Kombinatoryka & teoria grafów

by a fish

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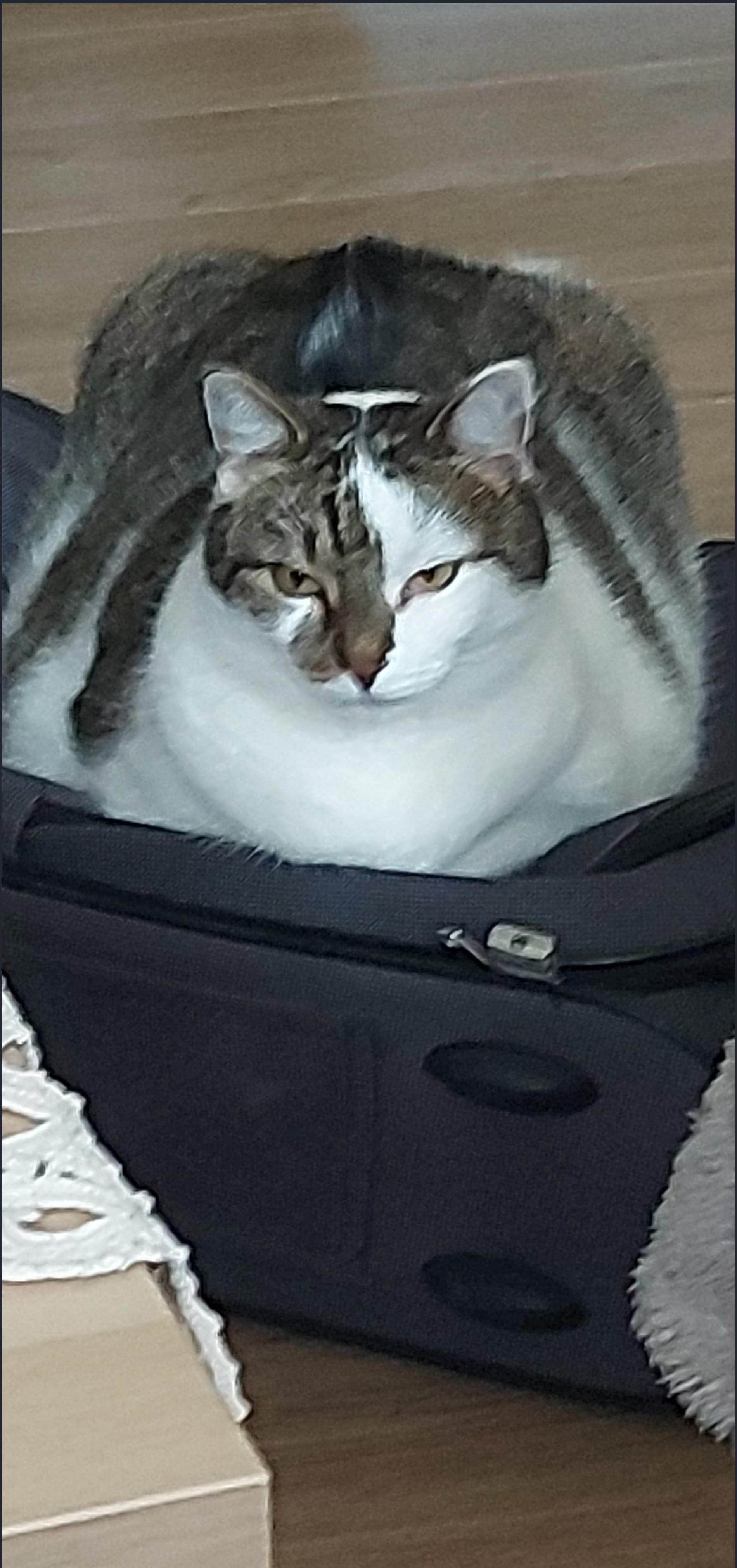

SYLABUS - teoria grafów:

