ZAD. 3.

Show that if $|G| = pq^2$, *where* p, q *are prime numbers, then* G *is solvable.*

First of all, we need to know what the heck does it mean that G is solvable. Definition says that a group is solvable if it has a subnormal series whose quotient groups are all abelian. Basically, we have a series of subgroups

$$1 = G_0 < G_1 < \ldots < G_j < G_{j+1} < \ldots < G_k = G$$

such that for all j $G_j \triangleleft G_{j+1}$ and G_{j+1}/G_j is abelian.

I guess we have to start from building such a chain of subgroups. Let G_1 be a p-subgroup of G. Then we know that G_1 is a normal subgroup in G and that G/G_1 has q^2 elements.

That means that $H = G/G_0$ is abelian, because every group of order p^k has nontrivial center, so $Z(H) \neq \{e\}$. So we have either $|Z(H)| = q$ or $|Z(H)| = q^2$. The latter suits us so we need to rule out the first option. This means that Z(H) is a q-subgroup so their number must divide q^2 and be $1 \mod q$, meaning that we have just one such q-subgroup and it is normal.

We end up with such a magnificent subnormal series:

$$1 = G_0 < G_1 \text{ - p-subgroup of } G < G$$

ZAD. 4.

Show that if $|G| = 200$, *then* G *is solvable.*

$$200 = 2^3 \cdot 5^2$$

Let us check how many 5-subgroups there are in G. We know that n_5 must divide 8 and be $1 \mod 5$. We can only have $n_5 = 1$ so the unique 5-subgroup is the normal subgroup of G.

What about 2-subgroups? We know that n_2 must divide 5^2 and that $n_2 \equiv 1 \mod 2$. If we had 5 2-subgroups then we have $5 \cdot 2^3$ unique Sylow 2-subgroups and they can only have the trivial element as their intersection. But we also need a 5-subgroup as we shown before and it also has to have a trivial intersection with all 2-subgroups. So we cannot have 5 2-subgroups and the same goes for 25. So we are left with just one 2-subgroup which has order 2^3.

Now let $G_0 = 1$ and G_1 = the unique 2-subgroup of G. We have $|G/G_1| = 5^2$ which is abelian and because G_1 is unique then $G_1 \triangleleft G$. And this is the end of my misery.

ZAD. 5

Show that if $|G| < 60$ *then* G *is solvable.*

Liczb pierwszych które się mieszczą do 60 jest dużo za dużo, to zacznijmy od kwadratów. Wiem, że $8^2 = 64$ i $7^2 = 49$, czyli możemy mieć co najwyżej 7 w kwadracie w rozkładzie na czynniki pierwsze. Sześcianów jest jeszcze mniej, bo tylko 2, 3 nie wypierdala poza 60. No to lecimy od góry, I guess?
Dla $|G| = 3^3 \cdot 2 = 27 \cdot 2 (= 54)$ mamy n_3 dzieli 2 i jest $1 \mod 3$, czyli $n_3 = 1$. Mamy więc unikalną 3-grupę Sylowa, więc jest ona dzielnikiem normalnym całości i ma rząd 27, czyli $|G/G_1| = 2$ co jest grupą abelową. Więc śmiga.

Niech $|G| = n$. Jeśli 7 jest w rozkładzie n na czynniki pierwsze, to nie możemy mieć tam również liczby większej niż 7 (bo $7^2 = 49$ i już większej liczby pierwszej nie wepchniemy), a więc liczba liczba n_7-podgrup musi dzielić liczbę co najwyżej 8 $(7 \cdot 8 = 56)$ i być $1 \mod 7$. No i jeśli wymagamy dzielenia liczby mniejszej niż 8 to musimy mieć dokładnie jedną KURWA NO NIE BĘDĘ TEGO ROBIĆ NO

ZAD. 6

Infinity cuz Feit-Thomson theorem.

ZAD. 7.

How many elements of order 7 *are in a simple group of order* 168?

$$168 = 2^3 \cdot 3 \cdot 7$$

We know thet G is simple so its only normal subgroups are a trivial group and G itself. So we cannot have only one p-subgroups.

Let us see how many of each p-subgroup can we have:

$$n_2 \in \{3, 7\}$$

$$n_3 \in \{4, 28\}$$

$$n_7 \in \{8\}$$

so we know that we have exactly 8 7-subgroups and all of them contain elements of order 7 or 1 - we have at least 48. And we do not have any more because all elements of order 7 must be in a subgroup of order 7 and the only such subgroups that we are considering are 7-subgroups.

ZAD. 8.

O co oni mnie kurwa pytają?

ZAD. 9.

Find a composition series of the group $\mathbb{Z}_n$

If n is a prime number then we kinda have G_0 and G. And that's it.

I guess it is gonna be easiest to start from the top? What is a normal subgroup in any $\mathbb{Z}_n$? Cuz it is obvious that $\mathbb{Z}_n$ is abelian. Let $n = pk$ for some prime p and $k \in \mathbb{N}$. Subgroup $H = \langle p \rangle$ is a subgroup of $\mathbb{Z}_n$ and we want that $H \triangleleft \mathbb{Z}_n$. So we take any $a \in \mathbb{Z}_n \setminus H$ and any $h \in H$. We know that $ah = ha \in Ha$ so $aH = Ha$ and this is normal.