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Multivariate Bicycle Codes

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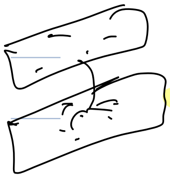

Quantum error correction suppresses noise in quantum systems to allow for high-precision computations. In this work, we introduce Multivariate Bicycle (MB) Quantum Low-Density Parity-Check (QLDPC) codes, via an extension of the framework developed by Bravyi *et al.* [Nature, 627, 778-782 (2024)] and particularly focus on Trivariate Bicycle (TB) codes. Unlike the weight-6 codes proposed in their study, we offer concrete examples of weight-4 and weight-5 TB-QLDPC codes which promise to be more amenable to near-term experimental setups. We show that our TB-QLDPC codes up to weight-6 have a bi-planar structure. Further, most of our new codes can also be arranged in a two-dimensional toric layout, and have substantially better encoding rates than comparable surface codes while offering similar error suppression capabilities. For example, we can encode 4 logical qubits with distance 5 into 30 physical qubits with weight-5 check measurements, while a surface code with these parameters requires 100 physical qubits. The high encoding rate and compact layout make our codes highly suitable candidates for near-term hardware implementations, paving the way for a realizable quantum error correction protocol.

Bivariate Bicycle codes



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High-threshold and low-overhead fault-tolerant quantum memory

Sergey Bravyi, Andrew W. Cross, Jay M. Gambetta, Dmitri Maslov, Patrick Rall & Theodore J. Yoder

Nature **627**, 778–782 (2024) | Cite this article

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ABSTRACT

Quantum error correction becomes a practical possibility only if the physical error rate is below a threshold value that depends on a particular quantum code, syndrome measurement circuit, and a decoding algorithm. Here we present an end-to-end quantum error correction protocol that implements fault-tolerant memory based on a family of LDPC codes with a high encoding rate that achieves an error threshold of 0.8% for the standard circuit-based noise model. This is on par with the surface code which has remained an uncontested leader in terms of its high error threshold for nearly 20 years. The full syndrome measurement cycle for a length-n code in our family requires n ancillary qubits and a depth-7 circuit composed of nearest-neighbor CNOT gates. The required qubit connectivity is a degree-6 graph that consists of two edge-disjoint planar subgraphs. As a concrete example, we show that 12 logical qubits can be preserved for ten million syndrome cycles using 288 physical qubits in total, assuming the physical error rate of 0.1%. We argue that achieving the same level of error suppression on 12 logical qubits with the surface code would require more than 4000 physical qubits. Our findings bring demonstrations of a low-overhead fault-tolerant quantum memory within the reach of near-term quantum processors.

LDPC CODES

- **Low-density parity check (LDPC) codes:** the number of bits involved in each check and the number of checks acting on each bit are bound by a constant for all members of the code family
- Many modern technologies, such as WiFi, DVB-T, and 5G, are error-corrected by LDPC codes

Fault-Tolerant Quantum Computation with Constant Overhead

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Abstract

What is the minimum number of extra qubits needed to perform a large fault-tolerant quantum circuit? Working in a common model of fault-tolerance, I show that in the asymptotic limit of large circuits, the ratio of physical qubits to logical qubits can be a constant. The construction makes use of quantum low-density parity check codes, and the asymptotic overhead of the protocol is equal to that of the family of quantum error-correcting codes underlying the fault-tolerant protocol.

QUANTUM LDPC CODES

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Abstract

What is the minimum number of extra qubits needed to perform a large fault-tolerant quantum circuit? Working in a common model of fault-tolerance, I show that in the asymptotic limit of large circuits, the ratio of physical qubits to logical qubits can be a constant. The construction makes use of quantum low-density parity check codes, and the asymptotic overhead of the protocol is equal to that of the family of quantum error-correcting codes underlying the fault-tolerant protocol.

Definition 1. An $[[n,k]]$ stabilizer code is an (r,c)-LDPC code if there exists a choice of generators $\{M_1,\ldots,M_{n-k}\}$ for the stabilizer Q of the code such that $\mathrm{wt}\, M_i \leq r$ for all i and the number of generators which act non-trivially (i.e., as X, Y, or Z) on qubit j is at most c for all $j \in [1,n]$.

