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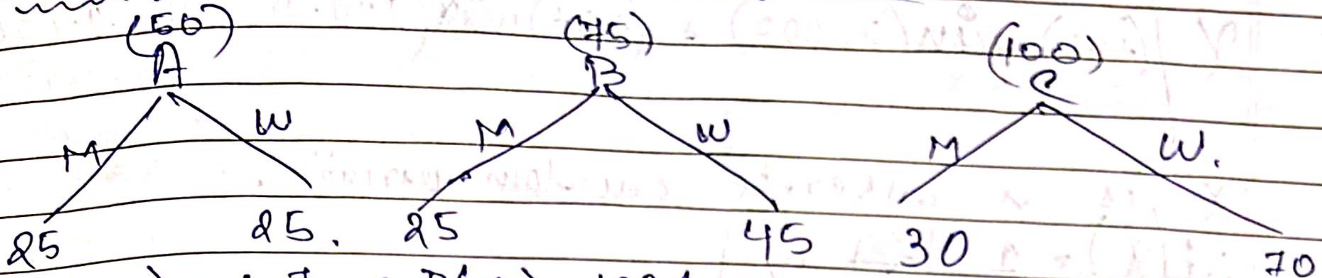
classmate
Date
Page

Stochastic Model. HW-1.

[1]



$P(W|C) = 0.7$ ✓ $P(C) = 100/225$ ✓

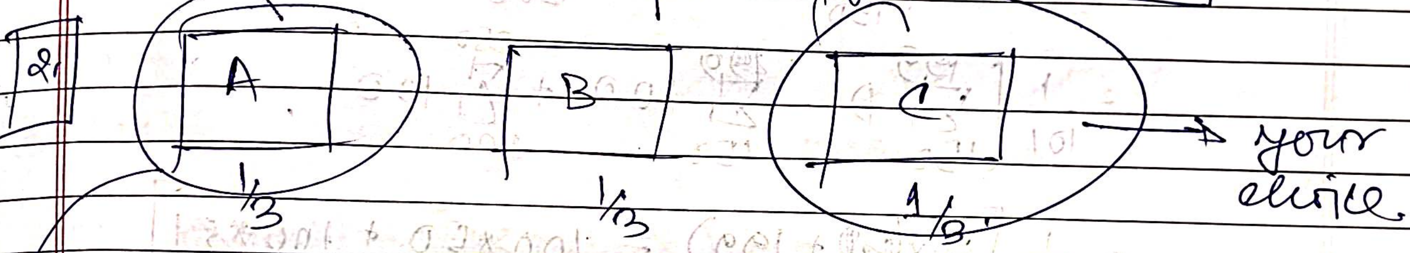
$P(C|W) \cdot P(W) = P(W|C) \cdot P(C)$.

$$\therefore P(C|W) = \frac{P(W|C) \cdot P(C)}{P(W|A) \cdot P(A) + P(W|B) \cdot P(B) + P(W|C) \cdot P(C)}$$

$$= \frac{0.7 \times \cancel{100/225}}{0.5 \times 50/\cancel{225} + 0.6 \times 75/\cancel{225} + 0.7 \times 100/\cancel{225}}$$

$$= \frac{70}{25 + 45 + 70} = \boxed{1/2}.$$

[2.]



Initial pick (probability) = 1/3. (equally likely).

The host picks A. A turns out to NOT have the prize.

This means B now has a probability 2/3 & therefore it is statistically more preferable or judicious to switch to B.

[3.]
(a) E(X) for billy is what we have to find.
X is the discrete variable for earnings.

~~$X = 1.5 \min(D, 200) + (0.5)(200 - D)$~~

$$\underset{\text{earnings}}{X} = \Big[\underbrace{1.5 \cdot \min(D, 200)}_{\text{profit factor}} + \underbrace{0.5 \cdot \max(200 - D, 0)}_{\text{loss factor}}\Big] - \underbrace{1 \cdot 200}_{\text{expenditure}}$$

Earnings

$$\therefore X = \left[\underbrace{(+1.5)\cdot\min(D, 200) + (+0.5)\max(200-D, 0)}_{\text{Net selling}}\right] - \underbrace{1\cdot(200)}_{\text{Net expenditure}}$$

X is a discrete random variable.

