Course 50.050/50.550
Advanced Algorithms

Week 1 – Cohort Class L01.03

SINGAPORE UNIVERSITY OF
TECHNOLOGY AND DESIGN

Outline of Cohort Class

- Counting methods, product rule, sum rule
- Permutations and Combinations
- Class Activity: Inclusion-exclusion principle

How many ways?

Given two shirts (S1, S2) and three pants (P1, P2, P3), how many ways can we create an outfit?

Answer: $2 \times 3 = 6$ ways.

Decomposing a task into subtasks

Task: Create an outfit.

- **Sub-task 1:** Choose a shirt for the outfit.
  - We could choose from 2 different shirts S1, S2.
  - There are $n_1 = 2$ ways complete this Sub-task 1.
- **Sub-task 2:** Choose a pants for the outfit.
  - We could choose from 3 different pants P1, P2, P3.
  - There are $n_2 = 3$ ways to complete this Sub-task 2.
    - Whatever shirt we had chosen in Sub-task 1, there are always 3 ways to complete Sub-task 2.
- We would complete the task after completing both subtasks.
  - The number of ways to complete Sub-task 2 does not depend on the choice made in Sub-task 1.
- Thus the total number of outfits we can create is $n_1 n_2 = 6$.

Another Example: A class has 12 boys and 18 girls. The teacher chooses a pair consisting of 1 boy and 1 girl.

- There are a total of $12 \times 18 = 216$ possible pairs.
  - The task of choosing a pair is decomposed into two subtasks: choosing the boy of the pair, and choosing the girl of the pair.

Product rule

Suppose a task can be decomposed into a sequence of k sub-tasks.

- Suppose the number of ways to complete a sub-task **remains invariant**, regardless of what specific choices we had made (or we would make) when completing the other sub-tasks.

Suppose that

- the 1st sub-task can be completed in n_1 ways;
- the 2nd sub-task can be completed in n_2 ways;
- $\vdots$
- the k-th sub-task can be completed in n_k ways.

Then the product rule says that there are $n_1 n_2 \cdots n_k$ possible ways to complete the task.

Useful Idea: Think of the choices made for the k sub-tasks as a

$$k\text{-tuple } \left(\substack{\text{choice made} \\ \text{for sub-task 1}}, \substack{\text{choice made} \\ \text{for sub-task 2}}, \ldots, \substack{\text{choice made} \\ \text{for sub-task } k} \right).$$

- **Product rule:** There are $n_1 n_2 \cdots n_k$ possible k-tuples.

Example 1

Suppose a home remodeling job involves purchasing appliances, plumbing and setting up electrical wires. There are 5 dealers for appliances, 12 plumbing contractors, and 9 electrical workers in the area to choose from.

How many ways are there to form a home remodeling team?

Answer:
We can think of the choices made for forming this team as a 3-tuple:

$$\left(\begin{array}{c}\text{name of chosen}\\\text{appliance dealer}\end{array}, \begin{array}{c}\text{name of chosen}\\\text{plumbing contractor}\end{array}, \begin{array}{c}\text{name of chosen}\\\text{electrical worker}\end{array}\right).$$

By the product rule (for 3-tuples), we can form this team in $5 \times 12 \times 9 = 540$ ways.

Sum rule

Idea: The different ways to complete a task can be split into cases.

- **Example:** Suppose to travel from city A to city B, there are:
  - 2 ways to travel by sea,
  - 3 ways to travel by air, and
  - 4 ways to travel by land/road.

  How many ways are there to travel from city A to city B?
  - **Answer:** By summing up the number of ways from the three cases (sea/air/land), there are a total of $2 + 3 + 4 = 9$ ways.

Let S be the set of all possible ways to complete a task, and suppose that S can be partitioned into k subsets.

- i.e. $S = S_1 \cup \cdots \cup S_k$, and $S_i \cap S_j = \emptyset$ for all i, j such that $i \neq j$.

Then the sum rule says that $|S| = |S_1| + \cdots + |S_k|$.

- **Note:** The sum rule does not involve sub-tasks!
  - Every element in each subset S_i represents a particular way to complete the entire task, not just some sub-task.

Permutations

Let S be a set with n objects, and consider the task of selecting k objects from S one at a time, without replacement.