Technically, based on this definition, every $[[n,k]]$ stabilizer code is automatically an $(n,n-k)$-LDPC code. However, typically, when we refer to an "LDPC" code, we are only considering the interesting cases, which arise when r and c are much smaller than n and $n-k$, particularly when r and c are constant for large n. Note that whenever the distance d of the LDPC code is greater than r, the code is degenerate.

QUANTUM LDPC CODES

Quantum Low-Density Parity-Check Codes

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Quantum error correction is an indispensable ingredient for scalable quantum computing. In this Perspective we discuss a particular class of quantum codes called "quantum low-density parity-check (LDPC) codes." The codes we discuss are alternatives to the surface code, which is currently the leading candidate to implement quantum fault tolerance. We introduce the zoo of quantum LDPC codes and discuss their potential for making quantum computers robust with regard to noise. In particular, we explain recent advances in the theory of quantum LDPC codes related to certain product constructions and discuss open problems in the field.

"Good"

$\frac{k}{n} = c_1$

$\frac{d}{n} = c_2$

TABLE I. The best proven parameters of quantum LDPC codes discussed in this paper. Some entries refer to a whole method of constructing codes. In these cases we cite the best known proven parameters of a family quantum LDPC codes constructed using this method.

Name	k	d	Section
Two-dimensional hyperbolic codes	$\Theta(n)$	$\Theta(\log n)$	III A
Four-dimensional hyperbolic codes	$\Theta(n)$	$\Omega(\sqrt[10]{n})$	III A
Freedman-Meyer-Luo codes	2	$\Omega\left(\sqrt[4]{\log n}\sqrt{n}\right)$	III B
Tensor products (good classical codes)	$\Theta(n)$	$\Theta(\sqrt{n})$	IV A
Tensor products (Ramanujan complexes)	$\Theta(\sqrt{n})$	$\Omega(\sqrt{n}\,\mathrm{polylog}\, n)$	IV A
Fibre bundle codes	$\Theta(n^{3/5}/\,\mathrm{polylog}\, n)$	$\Omega(n^{3/5}/\,\mathrm{polylog}\, n)$	IV B
Lifted product codes	$\Theta(n^{\alpha}\log n)$	$\Omega(n^{1-\alpha}/\log n)$	IV C
Balanced product codes	$\Theta(n^{4/5})$	$\Omega(n^{3/5})$	IV D

Not enough!

"GOOD" QUANTUM LDPC CODES

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Pavel Panteleev and Gleb Kalachev. Asymptotically good quantum and locally testable classical LDPC codes. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 375–388, 2022.

Anthony Leverrier and Gilles Zémor. Quantum Tanner codes. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 872–883. IEEE, 2022.

Asymptotically Good Quantum and Locally Testable Classical LDPC Codes

Pavel Panteleev and Gleb Kalachev*

January 24, 2022

Abstract

We study classical and quantum LDPC codes of constant rate obtained by the lifted product construction over non-abelian groups. We show that the obtained families of quantum LDPC codes are asymptotically good, which proves the qLDPC conjecture. Moreover, we show that the produced classical LDPC codes are also asymptotically good and locally testable with constant query and soundness parameters, which proves a well-known conjecture in the field of locally testable codes.

NLTS Hamiltonians from good quantum codes

Anurag Anshu[1], Nikolas P. Breuckmann[2], and Chinmay Nirkhe[3]

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$[[n,k,d]]$	Net Encoding Rate r	Circuit-level distance d_{circ}	Pseudo-threshold p_0	$p_L(0.001)$	$p_L(0.0001)$
$[[72,12,6]]$	$1/12$	≤ 6	$0.007(8)$	4×10^{-5}	3×10^{-8}
$[[90,8,10]]$	$1/23$	≤ 8	$0.007(3)$	2×10^{-6}	1×10^{-10}
$[[108,8,10]]$	$1/27$	≤ 8	$0.007(7)$	9×10^{-7}	5×10^{-11}
$[[144,12,12]]$	$1/24$	≤ 10	$0.008(3)$	4×10^{-8}	2×10^{-13}
$[[288,12,18]]$	$1/48$	≤ 18	$0.008(2)$	5×10^{-12}	3×10^{-21}

