Course 50.050/50.550
Advanced Algorithms

Week 3 – Cohort Class L03.03

SINGAPORE UNIVERSITY OF
TECHNOLOGY AND DESIGN

Outline of Cohort Class

- Quick review before Mini-quiz 1
- **Mini-quiz 1**
- Class activity: handshaking game
- Pigeonhole principle, Ramsey theory

Brief overview of sets and tuples

Let X, A, B, $X_1, \ldots, X_k$ be sets.

- Set operations: $A \cup B$, $A \cap B$, $A - B$, $A \backslash B$, $A \Delta B$.
  - If $A = \{1, 2\}$ and $B = \{2, 3\}$, then $A - B = \{1\}$ and $A \Delta B = \{1, 3\}$.
  - **Recall:** If $B \subseteq A$, then $A \backslash B$ is the set $A - B$.
- The power set of X, i.e. $\mathcal{P}(X)$, contains all possible subsets of X.
  - $|X| < |\mathcal{P}(X)| < |\mathcal{P}(\mathcal{P}(X))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(X)))| < \ldots$
- We say that A and B have the same cardinality, i.e. $|A| = |B|$, if there exists a bijection $f : A \to B$.
  - We say that $|A| \leq |B|$ if there exists an injection $f : A \to B$.
- X is called countably infinite if $|X| = |\mathbb{N}|$.
  - e.g. $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Z}^+| = |\mathbb{Q}^+| = |\mathbb{Q}| = \aleph_0$.
- The continuum hypothesis is the statement: There exists no set X such that $|\mathbb{N}| < |X| < |\mathbb{R}|$.
  - Gödel's theorem: The continuum hypothesis cannot be proven.
- The Cartesian product of $X_1, \ldots, X_k$ is

$$X_1 \times \cdots \times X_k := \{(x_1, \ldots, x_k) | x_1 \in X_1, \ldots, x_k \in X_k\}.$$

- A binary relation from A to B is a subset of $A \times B$.
  - A k-ary relation on $(X_1, \ldots, X_k)$ is a subset of $X_1 \times \cdots \times X_k$.

Brief overview of logic

- A proposition is a declarative statement that is true ($\top$) or false ($\perp$), but not both.
- Logical connectives for two propositions p and q:
  - **Disjunction**: $p \vee q$ is the proposition representing "p or q".
  - **Conjunction**: $p \wedge q$ is the proposition representing "p and q".
  - **Negation**: $\neg p$ is the proposition representing "not p".
  - **Implication**: $p \to q$ is the proposition representing "if p then q".
  - **Biconditional**: $p \leftrightarrow q$ is the proposition $(p \to q) \wedge (q \to p)$.
  - **Exclusive or**: $p \oplus q$ is the proposition "p or q, but not both".
    - i.e. $p \oplus q$ has the same meaning as $(p \vee q) \wedge \neg(p \wedge q)$.
    - $p \oplus q$ is also commonly denoted by $p \veebar q$.
- Suppose p is a compound proposition. Then p is a tautology if p is always true for all possible tuples of truth values.
  - If $p \leftrightarrow q$ is a tautology, then we write $p \equiv q$, and say that p, q are logically equivalent.
- A k-ary predicate is a k-ary relation on some tuple $(X_1, \ldots, X_k)$.
- Predicate logic (also first-order logic) consists of propositional logic, variables, predicates, and quantifiers.
  - universal quantifier $\forall$, existential quantifier $\exists$.

Brief overview of proofs

- Basic proof methods:
  - Proof by exhaustion: Check all (finitely many) cases.
  - Proof by counter-example: Find a single counter-example.
  - Proof by contradiction: Show that negation of proposition is false.
- Constructive vs non-constructive proofs
- Induction, strong induction, variants of induction/strong induction.
  - e.g. We can use induction to prove the well-ordering principle, which says that every non-empty set of $\mathbb{N}$ has a smallest element.
- Double counting: Count same value in two different ways.
- Fermat's method of infinite descent: If $P(n_0)$ is true implies $P(n_1)$, $P(n_2), \ldots$ are true for $n_1 > n_2 > \ldots$, then $P(n)$ is false for all n.
- Extremal principle: For a non-empty collection of objects with assigned values in $\mathbb{N}$, there is an object with the smallest value.
  - For a finite non-empty collection of objects, there is an object with the smallest/largest assigned value.
- Invariance: "Things that don't change will remain unchanged."

