Course 50.050/50.550 Advanced Algorithms

Week 2 – Cohort Class L02.03

SINGAPORE UNIVERSITY OF
TECHNOLOGY AND DESIGN

Outline of Cohort Class

▶ Extremal principle

▶ Invariance

▶ Class Activities: CSD game, checkerboard puzzle, etc.

Proof method: Extremal principle

Intuition: A "direct" proof method via the well-ordering principle.

- ▶ **Recall:** The well-ordering principle says that every **non-empty** subset $S \subseteq \mathbb{N}$ must contain a smallest element.
  - ▶ **Note:** If we replace $\mathbb{N}$ by $\mathbb{Z}$ or $\mathbb{R}_{\geq 0}$, then the statement is false.
    - ▶ e.g. $\{-k | k \in \mathbb{N}\} = \{0, -1, -2, \dots\} \subseteq \mathbb{Z}$ has no smallest element.
    - ▶ e.g. $\{10^{-k} | k \in \mathbb{Z}^+\} = \{0.1, 0.01, 0.001, 0.0001, \dots\} \subseteq \mathbb{R}_{\geq 0}$ has no smallest element.
    - ▶ An infinite set S may not have a smallest element.
- ▶ If we restrict to finite sets, then the following is true:
  - ▶ Every non-empty **finite** subset of $\mathbb{R}$ has a smallest element and a largest element.

Extremal Principle Let S be a **non-empty** collection of objects.

- ▶ If we assign each object in S to a (possibly repeated) value in $\mathbb{N}$, then there exists an object with the smallest assigned value.
- ▶ If S has **finitely** many objects, and if we assign each object in S to a (possibly repeated) real number, then there exists an object with the smallest assigned real number, and there exists an object with the largest assigned real number.

Example: Non-intersecting lines

Question: There are 20 points on a plane, where no 3 of them are collinear. 10 of them are colored red, and the other 10 are colored blue. Can we pair up the red points with the blue points such that if we draw a straight line segment joining the two points (one red and one blue) in each of the 10 pairs, then none of the 10 line segments intersect? Can we always do this?

Answer: Yes, it is possible to pair up, and we can always do this for any given configuration of 10 red points and 10 blue points.

Proof: By the **extremal principle**, among the 10! **finitely many** possible pairings, there is one with a **smallest** total sum of all the lengths of the line segments.

▶ This pairing cannot have any intersecting line segments.

  ▶ To see why, suppose instead this pairing has two red points R_1, R_2, and two blue points B_1, B_2, such that $R_1 B_1$ and $R_2 B_2$ are intersecting line segments.

  ▶ Then we can replace these two line segments by $R_1 B_2$ and $R_2 B_1$, thereby getting a strictly smaller total sum of all the lengths of the line segments, which is a contradiction. □

Example: One-way road

Question: There is a large country far far away, with multiple cities, where every road is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city that can be reached from every city directly or via at most one other city.

Proof: (by extremal principle).

- ▶ **Idea 1:** There can only be a **finite** number of cities, so by the **extremal principle**, there is some city (maybe more than one) with the maximum number of direct roads leading into it.
  - ▶ Pick any such city, and call it x.
- ▶ **Idea 2:** We claim that city x can be reached from every city directly or via at most one other city.
  - ▶ **Important Note:** Every pair of cities is connected by **exactly one** direct road.

Example: One-way road (continued)

Let S be the set of all cities in the country, let Dir be the subset of cities (that are not city x) with direct roads leading into city x, and let $Rest = S \backslash (Dir \cup \{x\})$ be the set of all remaining cities that are not city x, and that do NOT have direct roads leading into city x.
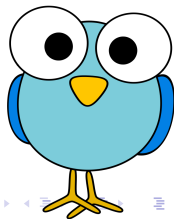
▶ For every city $y \in Rest$ (if any), we claim that there exists some city $z \in Dir$ such that there is a direct road from city y to city z. The existence of such a city z would then mean that city x is reachable from city y via city z.

Suppose on the contrary that no such city z exists.

▶ Since every pair of cities is connected by **exactly one** direct road, this implies that each of the $|Dir|$ cities in Dir has a direct road leading into y.

▶ Since $y \notin Dir$, it follows from definition that y does not have a direct road leading into x. But there is exactly one direct road between x and y, so x must have a direct road leading into y.