Table 1: Small examples of quasi-cyclic LDPC codes and their performance for the circuit-based noise model. All codes have weight-6 checks, thickness-2 Tanner graph, and a depth-7 syndrome measurement circuit. A code with parameters $[[n,k,d]]$ requires $2n$ physical qubits in total and achieves the net encoding rate $r=k/2n$ (we round r down to the nearest inverse integer). Circuit-level distance d_{circ} is the minimum number of faulty operations in the syndrome measurement circuit required to generate an undetectable logical error. The pseudo-threshold p_0 is a solution of the break-even equation $p_L(p)=kp$, where p and p_L are the physical and logical error rates respectively. The logical error rate p_L was computed numerically for $p\geq 10^{-3}$ and extrapolated to lower error rates.

QUASI-CYCLIC QUANTUM LDPC CODES

CSS

$H^x (H^z)^T = 0$

Let I_ℓ and S_ℓ be the identity matrix and the cyclic shift matrix of size $\ell \times \ell$ respectively. The i-th row of S_ℓ has a single nonzero entry equal to one at the column $i+1 \pmod{\ell}$. For example,

$$S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Consider matrices

$$x = S_\ell \otimes I_m \quad \text{and} \quad y = I_\ell \otimes S_m.$$

Note that $xy = yx$ and $x^\ell = y^m = I_{\ell m}$. A quasi-cyclic code is defined by a pair of matrices

$$A = A_1 + A_2 + A_3 \quad \text{and} \quad B = B_1 + B_2 + B_3 \tag{1}$$

where each matrix A_i and B_j is a power of x or y. Here and below the addition and multiplication of binary matrices is performed modulo two, unless stated otherwise. Thus, we also assume the A_i are distinct and the B_j are distinct to avoid cancellation of terms. For example, one could choose $A = x^3 + y + y^2$ and $B = y^3 + x + x^2$. Note that A and B have exactly three non-zero entries in each row and each column. Furthermore, $AB = BA$ since $xy = yx$. The above data defines a quasi-cyclic LDPC code denoted $\mathsf{QC}(A, B)$ with length $n = 2\ell m$ and check matrices

$$H^X = [A|B] \quad \text{and} \quad H^Z = [B^T|A^T]. \tag{2}$$

Here the vertical bar indicates stacking matrices horizontally and T stands for the matrix transposition. Both matrices H^X and H^Z have size $(n/2) \times n$. Each row $v \in \mathbb{F}_2^n$ of H^X defines an X-type check operator $X(v) = \prod_{i=1}^n X_i^{v_j}$. Each row $v \in \mathbb{F}_2^n$ of H^Z defines a Z-type check operator $Z(v) = \prod_{j=1}^n Z_j^{v_j}$. Any X-check and Z-check commute since they overlap on even number of qubits (note that $H^X(H^Z)^T = AB + BA = 0 \pmod 2$). To describe the code parameters we use certain linear subspaces associated with the check matrices, see Table 1 for our notations. Then the code $\mathsf{QC}(A, B)$ has parameters $[[n, k, d]]$ with

$$n = 2\ell m, \quad k = 2 \cdot \dim(\ker(A) \cap \ker(B)) \quad \text{and} \quad d = \min\{|v| : v \in \ker(H^X) \setminus \mathsf{rs}(H^Z)\}, \tag{3}$$

see Lemma 1. Here $|v| = \sum_{i=1}^n v_i$ is the Hamming weight of a vector $v \in \mathbb{F}_2^n$.

$H_x H_z^T$

$=$

$AB + BA$

$[A, B] = 0$

$\Downarrow$

$2AB$

$= 0$

QUASI-CYCLIC QUANTUM LDPC CODES

Notation	Name	Definition
$\mathsf{rs}(H)$	row space	Linear span of rows of H
$\mathsf{cs}(H)$	column space	Linear span of columns of H
$\mathsf{ker}(H)$	nullspace	Vectors orthogonal to each row of H
$\mathsf{rk}(H)$	rank	$\mathsf{rk}(H) = \dim(\mathsf{rs}(H)) = \dim(\mathsf{cs}(H))$

Table 2: Notations for linear spaces associated with a binary matrix H. Here the linear span, orthogonality, and dimension are computed over the binary field $\mathbb{F}_2 = \{0, 1\}$. If H has size $s \times n$ then $\mathsf{rs}(H) \subseteq \mathbb{F}_2^n$, $\mathsf{cs}(H) \subseteq \mathbb{F}_2^s$, and $\mathsf{ker}(H) \subseteq \mathbb{F}_2^n$.