$$\therefore E(X) = \frac{1}{101}\left[\sum_{150}^{250} X\right].$$

Case 1: $X(D)$

(i) $D \leq 199 \Rightarrow 1.5D + 0.5(200-D) - 200.$

$\Rightarrow (1.5-0.5)D + (100-200).$

$\Rightarrow D - 100,$

(ii) $D \geq 200 \Rightarrow 1.5(200) - 200.$

$\Rightarrow 100.$

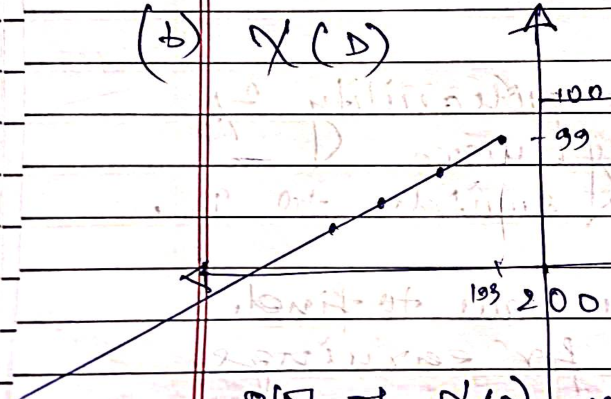
$$\therefore E(X) = \frac{1}{101}\left[\sum_{150}^{199}(D-100) + \sum_{200}^{250} 100\right].$$

$$= \frac{1}{101}\left[\sum_{150}^{199} D - \sum_{150}^{199} 100 + \sum_{200}^{250} 100\right].$$

$$= \frac{1}{101}\left[\frac{50\times(150+199)}{2} - \cancel{100\times 50 + 100\times 51}\right]$$

$$= \frac{1}{101}\left[\frac{50}{2}\times 349 + 100\right] \approx \boxed{87.3762}$$

(b) $X(D)$



Plot of $X(D)$ versus D.

since X does not change buying more newspapers is wasteful & does not make sense.

Q is variable.
D is NOT variable here.

$\text{Opt}[X(D)]$ happens $\forall D \in [200, 250]$.

Q variables means we are needing to 'buy' newspapers.

$\therefore Q(250) < Q(200)$

$Q(\text{opt}) = 200$ [exact].

(4.) Discrete random variables : X, Y.

$X = \text{Unif}(\{1,2,3,4,5\})$. $\therefore x \in \{1,2 \ldots 5\}$.
$Y = \text{Unif}(\{1,2 \ldots X\})$. $y \in \{1,2, \ldots X\}$.

(a) Joint pmf:

$$p_X(x,y) = p(x) \cdot p(y \mid x) = \frac{1}{5} \cdot \frac{1}{x}.$$

(b) conditional mass function $P(x \mid y=i)$.

$$\boxed{\cancel{\therefore P(X \mid Y=1) = \frac{P(Y \mid X) \cdot P(X)}{P(Y=1)} = .}}$$

General : $P(x \mid y=i) = \dfrac{P(y \cap x)}{P(y=i)} = \dfrac{\frac{1}{5x}}{\boxed{}}$ — joint pmf

$P(y=i) = P(y=i \mid x) \cdot P(x)$ — $1/x$, $1/5$

(*) Finding out what $P(y=i)$ means →

$\Rightarrow P(y=i) = P(y=i \mid x) \cdot P(x)$.

$x=1 \rightarrow P(Y=1 \cap x) = 1/5$.
$x=2 \rightarrow P(y=1 \cap x) = 1/5 \cdot 1/2$.
$x=3 \rightarrow P(y=1 \cap x) = 1/5 \cdot 1/3$
$\vdots$

$$\boxed{P(y=i) = \sum_{k=1}^{5} \frac{1}{5} \cdot \frac{1}{k}.}$$

$\therefore$ General : $P(x \mid y=i) = \dfrac{1/5x}{\sum_{1} \frac{1}{5} \cdot \frac{1}{k}} = \dfrac{1/x}{\sum_{k=i} 1/k}$.

Calculating denominators:

$k=1$ (start pt) $= 1/1 + 1/2 + 1/3 + 1/4 + 1/5 = 137/60$
$k=2$ (start pt) $= 1/2 + 1/3 + 1/4 + 1/5 = 77/60$
$k=3$ (start pt) $= 1/3 + 1/4 + 1/5 = 47/60$
$k=4$ (start pt) $= 1/4 + 1/5 = 9/20$
$k=5$ (start pt) $= 1/5 = 1/5$

(c) Absolutely NOT.

(i.) $P(X \cap Y) = \frac{1}{5X}$ → Not independent of X.

OR.

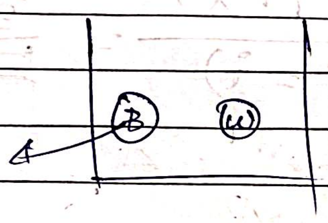
(ii.) $P(X \cap Y) = \boxed{P(Y|X)} \cdot P(X)$.

→ $P(Y|X) \neq P(Y)$.

(iii.) $P(Y = i)$ (for $i = 1, 2$) > 1.

Through (i), (ii) & (iii) we can see clearly that Y & X are NOT independent.

