Course 50.050/50.550
Advanced Algorithms

Week 3 – Lecture L03.02

SINGAPORE UNIVERSITY OF
TECHNOLOGY AND DESIGN

Outline of Lecture

- Special classes of simple undirected graphs
- Planar graphs, plane drawings, plane graphs
- Euler's formula
- Dual graphs

Finite graphs versus infinite graphs

Recall: A graph is a 3-tuple $G = (V, E, \phi)$, where V is the vertex set, E is the edge set, and ϕ is the incidence function.

- Whether G is undirected or directed depends on whether $\phi(E)$ consists of sets or tuples.

Definition: If both V and E are finite sets, then we say that G is a finite graph, otherwise we say that G is an infinite graph

- In general, the ideas needed to study/understand finite graphs versus infinite graphs are very different.
  - The kind of questions we ask are also very different for infinite graphs, and also possibly counter-intuitive.
  - Do we only allow countably infinite V? What about uncountable V?

Note: In this course, we will focus on finite graphs.

All graphs are assumed finite unless explicitly stated otherwise.

- In Weeks 3–6, we will build our intuition for finite graphs, especially for finite simple undirected graphs.

Basic properties of simple undirected graphs

Let $G = (V, E)$ be a simple undirected graph.

- **Recall:** This means that all edges of G are 2-subsets of V.
  - Every edge has the 2-subset of its incident vertices as the name of the edge.
- **Very useful notation:** Given any set S and $k \in \mathbb{N}$, we write $\binom{S}{k}$ to denote the set of all k-subsets of S.
  - Thus, all edges of G are elements of $\binom{V}{2}$, i.e. $E \subseteq \binom{V}{2}$.

Theorem: A simple undirected graph with n vertices has at most $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

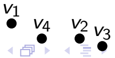
Proof: For a simple undirected graph $G = (V, E)$, we know that $E \subseteq \binom{V}{2}$, so $|E| \le \left|\binom{V}{2}\right|$. Since $\binom{V}{2}$ is the set of all 2-subsets of V, and since V has cardinality n, we get $\left|\binom{V}{2}\right| = \binom{n}{2} = \frac{n(n-1)}{2}$. $\square$

Note: It is possible for a graph to have no edges.

Definition A null graph is a graph with an empty edge set.

- **Warning:** Some authors define a null graph to have no vertices.

Example: A null graph with 4 vertices.



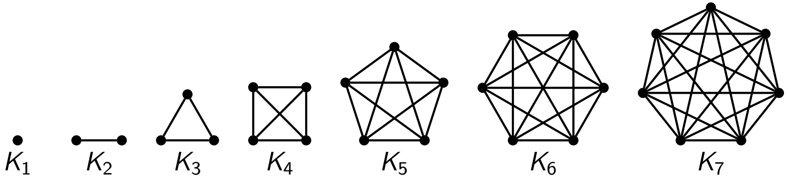
- Vertex set $V = \{v_1, v_2, v_3, v_4\}$, edge set $E = \emptyset$.

Complete graphs

Definition: A complete graph with n vertices is a simple undirected graph $G = (V, E)$ with $|E| = \binom{n}{2}$.

- **Standardized notation:** For $n \in \mathbb{N}$, we use the symbol K_n to denote a complete graph with n vertices.

Examples:



Complete graphs are examples of what are called regular graphs.

- A simple undirected graph is called regular if all vertices have the same degree. If this same degree is k, then we say that the graph is k-regular.

Question: Can you see why K_n is $(n-1)$-regular?

Bipartite graphs

Definition: Let $G = (V, E)$ be a simple undirected graph.

