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Assignment 2 — Stochastic Models.

1.) X_n, Y_n are independent & evolve independently.

$Z_n = (X_n, Y_n)$ is Markov on state space $I \times I$.

$$P(Z_{n+1} = (i,j) \mid Z_n = (i,j), Z_{n-1} = (i,j) \ldots) = P(X_{n+1} = i \mid X_n = i \ldots) \times P(Y_{n+1} = j \mid Y_n = j, Y_{n-1} = j \ldots)$$

Because $\underbrace{P((X_{n+1}, Y_{n+1}) = (i,j) \mid (X_n, Y_n) \ldots)}_{} = P(X_{n+1} \mid X_n) \cdot P(Y_{n+1} \mid Y_n)$

Joint probability is independent.

$\therefore$ Transition matrix for $Z_n = T_{ij}$.

$$\therefore \boxed{T_{ij} = P_{ii}\, Q_{jj}}$$

2.) (a)

$$T = \begin{array}{c|ccccc} & A & B & C & D & E \\ A & 0 & 1/2 & 1/2 & 0 & 0 \\ B & 1/3 & 0 & 1/3 & 1/3 & 0 \\ C & 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ D & 0 & 1/2 & 0 & 0 & 1/2 \\ E & 0 & 0 & 0 & 0 & 1 \end{array} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Hitting time $T = \min(n \geq 0 : X_n = E)$.

Let $K_x = E[T \mid X_0 = x]$ for $x \in \{A, B, C, D\}$, with $K_E = 0$.

$K_A = 1 + \frac{1}{2} K_B + \frac{1}{2} K_C$.

$K_B = 1 + \frac{1}{3} K_A + \frac{1}{3} K_C + \frac{1}{3} K_D$.

$K_C = 1 + \frac{1}{4} K_A + \frac{1}{4} K_B + \frac{1}{4} K_D + \cancel{\frac{1}{4} K_E}$

$\boxed{K_D = 1 + \frac{1}{2} K_B + \cancel{\frac{1}{2} K_E}}$

$K_E = 0$.

$2K_D = 2 + K_B$ ---- (i).

$K_C = 1 + \frac{K_A}{4} + \frac{K_B}{4} + \frac{K_D}{4}$.

$K_B = 1 + \frac{K_A}{3} + \frac{K_C}{3} + \frac{K_D}{3}$.

$K_A = 1 + \frac{K_B}{2} + \frac{K_C}{2}$.

$$K_A - \frac{K_B}{2} - \frac{K_C}{2} = 1$$
$$-\frac{K_A}{3} + K_B - \frac{K_C}{3} - \frac{K_D}{2} = 1.$$
$$-\frac{K_A}{4} - \frac{K_B}{4} + K_C - \frac{K_D}{4} = 1.$$
$$0 - \frac{K_B}{2} + 0 \quad K_D = 1$$

Gauss elimina^n.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & 1 \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & 1 \\ 0 & -\frac{1}{2} & 0 & 1 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 1+(-\frac{1}{6}) & -\frac{1}{3}+(-\frac{1}{6}) & -\frac{1}{3} & 1+\frac{1}{3} \\ 0 & -\frac{1}{4}+(-\frac{1}{8}) & 1+(-\frac{1}{8}) & -\frac{1}{4} & 1+\frac{1}{4} \\ 0 & -\frac{1}{2}+0 & 0+0 & 1 & 1+0 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & \frac{5}{6} & -\frac{1}{2} & -\frac{1}{3} & \frac{4}{3} \\ 0 & -\frac{3}{8} & \frac{7}{8} & -\frac{1}{4} & \frac{5}{4} \\ 0 & -\frac{1}{2} & 0 & 1 & 1 \end{array}\right] \end{array} \Rightarrow$$

$$\begin{array}{c} \begin{array}{cccc} A & B & B & D \end{array} \\ \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & \frac{13}{12}\ \frac{26}{24} & \frac{-10}{12} & \frac{[illegible]}{12} \\ 0 & 0 & -\frac{1}{2} & 1 & 1 \end{array}\right] \end{array} \Leftarrow \begin{array}{c} \begin{array}{cccc} A & C & B & D \end{array} \\ \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & -\frac{3}{8}+(\frac{35}{24}) & -\frac{1}{4}+(-\frac{7}{12}) & \frac{5}{4}-([illegible]) \\ 0 & 0 & -\frac{1}{2} & 1 & 1 \end{array}\right] \end{array}$$

$$\Downarrow$$

$$\begin{array}{c} \begin{array}{cccc} A & C & B & D \end{array} \\ \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & \frac{13}{12} & \frac{-10}{12} & -\frac{1}{2} \\ 0 & 0 & 0 & 1+\left(\frac{-10}{12} \times \frac{6}{13}\right) & 1+\left(-\frac{1}{2} \times \frac{6}{13}\right) \end{array}\right] \end{array}$$

