Constructing quantum codes from any classical code and their embedding in ground space of local Hamiltonians

Ramis Movassagh, Yingkai Ouyang
IBM Quantum Research, University of Sheffield

Main results

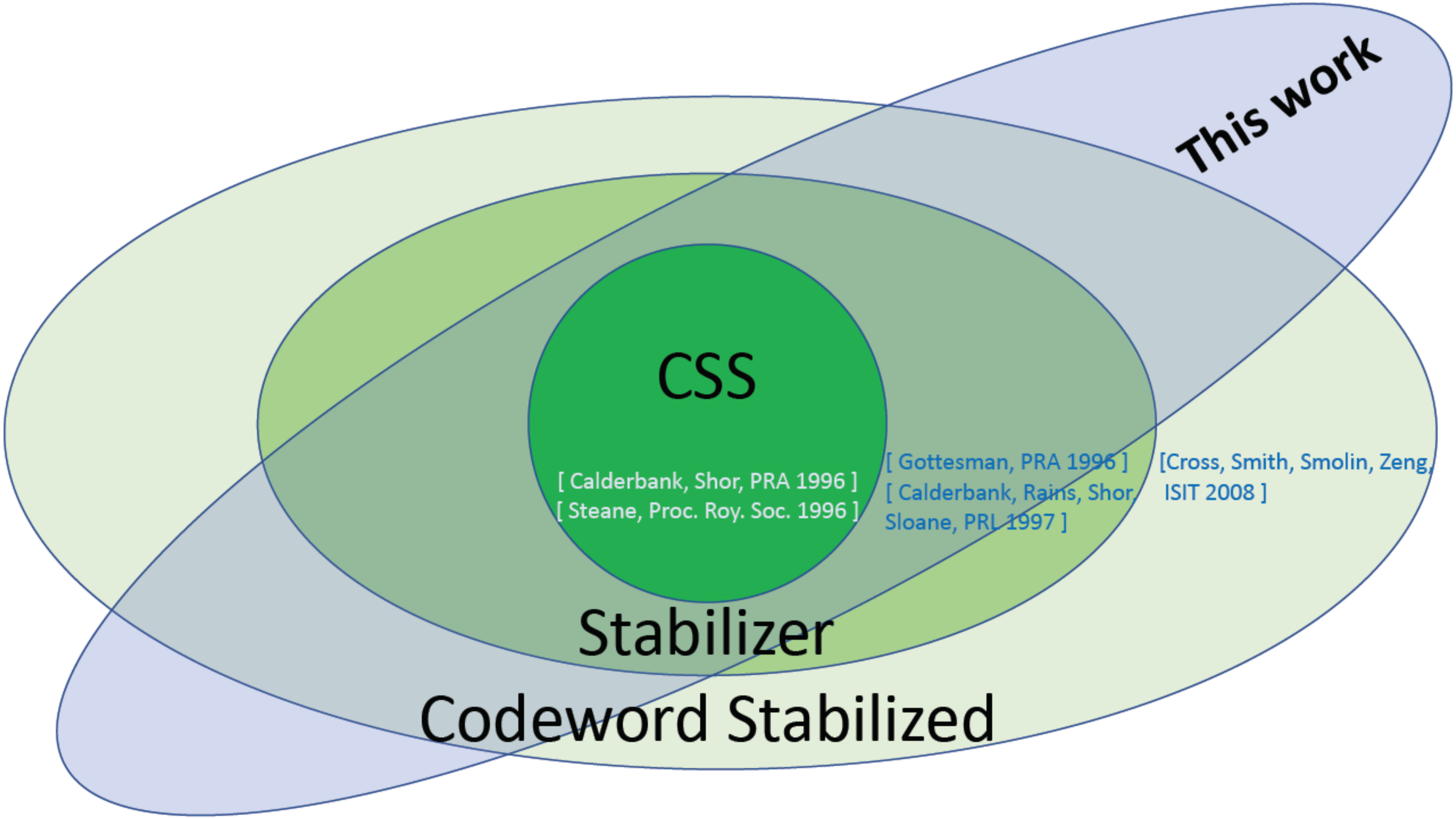
Part I

New Framework:
Any classical code to a quantum code.