Mini-quiz 1 (15 mins)

Only writing materials are allowed. No calculators, notes, books, or cheatsheets are allowed. Don't worry, you won't need calculators.

If you are not present in class at the start of the quiz, you will not be given additional time to finish the quiz.

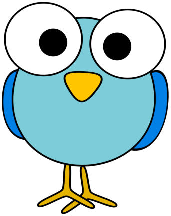
Remarks:

- There are no make-up mini-quizzes! If you arrive in class after the mini-quiz ends, or do not attend that cohort class, you will not have a chance to take the mini-quiz.
- To take into account unforeseen circumstances (e.g. mini-quiz missed due to illness), only the **best 3 of 4** mini-quiz scores will be counted towards your final grade.

Class activity: Handshaking game (5 mins)

To-Do List

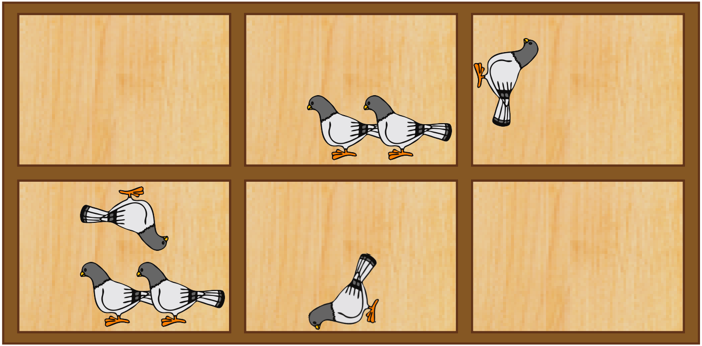
- Go around the classroom, shake hands with other students, and introduce your name. **Keep track of how many students you have shaken hands with.**
  - If you shake the same person's hands twice, it's still counted as 1 person in your total tally.
- When I say "Time's up!", choose a bird, and then place the chosen bird using Blu-tack in the region on the whiteboard labeled with the same number as the number of students you have shaken hands with.



Time's up!

Pigeons in Pigeonholes

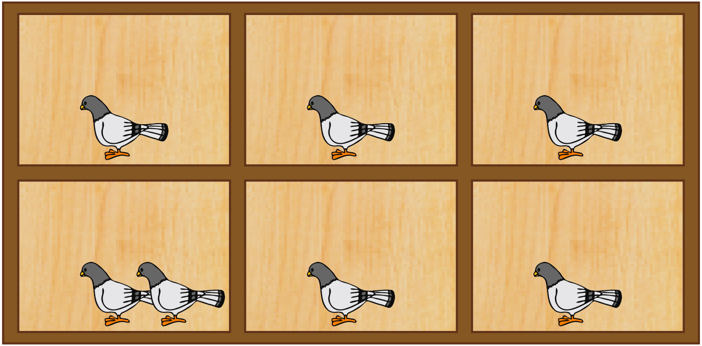
There are 7 pigeons resting in 6 pigeonholes.
Claim: There is at least one pigeonhole with at least 2 pigeons.



- The pigeons could try to squeeze together, so maybe there is more than one pigeonhole with at least 2 pigeons.
- Maybe the pigeons decide to rearrange themselves, so that there are no pigeonholes with exactly 2 pigeons.
- But no matter how the pigeons arrange themselves, there will always be **at least** one pigeonhole with **at least** 2 pigeons.

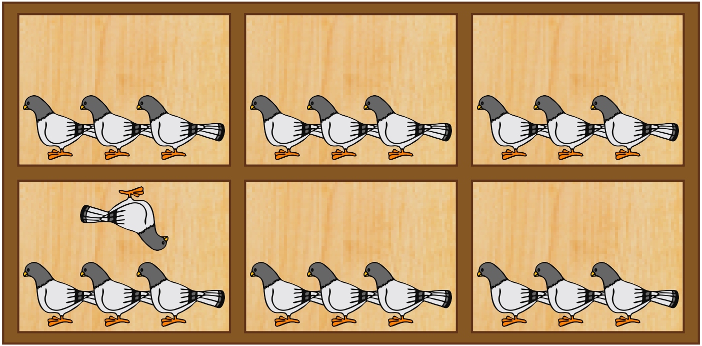
Pigeons in Pigeonholes (continued)

Pigeonhole principle (simplest version): Let $n \in \mathbb{Z}^+$. If there are $n+1$ pigeons in n pigeonholes, then at least one pigeonhole has at least 2 pigeons.



