Course 50.050/50.550
Advanced Algorithms

Week 4 – Cohort Class L04.03

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Outline of Cohort Class

- Exercise on adjacency matrices
- Class activity: Matching puzzle
- Hall's marriage theorem
- Discussion on matching algorithms

Recap: Adjacency matrix

Definition: Let $G = (V, E)$ be a **simple** graph, and suppose that $V = \{v_1, \dots, v_n\}$. The adjacency matrix of G is an n-by-n square matrix $A = [a_{i,j}]$, whose (i, j)-th entry $a_{i,j}$ is defined as follows:

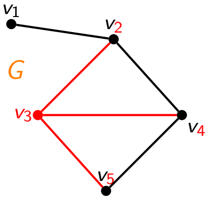
- (If G is undirected): $a_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$
- (If G is directed): $a_{i,j} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise.} \end{cases}$

Basic properties:

- Since G has no loops, the diagonal entries $a_{i,i}$ of A are all zeros.
- The adjacency matrix of an **undirected** graph is symmetric.
  - $A = [a_{i,j}]$ is called symmetric if $a_{i,j} = a_{j,i}$ for all $1 \le i, j \le n$.
- For an undirected graph G, we have:
  (row sum of i-th row) $=$ (column sum of i-th column) $= \deg(v_i)$.
- For a directed graph G, we have:
  - The row sum of the i-th row equals the out-degree $\deg^+(v_i)$.
  - The column sum of the j-th column equals the in-degree $\deg^-(v_j)$.

Examples of adjacency matrices

Undirected case:

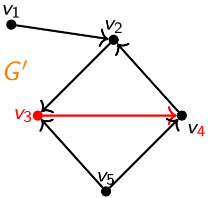


Adjacency matrix of G:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

- Row sums of the rows: $1, 3, 3, 3, 2$.
  - $\deg(v_1) = 1$, $\deg(v_2) = 3$, $\deg(v_3) = 3$, $\deg(v_4) = 3$, $\deg(v_5) = 2$.

Directed case:

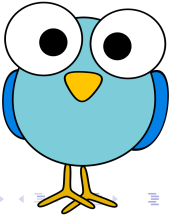


Adjacency matrix of G':

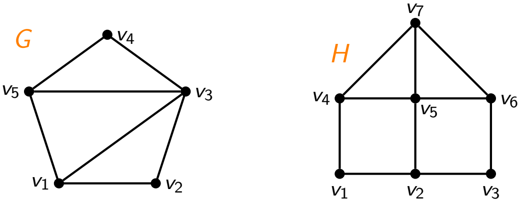
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

- Row sums of the rows: $1, 1, 1, 1, 2$.
  - $\deg^+(v_1) = 1$, $\deg^+(v_2) = 1$, $\deg^+(v_3) = 1$, $\deg^+(v_4) = 1$, $\deg^+(v_5) = 2$.
- Column sums of the rows: $0, 2, 2, 2, 0$.
  - $\deg^-(v_1) = 0$, $\deg^-(v_2) = 2$, $\deg^-(v_3) = 2$, $\deg^-(v_4) = 2$, $\deg^-(v_5) = 0$.

Let's be a mathematician for a day!
We have the chance to make the same discoveries that mathematicans made years ago.



Exercise 1 (20 mins)

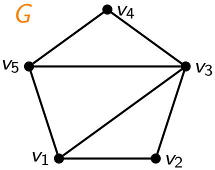


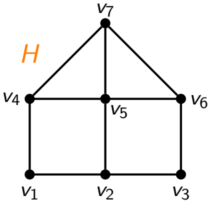
There are two simple undirected graphs G and H depicted above. Let A and B be the adjacency matrices of G of H, respectively.

1. What is A? What is B?
2. How many edges are there in G? How about H?
3. What is A^2? What is B^2?
   - *(Hint: How to compute the product of two n-by-n square matrices?)*
4. Can you relate the diagonal entries of A^2 to the number of edges in G? Similarly, can you relate the diagonal entries of B^2 to the number of edges in H? Justify your answer with a proof.

Challenge of the Day (Try at home)

Definition: Let G be an undirected graph. A triangle in G is a vertex-induced subgraph of G isomorphic to K_3.

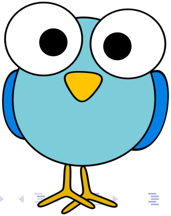




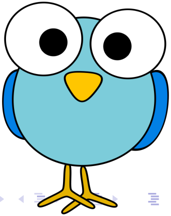
We have the same two simple undirected graphs G and H above. Again, let A and B be the adjacency matrices of G of H, respectively.

1. How many triangles are there in G? How about H?
2. What is A^3? What is B^3?
3. Can you relate the diagonal entries of A^3 to the number of triangles in G? Similarly, can you relate the diagonal entries of B^3 to the number of triangles in H? Justify your answer with a proof.

This week is bird conservation week!
Do you want to help out with my new
bird conservation initiative?



