Problem List 2 (Extremal problems)

Graph Theory, Winter Semester 2022/23, IM UWr

1. Let $n \geq 1$, and let $x_1, \ldots, x_{3n} \in \mathbb{R}^2$ be points such that $\|x_i - x_j\| \leq 1$ for all i and j. Prove that $\|x_i - x_j\| > 1/\sqrt{2}$ for at most $3n^2$ pairs (i, j) with $i < j$.
   [*Hint: can four of these points have all pairwise distances greater than* $1/\sqrt{2}$*?*]

2. Let G be a graph with $n \geq r + 2 \geq 4$ vertices and $t_r(n) + 1$ edges.
   (a) Show that for every p with $r + 1 \leq p \leq n$, G has a subgraph H with $|H| = p$ and $e(H) = t_r(p) + 1$.
   (b) Deduce that $K_{r+2} - \{e\} \leq G$, where $e \in E(K_{r+2})$.

3. For any integers $n \geq t \geq 1$, construct a graph G with $|G| = n$, $\Delta(G) = t - 1$, and $e(G) = \lfloor \frac{n(t-1)}{2} \rfloor$.

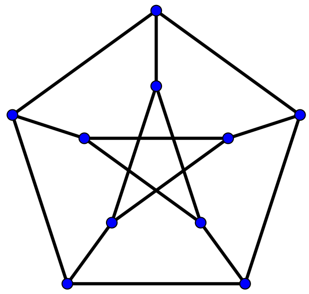
4. Show that $\mathrm{ex}(n; K_{t,t}) \leq \frac{1}{2} z_t(n)$ for all $n \geq t \geq 1$.

5. Let $p \geq 2$ be a prime number, and let $\mathbb{F}_p$ be the field with p elements. Consider the *projective plane over* $\mathbb{F}_p$, that is, $\mathbb{F}_p P^2 = (\mathbb{F}_p^3 \setminus \{(0,0,0)\})/\simeq$, where $\simeq$ is defined by setting $(x, y, z) \simeq (\lambda x, \lambda y, \lambda z)$ for $\lambda \in \mathbb{F}_p \setminus \{0\}$; we write $(x : y : z)$ for the equivalence class of $(x, y, z) \in \mathbb{F}_p^3$ in $\mathbb{F}_p P^2$. Let G be a graph with $V(G) = \mathbb{F}_p P^2$ such that $(x : y : z) \sim_G (x' : y' : z')$ if and only if $(x : y : z) \neq (x' : y' : z')$ and $xx' + yy' + zz' = 0$.
   (a) Show that G is C_4-free.
   [*Hint:* $\mathbb{F}_p$ *is a field, so you can use everything you know about vector spaces.*]
   (b) Compute the degree of any vertex $(x : y : z) \in V(G)$ (consider two cases, depending on whether or not $x^2 + y^2 + z^2 = 0$).
   (c) Compute $|G|$ and give a positive lower bound on $\frac{e(G)^2}{|G|^3}$.
   (d) Bertrand's Postulate, proved by P. L. Chebyshev, states that for any integer $n \geq 2$ there exists a prime number p with $n < p < 2n$. Use this to show that $\mathrm{ex}(n; C_4) = \Omega(n\sqrt{n})$.

6. Let G be an infinite graph. For $n \geq 2$, define $x_n = \max\{e(H)/\binom{n}{2} \mid H \leq G, |H| = n\}$. Show that the sequence $(x_n)_{n=2}^{\infty}$ is non-increasing, and deduce that it converges.

7. Let $c, \varepsilon > 0$ with $c + \varepsilon \leq 1$. Show that if $n \geq \frac{3}{\varepsilon}$, then every graph G with $|G| = n$ and $e(G) \geq (c + \varepsilon)\binom{n}{2}$ contains a subgraph $H \leq G$ with $\delta(H) \geq c|H|$ and $|H| \geq \varepsilon^{1/2} n$.
   [*Hint: consider a sequence of graphs obtained by starting with* G *and repeatedly removing a vertex of minimal degree, until the resulting graph* G' *has order* $\lfloor \varepsilon^{1/2} n \rfloor$*; then compute* $e(G')$.]

8. Determine the chromatic number of the following graphs.



   (a) P_n and C_n for any n.

   (b) The *Petersen graph*, displayed on the right.

   (c) The *join* K of graphs G and H, defined by setting $V(K) := V(G) \sqcup V(H)$ and $E(K) := E(G) \sqcup E(H) \sqcup \{vw \mid v \in G, w \in H\}$, where $\chi(G) = r$ and $\chi(H) = s$.

9. Let G be a graph with $|G| \geq 1$, and let $r = \chi(G)$.

   (a) Explain why $\mathrm{ex}(n; G) \geq \mathrm{ex}(n; K_r)$ for all $n \geq r$.

   (b) Show that if $\mathrm{ex}(n; G) = \mathrm{ex}(n; K_r)$ for some n, then there exists $e \in E(G)$ such that $G - \{e\}$ is $(r-1)$-partite.

10. For each $r \geq 3$, construct a K_r-free graph G with $\chi(G) = r$.
[*Hint: remove a certain collection of* $2r - 1$ *edges from* K_{2r-1}.]

11. (a) Let G be a connected graph of order $n \geq 1$, and let $k < n$ be such that for any $v, w \in G$ with $v \neq w$ and $v \nsim w$ we have $d(v) + d(w) \geq k$. Show that $P_k \leq G$.

    (b) Show that if G is P_k-free then $e(G) \leq \frac{|G|(k-1)}{2}$. Hence compute $\mathrm{ex}(nk, P_k)$ for all $n, k \geq 1$.

12. For every $k \geq 2$, give an example of a k-connected graph that is not Hamiltonian. What is the smallest possible order of such a graph?
[*Hint: see Problem 1.12.*]

13. Show that a graph G has an Euler trail if and only if it has at most 2 vertices of odd degree.

14. (*The Ten Bridges of Wrocław.*) You can see on the right a part of the Odra River in Wrocław, including some islands and bridges between them (the "dead end" islands which can only be accessed by one bridge are not displayed). Can you walk around the shown area by crossing each bridge exactly once? What about if you need to start and end at the same place? If the answer to either of these questions is "yes", exhibit such a walk explicitly.