

Course 50.050/50.550

Advanced Algorithms

Week 1 – Cohort Class L01.03



Outline of Cohort Class

- ▶ Counting methods, product rule, sum rule
- ▶ Permutations and Combinations
- ▶ Class Activity: Inclusion-exclusion principle



How many ways?

Given two shirts (S1, S2) and three pants (P1, P2, P3), how many ways can we create an outfit?



S1



S2



P1



P2



P3

Answer: $2 \times 3 = 6$ ways.



Decomposing a task into subtasks

Task: Create an outfit.

- ▶ **Sub-task 1:** Choose a shirt for the outfit.
 - ▶ We could choose from 2 different shirts S1, S2.
 - ▶ There are $n_1 = 2$ ways complete this Sub-task 1.
- ▶ **Sub-task 2:** Choose a pants for the outfit.
 - ▶ We could choose from 3 different pants P1, P2, P3.
 - ▶ There are $n_2 = 3$ ways to complete this Sub-task 2.
 - ▶ Whatever shirt we had chosen in Sub-task 1, there are always 3 ways to complete Sub-task 2.
- ▶ We would complete the task after completing both subtasks.
 - ▶ The number of ways to complete Sub-task 2 does not depend on the choice made in Sub-task 1.
- ▶ Thus the total number of outfits we can create is $n_1 n_2 = 6$.

Another Example: A class has 12 boys and 18 girls. The teacher chooses a pair consisting of 1 boy and 1 girl.

- ▶ There are a total of $12 \times 18 = 216$ possible pairs.
 - ▶ The task of choosing a pair is decomposed into two subtasks: choosing the boy of the pair, and choosing the girl of the pair.



Product rule

Suppose a task can be decomposed into a sequence of k sub-tasks.

- ▶ Suppose the number of ways to complete a sub-task **remains invariant**, regardless of what specific choices we had made (or we would make) when completing the other sub-tasks.

Suppose that

- ▶ the 1st sub-task can be completed in n_1 ways;
- ▶ the 2nd sub-task can be completed in n_2 ways;
- ▶ \vdots
- ▶ the k -th sub-task can be completed in n_k ways.

Then the **product rule** says that there are $n_1 n_2 \cdots n_k$ possible ways to complete the task.

Useful Idea: Think of the choices made for the k sub-tasks as a

k -tuple $\left(\begin{array}{c} \text{choice made} \\ \text{for sub-task 1} \end{array}, \begin{array}{c} \text{choice made} \\ \text{for sub-task 2} \end{array}, \dots, \begin{array}{c} \text{choice made} \\ \text{for sub-task } k \end{array} \right)$.

- ▶ **Product rule:** There are $n_1 n_2 \cdots n_k$ possible k -tuples.



Example 1

Suppose a home remodeling job involves purchasing appliances, plumbing and setting up electrical wires. There are 5 dealers for appliances, 12 plumbing contractors, and 9 electrical workers in the area to choose from.

How many ways are there to form a home remodeling team?

Answer:

We can think of the choices made for forming this team as a 3-tuple:

$$\left(\begin{array}{l} \text{name of chosen} \\ \text{appliance dealer} \end{array}, \begin{array}{l} \text{name of chosen} \\ \text{plumbing contractor} \end{array}, \begin{array}{l} \text{name of chosen} \\ \text{electrical worker} \end{array} \right).$$

By the product rule (for 3-tuples), we can form this team in $5 \times 12 \times 9 = 540$ ways.

Sum rule

Idea: The different ways to complete a task can be split into cases.

- ▶ **Example:** Suppose to travel from city A to city B , there are:
 - ▶ 2 ways to travel by sea,
 - ▶ 3 ways to travel by air, and
 - ▶ 4 ways to travel by land/road.

How many ways are there to travel from city A to city B ?

- ▶ **Answer:** By summing up the number of ways from the three cases (sea/air/land), there are a total of $2 + 3 + 4 = 9$ ways.

Let S be the set of all possible ways to complete a task, and suppose that S can be partitioned into k subsets.

- ▶ i.e. $S = S_1 \cup \dots \cup S_k$, and $S_i \cap S_j = \emptyset$ for all i, j such that $i \neq j$.

Then the **sum rule** says that $|S| = |S_1| + \dots + |S_k|$.

- ▶ **Note:** The sum rule does not involve sub-tasks!
 - ▶ Every element in each subset S_i represents a particular way to complete the entire task, not just some sub-task.

Permutations

Let S be a set with n objects, and consider the task of selecting k objects from S one at a time, without replacement.

- ▶ i.e. objects already selected cannot be selected subsequently.
- ▶ Outcome of task: a k -tuple $\left(\begin{smallmatrix} \text{1st object} \\ \text{selected} \end{smallmatrix}, \begin{smallmatrix} \text{2nd object} \\ \text{selected} \end{smallmatrix}, \dots, \begin{smallmatrix} \text{kth object} \\ \text{selected} \end{smallmatrix} \right)$.
- ▶ When selecting distinguishable objects, **the order matters!**
 - ▶ If we select object A first, followed by object B , that is not the same as selecting object B first, followed by object A .