- i.e. objects already selected cannot be selected subsequently.
- Outcome of task: a k-tuple $\left(\begin{smallmatrix}\text{1st object}\\\text{selected}\end{smallmatrix}, \begin{smallmatrix}\text{2nd object}\\\text{selected}\end{smallmatrix}, \ldots, \begin{smallmatrix}k\text{th object}\\\text{selected}\end{smallmatrix}\right)$.
- When selecting distinguishable objects, **the order matters!**
  - If we select object A first, followed by object B, that is not the same as selecting object B first, followed by object A.

Definition: Let S be a set, and let $k \in \mathbb{N}$. A k-permutation of S is a k-tuple with no repeated entries, whose entries are elements of S.

- A k-permutation is also called a permutation of S of size k.
- **Be careful!** Writing "permutation of S" without specifying size usually means "permutation of S of size $|S|$".
- **Notation:** If $|S| = n \in \mathbb{N}$, then the number of k-permutations of S is denoted by $P(n, k)$ *(following notation from course textbook).*
  - **Warning:** No fixed convention for denoting permutations.
    - Other possible notation used elsewhere: $P_{n,k}$, $P_{k,n}$, P^n_k.
- **Note:** $P(n, 0) = 1$ for all $n \in \mathbb{N}$ (including the case $n = 0$).

Question: Can you see how to calculate $P(n, k)$?

Example 2

In a group of 10 students, how many ways are there to select a list of 5 students, say for the five positions of "president", "vice-president", "secretary", "treasurer", and "logistics head"?

Answer: $P(10,5) = 10 \times 9 \times 8 \times 7 \times 6 = 30240$.

There are 10 ways to select the 1st student (as president).

There are 9 ways to select the 2nd student (as vice-president) from the remaining 9 students.

- Regardless of who the 1st student is, there are always 9 ways to select the 2nd student.

There are 8 ways to select the 3rd student (as secretary) from the remaining 8 students.

- Again, regardless of who the 1st and 2nd students are, there are always 8 ways to select the 3rd student.

So on and so forth.

Exercise 1 (10 mins)

There are 6 students: A, B, C, D, E and F.

Suppose we want to select a list of 4 students among these 6 students, for four **different** positions.

1. How many ways are there to select a list of 4 students?
2. How many ways are there to select a list of 4 students, such that student A is the second student of the list?
3. How many ways are there to select a list of 4 students, such that student A is one of the last two students of the list?

Exercise 1 - Solution

1. [Select a list of 4 students]
   The number of possible lists is $P(6,4) = 6 \times 5 \times 4 \times 3 = 360$.
2. [Select a list of 4 students with A as 2nd student in list]
   There are $P(5,3) = 5 \times 4 \times 3 = 60$ ways to select 3 students, to fill in the remaining entries of our list, so that:
   - the 1st selected student is the 1st student in our list;
   - the 2nd selected student is the 3rd student in our list; and
   - the 3rd selected student is the last/4th student in our list.
3. [Select a list of 4 students with A as 3rd or 4th student in list]
   First solution: We split into two cases.
   - **Case 1:** Student A is the 3rd student of the list.
     - There are $P(5,3) = 5 \times 4 \times 3 = 60$ ways to select 3 students, to fill in the 1st, 2nd and 4th entries of the list.
   - **Case 2:** Student A is the 4th student of the list.
     - There are $P(5,3) = 5 \times 4 \times 3 = 60$ ways to select 3 students, to fill in the 1st, 2nd and 3rd entries of the list.
   - By the **sum rule**, the total number of ways is $60 + 60 = 120$.

Exercise 1 - Solution (continued)

Second solution:
We can decompose our task into two sub-tasks.

- **Sub-task 1:** Select the position of student A in the list
  - There are 2 possible positions (3rd or 4th position) to select.
- **Sub-task 2:** Select the remaining students in the list.
  - There are $P(5,3) = 5 \times 4 \times 3 = 60$ ways to select 3 students, to fill in the 3 remaining entries in the list.

By the **product rule**, the total number of ways is $2 \times 60 = 120$.

Note: In both solutions, we get the same final answer of 120.

- In general, there could be multiple correct ways to count!

Combination

Definition: Let S be a set, and let $k \in \mathbb{N}$. A k-combination of S is a subset of S of cardinality k.

