# Course 50.050/50.550
# Advanced Algorithms

Week 2 – Cohort Class L02.03

SINGAPORE UNIVERSITY OF
TECHNOLOGY AND DESIGN

# Outline of Cohort Class

- Extremal principle
- Invariance
- Class Activities: CSD game, checkerboard puzzle, etc.

# Proof method: Extremal principle

**Intuition:** A “direct” proof method via the well-ordering principle.

- **Recall:** The well-ordering principle says that every **non-empty** subset $S \subseteq \mathbb{N}$ must contain a smallest element.
  - **Note:** If we replace $\mathbb{N}$ by $\mathbb{Z}$ or $\mathbb{R}_{\geq 0}$, then the statement is false.
    - e.g. $\{-k | k \in \mathbb{N}\} = \{0, -1, -2, \ldots\} \subseteq \mathbb{Z}$ has no smallest element.
    - e.g. $\{10^{-k} | k \in \mathbb{Z}^+\} = \{0.1, 0.01, 0.001, 0.0001, \ldots\} \subseteq \mathbb{R}_{\geq 0}$ has no smallest element.
    - An infinite set $S$ may not have a smallest element.
- If we restrict to finite sets, then the following is true:
  - Every non-empty **finite** subset of $\mathbb{R}$ has a smallest element and a largest element.

**Extremal Principle** Let $S$ be a **non-empty** collection of objects.

- If we assign each object in $S$ to a (possibly repeated) value in $\mathbb{N}$, then there exists an object with the smallest assigned value.
- If $S$ has **finitely** many objects, and if we assign each object in $S$ to a (possibly repeated) real number, then there exists an object with the smallest assigned real number, and there exists an object with the largest assigned real number.

# Example: Non-intersecting lines

**Question:** There are 20 points on a plane, where no 3 of them are collinear. 10 of them are colored red, and the other 10 are colored blue. Can we pair up the red points with the blue points such that if we draw a straight line segment joining the two points (one red and one blue) in each of the 10 pairs, then none of the 10 line segments intersect? Can we always do this?

**Answer:** Yes, it is possible to pair up, and we can always do this for any given configuration of 10 red points and 10 blue points.

**Proof:** By the **extremal principle**, among the 10! **finitely many** possible pairings, there is one with a **smallest** total sum of all the lengths of the line segments.

- This pairing cannot have any intersecting line segments.
  - To see why, suppose instead this pairing has two red points $R_1, R_2$, and two blue points $B_1, B_2$, such that $R_1B_1$ and $R_2B_2$ are intersecting line segments.
  - Then we can replace these two line segments by $R_1B_2$ and $R_2B_1$, thereby getting a strictly smaller total sum of all the lengths of the line segments, which is a contradiction. □

# Example: One-way road

**Question:** There is a large country far far away, with multiple cities, where every road is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city that can be reached from every city directly or via at most one other city.

**Proof:** (by extremal principle).

- **Idea 1:** There can only be a **finite** number of cities, so by the **extremal principle**, there is some city (maybe more than one) with the maximum number of direct roads leading into it.
  - Pick any such city, and call it $x$.
- **Idea 2:** We claim that city $x$ can be reached from every city directly or via at most one other city.
  - **Important Note:** Every pair of cities is connected by **exactly one** direct road.

# Example: One-way road (continued)

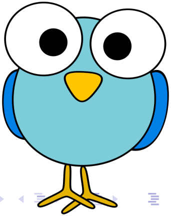
Let $S$ be the set of all cities in the country, let *Dir* be the subset of cities (that are not city $x$) with direct roads leading into city $x$, and let $Rest = S \backslash (Dir \cup \{x\})$ be the set of all remaining cities that are not city $x$, and that do NOT have direct roads leading into city $x$.

- For every city $y \in Rest$ (if any), we claim that there exists some city $z \in Dir$ such that there is a direct road from city $y$ to city $z$. The existence of such a city $z$ would then mean that city $x$ is reachable from city $y$ via city $z$.

Suppose on the contrary that no such city $z$ exists.

- Since every pair of cities is connected by **exactly one** direct road, this implies that each of the $|Dir|$ cities in *Dir* has a direct road leading into $y$.
- Since $y \notin Dir$, it follows from definition that $y$ does not have a direct road leading into $x$. But there is exactly one direct road between $x$ and $y$, so $x$ must have a direct road leading into $y$.
- Thus, a total of $|Dir| + 1$ cities have direct roads leading into $y$, which then contradicts the maximality of $|Dir|$.

My name is John.
Want to play a game I invented?
I call it the **CSD** Game.



