

## Problem List 3 (Ramsey theory)

GRAPH THEORY, WINTER SEMESTER 2022/23, IM UWR

1. (a) By constructing a suitable colouring of  $K_8$ , show that  $R(3, 4) \geq 9$ .  
 (b) Show that if  $R(s-1, t)$  and  $R(s, t-1)$  are both even, then we actually have  $R(s, t) \leq R(s-1, t) + R(s, t-1) - 1$ .  
 (c) Deduce that  $R(3, 4) = 9$ .
2. Consider the red/blue colouring of  $K_{17}$  such that the edge  $ij$  is coloured red if  $i - j \equiv \pm 1, \pm 2, \pm 4$  or  $\pm 8 \pmod{17}$ , and blue otherwise. Show that with this colouring  $K_{17}$  has no monochromatic  $K_4$ , and therefore  $R(4, 4) > 17$ .  
*[Hint: in order to simplify your proof, start by showing that the map  $[17] \rightarrow [17]$  sending  $i$  to  $3i$  (modulo 17) swaps the red edges with the blue edges.]*
3. Show that  $R(s, t) \leq \binom{s+t-2}{s-1}$  and that  $R(s, t) \leq 2^{s+t-3}$  for all  $s, t \geq 2$ . Deduce that  $R(s, s) = O(4^s)$ .
4. Given two graphs  $G$  and  $H$ , we write  $R(G, H)$  for the smallest  $n \geq 2$  such that any red/blue colouring of  $K_n$  has either a red subgraph isomorphic to  $G$ , or a blue subgraph isomorphic to  $H$ .  
 (a) Why does  $R(G, H)$  exist?  
 (b) Show that  $R(K_{1,t}, K_{r+1}) = rt + 1$  for all  $r, t \geq 1$ .  
*[Hint: use Turán's Theorem.]*  
 (c) For  $k \geq 1$ , let  $I_k$  be the “set of  $k$  disjoint edges”—that is, a graph with  $V(I_k) = \{v_i \mid i \in [k]\} \sqcup \{w_i \mid i \in [k]\}$  and  $E(I_k) = \{v_i w_i \mid i \in [k]\}$ . Show that  $R(I_k, K_r) = 2k + r - 2$  for all  $k \geq 1$  and  $r \geq 2$ .
5. Let  $k \geq 1$ .  
 (a) Show that every red/blue colouring of the edges of  $K_{2k-1, 2k-1}$  contains a monochromatic tree of order  $2k$  with two vertices of degree  $k$ .  
 (b) Give a red/blue colouring of the edges of  $K_{2k, 2k}$  with no connected monochromatic subgraphs of order  $2k + 1$ .
6. Given  $k, s \geq 2$ , we write  $R_k(s)$  for the Ramsey number  $R(\overbrace{s, \dots, s}^k)$ .  
 (a) By exhibiting a suitable colouring of  $K_{(s-1)^2}$ , show that  $R(s, s) = \Omega(s^2)$ .  
 (b) Show that  $R_k(s) = \Omega(s^k)$  for any fixed  $k \geq 2$ .

7. (a) Show that  $R_k(3) \leq k \cdot R_{k-1}(3)$  for all  $k \geq 3$ , and deduce that  $R_k(3) \leq 3 \cdot k!$  for all  $k \geq 2$ .
- (b) Let  $x_1, \dots, x_n \in \mathbb{R}^2$  be points such that no three of them lie on a straight line, where  $n = 3 \cdot k!$  for some  $k \geq 2$ . Show that some three of these points form an angle  $> \pi \left(1 - \frac{1}{k}\right)$ .
- (c) Show that if  $G_1, \dots, G_k \leq K_n$  are bipartite subgraphs and  $E(K_n) = \bigcup_{i=1}^k E(G_i)$ , then  $n \leq 2^k$ . Deduce that in the previous part of the problem we could take  $n = 2^k + 1$  instead of  $n = 3 \cdot k!$ .
8. Show that every sequence  $(x_n)_{n=1}^\infty$  of real numbers has a monotone (that is, non-increasing or non-decreasing) subsequence.
9. Let  $g_1, \dots, g_n: \mathbb{R} \rightarrow \mathbb{R}$  be bounded functions, and let  $\varepsilon, \delta > 0$ . Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is such that for all  $x, y \in \mathbb{R}$  with  $|f(x) - f(y)| > \delta$  we have  $|g_i(x) - g_i(y)| > \varepsilon$  for some  $i$ . Show that  $f$  is bounded.
10. (*Ramsey's Theorem for hypergraphs*). Let  $k \geq 2$ .
- (a) Let  $s, t \geq k$ . Show that there exists  $n \geq k$  with the following property: given any red/blue colouring of all  $k$ -element subsets of a set  $V$  with  $|V| = n$ , there exists either a subset  $W_1 \subseteq V$  with  $|W_1| = s$  all of whose  $k$ -element subsets are red, or a subset  $W_2 \subseteq V$  with  $|W_2| = t$  all of whose  $k$ -element subsets are blue.  
*[Hint: letting  $R^{(k)}(s, t)$  be the smallest such integer  $n$ , show that we have  $R^{(k)}(s, t) \leq R^{(k-1)}(R^{(k)}(s-1, t), R^{(k)}(s, t-1)) + 1$ .]*
- (b) Show that for any red/blue colouring of all  $k$ -element subsets of  $\mathbb{N}$ , there exists an infinite subset  $W \subseteq \mathbb{N}$  all of whose  $k$ -element subsets have the same colour.
- (c) Deduce that for any infinite subset  $A \subseteq \mathbb{R}^2$ , there exists an infinite subset  $B \subseteq A$  such that either  $B$  is contained in a line, or no three points in  $B$  are collinear.
- (d) Suppose we have a red/blue colouring of all infinite subsets of  $\mathbb{N}$ . Is there always an infinite subset  $W \subseteq \mathbb{N}$  all of whose infinite subsets have the same colour?  
*[Hint: could we have such a red/blue colouring so that  $A$  and  $A \cup \{n\}$  have different colours whenever  $n \notin A$ ? Feel free to use the Axiom of Choice.]*