MTP-2 Report

Nonequilibrium phenomena in Exciton-Polariton condensates

A numerical study

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CERTIFICATE

This is to certify that the project report entitled **Nonequilibrium phenomena in Exciton-Polariton Condensates** submitted by **Shobhit Saheb Dey (18PH20026)** to the Indian Institute of Technology, Kharagpur towards partial fulfilment of requirements for the award of the degree of **Bachelor of Science(Honours) and Master of Science in the Department of Physics,** is a record of bona fide work carried out by him under my supervision and guidance during Spring Semester 2022-23.

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1. Introduction

The construction of thermodynamic variables to study the macroscopic dynamics of any system while integrating out statistically equivalent spaces of microscopic variables is made upon the assumption of having equilibrium, i.e the microscopic statistics stay constant over time. But we often have systems that are slightly or far out-of-equilibrium. We can achieve this regime by applying small external perturbations and the microscopic variables start to deviate or fluctuate from their equilibrium values. But non-equilibrium dynamics also arise when the energy flows and dissipates along different parts of the same system or during the process of equilibration itself. Nonequilibrium dynamics are often hard to track analytically or numerically and even harder to probe in solid-state systems owing to the short time scale and exponential dimensionality of collective phenomena. After the first experimental realization of atomic Bose-Einstein Condensate[Dav+95], the field of simulating such non-equilibrium quantum phenomena using analogous systems of cold atoms has exploded[Gał+22],[Gre+02] and the development of quantum gas microscope[Bak+09] has increased the resolution to atomic scales. The idea if simply stated is as follows: if you want to probe the time evolution or any steady state of some quantum system, but the act of measuring any variable would make it collapse, we can prepare a large number of copies of the same quantum system and make the same measurement on each one of them giving us the expectation value. The natural way of preparing such a system is a Bose condensate. This idea can be further extended to *measure* the wavefunction in position basis by making such observations at every point in space, giving us the amplitude of the wavefunction while interferometry gives us the phase. In this project, we use a similar system to study nonequilibrium quantum phenomena. Although the measurements of individual atoms are not completely uncorrelated, the technique of weak measurements optimizes the accuracy[Bru+96].

Far before the first experimental realization of Bose condensate, the idea of condensing excitons, bound electron-hole pair, was proposed owing to its light effective mass leading to cryogenically achievable critical temperature. But the bottleneck had been to extend the lifetime of exciton beyond the thermalization time and achieve threshold number density suppressed due to the dipole-dipole interactions. Exciton-polariton on the other hand overcomes these difficulties and its nonequilibrium condensation was first proposed in the context of lasing without inversion[Ima+96]. The first comprehensive and irrefutable evidence of Bose condensation of polaritons was achieved in 2006[Den+02]. The later trajectory of polaritonic research has been similar to that of cold atoms. Superfluidity[Amo+09], spinorial BECs[Xu+17] and very recently polaritons have been confined into microcavity lattices to simulate materials[Rea+21].

Wave turbulence in quantum gas has recently been probed experimentally in 2D atomic condensate[Gał+22]. Another nonequilibrium phenomena of Kardar-Parisi-Zhang universality class of growing interfaces has been observed spin transport of 1D chain of cold atoms[Ima+96] and also 1D polaritonic lattice[Fon+22]. In this project, currently have demonstrated strong turbulence in exciton-polaritons we further plan to demonstrate KPZ in 2D on a similar setting.

2. Excitons

In the late 1920s narrow photoemission lines were observed in the spectra of organic molecular crystals by Kronenberger and Pringsheim and I. Obreimov and W. de Haas. These data were interpreted by the Russian theorist Frenkel who introduced the concept of excitation waves in crystals and later coined the term exciton. These excitations where due to bound pairs of electrons and holes. These can lead to two kinds of scenarios.

