Course 50.050/50.550
Advanced Algorithms

Week 2 – Lecture L02.01

SINGAPORE UNIVERSITY OF
TECHNOLOGY AND DESIGN

Outline of Lecture

- Pascal's identity, double counting, binomial theorem
- Pascal's triangle, basic properties of binomial coefficients
- Vandermonde's identity, hockey-stick identity
- Examples of proofs

Pascal's identity

Let $0 \leq k \leq n$ be integers, and let S be a set with n elements.

- **Recall:** The number of k-combinations of S is denoted by $\binom{n}{k}$.
  - A k-combination of S is a subset of S of cardinality k.
  - A k-combination is also sometimes called a k-subset.

Theorem (**Pascal's Identity**): Let $0 \leq k < n$ be integers. Then

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

- We can write this theorem as a statement T using logic notation:

$$\forall k \forall (n > k) \left(\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \right).$$

  - Domain of discourse: $k \in \mathbb{N}$.
  - Restricted domain of discourse: $n \in \mathbb{N}$ satisfying $n > k$.

Definition: Let $x_1, \ldots, x_k$ be variables. An identity is a proposition P of the form $\forall x_1 \forall x_2 \ldots \forall x_k \left(f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k) \right)$, where f and g are functions of $x_1, \ldots, x_k$, such that P is **true**.

- e.g. $\forall x \left(\cos^2 x + \sin^2 x = 1 \right)$; $\forall x \forall y \left((x+y)^2 = x^2 + 2xy + y^2 \right)$.
- Thus, if we can show that T is **true**, then T is an identity.

Definition: A proposition that is true is called a theorem.

- An identity is a theorem about the equality of two expressions.

Theorems and proofs

Question: How do we show that a theorem is true?

- We need a proof, i.e. a correct argument that justifies logically why the theorem is true, based on axioms and other theorems.
- **Important:** A proof should be as **clear** as possible, with all details explicitly and precisely indicated. Any intermediate statement that you declare to be true must be justified.
  - Good guideline: If you are not sure whether your proof is correct, it means your proof is not sufficiently clear!
- How do we write a proof? We need a proof method.

In 50.050, we will introduce many proof methods, starting with "double counting".

Double Counting: This is based on the idea that given a finite set S, if we (correctly) count the number of elements in S in two ways, to get two different expressions A and B for $|S|$, then $A = B$.

Interpreting Pascal's identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ in terms of sets.
Let $0 \leq k < n$ be integers. Let S be the set $\{1, 2, \ldots, n+1\}$.

- By definition, the number of $(k+1)$-combinations in S is $\binom{n+1}{k+1}$.
- **Question:** Can we count this number in another way?

Proof of Pascal's identity

Recall: A $(k+1)$-combination is a subset $A \subseteq S$ of cardinality $k+1$.

$$A \subseteq \{1, \quad 2, \quad \dots, \quad n, \quad n+1\}$$

Let's think about the task of choosing elements of A in terms of cases.
Idea: Either $n+1 \in A$, or $n+1 \notin A$.

- **Case:** $n+1 \in A$. We have to choose the remaining k elements in A from among elements in $\{1, 2, \dots, n\}$.
  - There are $\binom{n}{k}$ ways to choose these remaining k elements.
- **Case:** $n+1 \notin A$. The $k+1$ elements in A must then be elements from among $\{1, 2, \dots, n\}$.
  - There are $\binom{n}{k+1}$ ways to choose these $k+1$ elements.
- The two cases are disjoint. Hence, by the **sum rule**, the total number of ways to choose the elements of A is $\binom{n}{k} + \binom{n}{k+1}$.

Since the number of $(k+1)$-combinations of $\{1, 2, \dots, n+1\}$ is by definition $\binom{n+1}{k+1}$, we conclude that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$. $\square$

Other basic properties of binomial coefficients

Theorem: For all integers $0 \le k \le m \le n$, we have the following:

1. $\binom{n}{k} = \binom{n}{n-k}$.
2. $\binom{n}{0} = \binom{n}{n} = 1$.
3. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

Proof: Let S be the set $\{1, \ldots, n\}$.