Definition: Let S be a set, and let $k \in \mathbb{N}$. A k -permutation of S is a k -tuple with no repeated entries, whose entries are elements of S .

- ▶ A k -permutation is also called a **permutation of S of size k** .
- ▶ **Be careful!** Writing “permutation of S ” without specifying size usually means “permutation of S of size $|S|$ ”.
- ▶ **Notation:** If $|S| = n \in \mathbb{N}$, then the number of k -permutations of S is denoted by $P(n, k)$ (following notation from course textbook).
 - ▶ **Warning:** No fixed convention for denoting permutations.
 - ▶ Other possible notation used elsewhere: $P_{n,k}$, $P_{k,n}$, P_k^n .

- ▶ **Note:** $P(n, 0) = 1$ for all $n \in \mathbb{N}$ (including the case $n = 0$).

Question: Can you see how to calculate $P(n, k)$?



Example 2

In a group of 10 students, how many ways are there to select a list of 5 students, say for the five positions of “president”, “vice-president”, “secretary”, “treasurer”, and “logistics head”?

Answer: $P(10, 5) = 10 \times 9 \times 8 \times 7 \times 6 = 30240$.

There are 10 ways to select the 1st student (as president).

There are 9 ways to select the 2nd student (as vice-president) from the remaining 9 students.

- ▶ Regardless of who the 1st student is, there are always 9 ways to select the 2nd student.

There are 8 ways to select the 3rd student (as secretary) from the remaining 8 students.

- ▶ Again, regardless of who the 1st and 2nd students are, there are always 8 ways to select the 3rd student.

So on and so forth.



Exercise 1 (10 mins)

There are 6 students: A, B, C, D, E and F.

Suppose we want to select a list of 4 students among these 6 students, for four **different** positions.

1. How many ways are there to select a list of 4 students?
2. How many ways are there to select a list of 4 students, such that student A is the second student of the list?
3. How many ways are there to select a list of 4 students, such that student A is one of the last two students of the list?

Combination

Definition: Let S be a set, and let $k \in \mathbb{N}$. A k -combination of S is a subset of S of cardinality k .

- ▶ A combination of S is a k -combination for some $k \in \mathbb{N}$.
- ▶ **Notation:** If $|S| = n \in \mathbb{N}$, then the number of k -combinations of S is denoted by $C(n, k)$ or $\binom{n}{k}$.
 - ▶ When the notation $\binom{n}{k}$ is used, this number is called a **binomial coefficient**.
 - ▶ $\binom{n}{k}$ is usually read as “ n choose k ”.
 - ▶ We shall discuss some identities of binomial coefficients in Week 2.
- ▶ **Note:** $\binom{n}{0} = 1$ for all $n \in \mathbb{N}$ (including the case $n = 0$).

Formula to calculate binomial coefficients:

$$\binom{n}{k} = \frac{P(n, k)}{k!} = \frac{n!}{k!(n-k)!}$$

The number of k -combinations is the number of corresponding k -permutations disregarding the $k!$ different ways to order k entries.



Example 3

How many ways are there to select a list of 6 students, for six different positions, from a set of 10 students?

How many ways are there to select a list of 6 students, for six different positions, from a set of 6 students?

How many ways are there to select 6 students from a set of 10 students, to form a student committee?

- ▶ Assume that all members of a student committee have the same role called “member”.

Example 4

There are 10 students: A, B, C, D, E, F, G, H, I and J.

Suppose we want to select 4 students among these 10 students to form a student committee.

1. How many ways are there to select a committee of 4 students?
2. How many ways are there to select a committee of 4 students, such that student A is in the committee?
3. How many ways are there to select a committee of 4 students, such that both student A and student B are in the committee?



Example 4 - Solution

1. The number of possible ways is $\binom{10}{4} = \frac{P(10,4)}{4!} = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 210$.
▶ **Question:** Can you see why $\binom{10}{4}$ and $\binom{10}{6}$ have the same value?
2. If student A must be in the committee, then there are $\binom{9}{3} = \frac{P(9,3)}{3!} = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84$ ways to choose the remaining three students in the committee.
3. If student A and student B must both be in the committee, then there are $\binom{8}{2} = \frac{P(8,2)}{2!} = \frac{8 \times 7}{1 \times 2} = 28$ ways to choose the remaining two students in the committee.



Exercise 2 (10 mins)

Twelve different books, consisting of 5 math books, 4 science books, and 3 history books are arranged in order in a row on a bookshelf. Find the number of arrangements in each of the following cases:

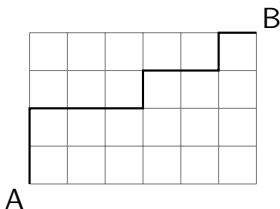
1. There are no restriction.
2. The books of each type must be together.
3. The math books must be together.

Note: Here, an arrangement means the placement of the twelve books in a row on the bookshelf, from left to right. If you change the order of the books, you get a different arrangement.

Example 5

There is an ant that wants to crawl from corner A to corner B via the line segments of the rectangular grid given below.

