40.520: Stochastic Models

Assignment 1

Due date: Feb 8th, 2026 before midnight

Problem 1. Stores A, B and C have 50, 75, and 100 employees, and, respectively, 50, 60, and 70 percent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in store C?

Problem 2. On a television game show, there are three boxes. Inside one of the boxes there is a check for $10,000. If you pick the box that contains the check, you keep the money. Suppose you pick one of the boxes at random. The host of the game opens one of the other boxes and reveals that it is empty. The host then offers you the chance to change your pick. Should you change your pick? If so, why? If not, why not?

Problem 3. This problem has a young Billy and an older and smarter Billy.

(a) Every morning, rain or shine, young Billy Gates can be found at the entrance to the metro, hawking copies of "The Morningstar." Demand for newspapers varies from day to day, but Billy's regular early morning haul yields him 200 copies. He purchases these copies for $1 per paper, and sells them for $1.50 apiece. Billy goes home at the end of the morning, or earlier if he sells out. He can return unsold papers to the distributor for $0.50 apiece. From experience, Billy knows that demand for newspapers on any given morning is uniformly distributed between 150 and 250, where each of the possible values 150, 151, ..., 250 is equally likely. What is the expected value of Billy's net earnings on any given day?

(b) Older Billy is still a newsboy, and has the choice of purchasing any number of newspaper copies as he wishes. What is the optimal number of newspaper copies that Billy should purchase to maximize his expected net earnings on any given day? Is it greater or less than 200?

Problem 4. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1, \ldots, X\}$. Call this second number Y.

(a) Find the joint mass function of X and Y.

(b) Find the conditional mass function of X given that $Y = i$. Do it for $i = 1, 2, 3, 4, 5$.

(c) Are X and Y independent? Why?

Problem 5 (Reinforced Urn with Random Resets). An urn initially contains 1 black ball and 1 white ball. At each discrete time step:

1. **Draw:** A ball is selected uniformly at random from the urn.
2. **Update:**
   - If the ball is black:
     - With probability $\frac{1}{2}$, add 0 new black balls.
     - With probability $\frac{1}{2}$, add 1 new black ball.
   - If the ball is white, remove that white ball from the urn.
3. **Reset:** After the update, with probability $\frac{1}{4}$ (independent of everything else), all balls are discarded and the urn returns to the initial state of 1 black and 1 white.

Let T denote the first time the urn contains no white balls.

(a) Provide a recursive for formulae to compute $E[T]$ (you do not need to solve for it).

Hint: define $E(b, w)$ as the expected number of steps to reach zero white balls starting with b black and w white balls.

(b) Compute $E[T] = E(1, 1)$ by solving the resulting system of equations.
Note: You may close the system by approximating $E(4, 1) \approx E(3, 1)$.

Problem 6 (Weak Law of Large Numbers). Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with finite mean $E[X]$ and finite variance σ^2. Your goal is to prove the Weak Law of Large Numbers:

$$\forall \epsilon > 0, \quad \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - E[X]\right| > \epsilon\right) = 0,$$

where $S_n = \sum_{i=1}^{n} X_i$.

(a) Start out by proving Markov's Inequality, which says: If X is non-negative then

$$P\{X > t\} \leq \frac{E[X]}{t}, \quad \forall t > 0.$$

(b) Now use Markov's Inequality to prove Chebyshev's Inequality, which says: Let Y be a random variable with finite mean $E[Y]$ and finite variance σ_Y^2. Then

$$P\{|Y - E[Y]| \geq t\} \leq \frac{\sigma_Y^2}{t^2}.$$

(c) Finally use Chebyshev's Inequality to prove the Weak Law of Large Numbers.