$[[n, k, d]]$	Net Encoding Rate r	ℓ, m	A	B
$[[72, 12, 6]]$	$1/12$	$6, 6$	$x^3 + y + y^2$	$y^3 + x + x^2$
$[[90, 8, 10]]$	$1/23$	$15, 3$	$x^9 + y + y^2$	$1 + x^2 + x^7$
$[[108, 8, 10]]$	$1/27$	$9, 6$	$x^3 + y + y^2$	$y^3 + x + x^2$
$[[144, 12, 12]]$	$1/24$	$12, 6$	$x^3 + y + y^2$	$y^3 + x + x^2$
$[[288, 12, 18]]$	$1/48$	$12, 12$	$x^3 + y^2 + y^7$	$y^3 + x + x^2$
$[[360, 12, \leq 24]]$	$1/60$	$30, 6$	$x^9 + y + y^2$	$y^3 + x^{25} + x^{26}$
$[[756, 16, \leq 34]]$	$1/95$	$21, 18$	$x^3 + y^{10} + y^{17}$	$y^5 + x^3 + x^{19}$

Not good!

Table 3: Small examples of quasi-cyclic LDPC codes and their parameters. All codes have weight-6 checks, thickness-2 Tanner graph, and a depth-7 syndrome measurement circuit. Code distance was computed by the mixed integer programming approach of Ref. [48]. Notation $\leq d$ indicates that only an upper bound on the code distance is known at the time of this writing. We round r down to the nearest inverse integer. The codes have check matrices $H^X = [A|B]$ and $H^Z = [B^T|A^T]$ with A and B defined in the last two columns. The matrices x, y obey $x^\ell = y^m = 1$ and $xy = yx$.

OPEN QUESTIONS

- Optimize for n,k and d
- Reduce reduce the weight of check operators

RELATED WORK

$[[n,k,d]]$	Net Encoding Rate r	Circuit-level distance d_{circ}	Pseudo-threshold p_0	$p_L(0.001)$	$p_L(0.0001)$
$[[72,12,6]]$	$1/12$	≤ 6	$0.007(8)$	4×10^{-5}	3×10^{-8}
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Fault-tolerant hyperbolic Floquet quantum error correcting codes

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(Dated: September 25, 2023)

Code	No. of Unit Cells	Genus	$[[n,k,d]]$
H16	1	2	[[16, 4, 2]]
H64	4	5	[[64, 10, 4]]
H144	9	10	[[144, 20, 6]]
H400	25	26	[[400, 52, 8]]
H2160	135	136	[[2160, 272, 10]]

$H_X = [A|B]$ $\qquad H_Z = [B^T|A^T]$

$$H_X H_Z^T = AB + BA$$
$$= 2AB \quad \text{if } AB = BA$$
$$= 0$$

$\boxed{R_2}$

$$R_r = \mathbb{F}_2[x_1, x_2, \ldots, x_r] / (x_1^{\ell_1} - 1, x_2^{\ell_2} - 1, \ldots, x_r^{\ell_r} - 1)$$

Choose an encoding for x_i using shift matrices

$x_1 =$

$x_1 = I \otimes S$

$x_2 = S \otimes I$

Let $x_1 = S_\ell \otimes I_m$

$x_2 = I_\ell \otimes S_m$

$x_3 = S_\ell \otimes S_m$

$n = 2\ell m$

$k = 2 \cdot \dim(\ker(A) \cap \ker(B))$

$d = \min\{|v| : v \in \ker(H_X) \setminus rs(H_Z)\}$

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0001 $\qquad I \otimes I \otimes I \otimes S$

