R is a commutative ring with 1 and $n \in \mathbb{N}_{>0}$.

ZAD. 1.

Let $r_1, \dots, r_n \in R$. *Show that* $(r_1, \dots, r_n) = r_1 R + \dots + r_n R$.

From what I gathered, $(r_1, \dots, r_n)$ is the minimal ring that contains $\{r_1, \dots, r_n\}$. Inclusion $\subseteq$ is quite trivial but the other way around is more difficult.

Ok, so we need to have that for all r_i, r_j that $r_i + r_j \in I$ so $\sum r_i \in I$. Furthermore, we need to have $r_i R \in I$. But that also means that $r_i R + r_j R \in I$, so we need to add all $r_i R$ in order to obtain I. And I guess that is the end.

ZAD. 2.

Let $I \triangleleft R$ *and* $\sqrt{I} := \{a \in R \ : \ (\exists\, n \in \mathbb{N})\ a^n \in I\}$. *Show that* $\sqrt{I} \triangleleft R$.

I is an ideal, which means that $a, b \in I \implies a + b \in I$ and $r \in R, a \in I \implies ra \in I$. This means that I is a normal subgroup of $(R, +)$ and that notations starts to make sens to me right now.

First, let us tacle the multiplication. We take any $a \in I$. This means that for some n $a^n \in I$ and so for any $r \in R$ we have $ra^n \in I$. So in particular for $r^n \in R$ we have $r^n a^n \in I$, which means that $r^n a^n = (ra)^n \in I$ and $ra \in \sqrt{I}$ for any $r \in R$.

Now, for the addition. This one is more difficult because we have to see that for $a^n, b^k \in I$ with assumption that $k \leq n$ we also have $(a + b)^n \in I$. But if $(a + b)^n \in I$ then in particular $(a + b)^{2n} \in I$ and the other way around. So let us start here.

$$
\begin{aligned}
(a+b)^{2n} &= a^{2n} + \binom{2n}{1} a^{2n-1} b + \dots + \binom{2n}{2n-1} a b^{2n-1} + b^{2n} = \\
&= a^n \left(a^n + \binom{2n}{1} a^{n-1} b + \dots + \binom{2n}{n} b^n\right) + b^k \left(\binom{2n}{n-1} a^{n-1} b^{n+1-k} + \dots + \binom{2n}{1} a b^{2n-1-k} + b^{2n-k}\right) = \\
&= a^n \cdot r_a + b^k \cdot r_b
\end{aligned}
$$

for $r_a, r_b \in R$ with those brutal formulas as seen above. Therefore $a^n r_a \in I$ and $b^k r_b \in I$ which means that $a^n r_a + b^k r_b \in I$ but this is equal to $(a + b)^n \in I$ so we have for $a, b \in \sqrt{I}$ $a + b \in \sqrt{I}$.

ZAD. 3.

Let $f : R \to S$ *be a homomorphism of commutative rings with* 1, $I \triangleleft R$, *and* $J \triangleleft S$. *Show the following:*

- $\hookrightarrow$ $f^{-1}(J) \triangleleft R$
- $\hookrightarrow$ *if* F *is and epimorphism (onto) then* $f(I) \triangleleft S$
- $\hookrightarrow$ *give an example of* f, I *such that* $f(I) \not\triangleleft S$

$f^{-1}(J) \triangleleft R$

So let us take any $a, b \in f^{-1}(J)$ we know that $f(a), f(b) \in f(J)$ so $f(a) + f(b) \in J$ as well. But f is a homomorphism, so it is additive or some bullshit and we can write

$$J \ni f(a) + f(b) = f(a + b)$$

so also $a + b \in f^{-1}(J)$, which gives as the addition thingy thing.

Now, the dreaded multiplication. We take any $a \in f^{-1}(J)$ and any $r \in R$. We know that $f(r) \in S$ and $f(a) \in J$, which means that $f(r)f(a) \in J$. But again, f is a homomorphism, so $f(r)f(a) = f(ra) \in J$ so $ra \in f^{-1}(J)$.

$f(I) \triangleleft S$ when f is onto

First of all, addition. We know that $a, b \in I$ implies that $a + b \in I$. So $f(a) + f(b) = f(a + b) \in f(I)$. Now, for the multiplication. We take any $r \in R$ and any $a \in I$ and we know that $ra \in I$ so $f(ra) = f(r)f(a) \in f(I)$ and $f(r)$ can be any element in S because f is onto.

EXAMPLE but im too lazy to think about it right now.

ZAD. 4.

Find $f \in \mathbb{Q}[X]$ *such that* $(f) = (X^2 - 1, x^3 + 1)$.

