ZAD. 1.

Find (by "drawing" pictures representing graphs) all pairwise non-simorfphic graphs of order 4

Trivial

ZAD. 2.

For a graph G, define a relation $\approx$ on $V(G)$ by saying $v \approx w$ iff there exists a path in G with endpoints v and w. Show that $\approx$ is an equivalence relation.

Reflexivity: $v \approx v$

Any path of length 0 begins and ends in one vertex without any in the middle.

Symmetry: $v \approx w \implies w \approx v$

Obviously if we have an undirected graph and path v...w, then we also have path going through the same vertices but in reversed order, that is w...v.

Transitivity: $(v \approx w \wedge w \approx z) \implies v \approx z$

We know that there exists a path v...w and another path w...z. The end of the first one is the same as the beggining of the second, so we can connect those two paths. That gives us a path v...ww...z = v...w...z.

ZAD. 3.

Given a graph G define its complement $\overline{G}$ as a graph with vertices $V(\overline{G}) = V(G)$ such that fiven $v, w \in V(G)$ with $v \neq w$ we have $vw \in E(\overline{G})$ *iff* $vw \notin E(G)$

(a) Show that if $G \simeq \overline{G}$, then $|G| \equiv 0$ or $1 \mod 4$

(b) Show that for any graph G, either G or $\overline{G}$ is connected.

(a) Let us assume that G has n vertices. If we want G to be isomorphic with $\overline{G}$, then they have to have the same number of edges. Between n vertices we can have no more than

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

We want to be able to divide this number into two equal parts, so $n(n-1)$ must be divisible by $2 \cdot 2 = 4$, so either n is divisible by 4 (then $n \equiv 0 \mod 4$) or $(n-1)$ is divisible by 4 ($n \equiv 1 \mod 4$).

(b) If G is connected then śmiga. Otherwise, G is not connected. Then there are at least two $U, W \subseteq V(G)$ such that for any $u \in U$ and $w \in W$ there is no $uw \in G$. Then we have $uw \in \overline{G}$ for every such u, w. So we can move between any two vertices just by jumping between U and W.

ZAD. 4.

Show that any graph of order at least 2 has two vertices of the same degree.

Let us take a graph of order n. Any of its vertices can have degree between 0 and $(n-1)$.

If at least one of those vertices has order 0, then no vertex can have order $(n-1)$, so we are left with $(n-2)$ possible degrees and $(n-1)$ vertices. By the pigeonhole principle, at least two vertices have the same degree.

If no vertex has order 0, then we have $(n-1)$ possible degrees and n vertices. Again, by the pigeonhole principle, at least two vertices have the same degree.

ZAD. 5.

(a) *Show that every connected graph G contains a vertex $v \in G$ such that $G - \{v\}$ is connected.*
Hint: pick v so that some connected component of $G - \{v\}$ is as big as possible.

We have a graph of order n and we want to take away one of its edges. Let v be a vertex such that $G - \{v\}$ contains a path as long as possible. Because G was a connceted graph, then this path spanned across all n of its vertices, so in $G - \{v\}$ the longest path would be $(n-1)$ vertices long. Which means, that the graph we obtained is connected.

(b) *A connected graph with at least one vertex is called a tree if it has no cycles. Show that every tree with ≥ 2 vertices has a vertex of degree 1 (such a vertex is called a leaf).*

Let us suppose that there is a tree with no vertices of degree 1. We will label it as G. Now, let P be the longest path in G. If we takie it out of G, we have a straight line with some vertex u as its beggining and another vertex v as its end. Now they are of degree 1. But if we put them back inside G, they cannot have degree 1, so they must be connected to some other vertex. But this is the longest path, so no new vertex can be added, therefore u must be connected with v, which gives us a cycle and a contradiction.

(c) Deduce that if T *is a tree then* $e(T) = |T| - 1$

A quick throwback to the definition of $e(G)$ for my dumb self:

$$e(G) = \frac{1}{2}\sum_{v \in G} d(v).$$

Now we gonna do induction on the number of vertices. For $n = 2$ we have a tree with $|T| = 2$ and $e(T) = \frac{1}{2}(1 + 1) = 1 = |T| - 1$.

Let us suppose that for a tree with n vertices this formula is true. We add one new vertex. Because in a tree we have at leas one (actually two but nevermind) vertex of degree one, by removing it we change $e(T)$ by only 1 - one less degree from the removed vertex and one less in degree of the only vertex it was connected to. At the same time, we change $|T|$ by one, so we have:

$$e(T) = e(T') + 1 = (|T'| - 1) + 1 = (|T| - 2) + 1 = |T| - 1.$$

The end <3

(d) Let G *be a graph with* $|G| = n$. *We say that a tuple* $(d_G(v_1), \ldots, d_G(v_n))$, *where* $\{v_1, \ldots, v_n\} = V(G)$ *is a degree sequence of* G. *Show that a given tuple* $(d_1, \ldots, d_n)$ *of integers, where* $n \geq 2$, *is a degree sequence of a tree* $\iff$ $d_i \geq 1$ *for all* i *and* $\sum_{i=1}^{n} d_i = 2n - 2$.