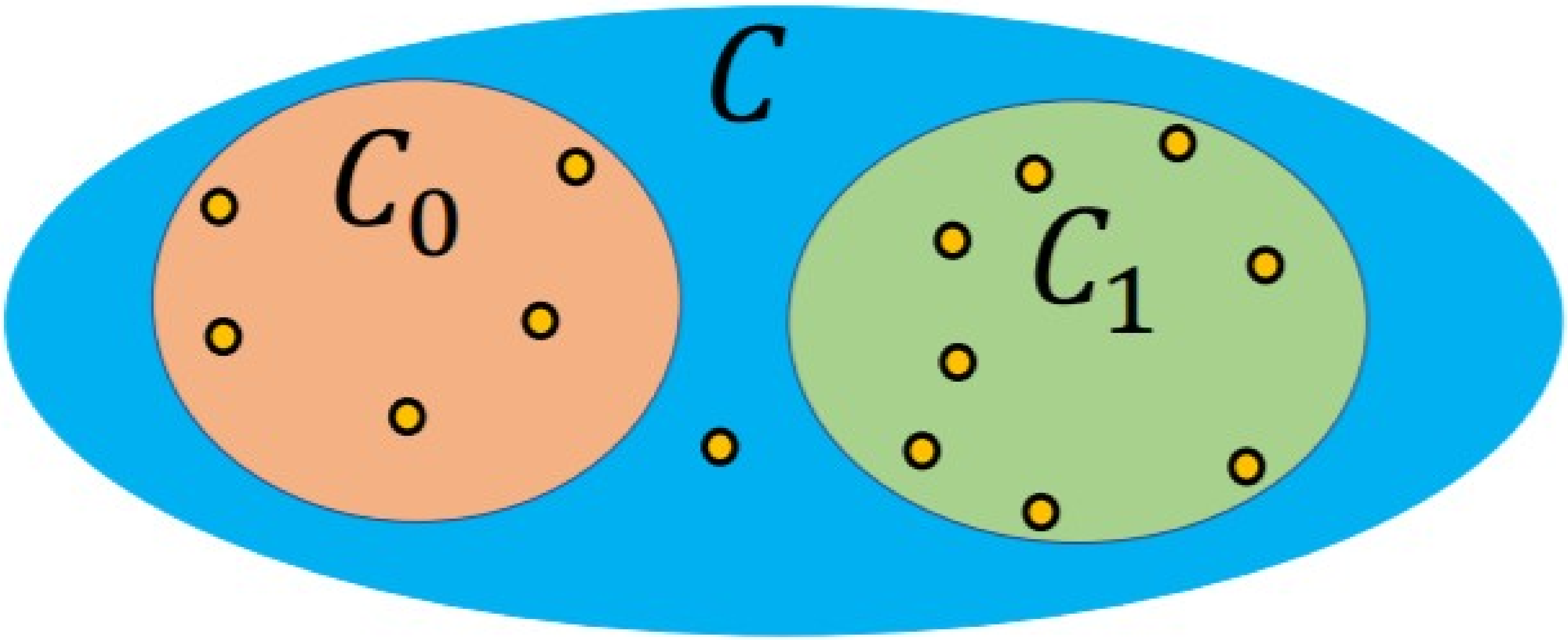
Part II

Linear distance quantum code in ground space of a new 2-local Hamiltonian.

Quantum coding formalisms



Quantum code's structure



$$|0_L\rangle = \sum_{\mathbf{c}\in C_0} a_{\mathbf{c}}|\mathbf{c}\rangle, \quad |1_L\rangle = \sum_{\mathbf{c}\in C_1} b_{\mathbf{c}}|\mathbf{c}\rangle$$

Non-negative

Quantum error correction criterion

$$\Pi P \Pi = c_P \Pi$$

$$\Pi = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$$

$$\begin{aligned} \langle 0_L|P|0_L\rangle &= \langle 1_L|P|1_L\rangle \\ \langle 0_L|P|1_L\rangle &= 0 \end{aligned}$$

$$\mathrm{mindist}(C) = d$$
$$C_0 \cap C_1 = \emptyset$$

- Non-deformation conditions
- Orthogonality conditions

Non-deformation conditions

$$\boxed{\langle 0_L | P | 0_L \rangle \quad = \quad \langle 1_L | P | 1_L \rangle}$$

$$\boxed{\mathrm{mindist}(C) = d}$$

$$\Downarrow$$

$$\boxed{\begin{gathered} \langle 0_L | P | 0_L \rangle = \langle 1_L | P | 1_L \rangle = 0 \\ 1 \leq \mathrm{wt}_X(P) \leq d - 1 \end{gathered}}$$