▶ Thus, a total of $|Dir| + 1$ cities have direct roads leading into y, which then contradicts the maximality of $|Dir|$.

My name is John.
Want to play a game I invented?
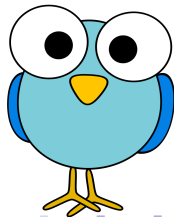I call it the **CSD** Game.

The CSD game (15 mins)

Rules of the game:

Our little blue friend 🐦 will start by thinking of a long string of letters using only **C**, **S**, and **D**. Once we are given this string of letters, we are allowed to do any of the following three moves:

- ▶ Strike off the letters **C** and **S**, then write the letter **D**.
- ▶ Strike off the letters **S** and **D**, then write the letter **C**.
- ▶ Strike off the letters **D** and **C**, then write the letter **S**.

Note: Letters to be striked off do NOT have to be consecutive. Can we find a sequence of moves such that exactly one and only one letter remains? Can this letter be **C**? What about **S**? What about **D**?

I am thinking of the string **CSDCSDCSDC**.

Three possible moves:

- ▶ Strike off the letters **C** and **S**, then write the letter **D**.
- ▶ Strike off the letters **S** and **D**, then write the letter **C**.
- ▶ Strike off the letters **D** and **C**, then write the letter **S**.

CSD game - Discussion

Let's look at the string **CSDCSDCSDC** that our little friend 🦉
gave us. At each step in the CSD game, let **c** be the number of
C's, let **s** be the number of **S**'s, and let **d** be the number of **D**'s.

- So, at the beginning of our game, we have $c = 4$, $s = 3$, $d = 3$.

The three possible moves in the CSD game:
- Strike off the letters **C** and **S**, then write the letter **D**.
- Strike off the letters **S** and **D**, then write the letter **C**.
- Strike off the letters **D** and **C**, then write the letter **S**.

Question: Will **c**, **s**, and **d** be odd or even after each move?

Answer: Odd becomes even, and even becomes odd.
- Initially, both **s** and **d** are odd, so after any sequence of
  moves, **s** and **d** will either be both odd, or both even.

Conclusion: We cannot end the game with a single letter **S** or **D**.
- To end with a single **S**, it means $c = 0$, $s = 1$, $d = 0$.
  - **s** is odd and **d** is even, which is impossible.
- To end with a single **D**, it means $c = 0$, $s = 0$, $d = 1$.
  - **s** is even and **d** is odd, which is impossible.

CSD game - Discussion (continued)

Question: What about ending the game with a single letter **C**? Is this possible?

- ► For **c** and **s**, one is odd, the other is even, so after any sequence of moves, one of them must be odd, and the other must be even.
- ► Our argument to show that we cannot end with a single **S** or a single **D**, does not apply here for **C**.

Note: In fact, it is possible to end the game with a single **C**!

- ► We would need a proof by construction.

Lemma: For the string **CSDC**, there exists a sequence of moves to change this string to **C**.

Proof: Consider the following sequence of moves:

$$\text{CSDC} \longrightarrow \text{DDC}, \qquad \text{DDC} \longrightarrow \text{DS}, \qquad \text{DS} \longrightarrow \text{C} \qquad \square.$$

By using this lemma repeatedly, we can change the original string to **C** as follows: (**CSDC**)**SDCSDC** $\longrightarrow$ (**C**)**SDCSDC**

$$(\text{CSDC})\text{SDC} \longrightarrow (\text{C})\text{SDC},$$

$$(\text{CSDC}) \longrightarrow (\text{C})$$

CSD game - Discussion (continued)

Definition: The parity of an integer n is even or odd, depending on whether n is even or odd.

- ▶ Two integers n, m are said to have the same parity if either m, n are both even, or m, n are both odd.

- ▶ Two integers n, m are said to have opposite parity if one of them is even, and the other is odd.

Thus, for the CSD game, the integers **s** and **d** always have the same parity, after any sequence of moves.

- ▶ The property "**s** and **d** have the same parity" is an **invariant** of the string, meaning that the property is always satisfied, independent of what sequence of moves we apply to the string.

- ▶ We used this invariant to show that it is impossible to end the CSD game with a single **S** or a single **D**.
  - ▶ We showed this using a **proof by invariance**.

