Numerical analysis
Nonlinear equations

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We consider the problem of finding a solution to the nonlinear equation

$$f(x) = 0, \tag{1}$$

where f is a nonlinear function from $\mathbb{R}^n$ to $\mathbb{R}^n$. $n = 1$

1 Fixed point iterative method

For simplicity, we assume $n = 1$ for a while.

Definition 1.1 (Fixed point iterative method)**.** *Let g be some function from some subset of $\mathbb{R}$ to $\mathbb{R}$. The iteration based on the following iteration is called* **Fixed point iterative method***:*

$$x_{k+1} = g(x_k), \quad k = 0, 1, 2, \cdots . \tag{2}$$

If the sequence $\{x_k\}_{k \geq 0}$ constructed by (2) converges to some $x^* \in \mathbb{R}$ and g is continuous, then it follows that

x^* g Fixed point of g

$$x^* = g(x^*), \tag{3}$$

and such x^* called *a fixed point* of g. From the same target equation ($\cancel{18}$ 1), we may construct several different iterative sequences defined by (2). In fact, Newton's method, the most important method in this chapter, is one of fixed point iterative methods.

Examples

$$f(x) = x^3 - 3x - 1 = 0$$

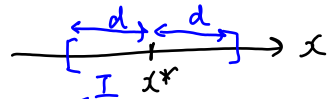
$$\Longleftrightarrow x = \frac{x^3 - 1}{3} =: g(x)$$

$$\boxed{x_{k+1} = \frac{x_k^3 - 1}{3} = g(x_k)}$$

$$\Longleftrightarrow x^3 = 3x + 1$$

$$\Longleftrightarrow x = \underbrace{\frac{3x+1}{x^2}}_{g(x)} \quad (x \neq 0)$$

$$\boxed{x_{k+1} = \frac{3x_k + 1}{x_k^2}}$$



Theorem 1.1 (Convergence of fixed point iterative method)**.** *Suppose that $g(x) = x$ has a solution x^* and that $g \in C^1(I)$, $I = [x^* - d, x^* + d]$ for some $d > 0$. If we have*

g, g' continuous over I

$$|g'(x)| \le \lambda < 1 \quad \text{for all } x \in I. \tag{4}$$

then:

(i) *The iteration defined by (2) converges to x^* for any initial value $x_0 \in I$.*
(ii) *x^* is the unique solution to $g(x) = x$ in I.*
(iii) *For each $k = 0, 1, 2, \cdots$, x_k satisfies*

$$\frac{\epsilon_k}{1+\lambda} \le |x_k - x^*| \le \frac{\epsilon_k}{1-\lambda} \le \frac{\lambda^k}{1-\lambda}\epsilon_0 = \frac{\lambda^k}{1-\lambda}|x_1 - x_0| \tag{5}$$

where $\epsilon_k = |x_{k+1} - x_k|$.

This theorem can also be interpreted as follows:

open set $(a,b) \ni x^*$, $a, b \in \mathbb{R}$

$\in C^1$

Corollary 1.2. *Suppose that $g(x) = x$ has a solution x^* and that g has a continuous derivative in a neighborhood around x^*. If $|g'(x^*)| < 1$, then we have, for some closed interval $I \ni x^*$ and $0 < \lambda < 1$,* (i)-(iii) *in Theorem* 1.1.

Proof. $\varepsilon := \dfrac{1 - |g'(x^*)|}{2}$

small $\delta > 0$

Because g' is continuous $|g'(x) - g'(x^*)| < \varepsilon$ for all $x \in [x^* - \delta, x^* + \delta] = I$

$$|g'(x)| = |g'(x) - g'(x^*) + g'(x^*)| \le \underbrace{|g'(x) - g'(x^*)|}_{<\varepsilon} + |g'(x^*)| < \frac{1 - |g'(x^*)|}{2} + |g'(x^*)|$$

$$= \frac{1 + |g'(x^*)|}{2} =: \lambda < 1$$

$\uparrow$ I ; <1

□

2 One-dimensional Newton's method

$\in C^2$

Definition 2.1 (Newton's method (one dimensional))**.** *Let f be some twice continuously differentiable function from some subset of $\mathbb{R}$ to $\mathbb{R}$. The iteration based on the following iteration is called* **Newton's method***:*

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \cdots.} \tag{6}$$

This is in fact a special case where $g(x_k) = x_k - f(x_k)/f'(x_k)$ in (2).

Once x_0 is given, one can calculate the iterative sequence based on (6) unless $f'(x_k) = 0$ for some k.

A lot of other iterative methods are known, for example:

- *Simplified Newton's method* defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{\boxed{f'(x_0)}}, \quad k = 0, 1, 2, \cdots. \tag{7}$$

fixed

This is the special case where $g(x_k) = x_k - f(x_k)/f'(x_0)$ in (2).

