UNIVERSITY OF MARYLAND

18 56

FINAL PROJECT

# Control of Robotic Systems

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ENPM667

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# 1 Introduction

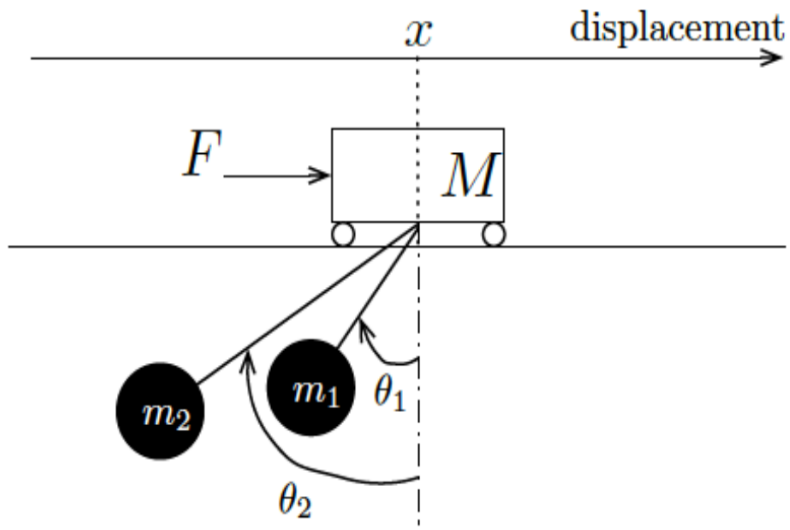


Figure 1: A crane that moves along 1 dimensional track and its configurations

In the given project statement, a crane moves along an 1-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m1 and m2, and the lengths of the cables are l1 and l2, respectively.

## 1.1 Given

- Mass of load denoted as (m1) $\rightarrow m1$.
- Mass of load denoted as (m2) $\rightarrow m2$.
- Length of cable connecting mass m1 to the crane $\rightarrow l1$.
- Angle made between load m1 and Y-axis $\rightarrow \theta_1$.
- Angle made between load m2 and Y-axis $\rightarrow \theta_2$.
- Length of cable connecting mass m2 to the crane $\rightarrow l2$.
- Applied force on cart $\rightarrow F$.
- Mass of the cart $\rightarrow M$.

## 1.2 Conditions and assumptions

- No friction across system.
- Cable does not stretch and mass is negligible when compared to mass of cart.
- Cart moves in X and load in X-Y directions.
- SIMULINK/MATLAB is required to run simulation code.

# 2 A. Equations of motion for the system and the corresponding nonlinear state-space representation

Modeling of the cart and load is achieved by using Euler-Lagrange method. This method uses kinetic and potential energies to derive the equations of motion for the given system. The Euler-Lagrange equation is given as,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = f \tag{1}$$

- Lagrange operator $\rightarrow$ $\mathcal{L}$ = $T$ - $V$.
- Generalized coordinates $\rightarrow q$.
- Kinetic energy $\rightarrow T$.
- Potential energy $\rightarrow V$.
- Force acting on the system $\rightarrow f$.

## 2.1 Lagrangian equation

The generalized coordinates of system are x, $\theta_1$ and $\theta_2$. Displacement of load 1 and load 2 can be written as,

$$x_{m_1} = x - l_1 s_1 \tag{2}$$

$$y_{m1} = -l_1 c_1 \tag{3}$$

$$x_{m2} = x - l_2 s_2 \tag{4}$$

$$x_{m2} = -l_1 c_1 \tag{5}$$

x is the displacement of cart in +ve x-direction.

Derivatives of above gives us the velocities of each load,

$$\dot{x}_{m_1} = \dot{x} - l_1 s_1 \dot{\theta}_1 \tag{6}$$

$$\dot{y}_{m1} = -l_1 c_1 \tag{7}$$

$$\dot{x}_{m_2} = \dot{x} - l_2 s_2 \dot{\theta}_2 \tag{8}$$

$$\dot{x}_{m2} = -l_1 c_1 \tag{9}$$

The magnitude of velocity vector for each load can be computed and is given by,

$$v_1^2 = \dot{x}_{m_1}^2 + \dot{y}_{m_1}^2 = (\dot{x} - l_1 c_1 \dot{\theta}_1)^2 + (l_1 s_1 \dot{\theta}_1)^2 = \dot{x}^2 + l_1 \dot{\theta_1}^2 - 2 l_1 c_1 \dot{x} \dot{\theta}_1 \tag{10}$$

$$v_2^2 = \dot{x}_{m_2}^2 + \dot{y}_{m_2}^2 = (\dot{x} - l_2 c_2 \dot{\theta}_2)^2 + (l_2 s_2 \dot{\theta}_2)^2 = \dot{x}^2 + l_2 \dot{\theta_2}^2 - 2 l_2 c_2 \dot{x} \dot{\theta}_2 \tag{11}$$

The velocity of cart is given by $\dot{x}$ .The sum of kinetic energies of cart and both load is shown as,

$$K.E = \frac{1}{2}\dot{x}^2(M) + \frac{1}{2}(m_1 v_1^2) + \frac{1}{2}(m_2 v_2^2) \tag{12}$$

Substitute the values of x, $v_1$, $v_2$,

$$K.E = \frac{1}{2}\dot{x}^2(M + m_1 + m_2) + \frac{1}{2}(m_1 l_1 \dot{\theta_1}^2) + \frac{1}{2}(m_2 l_2 \dot{\theta_2}^2) - m_1 l_1 c_1 \dot{x}\dot{\theta_1} - m_2 l_2 c_2 \dot{x}\dot{\theta_2} \tag{13}$$

For computing the Potential Energy of the system, the point where the the two loads are attached to the cart is taken as reference. Therefore, it consists of components from the pendulums only and is given by,

$$P.E = -m_1 g(l_1 c_1) - m_2 g(l_2 c_2) \tag{14}$$

Now, the Lagrange of the system can be calculated as,

$$\mathcal{L} = K.E - P.E \tag{15}$$

$$\mathcal{L} = \frac{1}{2}\dot{x}^2(M + m_1 + m_2) + \frac{1}{2}(m_1 l_1 \dot{\theta_1}^2) + \frac{1}{2}(m_2 l_2 \dot{\theta_2}^2) - m_1 l_1 c_1 \dot{x}\dot{\theta_1} - m_2 l_2 c_2 \dot{x}\dot{\theta_2} + m_1 g l_1 c_1 + m_2 g l_2 c_2 \tag{16}$$

