Problem List 5 (Random graphs)

Graph Theory, Winter Semester 2022/23, IM UWr

1.$^\circ$ By colouring the vertices of a graph G red/blue independently at random, show that $V(G)$ has a partition $V(G) = V_1 \sqcup V_2$ such that $e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G)$. Give also a constructive proof of the same fact.

2.$^\circ$ Let G be a graph with $|G| = n$ and $e(G) = m \geq 4n$. Show that the crossing number of G (see Problem 4.4) satisfies $\mathrm{cr}(G) > \frac{m^3}{64n^2}$.
[*Hint: draw G on a plane with $\mathrm{cr}(G)$ crossings, and apply Problem 4.4(b) to $G[W]$, where $W \subseteq V(G)$ is a random subset containing each vertex with probability $\frac{4n}{m}$.*]

3.$^\circ$ Show that for any $s, t \geq 2$ we have $R(s,t) \geq n - \binom{n}{s} p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for all $n \in \mathbb{N}$ and $p \in (0,1)$. By choosing $n = n(t)$ appropriately and taking $p = n^{-2/3}$, deduce that $R(4,t) = \Omega\left(\left(\frac{t}{\ln t}\right)^{3/2}\right)$.
[*Hint: by Stirling's Formula, $\ln(t!) = t\ln t + O(t)$; also, $\ln(1-p) < -p$ for $p \in (0,1)$.*]

4. Let $n \in \mathbb{N}$ players participate in a *tournament*, where each pair of players play a game and one of them beats the other (there are no draws).

   (a)$^\circ$ For every $k \geq 1$, prove that there exists a tournament in which for every k players, some other player beats all k of them.

   (b)$^\circ$ Construct such a tournament explicitly for $k = 2$.
   [*Hint: consider a tournament in which the players play rock-paper-scissors.*]

5.$^+$ Find a threshold function for $G \in \mathcal{G}(n,p)$ to contain a path of length 2.

6.$^\circ$ A vertex v of a graph G is said to be *isolated* if $d_G(v) = 0$. Show that $\frac{\ln n}{n}$ is a threshold function for $G \in \mathcal{G}(n,p)$ to have no isolated vertices.

7. We define the *Rado graph* R as an infinite graph with $V(R) = \mathbb{Z}_{\geq 0} = \{0, 1, \ldots\}$, so that given non-negative integers $x < y$, we have $x \sim_R y$ if and only if $\lfloor \frac{y}{2^x} \rfloor$ is odd.

   (a)$^-$ Show that R satisfies the *extension property*: given any finite disjoint subsets $U, W \subset V(R)$, there exists $v \in V(R) \setminus (U \cup W)$ such that $v \sim u$ for all $u \in U$ and $v \nsim w$ for all $w \in W$.

   (b)$^+$ Let G, H be two infinite graphs, with $V(G)$ and $V(H)$ countable, satisfying the extension property. Show that $G \cong H$.

   (c)$^\circ$ Given $p \in (0,1)$, let $\mathcal{G}(\infty, p)$ be the probability space of all infinite graphs G with vertex set $\mathbb{Z}_{\geq 0}$, with each edge appearing independently at random with probability p. Show that $G \in \mathcal{G}(\infty, p)$ is isomorphic to R with probability one.
   [*Hint: recall that probability measures are countably additive; you can assume without proof that all probabilities you are computing are well-defined.*]