Consider diagonal Paulis *P*

Sandwich evaluation

$$\langle 0_L | \, P \, | 0_L \rangle = \sum_{\mathbf{c} \in C_0} a_{\mathbf{c}}^2 \, \langle \mathbf{c} | \, P \, | \mathbf{c} \rangle$$

$$\langle 1_L | \, P \, | 1_L \rangle = \sum_{\mathbf{c} \in C_1} b_{\mathbf{c}}^2 \, \langle \mathbf{c} | \, P \, | \mathbf{c} \rangle$$

$$\langle 0_L | \, P \, | 0_L \rangle \; - \; \langle 1_L | \, P \, | 1_L \rangle$$

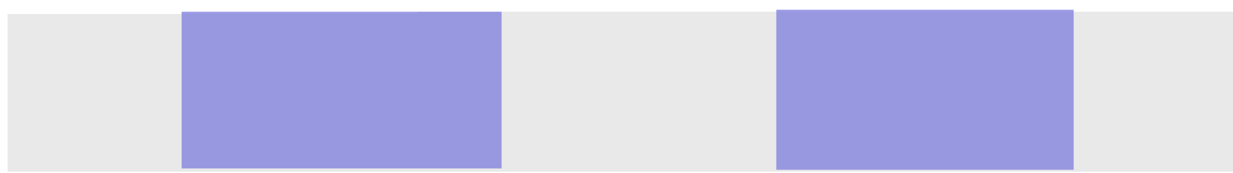
Quantum code from A's nullspace

- Make A-matrix

$$\mathrm{Re}\,\langle \mathbf{c}|\, P \,|\mathbf{c}\rangle \qquad \mathrm{Im}\,\langle \mathbf{c}|\, P \,|\mathbf{c}\rangle$$

- Find a real non-zero solution of

$$A\mathbf{x} = 0$$

- $\mathbf{x} =$ 

Non-negative, C_0

Negative, C_1

- $a_{\mathbf{c}} = \sqrt{x_{\mathbf{c}}} \qquad b_{\mathbf{c}} = \sqrt{-x_{\mathbf{c}}}$

Quantum coding implication

$|C|$ columns

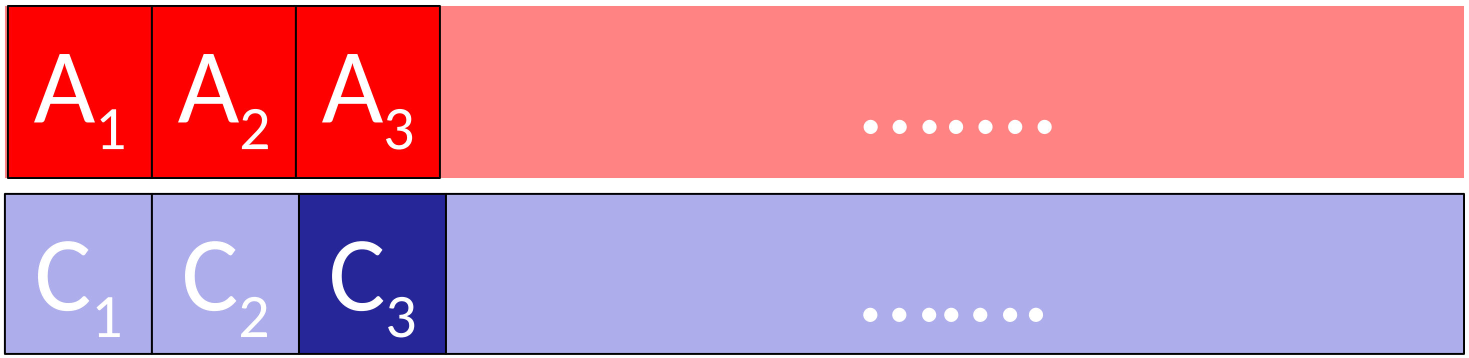
$2V_q(d-1)-1$ rows **A**

Theorem 1a: Using any non-zero solution of Ax = 0, we can derive a quantum code.

$$\boxed{|C| \geq 2V_q(d-1)}$$

Embedding more logical states

A-matrix: $\mathrm{Re}\langle \mathbf{c}| P |\mathbf{c}\rangle$ $\mathrm{Im}\langle \mathbf{c}| P |\mathbf{c}\rangle$



$$|j_L\rangle = \sum_{\mathbf{c}\in C_j} \alpha_{\mathbf{c}}^{(j)} |\mathbf{c}\rangle,$$

$$[A_1 A_2]\begin{bmatrix} x_1^+ \\ -x_1^- \end{bmatrix} = 0 \qquad \boxed{[A_2 A_3]\begin{bmatrix} -x_1^- \\ x_2 \end{bmatrix} = 0\,, \quad x_2 \geq 0}$$

Dines, Annals of Mathematics, 1926

Embedding more logical states

A-matrix: $\mathrm{Re}\,\langle \mathbf{c} | P | \mathbf{c} \rangle$ $\quad \mathrm{Im}\,\langle \mathbf{c} | P | \mathbf{c} \rangle$

Theorem 1b: Using this procedure, we can derive quantum codes with linear distance and constant rate.