Next, we compute the derivative of the Lagrangian with respect to $\dot{x}$, $\dot{\theta_1}$, $\dot{\theta_2}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F \tag{17}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \tag{18}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0 \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x}(M + m_1 + m_2) - m_1 l_1 c_1 \dot{\theta_1} - m_2 l_2 c_2 \dot{\theta_2} \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \tag{21}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \ddot{x}(M + m_1 + m_2) - m_1 l_1(\ddot{\theta_1} c_1 - s_1 \dot{\theta}_1^2) - m_2 l_2 c_2(\ddot{\theta_2} c_2 - s_2 \dot{\theta}_2^2) = F \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}} = m_1 l_1^2 \dot{\theta_1} - m_1 l_1 c_1 \dot{x} \tag{23}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}}\right) = m_1 l_1^2 \ddot{\theta_1} - m_1 l_1(c_1 \ddot{x} - s_1 \dot{x}\dot{\theta_1}) \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = s_1 m_1 l_1 m_1 \dot{x}\dot{\theta_1} - m_1 g l_1 s_1 \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}} = m_2 l_2^2 dot\theta_2 - m_2 l_2 c_2 \dot{x} \tag{26}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = s_2 m_2 l_2 m_2 \dot{x} dot\theta_2 - m_2 g l_2 s_2 \tag{27}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}}\right) = m_2 l_2^2 \ddot{\theta_2} - m_2 l_2 (c_2 \ddot{x} - s_2 \dot{x}\dot{\theta_2}) \tag{28}$$

Thus we get,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = m_1 l_1^2 \ddot{\theta_1} - m_1 l_1 (c_1 \ddot{x} - s_1 \dot{x}\dot{\theta_1}) - s_1 m_1 l_1 m_1 \dot{x}\dot{\theta_1} + m_1 g l_1 s_1 = 0 \tag{29}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_1}}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = m_1 l_1^2 \ddot{\theta_1} - m_1 l_1 c_1 \ddot{x} + m_1 g l_1 s_1 \tag{30}$$

Now we assume that for very small angle we take the following assumptions:

$$c_1 = 1, s_1 = \theta_1, \dot{\theta_1}^2 = 0, \dot{\theta_2}^2 = 0 \tag{31}$$

Thus upon linearizing the previous equation, we get:

$$m_1 l_1^2 \ddot{\theta_1} - m_1 l_1 \ddot{x} + m_1 g l_1 \theta_1 = 0 \tag{32}$$

Similarly

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_2^2 \ddot{\theta_2} - m_2 l_2 (c_2 \ddot{x} - s_2 \dot{x}\dot{\theta_2}) - s_2 m_2 l_2 m_2 \dot{x}\dot{\theta_2} + m_2 g l_2 s_2 = 0 \tag{33}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta_2}}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_2^2 \ddot{\theta_2} - m_2 l_2 c_2 \ddot{x} + m_2 g l_2 s_2 \tag{34}$$

linearizing the previous equation, we get:

$$m_2 l_2^2 \ddot{\theta_2} - m_2 l_2 \ddot{x} + m_2 g l_2 \theta_2 = 0 \tag{35}$$

$$F = \ddot{x}(M + m_1 + m_2) - m_1 l_1 (\ddot{\theta_1} c_1 - s_1 \dot{\theta}_1^2) - m_2 l_2 c_2 (\ddot{\theta_2} c_2 - s_2 \dot{\theta}_2^2) \tag{36}$$

$$m_1 l_1^2 \ddot{\theta_1} - m_1 l_1 c_1 \ddot{x} + m_1 g l_1 s_1 = 0 \tag{37}$$

$$m_2 l_2^2 \ddot{\theta_2} - m_2 l_2 c_2 \ddot{x} + m_2 g l_2 s_2 = 0 \tag{38}$$

$$\ddot{\theta_1} = \frac{C_1 \ddot{x} - g s_1}{l_1} \tag{39}$$

$$\ddot{\theta_2} = \frac{C_2 \ddot{x} - g s_2}{l_2} \tag{40}$$

Next, we substitute the obtained values of $\ddot{\theta_1}$, $\ddot{\theta_2}$ into the equation $\ddot{x}$ as follows:

$$\ddot{x} = \frac{F + m_1 l_1 (\ddot{\theta_1} c_1 - s_1 \dot{\theta_1}^2) + m_2 l_2 (\ddot{\theta_2} c_2 - s_2 \dot{\theta_2}^2)}{(M + m_1 + m_2)} \tag{41}$$

$$\ddot{x} = \frac{F + m_1 l_1 ((\frac{c_1 \ddot{x} - g s_1}{l_1}) c_1 - s_1 \dot{\theta_1}^2) + m_2 l_2 ((\frac{C_2 \ddot{x} - g s_2}{l_2}) c_2 - s_2 \dot{\theta_2}^2)}{(M + m_1 + m_2)} \tag{42}$$

$$\ddot{x} = \frac{F + m_1 (\ddot{x} c_1^2 - g S_1 c_1 - l_1 s_1 \dot{\theta_1}^2) + m_2 (\ddot{x} c_2^2 - g s_2 c_2 - l_2 s_2 \dot{\theta_2}^2)}{(M + m_1 + m_2)} \tag{43}$$

$$(M + m_1 + m_2)\ddot{x} - m_1 c_1^2 \ddot{x} - m_2 c_2^2 \ddot{x} = F - m_1(g s_1 c_1 + l_1 s_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 s_2 \dot{\theta_2}^2) \tag{44}$$

$$(M + m_1(1 - c_1^2) + m_2(1 - c_2^2))\ddot{x} = F - m_1(g s_1 c_1 + l_1 s_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 s_2 \dot{\theta_2}^2) \tag{45}$$

Upon separating  from the equation we get

$$\ddot{x} = \frac{F - m_1(g s_1 c_1 + l_1 s_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 s_2 \dot{\theta_2}^2)}{M + m_1 s_1^2 + m_2 s_2^2} \tag{46}$$

Next, we substitute the value of $\ddot{x}$ in the previously derived equations of $\ddot{\theta_1}$, $\ddot{\theta_2}$ we get

$$\ddot{\theta_1} = \frac{c_1}{l_1}\left(\frac{F - m_1(g s_1 c_1 + l_1 S_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 S_2 \dot{\theta_2}^2)}{M + m_1 s_1^2 + m_2 s_2^2}\right) - g\frac{s_1}{l_1} \tag{47}$$

$$\ddot{\theta_2} = \frac{c_2}{l_2}\left(\frac{F - m_1(g s_1 c_1 + l_1 s_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 s_2 \dot{\theta_2}^2)}{M + m_1 s_1^2 + m_2 s_2^2}\right) - g\frac{s_2}{l_2} \tag{48}$$