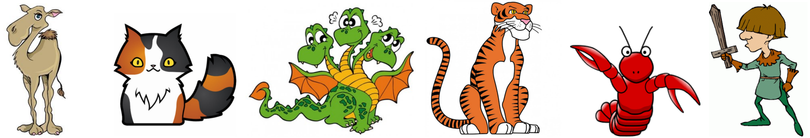
Actually, in my world, every week is
bird conservation week!



Matching Puzzle [Part I]

It is Day 1 of the bird conservation week.

Our little blue friend John  has six non-bird friends



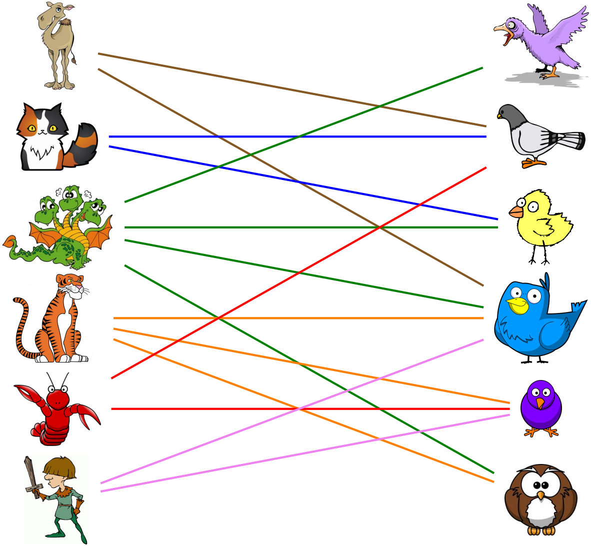
who have volunteered to help with the bird conservation initiative.

- Every volunteer has to be paired up with a bird to help with conservation efforts.
- The volunteers have preferences on who they want to work with.

John  has asked for your help to **match up** the volunteers with the birds, so that each volunteer is paired with a bird that he/she/it wants to work with.

Matching Puzzle [Part I] (10 mins)

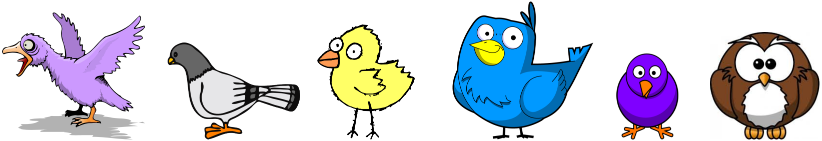
Here are the preferences of the volunteers.



Question: Can you find a matching? In a systematic way?

Matching Puzzle [Part II]

It is Day 2 of the bird conservation week.
For Day 2, our six bird friends now want to switch the volunteers they are paired up with.

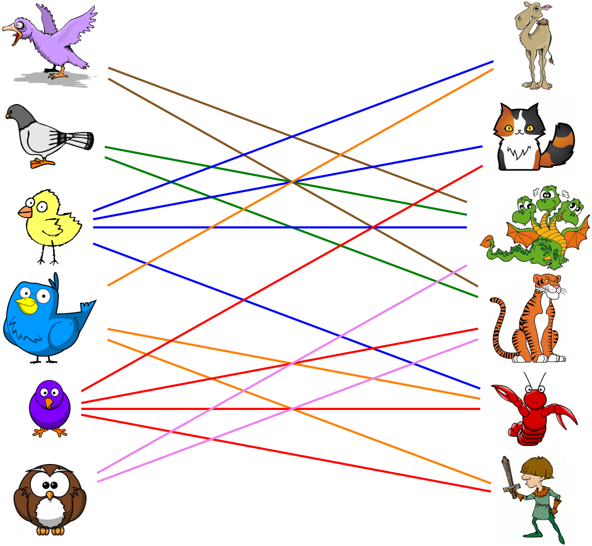


- Every bird friend has to be paired up with a volunteer.
- The bird friends have preferences on who they want to work with.

John  has asked for your help to **match up** the birds with the volunteers, so that each bird is paired with a volunteer that he/she/it wants to work with.

Matching Puzzle [Part II] (15 mins)

Here are the preferences of our bird friends.



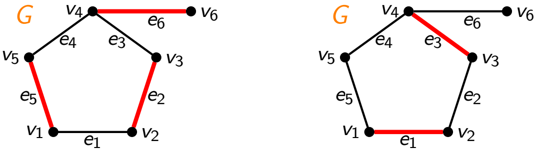
Question: Can you find a matching? Why or why not?

Matchings

Definition: Let $G = (V, E, \phi)$ be an undirected graph with no loops. A matching in G is a subset of edges $M \subseteq E$ such that no vertex $v \in V$ is incident to strictly more than one edge in M.

- A vertex $v \in V$ is called matched (w.r.t. M) if $v \in \phi(e)$ for some $e \in E$, and is called unmatched or free (w.r.t. M) otherwise.
- A maximal matching in G is a matching M such that there is no other possible matching M' in G such that $M \subsetneq M'$.
- A maximum matching in G is a matching M with the largest cardinality over all possible matchings in G.

Example: A graph G with two maximal matchings depicted.



- Both $M := \{e_2, e_5, e_6\}$ and $M' = \{e_1, e_3\}$ are maximal matchings.
- M is a maximum matching, but M' is not a maximum matching.
- v_5 is an unmatched (or free) vertex w.r.t. matching M'.