- If there exists a partition of V into two subsets V_1, V_2, such that every edge in G is of the form $\{v_1, v_2\}$ for some $v_1 \in V_1$, $v_2 \in V_2$, then we say that G is a bipartite graph.
  - **Recall:** "partition" implies $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.
  - **Intuition:** All edges must "cross" between V_1 and V_2.
    - There are no edges joining vertices only in V_1.
    - There are no edges joining vertices only in V_2.
- An ordered pair (V_1, V_2) satisfying the given conditions is called a bipartition of our bipartite graph G.
  - **Use of terminology:** We could write "Let G be a simple bipartite undirected graph with bipartition (V_1, V_2) ..."
  - **Note:** There could be multiple possible bipartitions.

Note: We will study bipartite graphs more in-depth in Week 4.

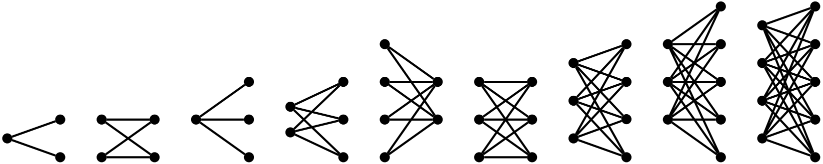
Example: A bipartite graph with two bipartitions (V_1, V_2), (V_1', V_2').

Complete bipartite graphs

Definition: A complete bipartite graph is a simple undirected bipartite graph $G = (V, E)$ with a bipartition (V_1, V_2), such that all possible pairs $\{v_1, v_2\}$ with $v_1 \in V_1$ and $v_2 \in V_2$ are edges in E.

- $E = \{\{v_1, v_2\} | v_1 \in V_1, v_2 \in V_2\}$ has a total of $|V_1| \cdot |V_2|$ edges.
- **Standardized notation:** For $m, n \in \mathbb{Z}^+$, we use the symbol $K_{m,n}$ to denote a complete bipartite graph with a bipartition (V_1, V_2) satisfying $|V_1| = m$ and $|V_2| = n$.

Examples:



$K_{1,2}$ $K_{2,2}$ $K_{1,3}$ $K_{2,3}$ $K_{4,2}$ $K_{3,3}$ $K_{3,4}$ $K_{3,5}$ $K_{4,5}$

- **Note:** K_2 and $K_{1,1}$ have the "same graph structure". 
  - In Week 4, we will learn what "same graph structure" means.

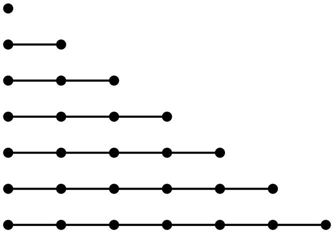
Path graphs

Definition: A path graph with $n \geq 1$ vertices is a simple undirected graph with n vertices and $n-1$ edges, such that the edge set can be written as $E = \Big\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\Big\}$ for some sequential order $v_1, \ldots, v_n$ of the n vertices.

- **Standardized notation:** For $n \in \mathbb{Z}^+$, we use the symbol P_n to denote a path graph with n vertices.

Examples:

P_1
P_2
P_3
P_4
P_5
P_6
P_7



Warning! A path graph is NOT the same as a path in a graph.

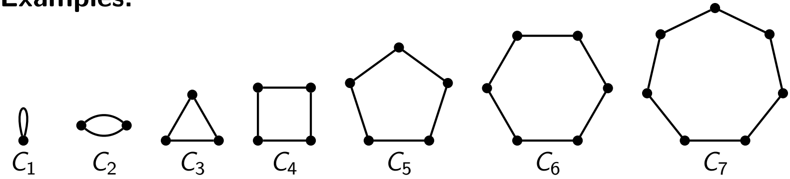
- We will study paths in graphs more in-depth in later weeks.

Cycle graphs

Definition: For integers $n \geq 3$, a cycle graph with n vertices is a simple undirected graph with n vertices and n edges, such that the edge set is $E = \Big\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\Big\}$ for some sequential order $v_1, \ldots, v_n$ of the n vertices.

