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Advanced Algorithms H-W/2.

1. Base case: a_0, a_1
Given → $a_0 = 5$, $a_1 = 4$.

$a_n = 2(5^n) + 3(-2)^n$

$\therefore a_0 = 2(5^0) + 3(-2)^0 = 2 + 3 = 5$.

$a_1 = 2(5^1) + 3(-2)^1 = 10 - 6 = 4$.

Now, we use the technique of induction to prove that the formula holds $\forall n \in \mathbb{N}$.

$a_n = 3a_{n-1} + 10a_{n-2}$

we need $(n-1)^{th}$ term & $(n-2)^{th}$ term for proof.

$a_{n-1} = 2(5^{n-1}) + 3(-2)^{n-1}$
$a_{n-2} = 2(5^{n-2}) + 3(-2)^{n-2}$

→ we must use the recursion formula to check if (i) is correct or NOT.

$\therefore a_n = 2(5^n) + 3(-2)^n$ ....(i)

$\therefore a_n = 3a_{n-1} + 10a_{n-2}$.

$= 3\left[2(5^{n-1}) + 3(-2)^{n-1}\right] + 10\left[2(5^{n-2}) + 3(-2)^{n-2}\right]$

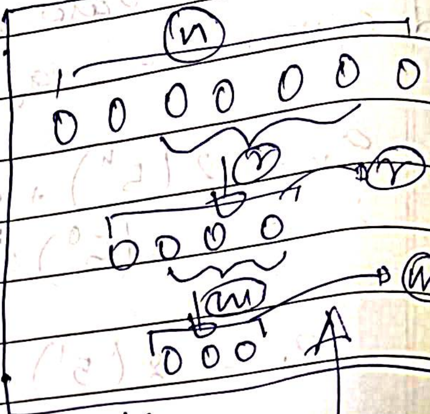
$= 5^{n-2}[30 + 20] + (-2)^{n-2}[-18 + 30]$.

$= 5^{n-2} \times 5^2 \times 2 + (-2)^{(n-2)} \cdot (-2)^2 \cdot 3$.

$= \boxed{2 \times 5^n + 3 \times (-2)^n}$ ---- (proved).

(2.) RTP:

$$\sum_{m}^{n} \binom{n}{r}\binom{r}{m} = 2^{n-m}\binom{n}{m}$$



Proof ⟨Double counting⟩

(I.) $m < r < n$ [Hard constraint].

"Technically" the expression is to count all the possible permutations of the following action:

" select 'r' items amongst 'n' items, out of the already selected 'r' items you can choose 'm' unique items. 'n' & 'm' are predefined natural numbers, with 'n' being necessarily greater than 'm'. 'r' must be at least 'm' & at max 'n' ".

(II) $m < r < n$

$0 < (r-m) < n-m$.

λ

Let's take (this) to be a discrete variable $\lambda(r)$ $(\lambda > 0)$.

$\therefore$ $\boxed{\lambda < n-m}$. λ has no lower bound, which means λ can go from 0 to '$n-m$'. This reduces the problem to all combinations of λ, going from 0 to $(n-m)$ which gives us simply 2^{n-m}. Now since r part was resolved, we still need to account for the 'm' which can be given by $\binom{n}{m}$ or ${}^{n}C_{m}$.

$$\therefore \boxed{\sum_{m}^{n} \binom{n}{r}\binom{r}{m} = 2^{n-m} \cdot \binom{n}{m}}.$$

Formal proof: Vandermonde's identity.

