INTRODUCTION

PEOPLE

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COURSE INFORMATION

- Office Hours
- Lessons
- Prerequisites
- Assessment
- Schedule
- Syllabus
- Project
- Homework
- eDimension
- Textbooks

WHAT IS MACHINE LEARNING?

Hard-Coded

Trained

Giving computers the ability to learn
without being explicitly programmed
– Arthur Samuel (1959)

WHAT IS MACHINE LEARNING?

Task

Performance

Experience

Algorithms that improve their performance
at some task with experience
– Tom Mitchell (1998)

TYPES OF MACHINE LEARNING

Supervised Learning

TYPES OF MACHINE LEARNING

Unsupervised Learning

TYPES OF MACHINE LEARNING

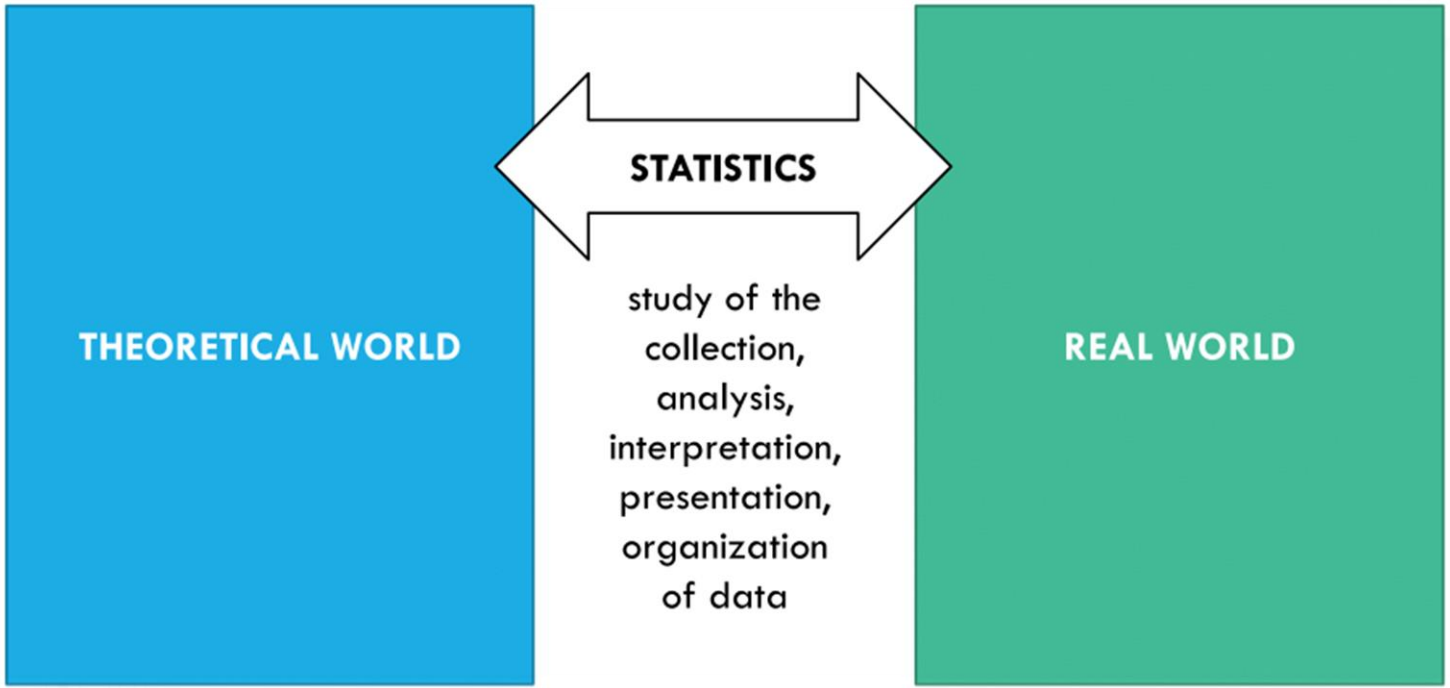
Reinforcement Learning

APPLICATIONS

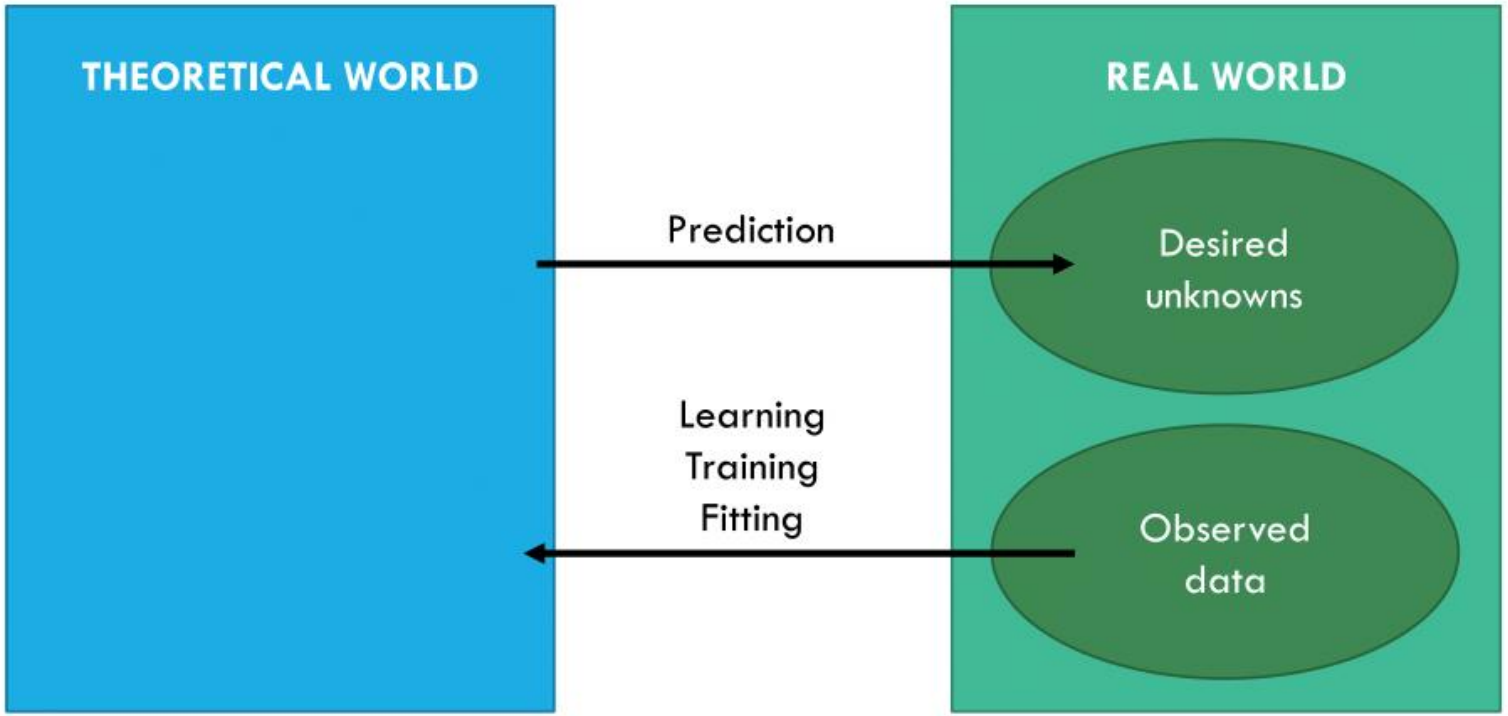
- Image Classification
- Spam filters
- Fraud Detection
- Face Recognition
- Speech Translation
- Healthcare
- Early Diagnosis
- Self-Driving Cars
- Recommender Systems
- Video Games
- Financial Analysis
- Retail Analysis
- Feature Extraction
- Event Prediction
- Hospitality
- System Management

