Optimal Bread Slicing

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1 Introduction

While countless[1] branches of mathematics abstract real-world phenomena to explore richer understandings, the same has yet to be done for slicing bread. For 95 years, sliced bread has remained a phenomenon investigated empirically by bakers and food engineers. Expanding sliced bread to mathematics, we present a formal construction of bread slicing. From there, we offer straightforward utility functions to optimize various aspects of bread slicing. The vast majority of this work follows intuitively from real-world bread, and most proofs in this work are proof sketches, since full proofs would distract from the important contributions of this work.

2 Construction of Problem

Before we can understand how to optimally slice a loaf of bread, we must first construct a loaf of bread. For our purposes, we only care about the shape of the loaf. As such, we provide the following definitions.

Definition 2.1. (Loaf Unit). A set B is a loaf unit if and only if it can be expressed as:

$$B = \left\{(x, y, z) \in \mathbb{R}^3 \middle|\ 0 \leq x \leq w,\ g(x) \leq x \leq f(x)\right\}$$

where $w > 0$ and $f : [0, w] \to \mathbb{R}$, $g : [0, w] \to \mathbb{R}$ are bounded, Lebesgue-integrable functions such that for all $x \in [0, w]$, $g(x) \leq f(x)$. Note that this definition extends to sets that become loaf units through rigid motion.

Loaf units are a type of prism in three dimensions with infinite length. Their face is the space bounded by the functions f and g on the interval $[0, w]$. Loaf units are equivalent up to rotation, reflection, and translation in $\mathbb{R}^3$, as their position is not relevant.

Definition 2.2. (Loaf). A set B is a loaf if and only if it can be expressed as the union of finitely-many, disjoint loaf units.

Our definition of a loaf allows for loaves with holes all the way through, or multiple, stacked overhangs. Without loss of generality, for the rest of this work, we consider a loaf made from only a single loaf unit, with no rotations, reflections, or translations.

Definition 2.3. (Slice). Given an angle $\theta \in \left(0, \frac{\pi}{2}\right]$ and a width $\delta > 0$, a (θ, δ, z_0)-slice is the set

$$S(\theta, \delta, z_0) = \left\{(x, y, z) \in \mathbb{R}^3 \middle|\ z_0 + x \cot\theta \leq z \leq z_0 + x \cot\theta + \delta\right\}$$

Definition 2.4. (Adjacent Slices). Without loss of generality, we say two slices $S(\theta_1, \delta_1, z_{0,1})$ and $S(\theta_2, \delta_2, z_{0,2})$ are adjacent if and only if $\theta_1 = \theta_2$ and $z_{0,1} = z_{0,2} + \delta_2$.

Theorem 2.5. *Adjacent Slices Touch. Given two adjacent slices, their intersection is the plane they share as a boundary.*

[1] a finite number

Proof. Take any two adjacent slices $S(\theta, \delta_1, z_0)$ and $S(\theta, \delta_2, z_0 + \delta_1)$. By definition, their intersection is the set of points $\{(x, y, z)\} \in \mathbb{R}^3$ such that

$$z_0 + x \cot\theta \leq z \leq z_0 + \delta_1 + x \cot\theta$$

and

$$z_0 + \delta_1 + x \cot\theta \leq z \leq z_0 + \delta_1 + x \cot\theta + \delta_2$$

This is only true for $z = z_0 + x \cot\theta + \delta_1$. As such, the intersection of the slices is the plane defined by that equation. □

Definition 2.6. (Slice of Bread). A slice of bread T is the intersection of a loaf B and a slice $S(\theta, \delta, z_0)$.

Theorem 2.7. *Angle Independence.The volume of a slice of bread is independent of the angle of the slice.*

Proof. Take any slice of bread T_1 as the intersection of a loaf B and a slice $S_1 = S(\theta_1, \delta, z_1)$. Let T_2 be the slice of bread of the same loaf and slice $S_2 = S(\theta_2, \delta, z_2)$ By definition, we can write $T_1 = \{(x, y, z) \in \mathbb{R}^3 |\ 0 \leq x \leq w,\ g(x) \leq y \leq f(x),\ z_1 + x \cot\theta_1 \leq z \leq z_1 + x \cot\theta_1 + \delta\}$. The volume of the slice can be expressed as

$$\begin{aligned}
m(T_1) &= m(S_1 \cap B) \\
&= \int_0^w \int_{g(x)}^{f(x)} \int_{z_1 + x\cot\theta_1}^{z_1 + x\cot\theta_1 + \delta} dzdydx \\
&= \delta \int_0^w \int_{g(x)}^{f(x)} dydx \\
&= \int_0^w \int_{g(x)}^{f(x)} \int_{z_2 + x\cot\theta_2}^{z_2 + x\cot\theta_2 + \delta} dzdydx \\
&= m(S_2 \cap B) \\
&= m(T_2)
\end{aligned}$$

As demonstrated, the volumes are equivalent. Therefore, the volume of a slice of bread is independent of the angle of the slice. □

Corollary 2.8. *Position Independence. The volume of a slice of bread is dependent only on the loaf and the width of the slice.*

Some may argue that this construction of the bread slice means we may double-count the intersection of adjacent slices when calculating volume. This is of no consequence.