$\mathbf{c} \in C_j$

$$[A_1 A_2] \begin{bmatrix} x_1^+ \\ -x_1^- \end{bmatrix} = 0 \qquad \boxed{[A_2 A_3] \begin{bmatrix} -x_1^- \\ x_2 \end{bmatrix} = 0\,, \quad x_2 \geq 0}$$

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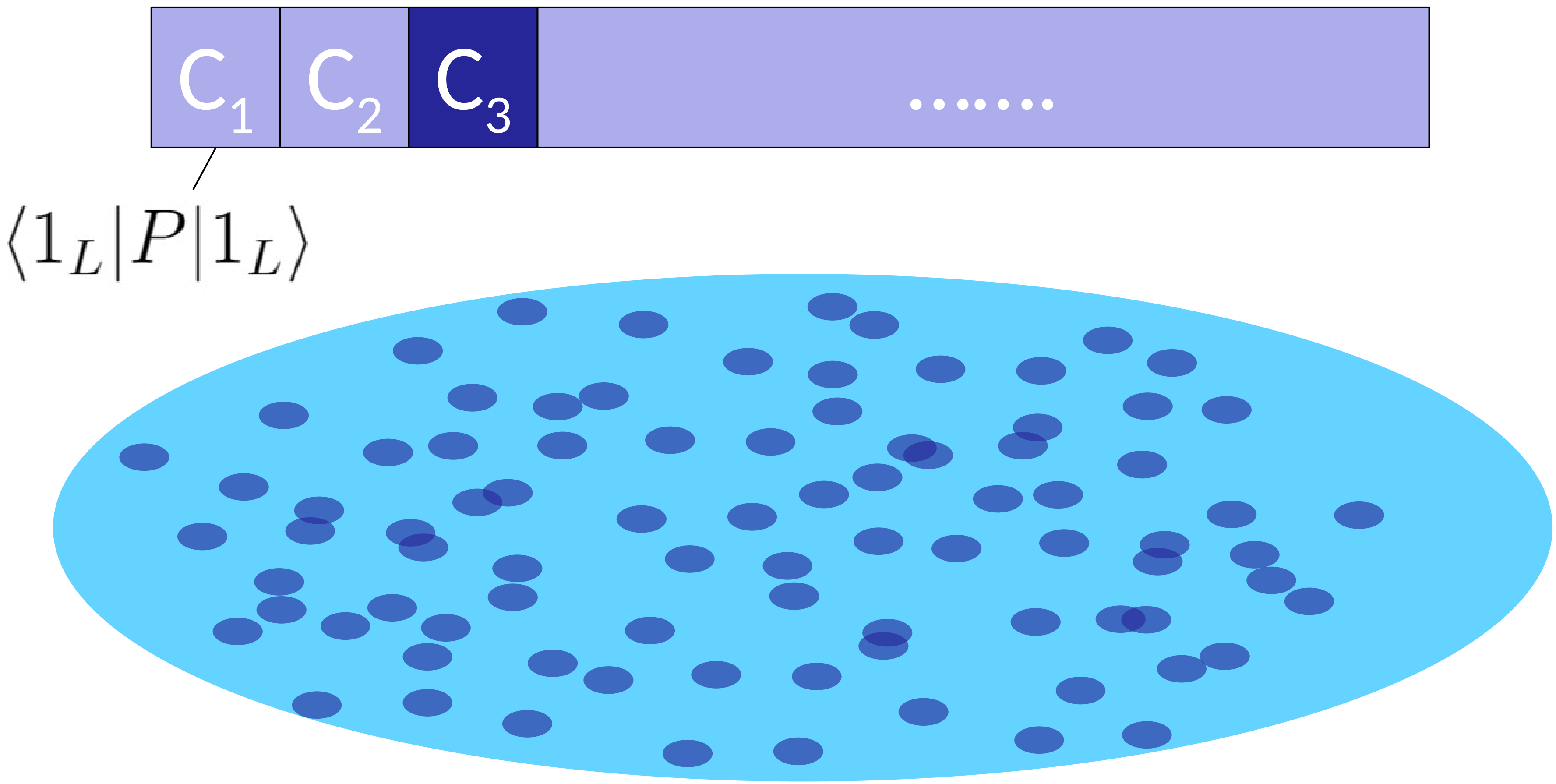
AQEC:Packing in a hypercube