(E.)



STEP 1

Prob(add black ball) $= \frac{1}{2} \cdot \frac{1}{2}$.

Prob(nothing happened) $= \frac{1}{2} \cdot \frac{1}{2}$.

Prob(white removed) $= \frac{1}{2}$.

STEP 2

Prob(reset) $= \frac{1}{4}$.

Prob(nothing happens) $= \frac{3}{4}$.

T → No white balls.

RTF: $E[t = T]$.

Hint: b black balls. w white balls, $E(b,w)$.

∴ Reset prob ⇒ $1/4$, inverse ⇒ $1 - 1/4 = 3/4$.

(*) white removal ∴ $\frac{w}{b+w}$ ⇒ $[b, w-1]$.
Termination criteria (for $w=1$).

(*) black drawn & NOT added back = $\frac{b}{2(b+w)}$ ⇒ $[b, w]$.

(*) black drawn & added ⇒ $\frac{b}{2(b+w)}$ ⇒ $[b+1, w]$.

→ possibility of process termination? $\boxed{No}$.

→ possibility of process termination? $\boxed{Yes}$.

criteria → $w = 1$. <end of loop>.

$$\therefore E(b,w) = 1 + \underbrace{\frac{b}{b+w}}_{\text{black branch}}\left[\overbrace{\frac{1}{2}}^{\text{black NOT added}}\left\{\overbrace{\frac{3}{4}E(b,w)}^{\text{continuation}} + \overbrace{\frac{1}{4}E(1,1)}^{\text{reset}}\right\} + \underbrace{\frac{1}{2}}_{\text{black added}}\left\{\underbrace{\frac{3}{4}E(b+1,w)}_{\text{continuation}} + \underbrace{\frac{1}{4}E(1,1)}_{\text{reset}}\right\}\right] + \underbrace{\frac{w}{b+w}}_{\text{white branch}}\Phi(b,w)$$

$$\Rightarrow \Phi(b,w) \begin{cases} 0 & w = 1. \\ \underbrace{\frac{3}{4}E(b, w-1)}_{\text{continuation}} + \underbrace{\frac{1}{4}E(1,1)}_{\text{reset}} & w \geq 2. \end{cases}$$

(*) Termination criteria: $\boxed{w=1}$.
continue if $\boxed{w > 1}$.

(*) $$\therefore E(b,w) = 1 + \underbrace{\left[\frac{b}{b+w}\right]}_{\text{black branch}}\left[\overbrace{\frac{1}{2}}^{\text{black NOT added}}\left[\underbrace{\frac{3}{4}}_{\text{continue prob coeff}}\text{continue} + \underbrace{\frac{1}{4}}_{\text{reset prob coeff}}\text{reset}\right] + \overbrace{\frac{1}{2}}^{\text{black added}}\left[\underbrace{\frac{3}{4}}_{\text{continue}}\text{continue} + \underbrace{\frac{1}{4}}_{\text{reset}}\text{reset}\right]\right]$$

$$+ \left(\frac{w}{b+w}\right)\left[\underbrace{0}_{\text{loop termination}} \quad \frac{3}{4}\text{continue} + \frac{1}{4}\text{reset}\right]$$

→ terminate ($b = b, w = 1$)
→ continue $\Phi(w > 1)$
→ $\Phi(b,w)$

(b) $E(4,1) \approx E(3,1)$.

For both $w=1$, therefore $f(b,1)=0$.

$\therefore$ Only part that survives is the black branch

$\therefore P(\text{draw black}) = \frac{b}{b+1}$.

$\therefore E_b = 1 + \frac{b}{b+1}\left(E\left[\text{remaining time}, b\right]\right)$.

→ Add 0 black $\Rightarrow \frac{3}{4}E_b + \frac{1}{4}E_1$.

→ Add 1 black $\Rightarrow \frac{3}{4}E_{b+1} + \frac{1}{4}E_1$.

$\therefore$ black branch $\Rightarrow \frac{1}{2}\left(\frac{3}{4}E_b + \frac{1}{4}E_1\right) + \frac{1}{2}\left(\frac{3}{4}E_{b+1} + \frac{1}{4}E_1\right)$.

$\Rightarrow \frac{3}{8}\left(E_{b+1} + E_b\right) + \frac{1}{4}E_1$.

white branch $\Rightarrow$ 0.

$$\underbrace{\therefore E_b = 1 + \frac{b}{b+1}\left(\frac{3}{8}\left(E_{b+1} + E_b\right) + \frac{1}{4}E_1\right)}_{\text{recurrence relation.}} \rightarrow \text{variable.}$$

Given : $E_4 = E_3$.

