_0^t y^3(s)\,ds.$$

If we plug the function $y(t)=t$ into this machine, (that is, if we compute $1+t^2+\int_0^t s^3\,ds$) then the machine returns to us the function $1+t^2+t^4/4$. If we plug the function $y(t)=\cos t$ into this machine, then it returns to us the function

$$1+\cos^2 t+\int_0^t \cos^3 s\,ds=1+\cos^2 t+\sin t-\frac{\sin^3 t}{3}.$$

According to this viewpoint, we can characterize all solutions $y(t)$ of (2) as those functions $y(t)$ which the machine L leaves unchanged. In other words, if we plug a function $y(t)$ into the machine L, and the machine returns to us this same function, then $y(t)$ is a solution of (2).

We can put the initial-value problem (1) into the special form (2) by integrating both sides of the differential equation $y'=f(t,y)$ with respect to t. Specifically, if $y(t)$ satisfies (1), then

$$\int_{t_0}^t \frac{dy(s)}{ds}\,ds=\int_{t_0}^t f(s,y(s))\,ds$$

so that

$$y(t)=y_0+\int_{t_0}^t f(s,y(s))\,ds. \tag{3}$$

Conversely, if $y(t)$ is continuous and satisfies (3), then $dy/dt=f(t,y(t))$. Moreover, $y(t_0)$ is obviously y_0. Therefore, $y(t)$ is a solution of (1) if, and only if, it is a continuous solution of (3).

Equation (3) is called an integral equation, and it is in the special form (2) if we set

$$L(t,y(t))=y_0+\int_{t_0}^{t}f(s,y(s))\,ds.$$

This suggests the following scheme for constructing a sequence of "approximate solutions" $y_n(t)$ of (3). Let us start by guessing a solution $y_0(t)$ of (3). The simplest possible guess is $y_0(t)=y_0$. To check whether $y_0(t)$ is a solution of (3), we compute

$$y_1(t)=y_0+\int_{t_0}^{t}f(s,y_0(s))\,ds.$$

If $y_1(t)=y_0$, then $y(t)=y_0$ is indeed a solution of (3). If not, then we try $y_1(t)$ as our next guess. To check whether $y_1(t)$ is a solution of (3), we compute

$$y_2(t)=y_0+\int_{t_0}^{t}f(s,y_1(s))\,ds,$$

and so on. In this manner, we define a sequence of functions $y_1(t)$, $y_2(t),\dots,$ where

$$y_{n+1}(t)=y_0+\int_{t_0}^{t}f(s,y_n(s))\,ds. \tag{4}$$

These functions $y_n(t)$ are called successive approximations, or Picard iterates, after the French mathematician Picard who first discovered them. Remarkably, these Picard iterates always converge, on a suitable interval, to a solution $y(t)$ of (3).

Example 1. Compute the Picard iterates for the initial-value problem

$$y'=y, \qquad y(0)=1,$$

and show that they converge to the solution $y(t)=e^t$.

Solution. The integral equation corresponding to this initial-value problem is

$$y(t)=1+\int_0^t y(s)\,ds.$$

Hence, $y_0(t)=1$

$$y_1(t)=1+\int_0^t 1\,ds=1+t$$

$$y_2(t)=1+\int_0^t y_1(s)\,ds=1+\int_0^t(1+s)\,ds=1+t+\frac{t^2}{2!}$$

and, in general,

$$y_n(t)=1+\int_0^t y_{n-1}(s)\,ds=1+\int_0^t\left[1+s+\ldots+\frac{s^{n-1}}{(n-1)!}\right]ds$$
$$=1+t+\frac{t^2}{2!}+\ldots+\frac{t^n}{n!}.$$

Since $e^t=1+t+t^2/2!+\ldots$, we see that the Picard iterates $y_n(t)$ converge to the solution $y(t)$ of this initial-value problem.

Example 2. Compute the Picard iterates $y_1(t),y_2(t)$ for the initial-value problem $y'=1+y^3$, $y(1)=1$.

