Problem List 4 (Drawings and colourings)

Graph Theory, Winter Semester 2022/23, IM UWr

1.$^\circ$ Which Turán graphs $T_r(n)$ for $n \geq r \geq 2$ are planar? For each planar one, draw it. [*Hint: exactly* 10 *of them are planar.*]

2.$^+$ Show that there exist exactly 5 (up to isomorphism) planar regular graphs with at least 3 faces that have a drawing in which each face has the same number of edges.

3.$^\circ$ Fix $n \geq 3$. Show that if G is a bipartite planar graph with $|G| = n$, then $e(G) \leq 2n-4$. Can we have the equality $e(G) = 2n - 4$ for some G?

4. Given a graph G, its *crossing number* $\mathrm{cr}(G)$ is the minimal number of crossings between edges in a "drawing" of G on the plane (formally, we may replace the condition "$\gamma_e((0,1)) \cap \gamma_f((0,1)) = \varnothing$ for all $e, f \in E(G)$ with $e \neq f$" in the definition of a drawing with the condition that $\sum_{\{e,f\}\subseteq E, e\neq f} |\gamma_e((0,1)) \cap \gamma_f((0,1))| \leq k$, and let $\mathrm{cr}(G)$ be the smallest k so that such a drawing exists). Thus, G is planar if and only if $\mathrm{cr}(G) = 0$.

   (a)$^\circ$ Compute $\mathrm{cr}(K_5)$, $\mathrm{cr}(K_{3,3})$ and $\mathrm{cr}(K_{3,4})$.

   (b)$^-$ If G is a graph with $|G| = n \geq 3$, show that $e(G) \leq 3n + \mathrm{cr}(G) - 6$.

5.$^\circ$ Following the proof of the Five Colour Theorem given in the lectures, try to give a proof of the Four Colour Theorem. Where does your argument fail?

6.$^+$ Show that $K_{4,4}$ can be drawn on $\mathbb{T}^2$. Can $K_{5,5}$ be drawn on $\mathbb{T}^2$? [*Hint: use the Euler–Poincaré Formula and the fact that $K_{5,5}$ is bipartite.*]

7. (a)$^-$ Explain why any graph that can be drawn on Σ_g (respectively N_g) can also be drawn on $\Sigma_{g'}$ (respectively $N_{g'}$) for any $g' \geq g$.

   (b)$^-$ Draw some non-planar graph on $\mathbb{R}P^2$.

8. Given a graph $G = (V, E)$, its *Mycielskian* $\mu(G)$ is a graph defined as follows. Let

$$V(\mu(G)) := V \sqcup \{u_v \mid v \in V\} \sqcup \{w\}$$

and

$$E(\mu(G)) := E \sqcup \{u_v v' \mid v, v' \in V, v \sim_G v'\} \sqcup \{u_v w \mid v \in V\},$$

so that $|\mu(G)| = 2|G| + 1$ and $e(\mu(G)) = 3e(G) + |G|$.

   (a)$^\circ$ Show that if G is triangle-free then so is $\mu(G)$.

   (b)$^+$ Show that $\chi(\mu(G)) = \chi(G) + 1$.

   (c)$^-$ Deduce that for every $k \geq 2$, there exists a triangle-free graph G with $\chi(G) = k$.

   (d)$^\circ$ Can $\mu(\mu(G))$ be planar for a graph G that has at least one edge?

9. Let G be a graph of order n, and let $\overline{G}$ be its complement.

   (a)$^\circ$ Show that $\chi(G) \cdot \chi(\overline{G}) \geq n$.

   (b)$^\circ$ Show that $\chi(G) + \chi(\overline{G}) \leq n+1$.

   (c)$^-$ Find all G for which $\chi(G) \cdot \chi(\overline{G}) = n$ and $\chi(G) + \chi(\overline{G}) = n+1$.

10.$^\circ$ Let G be a graph with $|G| = n$ and $e(G) = m$, and suppose that $p_G(x) = x^n - mx^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ is the chromatic polynomial of G. Show that $a_{n-2} = \binom{m}{2} - t$, where t is the number of triangles in G.

11.$^\circ$ Find the chromatic polynomial of C_n for $n \geq 3$.

12.$^\circ$ Give two different reasons why $p_G(x)$ is divisible by x for any graph G with $|G| \geq 1$ and any $x \geq 1$.

13. (*Kőnig's Edge Colouring Theorem.*) Let G be a bipartite graph with $e(G) > 0$.

   (a)$^\circ$ Show that G is a subgraph of a $\Delta(G)$-regular bipartite graph H.
   [*Hint: consider the graph obtained as follows: start with two disjoint copies of G, and add an edge between each pair of corresponding vertices of minimal degree.*]

   (b)$^\circ$ Show that $\chi'(H) = \Delta(H)$, and deduce that $\chi'(G) = \Delta(G)$.
   [*Hint: see Problem 1.8.*]

14. Given a graph G and an integer $x \geq 0$, let $p'_G(x)$ be the number of admissible x-edge-colourings of G.

   (a)$^\circ$ Show that p'_G is a polynomial.

   (b)$^-$ Assuming that $p'_G(x)$ has the form $x^{n'} - m'x^{n'-1} + \cdots$, give graph-theoretic interpretations of the numbers n' and m'.

The following problem will not be covered during recitation classes, unless the proof of Kuratowski's Theorem is covered in lectures at the end of the course. The problem is included here mostly for the interested students who have read Section 4.2 in the lecture notes.

15. Fill in the gaps in the proof of Kuratowski's Theorem given in the lectures:

   (a) Let G be a *minimal non-planar graph*—that is, a non-planar graph such that every subgraph $H \leq G$ with $H \neq G$ is planar. Explain (roughly) why G is 2-connected.

   (b) Let Q_1, Q_2, v, w and $H'_1, \ldots, H'_k$ be as in the proof of Kuratowski's Theorem given in the lectures. Show that no two interior H'_i overlap. Use this to give a rough explanation why there exists an interior H'_I containing vertices in both $Q_1 - \{v, w\}$ and $Q_2 - \{v, w\}$ that overlaps some exterior H'_O.
   [*Hint: explain why if this was not the case then G would be planar.*]

   (c) Show that one of the four configurations discussed at the end of the proof of Kuratowski's Theorem must appear in H'_I.
   [*Hint: split your argument into two cases, depending on whether or not we have $V(H'_I) \cap V(C) \subseteq \{v, w, u_1, u_2\}$.*]