0010 $\qquad I \otimes I \otimes S \otimes I$

0011

0100

Weight-4 QLDPC Codes

$[\![n,k,d]\!]$	r	l,m	$r_{\mathrm{BB}}/r_{\mathrm{SC}}$	p_0	$p_L(10^{-4})$	A	B	i,j,g,h	μ,λ	toric	bi-planar
$[\![112,8,5]\!]$	1/14	7, 8	1.8	0.0298	2×10^{-9}	z^2+z^6	$x+x^6$	×	×	×	✓
$[\![64,2,8]\!]$	1/32	8, 4	2.0	0.0767	4×10^{-13}	$x+x^2$	x^3+y	×	×	×	✓
$[\![72,2,8]\!]$	1/36	4, 9	1.8	0.0863	2×10^{-13}	$x+y^2$	x^2+y^2	×	×	×	✓
$[\![96,2,8]\!]$	1/48	6, 8	1.3	0.0911	4×10^{-16}	x^5+y^6	$z+z^4$	×	×	×	✓
$[\![112,2,10]\!]$	1/56	7, 8	1.8	0.097	2×10^{-16}	z^6+x^5	z^2+y^5	×	×	×	✓
$[\![144,2,12]\!]$	1/72	8, 9	2.0	0.1017	4×10^{-19}	x^3+y^7	$x+y^5$	×	×	×	✓

Weight-5 QLDPC Codes

$[\![n,k,d]\!]$	r	l,m	$r_{\mathrm{BB}}/r_{\mathrm{SC}}$	p_0	$p_L(10^{-4})$	A	B	i,j,g,h	μ,λ	toric	bi-planar
$[\![30,4,5]\!]$	1/7	3, 5	3.3	0.0437	6×10^{-10}	$x+z^4$	$x+y^2+z^2$	$(1,2,2,3)$	$(5,3)$	✓	✓
$[\![72,4,8]\!]$	1/18	4, 9	3.6	0.0785	8×10^{-14}	$x+y^3$	x^2+y+y^2	×	×	×	✓
$[\![96,4,8]\!]$	1/24	8, 6	2.7	0.0823	1×10^{-13}	x^6+x^3	z^5+x^5+y	$(1,2,1,2)$	$(8,6)$	✓	✓

Weight-6 QLDPC Codes

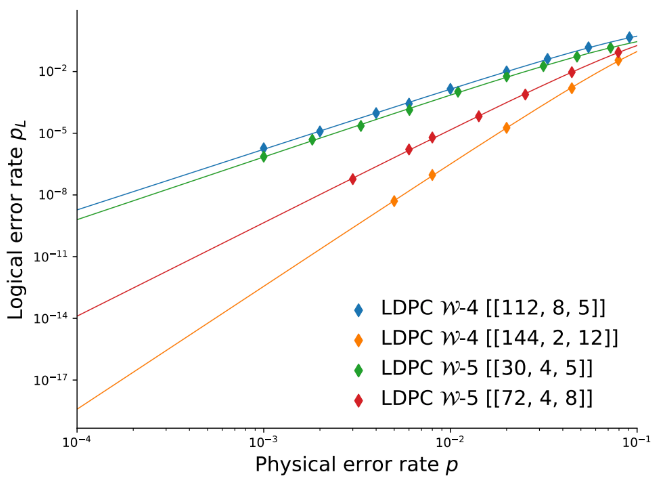
$[\![n,k,d]\!]$	r	l,m	$r_{\mathrm{BB}}/r_{\mathrm{SC}}$	p_0	$p_L(10^{-4})$	A	B	i,j,g,h	μ,λ	toric	bi-planar
$[\![30,6,4]\!]$	1/5	5, 3	3.2	0.0234	3×10^{-7}	x^4+z^3	x^4+x+z^4+y	$(1,2,1,3)$	$(5,3)$	✓	✓
$[\![48,6,6]\!]$	1/8	4, 6	4.5	0.0495	2×10^{-10}	x^2+y^4	$x^3+z^3+y^2+y$	×	×	×	✓
$[\![40,4,6]\!]$	1/10	4, 5	3.6	0.0588	7×10^{-11}	x^2+y	$y^4+y^2+x^3+x$	×	×	×	✓
$[\![48,4,6]\!]$	1/12	4, 6	3.0	0.0698	3×10^{-11}	x^3+y^5	$x+z^5+y^5+y^2$	$(1,2,3,4)$	$(12,2)$	✓	✓