Solution. The integral equation corresponding to this initial-value problem is

$$y(t)=1+\int_1^t\left[1+y^3(s)\right]ds.$$

Hence, $y_0(t)=1$

$$y_1(t)=1+\int_1^t(1+1)\,ds=1+2(t-1)$$

and

$$y_2(t)=1+\int_1^t\left\{1+\left[1+2(s-1)\right]^3\right\}ds$$
$$=1+2(t-1)+3(t-1)^2+4(t-1)^3+2(t-1)^4.$$

Notice that it is already quite cumbersome to compute $y_3(t)$.

(b) *Convergence of the Picard iterates*

As was mentioned in Section 1.4, the solutions of nonlinear differential equations may not exist for all time t. Therefore, we cannot expect the Picard iterates $y_n(t)$ of (3) to converge for all t. To provide us with a clue, or estimate, of where the Picard iterates converge, we try to find an interval in which all the $y_n(t)$ are uniformly bounded (that is, $|y_n(t)|\leqslant K$ for some fixed constant K). Equivalently, we seek a rectangle R which contains the graphs of all the Picard iterates $y_n(t)$. Lemma 1 shows us how to find such a rectangle.

Lemma 1. *Choose any two positive numbers a and b, and let R be the rectangle: $t_0\leqslant t\leqslant t_0+a, |y-y_0|\leqslant b$. Compute*

$$M=\max_{(t,y)\text{ in }R}|f(t,y)|,\quad\text{and set}\quad \alpha=\min\left(a,\frac{b}{M}\right).$$

Then,

$$|y_n(t)-y_0|\leqslant M(t-t_0)\tag{5}$$

for $t_0\leqslant t\leqslant t_0+\alpha$.

Lemma 1 states that the graph of $y_n(t)$ is sandwiched between the lines $y=y_0+M(t-t_0)$ and $y=y_0-M(t-t_0)$, for $t_0 \leqslant t \leqslant t_0+\alpha$. These lines leave the rectangle R at $t=t_0+a$ if $a \leqslant b/M$, and at $t=t_0+b/M$ if $b/M < a$ (see Figures 1a and 1b). In either case, therefore, the graph of $y_n(t)$ is contained in R for $t_0 \leqslant t \leqslant t_0+\alpha$.

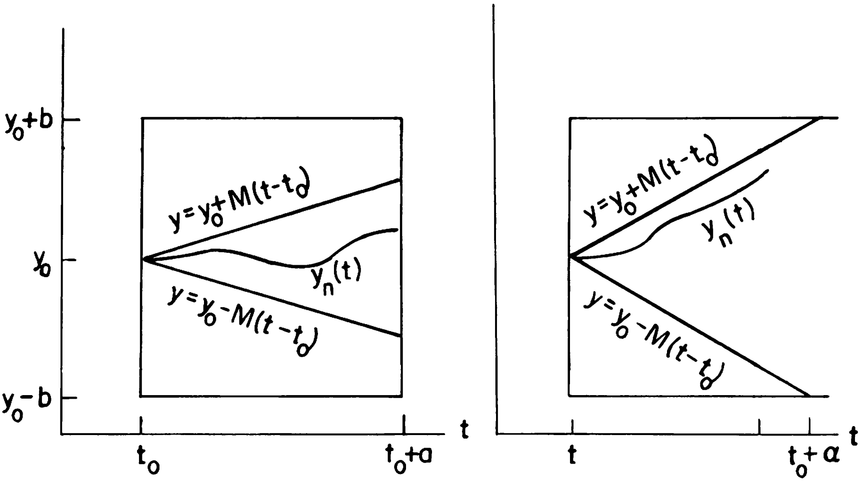

Figure 1. (a) $\alpha=a$; (b) $\alpha=b/M$

PROOF OF LEMMA 1. We establish (5) by induction on n. Observe first that (5) is obviously true for $n=0$, since $y_0(t)=y_0$. Next, we must show that (5) is true for $n=j+1$ if it is true for $n=j$. But this follows immediately, for if $|y_j(t)-y_0| \leqslant M(t-t_0)$, then

$$\begin{aligned} |y_{j+1}(t)-y_0| &= \left| \int_{t_0}^{t} f(s,y_j(s))\,ds \right| \\ &\leqslant \int_{t_0}^{t} |f(s,y_j(s))|\,ds \leqslant M(t-t_0) \end{aligned}$$

for $t_0 \leqslant t \leqslant t_0+\alpha$. Consequently, (5) is true for all n, by induction. □