$\frac{-21+15}{24}$

$\frac{-6}{12}$

$\frac{1}{12} \times 12 = \frac{1}{12}$

$$\begin{array}{c} \begin{array}{cccc} A & C & B & D \end{array} \\ \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & \frac{13}{12} & -\frac{10}{12} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{8}{13} & \frac{10}{13} \end{array}\right] \end{array}$$

(i) $\frac{8}{13} k_D = \frac{10}{13} \Rightarrow k_D = \frac{5}{4}$

(ii) $\frac{13}{12} k_B - \frac{10}{12} k_D = -\frac{1}{2}$

$\frac{13}{12} k_B = \frac{10}{12} \times \frac{5}{4} - \frac{1}{2}$

$\frac{13}{12} k_B = \frac{25 - 12}{24} \Rightarrow k_B = \frac{1}{2}$

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & -\frac{3}{5} & -\frac{2}{5} & \frac{8}{5} \\ 0 & -\frac{3}{8} & \frac{7}{8} & -\frac{1}{4} & \frac{5}{4} \\ 0 & -\frac{1}{2} & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{9}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{2}{5} & \frac{8}{5} \\ 0 & 0 & \frac{13}{20} & -\frac{2}{5} & \frac{37}{20} \\ 0 & 0 & -\frac{3}{10} & \frac{4}{5} & \frac{9}{5} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{113}{16} \\ 0 & 1 & 0 & 0 & \frac{53}{8} \\ 0 & 0 & 1 & 0 & \frac{11}{2} \\ 0 & 0 & 0 & 1 & \frac{69}{16} \end{array}\right] \Leftarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{9}{13} & \frac{53}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & \frac{43}{13} \\ 0 & 0 & 1 & -\frac{8}{13} & \frac{37}{13} \\ 0 & 0 & 0 & \frac{8}{13} & \frac{69}{26} \end{array}\right]$$

(ans)

$\therefore \boxed{K_A = \frac{113}{16}}$, $K_B = \frac{53}{8}$, $K_C = \frac{11}{2}$, $K_D = \frac{69}{16}$

(c) $h_x = P(\text{hit } E \text{ before returning to } A \mid X_0 = x)$, for $x \in \{B, C, D\}$

Find h_B, h_C, h_D.

$h_A = 0$ $\quad h_E = 0$

Therefore, $hit(E) = 1$, $hit(A) = 0$.

$\therefore h_B = \frac{1}{3}\cdot 0 + \frac{1}{3} h_C + \frac{1}{3} h_D.$

$h_C = \frac{1}{4}\cdot 0 + \frac{1}{4} h_B + \frac{1}{4} h_D + \frac{1}{4}\cdot 1 \rightarrow h_E.$

$h_D = \frac{1}{2} h_B + \frac{1}{2}\cdot 1.$

$\therefore h_B - \frac{h_C}{3} - \frac{h_D}{3} = 0.$ $\quad 3h_B - h_C - h_D = 0.$

$-\frac{h_B}{4} + h_C - \frac{h_D}{4} = \frac{1}{4}.$ $\Rightarrow -h_B + 4h_C - h_D = 1.$

$-\frac{h_B}{2} + 0 + h_D = \frac{1}{2}.$ $\quad -h_B + 0 + 2h_D = 1.$

$$\begin{array}{ccc|c} C & D & B & \\ \end{array}\left[\begin{array}{ccc|c} -1 & -1 & -3 & 0 \\ 4 & -1 & -1 & 1 \\ 0 & 2 & -1 & 1 \end{array}\right] \leftarrow \begin{array}{ccc} B & C & D \end{array}\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -1 & 4 & -1 & 1 \\ -1 & 0 & 2 & 1 \end{array}\right]$$