- *The secant method* defined by

$$\sim f'(x_k)^{-1}$$

$$x_{k+1} = x_k - \boxed{\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}} f(x_k), \quad k = 1, 2, \cdots . \tag{8}$$

  Note that two initial values x_0 and x_1 are required for the secant method.

- As one of the simplest methods, *bisection method* is well known. Although this method always converges as long as initial values x_0 and x_1 satisfy $x_0 x_1 < 0$, it is difficult to be generalized to a multi-dimensional method.

Theorem 2.1 (Convergence of Newton's method)**.** *Suppose that equation $f(x) = 0$ has a solution x^* satisfying $f'(x^*) \neq 0$ and that f has a continuous second derivative in a neighborhood around x^*. Then, Newton's method* (6) *converges to x^* if the initial value x_0 is sufficiently near to x^*.*

Proof. Let

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Then, $g(x)$ is C^1 in the neighborhood of x^*, and moreover

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2} \quad \text{in the neighborhood of } x^* \tag{9}$$

and

$$g'(x^*) = 0 \ (< 1). \qquad f(x^*) = 0 \tag{10}$$

Hence, Corollary 1.2 ensures the argument. $\square$

Theorem 2.2 ($\boxed{\text{Quadratic}}$ convergence of Newton's method)**.** *Under the same assumption in Theorem* 2.1*, the sequence of Newton's method* (6) *satisfies*

$$\lim_{k\to\infty} \frac{x_{k+1} - x^*}{(x_k - x^*)^2} = \frac{f''(x^*)}{2f'(x^*)} \ = \text{const.} \tag{11}$$

when $x_0 \neq x^*$.

Proof. Taylor's theorem ensures that

$$f(x^*) = f(x) + f'(x)(x^* - x) + \frac{1}{2}f''(\xi)(x^* - x)^2 = 0 \tag{12}$$

for ξ between x^* and any $x \in$ (the neighborhood of x^*). Since $f(x^*) = 0$, we have

$$f(x) + f'(x)(x^* - x) = -\frac{1}{2}f''(\xi)(x^* - x)^2. \tag{13}$$

Therefore,

$$\begin{aligned} x_{k+1} - x^* &= \overbrace{\boxed{x_k - \frac{f(x_k)}{f'(x_k)}}}^{= x_{k+1}} - x^* \\ &= -\frac{1}{f'(x_k)}\left(f(x_k) + f'(x_k)(x^* - x_k)\right) \\ &= \frac{f''(\xi_k)}{2f'(x_k)}(x_k - x^*)^2, \quad (\because (13)), \end{aligned}$$

$$\to x^* \qquad \to x^*$$

where ξ_k lies between x^* and x_k and therefore $\lim_{k\to\infty} \xi_k = x^*$. The continuity of f' and f'' ensures the argument. $\square$

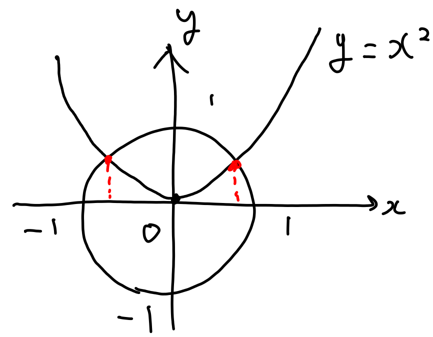
3 Multi-dimensional Newton's method

One-dimensional Newton's method (introduced in the previous section) can be extended to the multi-dimensional version, which is useful for solving systems of nonlinear equations. First, let us consider the system of two nonlinear equations:

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{14}$$

where f, g are functions from some $D \subset \mathbb{R}^2$ to $\mathbb{R}$.

$$\begin{cases} x^2 + y^2 = 1 \\ y = x^2 \end{cases} \iff \begin{cases} f(x,y) = x^2 + y^2 - 1 = 0 \\ g(x,y) = y - x^2 = 0 \end{cases}$$



For ensuring the convergence property of Newton's method, each f, g are required to be in $C^2(D)$ as well as the one-dimensional case. Newton's method for problem (15) is defined by

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \begin{bmatrix} f_x(x_k, y_k) & f_y(x_k, y_k) \\ g_x(x_k, y_k) & g_y(x_k, y_k) \end{bmatrix}^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}, \quad k = 0, 1, 2, \cdots, \tag{15}$$

where f_x denotes the partial derivative of f with respect to the first variable x; likewise f_y, g_x, and g_y are defined. Using the notation

$$\boldsymbol{x_k} = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad \boldsymbol{f}(\boldsymbol{x_k}) = \begin{bmatrix} f(\boldsymbol{x}_k) \\ g(\boldsymbol{x}_k) \end{bmatrix}, \quad \text{and} \quad \boldsymbol{f}'(\boldsymbol{x}_k) = \begin{bmatrix} f_x(\boldsymbol{x}_k) & f_y(\boldsymbol{x}_k) \\ g_x(\boldsymbol{x}_k) & g_y(\boldsymbol{x}_k) \end{bmatrix}, \tag{16}$$