Invariance

Invariance is a simple idea for problem-solving, yet it can be used as a very effective proof method.

- ▶ Suppose we have a "process" (e.g. the CSD game) that involves repeated "moves". There may be several possible "moves".

- ▶ If "something" (e.g. a property or value) does not change after every possible "move", then this "something" **will never change after any sequence of "moves"**. We call it an **invariant**.
    - ▶ More precisely, an invariant is a fixed proposition that is initially true, and remains true after any sequence of "moves", independent of the choice of such a sequence.

*(**Note:** In mathematics, the word 'invariant' can be either a noun or an adjective.)*

Example: If you are cutting a cake into several pieces, then however many pieces you cut it into, the total volume of all the pieces remains unchanged.

- ▶ What is an invariant in our cake-cutting example?

Invariants may not be obvious.
But if there is some kind of repetition,
look for what does not change!

Dragon puzzle (10 mins)

A dragon has 100 heads initially. A knight can cut off 15, 17, 32 or 5 heads each time with one blow of his sword. In each of these cases, 24, 2, 14, or 17 new heads grow on the dragon's shoulders. If all the heads are cut off, the dragon dies. Can the dragon ever be killed by our knight?

Question: What does not change?

Dragon puzzle - Solution

Recall: In 50.004 (hash tables), we looked at remainders after division.

- If r is the remainder when n is divided by m, then we say "n is congruent to r modulo m", and we write "$n \equiv r \pmod{m}$".
  - When we divide an integer n by some positive integer m, the possible remainders are $0, 1, \ldots, m-1$.
- **General Definition:** For $m \in \mathbb{Z}^+$, any $n, n' \in \mathbb{Z}$, we say that "n is congruent to n' modulo m", and we write "$n \equiv n' \pmod{m}$", if $n - n'$ is divisible by m. *(i.e. $n - n' = mk$ for some $k \in \mathbb{Z}$.)*

For our dragon puzzle, we start with 100 heads.

- The knight can cut off 15, 17, 32 or 5 heads.
- The dragon will grow 24, 2, 14, or 17 heads, respectively.
- Net change in number of heads: $+9, -15, -18, +12$ respectively.

Note: $+9, -15, -18, +12$ are all $\equiv 0 \pmod 3$.

Consequence: If N is the number of heads on the dragon after regrowing, then N modulo 3 is invariant.

- Since $100 \equiv 1 \pmod 3$, the number of heads after regrowing will always be $\equiv 1 \pmod 3$ after any number of cuts.

Dragon puzzle - Solution (continued)

Note: To kill the dragon, the number of heads must be exactly 0 immediately after the cut, so that the dragon cannot regrow any more heads.

- ▶ Let's think about this final fatal cut.
- ▶ Before the final fatal cut, the number of heads on the dragon must be $\equiv 1 \pmod 3$.
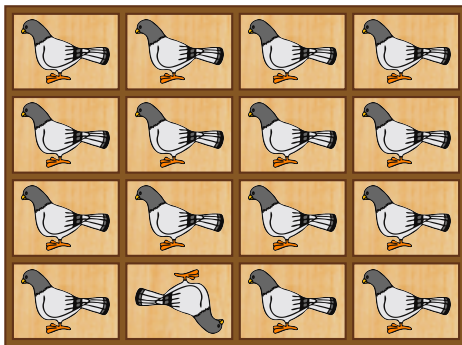- ▶ So in the final fatal cut, the number of heads cut off must be $\equiv 1 \pmod 3$.

Recall: The knight can cut off 15, 17, 32 or 5 heads.

- ▶ Among $15, 17, 32, 5$, none of them are $\equiv 1 \pmod 3$.
- ▶ Thus, it is NOT possible to have a final fatal cut.
- ▶ The dragon cannot be killed!

The Upside Down Pigeon Puzzle (15 mins)

There are 16 pigeons resting in 16 pigeonholes as shown below. One of them is upside down. You can switch the orientation of all the pigeons in a row, column, or any line parallel to one of the diagonals. You can also switch the orientation of any of the 4 corner pigeons. Right-side up pigeons become upside down, and upside down pigeons become right-side up. Is there a sequence of switches so that all the pigeons become right-side up?