Each point = a complex vector, components labelled by P
Number of points = number of subcodes

Illustrations (finite n)

Classical	Quantum
Repetition code	Nothing as expected
[7,4,3] Hamming code	Steane's code uniquely!
Nonlinear (4,8,2) cyclic code	((4,4,2)) CWS code

Permutation-invariant states, where product is not basis

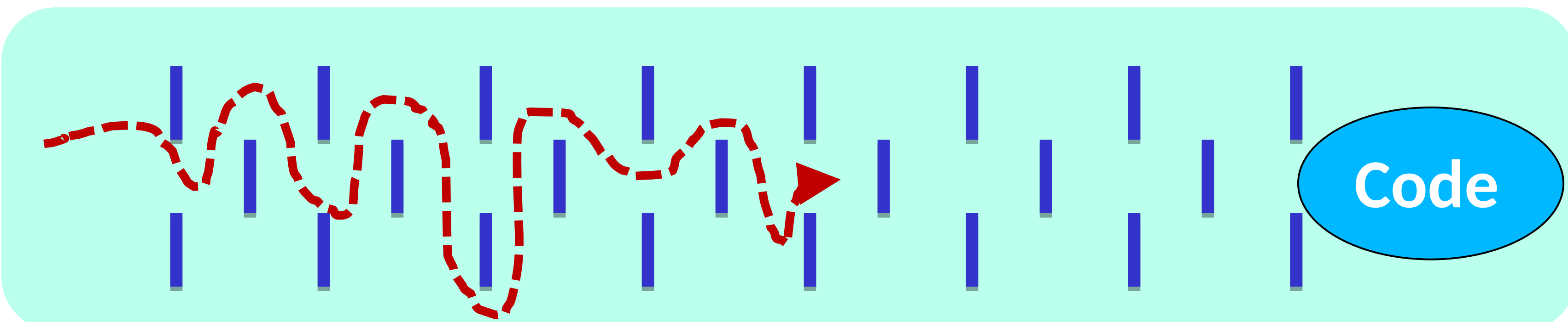
Ground states and quantum codes

- Heisenberg models, (Kitaev's code, compass model / XY model .

$$H = J \sum_{\langle j,k \rangle} (X_j X_k + Z_j Z_k)$$

Kitaev, Annals of Physics 2006
Dorier, Becca, Mila, PRB 2005
Li, Miller, Newman, Wu, Brown, PRX 2019

- **Engineer Hamiltonian** to **suppress** noise.

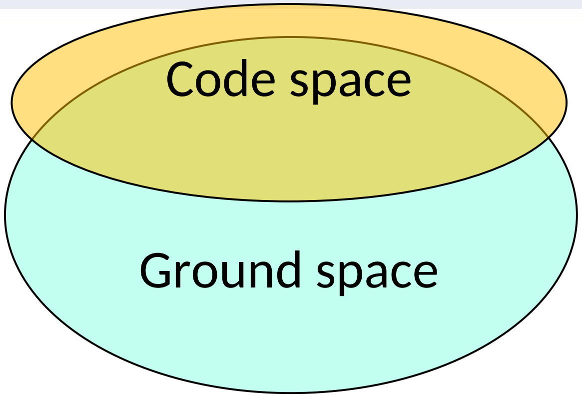


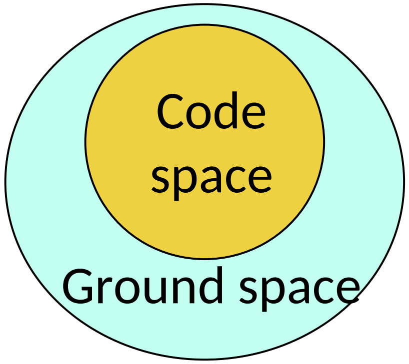
QECC in translation-invariant spin-chains

Brandao, Crosson, Sahinoglu, Bowen, PRL 2019

Properties:	Brandao et al (PRL 2018)	This work
QECC	Approximate with $\varepsilon = O(N^{-1/8})$	Exact
Distance d	$d = \Omega(\log(N))$	$d = \Theta(N)$
Rate	Vanishes	Vanishes
Error restriction	Consecutive spins	None
Code space	Low-energy eigenstates	Exact ground state
Translation invariance required?	Yes	No