## 2.2 Non-linear state space representation

Thus, the equations derived above are the non-linear equations of the system. Thus, nonlinear state space representation of the above given system is:

$$\dot{X} = AX + BU \tag{49}$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta_1} \\ \ddot{\theta_1} \\ \dot{\theta_2} \\ \ddot{\theta_2} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{F - m_1(g s_1 c_1 + l_1 s_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 s_2 \dot{\theta_2}^2)}{M + m_1 s_1^2 + m_2 s_2^2} \\ \dot{\theta_1} \\ \frac{c_1}{l_1}\left(\frac{F - m_1(g s_1 c_1 + l_1 s_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 s_2 \dot{\theta_2}^2)}{M + m_1 s_1^2 + m_2 s_2^2}\right) - g\frac{s_1}{l_1} \\ \dot{\theta_2} \\ \frac{c_2}{l_2}\left(\frac{F - m_1(g S_1 c_1 + l_1 s_1 \dot{\theta_1}^2) - m_2(g s_2 c_2 + l_2 s_2 \dot{\theta_2}^2)}{M + m_1 s_1^2 + m_2 s_2^2}\right) - g\frac{s_2}{l_2} \end{bmatrix} \tag{50}$$

# 3 B. Linearizing system around the equilibrium point and state-space representation of the linearized system.

Linearization is done by small angle approximation.
Now we take the non-linear equations derived earlier:

$$F = \ddot{x}(M + m_1 + m_2) - m_1 l_1 \ddot{\theta_1} - m_2 l_2 \ddot{\theta_2} \tag{51}$$

$$\ddot{\theta_1} = \frac{m_1 l_1 \ddot{x} - m_1 g l_1 \theta_1}{m_1 l_1^2} \tag{52}$$

$$\ddot{\theta_1} = \frac{\ddot{x} - g\theta_1}{l_1} \tag{53}$$

$$\ddot{\theta_2} = \frac{\ddot{x} - g\theta_2}{l_2} \tag{54}$$

$$\ddot{\theta_2} = \frac{m_2 l_2 \ddot{x} - m_2 g l_2 \theta_2}{m_2 l_2^2} \tag{55}$$

Upon substituting the values of $\ddot{\theta_1}$, $\ddot{\theta_2}$we get

$$F = \ddot{x}(M + m_1 + m_2) - m_1 l_1 \left(\frac{\ddot{x} - g\theta_1}{l_1}\right) - m_2 l_2 \left(\frac{\ddot{x} - g\theta_2}{l_2}\right) \tag{56}$$

$$F = M\ddot{x} + m_1 g\theta_1 + m_2 g\theta_2 \tag{57}$$

Thus,

$$\ddot{x} = \frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M} \tag{58}$$

$$\ddot{\theta_1} = \frac{\frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M}}{l_1} - \frac{g\theta_1}{l_1} \tag{59}$$

$$\ddot{\theta_1} = -\left(\frac{m_1 g}{M l_1} + \frac{g}{l_1}\right)\theta_1 - \left(\frac{m_2 g}{M l_1}\right)\theta_2 + \frac{F}{M l_1} \tag{60}$$

$$\ddot{\theta_2} = \frac{\frac{F - m_1 g\theta_1 - m_2 g\theta_2}{M}}{l_2} - \frac{g\theta_2}{l_2} \tag{61}$$

$$\ddot{\theta_2} = -\left(\frac{m_2 g}{M l_2} + \frac{g}{l_2}\right)\theta_2 - \left(\frac{m_1 g}{M l_2}\right)\theta_2 + \frac{F}{M l_2} \tag{62}$$

The state space representation of the above equations is as follows:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta_1} \\ \ddot{\theta_1} \\ \dot{\theta_2} \\ \ddot{\theta_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta_1} \\ \theta_2 \\ \dot{\theta_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F \tag{63}$$

# 4 C. Conditions for which the linearized system is controllable

Next, we obtain the conditions for which the system is controllable. The A and B matrices obtained above are independent of time and thus the system is an LTI system. An LTI system is controllable if the controllability matrix obtained has full rank condition. The dimensions of the controllability matrix 'C' are n x nm, hence its rank should be equal to n. Hence,

$$rank(C) = rank\begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} = n \tag{64}$$

Controllability matrix is obtained using MATLAB.

$$C = \begin{bmatrix} 0 & \frac{1}{M} & 0 & a_3 & 0 & a_6 \\ \frac{1}{M} & 0 & a_3 & 0 & a_6 & 0 \\ 0 & \frac{1}{Ml_1} & 0 & a_1 & 0 & a_4 \\ \frac{1}{Ml_1} & 0 & a_1 & 0 & a_4 & 0 \\ 0 & \frac{1}{Ml_2} & 0 & a_2 & 0 & a_5 \\ \frac{1}{Ml_2} & 0 & a_2 & 0 & a_5 & 0 \end{bmatrix} \tag{65}$$

Where ,

$$a_1 = -\frac{g(Ml_2 + l_1m_2 + l_2m_1)}{M^2l_1^2l_2} \tag{66}$$

$$a_2 = -\frac{g(Ml_1 + l_1m_2 + l_2m_1)}{M^2l_1l_2^2} \tag{67}$$

$$a_3 = -\frac{g(m_2l_1 + m_1l_2)}{M^2l_1l_2} \tag{68}$$

$$a_4 = \frac{g^2(M^2l_2^2 + Ml_1^2m_2 + Ml_1l_2m_2 + 2Ml_2^2m_1 + a_8 + 2l_1l_2m_1m_2 + a_7)}{M^3l_1^3l_2^2} \tag{69}$$

$$a_5 = \frac{g^2(M^2l_1^2 + 2Ml_1^2m_2 + Ml_1l_2m_1 + Ml_2^2m_1 + a_8 + 2l_1l_2m_1m_2 + a_7)}{M^3l_1^2l_2^3} \tag{70}$$

$$a_6 = \frac{g^2(a_8 + Ml_1^2m_2 + 2l_1l_2m_1m_2 + a_7 + Ml_2^2m_1)}{M^3l_1^2l_2^2} \tag{71}$$

$$a_7 = l_2^2m_1^2 \tag{72}$$

$$a_8 = l_1^2m_2^2 \tag{73}$$

For the above given controllability matrix to be of full rank, its determinant should not be equal to be zero i.e. det(C) $\neq 0 i.e.$

$$det(C) = \frac{-g^6(l_1^2 - l_2^2)}{M^6l_1^6l_2^6} \neq 0 \tag{74}$$

The determinant of C matrix won't be equal to zero only when $l_1 \neq l_2$. Thus, the given system is controllable only when the lengths of the cables of the crane isn't equal.

