

# Kombinatoryka & teoria grafów

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## SYLABUS – teoria grafów:

1. Basic concepts: graphs, paths and cycles, complete and bipartite graphs
2. Matchings: Hall's Marriage theorem and its variations
3. Forbidden subgraphs: complete bipartite and  $r$ -partite subgraphs, chromatic numbers, Turán's theorem, asymptotic behaviour of edge density, Erdős-Stone theorem
4. Hamiltonian cycles (Dirac's Theorem), Eulerian circuits
5. Connectivity: connected and  $k$ -connected graphs, Menger's theorem
6. Ramsey theory: edge colourings of graphs, Ramsey's theorem and its variations, asymptotic bounds on Ramsey numbers
7. Planar graphs and colourings: statements of Kuratowski's and Four Colour theorems, proof of Five Colour theorem, graphs on other surfaces and Euler characteristics, chromatic polynomial, edge colourings and Vizing's theorem
8. Random graphs: further asymptotic bounds on Ramsey numbers, Zarankiewicz numbers and their bounds, graphs of large first and high chromatic number, complete subgraphs in random graphs.
9. Algebraic methods: adjacency matrix and its eigenvalues, strongly regular graphs, Moore graphs and their existence.

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# 1 Structural properties

## 1.1 Basic definitions

**Graph** – an ordered pair  $G = (V, E)$ :  
 $\hookrightarrow$  **vertices**  $:= V$  [singular: *vertex*]  
 $\hookrightarrow$  **edges**  $:= E$ ,  $\{v, w\} := vw$

For an edge  $vw$ ,  $v \neq w$  we say that  $v, w$  are its **endpoints** and that it is **incident** to  $v$  (or  $w$ ).

Graphs  $G$  and  $H$  are **isomorphic** ( $G \simeq H$ ) if there exists  $f: V(G) \xrightarrow[1-1]{\text{on}} V(H)$  such that  
 $(\forall v, w \in V(G)) \quad vw \in E(G) \iff f(v)f(w) \in E(H)$   
*Meaning that edges are like an operation on a group of vertices*  
 $G$  is a **subgraph** of  $H$  [ $G \leq H$ ] if  $V(G) \subseteq V(H)$  and  $E(G) \subseteq E(H)$ .  
If  $G$  is **H-free** if it has no subgraphs isomorphic to  $H$ .

A **cycle** of length  $n \geq 3$  [ $C_n$ ] is a graph with vertices  
$$V(C_n) = [n]$$
  
and edges:  
$$E(C_n) = \{i(i+1) : 1 \leq i \leq n-1\} \cup \{1n\}.$$
  
A **path** of length  $n-1$  [ $P_{n-1}$ ] is a graph with vertices  
$$V(P_{n-1}) = [n]$$
  
and edges  
$$E(P_{n-1}) = \{i(i+1) : 1 \leq i \leq n-1\}.$$

An **induced** by  $A \subseteq V(G)$  subgraph of  $G$  is  
 $G[A] = (A, E_A)$   
A **connected component** of  $G$  is a subgraph  $G[W] \leq G$  where  $W \subseteq V$  is an equivalence class under  $\approx$  given by  
$$v \approx w \iff \text{exists a path } v \dots w \text{ in } G$$
  
A graph is **connected** if  $v \approx w$  for every  $v, w \in V$  ( $G$  has at most one connected component).

If  $v$  is a vertex in graph  $G$ , we say that its **neighbourhood** is  $N_G(v) = \{w \in G : vw \in E(G)\}$ . Furthermore, the **degree of**  $v$  is  $|N_G(v)|$ .  
If  $A \subseteq V$ , then  $N(A) := \bigcup_{v \in A} N(v)$ .

Dla krawedzi  $vw$ ,  $v \neq w$  mówimy, że  $v, w$  są jej **koncami** i że jest krawedzia **padająca** na  $v$  (lub  $w$ ).

Grafy  $G$  i  $G$  są **izomorficzne**, jeżeli istnieje  $f: V(G) \xrightarrow[1-1]{\text{na}} V(H)$  takie, że  
 $(\forall v, w \in V(G)) \quad vw \in E(G) \iff f(v)f(w) \in E(H)$   
 $G$  jest **podgrafem**  $H$  [ $G \leq H$ ] jeżeli  $V(G) \subseteq V(H)$  oraz  $E(G) \subseteq E(H)$ .  
 $G$  jest **H-free** (wolny od  $H$ ?), jeżeli nie ma podgrafów izomorficznych z  $H$ .