STATISTICS

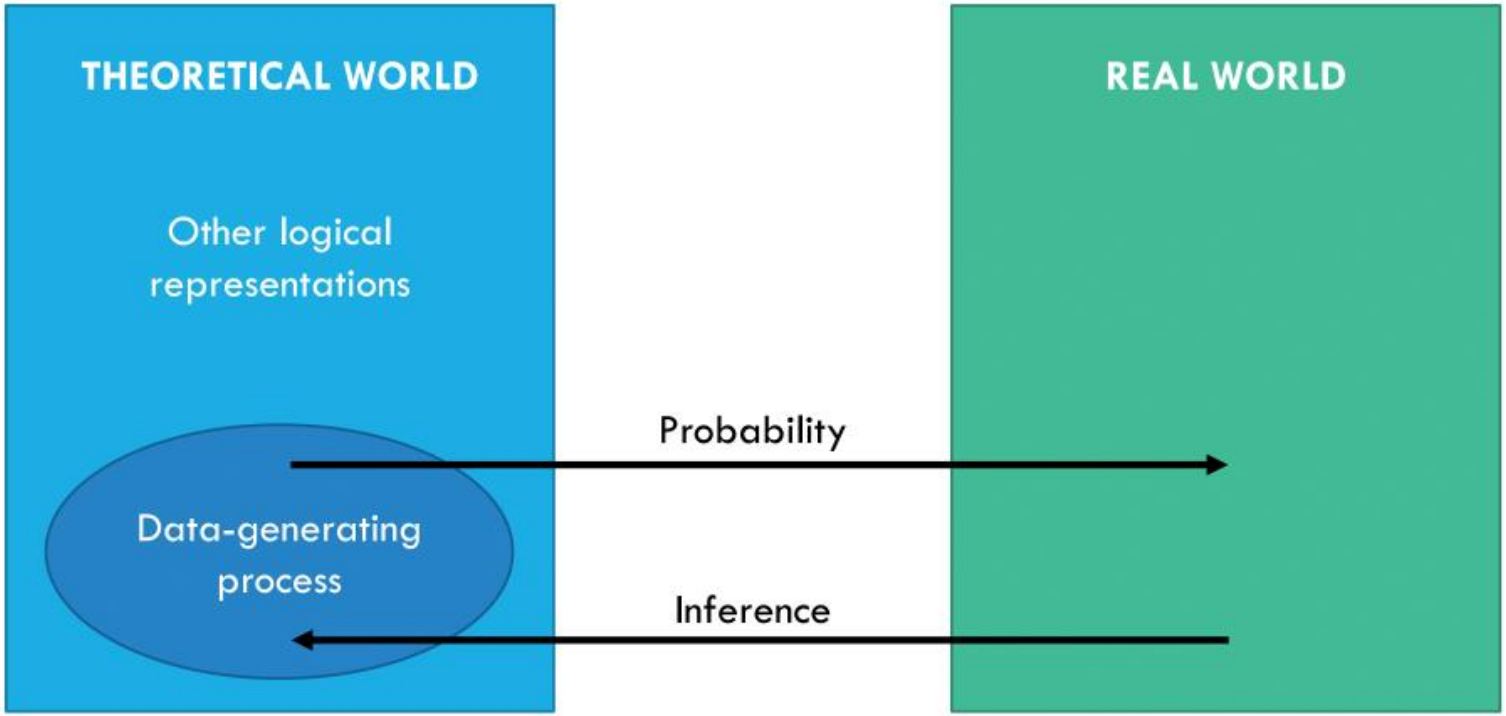
WHAT IS STATISTICS



PRACTICAL PERSPECTIVE



THEORETICAL PERSPECTIVE



NOTATION

- A state space is the group of outputs that we can observe from a probability distribution.
- If your probability distribution is derived from coin flips, then the state space is {heads, tails}. If it is the number of a die roll, the state space is {1,2,3,4,5,6}. If it is the height of a human being, then the state space is the continuous interval [0, 3 metres] (I’ve never seen anyone taller than 3 metres.)

PROBABILITY DENSITY FUNCTIONS

A random variable X on a discrete space is well-defined if

$$\sum_{x \in \mathcal{X}} P(X = x) = 1. \tag{1}$$

If the state space $\mathcal{X}$ is not discrete, for e.g. $\mathcal{X} = \mathbb{R}$ or $\mathbb{R}^n$, then a continuous random variable X is well-defined if there exists a probability density function (pdf) $f_X(x) \geq 0$ such that

$$\int_{\mathcal{X}} f_X(x)\,\mathrm{d}x = 1. \tag{2}$$

Its cumulative distribution function (cdf)

$$P(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x \tag{3}$$

is a function of a, and is also denoted by $F(a)$.