Pigeons in Pigeonholes (continued)

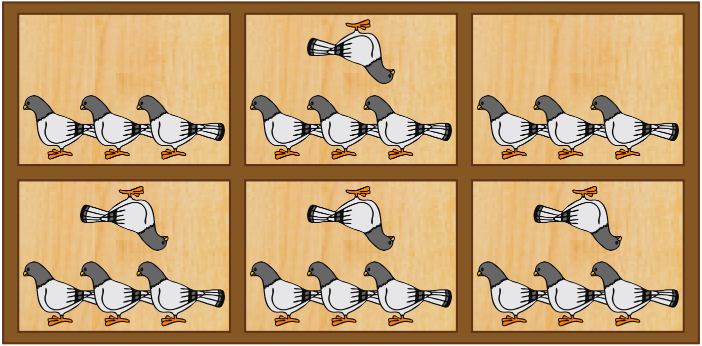
Pigeonhole principle (simple version): Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{N}$. If there are $kn + 1$ pigeons in n pigeonholes, then at least one pigeonhole has at least $k + 1$ pigeons.



Example: We have $19 = 3 \times 6 + 1$ pigeons and 6 pigeonholes, so at least one pigeonhole has at least $4 = 3 + 1$ pigeons.

Pigeons in Pigeonholes (continued)

Question: What if there are n pigeonholes, but the number of pigeons is not of the form $kn+1$?



- For example, what if there are 22 pigeons and 6 pigeonholes?
- What if there are 12 pigeons and 6 pigeonholes?
- What about 600 pigeons and 6 (very large) pigeonholes?
- Can we find a **general version** of the pigeonhole principle?

Quick recap: Floor function and ceiling function

Definition: Let $x \in \mathbb{R}$.

- The floor of x, denoted by $\lfloor x \rfloor$, is the largest integer at most x.
  - The function $x \mapsto \lfloor x \rfloor$ is called the floor function.
- The ceiling of x, denoted by $\lceil x \rceil$, is the smallest integer at least x.
  - The function $x \mapsto \lceil x \rceil$ is called the ceiling function.

Intuition: Think of the real number line vertically as a building.

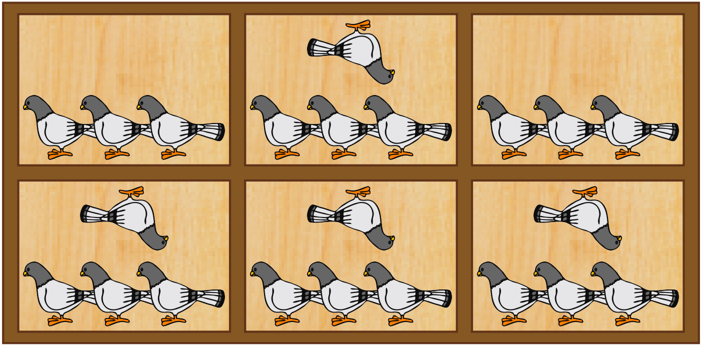
- The number 0 is at the ground level 0.
- The numbers $1, 2, 3, \dots$ are above the ground at levels $1, 2, 3, \dots$
- The numbers $-1, -2, -3, \dots$ are underground at levels $-1, -2, -3, \dots$

In this building, levels are labeled only by integers. Levels can be both the floor of the space above, and the ceiling of the space below.

- For $n \in \mathbb{Z}$, we have $\lfloor n \rfloor = \lceil n \rceil = n$.
- **Example:** $\lfloor \pi \rfloor = 3$, $\lceil \pi \rceil = 4$, $\lfloor -\pi \rfloor = -4$, $\lceil -\pi \rceil = -3$.
  - Note that $\pi \approx 3.14159\dots$

The Pigeonhole Principle

Pigeonhole principle (general version): Let $m, n \in \mathbb{Z}^+$. If there are n pigeons in m pigeonholes, then at least one pigeonhole has at least $\lceil \frac{n}{m} \rceil$ pigeons.



Example: There are 22 pigeons and 6 pigeonholes.

- By the pigeonhole principle, at least one pigeonhole has at least $\lceil \frac{22}{6} \rceil = \lceil 3.666\ldots \rceil = 4$ pigeons.