1. Basic concepts: graphs, paths and cycles, complete andbipartite graphs
2. Matchings: Hall's Marriage theorem and its variations
3. Forbidden subgraphs: complete bipartite and r-partite subgraphs, chromatic numbers, Tur"an's thorem, asymptotic behaviour og edge density, Erd"os-Stone theorem
4. Hamiltonian cycles (Dirac's Theorem), Eulerian circuits
5. Connectivity: connected and k-connected graphs, Menger's theorem
6. Ramsey theory: edge colourings of graphs, Ramsey's theorem and its variations, asymptotic bounds on Ramsey numbers
7. Planar graphs and colourings: statements of Kuratowski's and Four Colour theorems, proof of Five Colour theorem, graphs on other surfaces and Euler chracteristics, chromatic polynomial, edge colourings and Vizing's theorem
8. Random graphs: further asymptotic bounds on Ramsey numbers, Zarankiewicz numbers and their bounds, graphs of large firth and high chromatic number, cmplete subgraphs in random graphs.
9. Algebraic methods: adjavenvy matrix and its eigenvalues, strongly regular graphs, Moore graphs and their existence.

Contents

1 Structural properties

1.1 Basic definitions

Graph – an ordered pair $G = (V, E)$:
$\hookrightarrow$ vertices $:= V$ [singular: *vertex*]
$\hookrightarrow$ edges $:= E$, $\{v, w\} := vw$

For an edge vw, $v \neq w$ we say that v, w are its endpoints and that it is incident to v (or w).

Dla krawedzi vw, $v \neq w$ mowimy, ze v, w sa jej koncami i ze jest krawedzia padajaca na v (lub w).

Graphs G and H are isomorfic ($G \simeq H$) if there exists $f : V(G) \xrightarrow[1-1]{on} V(H)$ such that
$(\forall\, v, w \in V(G))\ vw \in E(G) \iff f(v)f(w) \in E(H)$

Meaning that edges are like an operation on a group of vertices

G is a subgraph of H $[G \leq H]$ if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

If G is H-free if it is has no subgraphs isomorfphic to H.

Grafy G i G sa izomorficzne, jezeli istnieje $f : V(G) \xrightarrow[\lbrack]{1} -1]\mathrm{na}V(H)$ takie, ze
$(\forall\, v, w \in V(G))\ vw \in E(G) \iff f(v)f(w) \in E(H)$

G jest podgrafem H $[G \leq H]$ jezeli $V(G) \subseteq V(H)$ oraz $E(G) \subseteq E(H)$.

G jest H-free (wolny od H?), jezeli nie ma podgrafow izomorficznych z H.

A cycle of length $n \geq 3$ $[C_n]$ is a graph with vertices
$$V(C_n) = [n]$$
and edges:
$$E(C_n) = \{i(i+1)\ :\ i \leq i \leq n-1\} \cup \{1n\}.$$
A path of length $n - 1$ $[P_{n-1}]$ is a graph with vertices
$$V(P_{n-1}) = [n]$$
and edges
$$E(P_{n-1}) = \{i(i+1)\ :\ 1 \leq i \leq n-1\}.$$

Cykl dlugosci $n \geq 3$ $[C_n]$ to graf z wierzcholkami
$$V(C_n) = [n]$$
i krawiedziami:
$$E(C_n) = \{i(i+1)\ :\ i \leq i \leq n-1\} \cup \{1n\}.$$
Sciezka dlugosci $n - 1$ $[P_{n-1}]$ to graf z wierzcholkami
$$V(P_{n-1}) = [n]$$
i krawedziami
$$E(P_{n-1}) = \{i(i+1)\ :\ 1 \leq i \leq n-1\}.$$

An induced by $A \subseteq V(G)$ subgraph of G is $G[A] = (A, E_A)$

A connected component of G is a subgraph $G[W] \leq G$ where $W \subseteq V$ is an equivalence class under $\approx$ given by
$$v \approx w \iff \text{exists a path } v\ldots w \text{ in } G$$
A graph is connected if $v \approx w$ for every $v, w \in V$ (G has at most one connected component).

If v is a vertex in graph G, we say that its neighbourhood is $N_G(v) = \{w \in G\ :\ vw \in E(G)\}$. Furthermore, the degree of v is $|N_G(v)|$.

If $A \subseteq V$, then $N(A) := \bigcup_{v \in A} N(v)$.

We define:

- ↪ minimal degree $\delta(G) = \min_{v \in G} d(v)$
- ↪ maximal degree $\Delta(G) = \max_{v \in G} d(v)$
- ↪ average degree $d(G) = \frac{\sum d(v)}{|G|}$.