We now show that the Picard iterates $y_n(t)$ of (3) converge for each t in the interval $t_0 \leqslant t \leqslant t_0+\alpha$, if $\partial f/\partial y$ exists and is continuous. Our first step is to reduce the problem of showing that the sequence of functions $y_n(t)$ converges to the much simpler problem of proving that an infinite series converges. This is accomplished by writing $y_n(t)$ in the form

$$y_n(t)=y_0(t)+\left[y_1(t)-y_0(t)\right]+\dots+\left[y_n(t)-y_{n-1}(t)\right].$$

Clearly, the sequence $y_n(t)$ converges if, and only if, the infinite series

$$\left[y_1(t)-y_0(t)\right]+\left[y_2(t)-y_1(t)\right]+\dots+\left[y_n(t)-y_{n-1}(t)\right]+\dots \tag{6}$$

converges. To prove that the infinite series (6) converges, it suffices to show that

$$\sum_{n=1}^{\infty} |y_n(t)-y_{n-1}(t)|<\infty. \tag{7}$$

This is accomplished in the following manner. Observe that

$$\begin{aligned}
|y_n(t)-y_{n-1}(t)| &= \left|\int_{t_0}^{t}\left[f(s,y_{n-1}(s))-f(s,y_{n-2}(s))\right]ds\right| \\
&\leqslant \int_{t_0}^{t}|f(s,y_{n-1}(s))-f(s,y_{n-2}(s))|\,ds \\
&= \int_{t_0}^{t}\left|\frac{\partial f(s,\xi(s))}{\partial y}\right||y_{n-1}(s)-y_{n-2}(s)|\,ds,
\end{aligned}$$

where $\xi(s)$ lies between $y_{n-1}(s)$ and $y_{n-2}(s)$. (Recall that $f(x_1)-f(x_2)=f'(\xi)(x_1-x_2)$, where ξ is some number between x_1 and x_2.) It follows immediately from Lemma 1 that the points $(s,\xi(s))$ all lie in the rectangle R for $s<t_0+\alpha$. Consequently,

$$|y_n(t)-y_{n-1}(t)|\leqslant L\int_{t_0}^{t}|y_{n-1}(s)-y_{n-2}(s)|\,ds, \qquad t_0\leqslant t\leqslant t_0+\alpha, \tag{8}$$

where

$$L=\max_{(t,y)\text{ in }R}\left|\frac{\partial f(t,y)}{\partial y}\right|. \tag{9}$$

Equation (9) defines the constant L. Setting $n=2$ in (8) gives

$$\begin{aligned}
|y_2(t)-y_1(t)| &\leqslant L\int_{t_0}^{t}|y_1(s)-y_0|\,ds\leqslant L\int_{t_0}^{t}M(s-t_0)\,ds \\
&= \frac{LM(t-t_0)^2}{2}.
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
|y_3(t)-y_2(t)| &\leqslant L\int_{t_0}^{t}|y_2(s)-y_1(s)|\,ds\leqslant ML^2\int_{t_0}^{t}\frac{(s-t_0)^2}{2}\,ds \\
&= \frac{ML^2(t-t_0)^3}{3!}.
\end{aligned}$$

Proceeding inductively, we see that

$$|y_n(t)-y_{n-1}(t)|\leqslant\frac{ML^{n-1}(t-t_0)^n}{n!}, \quad \text{for } t_0\leqslant t\leqslant t_0+\alpha. \tag{10}$$

Therefore, for $t_0 \leqslant t \leqslant t_0 + \alpha$,

$$
\begin{aligned}
&|y_1(t)-y_0(t)|+|y_2(t)-y_1(t)|+\dots\\
&\qquad \leqslant M(t-t_0)+\frac{ML(t-t_0)^2}{2!}+\frac{ML^2(t-t_0)^3}{3!}+\dots\\
&\qquad \leqslant M\alpha+\frac{ML\alpha^2}{2!}+\frac{ML^2\alpha^3}{3!}+\dots\\
&\qquad =\frac{M}{L}\left[\alpha L+\frac{(\alpha L)^2}{2!}+\frac{(\alpha L)^3}{3!}+\dots\right]\\
&\qquad =\frac{M}{L}(e^{\alpha L}-1).
\end{aligned}
$$