- Some authors also define cycle graphs with 1 or 2 vertices.
  - Cycle graphs with 1 or 2 vertices are non-simple graphs; see below.
- **Standardized notation:** For $n \in \mathbb{Z}^+$, we use the symbol C_n to denote a cycle graph with n vertices.
- A cycle graph with n vertices is also commonly called an n-cycle.

Examples:



Warning! A cycle graph is NOT the same as a cycle in a graph.

- We will study cycles in graphs more in-depth in later weeks.

Drawings of graphs

Recall: Formally, a graph (whether undirected or directed) is a 3-tuple $G = (V, E, \phi)$, where V and E are sets, and ϕ is a function on E.

- To help us understand the properties of G, we usually visualize G as a drawing.

Definition: Let $G = (V, E, \phi)$ be a (undirected or directed) graph. A drawing of G is a geometric diagram depicted on a 2D plane, where:

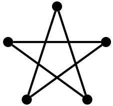
- The vertices in V are depicted as distinct points (e.g. $\bullet$);
  - The depicted points do not overlap.
- Every edge $e \in E$ is depicted by drawing a smooth curve joining the depicted endpoints of e, such that no other points on this curve are depicted points of vertices.
  - If G is undirected, and $\phi(e) = \{u, v\}$, then we could draw 

  - If G is directed, and $\phi(e) = (u, v)$, then we also include an arrowhead at the head v, and we could draw 

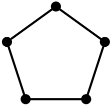
  - **Note:** The depictions of edges do NOT have to be line segments.

Important Note: Two drawings, even if they "look very different", could be two drawings of the same graph!

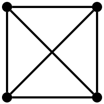
Different drawings of graphs



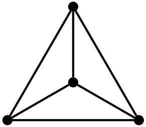
Another drawing of C_5



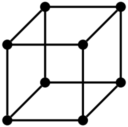
C_5



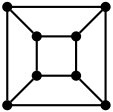
K_4



Another drawing of K_4



The cube graph



Another drawing of the cube graph

- Drawings on the left have depicted edges that cross each other.
- Drawings on the right have depicted edges that do not cross each other.

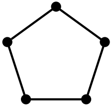
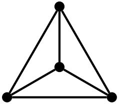
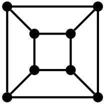
Planar graphs

Definition: Let $G = (V, E, \phi)$ be a (undirected or directed) graph.

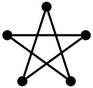
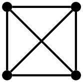
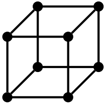
- A drawing of G is called a plane drawing of G, or a planar representation of G, if for every two distinct edges of G, their corresponding smooth curves depicted in the drawing do not have any common point of intersection, except possibly at the depicted endpoints of the edges.
- We say that G is planar if there exists a plane drawing of G.
  - A plane graph is a plane drawing of a planar graph.

Example: C_5, K_4, and the cube graph, are examples of planar graphs.

- The following drawings are plane graphs:

- The following drawings are NOT plane graphs, even though they are drawings of planar graphs:

How to define plane drawings mathematically?

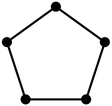
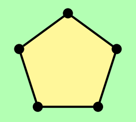
Intuition: To describe a plane drawing of a planar graph $G = (V, E, \phi)$, we need to specify coordinates for all vertices, and we need to specify how exactly to draw every smooth curve that depicts some edge of G.