- A combination of S is a k-combination for some $k \in \mathbb{N}$.
- **Notation:** If $|S| = n \in \mathbb{N}$, then the number of k-combinations of S is denoted by $C(n,k)$ or $\binom{n}{k}$.
  - When the notation $\binom{n}{k}$ is used, this number is called a binomial coefficient.
  - $\binom{n}{k}$ is usually read as "n choose k".
  - We shall discuss some identities of binomial coefficients in Week 2.
- **Note:** $\binom{n}{0} = 1$ for all $n \in \mathbb{N}$ (including the case $n = 0$).

Formula to calculate binomial coefficients:

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

The number of k-combinations is the number of corresponding k-permutations disregarding the $k!$ different ways to order k entries.

Example 3

How many ways are there to select a list of 6 students, for six different positions, from a set of 10 students?

Answer: $P(10, 6) = \dfrac{10!}{(10-6)!}$.

How many ways are there to select a list of 6 students, for six different positions, from a set of 6 students?

Answer: $P(6, 6) = 6!$.

How many ways are there to select 6 students from a set of 10 students, to form a student committee?

- Assume that all members of a student committee have the same role called "member".

Answer: $C(10, 6) = \binom{10}{6} = \dfrac{10!}{6!(10-6)!}$.

Example 4

There are 10 students: A, B, C, D, E, F, G, H, I and J.

Suppose we want to select 4 students among these 10 students to form a student committee.

1. How many ways are there to select a committee of 4 students?
2. How many ways are there to select a committee of 4 students, such that student A is in the committee?
3. How many ways are there to select a committee of 4 students, such that both student A and student B are in the committee?

Example 4 - Solution

1. The number of possible ways is $\binom{10}{4} = \frac{P(10,4)}{4!} = \frac{10\times9\times8\times7}{1\times2\times3\times4} = 210.$
   - **Question:** Can you see why $\binom{10}{4}$ and $\binom{10}{6}$ have the same value?

2. If student A must be in the committee, then there are $\binom{9}{3} = \frac{P(9,3)}{3!} = \frac{9\times8\times7}{1\times2\times3} = 84$ ways to choose the remaining three students in the committee.

3. If student A and student B must both be in the committee, then there are $\binom{8}{2} = \frac{P(8,2)}{2!} = \frac{8\times7}{1\times2} = 28$ ways to choose the remaining two students in the committee.

Exercise 2 (10 mins)

Twelve different books, consisting of 5 math books, 4 science books, and 3 history books are arranged in order in a row on a bookshelf. Find the number of arrangements in each of the following cases:

1. There are no restriction.
2. The books of each type must be together.
3. The math books must be together.

Note: Here, an arrangement means the placement of the twelve books in a row on the bookshelf, from left to right. If you change the order of the books, you get a different arrangement.

Exercise 2 - Solution

1. $12!$ (Permute all 12 books.)

2. $5! \cdot 4! \cdot 3! \cdot 3!$ (Permute within each type of books: $5! \cdot 4! \cdot 3!$; then permute the three types: $3!$.)

3. $5! \times 8!$ (Permute the math books: $5!$; then permute the rest of the books and the group of math books, by considering the group of math books as one big "book": $8!$.)

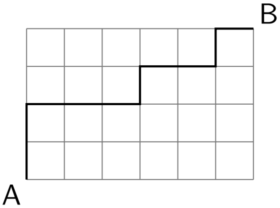
Important Insights:

- We can group objects into blocks.
- We can count blocks instead of individual objects.

Example 5

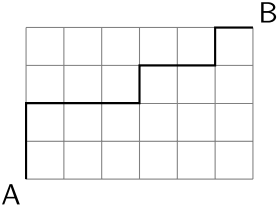
There is an ant that wants to crawl from corner A to corner B via the line segments of the rectangular grid given below.

- One possible shortest path from A to B has been shown in bold.



Question: How many possible shortest paths are there from A to B?

Example 5 - Solution



Key Idea: Any shortest path from A to B has 4 upwards line segments and 6 rightwards line segments, for a total of 10 line segments.

- We can represent a shortest path by a 10-tuple whose entries are "$\rightarrow$" or "$\uparrow$".
  - e.g. $(\uparrow, \uparrow, \rightarrow, \rightarrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$ for the path depicted above.
- We can equivalently represent a shortest path by a 10-tuple whose entries are 0 or 1.
  - e.g. $(1, 1, 0, 0, 0, 1, 0, 0, 1, 0)$ for the path depicted above.
- **Consequence:** Any shortest path is represented by a 10-tuple with six 0 entries and four 1 entries.
  - To choose one such 10-tuple, we have to choose four positions in the 10-tuple to place our 1 entries.