The Pigeonhole Principle

Theorem: (**Pigeonhole principle**) Let $m, n \in \mathbb{Z}^+$, and let S be a set with cardinality n. Suppose $S = S_1 \cup \cdots \cup S_m$ such that the m subsets $S_1, \ldots, S_m$ are pairwise disjoint, i.e. $S_i \cap S_j = \emptyset$ for all $i \neq j$. Then there exists some $1 \leq k \leq m$ such that $|S_k| \geq \lceil \frac{n}{m} \rceil$.

Proof: Suppose on the contrary that no such subset S_k exists.

- This means we must have $|S_i| \leq \lceil \frac{n}{m} \rceil - 1$ for all $1 \leq i \leq m$.

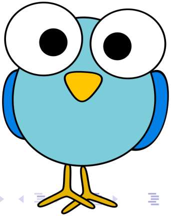
Key Idea: Note that $\lceil \frac{n}{m} \rceil - 1 < \frac{n}{m}$.

- Think "$\lceil \frac{n}{m} \rceil - 1$" and "$\frac{n}{m}$" in terms of the levels of a building.
  - Consider two cases: $\frac{n}{m} \in \mathbb{Z}$, and $\frac{n}{m} \notin \mathbb{Z}$.

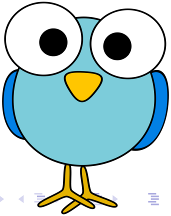
Consequence: $|S| = \sum_{i=1}^{m} |S_i| \leq m(\lceil \frac{n}{m} \rceil - 1) < m(\frac{n}{m}) = n$.

- But we cannot have $|S| < n$, since we assumed $|S| = n$. $\square$

For the handshaking game, I think there will always be two of you who would have shaken hands with the **same number** of students.



Can we show this using pigeonhole principle?
What are the "pigeons" and the "pigeonholes" on the whiteboard?



Another consequence of the pigeonhole principle

Proposition: There are at least **four**, non-bald, people in Singapore with exactly the same number of hairs on their heads.

- Singapore population ≈ 6.1 million.
- Research has shown that a person has on average ≈ 150000 hairs.
  - Most people are not bald.
  - We can assume that at least 3.05 million people in Singapore are not bald.
  - We can also safely assume that a person has at ≤ 1 million hairs.

"Pigeons" = non-bald people in Singapore. "Pigeonholes" are labeled by integers $1, 2, \ldots, 1000000$, representing the number of hairs.

By the **pigeonhole principle**, there are at least $\lceil \frac{3050000}{1000000} \rceil = 4$ non-bald people in Singapore with the same number of hairs (i.e. there are at least 4 "pigeons" in the same "pigeonhole").

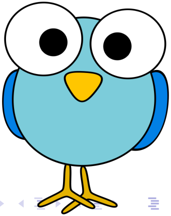
To use pigeonhole principle, ask yourself:
What are the “pigeons”? What are the “pigeonholes”?
What would be suitable “labels” for the “pigeonholes”?

Exercise 1 (15 mins)

Let S be a subset of $\{1, 2, \ldots, 20\}$ consisting of 11 integers.

1. Show that we can always find two consecutive integers in S.
2. Show that we can always find two integers in S whose sum is odd.
3. Show that we can always find two integers a and b in S, such that a divides b.
   - "a divides b" means "b is a multiple of a", or "there exists some $n \in \mathbb{Z}$ such that $b = na$".

Remember that the “pigeons” can be whatever objects you choose. Similarly, the “pigeonholes” can have whatever “labels” you want. Pigeonholes could come in different sizes!



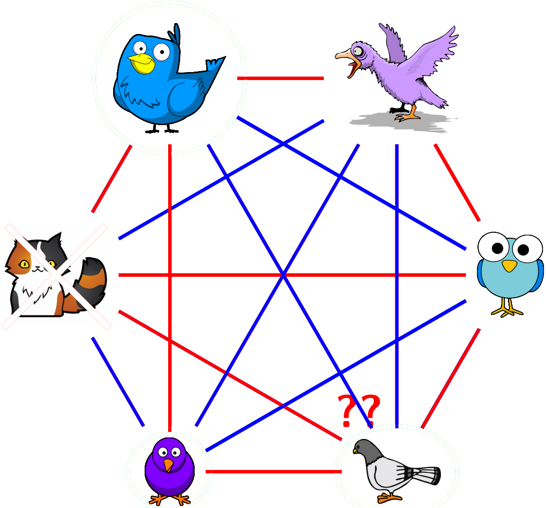
Exercise 2 (15 mins)

Let $x_1, x_2, \ldots, x_7$ be seven distinct real numbers. Show that we can always find two of them x_i, x_j such that

$$\left|\frac{x_i - x_j}{1 + x_i x_j}\right| < \frac{1}{\sqrt{3}}.$$

- **Hint 1:** $\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$.
- **Hint 2:** What has this got to do with the pigeonhole principle?