So we are looking for a function for which the smallest ideal that contains it is equal to the smallest ideal that contains $X^2 - 1$ and $X^3 - 1$. Maybe first let us write down how the RHS looks like

$$(x^2 - 1, x^3 - 1) = \{ra \ : \ r \in \mathbb{Q}[X], a = X^2 - 1, X^3 + 1\}.$$

Ok, now how about the LHS?

$$(f) = \{rf \ : \ r \in \mathbb{Q}[X]\}$$

Let us take any $r \in \mathbb{Q}[X]$ then we have $r(x^2-1) \in (f)$ and $r(x^3-1) \in (f)$. Furthermore, we have that $(x^2-1)+(x^3-1) = x^3+x^2 \in (f)$. Maybe $f = x + 1$? Yep.
We have that

$$(x+1)(x-1) = x^2 - 1 \in \text{RHS}$$

and also

$$(x^2 - x + 1)(x + 1) = x^3 + 1 \in \text{RHS}$$
$$x^2(x + 1) = x^3 + x^2 \in \text{RHS}$$

ZAD. 5.

Show that the ideal $(2, X) \triangleleft \mathbb{Z}[X]$ *is not principal.*

A principal ideal is an ideat I generated by one element $a \in R$ through multiplication of a by all elements of R.

$$(2, x) = \{ra \ : \ r \in \mathbb{Z}[X], a = 2, x\}$$

So what if $(2, x)$ was a principal ideal? We would have an $a \in \mathbb{Z}[x]$ such that

$$(\forall\ y \in (2, x))(\exists\ r \in \mathbb{Z}[x])\ ra = y$$

So let us start with 2 and x. We assumed that a as above existed, so

$$r_1 \cdot a = 2$$
$$r_2 \cdot a = x$$

and then

$$2 + x = r_1 \cdot a + r_2 \cdot a = (r_1 + r_2) \cdot a.$$

Now, because we only have polynomials with integer coefficients, we must have a of order 0, otherwise we could not obtain 2 by multiplying a by some other polynomial. We can have either $a = 1$ or $a = 2$. So for the second case we would need to find r_2 such that $r_2 \cdot 2 = x$ and we know that r_2 must be of order 1 so it must be $r_2 = r_2'x$ and $r_2'x \cdot 2 = x$ meaning, that $r_2' = \frac{1}{2}$ which cannot be. Therefore, we are left with $a = 1$ and $r_2 = 2$. Then, we have

$$2 + x = 2 \cdot 1 + x \cdot 1 = (2 + x) \cdot 2 = 4 + 2x$$

and this is a contradiction.

ZAD. 6.

Let $\phi : R \to S$ *be an epimorphism of rings, where* R *is Noetherian. Show that* S *is Noetherian as well.*

A ring is said to be Noehterian if every sequence of ideals stabilizes, that is if $I_1 \subseteq I_2 \subseteq \ldots$ is a sequence of ideals then there exists an N such that for all $n \geq N$ we have $I_n = I_{n+1}$. Alternatively, we can say that every ideal is finitely generated (it is $I = (a_1, \ldots, a_k)$ generated by a finite set).
Let us take any $I \triangleleft R$. From ex. 3 we know that $\phi(I) \triangleleft S$. So now let $I_1 \subseteq I_2 \subseteq \ldots$ be any sequence of ideals in R. Let N be the point from which they begin to stabilize. Then also $f(I_N)$ is the point from which they stabilize, because if $A = B$ then $f(A) = f(B)$.
But what if in S we had a sequence of ideals $J_1 \subseteq J_2 \subseteq \ldots$ that did not stabilize? Then we would have $f^{-1}(J_1) \subseteq f^{-1}(J_2) \subseteq \ldots$ a sequence of ideals in R. Now, let us take any $J_n = (a_1, \ldots, a_k)$ and $J_{n+1} = (a_1, \ldots, a_k, \ldots a_m)$.

$$f^{-1}(J_n) = f^{-1}(a_1, \ldots, a_k) = (f^{-1}(a_1), \ldots, f^{-1}(a_k))$$

$$f^{-1}(J_{n+1}) = f^{-1}(a_1, \ldots, a_k, \ldots, a_m) = (f^{-1}(a_1), \ldots, f^{-1}(a_k), \ldots, f^{-1}(a_m))$$

and because $J_n \neq J_{n+1}$ then $(f^{-1}(a_1), \ldots, f^{-1}(a_k)) \neq (f^{-1}(a_1), \ldots, f^{-1}(a_k), \ldots, f^{-1}(a_m))$ so for each J_n we have a new ideal in R and that is a contradiction.

ZAD. 8.

Show that the rignt $\mathbb{Z}[\sqrt{2}]$ is Euclidean.

I don't really understand the definition.