If there exists an $r \geq 0$ such that

$$\delta(G) = \Delta(G) = d(G) = r$$

then we say that the graph is r-regular or, more generally, it is regular for some r.

Handshaking Lemma: for any graph G we have $e(G) = \frac{1}{2} \sum d(v) = \frac{|G|}{2} d(G)$

1.2 Hall's Marriage Theorem

Graph G is bipartite with vertex classes U and W if $V = U \cup W$ so that every edge has form uw for some $u \in U$ and $w \in W$.

G is bipartite iff it has no cycles of odd length.

Graf G jest dwudzielny z klasami wierzcholkow U i W, jesli $V = U \cup W$ takimi, ze kazda krawedz jest formy uw dla pewnych $u \in U$ oraz $w \in W$.

G jest dwudzielny wtw kiedy nie ma cykli o nieparzystej dlugosci.

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$\Longrightarrow$

Let U, W be the vertex classes and $v_1, v_2, \ldots, v_n, v_1$ be a cycle in G. WLG suppose that $v_1 \in U$. Then $v_2 \in W$ etc. Specifically we have $v_i \in U$ if i is odd and $v_i \in W$ if i is even. Then, we have $v_n v_i$, so n must be even.

$\Longleftarrow$

Suppose G has no cycles of odd length. WLOG, assume that $V(G) \neq \emptyset$ and that G is connected, because G will be bipartite if all its connected components are bipartite. Fix $v \in G$ and for all other $w \in G$ define distance $\mathrm{dist}(v, w)$ as the smallest $n \geq 0$ such that there exists a path $v \ldots w$ in G of length n.

Now, let $V_n := \{w \in G : \mathrm{dist}(v, w) = n\}$ and set

$$U = V_0 \cup V_2 \cup V_4 \cup \ldots$$
$$W = V_1 \cup V_3 \cup V_5 \cup \ldots$$

We want to show that there are no edges in G of the form $v'v''$ where $v', v'' \in U$ or $v', v'' \in W$.

Suppose that $v'v'' \in E(G)$ with $v' \in V_m, v'' \in V_n$ and $m \leq n$. Then, we have a path

$$v \ldots v'v'' \in G$$

of length $m + 1$, implying that

$$n \in \{m, m + 1\}.$$

Supose that $n = m$. Let $v'_0 v'_1 \ldots v'_m$ and $v''_0 v''_1 \ldots v''_m$ be paths in G with $v = v'_0 = v''_0$, $v' = v'_m$ and $v'' = v''_m$. Note that $v'_i, v''_i \in V_i$ for $0 \leq i \leq m$. Let $k \geq 0$ be largest such that

$$v'_k = v''_k$$

and note that $k \leq m - 1$ as $v' \neq v''$. Then

$$v'_k v'_{k+1} \ldots v'_m v''_m v''_{m-1} \ldots v''_k$$

is a cycle of odd length, which is a contradiction.

Therefore, we can only have $n = m + 1$ and then exactly one of n, m is even meaning that exactly one of v' and v'' is in U as required for G to be bipartite.

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$\Longrightarrow$

Niech U, W beda klasami wierzcholkow oraz niech $v_1, v_2, \ldots, v_n, v_1$ niech bedzie cyklem w G. BSO zalozmy, ze $v_1 \in U$. W takim razie, $v_2 \in W$ etc. W szczegolnosci, mamy $v_i \in U$ jezeli i jest nieparzyste oraz $v_i \in W$ jezeli i jest parzyste. W takim razie, skoro $v_n v_1$, to n musi byc parzyste.

$\Longleftarrow$

Zalozmy, ze G nie ma cykli o nieparzystej dlugosci. BSO zalozmy, ze $V(G) \neq \emptyset$ i ze G jest spojny, poniewaz G bedzie dwudzielny, wtw gdy wszystkie jego skladowe spojne (????) beda dwudzielne. Ustalmy $v \in G$ i dla kazdego innego $w \in G$ zdefiniujmy dystans $\mathrm{dist}(v, w)$ jako najmniejsze $n \geq 0$ takie, ze istnieje sciezka $v \ldots w$ w G o dlugosci n.