This quantity, obviously, is less than infinity. Consequently, the Picard iterates $y_n(t)$ converge for each t in the interval $t_0 \leqslant t \leqslant t_0 + \alpha$. (A similar argument shows that $y_n(t)$ converges for each t in the interval $t_0 - \beta \leqslant t \leqslant t_0$, where $\beta = \min(a, b/N)$, and N is the maximum value of $|f(t,y)|$ for (t,y) in the rectangle $t_0 - a \leqslant t \leqslant t_0, |y - y_0| \leqslant b$.) We will denote the limit of the sequence $y_n(t)$ by $y(t)$. □

(c) *Proof that $y(t)$ satisfies the initial-value problem* (1)

We will show that $y(t)$ satisfies the integral equation

$$y(t)=y_0+\int_{t_0}^{t} f(s,y(s))\,ds \tag{11}$$

and that $y(t)$ is continuous. To this end, recall that the Picard iterates $y_n(t)$ are defined recursively through the equation

$$y_{n+1}(t)=y_0+\int_{t_0}^{t} f(s,y_n(s))\,ds. \tag{12}$$

Taking limits of both sides of (12) gives

$$y(t)=y_0+\lim_{n\to\infty}\int_{t_0}^{t} f(s,y_n(s))\,ds. \tag{13}$$

To show that the right-hand side of (13) equals

$$y_0+\int_{t_0}^{t} f(s,y(s))\,ds,$$

(that is, to justify passing the limit through the integral sign) we must show that

$$\left|\int_{t_0}^{t} f(s,y(s))\,ds-\int_{t_0}^{t} f(s,y_n(s))\,ds\right|$$

approaches zero as n approaches infinity. This is accomplished in the following manner. Observe first that the graph of $y(t)$ lies in the rectangle R for $t \leqslant t_0 + \alpha$, since it is the limit of functions $y_n(t)$ whose graphs lie in R.

Hence

$$\left|\int_{t_0}^{t} f(s,y(s))\,ds-\int_{t_0}^{t} f(s,y_n(s))\,ds\right|$$

$$\leqslant \int_{t_0}^{t} |f(s,y(s))-f(s,y_n(s))|\,ds \leqslant L\int_{t_0}^{t} |y(s)-y_n(s)|\,ds$$

where L is defined by Equation (9). Next, observe that

$$y(s)-y_n(s)=\sum_{j=n+1}^{\infty}\left[y_j(s)-y_{j-1}(s)\right]$$

since

$$y(s)=y_0+\sum_{j=1}^{\infty}\left[y_j(s)-y_{j-1}(s)\right]$$

and

$$y_n(s)=y_0+\sum_{j=1}^{n}\left[y_j(s)-y_{j-1}(s)\right].$$

Consequently, from (10),

$$|y(s)-y_n(s)| \leqslant M\sum_{j=n+1}^{\infty} L^{j-1}\frac{(s-t_0)^j}{j!}$$

$$\leqslant M\sum_{j=n+1}^{\infty}\frac{L^{j-1}\alpha^j}{j!}=\frac{M}{L}\sum_{j=n+1}^{\infty}\frac{(\alpha L)^j}{j!}, \tag{14}$$

and

$$\left|\int_{t_0}^{t} f(s,y(s))\,ds-\int_{t_0}^{t} f(s,y_n(s))\,ds\right| \leqslant M\sum_{j=n+1}^{\infty}\frac{(\alpha L)^j}{j!}\int_{t_0}^{t} ds$$

$$\leqslant M\alpha\sum_{j=n+1}^{\infty}\frac{(\alpha L)^j}{j!}.$$

This summation approaches zero as n approaches infinity, since it is the tail end of the convergent Taylor series expansion of $e^{\alpha L}$. Hence,

$$\lim_{n\to\infty}\int_{t_0}^{t} f(s,y_n(s))\,ds=\int_{t_0}^{t} f(s,y(s))\,ds,$$

and $y(t)$ satisfies (11).