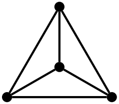
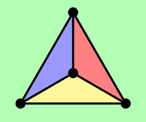
- **Note:** We treat the 2D plane as the Euclidean plane $\mathbb{R}^2$.
- The information about the Cartesian coordinates of all vertices in V, can be encoded as a function $\varphi : V \to \mathbb{R}^2$.
  - For each vertex $v \in V$, its output value $\varphi(v) = (x_v, y_v)$ is the pair of coordinates of the corresponding depicted point in $\mathbb{R}^2$.
- For each edge $e \in E$, the information about its corresponding smooth curve can be encoded as a function $\psi_e : [0, 1] \to \mathbb{R}^2$.
  - **Intuition:** Imagine for the edge $\{u, v\}$, we are drawing its corresponding smooth curve in 1 second. Suppose at time $t = 0$, we start at the point (x_u, y_u) in $\mathbb{R}^2$ (a point in $\mathbb{R}^2$ depicting vertex u). As time progresses, we traverse along our smooth curve in $\mathbb{R}^2$, until at time $t = 1$, we reach the point (x_v, y_v) in $\mathbb{R}^2$ (a point in $\mathbb{R}^2$ depicting vertex v).
- Thus, we *could define* a plane drawing of G to be an ordered pair $\Psi = (\varphi, \{\psi_e\}_{e \in E})$, and we *could define* a plane graph to be an ordered pair (G, Ψ).
  - **Warning:** In almost all sources, definitions for "plane drawing" or "plane graph" are left to the reader to figure out.

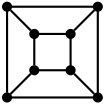
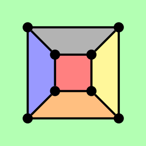
Faces of plane graphs

Note: A plane graph divides the 2D plane into regions called faces.

- Every face has a boundary formed by several edges.

 has two faces: 

 has four faces: 

 has six faces: 

- The faces together must cover the entire 2D plane.
  - Remember the “outer” face too!

Euler's formula

Definition: Let $G = (V, E, \phi)$ be an **undirected** graph.

- Let u, v be distinct vertices in V. We say that u and v are connected if for some $k \in \mathbb{Z}^+$, there exists a sequence of k vertices $v_0, v_1, \ldots, v_k$, such that $v_0 = u$, $v_k = v$, and every two consecutive vertices v_{i-1}, v_i (for $1 \leq i \leq k$) are adjacent.
- We say that G is connected if every two distinct vertices of G (if any) are connected. (If G has ≤ 1 vertices, then G is connected.)

Note: We will learn more about graph connectivity in Week 5.

- **Warning:** Terminology for "graph connectivity" is generally messy!

Theorem: (**Euler's formula**) Let G be a connected planar undirected graph with n vertices and m edges. Let f be the number of faces in a plane drawing of G. Then $f = m - n + 2$.

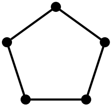
Proof Idea: By induction. See proof in course textbook. $\square$

Corollary: Let G be a connected planar undirected graph. Then the number of faces f, in any plane drawing of G, is invariant over all plane drawings of G.

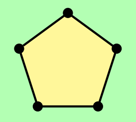
- Thus, it make sense to *define* f as the number of faces of G.

Examples of Euler's formula

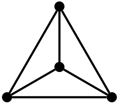
Given a connected planar undirected graph G, let n, m, f be the numbers of vertices, edges, and faces, respectively, of G.



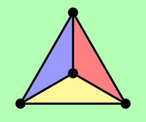
$n = 5$ vertices
$m = 5$ edges
$f = 2$ faces



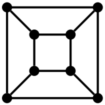
$n - m + f = 2.$



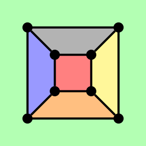
$n = 4$ vertices
$m = 6$ edges
$f = 4$ faces



$n - m + f = 2.$



$n = 8$ vertices
$m = 12$ edges
$f = 6$ faces



$n - m + f = 2.$

- **Intuition:** The number 2 can be interpreted as an **invariant** associated to the **2D plane**, where no matter what plane graph we draw, if our graph is connected, then the value of $n - m + f$ is always 2.

Number of edges of planar graphs

Theorem: Let $G = (V, E)$ be a **simple** connected planar undirected graph with $n \geq 3$ vertices, and m edges. Then $m \leq 3n - 6$.

Proof: Let f be the number of faces of G. Consider a plane drawing of G, and let $F_1, F_2, \ldots, F_f$ be the faces of the drawing.

Key Idea 1: For each $1 \leq i \leq f$, let m_i be the number of edges on the boundary of face F_i. Then $m_i \geq 3$.