RTP: $$\sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} = \sum_{r=m}^{n} \binom{n}{m}\binom{n-m}{r-m}.$$

LHS ⇒ Dropping the sum for now

the general term expansion is:

$$(i) \ldots \binom{n}{r}\binom{r}{m} = \frac{n!}{\cancel{r!}\,(n-r)!} \cdot \frac{\cancel{r!}}{m!\,(r-m)!}$$

$$= \frac{n!}{(n-r)!\,(r-m)!\,m!}.$$

RHS ⇒ Again dropping the sum, the general term:

$$\binom{n}{m}\binom{n-m}{r-m} \ldots (ii).$$

$$= \frac{n!}{m!\,\cancel{(n-m)!}} \cdot \frac{\cancel{(n-m)!}}{(r-m)!\,(n-\cancel{m}-r+\cancel{m})!}$$

$$= \frac{n!}{m!\,(r-m)!\,(n-r)!}$$

∴ (i) & (ii) are equal.

$$\therefore \sum_{r=m}^{n} \binom{n}{r}\binom{r}{m} = \boxed{\sum_{r=m}^{n} \binom{n}{m}\binom{n-m}{r-m}}.$$

evaluating RHS ⟶

$$\binom{n}{m}\sum_{r=m}^{n}\binom{n-m}{r-m}$$

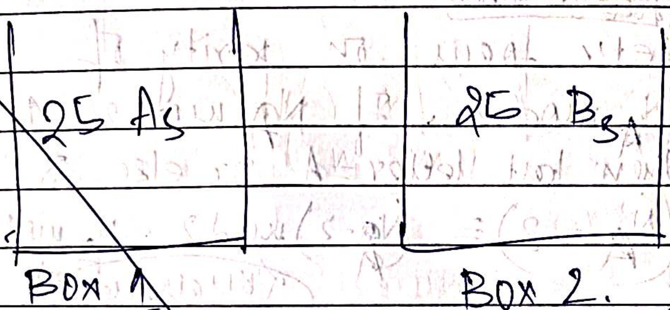
$\left\{ m < r < n \right.$
$0 < r-m < n-m$
$\left. 0 < K < n-m \right\}$

let $r-m = K$, K goes $(0 \longrightarrow n-m)$

$$\therefore \binom{n}{m}\sum_{K=0}^{n-m}\binom{n-m}{K}$$

$$\Rightarrow \boxed{\binom{n}{m} 2^{n-m}} \text{ (proved).}$$

(B)



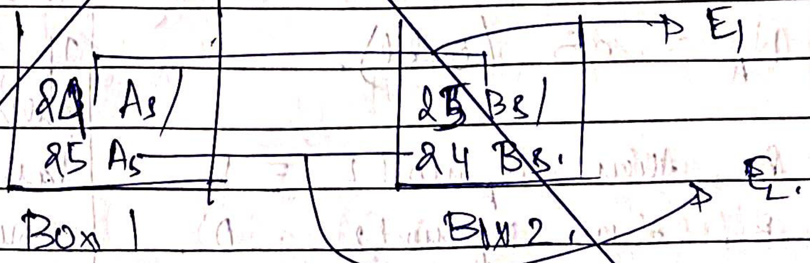
Box 1 $\longrightarrow$ A_s.
Box 2 $\longrightarrow$ B_s.

Pick two letters $\longrightarrow$ AA (I), AB (x2) (II), BB (III).

Iteration 1 $\longrightarrow$ Case I / III

$P(E_1) = \left(\frac{1}{2}\right)^2$.

$P(E_2) = \left(\frac{1}{2}\right)^2$.



(3) unequal → write A. } parity problem.
equal → write B.

$x \oplus y$ $\rightarrow$ A → 1. B → 0.
Operation.

XOR.

$\therefore$ write $= f(x, y) = x \oplus y$

x → item erased first
y → item erased second

$N_A = 25.$
$N_B = 25.$

Let's focus on parity of $N_A \bmod 2$. If $N_A \bmod 2 = 1$ then last letter = A or else B.

Cases

(i.) AA → $(N_A - 2) \equiv (N_A - 2) \bmod 2 = N_A \bmod 2$ (unchanged)

(ii.) BB → $N_A \bmod 2$ (unchanged)

(iii.) AB → $(N_A - 1) + 1$ [given according to question] ⇒ (unchanged).

(iii.) BA ≅ AB → $N_A \bmod 2$ (unchanged)

Therefore, N_A is invariant to any sequence of events.

$\therefore N_A = 25$ (odd).

* If A, then $N_A \bmod 2 = 1$
* If B, then $N_A \bmod 2 = 0$.

