Problem-1: Let G be a group of ab order pq, where p and q are distinct primes. Prove that G is abelian.

Answer:

The product of subgroups P and Q, dendoned PQ, is a subgroup of G with order $|PQ| = \frac{|P||Q|}{|P\cap Q|} = \frac{pq}{1} = pq$.
Thus, $G = PQ$. Since Q is a normal subgroup, G is the internal direct product of P and Q, which means $G \cong P \times Q$.
Since ~~Q is a~~ ~~Aorn~~ P and Q are both cyclic groups of prime order, they are abelian. The ~~produ~~ direct product of abelian groups is abelian, so G is an abelian groups.

Problem-2: Prove that if G group of order p^2, where p is prime, then G is abelian if and only if it has p+1 subgroups of order p.

Answer:

Every group of order p^2 is either cyclic Z_{p^2} or elementary abelian $Z_p \times Z_p$.

If $G \cong Z_{p^2}$, there is exactly one subgroup of order p.

If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$, it is a 2-dimentional vector space over $\mathbb{F}_p$. One-dimentional subspaces correspond to subgroup of order p. Number of 1-dimensional subspaces $= \frac{p^2-1}{p-1} = p+1$.

Conversely, if a group of order p^2 has $p+1$ subgroups of order p, it must be the elementary abelian case $\mathbb{Z}_p \times \mathbb{Z}_p$, which is abelian.

Conclusion: True - The equivalence holds; the "p+1" case corresponds exactly to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Problem 3: Let G be a finite group and N be a normal subgroup of G. Prove that the union of all conjugates of H cannot be equal to G.

Answer: Let G be a finite group and H be a proper subgroup of G. The number of distinct conjugates of H is $k = [G : N_G(H)]$, where $N_G(H)$ is the normalizer of H. Each conjugate has size $|H|$. The union of all conjugates, $\bigcup_{g \in G} gHg^{-1}$, can be written as a union of k subgroups, $H_1, \dots, H_k$. All these subgroups share the identity element. The total number of elements in the union is at most the sum of the non-identity elements of such subgroup plus the identity element.

$$\left|\bigcup_{g\in G} gHg^{-1}\right| \leq k(|H|-1)+1 = [G:N_G(H)](|H|-1)+1$$

Since H is a subgroup of $N_G(H)$, we have $|H| \leq |N_G(H)|$, which implies $[G:N_G(H)] \leq [G:H]$.

$$[G:H] = \frac{|G|}{|H|}$$

$$\left|\bigcup_{g\in G} gHg^{-1}\right| \leq [G:H](|H|-1)+1 = \frac{|G|}{|H|}(|H|-1)+1 = |G| - \frac{|G|}{|H|}+1$$

Since H is a proper subgroup, $|H|<|G|$, so $|G|/|H| \geq 2$.

Therefore, $\frac{|G|}{|H|} - 1 \geq 1$,

and ;

$$\left|\bigcup_{g\in G} gHg^{-1}\right| \leq |G| - \left(\frac{|G|}{|H|}-1\right) < |G|$$

– Thus, the union of all conjugates of H cannot be equal to G, as the union is a proper subset of G.

– Therefore, the given problem statement is true.

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Problem-04: Let G be a group and N be a normal subgroup of G. If G/N is cyclic and N is cyclic, prove that G is abelian.

Answer: This statement as written is false.

Counterexample: Dihedral group D_{2n} (symmetries of an n-gon).

Let R be the cyclic rotation subgroup of order n (so R is cyclic and normal), and $D_{2n}/R \cong C_2$ is cyclic. But D_{2n} is non-abelian for $n \geq 3$.

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Problem-06: Let G be a finite group and p be the smallest prime dividing $|G|$. Prove that any subgroup of index p in G is normal.

Solution: Let G act on left cosets of H, giving $\phi: G \to S_p$.

Then $|\varphi(G)|$ divides $|G|$ and also divides $p!$.

Any prime divisor of $|\varphi(G)|$ is a prime divisor of $|G|$. Because p is the smallest prime dividing $|G|$, the only possible primes dividing $|\varphi(G)|$ are $\geq p$. That forces $|\varphi(G)|$ to be either 1 or divisible by p.

If $|\varphi(G)| = 1$, kernel $= G$, index 1 contradicts index p. So $|\varphi(G)|$ is divisible by p. The only subgroup of S_p of order divisible by p and a divisor of $|G|$ under the minimality hypothesis must have order p. So $|\varphi(G)| = p$. And the kernel has index p, hence equals H. The kernel is normal, so H is normal.

Hence, True - subgroup of index p is normal when p is the smallest prime divisor.

(i) Problem 07: Let G be a group and $a, b \in G$. Prove that if $a^4 = b^2$ and $ab = ba$, then $(ab)^6 = e$.

Answer: statement false as given.

From commutativity $(ab)^6 = a^6 b^6$

Using $b^2 = a^4$ gives $b^6 = (b^2)^3 = (a^4)^3 = a^{12}$

Thus $(ab)^6 = a^6 a^{12} = a^{18}$

Nothing in the hypotheses forces $a^{18} = e$. So the claim is not generally true.