- If F_i is not the "outer face", then we must have $m_i \geq 3$, since the only way to have $m_i \leq 2$ is if F_i looks like 

or 

, which are not possible since G is simple by assumption.
- If F_i is the "outer face", then since G has ≥ 3 vertices, there are at least 3 "outer edges" that are part of the boundary of F_i.

Key Idea 2: $\sum_{i=1}^{f} m_i \leq 2m$.

- Every edge of G is always a boundary edge of at most 2 faces.
  - e.g. every edge of a path graph is a boundary edge of 1 face.
- So in the sum $\sum_{i=1}^{f} m_i$, every edge is counted at most twice.
  - Thus, $\sum_{i=1}^{f} m_i \leq 2|E| = 2m$.

Number of edges of planar graphs (continued)

So far: We know $\sum_{i=1}^{f} m_i \leq 2m$, and $m_i \geq 3$ for all $1 \leq i \leq f$.

- **Consequence:** $3f \leq 2m$.

Key Idea 3: Use Euler's formula $f = m - n + 2$.

- Note that $3f = 3m - 3n + 6$, so it follows from $3f \leq 2m$ that $3m - 3n + 6 \leq 2m$, or equivalently, $m \leq 3n - 6$. $\square$

Intuition: If G is a simple connected planar undirected graph, then this theorem gives an upper bound on the maximum possible number of edges, in terms of n, the number of vertices.

- Intuitively, "planarity" implies the number of edges is at most $O(n)$.
  - In contrast, without "planarity", a simple connected undirected graph has at most $\binom{n}{2} = \frac{n(n-1)}{2} = O(n^2)$ edges.
- The "**simple**" condition is necessary, since for non-simple graphs, the number of edges could be made arbitrarily large without changing the number of vertices, via "repeated edges".

Number of edges of planar graphs (continued)

Definition: Let $G = (V, E)$ be a **simple** undirected graph. We say that g is triangle-free if for every 3 distinct vertices v_1, v_2, v_3 in V, the three pairs $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$ are not all edges in E.

- **Intuition:** G is triangle-free if there are no "triangles".

Theorem: Let $G = (V, E)$ be a **simple** connected planar **triangle-free** undirected graph with $n \geq 3$ vertices, and m edges. Then $m \leq 2n - 4$.

Proof: Let f be the number of faces of G. Consider a plane drawing of G, and let $F_1, F_2, \ldots, F_f$ be the faces of the drawing.

Modified Key Idea 1: For each $1 \leq i \leq f$, let m_i be the number of edges on the boundary of face F_i. Then $m_i \geq 4$.

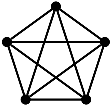
- We have $m_i \geq 3$ via the same reasoning as in previous theorem.
- G is triangle-free, so we cannot have $m_i = 3$.

Same Key Idea 2: $\sum_{i=1}^{f} m_i \leq 2m$. (**Consequence:** $4f \leq 2m$.)

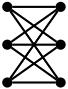
Similar Key Idea 3: Use Euler's formula $f = m - n + 2$.

- Note that $4f = 4m - 4n + 8$, so it follows from $4f \leq 2m$ that $4m - 4n + 8 \leq 2m$, or equivalently, $m \leq 2n - 4$. □

Examples of non-planar graphs



K_5



$K_{3,3}$

Theorem: K_5 is a non-planar graph.

Proof: K_5 has $n = 5$ vertices, and $m = \binom{5}{2} = 10$ edges. Note also that K_5 is a simple connected undirected graph.

- If K_5 is planar, then by the theorem on Slide 16, we must have $m \leq 3n - 6$. However, $3n - 6 = 9$, and $10 \not\leq 9$, so $m \not\leq 3n - 6$.
- Therefore K_5 cannot be planar. $\square$

Theorem: $K_{3,3}$ is a non-planar graph.

Proof: $K_{3,3}$ has $n = 6$ vertices, and $m = (3)(3) = 9$ edges. Note also that $K_{3,3}$ is a simple connected undirected **triangle-free** graph.