$$\begin{array}{ccc} C & D & B \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1-(4\cdot 1) & -1+(4\cdot 3) & 1-(4\cdot 0) \\ 0 & 2+(0) & -1+0 & 1 \end{array}\right] \rightarrow \begin{array}{ccc} C & D & B \end{array}\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -5 & 11 & 1 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$\begin{array}{ccc} C & D & B \end{array}\left[\begin{array}{ccc|c} 1 & 0 & -3-(-\frac{11}{5}) & 0-(-\frac{1}{5}) \\ 0 & 1 & -\frac{11}{5} & -\frac{1}{5} \\ 0 & 0 & -1+(2\cdot\frac{11}{5}) & 1+(2\times -\frac{1}{5}) \end{array}\right] \leftarrow \begin{array}{ccc} C & D & B \end{array}\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -\frac{11}{5} & -\frac{1}{5} \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$\begin{array}{ccc} C & D & B \end{array}\left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{11}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{17}{5} & \frac{4}{5} \end{array}\right] \rightarrow \begin{array}{ccc} C & D & B \end{array}\left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{11}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{4}{17} \end{array}\right] \rightarrow \begin{array}{ccc} C & D & B \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{5}+(\frac{4}{5}\times\frac{4}{17}) \\ 0 & 1 & 0 & -\frac{1}{5}+(\frac{4}{17}\times\frac{11}{5}) \\ 0 & 0 & 1 & \frac{4}{17} \end{array}\right]$$

$u_B = 7/17$, $u_D = -\frac{1}{5} + \frac{77}{85}$, $u_C = \frac{1}{5} + \left(\frac{28}{85}\right)$

$u_B = 7/17$, $u_D = \frac{\cancel{60}^{12}}{\cancel{85}_{17}}$, $u_C = \frac{\cancel{45}^{9}}{\cancel{85}_{17}}$.

$$\boxed{u_B = 7/17, \quad u_D = \frac{12}{17}, \quad u_C = 9/17.}$$

(d) $u_x = P(\text{hit } D \text{ before sinking to } E \mid X_0 = x)$.

$x \in \{A, B, C\}$, Bdry vals $\Rightarrow$ $u_D = 1$, $u_E = 0$.

$u_A = \frac{u_B}{2} + \frac{u_C}{2}$.

$u_B = \frac{u_A}{3} + \frac{u_C}{3} + \frac{1}{3} \cdot \underbrace{1}_{u_D}$

$u_C = \frac{u_A}{4} + \frac{u_B}{4} + \frac{1}{4} \cdot \underbrace{1}_{u_D} + \frac{1}{4} \cdot \underbrace{0}_{u_E}$.

$$\left.\begin{aligned} 2u_A - u_B - u_C &= 0 \\ u_A - 3u_B + u_C &= -1 \\ u_A + u_B - 4u_C &= -1 \end{aligned}\right\} \begin{array}{ccc} A & B & C \end{array} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -3 & 1 & -1 \\ 1 & 1 & -4 & -1 \end{array}\right].$$

$$\begin{array}{ccc} A & B & C \end{array} \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & -5/2 & 3/2 & -1 \\ 0 & 3/2 & -7/2 & -1 \end{array}\right] \leftarrow \begin{array}{ccc} A & B & C \end{array} \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 1 & -3 & 1 & -1 \\ 1 & 1 & -4 & -1 \end{array}\right].$$

$$\begin{array}{ccc} A & B & C \end{array} \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -3/5 & 2/5 \\ 0 & 3/2 & -7/2 & -1 \end{array}\right] \to \begin{array}{ccc} A & B & C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2}+\left(\frac{3}{10}\right) & 0+\left(\frac{1}{2}\cdot\frac{2}{5}\right) \\ 0 & 1 & -3/5 & 2/5 \\ 0 & 0 & -\frac{7}{2}+\left(\frac{9}{10}\right) & -1-\left(\frac{3}{2}\cdot\frac{2}{5}\right) \end{array}\right] \to \begin{array}{ccc} A & B & C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{8}{10} & 1/5 \\ 0 & 1 & -3/5 & 2/5 \\ 0 & 0 & -\frac{26}{10} & -8/5 \end{array}\right]$$

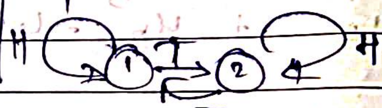
$$\left[\begin{array}{ccc|c} 1 & 0 & 4/5 & 1/5 \\ 0 & 1 & -3/5 & 2/5 \\ 0 & 0 & 1 & 8/13 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9/13 \\ 0 & 1 & 0 & 10/13 \\ 0 & 0 & 1 & 8/13 \end{array}\right].$$

$u_C = \frac{8}{13}$, $u_B = \frac{10}{13}$, $\boxed{u_A = \frac{9}{13}}$ (ans i)

(4) C_1: $P(H) = .6$, $P(T) = .4$.