# The CSD game (15 mins)

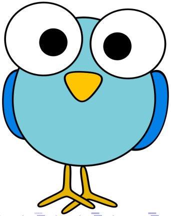
**Rules of the game:**

Our little blue friend  will start by thinking of a long string of letters using only **C**, **S**, and **D**. Once we are given this string of letters, we are allowed to do any of the following three moves:

- Strike off the letters **C** and **S**, then write the letter **D**.
- Strike off the letters **S** and **D**, then write the letter **C**.
- Strike off the letters **D** and **C**, then write the letter **S**.

**Note:** Letters to be striked off do NOT have to be consecutive.

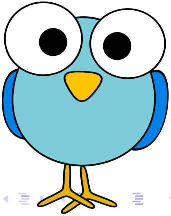
Can we find a sequence of moves such that exactly one and only one letter remains? Can this letter be **C**? What about **S**? What about **D**?



I am thinking of the string CSDCSDCSDC.

**Three possible moves:**

- Strike off the letters C and S, then write the letter D.
- Strike off the letters S and D, then write the letter C.
- Strike off the letters D and C, then write the letter S.



# Invariance

Invariance is a simple idea for problem-solving, yet it can be used as a very effective proof method.

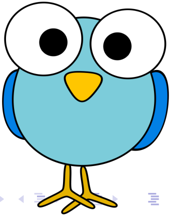
- Suppose we have a "process" (e.g. the CSD game) that involves repeated "moves". There may be several possible "moves".
- If "something" (e.g. a property or value) does not change after every possible "move", then this "something" **will never change after any sequence of "moves"**. We call it an **invariant**.
  - More precisely, an invariant is a fixed proposition that is initially true, and remains true after any sequence of "moves", independent of the choice of such a sequence.

*(**Note:** In mathematics, the word 'invariant' can be either a noun or an adjective.)*

**Example:** If you are cutting a cake into several pieces, then however many pieces you cut it into, the total volume of all the pieces remains unchanged.

- What is an invariant in our cake-cutting example?

Invariants may not be obvious.
But if there is some kind of repetition,
look for what does not change!



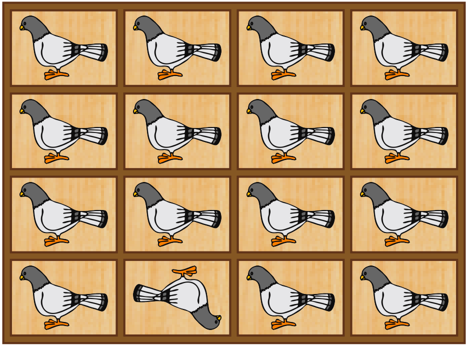
# Dragon puzzle (10 mins)

A dragon has 100 heads initially. A knight can cut off 15, 17, 32 or 5 heads each time with one blow of his sword. In each of these cases, 24, 2, 14, or 17 new heads grow on the dragon's shoulders. If all the heads are cut off, the dragon dies. Can the dragon ever be killed by our knight?

**Question:** What does not change?

# The Upside Down Pigeon Puzzle (15 mins)

There are 16 pigeons resting in 16 pigeonholes as shown below. One of them is upside down. You can switch the orientation of all the pigeons in a row, column, or any line parallel to one of the diagonals. You can also switch the orientation of any of the 4 corner pigeons. Right-side up pigeons become upside down, and upside down pigeons become right-side up. Is there a sequence of switches so that all the pigeons become right-side up?

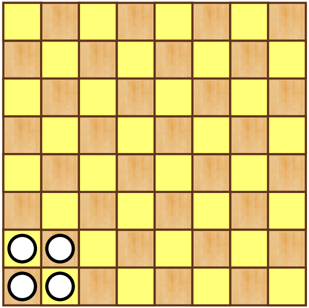


# Checkerboard puzzle (20mins)

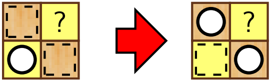
Four chips are placed at the bottom left part of an $8 \times 8$ checkerboard in a $2 \times 2$ area as shown below. We are allowed the following move:

- If a square has a chip, while the squares above and to the right of it are empty, then we can remove the chip and place new chips on each of the other two squares.

Can you find a sequence of moves such that the bottom left $2 \times 2$ area no longer have chips?



**Allowed Move**




Square with '?' can be empty or non-empty.

# Summary

- Extremal principle
- Invariance
- Class Activities: CSD game, checkerboard puzzle, etc.

**Announcement:** Mini-quiz 1 held next week during cohort class.

**Reminder:** Homework 1 due tomorrow 1pm (online submission).