To show that $y(t)$ is continuous, we must show that for every $\varepsilon>0$ we can find $\delta>0$ such that

$$|y(t+h)-y(t)|<\varepsilon \quad \text{if } |h|<\delta.$$

Now, we cannot compare $y(t+h)$ with $y(t)$ directly, since we do not know $y(t)$ explicitly. To overcome this difficulty, we choose a large integer N and observe that

$$\begin{aligned} y(t+h)-y(t) = & \left[y(t+h)-y_N(t+h)\right] \\ & + \left[y_N(t+h)-y_N(t)\right] + \left[y_N(t)-y(t)\right]. \end{aligned}$$

Specifically, we choose N so large that

$$\frac{M}{L} \sum_{j=N+1}^{\infty} \frac{(\alpha L)^j}{j!} < \frac{\varepsilon}{3}.$$

Then, from (14),

$$|y(t+h)-y_N(t+h)| < \frac{\varepsilon}{3} \quad \text{and} \quad |y_N(t)-y(t)| < \frac{\varepsilon}{3},$$

for $t < t_0+\alpha$, and h sufficiently small (so that $t+h < t_0+\alpha$.) Next, observe that $y_N(t)$ is continuous, since it is obtained from N repeated integrations of continuous functions. Therefore, we can choose $\delta > 0$ so small that

$$|y_N(t+h)-y_N(t)| < \frac{\varepsilon}{3} \quad \text{for } |h| < \delta.$$

Consequently,

$$\begin{aligned} |y(t+h)-y(t)| \leqslant & |y(t+h)-y_N(t+h)| + |y_N(t+h)-y_N(t)| \\ & + |y_N(t)-y(t)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for $|h| < \delta$. Therefore, $y(t)$ is a continuous solution of the integral equation (11), and this completes our proof that $y(t)$ satisfies (1). □

In summary, we have proven the following theorem.

Theorem 2. *Let f and $\partial f/\partial y$ be continuous in the rectangle $R: t_0 \leqslant t \leqslant t_0+a$, $|y-y_0| \leqslant b$. Compute*

$$M = \max_{(t,y) \text{ in } R} |f(t,y)|, \quad \textit{and set} \quad \alpha = \min\left(a, \frac{b}{M}\right).$$

Then, the initial-value problem $y' = f(t,y)$, $y(t_0) = y_0$ has at least one solution $y(t)$ on the interval $t_0 \leqslant t \leqslant t_0+\alpha$. A similar result is true for $t < t_0$.

Remark. The number α in Theorem 2 depends specifically on our choice of a and b. Different choices of a and b lead to different values of α. Moreover, α doesn't necessarily increase when a and b increase, since an increase in a or b will generally result in an increase in M.

Finally, we turn our attention to the problem of uniqueness of solutions of (1). Consider the initial-value problem

$$\frac{dy}{dt} = (\sin 2t) y^{1/3}, \qquad y(0) = 0. \tag{15}$$

One solution of (15) is $y(t) = 0$. Additional solutions can be obtained if we ignore the fact that $y(0)=0$ and rewrite the differential equation in the form

$$\frac{1}{y^{1/3}}\frac{dy}{dt}=\sin 2t,$$

or

$$\frac{d}{dt}\frac{3y^{2/3}}{2}=\sin 2t.$$

Then,

$$\frac{3y^{2/3}}{2}=\frac{1-\cos 2t}{2}=\sin^2 t$$

and $y=\pm\sqrt{8/27}\,\sin^3 t$ are two additional solutions of (15).

Now, initial-value problems that have more than one solution are clearly unacceptable in applications. Therefore, it is important for us to find out exactly what is "wrong" with the initial-value problem (15) that it has more than one solution. If we look carefully at the right-hand side of this differential equation, we see that it does not have a partial derivative with respect to y at $y=0$. This is indeed the problem, as the following theorem shows.

Theorem 2′. *Let f and $\partial f/\partial y$ be continuous in the rectangle $R: t_0 \leqslant t \leqslant t_0+a$, $|y-y_0|\leqslant b$. Compute*

$$M=\max_{(t,y)\text{ in }R}|f(t,y)|, \quad \textit{and set} \quad \alpha=\min\left(a,\frac{b}{M}\right).$$

Then, the initial-value problem

$$y'=f(t,y), \qquad y(t_0)=y_0 \tag{16}$$

has a unique solution $y(t)$ on the interval $t_0\leqslant t\leqslant t_0+\alpha$. In other words, if $y(t)$ and $z(t)$ are two solutions of (16), *then $y(t)$ must equal $z(t)$ for $t_0\leqslant t \leqslant t_0+\alpha$.*