EXAMPLES

- Discrete:
  - Bernoulli
  - Binomial
  - Geometric
  - Poisson
  - Multinomial
- Continuous:
  - Gaussian/Normal
  - Exponential
  - Chi-squared
  - Gamma
  - Uniform

EXAMPLES

Bernoulli(p)

$$f(0) = 1 - p, f(1) = p$$
$$\mu = p. \quad \sigma^2 = p(1-p) = pq$$
$$m(t) = pe^t + q$$

Poisson(λt)

$$f(x) = \frac{1}{x!}(\lambda t)^x e^{-\lambda t}, \text{ for } x = 0, 1, \ldots$$
$$\mu = \lambda t. \quad \sigma^2 = \lambda t$$
$$m(s) = e^{\lambda t(e^s - 1)}$$

Binomial(n, p)

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \ldots, n$$
$$\mu = np. \quad \sigma^2 = np(1-p) = npq$$
$$m(t) = (pe^t + q)^n$$

Geometric(p)

$$f(x) = q^{x-1}p, \text{ for } x = 1, 2, \ldots$$
$$\mu = \frac{1}{p}. \quad \sigma^2 = \frac{1-p}{p^2}$$
$$m(t) = \frac{pe^t}{1 - qe^t}$$

EXAMPLES

$\textsc{Uniform}(a, b)$

$$f(x) = \frac{1}{b-a}, \text{ for } x \in [a, b]$$
$$\mu = \frac{a+b}{2}. \quad \sigma^2 = \frac{(b-a)^2}{12}$$
$$m(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$$

$\textsc{Normal}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \text{ for } x \in \mathbf{R}$$
$$\mu = \mu. \quad \sigma^2 = \sigma^2$$
$$m(t) = \exp(\mu t + t^2\sigma^2/2)$$

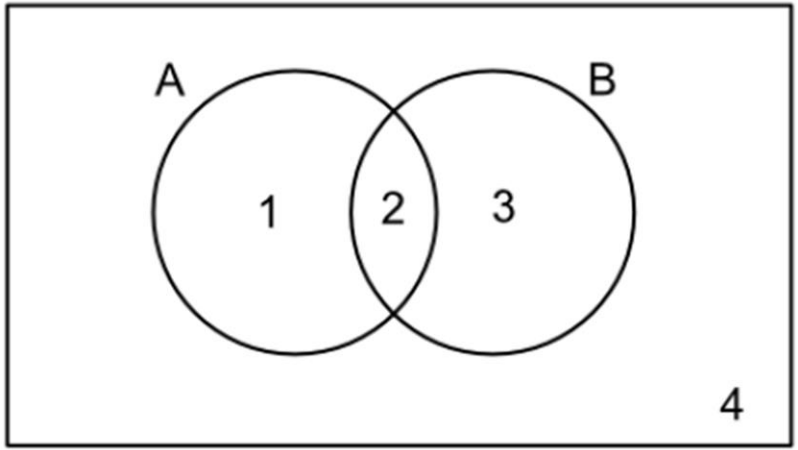
$\textsc{Exponential}(\lambda)$

$$f(x) = \lambda e^{-\lambda x}, \text{ for } x \in [0, \infty)$$
$$\mu = 1/\lambda. \quad \sigma^2 = 1/\lambda^2$$
$$m(t) = (1 - t/\lambda)^{-1}$$

$\textsc{ChiSquared}(\nu)$

$$f(x) = \frac{x^{\nu/2-1}e^{x/2}}{2^{\nu/2}\Gamma(\nu/2)}, \text{ for } x \geq 0$$
$$\mu = \nu. \quad \sigma^2 = 2\nu$$
$$m(t) = (1 - 2t)^{\nu/2}$$

UNION BOUND



When events A and B don't intersect, they are known to be mutually exclusive.

$$P(A^c) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) \leq P(A) + P(B)$$

JOINT DENSITY FUNCTIONS

- Sometimes, we don’t care about only one die roll. Sometimes we care about the output of two die rolls or three die rolls. (I will be using a lot of casino analogies.)
- The output of two coins can be (H,H), (H,T), (T,H) and (T,T). This is known as a joint probability distribution.