Niech $V_n := \{w \in G : \mathrm{dist}(v, w) = n\}$ i zbiory

$$U = V_0 \cup V_2 \cup V_4 \cup \ldots$$
$$V = V_1 \cup V_3 \cup V_5 \cup \ldots$$

Chcemy pokazac, ze nie istnieja w G krawedzie postaci $v'v''$, gdzie $v', v'' \in U$ lub $v', v'' \in W$.

Zalozmy, ze $v'v'' \in E(G)$ z $v' \in V_m, v'' \in V_n$ oraz $m \leq n$. Wtedy istnieje sciezka

$$v \ldots v'v'' \in G$$

dlugosci $m+1$, co implikuje, ze
$$n \in \{m, m+1\}.$$

Zalozmy, ze $n = m$. Niech $v_0'v_1' \ldots v_m'$ oraz $v_0''v_1'' \ldots v_m''$ sa sciezkami w G takimi, ze $v = v_0'v_0''$, $v' = v_m'$ oraz $v'' = v_m''$. Zauwazmy, ze $v_i', v_i'' \in V_i$ dla $0 \leq i \leq m$. Niech $k \geq 0$ bedzie najwiksze takie, ze

$$v_k' = v_k''$$

i zauwazmy, ze $k \leq m-1$ poniewaz $v' \neq v''$. Wtedy

$$v_k' v_{k+1}' \ldots v_m' v_m'' v_{m-1}'' \ldots v_k''$$

jest cyklem o nieparzystej dlugosci, co daje nam sprzecznosc.

W takim raize, mozemy miec tylko $n = m+1$ i wtedy dokladnie jedno z n, m moze byc parzystem, co daje nam dokladnie jedno z v' i v'' w U tak, jak jest wymagane zeby to byl graf dwudzielny.

If G is a bipartite graph with $V = W \cup M$ and $W' \subseteq W$, a partial matching in G from W' to M is
$$\{wv_w : w \in W'\} \subseteq E(G)$$
for some $v_w \in M$ such that $w \neq w' \Longrightarrow v_w \neq v_{w'}$. A partial matching from W to M is called a matching.

Sufficient condition:

$$|N(A)| \geq |A| \quad (☕)$$

for every $A \subseteq W$

Jesli G jest grafem dwudzielnym z $V = W \cup M$ oraz $W' \subseteq W$, wtedy czesciowe skojarzenie w G z W' do M to

$$\{wv_w : w \in W'\} \subseteq E(G)$$

dla pewnych $v_w \in M$ takich, ze $w \neq w' \Longrightarrow v_w \neq v_{w'}$. Czesciowe kojarzenie z W do M jest nazywane kojarzeniem.

Wystarczajacy warunek:

$$|N(A)| \geq |A| \quad (☕)$$

dla kazdego $A \subseteq W$

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A bipartite graf G contains a matching from W to M iff (G, W) satisfies Hall's condition (☕).

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Dwudzielny graf G zawiera kojarzeniem iff gdy (G, W) zadowala warunek Halla (☕).

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$\Longrightarrow$

Trivial.

$\Longleftarrow$

Using induction on $|W|$. For $|W| = 0, 1$ it is trivial.

We gonna break it into parts: $|N(A)| > |A|$ and $|N(A)| = |A|$

Suppose that $|N(A)| > |A|$ for every non-empty subset $A \subsetneq W$. Take any $w \in W$ and $v \in N(w)$ and construct a new graph

$$G_0 = G - \{w, v\}.$$

For any non-empty $B \subseteq W - \{w\}$ we have

$$N_{G_0}(B) = N_G(B) - \{v\}$$

and therefore

$$|N_{G_0}(B)| \geq |N_G(B)| - 1 \geq |B|$$

and so $(G_0, W - \{w\})$ satisfies Hall's condition. From induction we have a matching P in G_0 from $W - \{w\}$ to $M - \{v\}$ and so $P \cup \{wv\}$ is a matching from W to M.