Proof. Theorem 2 guarantees the existence of at least one solution $y(t)$ of (16). Suppose that $z(t)$ is a second solution of (16). Then,

$$y(t)=y_0+\int_{t_0}^{t} f(s,y(s))\,ds \quad\text{and}\quad z(t)=y_0+\int_{t_0}^{t} f(s,z(s))\,ds.$$

Subtracting these two equations gives

$$\begin{aligned}|y(t)-z(t)| &= \left|\int_{t_0}^{t}\left[f(s,y(s))-f(s,z(s))\right]ds\right|\\ &\leqslant \int_{t_0}^{t}|f(s,y(s))-f(s,z(s))|\,ds\\ &\leqslant L\int_{t_0}^{t}|y(s)-z(s)|\,ds\end{aligned}$$

where L is the maximum value of $|\partial f/\partial y|$ for (t,y) in R. As Lemma 2 below shows, this inequality implies that $y(t)=z(t)$. Hence, the initial-value problem (16) has a unique solution $y(t)$. □

Lemma 2. *Let $w(t)$ be a nonnegative function, with*

$$w(t) \leqslant L\int_{t_0}^{t} w(s)\,ds. \tag{17}$$

Then, $w(t)$ is identically zero.

FAKE PROOF. Differentiating both sides of (17) gives

$$\frac{dw}{dt} \leqslant Lw(t), \quad \text{or} \quad \frac{dw}{dt} - Lw(t) \leqslant 0.$$

Multiplying both sides of this inequality by the integrating factor $e^{-L(t-t_0)}$ gives

$$\frac{d}{dt}e^{-L(t-t_0)}w(t) \leqslant 0, \quad \text{so that} \quad e^{-L(t-t_0)}w(t) \leqslant w(t_0)$$

for $t \geqslant t_0$. But $w(t_0)$ must be zero if $w(t)$ is nonnegative and satisfies (17). Consequently, $e^{-L(t-t_0)}w(t) \leqslant 0$, and this implies that $w(t)$ is identically zero.

The error in this proof, of course, is that we cannot differentiate both sides of an inequality, and still expect to preserve the inequality. For example, the function $f_1(t)=2t-2$ is less than $f_2(t)=t$ on the interval $[0,1]$, but $f_1'(t)$ is greater than $f_2'(t)$ on this interval. We make this proof "kosher" by the clever trick of setting

$$U(t) = \int_{t_0}^{t} w(s)\,ds.$$

Then,

$$\frac{dU}{dt} = w(t) \leqslant L\int_{t_0}^{t} w(s)\,ds = LU(t).$$

Consequently, $e^{-L(t-t_0)}U(t) \leqslant U(t_0)=0$, for $t \geqslant t_0$, and thus $U(t)=0$. This, in turn, implies that $w(t)=0$ since

$$0 \leqslant w(t) \leqslant L\int_{t_0}^{t} w(s)\,ds = LU(t) = 0.$$ □

Example 3. Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = t^2 + e^{-y^2}, \qquad y(0)=0$$

exists for $0 \leqslant t \leqslant \frac{1}{2}$, and in this interval, $|y(t)| \leqslant 1$.

Solution. Let R be the rectangle $0 \leqslant t \leqslant \frac{1}{2}$, $|y| \leqslant 1$. Computing

$$M = \max_{(t,y) \text{ in } R} t^2 + e^{-y^2} = 1 + \left(\tfrac{1}{2}\right)^2 = \tfrac{5}{4},$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \min\left(\frac{1}{2}, \frac{1}{5/4}\right) = \frac{1}{2},$$

and in this interval, $|y(t)| \leqslant 1$.

Example 4. Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \qquad y(0) = 1$$

exists for $0 \leqslant t \leqslant 1/9$, and in this interval, $0 \leqslant y \leqslant 2$.
Solution. Let R be the rectangle $0 \leqslant t \leqslant \frac{1}{9}$, $0 \leqslant y \leqslant 2$. Computing

$$M = \max_{(t,y) \text{ in } R} e^{-t^2} + y^3 = 1 + 2^3 = 9,$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \min\left(\tfrac{1}{9}, \tfrac{1}{9}\right)$$

and in this interval, $0 \leqslant y \leqslant 2$.