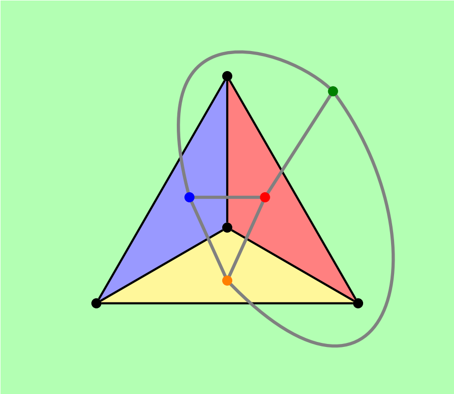
- If $K_{3,3}$ is planar, then by the theorem on Slide 18, we must have $m \leq 2n - 4$. However, $2n - 4 = 8$, and $9 \not\leq 8$, so $m \not\leq 2n - 4$.
- Therefore $K_{3,3}$ cannot be planar.

Intuition of dual graphs

Idea: The faces of a plane graph could be vertices of a new graph.

- **Recall:** The definition of "vertex set" has no restrictions on what objects could be considered vertices.
- For every two faces that share a common boundary edge, we draw a new edge.

Example: We introduce 4 new vertices, then introduce 6 new edges.



Definition of dual graphs

Let (G, Ψ) be a plane graph.

- $G = (V, E, \phi)$ is a planar undirected graph.
  - Let $n = |V|$, $m = |E|$, and let f be the number of faces of G.
- Ψ is a plane drawing of G.
  - Let $F_1, \ldots, F_f$ be the faces of Ψ.

Definition: The dual graph of (G, Ψ) is a graph $G^* = (V^*, E^*, \phi^*)$ constructed from G as follows:

- $V^* = \{F_1^*, \ldots, F_f^*\}$ is a set with f vertices.
  - We are treating "F_1^*", "F_2^*", ..., "F_f^*" as f symbols, to be used as the names for the f vertices in V^*.
    - If we prefer, we could instead, e.g., name the f vertices $u_1, \ldots, u_f$.
- $E^* = \{e_1^*, \ldots, e_m^*\}$ is a set with m edges.
  - We are treating "e_1^*", "e_2^*", ..., "e_m^*" as m symbols, to be used as the names for the m edges in E^*.
    - If we prefer, we could use other names for the m edges.
- ϕ^* is a function on E^*, defined as follows:
  - For every edge $e_i^* \in E^*$, consider its corresponding edge $e_i \in E$.
  - **Note:** e_i is the boundary edge of at most two faces.
    - If there are two faces $F_j, F_{j'}$, then define $\phi^*(e_i^*) = \{F_j^*, F_{j'}^*\}$.
    - If there is only one face F_j, then define $\phi^*(e_i^*) = \{F_j^*\}$.

Some remarks on dual graphs

- The notion of "dual graph" depends on the given plane drawing of a planar graph.
  - Even for the same planar graph G, two different plane drawings Ψ, Ψ' of G could result in two different dual graphs with different "graph structure".
  - **Challenge:** Can you find an example of two such plane drawings?
- The dual graph of a simple plane graph is not necessarily simple.
  - **Example:** For any plane drawing Ψ of the path graph P_n, the dual graph of (P_n, Ψ) has n loops.
  - **Example:** For any plane drawing Ψ of the cycle graph C_n, the dual graph of (C_n, Ψ) has n edges with the same two endpoints.

Theorem: The dual graph of a plane graph is planar.

- **Challenge:** Can you prove why this is true?
  - (Hint: Think about how to draw a dual graph. Can you ensure that the depicted edges do not "cross"?)

Summary

- Special classes of simple undirected graphs
- Planar graphs, plane drawings, plane graphs
- Euler's formula
- Dual graphs

Reminder:
Our first mini-quiz will be held during cohort class this Thursday!