Problem List $\frac{3}{2}$ (extra problems)

Graph Theory, Winter Semester 2022/23, IM UWr

1. Let $n \geq 1$.
   (a) Construct a graph G of order $4n$ such that $G \cong \overline{G}$ (see Problem 1.3).
   [*Hint: take inspiration from the fact that* P_3 *has order* 4 *and* $P_3 \cong \overline{P_3}$.]
   (b) Modify your construction to obtain a graph H of order $4n+1$ such that $H \cong \overline{H}$.

2. Let G be a bipartite graph with vertex classes W and M, and suppose that G contains a matching from W to M.
   (a) Show that there exists a vertex $w \in W$ such that for all $v \in N(w)$ there exists a matching containing the edge wv.
   (b) Deduce that if $d(w) \geq r$ for all $w \in W$, then G contains at least $r!$ matchings if $r \leq |W|$, and at least $\frac{r!}{(r-|W|)!}$ matchings if $r > |W|$.

3. Prove that an incomplete regular graph of order n cannot contain a complete subgraph of order $> \frac{n}{2}$.

4. Show that any connected regular bipartite graph is 2-connected.

5. Let $k \geq 2$. Give an example of a graph G such that $G - \{v\}$ is not 2-edge-connected but $G - \{vw\}$ is k-edge-connected for some $v \in G$ and $w \in N_G(v)$.

6. Let T be a tree, and let φ be an *automorphism* of T, i.e. a bijection $\varphi \colon V(T) \to V(T)$ such that $v \sim w$ if and only if $\varphi(v) \sim \varphi(w)$. Show that either $\varphi(v) = v$ for some $v \in T$, or $\varphi(v) = w$ and $\varphi(w) = v$ for some $vw \in E(T)$.