ZAD 1.

$$w(x) = \frac{1}{2}c_0T_0(x) + c_1T_1(x) + ... + c_nT_n(x)$$

$$B_{n+2} := B_{n+1} := 0$$
$$B_k := 2xB_{k+1} - B_{k+2} + c_k$$

wtedy $w(x) = \frac{1}{2}(B_0 - B_2)$.

Wiemy, ze

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

Indukcja po n? Dla $n = 2$ mamy

$$w(x) = \frac{1}{2}c_0T_0(x) + c_1T_1(x) + c_2T_2(x) = \frac{1}{2}c_0 + c_1x + c_2(2x^2 - 1)$$

$$B_4 = B_3 = 0$$
$$B_2 = 2xB_3 - B_4 + c_2 = c_2$$
$$B_1 = 2xB_2 - B_3 + c_1 = 2xc_2 + c_1$$
$$B_0 = 2xB_1 - B_2 + c_0 = 4x^2c_2 + 2xc_1 - c_2 + c_0$$
$$w(x) = \frac{1}{2}(B_0 - B_2) = \frac{1}{2}(4x^2c_2 + 2xc_1 - c_2 + c_0 - c_2) = 2x^2c_2 + xc_1 - c_2 + \frac{1}{2}c_0$$

więc śmiga.
Załóżmu indukcyjnie, że algorytm działa dla dowolnego algorytmu zawierającego $T_0(x), ..., T_n(x)$.
Pokażemy, że działa wtedy też dla wielomianu z doklejonym $T_{n+1}(x)$.

$$\begin{aligned}
w(x) &= \frac{1}{2}T_0(x) + ... + c_nT_n(x) + c_{n+1}T_{n+1}(x) = \\
&= \frac{1}{2}T_0(x) + ... + c_nT_n(x) + c_{n+1}(2xT_n(x) - T_{n-1}(x)) = \\
&= \frac{1}{2}T_0(x) + ... + T_{n-1}(x)(c_{n-1} - c_{n+1}) + T_n(c_n + 2xc_{n+1})
\end{aligned}$$

Taki wielomian z założenia indukcyjnego można rozwiązać za pomocą algorytmu, więc mamy

$$\begin{aligned}
B_{n+2} &= B_{n+1} = 0 \\
B_n &= 2xB_{n+1} - B_{n+2} + c_n + 2xc_n = c_n + 2xc_{n+1} \\
B_{n-1} &= 2xB_n - B_{n+1} + c_{n-1} - c_{n+1} = 4x^2c_{n+1} + 2xc_n + c_{n-1} - c_{n+1} \\
B_{n-2} &= 2xB_{n-1} - B_nc_{n-2}...
\end{aligned}$$

Rozważmy więc nowy ciąg, C, zdefiniowany rekurencyjnie:

$$\begin{aligned}
C_{n+3} &= C_{n+2} = 0 \\
C_{n+1} &= c_{n+1} \\
C_n &= 2xc_{n+1} + c_n = B_n \\
C_{n-1} &= 2xC_n - C_{n+1} = 4x^2c_{n+1} + 2xc_n - c_{n+1} + c_{n-1} = B_{n-1} \\
C_k &= 2xC_{k+1} - C_{k+2} + c_k
\end{aligned}$$

Ponieważ C_n i C_{n-1} odpowiadają B_n i B_{n-1} i oba ciągi mają tę samą definicję rekurencyjną, to są sobie równe od n w dół. Skoro C to algorytm dla w(x) w całość, to

$$w(x) = \frac{1}{2}(C_0 - C_2)$$

i koniec.

ZAD. 3

$$H_{2n+1}(x_i) = f(x_i)$$

Tutaj zauwazamy, ze drugi wyraz sumy zeruje sie dla kazdego x_k, bo

$$\sum_{k=0}^{n} f'(x_k)\bar{h}_k(x_i) = \sum_{k=0}^{n} f'(x_k)(x_i - x_k)\frac{(x_i - x_0)\ldots(x_i - x_i)\ldots(x_i - x_n)}{(x_i - x_k)p'_{n+1}(x_k)} = 0$$

Teraz peirwszy wyraz, on chcialabym zeby sie uproscil do $f(x_i)$.

$$h_k(x) = [1 - 2(x - x_k)\lambda'_k(x_k)]\lambda_k^2(x_k)$$

Dla $k \neq i$ mamy

$$h_k(x_i) = [\ldots]\lambda_i^2(x_i) = [\ldots]\frac{(x_i - x_0)\ldots(x_i - x_{k-1})(x_i - x_{k+1})\ldots(x_i - x_n)}{p'_{n+1}(x_i)} = [\ldots]\cdot 0 = 0$$

wiec wszystko poza $f(x_i)h_i(x_i)$ sie zeruje (winko rowniez).

