Numerical analysis System of linear equations

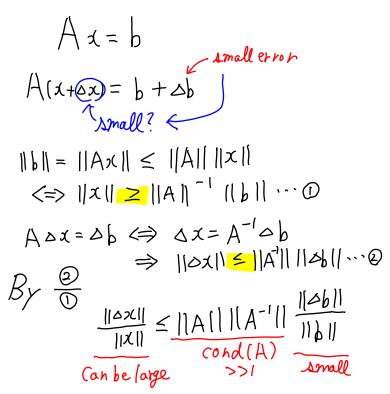
Kazuaki Tanaka $| = || \mathbf{I} || = || \mathbf{A} \cdot \mathbf{A}^{-1} ||$ $\leq || \mathbf{A} || \mathbf{A}^{-1} ||$ Condition number $= \operatorname{cond}(\mathbf{A})$

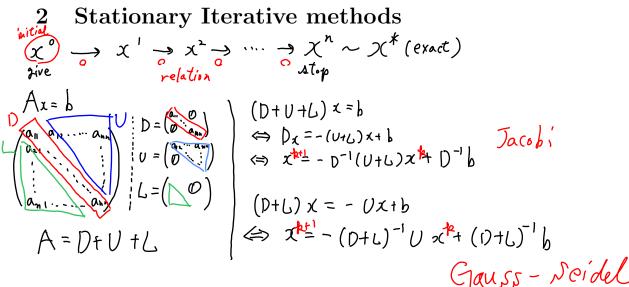
Definition 1.1. The condition number cond(A) of nonsingular matrix A is defined by

$$\operatorname{cond}(A) = \underline{\|A\| \|A^{-1}\|}. \ge \tag{1}$$

Note: Condition numbers depend on the norm imposed on spaces (e.g., $l^1,\ l^2$ or l^∞).

1.1 Impact on the stability of linear equations





Jacobi method

$$x^{k+1} = -D^{-1}(L+U)x^k + D^{-1}b, \quad k = 0, 1, 2, \cdots$$
(2)

(3)

This can be written in the component form



Gauss-Seidel method

$$x^{k+1} = -(L+D)^{-1}Ux^k + (L+D)^{-1}b, \quad k = 0, 1, 2, \cdots$$
(4)

This can be written in the component form

$$C_{x_i^{k+1}} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k \right), \quad i = 1, 2, \dots, n, \quad k = 0, 1, 2, \dots$$
 (5)

2.1Stopping criterion

For small $\varepsilon > 0$,

•
$$\frac{\left\|x^{k+1} - x^k\right\|}{\left\|x^k\right\|} < \varepsilon$$
 then stop

Very easy, but it sometimes does not converge enough.

$$\bullet \frac{\|Ax^k - b\|}{\|b\|} < \underbrace{\epsilon = |e - 10|}_{\text{then stop}}$$

Stable, but computational cost is higher then the above.

2.2Convergence analysis

The spectral radius of matrices is important for analyzing convergence property of iterative methods.

Definition 2.1. The spectral radius $\rho(A)$ of an $n \times n$ matrix A is defined by

$$\rho(A) := \max_{1 \le i \le n} |\lambda_i|,$$
(6)

where λ_i is the i-th eigenvalue of A.

Lemma 2.2. For any matrix norm ||A|| induced from by the vector norm, we have

$$\rho(A) \le ||A||. \tag{7}$$

Proof.
$$A_{X} = \lambda X$$
, $X \neq 0$

Therefore

 $\|\lambda\| \|x\| = \|\lambda x\| = \|Ax\| \le \|A\| \|\lambda x\|$

So, we have
 $\|\lambda\| \le \|A\|$ for all eigenvalues λ .

Both Jacobi and Gauss-Seidel methodacan be written in the general form

$$x^{k+1} = Mx^k + c, \quad k = 0, 1, 2, \cdots$$
 (8)

for finding a solution of

$$x = Mx + c, (9)$$

where

[Jacobi method]
$$M = -D^{-1}(L+U), \quad c = D^{-1}b,$$

[Gauss-Seidel method] $M = -(L+D)^{-1}U, \quad c = (L+D)^{-1}b.$

Theorem 2.3. Equation (9) has a unique solution and the sequence $\{x^k\}$ defined by (8) converges to the unique solution from any starting point x^0 if and only if $\rho(M) < 1$.

Proof. See, for example, Section 8.2 in [J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, Springer, 2002].

Corollary 2.4. If ||M|| < 1, then the sequence $\{x^k\}$ defined by (8) converges to the unique solution of (9) from any starting point x^0 .

Proof. This follows from Lemma 2.2 and Theorem 2.3.

$$\mathcal{C}(M) \leq ||M|| < |$$

Definition 2.5. An $n \times n$ matrix A is called strictly diagonally dominant if

$$\left(|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|\right) \tag{10}$$

for all $i = 1, 2, \dots, n$.