Example 5. What is the largest interval of existence that Theorem 2 predicts for the solution $y(t)$ of the initial-value problem $y' = 1 + y^2$, $y(0) = 0$?
Solution. Let R be the rectangle $0 \leqslant t \leqslant a$, $|y| \leqslant b$. Computing

$$M = \max_{(t,y) \text{ in } R} 1 + y^2 = 1 + b^2,$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \alpha = \min\left(a, \frac{b}{1+b^2}\right).$$

Clearly, the largest α that we can achieve is the maximum value of the function $b/(1+b^2)$. This maximum value is $\frac{1}{2}$. Hence, Theorem 2 predicts that $y(t)$ exists for $0 \leqslant t \leqslant \frac{1}{2}$. The fact that $y(t) = \tan t$ exists for $0 \leqslant t \leqslant \pi/2$ points out the limitation of Theorem 2.

Example 6. Suppose that $|f(t,y)| \leqslant K$ in the strip $t_0 \leqslant t < \infty$, $-\infty < y < \infty$. Show that the solution $y(t)$ of the initial-value problem $y' = f(t,y)$, $y(t_0) = y_0$ exists for all $t \geqslant t_0$.
Solution. Let R be the rectangle $t_0 \leqslant t \leqslant t_0 + a$, $|y - y_0| \leqslant b$. The quantity

$$M = \max_{(t,y) \text{ in } R} |f(t,y)|$$

is at most K. Hence, $y(t)$ exists for

$$t_0 \leqslant t \leqslant t_0 + \min(a, b/K).$$

Now, we can make the quantity $\min(a,b/K)$ as large as desired by choosing a and b sufficiently large. Therefore $y(t)$ exists for $t \geqslant t_0$.

Exercises

1. Construct the Picard iterates for the initial-value problem $y' = 2t(y+1)$, $y(0) = 0$ and show that they converge to the solution $y(t) = e^{t^2} - 1$.
2. Compute the first two Picard iterates for the initial-value problem $y' = t^2 + y^2$, $y(0) = 1$.
3. Compute the first three Picard iterates for the initial-value problem $y' = e^t + y^2$, $y(0) = 0$.

In each of Problems 4–15, show that the solution $y(t)$ of the given initial-value problem exists on the specified interval.

4. $y' = y^2 + \cos t^2, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \frac{1}{2}$
5. $y' = 1 + y + y^2 \cos t, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \frac{1}{3}$
6. $y' = t + y^2, y(0) = 0; 0 \leqslant t \leqslant (\frac{1}{2})^{2/3}$
7. $y' = e^{-t^2} + y^2, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \frac{1}{2}$
8. $y' = e^{-t^2} + y^2, y(1) = 0; 1 \leqslant t \leqslant 1 + \sqrt{e}/2$
9. $y' = e^{-t^2} + y^2, \quad y(0) = 1; \quad 0 \leqslant t \leqslant \dfrac{\sqrt{2}}{1 + (1+\sqrt{2})^2}$
10. $y' = y + e^{-y} + e^{-t}, \quad y(0) = 0; \quad 0 \leqslant t \leqslant 1$
11. $y' = y^3 + e^{-5t}, \quad y(0) = 0.4; \quad 0 \leqslant t \leqslant \frac{3}{10}$
12. $y' = e^{(y-t)^2}, \quad y(0) = 1; \quad 0 \leqslant t \leqslant \dfrac{\sqrt{3}-1}{2} e^{-((1+\sqrt{3})/2)^2}$
13. $y' = (4y + e^{-t^2})e^{2y}, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \dfrac{1}{8\sqrt{e}}$
14. $y' = e^{-t} + \ln(1 + y^2), \quad y(0) = 0; \quad 0 \leqslant t < \infty$
15. $y' = \frac{1}{4}(1 + \cos 4t)y - \frac{1}{800}(1 - \cos 4t)y^2, \quad y(0) = 100; \quad 0 \leqslant t \leqslant 1$
16. Consider the initial-value problem

$$y' = t^2 + y^2, \qquad y(0) = 0, \tag{*}$$

and let R be the rectangle $0 \leqslant t \leqslant a$, $-b \leqslant y \leqslant b$.

(a) Show that the solution $y(t)$ of (*) exists for

$$0 \leqslant t \leqslant \min\left(a, \frac{b}{a^2 + b^2}\right).$$