If the Coloumbic attraction between the electron and hole is so high that they reside in neighbouring unit cells and the hopping is also suppressed, then the combined 2-body wavefunction can be pictured to be effectively localized. In this regime, the long wavelength approximation is no longer valid to allow us to average out the effect of the positive periodic potential. In other words, the exciton is not *free enough* to be assigned an effective mass moving around the crystal. This is called the **Frenkel exciton** and is relevant in organic semiconductors where the binding energy is high. The other class is called the **Wannier-Mott exciton**, where the intersite hopping is fast enough owing to the low-binding energy commonly present in inorganic semiconductors. In this case, we can assign an effective mass to the exciton while the internal configurations can be seen as 2-bodies of masses m_e and m_h with Coulomb attraction between them, hence the effective Hamiltonian becomes:

$$H = \frac{P^2}{2M} + \frac{L^2}{2\mu r^2} - \frac{e^2}{4\pi\epsilon_0 r} \tag{1}$$

Here P and M is the total momentum operator and effective mass of the exciton while L, μ, r is the internal angular momentum operator, reduced mass of the e-h pair and their distance of separation. Now again invoking the large wavelength approximation, we get translational symmetry over the space thus $[H, P] = [\frac{L^2}{2\mu r^2} - \frac{e^2}{4\pi\epsilon_0 r}, P] = 0$. Thus we can independently solve for the eigenvalues of the Kinetic Energy of COM and the remain of the hamiltonian that is the internal energy. Due to complete similarity with the Hydrogen atom hamiltonian, we get the exact same internal eigenenergies with only m_e replaced with μ, while the COM Kinetic Energy eigenstates are just plane waves. Hence the total eigenenergy spectrum in theis:

$$E(n, k) = \frac{\hbar^2 k^2}{2M} - \frac{\mu}{m_e}\frac{R}{n^2} \tag{2}$$

Here R is the Rydberg's constant and the prefactor is to adjust for the electron's mass. Now if the resultant excitonic radius is large enough compared to the unit cell's dimension, which will be the case anyway because the bond lengths are shorter due to atomic numbers $>> 1$ of the ionic centers, we can ignore the effects of different internal states on the effective mass of the exciton. Thus we can approximate upto first order $M = m_e + m_h$.

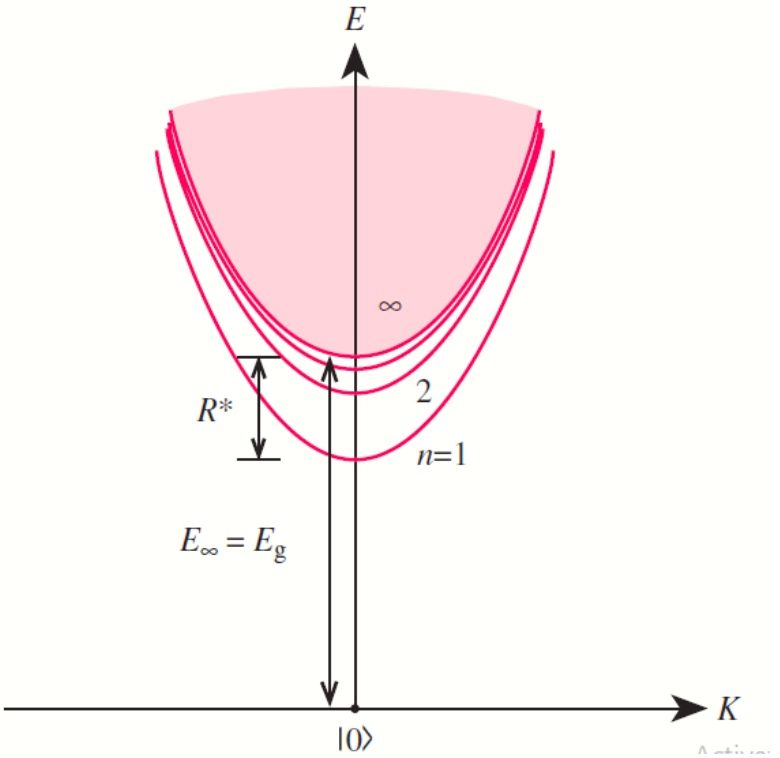

Figure 1: Dispersion curves for excitons at different Hydrogen atom like states labeled by n[PC10]

