40.520: Stochastic Models

Assignment 2

Due date: Mar 1st, 2026 before midnight

Problem 1. Consider two independent Markov chains $(X_n : n \geq 0)$ and $(Y_n : n \geq 0)$ defined on the state space I. Suppose (X_n) has transition matrix P and (Y_n) has transition matrix Q, and their evolutions are independent. Is the process $Z_n = (X_n, Y_n)$ a Markov chain? If so, find its transition probabilities.

Problem 2. Consider a Markov chain with state space $S = \{A, B, C, D, E\}$ and the following transition rules:

- From A: Go to B with probability 1/2, go to C with probability 1/2.
- From B: Go to A with probability 1/3, go to C with probability 1/3, go to D with probability 1/3.
- From C: Go to A with probability 1/4, go to B with probability 1/4, go to D with probability 1/4, go to E with probability 1/4.
- From D: Go to B with probability 1/2, go to E with probability 1/2.
- From E: Absorbing state (once entered, never leave).

Let X_n denote the state at time $n \geq 0$.

(a) Write the transition matrix P.

(b) Let $T = \min\{n \geq 0 : X_n = E\}$ be the hitting time of E. Define $k_X = \mathbb{E}[T \mid X_0 = X]$ for $X \in \{A, B, C, D\}$ (with $k_E = 0$). Write equations for k_A, k_B, k_C, k_D and solve for k_A.

(c) Define $h_X = \mathbb{P}(\text{hit } E \text{ before returning to } A \mid X_0 = X)$ for $X \in \{B, C, D\}$. Write equations for h_B, h_C, h_D and solve them.

(d) Define $u_X = \mathbb{P}(\text{hit } D \text{ before } E \mid X_0 = X)$ for $X \in \{A, B, C\}$. Write equations for u_X and solve for u_A.

Problem 3. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

(a) $\mathbf{P}_1 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ (b) $\mathbf{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(c) $\mathbf{P}_3 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (d) $\mathbf{P}_4 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

Problem 4. Coin 1 comes up heads with probability 0.6 and coin 2 with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

(a) What proportion of flips use coin 1?

(b) If we start the process with coin 1, what is the probability that coin 2 is used on the fifth flip?

Problem 5. A flea moves around the vertices of a triangle in the following manner: Whenever it is at vertex i it moves to its clockwise neighbor vertex with probability p_i and to the counterclockwise neighbor with probability $q_i = 1 - p_i$, $i = 1, 2, 3$.

(a) Find the proportion of time that the flea is at each of the vertices.

(b) How often does the flea make a counterclockwise move that is then followed by five consecutive clockwise moves?

Problem 6. A weather system follows these rules each day:

- If it rained on **all** of the past three days, it will rain today with probability 0.8.
- If it rained on **none** of the past three days, it will rain today with probability 0.2.
- In **all other cases**, the weather today is the same as yesterday's weather with probability 0.6, and opposite with probability 0.4.

Define a Markov chain where the state represents the weather pattern of the past three days. Determine the transition probability matrix $\mathbf{P}$ for this chain.

Problem 7. Let X_n be the number of families that check into a hotel on day n, where $\{X_n\}_{n\geq 1}$ are independent Poisson(λ) random variables. Each family that checks in stays for a random number of days that follows a geometric distribution with parameter p, ($P(\text{stay} = k) = p(1-p)^{k-1}$ for $k \geq 1$), with all stay durations independent of each other and of the arrivals process. Let Y_n be the number of families present in the hotel at the *beginning* of day n (before new arrivals on day n).

(a) Show that $\{Y_n\}_{n\geq 1}$ is a Markov chain. Identify the state space and give an intuitive explanation.

(b) Given $Y_n = i$, find the distribution of Y_{n+1}. That is, determine the transition probabilities $P_{ij} = P(Y_{n+1} = j \mid Y_n = i)$.