**Cykl** długości  $n \geq 3$  [ $C_n$ ] to graf z wierzchołkami  
$$V(C_n) = [n]$$
  
i krawędziami:  
$$E(C_n) = \{i(i+1) : 1 \leq i \leq n-1\} \cup \{1n\}.$$
  
**Ścieżka** długości  $n-1$  [ $P_{n-1}$ ] to graf z wierzchołkami  
$$V(P_{n-1}) = [n]$$
  
i krawędziami  
$$E(P_{n-1}) = \{i(i+1) : 1 \leq i \leq n-1\}.$$

We define:

$$\hookrightarrow \text{minimal degree } \delta(G) = \min_{v \in G} d(v)$$

$$\hookrightarrow \text{maximal degree } \Delta(G) = \max_{v \in G} d(v)$$

$$\hookrightarrow \text{average degree } d(G) = \frac{\sum d(v)}{|G|}.$$

If there exists an  $r \geq 0$  such that

$$\delta(G) = \Delta(G) = d(G) = r$$

then we say that the graph is **r-regular** or, more generally, it is **regular** for some  $r$ .

**Handshaking Lemma:** for any graph  $G$  we have  $e(G) = \frac{1}{2} \sum d(v) = \frac{|G|}{2} d(G)$



## 1.2 Hall's Marriage Theorem

Graph  $G$  is **bipartite** with vertex classes  $U$  and  $W$  if  $V = U \cup W$  so that every edge has form  $uw$  for some  $u \in U$  and  $w \in W$ .

$G$  is bipartite iff it has no cycles of odd length.

Graf  $G$  jest **dwudzielny** z klasami wierzchołków  $U$  i  $W$ , jeśli  $V = U \cup W$  takimi, że każda krawędź jest formy  $uw$  dla pewnych  $u \in U$  oraz  $w \in W$ .

$G$  jest dwudzielny wtw kiedy nie ma cykli o nieparzystej długości.

[  ] [  ]

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$\Rightarrow$

Let  $U, W$  be the vertex classes and  $v_1, v_2, \dots, v_n, v_1$  be a cycle in  $G$ . WLOG suppose that  $v_1 \in U$ . Then  $v_2 \in W$  etc. Specifically we have  $v_i \in U$  if  $i$  is odd and  $v_i \in W$  if  $i$  is even. Then, we have  $v_n v_1$ , so  $n$  must be even.

$\Leftarrow$

Suppose  $G$  has no cycles of odd length. WLOG, assume that  $V(G) \neq \emptyset$  and that  $G$  is connected, because  $G$  will be bipartite if all its connected components are bipartite. Fix  $v \in G$  and for all other  $w \in G$  define distance  $\text{dist}(v, w)$  as the smallest  $n \geq 0$  such that there exists a path  $v \dots w$  in  $G$  of length  $n$ .

Now, let  $V_n := \{w \in G : \text{dist}(v, w) = n\}$  and set

$$U = V_0 \cup V_2 \cup V_4 \cup \dots$$

$$W = V_1 \cup V_3 \cup V_5 \cup \dots$$

We want to show that there are no edges in  $G$  of the form  $v'v''$  where  $v', v'' \in U$  or  $v', v'' \in W$ . Suppose that  $v'v'' \in E(G)$  with  $v' \in V_m, v'' \in V_n$  and  $m \leq n$ . Then, we have a path

$$v \dots v'v'' \in G$$

of length  $m+1$ , implying that

$$n \in \{m, m+1\}.$$

Suppose that  $n = m$ . Let  $v'_0 v'_1 \dots v'_m$  and  $v''_0 v''_1 \dots v''_m$  be paths in  $G$  with  $v = v'_0 = v''_0$ ,  $v' = v'_m$  and  $v'' = v''_m$ . Note that  $v'_i, v''_i \in V_i$  for  $0 \leq i \leq m$ . Let  $k \geq 0$  be largest such that

$$v'_k = v''_k$$

and note that  $k \leq m-1$  as  $v' \neq v''$ . Then

$$v'_k v'_{k+1} \dots v'_m v''_m v''_{m-1} \dots v''_k$$

is a cycle of odd length, which is a contradiction.

Therefore, we can only have  $n = m+1$  and then exactly one of  $n, m$  is even meaning that exactly one of  $v'$  and  $v''$  is in  $U$  as required for  $G$  to be bipartite.

[  ]

⇒

Niech  $U, W$  będą klasami wierzchołków oraz niech  $v_1, v_2, \dots, v_n, v_1$  niech będzie cyklem w  $G$ . BSO założymy, że  $v_1 \in U$ . W takim razie,  $v_2 \in W$  etc. W szczególności, mamy  $v_i \in U$  jeżeli  $i$  jest nieparzyste oraz  $v_i \in W$  jeżeli  $i$  jest parzyste. W takim razie, skoro  $v_n v_1$ , to  $n$  musi być parzyste.