Thus, the total number of shortest paths from A to B is $\binom{10}{4} = 210$.

How to think?

It takes a lot of practice to extract the "mathematical core" beneath a physical description, which may be covered by various kinds of fancy stories.

To count the number of outcomes satisfying certain given conditions, think of how you can reinterpret the condition in terms of **k-tuples**, **k-permutations**, and/or **k-combinations**, and how you can use the **product rule** and/or the **sum rule**.

Try to convince your friends that your counting method is correct. If you can do so, then you are probably correct.

Remember, there could be more than one method to count!

Exercise 3 (15 mins)

At a restaurant, there are three types of sandwiches: beef (B), chicken (C), and fish (F). John wishes to place an order of 6 sandwiches. Assuming that there is no limit to the supply of each type of sandwich, how many different possible orders can John place?

Note 1: An order of 2 beef sandwiches, 3 chicken sandwiches, and 1 fish sandwich, would be the same as an order of 3 chicken sandwiches, 1 fish sandwich, and 2 beef sandwiches.

Note 2: An order of 2 beef sandwiches, 1 chicken sandwich, 2 beef sandwiches, and 1 chicken sandwich, would be the same as an order of 4 beef sandwiches and 2 chicken sandwiches.

Note 3: An order does not have to include sandwiches of all types. An order could have only one type of sandwich, or two types of sandwiches.

Exercise 3 - Solution

Key Observation: Whatever order John places, the order can be "organized" into "n_1 beef sandwiches, n_2 chicken sandwiches, and n_3 fish sandwiches, where $n_1 + n_2 + n_3 = 6$.

- Two orders are exactly the same if and only if their corresponding 3-tuples (n_1, n_2, n_3) are exactly the same.
  - The order of the entries in this 3-tuple has been fixed, 1st entry for B, 2nd entry for C, 3rd entry for F.
- For each 3-tuple (n_1, n_2, n_3), we can write a corresponding string of 6 characters, where the characters are from among B, C, F.
  - **Example:** $(2, 3, 1)$ corresponds to the string 'BBCCCF'.
  - **Example:** $(2, 2, 2)$ corresponds to the string 'BBCCFF'.
  - **Example:** $(3, 0, 3)$ corresponds to the string 'BBBFFF'.

Key Idea 1: For better "organization" or "clarity", we could insert two '|' characters between the three groups of same characters.

- **Example:** 'BBCCCF' is rewritten as 'BB|CCC|F'.
- **Example:** 'BBCCFF' is rewritten as 'BB|CC|FF'.
- **Example:** 'BBBFFF' is rewritten as 'BBB||FFF'.

Exercise 3 - Solution (continued)

Question: Once we have inserted our two '|' characters, does the letters B, C,F really matter?

Key Idea 2: Replace the letters B, C, F by the common symbol 🥪.

- **Example:** 'BB|CCC|F' is rewritten as '🥪🥪|🥪🥪🥪|🥪'.
- **Example:** 'BB|CC|FF' is rewritten as '🥪🥪|🥪🥪|🥪🥪'.
- **Example:** 'BBB||FFF' is rewritten as '🥪🥪🥪||🥪🥪🥪'.

For each string, we can further rewrite by replacing the characters '🥪' and '|' by 0 and 1 respectively.

- **Example:** '🥪🥪|🥪🥪🥪|🥪' is rewritten as '00100010'.
- **Example:** '🥪🥪|🥪🥪|🥪🥪' is rewritten as '00100100'.
- **Example:** '🥪🥪🥪||🥪🥪🥪' is rewritten as '00011000'.

Question: What would an order of 2 fish sandwiches, 3 chicken sandwiches, and 1 beef sandwich correspond to?

- **Note:** Be careful of the order of the types of sandwiches!

Answer: '01000100'.

Exercise 3 - Solution (continued)

Key Idea 3: Every order that John places can be represented by an 8-character string consisting of two 1s and six 0s, and vice versa.