C_2: $P(H) = P(T) = .5$.

Define:
H → stay / self
T → switch



2 state Markov chain:
$X_n \in \{1, 2\}$.

One step transition probability
* from coin 1: $P(1 \to 1) = 0.6$, $P(1 \to 2) = 0.4$.
* from coin 2: $P(2 \to 1) = 0.5$, $P(2 \to 2) = 0.5$.

$$\therefore P = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix}.$$

(a) Long run proportion of time spent in each state equals the stationary distribution:

$\pi_j = \sum_i \pi_i P_{ij}$ → $\boxed{\text{theorem}}$

$\therefore \pi_1 = .6\pi_1 + .5\pi_2$
$\pi_2 = .5\pi_2 + .4\pi_1$ $\Big\}$ $\pi_2 = 1 - \pi_1$.

$\pi_1 = .6\pi_1 + .5 - .5\pi_1$.
$\pi_1 = .1\pi_1 + .5$.

$\pi_1 (0.9) = 0.5$.

$\boxed{\pi_1 = 5/9}$ $\boxed{\pi_2 = 4/9}$

$\boxed{\text{Inference: chain spends more time in coin 1.}}$

(b) Start with c_1.

$P(c_2 \mid T=5) = (P^4)_{12}$.

$P = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix}$. Diagonalize $\longrightarrow \lambda_1 \lambda_2 = \frac{Tr}{2} \pm \sqrt{\left(\frac{Tr}{2}\right)^2 - \Delta}$.

$\boxed{\frac{Tr}{2} = .55}$, $\boxed{\Delta = .3 - .2 = .1}$

$= .55 \pm \sqrt{.3025 - .1}$
$= .55 \pm \sqrt{.2025}$
$= .55 \pm .45$
$= \boxed{1 / .1}$.

$\lambda_1 = .1$, $|\lambda_1\rangle = \begin{bmatrix} -4/5 \\ 1 \end{bmatrix}$.
$\lambda_2 = 1$, $|\lambda_2\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\Big\} \longrightarrow P = \begin{bmatrix} 1 & -4/5 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$.

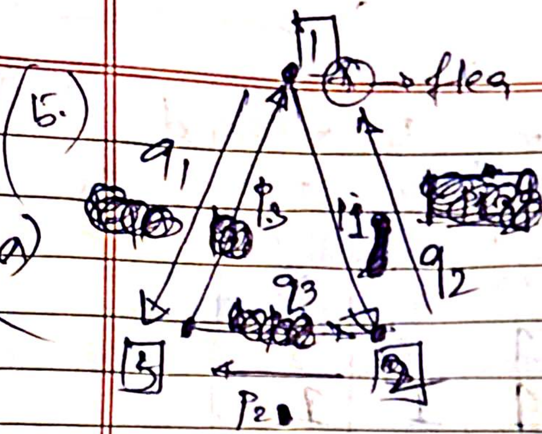
$P^{-1} = \begin{bmatrix} 5/9 & 4/9 \\ -5/9 & 5/9 \end{bmatrix}$.

$\therefore P = (P D P^{-1})$.
$P^4 = (P \cdot D^4 \cdot P^{-1})$.

$\therefore P^4 = \begin{bmatrix} 1 & -4/5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .1^4 \end{bmatrix} \begin{bmatrix} 5/9 & 4/9 \\ -5/9 & 5/9 \end{bmatrix} = \begin{bmatrix} .5556 & 0.4444 \\ .5555 & .4445 \end{bmatrix}$.

$\boxed{\text{Ans} \longrightarrow 0.4444}$

5. (a)



proportion of time flea is @ vertex ?

Let's build the p matrix first.