# 5 D. Check the system is controllable and obtain an LQR controller

We are given that $M$ = 1000kg, $m_1$ = $m_2$ = 100kg, $l_1$ = 20m, $l_2$ = 10m.Thus, we get the values of A matrix as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.98 & 0 & -9.98 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -5.39 & 0 & 0.049 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.098 & 0 & -1.078 & 0 \end{bmatrix} \tag{75}$$

The eigen values of A matrix before applying the LQR controller are:

$$0.0000 + 0.0000i$$

$$0.0000 + 0.0000i$$

$$-0.0000 + 0.7285i$$

$$-0.0000 - 0.7285i$$

$$0.0000 + 1.0430i$$

$$0.0000 - 1.0430i$$

We know that Q and R are symmetric positive definite matrices. So, using Ricatti Equation given below, we compute P matrix.

Q and R are the weight matrices that can be manipulated by us.

$$A^T + PA - PBR^-1B^TP + Q = 0$$

where,

$$K = R^-1B^TP$$

also,

$$U = -KX$$

Upon substituting the value of K obtained in (A-BK), we get a stabilizable controller that has proper system eigen values, which also minimizes the cost J calculated earlier.

The Q and R are selected on the basis of trial and error. The Q matrix chosen is:

$$Q = \begin{bmatrix} 0.0833 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 312.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 833.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 312.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 312.5 \end{bmatrix} \tag{76}$$

The eigen values after applying the LQR controller are:

$$-0.8934 + 0.0000i$$

$$-0.1013 + 0.0000i$$

$$-0.3929 + 0.8445i$$

$$-0.3929 - 0.8445i$$

$$-0.1279 + 0.7370i$$

$$-0.1279 - 0.7370i$$

Since the eigen values of (A-B*K) matrix have negative real part, the original system is locally stable around equilibrium point (0, 0). In this case, a Lyapunov function for the linearized system will be valid at least locally. The initial states of the system were taken to be 5 degrees for mass 1 and 5 degrees for mass 2.