NOTE:
If N_A was even, then, the last letter would be B.

A → 1, B → 0.

$\underbrace{(1 \oplus 1 \oplus 1 \oplus 1 \oplus \ldots)}_{25 \text{ times}} \oplus \underbrace{(0 \oplus 0 \oplus 0 \oplus 0 \ldots)}_{25 \text{ times}}$

→ 1 → A → last element (ans) will be Always A.

4 10 castles, 500 immortal warriors.

$500 = N_1 + N_2 + N_3 \cdots + N_{10}$.

→ N_i has no obligation to be equal to N_j. $(i \neq j)$

$N_i \longrightarrow N_j$ (relocation)

→ condition $N_i - 1 \geq N_j + 1$.

$\therefore N_i - N_j \geq 2$.

$\therefore \frac{N_i - N_j}{2} \geq 1$.

Therefore relocation can happen only when their is an unequal distribution of the soldier.

Lets define $\Phi = \sum_1 N_i^2$.

& $S = \sum_1 N_i$

Now $S = 500$ (fixed).

Φ be their nonzero function. Φ_n be the value at the n^{th} relocation event.

$\therefore \Phi_{n+1} = \sum_1 N_k^2 + (N_i+1)^2 + (N_j-1)^2$

$\Phi_n = \sum_1 N_i^2$

$\therefore \Phi_{n+1} - \Phi_n = (N_i+1)^2 + (N_j-1)^2 - N_i^2 - N_j^2$.

$= 2N_i + 1 - 2N_j + 1 = 2(N_i - N_j + 1)$.

we know $N_i - N_j \geq 2$.

$2(N_i - N_j) + 2 \geq 6$.

$\therefore \Phi$ is a strictly decreasing function for every relocation event.

$\therefore$ This relocation sequence must terminate @ a configuration where relocation is impossible.

This statement can be true iff $N_i = N_j$ $\forall \{i, j\} \in \{1,2,3,4,5,6,7,8,9,10\}$.

(5) Consider the procedure fuse(n, m). Let $m, n \in \mathbb{Z}_{\geq 0}$. The algorithm initializes $A \leftarrow \emptyset$, $a \leftarrow 2n$, $b \leftarrow m$, then iterates while $a > 1$: set $a \leftarrow \lfloor a/2 \rfloor$, if a is odd append b to A, set $b \leftarrow 2b$. Finally it returns $\sum A$.

(i) Closed form of returned value: mn.

fuse$(m, n) = mn$.

(ii) Proof by loop invariant.

Let $S = \sum A$ denote the sum of the elements currently stored in array A. We prove that algorithm returns mn by exhibiting an invariant that exactly tracks the missing contribution to the product.

Loop invariant:

(*) predicate $\longrightarrow \mathcal{J}(a, b, S) \Rightarrow mn = S + \lfloor a/2 \rfloor b$.

(*) Initialization $\longrightarrow$ initially $A = \emptyset \Rightarrow S = 0$, $a = 2n$, $b = m$

Then, $S + \lfloor a/2 \rfloor b = 0 + \lfloor \frac{2n}{2} \rfloor m = mn$ {Satisfied}.

(*) Maintenance:

Case 1: a is even $\therefore \lfloor a/2 \rfloor = a/2$

$\therefore s + \lfloor a/2 \rfloor b = s + (a/\cancel{2})(\cancel{2}b) = s + ab.$

Case 2: a is odd $\therefore \lfloor a/2 \rfloor = \frac{a-1}{2}.$

$\therefore s + \lfloor a/2 \rfloor b = (s+b) + \left(\frac{a-1}{\cancel{2}}\right)(\cancel{2}b)$

$= s + b[\cancel{1} + (a - \cancel{1})] = s + ab.$

(satisfied).

(*) Termination & correctness

Loop terminates when $a < 1$. At termination, a is exactly 1.

$\therefore mn = s + \cancel{\lfloor 1/2 \rfloor b}^{0} = s.$

$\therefore$ Algorithm returns s. Returned value equals mn.