Weight-7 QLDPC Codes

$[\![n,k,d]\!]$	r	l,m	$r_{\mathrm{BB}}/r_{\mathrm{SC}}$	p_0	$p_L(10^{-4})$	A	B	i,j,g,h	μ,λ	toric	bi-planar
$[\![30,4,5]\!]$	1/7	5, 3	3.3	0.0507	5×10^{-10}	x^4+x^2	$x+x^2+y+z^2+z^3$	$(1,2,2,4)$	$(5,3)$	✓	×

TABLE II. Summary of QLDPC codes sorted by encoding rate $r = k/n$. For each code, we provide the improvement in qubit-efficiency $r_{\mathrm{BB}}/r_{\mathrm{SC}}$ compared to the surface code, the physical error-rate below which error correction becomes a net-gain also known as the pseudo-threshold p_0 and the respective code polynomials A and B. i, j, g, h refer to the indices chosen to satisfy a toric layout, with μ, λ as the torus parameters such that the torus is embedded on a $2\mu \times 2\lambda$ grid with periodic boundary conditions. If no suitable indices could be found that allow for a toric layout, the column value is set to $\times$.

$[[n,k,d]]$	$\mathcal{W}$	$r = k/n$	$r_{\mathrm{TB}}/r_{\mathrm{SC}}$	p_0	$p_{\mathrm{L}}(10^{-4})$	toric	bi-planar
$[[144, 2, 12]]$	4	1/72	2.0	0.1017	4×10^{-19}	×	✓
$[[30, 4, 5]]$	5	1/7	3.3	0.0437	6×10^{-10}	✓	✓
$[[30, 6, 4]]$	6	1/5	3.2	0.0234	3×10^{-7}	✓	✓
$[[30, 4, 5]]$	7	1/7	3.3	0.0507	5×10^{-10}	✓	unknown



Proposition 1 (Bi-planar Architecture)**.** *All TB-QLDPC codes of weight-4, where $A = A_1 + A_3$, $B = B_1 + B_3$, all codes of weight-5 such that $A = A_1 + A_3$, $B = B_1 + B_2 + B_3$, and all codes of weight-6 such that $A = A_1 + A_2 + A_3$, $B = B_1 + B_2 + B_3$ (the case presented in Ref. [20]), or $A = A_1 + A_3$, $B = B_1 + B_2 + B_3 + B_4$ allow for a bi-planar architecture of thickness $\theta = 2$. The bi-planar decomposition can be computed in time $O(n)$.*

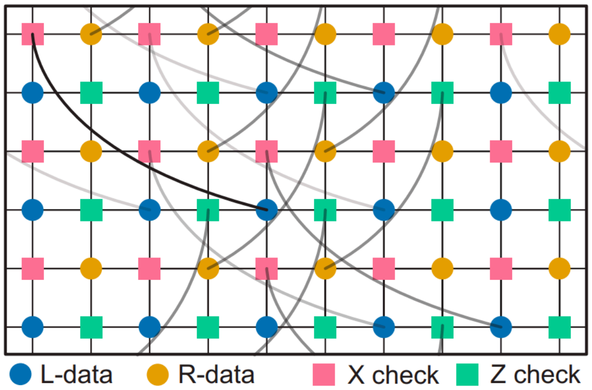


FIG. 1. Our weight-5 $[\![30,4,5]\!]$ TB-QLDPC code can be arranged in a bi-planar grid with periodic boundary conditions (PBC) using 30 data qubits (which belong either to the L or R sublattice) and 30 qubits for parity check measurements. Each parity check involves four neighbouring data qubits, and one non-local interaction described by a translational invariant lattice vector (x, y). For X-checks, the interaction vector is $(-4, -3)$, while for Z-checks we have $(2, 3)$. For better visibility, we only plot representatives of the non-local interactions at the center of the grid.