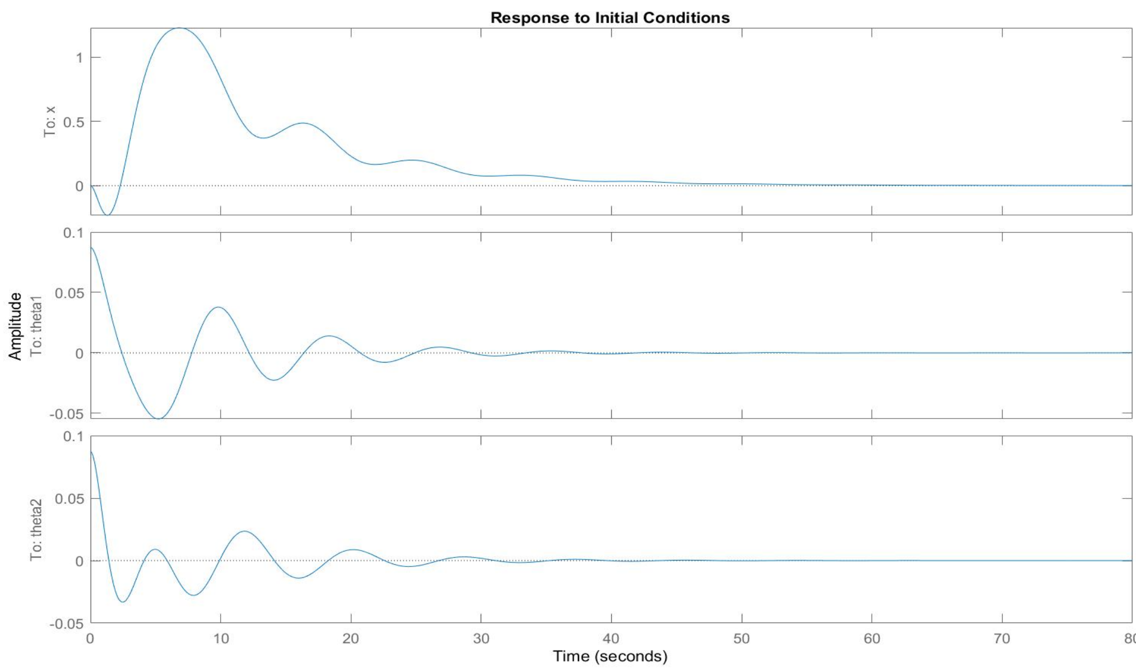


Figure 2: Response to initial conditions of linearized system

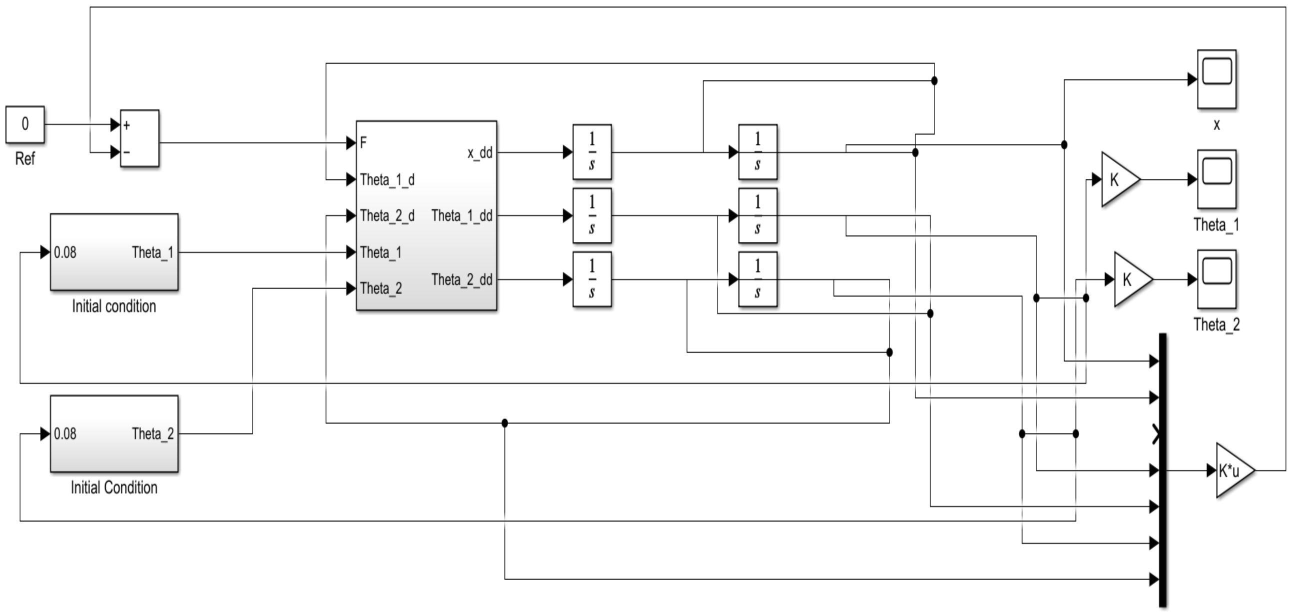


Figure 3: LQR Controller for Non-Linear system

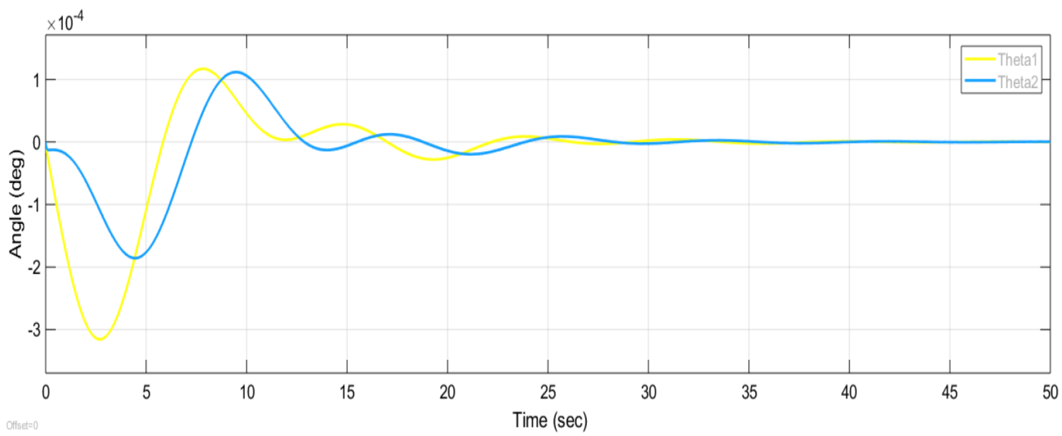


Figure 4: Angle of load on NonLinear system

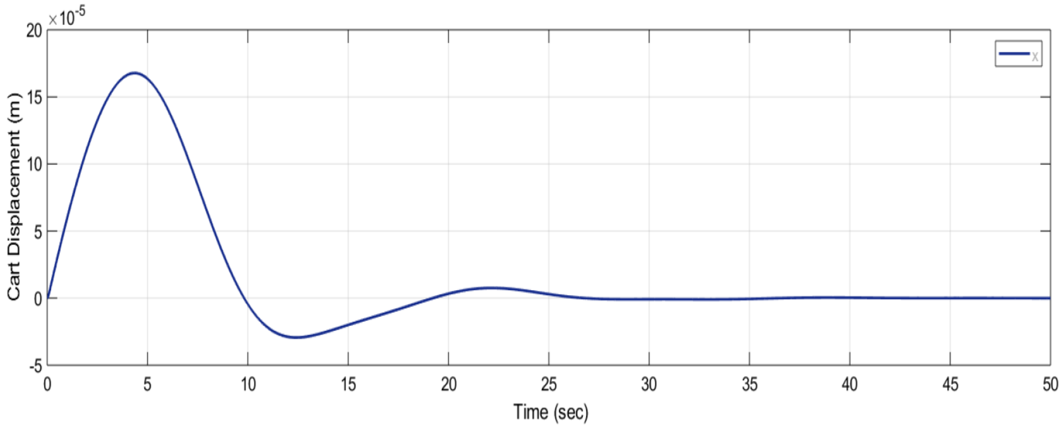


Figure 5: Position of cart on NonLinear system

# 6 E. Determine the observability of the linearized system for given output vectors

Here we have four output vectors: $x(t)$, $(\theta_1(t);\theta_2(t))$, $(x(t);\theta_2(t))$ and $(x(t);\theta_1(t);\theta_2(t))$
For vextor: $x(t)$,

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{77}$$

For vector: $(\theta_1(t);\theta_2(t))$,

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{78}$$

For vector: $(x(t);\theta_2(t))$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{79}$$

For output vector: $(x(t);\theta_1(t);\theta_2(t))$

$$C_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{80}$$

From the code, we get the rank for respective output vector as 6,4,6,6. So the output vector $\theta_1(t);\theta_2(t)$ is not observable.

```matlab
syms l1 l2 m1 m2 g M
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M*l1) 0 -g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*l2) 0];
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];
C1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
C2 = [0 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 1 0];
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
Observer1 = rank([C1' A'*C1' ((A')^2)*C1' ((A')^3)*C1' ((A')^4)*C1' ((A')^5)*C1'])
Observer2 = rank([C2' A'*C2' ((A')^2)*C2' ((A')^3)*C2' ((A')^4)*C2' ((A')^5)*C2'])
Observer3 = rank([C3' A'*C3' ((A')^2)*C3' ((A')^3)*C3' ((A')^4)*C3' ((A')^5)*C3'])
Observer4 = rank([C4' A'*C4' ((A')^2)*C4' ((A')^3)*C4' ((A')^4)*C4' ((A')^5)*C4'])
```

Command Window

New to MATLAB? See resources for Getting Started.

```
>> obs

Observer1 =

     6


Observer2 =

     4


Observer3 =

     6


Observer4 =

     6
```

Figure 6: Ranks of given output vectors

# 7 F. Obtain Luenberger observer for the output vectors for which the system is observable

The Luenberger Observer is written in state-space representation as:

$$\dot{\hat{X}}(t) = A\hat{x} + B_k U_k(t) + L(Y(t) - C\hat{x}(t))$$

Here, $\hat{x}(t)$ is state estimator, $L$ is observer gain matrix, $Y(t) - C\hat{x}(t)$ is correction term and $\hat{x}(0) = 0$. The estimation error $X_e(t) = X(t) - \hat{X}(t)$ has the following state space representation:

$$\dot{X}_e(t) = \dot{X}(t) - \dot{\hat{X}}(t)$$

$$\dot{X}_e(t) = AX_e(t) - L(Y(t) - C\hat{x}(t)) + B_d U_d(t)$$

Here, we assume $D = 0$, $Y(t) = Cx(t)$.Therefore, the equation can be written as

$$\dot{X}_e(t) = (A - LC)X_e(t) + B_d U_d(t)$$

Now, let's see the MATLAB code and its output respectively
The code for the above response can be found in the section MATLAB CODE - 1

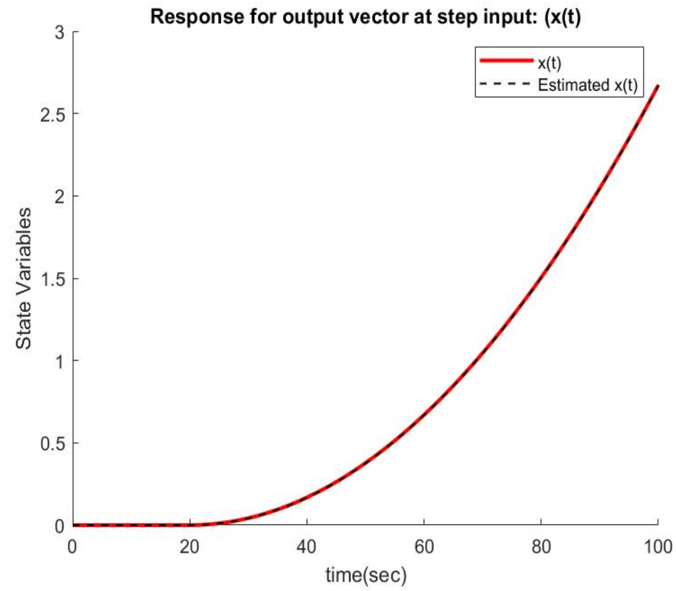


Figure 7:

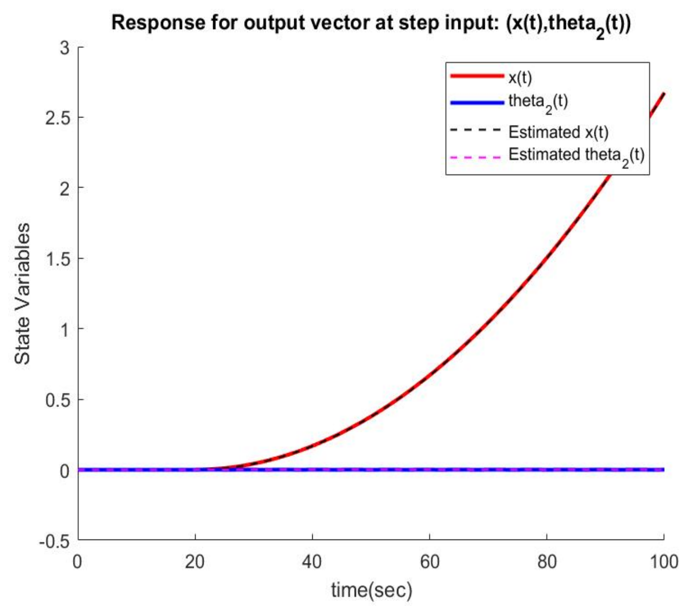


Figure 8:

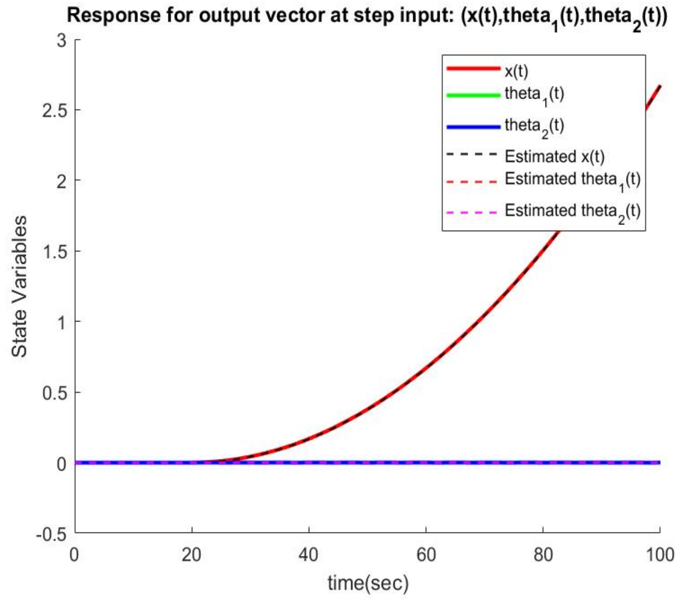


Figure 9:

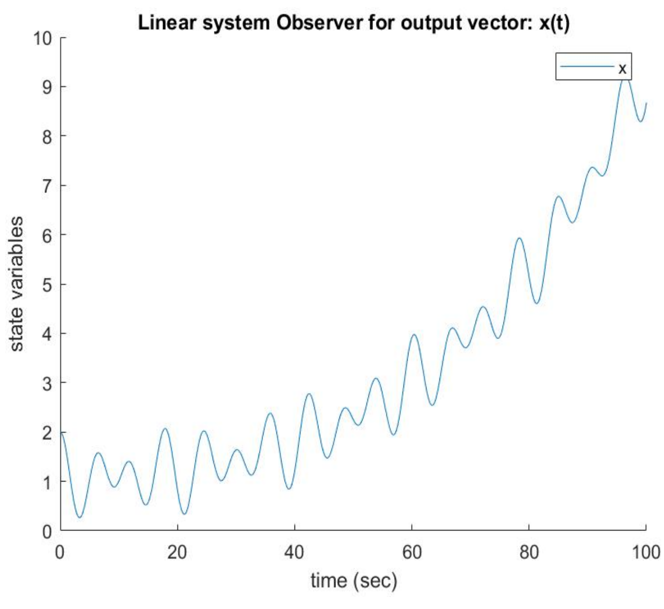


Figure 10:

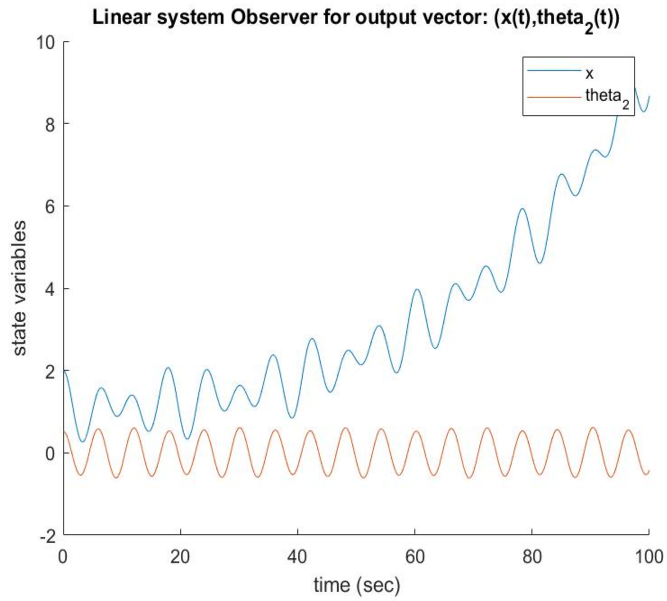


Figure 11:

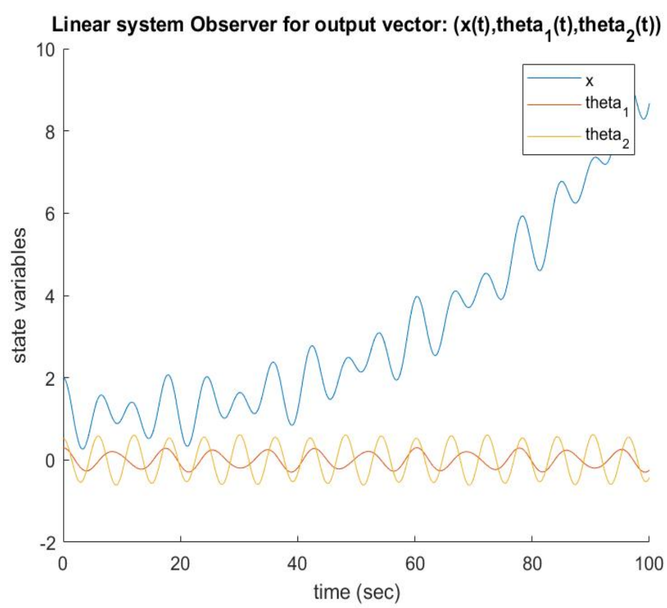


Figure 12:

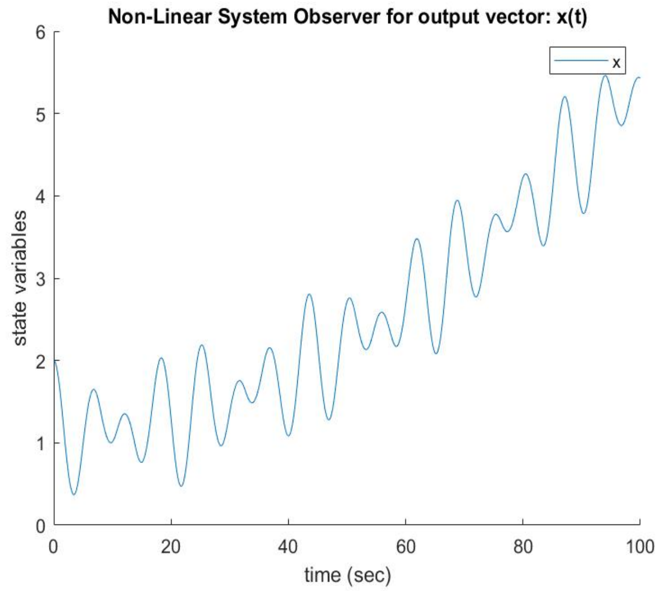


Figure 13:

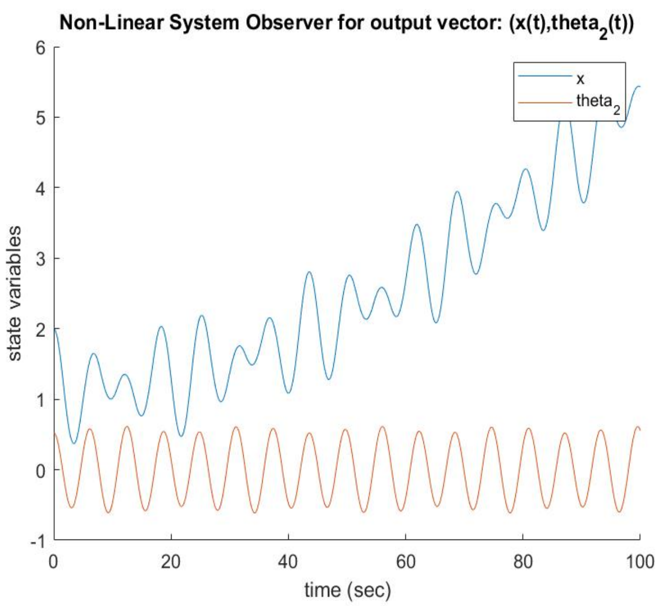


Figure 14:

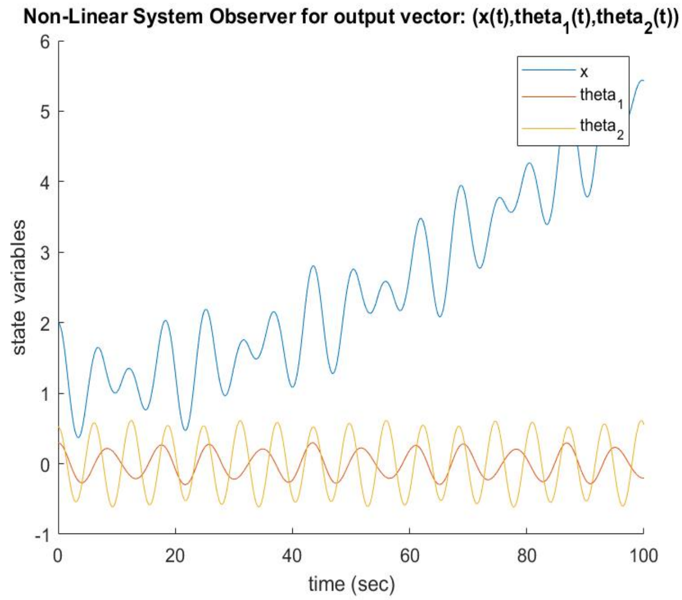


Figure 15:

# 8 G. Design an output feedback controller - LQG

The LQG controller is blend of LQR controller and the Kalman filter. On substituting Q = 1, R = 0.1, noise Bd = 0.1 and Vd = 0.01, we get the following response curve of the output. The code for the following response is in the section MATLAB CODE - 2.

We will reconfigure our controller by providing a desired x vector and tune the LQR controller accordingly to get the feedback according to the desired output. Yes, our design can reject constant force disturbances applied on the cart. If you increase the disturbance noise in the Kalman filter, you will notice that the LQR controller is strong enough to stabilize the x(t) cart position within few seconds.

# 9 MATLAB CODE - 1

```
syms l1 l2 m1 m2 g M
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;
q0 = [2 0 deg2rad(17) 0 deg2rad(30) 0];
tspan = 0:0.1:100;
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M
    *l1) 0 -g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*
    l2) 0];
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];
C1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
C2 = [0 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 1 0];
```