Examples:
. 1D ferromagnetic Heisenberg
. Motzkin spin chain (s=1)



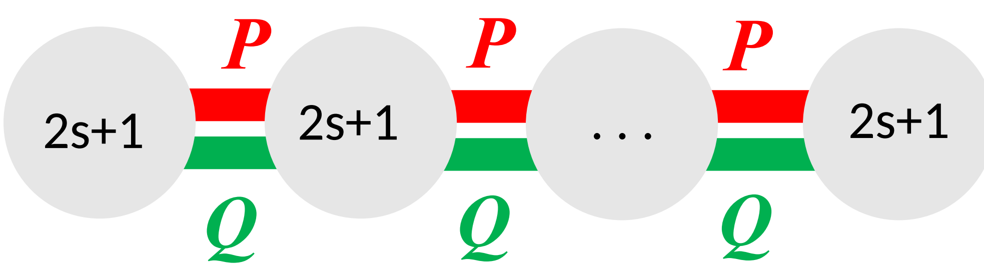


2-local Hamiltonian

$$H_n^s = \sum_{k=1}^{n-1} \left\{ \sum_{m=-s}^{s} P_{k,k+1}^m + \sum_{m=1}^{s} Q_{k,k+1}^m \right\}$$

$$H_n^J = J \sum_{k=1}^{n} \left(|0\rangle\langle 0| \right)_k \qquad J > 0$$

$$\boxed{H = H_n^J + H_n^s}$$



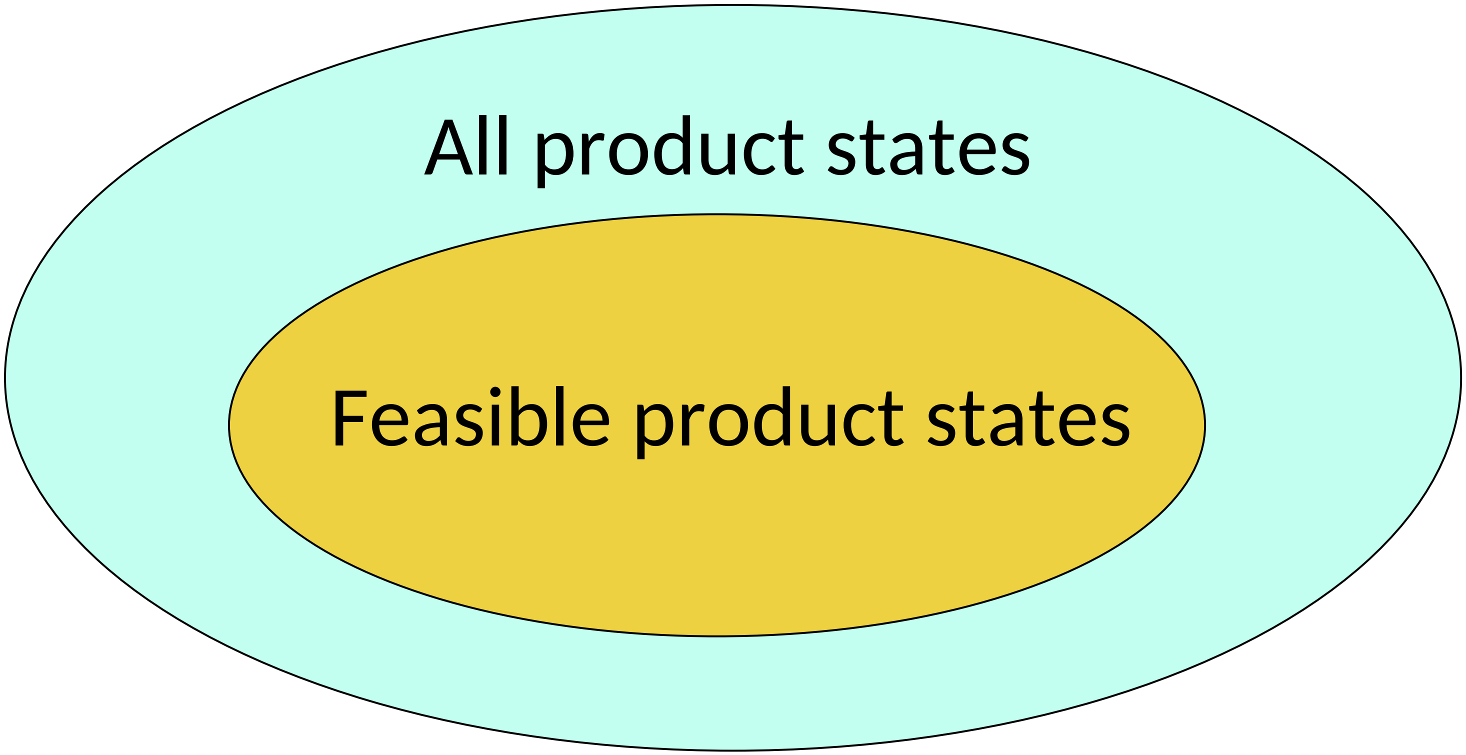