- We have constructed a bijection

$$\left\{\begin{matrix}\text{all possible orders}\\\text{that John could place}\end{matrix}\right\} \to \left\{\begin{matrix}\text{all 8-character strings consisting}\\\text{of two 1s and six 0s}\end{matrix}\right\}.$$

Thus, to count the number of possible orders, it suffices to count the number of 8-character strings with two 1s and six 0s.

- There are 8 positions in our string, of which we want to choose 2 positions to fill in with the character 1.
- Therefore, the total number of possible orders is $\binom{8}{2} = 28$.

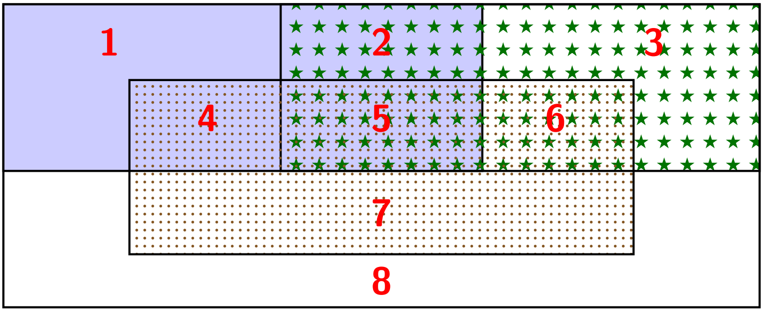
Important Insight: The idea of inserting '|' to separate groups of objects is known as the stars and bars method, or sticks and stones method, or balls and bars method.

- Or in this cohort class, the "sandwiches and bars method".

Class Activity (15 mins)

It's time to get up!

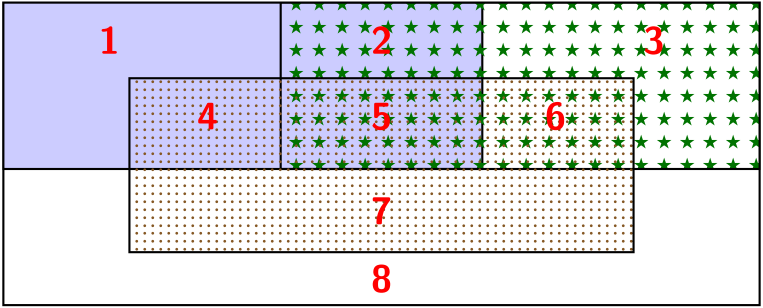
Front whiteboard



There are a total of 8 zones. Place your avatar (to be provided) on your zone, according to the following criteria:

- **Blue Zones:** Your birthday falls within 1 January–30 June.
- **Star Zones:** You have been to Macdonalds in the past month.
- **Dotted Zones:** You have an elder brother or elder sister.

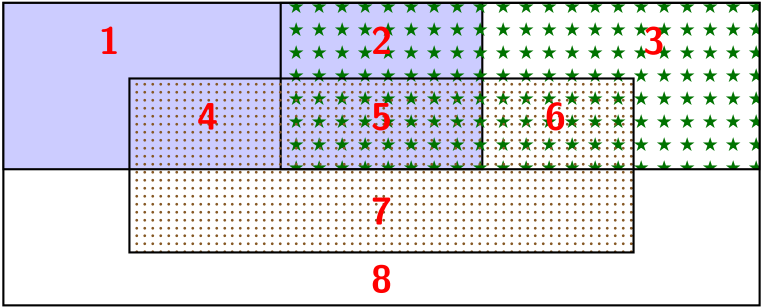
Class Activity (15 mins)



Question: How many of you are born in the first half of the year, **or** have been to Macdonalds in the past month, **or** have an elder brother or elder sister? (i.e. How many of you are in zones **1**–**7**?)

- First, let's count the number of students in blue zones, star zones, and dotted zones.
- Next, let's count the number of students in blue star zones, blue dotted zones, and star dotted zones.
- What about the number of students in all three zones?

Class Activity (15 mins)



Question: How many students are there in zones **1–7**?