1. (Proof by double counting.) Choosing the k elements for a k-combination $A \subseteq S$ is equivalent to choosing $n-k$ elements of S to exclude from A. The number of ways to do this is $\binom{n}{n-k}$.
2. (Proof by definition.) The only 0-combination of S is $\emptyset$. The only n-combination of S is S itself.
3. (Proof by double counting.) Consider the task of choosing an ordered pair (A, B) of disjoint subsets of S such that $|A| = k$, $|A \cup B| = m$. There are 2 ways to split this task into sub-tasks:
   - Choose m elements for $A \cup B$, then from these m elements, choose k elements for A. [Total of $\binom{n}{m}\binom{m}{k}$ ways.]
   - Choose k elements for A, then from the remaining $n-k$ elements in S, choose $m-k$ elements for B. [Total of $\binom{n}{k}\binom{n-k}{m-k}$ ways.] $\square$

Binomial coefficients and binomial theorem

Let $0 \le k \le n$ be integers. $\binom{n}{k}$ is called a **binomial** coefficient, since it appears as the coefficients of terms in the **binomial** theorem.

Binomial Theorem. Let $n \in \mathbb{N}$, and let $x, y \in \mathbb{R}$. Then

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n.$$

- We can write this theorem as a statement T using logic notation:
  $$\forall n \forall x \forall y \left((x+y)^n = \textstyle\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\right).$$
  - Domain of discourse: $n \in \mathbb{N}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.

Question: How would you prove the binomial theorem?

- *Prior to 50.050, you already know a proof method called **induction**.*

What is induction? It is used to prove propositions of the form $\forall n\, P(n)$, where P is a predicate, and the domain of discourse is $n \in \mathbb{N}$.

- Intuitively, you have a statement $P(n)$ in terms of n, and you want to show that $P(n)$ is true for all $n \in \mathbb{N}$.
- **Two steps of induction:** 1) Base case; 2) Induction step.
  1. Show that $P(0)$ is true.
  2. Show that if $P(n)$ is true for some $n \in \mathbb{N}$, then $P(n+1)$ is true.
     - i.e. show that for any $n \in \mathbb{N}$, we have "$P(n) \Rightarrow P(n+1)$".

Proof of binomial theorem by induction

We can re-write the assertion of binomial theorem as $\forall (n \in \mathbb{N})\, P(n)$, where $P(n)$ is "$\forall x \forall y\, \left((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\right)$"

- Intuitively, we want to show that the statement

  "$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n} x^0 y^n$ is true for all $x, y \in \mathbb{R}$"

  is true for all non-negative integers n.

Base case: $P(0)$ is the statement "$(x+y)^0 = \binom{0}{0} x^0 y^0$".

- **Note:** $(x+y)^0 = 1$, and $\binom{0}{0} x^0 y^0 = 1$. Thus, $P(0)$ is true.
  - Recall that $\binom{0}{0} = 1$. (It counts the single 0-combination $\emptyset$.)

Induction step: We want to show that $P(n) \Rightarrow P(n+1)$.

- i.e. we want to show the implication $P(n) \to P(n+1)$ is true.
  - **Recall:** The proposition p is called the hypothesis of $p \to q$.
  - In our induction step, $P(n)$ is called the induction hypothesis.
- Our induction hypothesis $P(n)$ (for this specified n) is the statement that "$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ is true for all $x, y \in \mathbb{R}$."
- We want to show the statement $P(n+1)$ (for the same n as before) that "$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$ is true for all $x, y \in \mathbb{R}$."

Proof of binomial theorem by induction (continued)

- By definition, $(x+y)^{n+1} = (x+y)(x+y)^n$, so using the **induction hypothesis**, we get that $(x+y)^{n+1}$ equals

$$(x+y)\Big(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\Big)$$

$$= x\Big(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\Big) + y\Big(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\Big)$$

$$= \binom{n}{0}x^{n+1} + \binom{n}{1}x^n y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}xy^n$$

$$+ \binom{n}{0}x^n y + \binom{n}{1}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^n + \binom{n}{n}y^{n+1}$$

$$= \binom{n}{0}x^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k-1} + \binom{n}{k}\right)x^{n+1-k}y^k + \binom{n}{n}y^{n+1}.$$

- By **Pascal's identity**, we have $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.
  - Note that also $\binom{n}{0} = 1 = \binom{n+1}{0}$ and $\binom{n}{n} = 1 = \binom{n+1}{n+1}$.

Conclusion: $(x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k$. $\square$

Proving binomial theorem via counting

Let $0 \leq k \leq n$ be integers. Consider the expansion of $(x+y)^n$. This expansion would look like

$$(x+y)^n = c_0 x^n y^0 + c_1 x^{n-1} y + \cdots + c_{n-1} x^1 y^{n-1} + c_n x^0 y^n,$$

where c_k is a constant representing the coefficient of $x^{n-k} y^k$.