Chcemy teraz sprawdzic, czy $h_i(x_i) = 1$

$$\begin{aligned} h_i(x_i) &= [1 - 2(x_i - x_i)\lambda'_i(x_i)]\lambda_i^2(x_i) = \\ &= [1 - 0]\frac{(x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}{\sum_{l=0}^{n}\prod_{j=0, j\neq i}^{n}(x_i - x_j)} = \\ &= \frac{(x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}{(x_i - x_0)\ldots(x_i - x_i)\ldots(x_i - x_n) + \ldots + (x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)} = 1. \end{aligned}$$

$$H'_{2n+1}(x_i) = f'(x_i)$$

$$H'_{2n+1}(x) = \sum_{k=0}^{n} f(x_k)h'_k(x) + \sum_{k=0}^{n} f'(x_k)\bar{h}'(x)$$

Wpierw warto pokazac, ze

$$h'_k(x_i) = [1 - 2(x_i - x_k)\lambda'_k(x_k)]2\lambda_k(x_i)\lambda'_k(x_i) - 2\lambda'_k(x_k)\lambda_k^2(x_i)$$

Gdy $k = i$, to wtedy

$$\begin{aligned} h'_i(x_i) &= [1 - 0]2\lambda_i(x_i)\lambda_i(x_i) - 2\lambda'_i(x_i)\lambda_i^2(x_i) = \\ &= 2\lambda'_i(x_i)\lambda_i(x_i)[1 - \lambda_i(x_i)] = \\ &= (\text{costam})[1 - 1] = 0 \end{aligned}$$

w przeciwnym wypadku

$$\begin{aligned} h_i(x_i) &= [1 - 2(x_i - x_k)\lambda'_k(x_k)]2\lambda_k(x_i)\lambda'_k(x_i) - 2\lambda'_k(x_k)\lambda_k^2(x_i) = \\ &= 2\lambda_k(x_i)[\ldots] = 2\cdot 0 = 0 \end{aligned}$$

$$\bar{h}'_k(x) = \lambda_k^2(x) + 2\lambda_k(x)\lambda'_k(x)(x - x_k)$$

dla $k = i$ mamy

$$\bar{h}'_i(x_i) = \lambda_i^2(x_i) + 2\lambda_i(x_i)\lambda'_i(x_i)(x_i - x_i) = \lambda_i^2(x_i) = 1$$

dla $k \neq i$ mamy

$$\begin{aligned} \bar{h}'_k(x_i) &= \lambda_k^2(x_i) + 2\lambda_k(x_i)\lambda'_k(x_i)(x_i - x_k) = \\ &= 0 + 2\cdot 0\cdot\ldots = 0. \end{aligned}$$

ZAD. 5

Tabelka w ramach pomocy:

x_0		x_0		x_1		x_1		x_2		x_2
7		7		6		6		22		22
	-1		-1		0		16		56	
		0		1		16		40		
			1		$\frac{15}{2}$		24			
				$\frac{13}{4}$		$\frac{33}{4}$				
					$\frac{5}{2}$					

$$p(x) = 7 - (x+1) + x(x+1)^2 + \frac{13}{4}x^2(x+1)^2 + \frac{5}{2}x^2(x+1)^2(x-1)$$

$$p'(x) = \frac{1}{2}x(25x^3 + 46x^2 + 30x + 11)$$

ZAD. 7

Chcemy pokazac, ze

$$\int_a^b [s''(x)]^2 dx = \sum_{k=1}^{n-1} s''(x_k)(f[x_k, x_{k+1}] - f[x_{k-1}, x_k])$$

Zacznijmy od calki, calka na przedziale jest suma calek na podprzedzialach, czyli