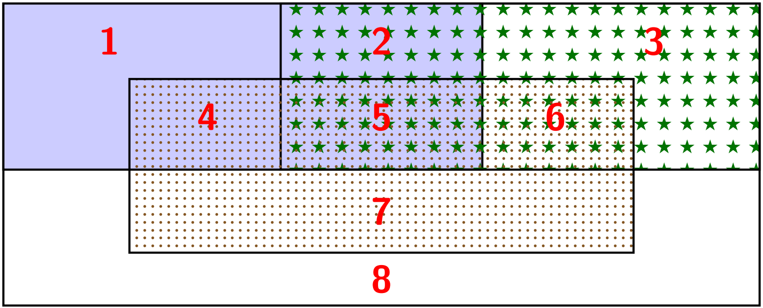
Answer: A first estimate..

$$\binom{\text{Number in}}{\textbf{blue}\text{ zones}} + \binom{\text{Number in}}{\textbf{star}\text{ zones}} + \binom{\text{Number in}}{\textbf{dotted}\text{ zones}}$$

Some students **included** are counted more than once!

- Those in **blue star** zones are counted at least twice!
- Those in **blue dotted** zones are counted at least twice!
- Those in **star dotted** zones are counted at least twice!

Class Activity (15 mins)



Question: How many students are there in zones **1–7**?

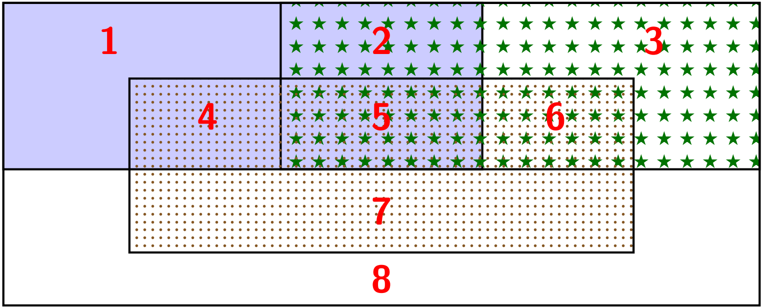
Answer: A better estimate..

$$\begin{aligned} &\binom{\text{Number in}}{\textbf{blue}\text{ zones}} + \binom{\text{Number in}}{\textbf{star}\text{ zones}} + \binom{\text{Number in}}{\textbf{dotted}\text{ zones}} \\ &- \binom{\text{Number in}}{\textbf{blue star}\text{ zones}} - \binom{\text{Number in}}{\textbf{blue dotted}\text{ zones}} - \binom{\text{Number in}}{\textbf{star dotted}\text{ zones}} \end{aligned}$$

We have **excluded** all over-counted zones.

- Did we exclude/subtract too much?
- How many times is Zone **5** counted? **Answer:** 0 times!

Class Activity (15 mins)



Question: How many students are there in zones **1–7**?

Answer: We have to **include** back those in **blue star dotted** zones:

$$\begin{aligned}
&\binom{\text{Number in}}{\textbf{blue}\text{ zones}} + \binom{\text{Number in}}{\textbf{star}\text{ zones}} + \binom{\text{Number in}}{\textbf{dotted}\text{ zones}} \\
&- \binom{\text{Number in}}{\textbf{blue star}\text{ zones}} - \binom{\text{Number in}}{\textbf{blue dotted}\text{ zones}} - \binom{\text{Number in}}{\textbf{star dotted}\text{ zones}} \\
&+ \binom{\text{Number in}}{\textbf{blue star dotted}\text{ zones}}
\end{aligned}$$

We found the exact value by a series of **inclusions** and **exclusions**.

Inclusion-exclusion principle

Inclusion-exclusion principle: Given any three sets $\mathbf{A}, \mathbf{B}, \mathbf{C}$,

$$\begin{aligned}
|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}| = |\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}| & \quad \text{(include all individual sets)} \\
- |\mathbf{A} \cap \mathbf{B}| - |\mathbf{A} \cap \mathbf{C}| - |\mathbf{B} \cap \mathbf{C}| & \quad \text{(exclude all pairs of sets)} \\
+ |\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}| & \quad \text{(include all triples of sets).}
\end{aligned}$$

More generally, given any n sets $A_1, \ldots, A_n$,

$$\begin{aligned}
|A_1 \cup \cdots \cup A_n| = {} & |A_1| + \cdots + |A_n| \\
& - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\
& + |A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\
& \ldots \\
& + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.
\end{aligned}$$

Note: The intersection of every k-tuple $(A_{i_1}, \ldots, A_{i_k})$ of sets is included if k is odd, and excluded if k is even.

Summary

- Counting methods, product rule, sum rule
- Permutations and Combinations
- Class Activity: Inclusion-exclusion principle