Question: How do we compute the coefficient for $x^{n-k} y^k$?

- e.g. For $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, how do we get three x^2y terms in the expansion $(x+y)(x+y)(x+y)$?

Note: $(x+y)^n = (x+y)\cdots(x+y)$ has n factors of $(x+y)$.
Idea: As we expand $(x+y)^n$, each term corresponds to choosing either x or y from each of these n factors.

- To get a term $x^{n-k}y^k$, we need to choose 'y' from k of these n factors, and choose 'x' from the remaining $n-k$ factors.
  - There are $\binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$ ways to choose, hence $c_k = \binom{n}{k}$.

Therefore, $(x+y)^n = \sum_{k=0}^{n} c_k x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. $\square$

Why have multiple proofs for the same theorem?

We want to prove a theorem, not just because we are interested in that one theorem, but also because we want to better understand the topic of the theorem. Different proofs give different insights.

Some proofs are more intuitive than others.

- The proof of the binomial theorem via counting is more intuitive.
  - If we had stopped with the induction proof of binomial theorem, and not prove the theorem via counting, then our understanding of **exactly why** the binomial theorem is true (i.e. the intuition on why the theorem has to be true) would not be as complete.

Some proofs can be generalized. Some proofs are easier to generalize.

- A generalization of a theorem is an extension of a theorem, so the original theorem becomes a special case of the generalization.
- **Example:** Consider the following two theorems:
  - Theorem 1: For all $x, y \in \mathbb{R}$, $(x+y)^2 = x^2 + 2xy + y^2$.
  - Theorem 2: For all $x, y \in \mathbb{R}$, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
  - The binomial theorem is a (common) generalization of these two theorems.

Pascal's triangle

Idea: Write down all binomial coefficients in a triangular array.

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \boxed{\binom{4}{2}} & & \boxed{\binom{4}{3}} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \boxed{\binom{5}{3}} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} \\
\end{array}$$

$10 = 4 + 6$ $\quad$ $15 = 10 + 5$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & \boxed{4} & & \boxed{6} & & 4 & & 1 & & \\
 & 1 & & 5 & & \boxed{10} & & \boxed{10} & & \boxed{5} & & 1 & \\
1 & & 6 & & 15 & & 20 & & \boxed{15} & & 6 & & 1 \\
\end{array}$$

For each $n \in \mathbb{N}$, the n-th row has $n+1$ numbers $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

- This infinite triangular array is called Pascal's triangle.

Pascal's identity can be easily visualized in Pascal's triangle.

- Every entry is the sum of the above two entries. ("Blank" entries are 0.)

Basic properties:

- Entries in each row are symmetric.
  - Due to identity $\binom{n}{k} = \binom{n}{n-k}$.
- Entries in outer leftward/rightward diagonals are all 1's.
  - Due to $\binom{n}{0} = \binom{n}{n} = 1$.
- Entries in next leftward/rightward diagonals are consecutive integers.
  - Due to $\binom{n}{1} = \binom{n}{n-1} = n$.

More observations from Pascal's triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & \boxed{1} & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\iddots & & \iddots & & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

Patterns observed: For each $n \in \mathbb{N}$,

- Sum of entries in n-th row equals 2^n.
  - e.g. In Row 4: $1 + 4 + 6 + 4 + 1 = 16 = 2^4$.
- Alternating sum of entries in n-th row equals 0.
  - e.g. In Row 4: $1 - 4 + 6 - 4 + 1 = 0$.

Theorem: For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof: By the binomial theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ is an identity that holds for all $x, y \in \mathbb{R}$. Substituting $x = 1$, $y = 1$ into this identity, we get $2^n = \sum_{k=0}^{n} \binom{n}{k}$. $\square$

Theorem: For all $n \in \mathbb{N}$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Proof: By the binomial theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ is an identity that holds for all $x, y \in \mathbb{R}$. Substituting $x = 1$, $y = -1$ into this identity, we get $0 = (1-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$. $\square$

Hockey-stick identity

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\iddots & \iddots & \iddots & \iddots & & & & & & & \ddots & \ddots & \ddots & \ddots &
\end{array}
$$

Pattern: For all $k, m \in \mathbb{N}$,

- Sum of $\binom{k}{k}, \binom{k+1}{k}, \ldots, \binom{k+m}{k}$, i.e. the first $m+1$ entries in the k-th leftward diagonal, equals the $(m+1)$-th entry in the $(k+1)$-th leftward diagonal.
  - e.g. In 1st leftward diagonal: $\binom{1}{1} + \binom{2}{1} + \cdots + \binom{6}{1} = \binom{7}{2}$. $1+2+3+4+5+6 = 21$.