Now, suppose that $|N(A) = |A|$ for some non-empty subset $A \subsetneq W$. Let

$$G_1 = G[A \cup N(A)]$$

and

$$g_2 = G[(W - A) \cup (M - N(A))].$$

We will show that both those graphs satisfy Hall's condition.
Let us take any $B \subseteq A$ in G_1. We have

$$N_G(B) \subseteq N_G(A) \subseteq V(G_1)$$

$$|N_{G_1}(B)| = |N_G(B)| \geq |B|$$

and so graph G_1 satisfies Hall's condition.
Now, let us take any $B \subseteq W - A$ in G_2. We know that $N_{G_2}(B) \subseteq M - N(A)$ so

$$N_{G_2}(B) = N_G(B) - N_G(A) = N_G(A \cup B) - N_G(A)$$

$$|N_{G_2}(B)| = |N_G(A \cup B) - N_G(A)| \geq |N_G(A \cup B)| - |N_G(A)| \geq |A \cup B| - |A| = |A| + |B| - |A| = |B|$$

Therefore, graph G_2 also satisfies Hall's condition.
Using inductive hypothesis, we have that there exists a matching P_1 in G_1 and a matching P_2 in G_2. The first one is from A to $N_G(A)$ while the second is from $W - A$ to $M - N_G(A)$, so they are disjoint. Therefore, $P_1 \cup P_2$ is a matching in G from W to M.

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$\Longrightarrow$

Trywialne.

$\Longleftarrow$

Uzyjemy indukcji na $|W|$. Dla $|W| = 0, 1$ jest trywialne.

Podzielimy dowod na dwie czesci: $|N(A)| > |A|$ oraz $|N(A)| = |A|$.

Zalozmy, że $|N(A)| > |A|$ dla kazdego niepustego podzbioru $A \subsetneq W$. Wezmy dowolne $w \in W$ oraz $v \in N(w)$ i skonstruujmy nowy graf

$$G_0 = G - \{w, v\}.$$

Dla kazdego niepustego $B \subseteq W - \{w\}$ mamy

$$N_{G_0}(B) = N_G(B) - \{v\}$$

i w takim razie

$$|N_{G_0}(B)| \geq |N_G(B)| - 1 \geq |B|,$$

czyli $(G_0, W - \{w\})$ spelnia warunek Halla. Z zalozenia indukcyjnego istnieje kojarzenie P w G_0 z $W - \{w\}$ do $M - \{v\}$, w takim razie $P \cup \{wv\}$ jest kojarzeniem z W do M.

Zalozmy teraz, ze $|N(A) = A|$ dla pewnego niepustego podzbioru $A \subsetneq W$. Niech

$$G_1 = G[A \cup N(A)]$$

oraz

$$g_2 = G[(W - A) \cup (M - N(A))].$$

Pokazemy, ze oba te grafy zaspokajaja warunek Halla.
Wezmy dowolny $B \subseteq A$ w G_1. Mamy

$$N_G(B) \subseteq N_G(A) \subseteq V(G_1)$$

$$|N_{G_1}(B)| = |N_G(B)| \geq |B|$$

a wiec graf G_1 zaspokaja warunek Halla.
Teraz, wezmy dowolny $B \subseteq W - A$ w G_2. Wiemy, ze $N_{G_2}(B) \subseteq M - N(A)$, a wiec

$$N_{G_2}(B) = N_G(B) - N_G(A) = N_G(A \cup B) - N_G(A)$$

$$|N_{G_2}(B)| = |N_G(A \cup B) - N_G(A)| \geq |N_G(A \cup B)| - |N_G(A)| \geq |A \cup B| - |A| = |A| + |B| - |A| = |B|$$

W takim razie G_2 spelnia warunek Halla.

Z zalozenia indukcyjnego wiemy, ze istnieje kojarzenie P_1 w G_1 oraz P_2 w G_2. Pierwsze jest z A do $N_G(A)$, natomiast drugie jest z $W - A$ do $M - N_G(A)$, czyli sa rozlaczne. W takim razie $P_1 \cup P_2$ jest kojarzeniem w G z W do M.

Let G be a finite group and let $H \leq G$ be a subgroup with $\frac{\|G\|}{\|H\|} = k$, then $g_1 H \cup \ldots \cup g_k H = G = Hg_1 \cup \ldots \cup Hg_k$ for some $g_1, \ldots, g_k \in G$.	Niech G bedzie skonczona grupa i niech $H \leq G$ bedzie podgrupa z $\frac{\|G\|}{\|H\|} = k$, wtedy $g_1 H \cup \ldots \cup g_k H = G = Hg_1 \cup \ldots \cup Hg_k$ fdla pewnych $g_1, \ldots, g_k \in G$.