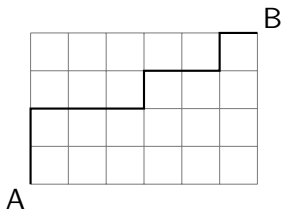
- ▶ One possible shortest path from A to B has been shown in bold.



Question: How many possible shortest paths are there from A to B?



Example 5 - Solution



Key Idea: Any shortest path from A to B has 4 upwards line segments and 6 rightwards line segments, for a total of 10 line segments.

- ▶ We can represent a shortest path by a 10-tuple whose entries are “→” or “↑”.
 - ▶ e.g. $(\uparrow, \uparrow, \rightarrow, \rightarrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$ for the path depicted above.
- ▶ We can equivalently represent a shortest path by a 10-tuple whose entries are 0 or 1.
 - ▶ e.g. $(1, 1, 0, 0, 0, 1, 0, 0, 1, 0)$ for the path depicted above.
- ▶ **Consequence:** Any shortest path is represented by a 10-tuple with six 0 entries and four 1 entries.
 - ▶ To choose one such 10-tuple, we have to choose four positions in the 10-tuple to place our 1 entries.

Thus, the total number of shortest paths from A to B is $\binom{10}{4} = 210$.



How to think?

It takes a lot of practice to extract the “mathematical core” beneath a physical description, which may be covered by various kinds of fancy stories.

To count the number of outcomes satisfying certain given conditions, think of how you can reinterpret the condition in terms of **k-tuples**, **k-permutations**, and/or **k-combinations**, and how you can use the **product rule** and/or the **sum rule**.

Try to convince your friends that your counting method is correct. If you can do so, then you are probably correct.

Remember, there could be more than one method to count!

Exercise 3 (15 mins)

At a restaurant, there are three types of sandwiches: beef (B), chicken (C), and fish (F). John wishes to place an order of 6 sandwiches. Assuming that there is no limit to the supply of each type of sandwich, how many different possible orders can John place?

Note 1: An order of 2 beef sandwiches, 3 chicken sandwiches, and 1 fish sandwich, would be the same as an order of 3 chicken sandwiches, 1 fish sandwich, and 2 beef sandwiches.

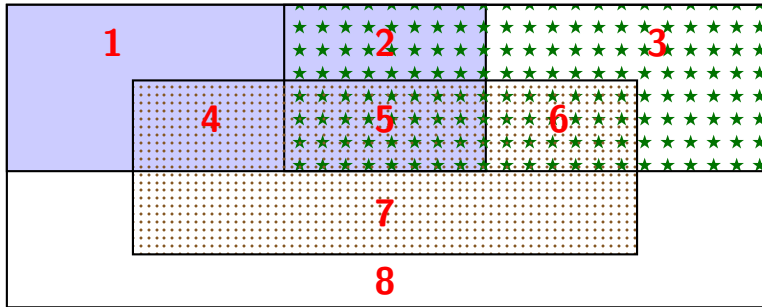
Note 2: An order of 2 beef sandwiches, 1 chicken sandwich, 2 beef sandwiches, and 1 chicken sandwich, would be the same as an order of 4 beef sandwiches and 2 chicken sandwiches.

Note 3: An order does not have to include sandwiches of all types. An order could have only one type of sandwich, or two types of sandwiches.

Class Activity (15 mins)

It's time to get up!

Front whiteboard

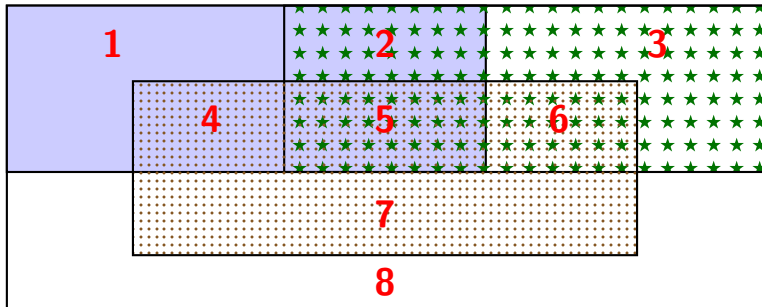


There are a total of 8 zones. Place your avatar (to be provided) on your zone, according to the following criteria:

- ▶ **Blue Zones:** Your birthday falls within 1 January–30 June.
- ▶ **Star Zones:** You have been to Macdonalds in the past month.
- ▶ **Dotted Zones:** You have an elder brother or elder sister.



Class Activity (15 mins)



Question: How many of you are born in the first half of the year, **or** have been to Macdonalds in the past month, **or** have an elder brother or elder sister? (i.e. How many of you are in zones **1–7**?)

- ▶ First, let's count the number of students in blue zones, star zones, and dotted zones.
- ▶ Next, let's count the number of students in blue star zones, blue dotted zones, and star dotted zones.
- ▶ What about the number of students in all three zones?



Summary

- ▶ Counting methods, product rule, sum rule
- ▶ Permutations and Combinations
- ▶ Class Activity: Inclusion-exclusion principle