$$\begin{aligned}
\int_{x_0}^{x_n} [s''(x)]^2 dx &= \sum \int_{x_{k-1}}^{x_k} [s''(x)]^2 dx = \\
&= \sum \int_{x_{k-1}}^{x_k} s''(x)s''(x)dx = \\
&= \sum \begin{bmatrix} u = s''(x) & du = s''(x) \\ v = s'(x) & dv = s''(x) \end{bmatrix} = \\
&= \sum [s''(x_k)s'(x_k) - s''(x_{k-1})s'(x_{k-1}) - \int_{x_{k-1}}^{x_k} s'(x)(x_k - x_{k-1})^{-1}(s''(x_k) - s''(x_{k-1}))] = \\
&= \sum [s''(x_k)s'(x_k) - s''(x_{k-1})s'(x_{k-1})] - \sum (x_k - x_{k-1})^{-1} \int_{x_{k-1}}^{x_k} s'(x)(s''(x_k) - s''(x_{k-1}))dx = \\
&= s''(x_1)s'(x) - s''(x_0)s'(x_0) + s''(x_2)s'(x_2) - s''(x_1)s'(x_1) + \ldots + \\
&- \sum (x_k - x_{k-1})^{-1}(s''(x_k) - s''(x_{k-1})) \int_{x_{k-1}}^{x_k} s'(x)dx = \\
&= -\sum (x_k - x_{k-1})^{-1}(s''(x_k) - s''(x_{k-1}))(s(x_k) - s(x_{k-1})) = \\
&= -(x_1 - x_0)^{-1}(s''(x_1) - s''(x_0))(s(x_1) - s(x_0)) - (x_2 - x_1)^{-1}(s''(x_2) - s''(x_1))(s(x_2) - s(x_1)) - \ldots + \\
&- (x_n - x_{n-1})^{-1}(s''(x_n) - s''(x_{n-1}))(s(x_n) - s(x_{n-1})) = \\
&= s''(x_1)\left(\frac{s(x_2) - s(x_1)}{x_2 - x_1} - \frac{s(x_1) - s(x_0)}{x_1 - x_0}\right) + \ldots + s''(x_{n-1})\left(\frac{s(x_n) - s(x_{n-1})}{x_n - x_{n-1}} - \frac{s(x_{n-1}) - s(x_{n-2})}{x_{n-1} - x_{n-2}}\right) = \\
&= \sum_{k=1}^{n-1} s''(x_k)[\frac{s(x_{k+1}) - s(x_k)}{x_{k+1} - x_k} - \frac{s(x_k) - s(x_{k-1})}{x_k - x_{k-1}}] = \sum_{k=1}^{n-1} M_k(f[x_k, x_{k+1}] - f[x_k, x_{k-1}])
\end{aligned}$$

ZAD. 8

$$\int_a^b s(x)dx = h\sum_{k=0}^{n}{}'' f(x_k) - \frac{h^3}{12}\sum_{k=0}^{n}{}'' M_k$$

$$\int M_{k-1}(x_k - x)^3 dx = M_{k-1}\int (x_k - x)^3 dx =$$

$$M_{k-1}\frac{-(x_k - x)^4}{4} =$$

$$= M_{k-1}\frac{(x_k - x_{k-1})^4}{4} =$$

$$= M_{k-1}\frac{(a + kh - a - (k-1)h)^4}{4} =$$

$$= M_{k-1}\frac{h^4}{4}$$

$$\int M_k(x - x_{k-1})^3 dx = M_k\int (x - x_{k-1})^3 =$$

$$= M_k\frac{(x - x_{k-1})^4}{4} =$$

$$= M_k\frac{(x_k - x_{k-1})^4}{4} =$$

$$= M_k\frac{(a + kh - a - (k-1)h)^4}{4} =$$

$$= M_k\frac{h^4}{4}$$

$$\int (6f(x_{k-1}) - M_{k-1}(x_k - x_{k-1})^2)(x_k - x) = (6f(x_{k-1}) - M_{k-1}h^2)\int (x_k - x)dx =$$

$$= (6f(x_{k-1}) - M_{k-1}h^2)\frac{-(x_k - x)^2}{2} =$$

$$= (6f(x_{k-1}) - M_{k-1}h^2)\frac{h^2}{2}$$

$$\int (6f(x_k) - M_k h^2)(x - x_{k-1})dx = (6f(x_k) - M_k h^2)\frac{(x - x_{k-1})^2}{2} =$$

$$= (6f(x_k) - M_k h^2)\frac{h^2}{2}$$

$$\int_a^b s(x)dx = \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} s(x)dx =$$

$$= \sum_{k=1}^{n} h^{-1}\frac{1}{6}[M_{k-1}\frac{h^4}{4} + M_k\frac{h^4}{4} + 6f(x_{k-1} - M_{k-1}h^2)\frac{h^2}{2} + (6f(x_k) - M_k h^2)\frac{h^2}{2}] =$$

$$= \sum_{k=1}^{n} h^{-1}\frac{1}{6}[-M_{k-1}\frac{h^4}{4} - M_k\frac{h^4}{4} + 3(f(x_{k-1}) + f(x_k))h^2] =$$

$$= \frac{1}{2}f(x_0)h + \frac{1}{2}f(x_1)h + \frac{1}{2}f(x_1)h + \frac{1}{2}f(x_2)h + \ldots + \frac{1}{2}f(x_{n-1})h + \frac{1}{2}f(x_n)h +$$

$$- \frac{1}{24}M_0h^3 - \frac{1}{24}M_1h^3 - \frac{1}{24}M_1h^3 - \frac{1}{24}M_2h^3 - \ldots - \frac{1}{24}M_{n-1}h^3 - \frac{1}{24}M_nh^3 =$$

$$= \frac{1}{2}f(x_0)h + h\sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}hf(x_n) - \frac{1}{24}M_0h^3 - \frac{1}{12}h^3\sum_{k=1}^{n-1} M_k - \frac{1}{24}M_nh^3$$