Co mam: 1, 4, 6, 8 (4)
MAX: 10

ZAD. 1

Chcemy udowodnić, że

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_k(x)T_l(x)dx = \begin{cases} 0 & k \neq l \\ \pi & k = l = 0 \\ \frac{\pi}{2} & k = l \neq 0 \end{cases}$$

Po pierwsze zauważmy, że

$$\int_{-\pi}^{\pi}\cos(k\theta)\cos(l\theta)d\theta = \begin{cases} 0 & k \neq l \\ 2\pi & k = l = 0 \\ \pi & \end{cases}$$

Korzystając z zależności

$$\frac{1}{2}(\cos(x-y)+\cos(x+y)) = \frac{1}{2}(\cos(x)\cos(y)-\sin(x)\sin(y)+\cos(x)\cos(y)+\sin(x)\sin(y)) =$$
$$= \frac{1}{2}(2\cos(x)\cos(y)) = \cos(x)\cos(y)$$

Jeśli $k = l = 0$, to mamy

$$\int_{-\pi}^{\pi}\cos^2(0\cdot\theta)d\theta = \int_{-\pi}^{\pi}1d\theta = \pi-(-\pi) = 2\pi$$

natomiast jeśli $k = l \neq 0$, to jest

$$\int_{-\pi}^{\pi}\cos^2(k\theta)d\theta = \frac{2k(\pi+\pi)+\sin(2k\pi)-\sin(-2k\pi)}{4k} = \frac{4k\pi}{4k} = \pi$$

Jeśli $k \neq l$:

$$\int_{-\pi}^{\pi}\cos(k\theta)\cos(l\theta)d\theta = \int_{-\pi}^{\pi}\frac{1}{2}(\cos((k-l)\theta)+\cos((k+l)\theta))d\theta =$$
$$= \frac{1}{2}\int_{-\pi}^{\pi}\cos((k-l)\theta)d\theta + \frac{1}{2}\int_{-\pi}^{\pi}\cos((k+l)\theta)d\theta =$$
$$= \frac{1}{2}\frac{\sin((k-l)\theta)}{k-l} + \frac{1}{2}\frac{\sin((k+l)\theta)}{k-l} =$$
$$= \frac{1}{2}\left(\frac{\sin((k-l)\pi)-\sin((l-k)\pi)}{k-l} + \frac{\sin((k+l)\pi)-\sin((-k-l)\pi)}{k+l}\right) =$$
$$= \frac{\sin(\pi\frac{k-l+l-k}{2})\cos(\pi\frac{k-l+l-k}{2})}{k-l} + \frac{\sin(\pi\frac{k+l-k-l}{2})\cos(\pi\frac{k+l-k-l}{2})}{k+l} =$$
$$= 0 + 0 = 0$$

Wracając do Czebyszewa, wiemy, że $T_n(\cos(\theta)) = \cos(n\theta)$. W takim razie

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_k(x)T_l(x)dx = \begin{bmatrix} x = \cos(\theta) \\ dx = -\sin(\theta)d\theta = -\sqrt{1-\cos^2(\theta)}d\theta \end{bmatrix} = -\int_{\pi}^{0}\cos(k\theta)\cos(l\theta)d\theta =$$
$$= \int_{0}^{\pi}\cos(k\theta)\cos(l\theta)d\theta$$

A ponieważ $\cos(x)$ jest funkcją parzystą, to

$$\int_{-\pi}^{\pi}\cos(k\theta)\cos(l\theta)d\theta = 2\int_{0}^{\pi}\cos(k\theta)\cos(l\theta)d\theta$$

ZAD. 4.

$$\Big\|\sum_{j=0}^{n} c_j f_j\Big\|_2^2 = \sum_{j=0}^{n} |c_j|^2 \|f_j\|_2^2$$

Norma:

$$\|f\|_2 = \left(\int_{-1}^{1} f^2(x)(1-x^2)^{-\frac{1}{2}}dx\right)^{\frac{1}{2}}$$

$$\|\sum_{j=0}^{n} c_j f_j\|_2^2 = \langle\sum_{j=0}^{n} c_j f_j, \sum_{i=0}^{n} c_i f_i\rangle = \sum_{j=0}^{n} |c_j|\langle f_j, \sum_{i=0}^{n} c_i f_j\rangle =$$
$$= \sum_{j=0}^{n} |c_j| \sum_{i=0}^{n} |\overline{c_i}|\langle f_j, f_i\rangle = \sum_{j=0}^{n}\sum_{i=0}^{n} |c_j||\overline{c_i}|\langle f_j, f_i\rangle =$$
$$= \sum_{j=0}^{n} |c_j|^2\langle f_j, f_j\rangle = \sum_{j=0}^{n} |c_j|^2 \|f_j\|_2^2$$

ZAD. 6.

Wiem, że $\langle f_i, f_j\rangle = 0$ dla każdego $i \neq j$. Chce pokazać, że

$$\sum_{i=0}^{m} a_i f_i = 0 \iff a_i = 0$$

Załóżmy nie wprost, że co najmniej jedno $a_k \neq 0$. Weźmy to k i wtedy:

$$0 = \langle 0, f_k\rangle = \langle\sum_{i=0}^{m} a_i f_i, f_k\rangle = \sum_{i=0}^{n} |a_i|\langle f_i, f_k\rangle = |a_k|\langle f_k, f_k\rangle = |a_k|\|f_k\|^2 \neq 0$$

co daje sprzeczność.

ZAD. 8.

T	0	10	20	30	40	80	90	95
S	68	67.1	66.4	65.6	64.6	61.8	61.0	60

Średnia wartość x to $\overline{x} = 45.625$, wartość średnia y - $\overline{y} = 64.3125$, średnie nachylenie:

$$\overline{a} = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(x_i - \overline{x})^2} = -0.07993035770813546$$

I chcemy, żeby to przechodziło przez średni punkt:

$$64.3125 = -0.07993 \cdot 45.625 + b \implies b = 67.9593$$

a więc

$$S = -0.07993T + 67.9593$$