Theorem: (**Hockey-stick identity**) For all integers $0 \le k \le n$,

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Equivalently, for all $k, m \in \mathbb{N}$,

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{k+m}{k} = \binom{k+m+1}{k+1}.$$

- This identity is also called the christmas stocking identity.

Hockey-stick identity (first proof)

Hockey-stick identity: For all integers $0 \leq k \leq n$, we have

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Proof: (by induction) For $n \in \mathbb{N}$, let $P(n)$ be the statement "$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ is true for all integers $0 \leq k \leq n$."
We want to show by induction that $P(n)$ is true for all $n \in \mathbb{N}$.

- **Base case:** When $n = 0$, the only integer k satisfying $0 \leq k \leq n$ is $k = 0$, so $P(0)$ becomes the statement "$\binom{0}{0} = \binom{1}{1}$ is true", which is true since $\binom{0}{0} = \binom{1}{1} = 1$. Thus, $P(0)$ is true.
- **Induction step:** Suppose that $P(n)$ is true for some $n \in \mathbb{N}$.
  - **Goal:** Show that $P(n+1)$ is true, i.e. we want to show that $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} + \binom{n+1}{k}$ equals $\binom{n+2}{k+1}$, for all integers $0 \leq k \leq n+1$.
  - By **induction hypothesis**,
    $$\left(\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}\right) + \binom{n+1}{k} = \binom{n+1}{k+1} + \binom{n+1}{k}.$$
  - By **Pascal's identity**, $\binom{n+1}{k+1} + \binom{n+1}{k} = \binom{n+2}{k+1}$.

  Thus, $P(n) \Rightarrow P(n+1)$, which completes the induction step.

Therefore, by induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Hockey-stick identity (second proof)

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\iddots & & \iddots & & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

Hockey-stick identity: For all integers $0 \le k \le n$, $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$.

- e.g. In 1st leftward diagonal:
  $\binom{1}{1} + \binom{2}{1} + \cdots + \binom{6}{1} = \binom{7}{2}$.
  $1 + 2 + 3 + 4 + 5 + 6 = 21$.

Key Proof Idea: $\binom{k}{k} = \binom{k+1}{k+1}$.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\iddots & & \iddots & & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

Example: $n = 6$, $k = 1$. By Pascal's identity, $1 + 2 + 3 + 4 + 5 + 6$ equals

$$
\begin{aligned}
& \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} + \binom{6}{1} \\
= & \binom{2}{2} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} + \binom{6}{1} \\
= & \binom{3}{2} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} + \binom{6}{1}
\end{aligned}
$$

Hockey-stick identity (second proof continued)

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\end{array}$$

By Pascal's identity,

$$\begin{aligned} & 3 + 3 + 4 + 5 + 6 \\ = & \binom{3}{2} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} + \binom{6}{1} \\ = & \binom{4}{2} + \binom{4}{1} + \binom{5}{1} + \binom{6}{1} \end{aligned}$$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\end{array}$$

By Pascal's identity,

$$\begin{aligned} & 6 + 4 + 5 + 6 \\ = & \binom{4}{2} + \binom{4}{1} + \binom{5}{1} + \binom{6}{1} \\ = & \binom{5}{2} + \binom{5}{1} + \binom{6}{1} \end{aligned}$$

Hockey-stick identity (second proof continued)

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\iddots & & \iddots & & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

By Pascal's identity,

$$
\begin{aligned}
& 10 + 5 + 6 \\
&= \binom{5}{2} + \binom{5}{1} + \binom{6}{1} \\
&= \binom{6}{2} + \binom{6}{1}
\end{aligned}
$$

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
\iddots & & \iddots & & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

By Pascal's identity,

$$
\begin{aligned}
& 15 + 6 \\
&= \binom{6}{2} + \binom{6}{1} \\
&= \binom{7}{2}
\end{aligned}
$$

Hockey-stick identity (second proof continued)

Proof: For any given $k \in \mathbb{N}$, we want to show that
"$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ is true for all integers $n \geq k$."
From $\binom{k}{k} = \binom{k+1}{k+1}$, and an iterated use of Pascal's identity, we have

$$\begin{aligned}
& \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \binom{k+3}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} \\
= & \left[\binom{k+1}{k+1} + \binom{k+1}{k}\right] + \binom{k+2}{k} + \binom{k+3}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} \\
= & \left[\binom{k+2}{k+1} + \binom{k+2}{k}\right] + \binom{k+3}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} \\
= & \left[\binom{k+3}{k} + \binom{k+3}{k}\right] + \cdots + \binom{n-1}{k} + \binom{n}{k} \\
= & \binom{k+4}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} \\
= & \vdots \\
= & \binom{n-1}{k} + \binom{n}{k} \\
= & \binom{n+1}{k+1}
\end{aligned}$$

This iterated use of Pascal's identity works for any $n \geq k$.