JOINT DENSITY FUNCTIONS

A multivariate random variable $\boldsymbol{X} = (X_1, \ldots, X_n)$ with state space $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is a joint distribution if

$$\sum_{x_1 \in \mathcal{X}_1, \ldots, x_n \in \mathcal{X}_n} P(X_1 = x_1, \ldots, X_n = x_n) = 1, \tag{4}$$

for discrete random variables. For continuous random variables, there exists a density function $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) \geq 0$ such that

$$\int_{x_1 \in \mathcal{X}_1, \ldots, x_n \in \mathcal{X}_n} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, \mathrm{d}\mathbf{x} = 1. \tag{5}$$

MARGINAL DISTRIBUTIONS

With the joint distribution probabilities, one can derive the distribution of each individual X_i, or a subset of them. These distributions are known as marginal distributions.

For discrete random variables, the multivariate random variable $(X_1, \ldots, X_{n-1})$ has the probability distribution

$$P(X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) = \sum_{x_n \in \mathcal{X}_n} P(X_1 = x_1, \ldots, X_n = x_n), \quad (6)$$

while the random variable X_1 has the density function

$$P(X_1 = x_1) = \sum_{x_2 \in \mathcal{X}_2, \ldots, x_n \in \mathcal{X}_n} P(X_1 = x_1, \ldots, X_n = x_n). \quad (7)$$

For continuous random variables, the random variable X_1 has the density function

$$f_{X_1}(x_1) = \int_{x_2 \in \mathcal{X}_2, \ldots, x_n \in \mathcal{X}_n} f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) \, \mathrm{d}x_2 \ldots \mathrm{d}x_n. \quad (8)$$

CONDITIONAL DISTRIBUTIONS

Given a joint discrete distribution (X, Y), the conditional probability function of X given Y is given by

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}. \tag{9}$$

When (X, Y) is continuous, the probability density function, $f_{X|Y}(x \mid y)$, of X given Y has the expression

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}. \tag{10}$$

Thus, the conditional distributions can be computed from the joint distributions and the marginal distributions.

EXPECTATION

The expectation (mean) of a random variable X can be expressed as

$$E(X) = X_{\text{mean}} = \sum_{x \in \mathcal{X}} xP(X = x). \tag{4}$$

The variance and covariance can be defined therefore in terms of the expectation, where

$$Var(X) = E((X - E(X))^2) = E(X^2) - E(X)^2, \tag{5}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y). \tag{6}$$

Conditional expectations and variances follow from the conditional distributions

$$E(X \mid Y = y) = \sum_{x \in \mathcal{X}} xP(X = x \mid Y = y). \tag{7}$$

EXPECTATION AND VARIANCE OF IMPORTANT RANDOM VARIABLES

Distribution	Mean	Variance
Point mass at a	a	0
Bernoulli(p)	p	$p(1-p)$
Binomial(n,p)	np	$np(1-p)$
Geometric(p)	$1/p$	$(1-p)/p^2$
Poisson(λ)	λ	λ
Uniform(a,b)	$(a+b)/2$	$(b-a)^2/12$
Normal(μ,σ^2)	μ	σ^2
Exponential(β)	β	β^2
Gamma(α,β)	$\alpha\beta$	$\alpha\beta^2$
Beta(α,β)	$\alpha/(\alpha+\beta)$	$\alpha\beta/((\alpha+\beta)^2(\alpha+\beta+1))$
t_ν	0 (if $\nu>1$)	$\nu/(\nu-2)$ (if $\nu>2$)
χ^2_p	p	$2p$
Multinomial(n,p)	np	see below
Multivariate Normal(μ,Σ)	μ	Σ

$n(I-pp^T)$

MAXIMUM LIKELIHOOD

- The Maximum Likelihood Estimation (MLE) is a method of estimating the parameters of a model. This estimation method is one of the most widely used.
- The method of maximum likelihood selects the set of values of the model parameters that maximizes the likelihood function. Intuitively, this maximizes the "agreement" of the selected model with the observed data.
- The Maximum-likelihood Estimation gives an unified approach to estimation.

MAXIMUM LIKELIHOOD

Definition

This joint probability is a function of θ (the unknown parameter) and corresponds to the **likelihood of the sample** $\{x_1, .., x_N\}$ denoted by

$$L_N(\theta; x_1.., x_N) = \Pr((X_1 = x_1) \cap ... \cap (X_N = x_N))$$

Question: What value of θ would make this **sample most probable**?

EXAMPLE: BERNOULLI

Example 1. A coin is flipped 100 times. Given that there were 55 heads, find the maximum likelihood estimate for the probability p of heads on a single toss.

