

# Kombinatoryka & teoria grafów

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## SYLABUS – teoria grafów:

1. Basic concepts: graphs, paths and cycles, complete and bipartite graphs
2. Matchings: Hall's Marriage theorem and its variations
3. Forbidden subgraphs: complete bipartite and  $r$ -partite subgraphs, chromatic numbers, Turán's theorem, asymptotic behaviour of edge density, Erdős-Stone theorem
4. Hamiltonian cycles (Dirac's Theorem), Eulerian circuits
5. Connectivity: connected and  $k$ -connected graphs, Menger's theorem
6. Ramsey theory: edge colourings of graphs, Ramsey's theorem and its variations, asymptotic bounds on Ramsey numbers
7. Planar graphs and colourings: statements of Kuratowski's and Four Colour theorems, proof of Five Colour theorem, graphs on other surfaces and Euler characteristics, chromatic polynomial, edge colourings and Vizing's theorem
8. Random graphs: further asymptotic bounds on Ramsey numbers, Zarankiewicz numbers and their bounds, graphs of large first and high chromatic number, complete subgraphs in random graphs.
9. Algebraic methods: adjacency matrix and its eigenvalues, strongly regular graphs, Moore graphs and their existence.

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Definicja – to jest bardzo wye-  
dukowany tekst w ramce

Troszke mniej wyedukowany tekst, bo jest poza  
fajna ramka ktora wyglada jak guwno i w sumie  
to nie wiem czemu ja ja w ten sposob ranie

# 1 Motivation

**Graph** – collection of vertices joined by edges.

**Seven bridges of Königsberg** – can one walk around the city crossing each bridge only once? NO

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**Simultaneous representation of cosets:** let  $G$  be a finite group and  $H \subseteq G$  be a subgroup of  $G$ , then there exists elements  $a_1, \dots, a_k \in G$  such that  $G = a_1H \cup \dots \cup a_kH$  and there exists  $b_1, \dots, b_k \in G$  such that  $G = Hb_1 \cup \dots \cup Hb_k$ .

Can we take  $a_i = b_i$ ? YES (Hall's Marriage Theorem)

**Map coloring problem**

Suppose we have a political map and we want to color its regions so that given any two regions sharing a border they have different colors.

How many colors do we need? 4 will always be enough.

Fermat's Last Theorem modulo  $p$  (where  $p$  is prime) – FLT stated that if  $x^n + y^n = z^n$ ,  $x, y, z, n \in \mathbb{Z}$  and  $n \geq 3$  then  $x = 0 \vee y = 0 \vee z = 0$ . Given a prime  $p$  do there exist  $x, y, z \in \mathbb{Z}$  such that  $x^n + y^n = z^n \pmod{p}$  but  $x, y, z \not\equiv 0 \pmod{p}$ ? YEP for  $p$  large enough.

PROOF

Let  $G = (\mathbb{Z}/p\mathbb{Z})^X$  and  $H = \{g^n : g \in G\} \leq G$ . Given  $h \in H$ , the polynomial  $x^n - h \in (\mathbb{Z}/p\mathbb{Z})[X]$  (ring of remainder modulo  $p$ ????) has degree  $n$  so it has at most  $n$  roots. Therefore,  $\frac{|G|}{|H|} \leq n$ , so there are at most  $n$  left cosets of  $H$  in  $G$ . Suppose there exists  $a, b, c \in gH$  such that  $a + b = c$ . Then  $g^{-1}a + g^{-1}b = g^{-1}c$  and  $g^{-1}a, \dots \in H$ , which implies that  $g^{-1}a = x^n, \dots$ . It is enough to show that for any integer  $k$  that is large enough if the set  $\{1, 2, \dots, k - 1\}$  is partitioned into  $n$  parts, then some part contains  $a, b, c$  such that  $a + b = c$ . This will be shown using graph theory and it's such a fucking mess.

**Graph** is an ordered pair  $G = (V, E)$  where

$\hookrightarrow V$  is a set of vertices  $\hookrightarrow E$  is a set of edges (unordered pairs  $\{v, w\}$  where  $v, w \in V$  and  $v \neq w$ )

We write  $v \in G$  to write  $v \in V(G)$  and denote  $\{v, w\} \in E(G)$  as  $vw$ :

$\hookrightarrow v, w$  are endpoints of  $vw \in E(G)$

$\hookrightarrow vw$  is incident to  $v$

Unless specified otherwise, graph  $G$  is finite. If  $|V| = \infty$  we say that  $G$  is an infinite graph.

**Order of  $G$**  is  $|G| := |V(G)|$

**Size of  $G$**  is  $e(G) := |E(G)|$