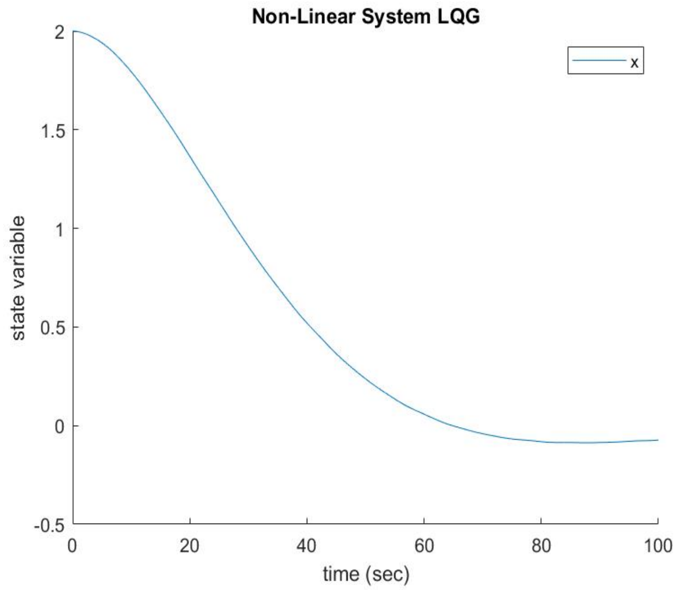


Figure 16: Response for output vector: x(t)

```
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
D = [0; 0; 0];
Observer1 = rank([C1' A'*C1' ((A')^2)*C1' ((A')^3)*C1' ((A')^4)*C1' ((A')^5)*
    C1']);
Observer2 = rank([C2' A'*C2' ((A')^2)*C2' ((A')^3)*C2' ((A')^4)*C2' ((A')^5)*
    C2']);
Observer3 = rank([C3' A'*C3' ((A')^2)*C3' ((A')^3)*C3' ((A')^4)*C3' ((A')^5)*
    C3']);
Observer4 = rank([C4' A'*C4' ((A')^2)*C4' ((A')^3)*C4' ((A')^4)*C4' ((A')^5)*
    C4']);
SYS1 = ss(A,B,C1,D);
SYS3 = ss(A,B,C3,D);
SYS4 = ss(A,B,C4,D);
Bd = 0.1*eye(6);
Vn = 0.01;
[Lue_1,P,E] = lqe(A,Bd,C1,Bd,Vn*eye(3));
[Lue_3,P,E] = lqe(A,Bd,C3,Bd,Vn*eye(3));
[Lue_4,P,E] = lqe(A,Bd,C4,Bd,Vn*eye(3));
AC1 = A-(Lue_1*C1);
AC3 = A-(Lue_3*C3);
AC4 = A-(Lue_4*C4);
sys1 = ss(AC1,[B Lue_1],C1,0);
sys3 = ss(AC3,[B Lue_3],C3,0);
sys4 = ss(AC4,[B Lue_4],C4,0);
unitStep = 0*tspan;
unitStep(200:length(tspan)) = 1;
[y1,t] = lsim(SYS1,unitStep,tspan);
[x1,t] = lsim(sys1,[unitStep;y1'],tspan);
[y3,t] = lsim(SYS3,unitStep,tspan);
```

```
[x3,t] = lsim(sys3,[unitStep;y3'],tspan);
[y4,t] = lsim(SYS4,unitStep,tspan);
[x4,t] = lsim(sys4,[unitStep;y4'],tspan);
figure();
hold on
plot(t,y1(:,1),'r','Linewidth',2)
plot(t,x1(:,1),'k--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','Estimated x(t)')
title('Response for output vector at step input: (x(t)')
hold off
figure();
hold on
plot(t,y3(:,1),'r','Linewidth',2)
plot(t,y3(:,3),'b','Linewidth',2)
plot(t,x3(:,1),'k--','Linewidth',1)
plot(t,x3(:,3),'m--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','theta_2(t)','Estimated x(t)','Estimated theta_2(t)')
title('Response for output vector at step input: (x(t),theta_2(t))')
hold off
figure();
hold on
plot(t,y4(:,1),'r','Linewidth',2)
plot(t,y4(:,2),'g','Linewidth',2)
plot(t,y4(:,3),'b','Linewidth',2)
plot(t,x4(:,1),'k--','Linewidth',1)
plot(t,x4(:,2),'r--','Linewidth',1)
plot(t,x4(:,3),'m--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','theta_1(t)','theta_2(t)','Estimated x(t)','Estimated theta_1(t)
    ','Estimated theta_2(t)')
title('Response for output vector at step input: (x(t),theta_1(t),theta_2(t))'
    )
hold off
%% Linear system response
[t,q1] = ode45(@(t,q)linearObs1(t,q,Lue_1),tspan,q0);
figure();
hold on
plot(t,q1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: x(t)')
legend('x')
hold off
[t,q3] = ode45(@(t,q)linearObs3(t,q,Lue_3),tspan,q0);
figure();
hold on
```

```
plot(t,q3(:,1))
plot(t,q3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: (x(t),theta_2(t))')
legend('x','theta_2')
hold off
[t,q4] = ode45(@(t,q)linearObs4(t,q,Lue_4),tspan,q0);
figure();
hold on
plot(t,q4(:,1))
plot(t,q4(:,3))
plot(t,q4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: (x(t),theta_1(t),theta_2(t))'
    )
legend('x','theta_1','theta_2')
hold off
%% Non-linear system observer response
[t,q1] = ode45(@(t,q)nonLinearObs1(t,q,1,Lue_1),tspan,q0);
figure();
hold on
plot(t,q1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: x(t)')
legend('x')
hold off
[t,q3] = ode45(@(t,q)nonLinearObs3(t,q,1,Lue_3),tspan,q0);
figure();
hold on
plot(t,q3(:,1))
plot(t,q3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: (x(t),theta_2(t))')
legend('x','theta_2')
hold off
[t,q4] = ode45(@(t,q)nonLinearObs4(t,q,1,Lue_4),tspan,q0);
figure();
hold on
plot(t,q4(:,1))
plot(t,q4(:,3))
plot(t,q4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: (x(t),theta_1(t),theta_2(
    t))')
legend('x','theta_1','theta_2')
hold off
```

# 10 MATLAB CODE - 2

```
syms m1 m2 g M l1 l2 x dx
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;
tspan = 0:0.1:100;
q_init = [2 0 deg2rad(0) 0 deg2rad(0) 0];
A = [0 1 0 0 0 0; 0 0 -m1*g/M 0 -m2*g/M 0; 0 0 0 1 0 0; 0 0 -((M*g)+(m1*g))/(M
    *l1) 0 -g*m2/(M*l1) 0; 0 0 0 0 0 1; 0 0 -m1*g/(M*l2) 0 -((M*g)+(m2*g))/(M*
    l2) 0];
B = [0; 1/M; 0; 1/(l1*M); 0; 1/(l2*M)];
C1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
D = [1;0;0];
sys = ss(A,B,C1,D);
Q = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0
    0 0];
R = 0.1;
[K,S,P] = lqr(A,B,Q,R);
sys = ss(A-B*K,B,C1,D);
Bd = 0.1*eye(6);
Vn = 0.01;
[Lue,P,E] = lqe(A,Bd,C1,Bd,Vn*eye(3));
AC1 = A-(Lue*C1);
sys_ob = ss(AC1,[B Lue],C1,0);
[t,q1] = ode45(@(t,q)nonLinearObs1(t,q,-K*q,Lue),tspan,q_init);
figure();
hold on
plot(t,q1(:,1))
ylabel('state variable')
xlabel('time (sec)')
title('Non-Linear System LQG')
legend('x')
hold off
```