Spin transport, spin interaction

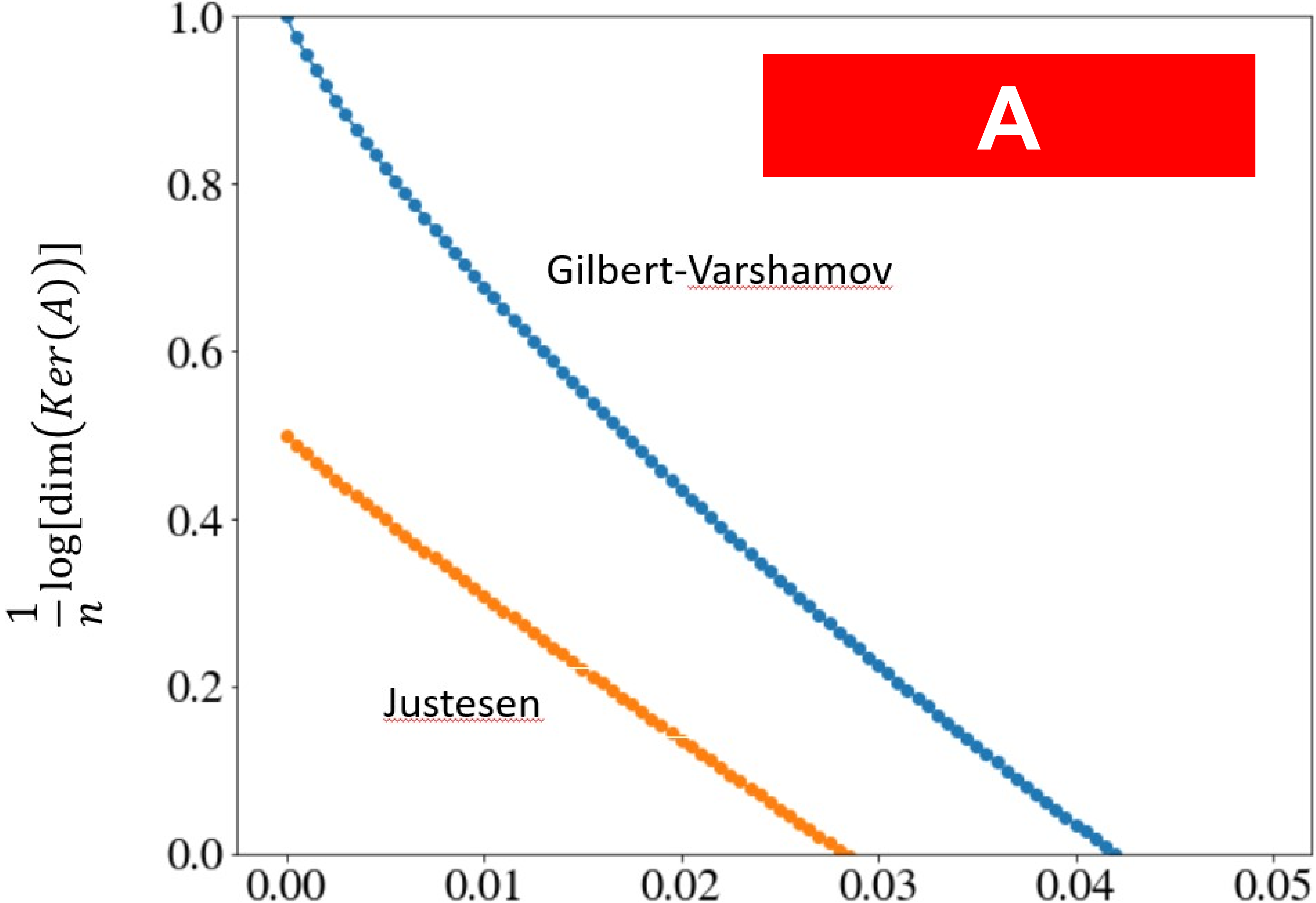
- $$P^m = |0 \leftrightarrow m\rangle\langle 0 \leftrightarrow m|$$
- $$Q^m = |00 \leftrightarrow \pm m\rangle\langle 00 \leftrightarrow \pm m|$$
- $$|0 \leftrightarrow m\rangle \equiv \frac{1}{\sqrt{2}}\left[|0,m\rangle - |m,0\rangle\right]$$
  $$|00 \leftrightarrow \pm m\rangle \equiv \frac{1}{\sqrt{2}}\left[|0,0\rangle - |m,-m\rangle\right]$$