Hockey-stick identity (third proof)

Hockey-stick identity: For all integers $0 \le k \le n$, we have
$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Proof: (by double counting) Let S be the set $\{1, 2, \ldots, n+1\}$. Consider the task of choosing a $(k+1)$-combination A of S.

- By definition, the number of ways to complete this task is $\binom{n+1}{k+1}$.

We shall count differently by splitting the task into the following cases:

Case 1: $n+1 \in A$.

- $\binom{n}{k}$ ways to choose rest of k elements for A from $\{1, \ldots, n\}$.

Case 2: $n+1 \notin A$ and $n \in A$.

- $\binom{n-1}{k}$ ways to choose rest of k elements for A from $\{1, \ldots, n-1\}$.

Case 3: $n+1, n \notin A$ and $n-1 \in A$.

- $\binom{n-2}{k}$ ways to choose rest of k elements for A from $\{1, \ldots, n-2\}$.

$\vdots$

Case $(n-k+1)$: $n+1, n, \ldots, k+2 \notin A$ and $k+1 \in A$.

- $\binom{k}{k}$ ways to choose rest of k elements for A from $\{1, \ldots, k\}$.

Thus, by the **sum rule**, the total number of ways is $\sum_{m=k}^{n} \binom{m}{k}$. $\square$

Mirror-image hockey-stick identity

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1 \\
\end{array}$$

Pattern: For all $k, m \in \mathbb{N}$,

- Sum of $\binom{m}{0}, \binom{m+1}{1}, \ldots, \binom{m+k}{k}$, i.e. the first $k+1$ entries in the m-th rightward diagonal, equals the $(k+1)$-th entry in the $(m+1)$-th rightward diagonal.
  - e.g. In 1st rightward diagonal: $\binom{1}{0} + \binom{2}{1} + \cdots + \binom{6}{5} = \binom{7}{5}$. $1 + 2 + 3 + 4 + 5 + 6 = 21$.

Corollary: (**Mirror-image hockey-stick identity**)
For all integers $0 \leq k \leq n$,

$$\binom{n-k}{0} + \binom{n-k+1}{1} + \cdots + \binom{n}{k} = \binom{n+1}{k}.$$

Equivalently, for all $k, m \in \mathbb{N}$,

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+k}{k} = \binom{m+k+1}{k}.$$

Proof of mirror-image hockey-stick identity

Mirror-image hockey-stick identity says that for all integers $0 \le k \le n$, we have $\binom{n-k}{0} + \binom{n-k+1}{1} + \cdots + \binom{n}{k} = \binom{n+1}{k}$.

Note: This identity is a corollary of the hockey-stick identity.

- A corollary is true proposition that is a (usually rather easy) consequence of a theorem.

Proof of corollary: The left-hand side (LHS) of the mirror-image hockey-stick identity is $\sum_{i=0}^{k} \binom{n-k+i}{i}$. By the symmetry of binomial coefficients, we know that $\binom{n-k+i}{i} = \binom{n-k+i}{(n-k+i)-i} = \binom{n-k+i}{n-k}$. Thus, the LHS equals

$$\binom{n-k}{n-k} + \binom{n-k+1}{n-k} + \binom{n-k+2}{n-k} + \cdots + \binom{n}{n-k}.$$

- By the hockey-stick identity, this sum equals $\binom{n+1}{n-k+1}$.
- By the symmetry of binomial coefficients again, we have $\binom{n+1}{n-k+1} = \binom{n+1}{(n+1)-(n-k+1)} = \binom{n+1}{k}$. $\square$

Challenge: Can you prove the mirror-image hockey-stick identity **directly via double counting**? What would you be counting?

Vandermonde's identity

Theorem: (**Vandermonde's identity**) For all $m, n, r \in \mathbb{N}$,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

- **Note:** For $p, q \in \mathbb{N}$, it follows from definition that $\binom{p}{q} = 0$ whenever $q > p$.
  - Not possible to select a q-subset from a set of cardinality p, if $q > p$.