Before actually solving the problem, let's establish some notation and terms.

We can think of counting the number of heads in 100 tosses as an experiment. For a given value of p, the probability of getting 55 heads in this experiment is the binomial probability

$$P(55 \text{ heads}) = \binom{100}{55} p^{55}(1-p)^{45}.$$

The probability of getting 55 heads depends on the value of p, so let's include p in by using the notation of conditional probability:

$$P(55 \text{ heads} \,|\, p) = \binom{100}{55} p^{55}(1-p)^{45}.$$

You should read $P(55 \text{ heads} \,|\, p)$ as:

'the probability of 55 heads given p,'

or more precisely as

'the probability of 55 heads given that the probability of heads on a single toss is p.'

EXAMPLE: BERNOULLI

- Likelihood, or likelihood function: this is $P(\text{data}\,|\,p)$. Note it is a function of both the data and the parameter p. In this case the likelihood is

$$P(55 \text{ heads}\,|\,p) = \binom{100}{55} p^{55}(1-p)^{45}.$$

Definition: Given data the maximum likelihood estimate (MLE) for the parameter p is the value of p that maximizes the likelihood $P(\text{data}\,|\,p)$. That is, the MLE is the value of p for which the data is most likely.

EXAMPLE: BERNOULLI

answer: For the problem at hand, we saw above that the likelihood

$$P(55 \text{ heads} \,|\, p) = \binom{100}{55} p^{55}(1-p)^{45}.$$

We'll use the notation $\hat{p}$ for the MLE. We use calculus to find it by taking the derivative of the likelihood function and setting it to 0.

$$\frac{d}{dp} P(\text{data} \,|p) = \binom{100}{55} (55p^{54}(1-p)^{45} - 45p^{55}(1-p)^{44}) = 0.$$

Solving this for p we get

$$55p^{54}(1-p)^{45} = 45p^{55}(1-p)^{44}$$
$$55(1-p) = 45p$$
$$55 = 100p$$
$$\text{the MLE is } \hat{p} = .55$$

This is known as the estimator.

EXAMPLE: POISSON

Example

Suppose that $X_1, X_2, \cdots, X_N$ *are i.i.d. discrete random variables, such that* $X_i \sim Pois(\theta)$ *with a* **pmf** (probability mass function) defined as:

$$\Pr(X_i = x_i) = \frac{\exp(-\theta)\,\theta^{x_i}}{x_i!}$$

where θ is an unknown parameter to estimate.

EXAMPLE: POISSON

Question: What is the probability of observing the **particular sample** $\{x_1, x_2, .., x_N\}$, assuming that a Poisson distribution with as yet unknown parameter θ generated the data?

This probability is equal to

$$\Pr\left((X_1 = x_1) \cap ... \cap (X_N = x_N)\right)$$

EXAMPLE: POISSON

Since the variables X_i are *i.i.d.* this joint probability is equal to the product of the marginal probabilities

$$\Pr\left((X_1 = x_1) \cap \ldots \cap (X_N = x_N)\right) = \prod_{i=1}^{N} \Pr\left(X_i = x_i\right)$$

Given the pmf of the Poisson distribution, we have:

$$\begin{aligned} \Pr\left((X_1 = x_1) \cap \ldots \cap (X_N = x_N)\right) &= \prod_{i=1}^{N} \frac{\exp\left(-\theta\right)\theta^{x_i}}{x_i!} \\ &= \exp\left(-\theta N\right) \frac{\theta^{\sum_{i=1}^{N} x_i}}{\prod_{i=1}^{N} x_i!} \end{aligned}$$

EXAMPLE: POISSON

Calculus Recap!

- If you want to find the value of x that gives you the highest value of f(x), we can use derivatives.
- This value of x is also know as argmax f(x).
- Recall the first derivative gives you the gradient function. A maximum or minimum of a continuous function must have gradient zero.
- To determine if it's a max, min or saddle point, we can use second derivatives.