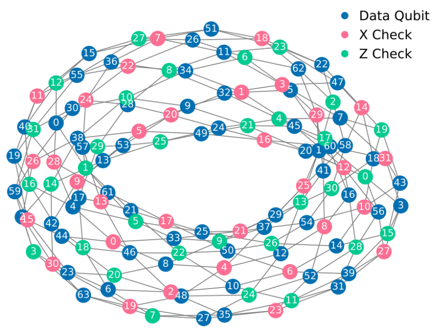


FIG. 4. Tanner Graph of the weight-4 $[\![64,2,8]\!]$ code, as a tangled torus with layout parameters $\mu = l = 8, \lambda = m = 4$. The usual torus topology is disrupted by a tangle on the right side of the figure.

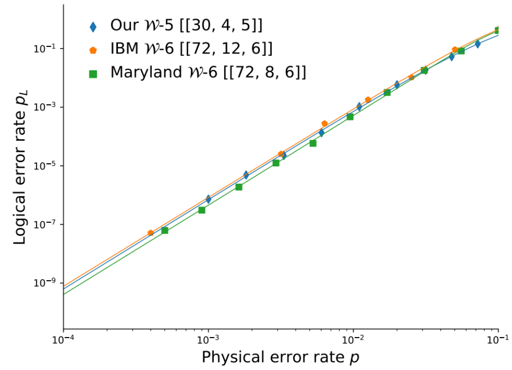


FIG. 3. Logical error rate of our weight-5 $[\![30,4,5]\!]$ TB code, weight-6 $[\![72,8,6]\!]$ BB code of Ref. [23], and weight-6 $[\![72,12,6]\!]$ BB code of Ref. [20].

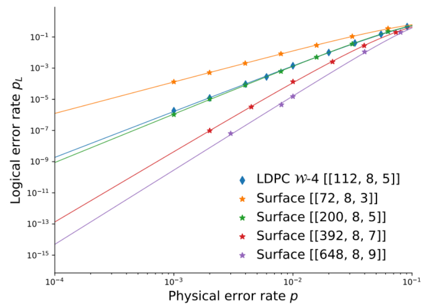


FIG. 5. Logical error rate p_L against physical error rate p for our weight-4 code $[\![112,8,5]\!]$, and surface codes with similar code parameters. The error curves are fitted with Eq. (4), yielding a fitted code distance of $d_{\rm fit} = 5.9$.

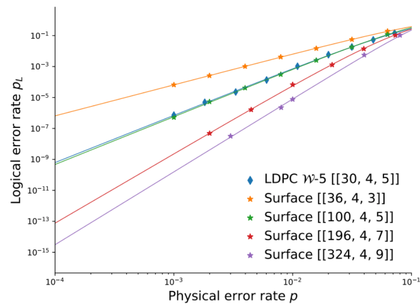


FIG. 6. Logical error rate p_L against physical error rate p for our weight-5 code $[\![30,4,5]\!]$. We show surface codes with comparable code parameters as reference. We find $d_{\rm fit} = 6.1$ for our QLDPC code.

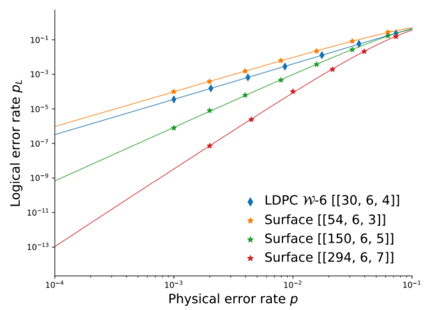


FIG. 7. Logical error rate of our weight-6 code $[\![30,6,4]\!]$ against comparable surface codes. We find $d_{\rm fit} = 4.1$.

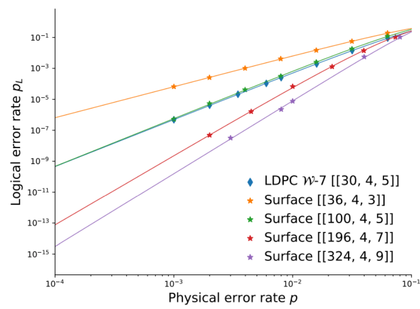


FIG. 8. Our weight-7 code $[\![30,4,5]\!]$ compared against surface codes with the same number of logical qubits. We find $d_{\rm fit} = 6.1$.