Friends and Strangers - Discussion



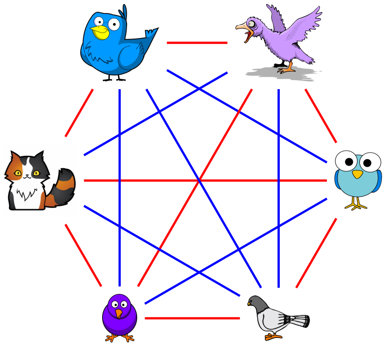
Question

Among the 6 of them, can we always find 3 of them that are either all friends with each other or all strangers with each other?

Friends

Strangers

Friends and Strangers



What we have shown

In any group of "birds", if the total number of "birds" is **at least 6**, then we can **always** find either 3 of them who are all friends with each other, or 3 of them who are all strangers with each other.

This **minimum** total number of "birds" for always finding **3** mutual friends or **3** mutual strangers is written as $R(\mathbf{3}, \mathbf{3})$.

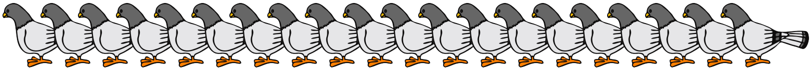
In other words, we have shown that $R(3, 3) = 6$.

Question: What about having 4 of them who are all friends with each other? Or 5 of them who are all strangers with each other?

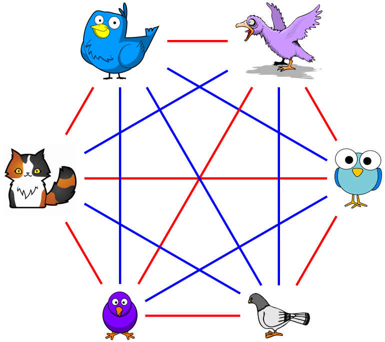
Ramsey's theorem

Theorem: (**Ramsey's theorem**) Let $r, s \in \mathbb{Z}^+$. In a **sufficiently large** community of people, no matter who are friends or who are strangers, there will always be either r people who are all friends with each other, or s people who are all strangers with each other.

- You may need to have a very very large number of people to guarantee that you can always find r mutual friends, or s mutual strangers.
- The minimum total number of people in the community for this to be always true is called a **Ramsey number**, and it is written as $R(r, s)$.



Other Ramsey numbers



We have shown that $R(3,3) = 6$.

Other known exact values

$R(3,4) = 9$

$R(4,4) = 18$ (proven in 1979)

$R(4,5) = 25$ (proven in 1995)

$R(5,5) = ??$

$R(6,6) = ??$

Known inequalities

$43 \leq R(5,5) \leq 46$.

(Upper bound reduced from 48 to 46 in Sept 2024!)

$102 \leq R(6,6) \leq 165$.

> *"Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6,6)$. In that case, we should attempt to destroy the aliens."* – Joel Spencer (1994)

What is Ramsey theory?

Ramsey's theorem is the starting point of Ramsey theory.

- **Intuition:** Ramsey theory is the study of conditions that would guarantee the existence of substructures with regular properties amidst "disorderly" structures.
  - Guiding question: Is there structure amidst apparent randomness, when some quantity is sufficiently large?
- Ramsey theory is full of non-constructive proofs.

Only if you are interested: A good representative of what another result in Ramsey theory (other than Ramsey's theorem) looks like is the following result:

Theorem: (**Van der Waerden's theorem**) Let $r, s \in \mathbb{Z}^+$, and fix some set of r colors. For a sufficiently large $N \in \mathbb{Z}^+$, if the integers in the set $\{1, 2, \ldots, N\}$ are colored, each with one of the fixed r colors, then there exists s integers in $\{1, 2, \ldots, N\}$ in **arithmetic progression**, such that all s of these integers have the same color.

Summary

- Quick review before Mini-quiz 1
- **Mini-quiz 1**
- Class activity: handshaking game
- Pigeonhole principle, Ramsey theory