$p_{11} = 0$, $p_{22} = 0$, $p_{33} = 0$

$p_{12} = p_1$, $p_{13} = q_1$

$p_{21} = q_2$ $p_{23} = p_2$

$p_{31} = p_3$ $p_{32} = q_3$

$$\begin{bmatrix} 0 & q_1 & p_1 \\ p_2 & 0 & q_2 \\ q_3 & p_3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & q_1 & p_1 \\ q_2 & 0 & p_2 \\ p_3 & q_3 & 0 \end{bmatrix}$$

$\pi_1 = \pi_1 \cdot 0 + q_2 \cdot \pi_2 + p_3 \pi_3$

$\pi_2 = \pi_1 q_2 + \pi_2 \cdot 0 + \pi_3 p_2$

$\pi_3 = \pi_1 p_3 + \pi_2 q_3 + \pi_3 \cdot 0$

$$\Rightarrow \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 & q_1 & p_1 \\ q_2 & 0 & p_2 \\ p_3 & q_3 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

$\vec{\pi} = P \cdot \vec{\pi}$ (theorem).

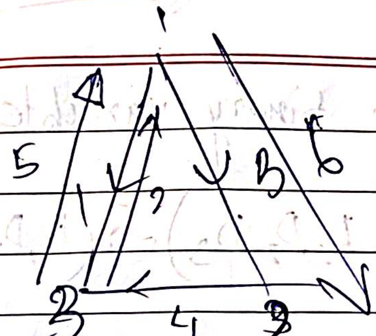
$$\pi_1 = \frac{p_3 + q_2 q_3}{(p_3 + q_2 q_3) + (q_2 + p_1 p_3) + (p_2 + q_1 p_2)}$$

$$\pi_2 = \frac{q_3 + p_1 p_3}{(p_3 + q_2 q_3) + (q_3 + p_1 p_3) + (p_2 + q_1 q_2)}$$

$$\pi_3 = \frac{p_2 + q_1 q_2}{(p_3 + q_2 q_3) + (q_3 + p_1 p_3) + (p_2 + q_1 q_2)}$$

$\pi_1 : \pi_2 : \pi_3 = (p_3 + q_2 q_3) : (q_3 + p_1 p_3) : (p_2 + q_1 q_2)$.

(b)



For starting index i:
Total 6 moves.
1 ccw, 5 cw,

$$q_i \times p_{i+2} \times p_i \times p_{i+1} \times p_{i+2} \times p_i$$

Therefore probability $= q_i \cdot p_i^2 \, p_{i+2}^2 \cdot p_{i+1}$.

Long run frequency : $\boxed{\sum_{1}^{3} \pi_i \, p_i^2 \cdot p_{i+2}^2 \cdot p_{i+1}}$.

$i+2 = i-1$ (cyclic)

(c) Define string $D_1 D_2 D_3$. D_i are binary variables denoting rain = 1, no rain = 0.

$P(1 \mid D_1, D_2, D_3) = r(D_1, D_2, D_3)$, $P(0 \mid D_1, D_2, D_3) = \bar{r}(D_1, D_2, \ldots$

If: $D_1 D_2 D_3 = 111$, $r = 0.8$.
$D_1 D_2 D_3 = 000$, $r = 0.2$.
$D_1 D_2 D_3 = D_1 D_2 0$, $r = 0.4$
$D_1 D_2 D_3 = D_1 D_2 1$, $r = 0.6$.

States:

$000 \Rightarrow r = .2, \bar{r} = 0.8$	$100 \Rightarrow r = 0.4, \bar{r} = .6$
$001 \Rightarrow r = 0.6, \bar{r} = .4$	$101 \Rightarrow r = 0.6, \bar{r} = .4$
$010 \Rightarrow r = 0.4, \bar{r} = 0.6$	$110 \Rightarrow r = 0.4, \bar{r} = .6$
$011 \Rightarrow r = 0.6, \bar{r} = .4$	$111 \Rightarrow r = .8, \bar{r} = 0.2$

Transition Matrix $\Rightarrow$ abc $\rightarrow$ bcx $\rightarrow$ Next day.

	bcx = 000 ($\bar{r}$)	bcx = 001 (r)	bcx 010 ($\bar{r}$)	bcx 011 (r)	bcx 100 ($\bar{r}$)	bcx 101 (r)	bcx 110 ($\bar{r}$)	bcx 111 (r)
abc = 000	0.8	0.2	0	0	0	0	0	0
abc = 001	0	0	0.4	0.6	0	0	0	0
abc = 010	0	0	0	0	0.6	0.4	0	0
abc = 011	0	0	0	0	0	0	0.4	0.6
abc = 100	0.6	0.4	0	0	0	0	0	0
abc = 101	0	0	0.4	0.6	0	0	0	0
abc = 110	0	0	0	0	0.6	0.4	0	0
abc = 111	0	0	0	0	0	0	0.2	0.8

(Ans.)