$\boxed{\text{Case 1}}$: $(b=1)$.

$$\boxed{11E_1 = 3E_2 + 16} \quad \text{(i)}$$

$E_1 = 1 + \frac{1}{2}\left(\frac{3}{8}\left(E_2 + E_1\right) + \frac{E_1}{4}\right)$.

$E_1 = 1 + \frac{3}{16}E_2 + \underbrace{\left(\frac{3}{16} + \frac{2}{16}\right)}_{5/16}E_1 \Rightarrow \frac{11}{16}E_1 = \frac{3}{16}E_2 + 1$.

case 2 : $(b=2)$.

$$E_2 = 1 + \frac{2}{3}\left[\frac{3}{8}(E_3 + E_2) + \frac{1}{4}E_1\right].$$

$$E_2 = 1 + \frac{E_3}{4} + E_2\left(\frac{1}{4}\right) + \frac{E_1}{6}.$$

$$\frac{3}{4}E_2 = 1 + \frac{E_3}{4} + \frac{E_1}{6}.$$

$$\Rightarrow \boxed{9E_2 = 12 + 3E_3 + 2E_1} \quad \text{---- (ii)}$$

case 3 : $(b=3)$. $\Rightarrow E_4 = E_3$ (given).

$$E_3 = 1 + \frac{2}{4}\left[\frac{3}{8}(E_4 + E_3) + \frac{1}{4}E_1\right].$$

$$E_3 = 1 + \frac{3}{4}\left[\frac{3}{4}E_3 + \frac{1}{4}E_1\right].$$

$$E_3 = 1 + \frac{9}{16}E_3 + \frac{3}{16}E_1.$$

$$\frac{7}{16}E_3 = 1 + \frac{3}{16}E_1 \Rightarrow \boxed{7E_3 = 16 + 3E_1} \quad \text{--- (iii)}$$

$\boxed{3(i) + (ii).}$

$$33E_1 - \cancel{9E_2} - 48 = 0$$
$$-2E_1 + \cancel{9E_2} - 12 - 3E_3 = 0.$$

$$31E_1 - 60 - 3E_3 = 0 \quad \text{---- (iv).}$$

$\boxed{7(iv) + 3(iii).}$

$$217E_1 - 420 - \cancel{21E_3} = 0.$$
$$-9E_1 - 48 + \cancel{21E_3}$$

$$208E_1 = 468. \quad (v)$$

$$\Rightarrow E_1 = \frac{468}{208} = 2.25$$

(answer).

(6)

(a) Let $X \geq 0$ be a non negative random variable and let $t > 0$.

$$\Phi(X \geq t) \begin{cases} 1 & \text{if } X \geq t. \\ 0 & \text{otherwise} \end{cases}$$

$\therefore X \geq t.\Phi$

$E(X) \geq E(t.\Phi) \Rightarrow E(X) \geq t(E(\Phi))$ → $P(X \geq t)$.

$\therefore \boxed{E(X) \geq t.P(X \geq t).}$ (proof).

$\therefore P(X \geq t) \leq \frac{E(X)}{t}$. (markov's inequality).

(b) $E[Y] = \mu$

$E[(Y-\mu)^2] = \sigma^2$. } → let Z be equal to $(Y-\mu)^2$.

$\therefore P(|Y-\mu| \geq t) = P(|Y-\mu|^2 \geq t^2) = P(Z \geq t^2)$.

$P(Z \geq t^2) \leq \frac{E(Z)}{t^2} \Rightarrow P(Z \geq t^2) \leq \frac{E[(Y-\mu)^2]}{t^2}$.

$\therefore P(Z \geq t^2) \leq \frac{\sigma^2}{t^2} \Rightarrow \boxed{P(|Y-\mu| \geq t) \leq \frac{\sigma^2}{t^2}}$ (proof)

CHEBYSHEV.

(c) Law of large numbers

let $X_1, X_2 \ldots$ be i.i.d with $\boxed{E(X_i) = \mu, Var(X_i) = \sigma^2}$

$\sigma_i^2 < \infty$. || $S_n = \sum_1^n X_i$

$\bar{X}_n = \frac{S_n}{n} = \frac{\sum_1^n X_i}{n}$. $\therefore E[\bar{X}_n] = \frac{1}{n} E[S_n]$

$= \frac{1}{n}.\mu.n = \mu$.

$Var(\bar{X}_n) = Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2} Var(S_n)$

$Var(\bar{X}_n) = \frac{1}{n^2}.n\sigma^2 = \frac{\sigma^2}{n}$.

$\lim_{n \to \infty} \left(\frac{\sigma^2}{n\epsilon}\right) = \boxed{0}$

If $Y = \bar{X}_n$ [CHEBYCHEV invoke] $\Rightarrow P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon n}$ $\therefore P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) = 0$ (proof)