#1) Question: Is set of odd numbers with binary operations (+), i.e., $\langle O,+\rangle$ an abelian group? If not explain the resone with necessary notations.

Answer:

The set of odd numbers with binary operations (+) is denoted as: $(O,+)$. Where $O = \{\dots -3, -1, 1, 3, 5 \dots\}$.

For a non-empty set S, $(S,*)$ is called a Abelian group if it follows the following axiom.

1. Closure: $(a*b)$ belongs to S for all $a, b \in S$.
2. Associativity: $a*(b*c) = (a*b)*c$ where $a, b, c \in S$.
3. Identity Element: There exists $e \in S$ such that $a*e = e*a = a$ where $a \in S$.
4. Inverses: For $a \in S$ there exists $a^{-1} \in S$ such that $a*a^{-1} = a^{-1}*a = e$
5. Commutative: $a*b = b*a$ for all $a, b \in S$.

Step 1 (closure):

Take any two odd numbers, say $a = 2m+1$ and $b = 2n+1$.

$a+b = 2(m+n+1)$

This result is an even number, not an odd number.

Therefore, $(O,+)$ is not closed under addition. Closure fails

Step 2 (Associativity):

Addition of integers is associative : $(a+b)+c = a+(b+c)$ for all integers a, b, c.

So associativity holds.

However, since closure already fails, this property cannot save the structure.

Step 3 (Identity Element):

For a group, there must exist an identity element e such that $a+e = e+a = a$ for all a in O.

For integers under addition, the identity is 0.

But $0 \notin O$ (since 0 is even, not odd).

Therefore, no identity element exists in $(O, +)$.

Step 4 (Inverse Element):

Without an identity element, inverses cannot exist.

Even though the inverse of an odd integer is also odd (e.g., the inverse of 3 is -3), the lack of an identity element disqualifies $(O, +)$ from being a group.

Step 5 (Commutativity):

Addition is commutative: $a+b=b+a$.

Thus, if the other properties were satisfied, $(O,+)$ could would be Abelian.

As,

Closure fails: odd + odd = even $\notin O$.

Identity element fails: $0 \notin O$.

Therefore, $(O,+)$ is not a group, and hence not an Abelian group.