$\rightarrow$ Transition matrix $[D_1 D_2 D_3 \Rightarrow D_2 D_3 D_4]$.

8(a)

classmate

Date

Page

(7) Arrivals $\Rightarrow X_n \sim \text{Poisson}(\lambda) \Rightarrow$ $\boxed{X(n) = \frac{e^{-\lambda}\lambda^n}{n!}}$ "arrival func".

Each family stay for $k \geq 1$ days.

stay function $\rightarrow S(k) = p(1-p)^{k-1}$.

a) (p denotes the "leaving probability").

At any day, $Y(n)$ i.e families available at any day in the hotel, is then:

we already know.

$\boxed{Y_{n+1} = S_n + X_n.}$ (since number of days of staying is ALWAYS ≥ 1).

To prove Y_n is markov, it is enough to show that conditional on $Y_n = i$, (S_n, X_n) is NOT dependent on Y_{n-1} & Y_{n-2} ......

$\boxed{1}$ $P(S_n \mid Y_n = i)$ depends only on i :-

S_n is geometric distribution. It is memoryless which means probability of surviving the NEXT day is always $(1-p)$ and does not depend on how long they have been staying.
$P(S_{n+1} \mid S_n) = (1-p)$. Independence is also given across other families.

$\boxed{2}$ $P(R_n \mid Y_n = i) \longrightarrow$

Arrivals on day: $X_n \sim \text{Poisson}(\lambda)$ independent of EVERYTHING in the past.

$\rightarrow$ Each arriving family lives AT LEAST 2 days.
$\therefore P(L \geq 2) = 1-p$.

$\rightarrow$ Conditioned on X_n, the number that survive to day $n+1$, is Binomial $(X_n, 1-p)$: $X_n' \sim \text{Poisson}(\lambda(1-p))$.
Also no dependence on History.

(b) $P_{ij} = P(X_{n+1}=j \mid X_n = i)$

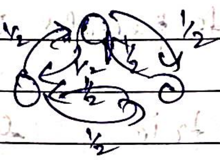
~~$P(X_{n+1} \mid i) \sim$ Bino~~

$$P(j \mid i) = \underbrace{\text{Binomial}(i, 1-p) + \text{Poisson}(\lambda(1-p))}_{\text{convolution}}.$$

~~$P_{ij} = \text{Conv}\left[\binom{i}{s}(1-p)^s p^{i-s}\right],$~~

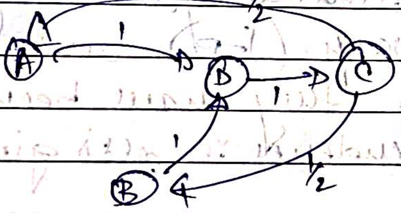
$$P_{ij} = \text{Conv}\left[\left\{ {}^{i}C_{s}\,(1-p)^{s}\, p^{i-s} \right\}, \left\{ \frac{e^{-\lambda(1-p)}\,[\lambda(1-p)]^{j-s}}{(j-s)!} \right\}\right]$$

3. [a] $P_1 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$



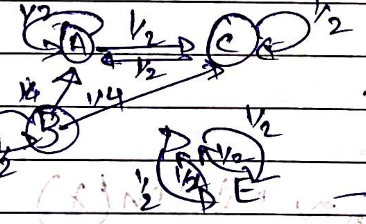
No sink hole. $\boxed{\text{Recurrent}}$

[b]



$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\boxed{\text{Recurrent}}$

[c] $\begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$

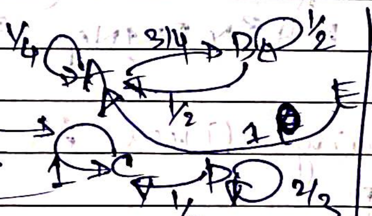


$\rightarrow$ $\boxed{\text{Transient}} \rightarrow \{B\}$

Recurrent $\rightarrow \{(A, C), (D, E)\}$.

[d]

$\begin{array}{c} \\ \\ C \\ D \\ E \end{array} \begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$



Transient $\rightarrow \{D\}, \{E\}$

Recurrent $\rightarrow \{\{C\}, \{A, B\}\}$