EXAMPLE: POISSON

Consider maximizing the likelihood function $L_N(\theta; x_1.., x_N)$ with respect to θ. Since the log function is monotonically increasing, we usually maximize $\ln L_N(\theta; x_1.., x_N)$ instead. In this case:

$$\ln L_N(\theta; x_1.., x_N) = -\theta N + \ln(\theta) \sum_{i=1}^{N} x_i - \ln\left(\prod_{i=1}^{N} x_i!\right)$$

$$\frac{\partial \ln L_N(\theta; x_1.., x_N)}{\partial \theta} = -N + \frac{1}{\theta} \sum_{i=1}^{N} x_i$$

$$\frac{\partial^2 \ln L_N(\theta; x_1.., x_N)}{\partial \theta^2} = -\frac{1}{\theta^2} \sum_{i=1}^{N} x_i < 0$$

EXAMPLE: POISSON

Under suitable regularity conditions, the maximum likelihood estimate (estimator) is defined as:

$$\widehat{\theta} = \underset{\theta \in \mathbb{R}^+}{\arg\max} \ln L_N(\theta; x_1.., x_N)$$

$$FOC: \left. \frac{\partial \ln L_N(\theta; x_1.., x_N)}{\partial \theta} \right|_{\widehat{\theta}} = -N + \frac{1}{\widehat{\theta}} \sum_{i=1}^{N} x_i = 0$$

$$\iff \widehat{\theta} = (1/N) \sum_{i=1}^{N} x_i$$

$$SOC: \left. \frac{\partial^2 \ln L_N(\theta; x_1.., x_N)}{\partial \theta^2} \right|_{\widehat{\theta}} = -\frac{1}{\widehat{\theta}^2} \sum_{i=1}^{N} x_i < 0$$

$\widehat{\theta}$ is a maximum.

CONTINUOUS MLE

Continuous variables

- The reference to the probability of observing the given sample is not exact in a continuous distribution, since a particular sample has probability zero. Nonetheless, the principle is the same.
- The likelihood function then corresponds to the pdf associated to the **joint distribution** of $(X_1, X_2, .., X_N)$ evaluated at the point $(x_1, x_2, .., x_N)$:

$$L_N(\theta; x_1.., x_N) = f_{X_1,..,X_N}(x_1, x_2, .., x_N; \theta)$$

CONTINUOUS MLE

Continuous variables

- If the random variables $\{X_1, X_2, .., X_N\}$ are $i.i.d.$ then we have:

$$L_N(\theta; x_1.., x_N) = \prod_{i=1}^{N} f_X(x_i; \theta)$$

where $f_X(x_i; \theta)$ denotes the pdf of the marginal distribution of X (or X_i since all the variables have the same distribution).

- The values of the parameters that maximize $L_N(\theta; x_1.., x_N)$ or its log are the maximum likelihood estimates, denoted $\widehat{\theta}(x)$.

INDEPENDENCE

Two random variables are independent when the probability distribution of one random variable does not affect the other. More concretely, two random variables X and Y are independent, that is, $X \perp Y$, if and only if

$$P(X = x, Y = y) = P(X = x)P(Y = y), \quad \text{for all } x \in \mathcal{X}, y \in \mathcal{Y}. \tag{8}$$

If X and Y are continuous with joint density function $f_{X,Y}(x, y)$, then the above condition reduces to finding functions $h(x)$ and $g(y)$ such that

$$f_{X,Y}(x, y) = h(x)g(y). \tag{9}$$

CONDITIONAL INDEPENDENCE

Two random variables X and Y are conditionally independent given a third variable Z, denoted as $X \perp Y \mid Z$, if and only if

$$P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z)P(Y = y \mid Z = z), \quad (10)$$

for all $x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{Z}$.
This is equivalent to saying

$$P(X = x \mid Y = y, Z = z) = P(X = x \mid Z = z).$$

Note that $X \perp Y \mid Z$ does not imply that $X \perp Y$, and vice versa.

C.I. RELATIONS

- Symmetry:

$$X \perp Y \,|\, Z \implies Y \perp X \,|\, Z$$

- Decomposition:

$$X \perp Y, W \,|\, Z \implies X \perp Y \,|\, Z \quad (\text{and } X \perp W \,|\, Z)$$

- Weak union:

$$X \perp Y, W \,|\, Z \implies X \perp Y \,|\, Z, W \quad (\text{and } X \perp W \,|\, Y, Z)$$

- Contraction:

$$X \perp Y \,|\, Z \text{ and } X \perp W \,|\, Y, Z \implies X \perp Y, W \,|\, Z$$