More matching terminology

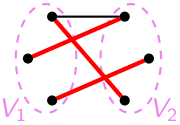
Definition: Let $G = (V, E, \phi)$ be an undirected graph with no loops. A matching in G is called perfect if no vertex is unmatched w.r.t. M.

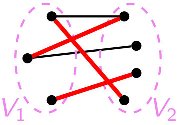
- If G has a perfect matching, then $|V|$ must be even, and $|M| = \frac{|V|}{2}$.

Definition: Let $G = (V, E, \phi)$ be an undirected **bipartite** graph with bipartition (V_1, V_2). A complete matching **from** V_1 **to** V_2 is a matching M such that every $v \in V_1$ is incident to some edge in M.

- **Note:** A complete matching from V_1 to V_2 is perfect if and only if $|V_1| = |V_2|$.

Example: Two bipartite graphs with complete matchings from V_1 to V_2 depicted.





- Left depicted matching is a perfect matching.
- Right depicted matching is not a perfect matching.

Hall's marriage theorem

Let $G = (V, E, \phi)$ be an undirected graph, and let $x \in V$.

- **Recall:** The neighborhood (or open neighborhood) of x is the set $N_G(x)$ of all neighbors of x.
  - i.e. $N_G(x)$ contains the set of all vertices adjacent to x.
- Given a subset $U \subseteq V$, the neighborhood (or open neighborhood) of U is the set $N_G(U) := \cup_{x \in U} N_G(x)$.
  - **Intuition:** $N_G(U)$ contains all vertices adjacent to some vertex in U.
  - If the context of the graph G is clear, we could simply write $N(U)$.
  - By default, $N_G(\emptyset)$ is defined to be $\emptyset$.

Theorem: (**Hall's marriage theorem** (1935)) Let G be an undirected **bipartite** graph with bipartition (V_1, V_2). The following are equivalent:

- There exists a complete matching from V_1 to V_2.
- $|N_G(U)| \geq |U|$ for all subsets $U \subseteq V_1$.

This theorem is called "Hall's marriage theorem" because its original formulation was in terms of matching women to men in the "marriage problem", stated as follows: There are n women and $m \geq n$ men. For each woman, there is a subset of the men, any of whom she will be happy to marry. Any man will be happy to marry. Is it possible to pair up the n women in n marriages?

Matchings are important

Numerous applications in various areas.

- Scheduling applications
  - Exam scheduling, airline scheduling, sports match scheduling, hospital service scheduling, etc.
- Applications involving matching people or things
  - Online dating apps, recruitment and job matching, etc.
- Multimodal data analysis and tracking applications
  - e.g. match geolocation with other user attributes
  - e.g. checking for consistency in census records

Note: Matchings are frequently considered in the context of bipartite graphs.

Matching algorithms

Goal: Given an input graph G, returns as output a matching M in G.

- Variant 1: Return a maximal matching M in G.
- Variant 2: Return a maximum matching M in G.

Note: Matching algorithms that return maximal matchings as outputs are usually based on greedy algorithms.

- e.g. find an unmatched vertex v, check the vertices in $N_G(v)$; if there exists an unmatched vertex $u \in N_G(v)$, insert new edge (that is incident to u and v) into the current matching.

Intuition: Maximal matchings could be "very far from" maximum matchings.

- e.g. For a cycle graph C_{3n} with $3n$ vertices, there exists a maximal matching with n edges, while in contrast, a **maximum** matching of C_{3n} must have $\lfloor \frac{3n}{2} \rfloor$ edges.

Matching algorithms (continued)

Note: Matching algorithms that return maximum matchings are usually based on the following important lemma.

Lemma Let $G = (V, E, \phi)$ be an undirected graph with no loops, and let M be a matching in G (possibly $M = \emptyset$). Let $\ell \in \mathbb{N}$, and let $(v_0, v_1, \ldots, v_{2\ell+1})$ be a sequence of pairwise adjacent vertices, such that v_0 and $v_{2\ell+1}$ are **distinct** **unmatched** vertices, and such that there exist ℓ **distinct** edges $e_1, \ldots, e_\ell$ in M satisfying the following:

- $\phi(e_i) = \{v_{2i-1}, v_{2i}\}$ for every $1 \leq i \leq \ell$ (if $\ell > 0$).

Then there exists $\ell + 1$ distinct edges $e'_0, e'_1, \ldots, e'_\ell$ in G satisfying the following:

- $\phi(e'_i) = \{v_{2i}, v_{2i+1}\}$ for every $0 \leq i \leq \ell$.
- $(M \backslash \{e_1, \ldots, e_\ell\}) \cup \{e'_0, e'_1, \ldots, e'_\ell\}$ is a matching in G.

Intuition: We can apply this lemma iteratively to construct matchings of larger and larger cardinalities.

- We would need to find the distinct unmatched vertices $v_0, v_{2\ell+1}$.
- We would need to find the edges $e_1, \ldots, e_\ell$ from the currently "available" matching.

Summary

- Exercise on adjacency matrices
- Class activity: Matching puzzle
- Hall's marriage theorem
- Discussion on matching algorithms