How do we prove this identity via a counting argument?

- **Idea 1:** The product $\binom{m}{r-k}\binom{n}{k}$ suggests that we are choosing from two sets, one with m elements, and one with n elements.
  - In fact, we are choosing $r-k$ elements from the first set, and k elements from the second set, for a total of r elements.
  - We want to relate this product to the **product rule**, so we want:
    - Choosing from the first set to be the first sub-task;
    - Choosing from the second set to be the second sub-task.
- **Idea 2:** The sum over k suggests that we have multiple cases that depend on the value of k.
  - We want to relate this sum over k to the **sum rule**, so we want disjoint cases that correspond to different values of k.

Vandermonde's identity (first proof)

Vandermonde's identity: For all $m, n, r \in \mathbb{N}$, $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$.

Proof: (By double counting) Let A and B be disjoint sets such that $|A| = m$, $|B| = n$. Consider the task of choosing a subset $C \subseteq A \cup B$ of cardinality r. Note that $|C \cap B| \in \{0, 1, \ldots, r\}$, i.e. there are $r + 1$ possible values for $|C \cap B|$, so we can split into these $r + 1$ cases.

- If $|C \cap B| = k$ for some $k \in \{0, 1, \ldots, r\}$, then there are:
  - $\binom{n}{k}$ ways to choose the k elements for C from the n elements in B;
  - $\binom{m}{r-k}$ ways to choose the remaining $r - k$ elements for C from the m elements in A.
- By the **product rule**, there are a total of $\binom{n}{k}\binom{m}{r-k}$ ways to choose the r elements for C, for this case when $|C \cap B| = k$.

Thus, by the **sum rule**, the total possible number of ways to choose our subset C (across all $r + 1$ cases) is $\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$.

Finally, since $|A \cup B| = m + n$, there are $\binom{m+n}{r}$ ways to choose $C \subseteq A \cup B$ of cardinality r, therefore $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$. $\square$

Vandermonde's identity (second proof)

Vandermonde's identity: For all $m, n, r \in \mathbb{N}$, $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$.

- **Idea:** Use the binomial theorem. Treat binomial coefficients as the coefficients of terms in some expansion.
  - For all $n \in \mathbb{N}$, the binomial theorem yields $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Proof: By the binomial theorem, $(x+1)^{m+n} = \sum_{r=0}^{m+n} \binom{m+n}{r} x^r$.

- The coefficient of x^r in the expansion of $(x+1)^{m+n}$ equals $\binom{m+n}{r}$.
  - This coefficient is the LHS of Vandermonde's identity.

Next, consider the factorization $(x+1)^{m+n} = (x+1)^m (x+1)^n$.

- We know that $(x+1)^m = \sum_{i=0}^{m} \binom{m}{i} x^i$ and $(x+1)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$.
  - To get an x^r term, we need to get an x^i term from $(x+1)^m$ and an x^j term from $(x+1)^n$, such that $i + j = r$.
  - Possible values for j: $0, 1, \ldots, r$. Note: $i = r - j$.
  - Thus, coefficient of x^r in $(x+1)^m (x+1)^n$ equals $\sum_{j=0}^{r} \binom{m}{r-j}\binom{n}{j}$.

Therefore, renaming j as k, we get $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$. $\square$

General strategies for proving identities

How do we prove an identity via a counting argument?

- We always go back to basic counting principles: sum rule, product rule, k-permutations, k-combinations, k-tuples.
- **Note:** An expression, no matter how complicated, is formed by the addition of terms, and the multiplication of factors.
  - Can the addition of terms be interpreted as a use of the sum rule? Can you think of a task split into cases, where the number of ways for each case is counted by each term?
  - Can the multiplication of factors be interpreted as a use of the product rule? Can you think of a task decomposed into subtasks, where the number of ways to complete each subtask is counted by each factor?
- **Suggestion:** Get exposed to as many proofs of identities as possible (not just counting proofs), and look out for common ideas. Many identities are proven using the same "few" ideas.
  - The course textbook has several examples, and a long list of textbook exercises for your practice.
  - **Double counting** is a very powerful proof method to prove identities involving binomial coefficients!

Summary

- Pascal's identity, double counting, binomial theorem
- Pascal's triangle, basic properties of binomial coefficients
- Vandermonde's identity, hockey-stick identity
- Examples of proofs