$\Longrightarrow$

It is simply from the previous excercise. For a tree with n degrees we have

$$\frac{1}{2}\sum d_G(v_i) = n - 1$$

$$\sum d_G(v_i) = 2n - 2.$$

$\Longleftarrow$

Induction on n. For $n = 2$ we have that $\sum d_i = 2$, so the edges are connected by one edge and this is a tree.
Now, let us take any graph with $n + 1$ vertices that has $\sum d_i = 2n$. Then we cannot have all vertices of even degree because at least one would have degree 0. We have n edges and so at least one vertex has degree 1. We can cut it out to obtain a graph of n vertices and $\sum d_i = 2n - 2$, which is a tree. Now, if we add a vertex of degree 1 we still will have a tree which concludes my very insightful investigation.

ZAD. 6.

Let $G = (V, E)$ *be a graph. Show that there exists a partition* $V = A \cup B$ *such that all vertices of* $G[A]$ *and of* $G[B]$ *have even degree.*
Hint: consider what happens when we remove a vertex x of odd degree and "invert" adjacency between the neighbors of v.

Ok, so we doing induction once again. So for $n = 1$ it is quite obvious.
So let us assume that for all graphs that have n vertices this works. Now, let us take a graph G with $n + 1$ vertices. If there are no vertices of odd degree then we all good. So let us assume that there exists at least one vertex of odd degree, we gonna name him Krzys. To simplify the upcoming struggles, we gonna say that Krzys' signature is x. And that he is a priest. Now, let us say that Krzys is a pedophile and that he is currently serving his sentence in jail. However, we cannot let the sheep wander without a shepherd, so we want to rewire his believers so that they are connected with each other via a priest. That is, we make new graph G' where $V(G') = V(G') \setminus x$ and for all $a, b \in N_G(x)$ if $ab \in G$ we remove this connection and if $ab \notin G$ we add this connection. This graph has n vertices, so we can divide it into two vertex classes A and B such that all vertices in them have even degrees. Now, let 11 years pass and Krzys is out of the prison. We need to let him back to the church. Therefore, we must choose if he joins the vertex class A or B. We still want the degrees to be even and since Krzys had odd degree, he must have an even degree of neighbors in one of A or B. Therefore, we need to add him to that group. We need to severe the fragile, artificial connections that were made and revive those that were lost but without connecting A and B.
Such operation does not change the parity of degrees. Let us assume that in A we have even number of neighbors of Krzys, let us name this number k. By adding x without changing anything else we make them be of odd degree. We had k neighbors of x in A and this is even. One vertex could have a total of $k - 1$ edges with its neighbors, which would be an odd number so $k - 1$ minus an even number would give us an odd number plus one from Krzys would be even.
In B let us say we had m neighbors of Krzys, which is an odd number. Each could have been connected with a maximum of $m - 1$ neighbors, which is an even number. So now after reversing it we would have $m - 1$ minus an even number, yield once again an even number.
This way, we divided G into two classes of vertices, both containing only vertices of even degree.

ZAD. 7.

Suppose G *is a graph that has no induced cycles of odd length - that is, for any* $A \subseteq V(G)$, *the graph* $G[A]$ *is not a cycle of odd length. Show that* G *is bipartite.*

Suppose that G is a graph such that it does not contain induced cycles of odd length. Without loss of generality, take G that is connected, otherwise we would have several connected components each being bipartite. Let us choose one vertex, $v_0 \in G$ and divide all the other vertices into two sets:

$$A = \{v \ : \ \text{the shortest } v...v_0 \in G \text{ is of odd length}\}$$
$$B = \{v \ : \ \text{the shortest } v...v_0 \in G \text{ is of even length}\}$$

We will show that A and B are vertex classes of graph G.
Suppose that there exists an edge $wu \in A$. Then we have a cycle

$$w...v...uw \in G$$

consisting of a path $w...v$ of length $2k$ for some k and $v...u$ of length $2n$. Additionally, we have the last path wu of length 1, which gives us a cycle of length $2(k+n)+1$, which certainly cannot be classified as even.
Now, if there exists an edge $wu \in B$, then we have

$$w...v...uw \in G$$

with $w...v$ having length $2k+1$ and $v...u$ of length $2n+1$, adding up to a cycle of length $2(k+n+1)+1$.
Therefore, G does not contain wv for $w, v \in A$ or $w, v \in B$ and is bipartite.

ZAD. 8.

Let G *be a regular bipartite graph with vertex classes* W *and* M. *Show that* G *contains a matching from* W *to* M.

A graph G is regular if there exists an r such that

$$(\forall\ v \in G)\ d(v) = r.$$

A graph G contains a matching if for any $A \subseteq V$ we have $|N(A)| \geq |A|$.

Let us take a r-regular bipartite graph G with vertex classes W and M. Without the loss of generality, let us take any $A \subseteq W$. We know, that every $a \in A$ must have r neighbors, each in M. We must have $r|A|$ edges leaving A and we know that $|N(A)| \geq r$. If $|A| \leq r$, then we know that $|N(G)| \geq r \geq |A|$ and the Hall's condition is satisfied. Otherwise, we would have $|A| > r$ and if $n = |A| > |N(A)| = m$, then we nr edges going to $m < n$ vertices, implying that some vertex from $N(A)$ has degree larger than r.