The Upside Down Pigeon Puzzle - Solution

Key Idea: Consider the 8 boundary non-corner pigeonholes.

- ▶ **Important Observation:** The parity of the number of right-side up pigeons in these 8 pigeonholes is an invariant.
    - ▶ Initially, there are 7 right-side up pigeons in these 8 pigeonholes.
    - ▶ For all the pigeons to become right-side up, all 8 pigeons in these 8 pigeonholes must also be right-side up.
    - ▶ However, 7 and 8 have opposite parity.
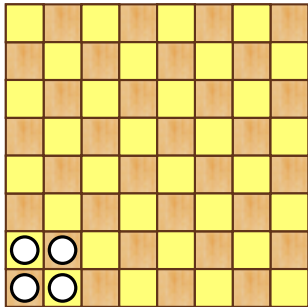- ▶ **Conclusion:** There is no possible sequence of switches.

Checkerboard puzzle (20mins)

Four chips are placed at the bottom left part of an 8×8 checkerboard in a 2×2 area as shown below. We are allowed the following move:

- ▶ If a square has a chip, while the squares above and to the right of it are empty, then we can remove the chip and place new chips on each of the other two squares.

Can you find a sequence of moves such that the bottom left 2×2 area no longer have chips?

Allowed Move

Square with '?' can be empty or non-empty.

Checkerboard puzzle - Solution

Key Idea: Assign weights to the squares.

$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{1}{4096}$	$\frac{1}{8192}$	$\frac{1}{16384}$
$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{1}{4096}$	$\frac{1}{8192}$
$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{1}{4096}$
$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$

Total weight of starting squares $= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = 2\frac{1}{4}$.

Question: What happens to the weight every time we make a move?

Checkerboard puzzle - Solution (continued)

Making things precise:

▶ Introduce an array $W[0..7][0..7]$, where each $W[i][j]$ represents the weight assigned to the checkerboard square in the (i,j)-th position.

  ▶ For each $0 \le i, j \le 7$, define $W[i][j] := \frac{1}{2^{i+j}}$.
  ▶ Total sum of weights in j-th row (for $0 \le i \le 7$):

$$\sum_{i=0}^{7} \frac{1}{2^{i+j}} = \frac{\frac{1}{2^j}\left(1 - \frac{1}{2^8}\right)}{\left(1 - \frac{1}{2}\right)} = \frac{1}{2^{j-1}}\left(1 - \frac{1}{2^8}\right)$$

  ▶ Total sum of weights in all 8^2 squares:

$$\sum_{j=0}^{7} \frac{1}{2^{j-1}}\left(1 - \frac{1}{2^8}\right) = \frac{2\left(1 - \frac{1}{2^8}\right)}{\left(1 - \frac{1}{2}\right)}\left(1 - \frac{1}{2^8}\right) = 4\left(1 - \frac{1}{2^8}\right)^2 \approx 3.9688 < 4.$$

  ▶ **Note:** Here, we use the geometric progression formula:
    $a + ar + \cdots + ar^n = \frac{a(1-r^{n+1})}{(1-r)}$.

▶ Thus, the total sum of weights in the squares excluding the squares in the bottom-left 2×2 area is $\approx 1.7188 < 1.75$.

Checkerboard puzzle - Solution (continued)

Important Observation: Every move involves removing a chip of weight $\frac{1}{2^k}$, and placing two new chips, each of weight $\frac{1}{2^{k+1}}$.

▶ Since $\frac{1}{2^k} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}}$, the **total weight of all chips** on the checkerboard is an **invariant**.

Initial total weight of all chips: $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = 2\frac{1}{4}$.

▶ If there are no chips in the bottom-left 2×2 area, then the maximum possible total weight of the chips is $\approx 1.7188 < 1.75$, which is less than the initial total weight of all chips.

▶ **Conclusion:** It is impossible to have a sequence of moves such that there are no chips in the bottom-left 2×2 area. □

From the Checkerboard Puzzle, we learned that sometimes we can assign **weights** and find an invariant based on these weights.

Summary

▶ Extremal principle

▶ Invariance

▶ Class Activities: CSD game, checkerboard puzzle, etc.

Announcement: Mini-quiz 1 held next week during cohort class.
Reminder: Homework 1 due tomorrow 1pm (online submission).