Theorem 2.9. *No Added Volume. The volume of any finite number of adjacent slices is equivalent to the volume of a single slice that spans exactly those slices.*

Proof. Consider any two adjacent slices $S_1 = S(\theta, \delta_1, z_0)$ and $S_2 = S(\theta, \delta_2, z_0 + \delta_1)$ and a loaf B. We want to show $m((S_1 \cap B) \cup (S_2 \cap B)) = m(S_1 \cap B) + m(S_2 \cap B)$. By theorem 2.5, the intersection of S_1 and S_2 is a plane, so $m(S_1 \cap S_2) = 0$. Thus, $m((S_1 \cap B) \cap (S_2 \cap B)) = m(S_1 \cap S_2 \cap B) = 0$. From here, we can consider the original equation.

$$\begin{aligned}
m((S_1 \cap B) \cup (S_2 \cap B)) &= m(S_1 \cap B) + m(S_2 \cap B) - m((S_1 \cap B) \cap (S_2 \cap B)) \\
&= m(S_1 \cap B) + m(S_2 \cap B)
\end{aligned}$$

□

Informally, this theorem allows us to add back the faces of each slice with no meaningful change in the quantity of bread.

Definition 2.10. (Slice Face). The face of a slice of bread is the intersection of a loaf and one bounding plane of a slice. Given a slice $S = S(\theta, \delta, z_0)$, and a loaf B, the face of a slice can be expressed as $B \cap \{(x, y, z) \in S |\ z = z_0 + x \cot\theta\}$.

Theorem 2.11. *Area of Slice Face. Given a slice of bread from the slice $S(\theta, \delta, z_0)$ and loaf B, the area of one face of the bread is as follows.*

$$A_\theta = \csc\theta \int_0^w f(x) - g(x)dx$$

Proof. Take any slice $S = S(\theta, \delta, z_0)$ and loaf B. By definition, the slice is of the form $\{(x, y, z) \in S |\ 0 \le x \le w,\ g(x) \le y \le f(x),\ z = z_0 + x \cot\theta\}$. This is a region of the plane defined by the equation $z = z_0 + x\cot\theta$. Let this plane be denoted P. Note that the xz-region of the plane is defined as $\{(x, y, z) \in P |\ 0 \le x \le w, z = z_0 + x\cot\theta\}$. Call this region F. Let f_F and g_F on this region be the real-valued functions that give the bounds of y.

We can map this plane P onto the xy-plane by mapping the xy-plane onto P and subsequently taking the inverse. First, consider the shear $S : \mathbb{R}^3_{\{z=0\}} \to \mathbb{R}^3$ such that $S(x, y, 0) \mapsto (x, y, x\cot\theta)$. Next, consider the rigid translation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $(x, y, z) \mapsto (x, y, z_0 + z)$. We can combine the x and z coordinates to a new coordinate direction on P in which $x \mapsto z_0 + \sqrt{x^2 + (x\cot\theta)^2} = z_0 + x\csc\theta$. Note the mappings are invertible, and $T \circ S$ is a C^1-diffeomorphism from the xy-plane to P, so $S^{-1} \circ T^{-1}$ maps P to the plane. By change of variables,

$$\int_F f_F - g_F = \csc\theta \int_0^w f(x) - g(x)dx$$

Since the left-hand side calculates the face area, so does the right-hand side. □

Corollary 2.12. *Thickness of Slice. Given a slice $S(\theta, \delta, z_0)$ and a loaf B, the thickness h of the slice of bread is $\delta \sin\theta$.*

Proof. This follows from knowing the volume of the slice and the face area. Under the assumption that a slice of bread is an oblique prism, the volume is the product of the base and height. Solving for height results in $\delta\sin\theta$. □

Lemma 2.13. *Adjacent Slice Combined Thickness. For a finite number of adjacent slices $S_1 = S(\theta, \delta_1, z_1), \ldots, S_k = S(\theta, \delta_k, z_k)$, the thickness of a combined slice $S(\theta, \sum \delta_i, z_1)$ is the same as the sum of the thickness of each slice.*

Given a fixed volume, we can vary θ, creating a trade-off between surface area and thickness. This next section will focus on optimizing this trade-off under various constraints.