Local Projector	Local moves
P^m	$0m \longleftrightarrow m0$
Q^m	$00 \longleftrightarrow m,-m$

Ground space

Count all n-strings with no 0's and no (m,-m) substrings.




A
Gilbert-Varshamov
Justesen
$\frac{1}{n}\log[\dim(Ker(A))]$
fraction of correctible errors τ
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.01
0.02
0.03
0.04
0.05

Embedding more logical states

A-matrix: $\mathrm{Re}\,\langle \mathbf{c}|\,P\,|\mathbf{c}\rangle \qquad \mathrm{Im}\,\langle \mathbf{c}|\,P\,|\mathbf{c}\rangle$

Theorem 1b: Using this procedure, we can derive quantum codes with linear distance and constant rate.

$\mathbf{c}\in C_j$

$$[A_1 A_2]\begin{bmatrix} x_1^+ \\ -x_1^- \end{bmatrix} = 0 \qquad \boxed{[A_2 A_3]\begin{bmatrix} -x_1^- \\ x_2 \end{bmatrix} = 0\,, \quad x_2 \geq 0}$$

Dines, Annals of Mathematics, 1926

Example: An 8-qudit code

$$|0_L\rangle = (|\phi_0\rangle|\theta_0\rangle + |\phi_1\rangle|\theta_1\rangle + |\phi_2\rangle|\theta_2\rangle + |\phi_3\rangle|\theta_3\rangle + |\phi_4\rangle|\theta_4\rangle + |\phi_5\rangle|\theta_5\rangle)/\sqrt{6}$$

$$|1_L\rangle = (|\phi_1\rangle|\theta_4\rangle + |\phi_0\rangle|\theta_3\rangle + |\phi_3\rangle|\theta_0\rangle + |\phi_2\rangle|\theta_5\rangle + |\phi_5\rangle|\theta_2\rangle + |\phi_4\rangle|\theta_1\rangle)/\sqrt{6}.$$

$$|\phi_0\rangle = |1, 1, 1, -2\rangle, \quad |\theta_0\rangle = |-2, 2, 2, 1\rangle,$$
$$|\phi_1\rangle = |1, -2, -1, -1\rangle, \quad |\theta_1\rangle = |1, -2, -2, -2\rangle,$$
$$|\phi_2\rangle = |-1, -2, -2, -1\rangle, \quad |\theta_2\rangle = |-1, 2, 2, 1\rangle,$$
$$|\phi_3\rangle = |-1, -1, 1, 1\rangle, \quad |\theta_3\rangle = |-2, -2, -2, -2\rangle,$$
$$|\phi_4\rangle = |2, -1, -1, 1\rangle, \quad |\theta_4\rangle = |1, 2, 2, 1\rangle,$$
$$|\phi_5\rangle = |2, 1, -2, -2\rangle, \quad |\theta_5\rangle = |-1, -2, -2, -2\rangle.$$

Error-detecting code

$$|0_L\rangle = \frac{|1,1,2,1,-2,1\rangle + |-2,1,-2,-2,2,2\rangle}{\sqrt{2}}$$

$$|1_L\rangle = \frac{|1,1,-2,-2,2,1\rangle + |-2,1,2,1,-2,2\rangle}{\sqrt{2}}.$$

Logical X:

3rd and 5th spin $\longleftrightarrow$ -2

4th spin $\longleftrightarrow$ -2

Quantum LDPC with linear distance, TQO in 1D?

Subspace LDPC?

Bravyi-Terhal no-go, $d=O(L^{D-1})$. In 1D, Stabilizer and subsystem codes have $d=O(1)$. Bravyi, Terhal NJP(2009)
We sidestep this no-go by relaxing stabilizer constraint.

Approximate quantum LDPC in 10-local Hamiltonian.
Ours is 2-local and exact. Bohdanowicz, Crosson, Nirkhe, Yuen, ACM SIGACT Symposium on Theory of Computing (2019)

Topological order in 1D, but we do not use all of the ground space. Bravyi, Hastings, Michalakis JMP (2010)