⇐

Założymy, że  $G$  nie ma cykli o nieparzystej długości. BSO założymy, że  $V(G) \neq \emptyset$  i że  $G$  jest spójny, ponieważ  $G$  będzie dwudzielny, wtw gdy wszystkie jego składowe spójne (????) będą dwudzielne. Ustalmy  $v \in G$  i dla każdego innego  $w \in G$  zdefiniujmy dystans  $\text{dist}(v, w)$  jako najmniejsze  $n \geq 0$  takie, że istnieje ścieżka  $v \dots w$  w  $G$  o długości  $n$ .

Niech  $V_n := \{w \in G : \text{dist}(v, w) = n\}$  i zbiory

$$U = V_0 \cup V_2 \cup V_4 \cup \dots$$

$$W = V_1 \cup V_3 \cup V_5 \cup \dots$$

Chcemy pokazać, że nie istnieją w  $G$  krawędzie postaci  $v'v''$ , gdzie  $v', v'' \in U$  lub  $v', v'' \in W$ .

Założymy, że  $v'v'' \in E(G)$  z  $v' \in V_m, v'' \in V_n$  oraz  $m \leq n$ . Wtedy istnieje ścieżka

$$v \dots v'v'' \in G$$

długości  $m+1$ , co implikuje, że

$$n \in \{m, m+1\}.$$

Założymy, że  $n = m$ . Niech  $v'_0 v'_1 \dots v'_m$  oraz  $v''_0 v''_1 \dots v''_m$  są ścieżkami w  $G$  takimi, że  $v = v'_0 v''_0$ ,  $v' = v'_m$  oraz  $v'' = v''_m$ . Zauważmy, że  $v'_i, v''_i \in V_i$  dla  $0 \leq i \leq m$ . Niech  $k \geq 0$  będzie największe takie, że

$$v'_k = v''_k$$

i zauważmy, że  $k \leq m-1$  ponieważ  $v' \neq v''$ . Wtedy

$$v'_k v'_{k+1} \dots v'_m v''_m v''_{m-1} \dots v''_k$$

jest cyklem o nieparzystej długości, co daje nam sprzeczność.

W takim razie, możemy mieć tylko  $n = m+1$  i wtedy dokładnie jedno z  $n, m$  może być parzyste, co daje nam dokładnie jedno z  $v'$  i  $v''$  w  $U$  tak, jak jest wymagane żeby to był graf dwudzielny.

If  $G$  is a bipartite graph with  $V = W \cup M$  and  $W' \subseteq W$ , a **partial matching** in  $G$  from  $W'$  to  $M$  is

$$\{wv_w : w \in W'\} \subseteq E(G)$$

for some  $v_w \in M$  such that  $w \neq w' \implies v_w \neq v_{w'}$ . A partial matching from  $W$  to  $M$  is called a **matching**.

Sufficient condition:

$$|N(A)| \geq |A| \quad (\text{☕})$$

for every  $A \subseteq W$

.....  
A bipartite graph  $G$  contains a matching from  $W$  to  $M$  iff  $(G, W)$  satisfies Hall's condition (☕).

Jesli  $G$  jest grafem dwudzielnym z  $V = W \cup M$  oraz  $W' \subseteq W$ , wtedy **czesciowe skojarzenie** w  $G$  z  $W'$  do  $M$  to

$$\{wv_w : w \in W'\} \subseteq E(G)$$




dla pewnych  $v_w \in M$  takich, że  $w \neq w' \implies v_w \neq v_{w'}$ . Czesciowe kojarzenie z  $W$  do  $M$  jest nazywane **kojarzeniem**.

Wystarczajacy warunek:

$$|N(A)| \geq |A| \quad (\text{☕})$$

dla kazdego  $A \subseteq W$

.....  
Dwudzielny graf  $G$  zawiera kojarzeniem iff gdy  $(G, W)$  zadowala warunek Halla (☕).

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⇒

Trivial.

←

Using induction on  $|W|$ . For  $|W| = 0, 1$  it is trivial.

We gonna break it into parts.

Suppose that  $|N(A)| \geq |A|$  for every non-empty subset  $A \subseteq W$ . Take any  $w \in W$  and  $v \in N(w)$  and construct a new graph

$$G_0 = G - \{w, v\}.$$

For any non-empty  $B \subseteq W - \{w\}$  we have

$$N_{G_0}(B) = N_G(B) - \{v\}$$

and therefore

$$|N_{G_0}(B)| \geq |N_G(B)| - 1 \geq |B|$$

and so  $(G_0, W - \{w\})$  satisfies Hall's condition. From induction we have a matching  $P$  in  $G_0$  from  $W - \{w\}$  to  $M - \{v\}$  and so  $P \cup \{wv\}$  is a matching from  $W$  to  $M$ .