3 Optimization

First, as a bit of notation, since A_θ represents the face area of a slice, let A be the area for a slice where $\theta = \frac{\pi}{2}$. This way, $A_\theta = A\csc\theta$.

Definition 3.1. Bread Optimization. Given two utility functions u_A and u_h, the utility of a slice $S = S(\theta, \delta, z_0)$ of a loaf B is $u(S, B) = u_A(A\csc\theta) + u_h(\delta\sin\theta)$.

Definition 3.2. Bruschetta Optimization. Given a loaf B, a bruschetta optimization is a bread optimization such that there exists a minimum thickness h_{min} such that for $h \ge h_{min}$, $u_h(h) = 1$, otherwise $u_h(h) = -\infty$. Also, $u_A(A\csc\theta) = cA\csc\theta$ for some constant $c > 0$.

Theorem 3.3. *Bruschetta Impossibility. Given a fixed volume V, a loaf B, and a bruschetta optimization, there is a maximum number of slices that can be constructed such that $u_h \not\equiv -\infty$.*

Proof. Note that for a fixed volume of bread, the maximum thickness is $\max_\theta \delta \sin\theta = \delta$, so $\theta = \frac{\pi}{2}$. Given a large slice $S = S(\frac{\pi}{2}, l, z_0)$ of a loaf B, let $V < \infty$ be the total bread volume, and let h_{min} be the minimum bruschetta thickness. For the sake of clarity, we will call the large slice S a finite loaf. Note that $V = A \cdot l$.

Let $n = \left\lfloor \frac{l}{h_{min}} \right\rfloor$, so $l - h_{min} < nh_{min} \leq l$. By contradiction, assume you can slice the finite loaf S into some $n+k > n$ adjacent slices such that $u_h \not\equiv -\infty$. Let those adjacent slices be $S_1 = S(\frac{\pi}{2}, \delta_1, z_0), \ldots, S_{n+k} = S(\frac{\pi}{2}, \delta_{n+k}, z_0 + \sum_{i=1}^{n+k-1} \delta_i)$. The combined thickness of the adjacent slices is $\sum_{i=1}^{n+1} \delta_i$. By lemma 2.13, since S is a slice representing the combined adjacent slices, $l \geq \sum_{i=1}^{n+k} \delta_i$. By the bruschetta optimization, for all slices S_i, $\delta_i \geq h_{min}$, so

$$\begin{aligned}
\sum_{i=1}^{n+k} \delta_i &\geq \sum_{i=1}^{n+1} \delta_i \text{ WLOG} \\
&\geq \sum_{i=1}^{n+1} h_{min} \\
&= (n+1)h_{min} \\
&= nh_{min} + h_{min} \\
&> l - h_{min} + h_{min} \\
&= l
\end{aligned}$$

which is a contradiction. Therefore, there exists a maximum number of slices that can be constructed such that $u_h \not\equiv -\infty$. In particular, that maximum is $n = \left\lfloor \frac{l}{h_{min}} \right\rfloor$. □

Theorem 3.4. *Optimal Bruschetta Slice. Given a fixed width of bread l and a number of slices n, if n does not exceed the Bruschetta Impossibility bound, then the bruschetta optimal angle is $\theta = \arcsin\left(\frac{nh_{min}}{l}\right)$.*

Proof. Clearly, bruschetta optimization requires maximum surface area given the h_{min} thickness constraint. Maximizing surface area with respect to θ requires minimizing θ under this constraint. For n slices of thickness h, since h evenly divides the thickness the bread n times, $h = \frac{l \sin\theta}{n} \implies \theta = \arcsin\left(\frac{nh}{l}\right)$. Minimizing θ under the h_{min} constraint results in $h = h_{min}$, so optimally, $\theta = \arcsin\left(\frac{nh_{min}}{l}\right)$. □

Definition 3.5. Sandwich Optimization. Given a loaf B, a sandwich optimization is a bread optimization such that $u_A(A \csc\theta) = \frac{c_A}{A \csc\theta}$ and $u_h(\delta \sin\theta) = c_h \delta \sin\theta$ for some constants $c_h > 0, c_A > 0$.

Theorem 3.6. *Optimal Sandwich. The sandwich optimal angle is always $\theta = \frac{\pi}{2}$.*

Proof. Note that the slice angle must be in the interval $\left(0, \frac{\pi}{2}\right]$. Maximizing u_A with respect to θ requires maximizing θ. Furthermore, maximizing u_h with respect to θ requires maximizing θ. Therefore, the sandwich optimal angle is $\theta = \frac{\pi}{2}$. □

4 Conclusion

The conclusion is left as an exercise to the reader, preferably done with a loaf in hand.

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