Theorem 2.6. If A is strictly diagonally dominant, both Jacobi and Gauss-Seidel methods generate sequences $\{x^k\}$ that converge to unique solution of Ax = b for any starting point x^0 .

Proof. [Jacobi] We want to prove || M || = || D-1 (L+U) || < |

$$M = \begin{pmatrix} 0 & \frac{\alpha_{ij}}{\alpha_{ij}} \\ \frac{\alpha_{ij}}{\alpha_{ij}} \end{pmatrix} \qquad ||M||_{\infty} = \max_{1 \leq i \leq n} \sum_{\substack{j=1 \ j \neq i}}^{n} \left| \frac{\alpha_{ij}}{\alpha_{ij}} \right| = \max_{1 \leq i \leq n} \left(\frac{1}{|\alpha_{ij}|} \sum_{\substack{j=1 \ j \neq i}}^{n} |\alpha_{ij}| \right)$$

[Gause - Seidel] Ex 6.1

Hind: proing by induction
$$MJ = : Z$$

$$\|M\|_{\infty} = \|(L+D)^{-1}U\|_{\infty} = \max_{\|B\|_{\infty}^{-1}} \|(L+D)^{-1}UJ\|_{\infty} (< 1)$$
We want

Prove, by induction, that

- (i) |x| = 7 < 1
- (ii) Prove | Ziti | < 1 when | Zi | < 1

$$(x, y) = x \cdot y$$

$$:= x^{T} A y = (x, y)_{A}$$
when $A = 1$

$$(x, y) = x^{T} y$$

Remark 2.7. In fact, if A is a <u>symmetric positive-definite</u> matrix $(A = A^T)$ and $x^T Ax > 0$ for all $x \neq 0$), Gauss-Seidel method generates sequences $\{x^k\}$ that converge to unique solution of Ax = b for any starting point \mathbf{x}

$$A = b$$

$$A' = A' b$$

$$A' x = b'$$

$$A' x = b$$

3 Nonstationary iterative methods — Conjugate gradient method (CG method)

One of the top 10 algorithms in the 20th century for solving linear equation Ax = b, where $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^n \times \mathbb{R}^n$ is a symmetric positive-definite matrix $(A = A^T \text{ and } x^T Ax > 0 \text{ for all } x \neq 0)$.

Algorithm

Compute $p_0 = r_0 = b - Ax_0$. For $k = 0, 1, 2, \cdots$

If η is small enough, break.

$$\alpha_k = \frac{(r_k, p_k)}{(Ap_k, p_k)} \left(= \frac{\|r_k\|_2^2}{(Ap_k, p_k)} \right)$$

$$x_{k+1} = x_k + \alpha_k p_k$$

$$r_{k+1} = r_k - \alpha_k A p_k$$

$$\beta_k = -\frac{(Ap_k, r_{k+1})}{(Ap_k, p_k)} \left(= \frac{\|r_{k+1}\|_2^2}{\|r_k\|_2^2} \right)$$

$$p_{k+1} = r_{k+1} + \beta_k p_k$$

End

Theorem 3.1. The sequence generated by the conjugate gradient method converges to the solution of Ax = b at most n steps.

Theorem 3.2. Let

$$\kappa := \frac{\sqrt{\rho} - 1}{\sqrt{\rho} + 1}, \quad \rho = \operatorname{cond}_2(A) = \|A\|_2 \|A^{-1}\|_2$$

Then, we have

exact while of Ax = b
$$\|\underline{x}^* - x_k\|_2 \leq \frac{2\rho\kappa^k}{1+\kappa^{2k}} \, \underline{\|x^* - x_0\|_2} \,, \quad k = 0, 1, 2, \cdots, n.$$

3.1 Preconditioned conjugate gradient method (PCG method)

Theorem 3.3. If A is a symmetric positive-definite matrix, then there exists a unique real lower triangular matrix L with positive diagonals such that

$$A = LL^{T} = \left(\underbrace{\mathbb{N}^{0}} \right) \left(\underbrace{\mathbb{N}^{0}} \right)$$

$$\tag{11}$$

This factorization is called Cholesky's factorization or Cholesky's decomposition.

$$A' = b, \quad \text{cond}(A) >> 1 \quad \text{very large}$$

$$(L^{-1}AL^{-T})(L^{T}x) = L^{-1}b$$

$$A' = b' \quad (\Rightarrow) \quad Ax = b$$

$$A' = L^{-1}AL^{-T} = L^{-1}LL^{T}L^{-T} = I$$

$$A = LL^{T} + E \Rightarrow A' \sim I$$
Incomplete cholerky's factorization fill-in

to avoid fill-in A~ LL.T