(15) is written in the following form similar to (6)

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \underbrace{\boldsymbol{f}'(\boldsymbol{x_k})^{-1}}_{A^{-1}} \underbrace{\boldsymbol{f}(\boldsymbol{x}_k)}_{b}, \quad k = 0, 1, 2, \cdots. \tag{17}$$

$$= x \rightarrow A \backslash b$$

Here, the matrix $\boldsymbol{f}'(\boldsymbol{x}_k)$ is called *the Jacobian matrix* of $\boldsymbol{f}$ at $\boldsymbol{x}_k$; it is often denoted, e.g., by $D\boldsymbol{f}(\boldsymbol{x}_k)$ or $J_{\boldsymbol{f}}(\boldsymbol{x}_k)$.

$$\boxed{Ax = b}$$

4 Zeros of polynomials

Consider the problem of finding all solutions of the equation

$$f(x) := a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0, \tag{18}$$

where $a_i \in \mathbb{C}$ $(i = 0, 1, \cdots, n)$ and $a_n \neq 0$. $\hookrightarrow$ has n solutions.

$x = x_1, x_2, \cdots, x_n$

4.1 Durand-Kerner method

The iterative method based on the following iteration is called the **Durand-Kerner method (DK method)**:

$$x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{pmatrix}$$

$$\underbrace{x_{i}^{\boxed{(k+1)}}}_{\text{step: }(k+1),\ \text{solution: } i} = x_i^{(k)} - \frac{f\left(x_i^{(k)}\right)}{\boxed{a_n \prod_{j \neq i} \left(x_i^{(k)} - x_j^{(k)}\right)}}, \quad k = 0, 1, 2, \cdots, \tag{19}$$

$$\sim f'(x_i^{(k)})$$

where, when $x_i^{(k)} = x_j^{(k)}$ for some $j \neq i$, we set $x_i^{(k+1)} = x_i^{(k)}$ afterward.

Assume that $x_1, x_2, \cdots, x_n$ are good approximations of solutions of $f(x) = 0$.

So, $f(x_i) \sim 0$.

We have $f(x) \sim a_n (x - x_1)(x - x_2) \cdots\cdots (x - x_n)$ $\quad (x - x_i)$

$$= a_n \prod_{i=1}^{n} (x - x_i) = a_n \underbrace{(x - x_i)}_{A(x)} \underbrace{\prod_{j \neq i} (x - x_j)}_{B(x)} \quad (i = 1, 2, \cdots, n)$$

$$f'(x) \sim a_n \{A'(x) B(x) + A(x) B'(x)\}$$

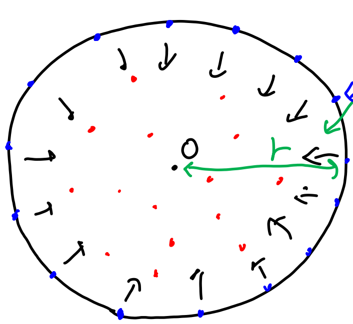
$$x_i: \quad = a_n \Big\{ \prod_{j \neq i} (x_i - x_j) + \cancel{(x_i - x_i) B'(x)} \Big\}$$

$$\boxed{f'(x_i) \sim a_n \prod_{j \neq i} (x_i - x_j)}$$

4.2 A simple initial value

Since the Durand-Kerner method is an iterative method, setting a "good" starting point important. One possibility is putting the points on a circle including all the solutions of the target equation. Such a circle can be given with the center $(0, 0)$ and the radius

$$r := \max_{0 \leq i \leq n-1} \left(n \left| \frac{a_i}{a_n} \right| \right)^{\frac{1}{n-i}}.$$



$$x_i^0 = r \exp\left[\left(\frac{2(i-1)\pi}{\cancel{x}\, n} + \frac{\pi}{2n}\right)\sqrt{-1}\right]$$

$a = 1:10;$

$f(x) = 1 + 2x + 3x^2 + \cdots + 10x^9$

4.3 Aberth's initial values

A very effective choice is known as Aberth's starting point; see [1, Section 4]. The DK method with the Aberth's starting point is called the **Durand-Kerner-Aberth method (DKA method)**.

References

[1] Oliver Aberth, Iteration methods for finding all zeros of a polynomial simultaneously Mathematics of computation 27, (122), 339-344 (1973).
https://www.ams.org/journals/mcom/1973-27-122/S0025-5718-1973-0329236-7/