3. Semiclassical theory of light-matter interaction

In the classical limit, where we have a continuum of energies for both the bound exciton(equivalent to having a large value of principal quantum number) as well the excitation light, we can suitably use the Lorentz model where the bound pair is assumed to be a damped harmonic oscillator and the applied electromagnetic field is the driving force:

$$m\ddot{r} + 2m\gamma\dot{r} + m\omega_0^2 r = -eE(t) \tag{3}$$

Using a $E(t) = e^{\iota\omega t}$, and further assuming the applied frequency ω close to the natural frequency of excitonic vibration(as we shall also experimentally impose), then the amplitude spectrum comes out to be:

$$A(\omega) = \frac{eE_0}{m}\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2} \tag{4}$$

This amplitude function spectrum is famously known to be the Lorentzian curve characterized by two essential features, the resonating frequency ω_0 and the linewidth γ that stems from the damping. Extending this to the semi-classical case where the energy levels of the atoms are effectively discretized while the applied field remains to be classical, we get the Rabi picture. In this case, selection rules also should be considered to identify pair of states among which we can have transitions. When an

oscillating electric field is applied, the system oscillates between the 2-states, called Rabi oscillation where the occupation probability of any one state changes with time as follows:

$$P(t) = \frac{\left(\frac{V}{\hbar}\right)^2}{\left(\frac{V}{\hbar}\right)^2 + (\omega - \omega_0)^2/4} \sin^2\left[\left(\left(\frac{V}{\hbar}\right)^2 + \frac{(\omega-\omega_0)^2}{4}\right)^{1/2} t\right] \tag{5}$$

Here V is the absolute value of the matrix element of the 2-states among which Rabi oscillation is considered.

4. Jaynes-Cummings Model

If we study the full quantum picture of light-matter interaction, then the applied electric is in the photonic regime, in the sense that the energy of the total em-field is comparable to the energy of a single photon. While the atomic transitions' energies are already comparable to the total energy. So the total Hilbert space is the tensor product of the excitonic and photonic states and the total Hamiltonian is written as:

$$H = H_A + H_F - e\mathbf{r} \cdot \mathbf{E} \tag{6}$$

Where, $H_A = \sum_i E_i |i\rangle \langle i|$ with $|i\rangle = |n, l, m\rangle$ being the states of exciton, $H_F = \sum_{\mathbf{k}} \hbar\omega_k (a_k^\dagger a_k + \frac{1}{2})$ and $\mathbf{r}$ is the vector operator corresponding to the relative position of electron from hole.
Now in order to express the interaction term in the basis of eigenstates of H_A and H_B we write $e\mathbf{r} = \sum_{ij} e |i\rangle \langle i| \mathbf{r} |j\rangle \langle j| = \sum_{ij} \rho_{ij}\sigma_{ij}$ where we just identify $\sigma_{ij} = |i\rangle \langle j|$ and ρ_{ij} is the dipole matrix element. And we also use the standard equation for electric field operator $\mathbf{E} = \sum_{\mathbf{k}} \hat{\epsilon}_k \mathcal{E}_k (a_k + a_k^\dagger)$ where $\mathcal{E}_k$ is $[\hbar\omega_k/2\epsilon_0 V]^{1/2}$. So re-writing the total hamiltonian in the chosen basis:

$$H = \sum_i E_i \sigma_{ii} + \sum_k \hbar\omega_k (a_k^\dagger a_k + 1/2) - \sum_{ij} \sum_k \rho_{ij} \cdot \hat{\epsilon}_k \sigma_{ij} (a_k + a_k^\dagger) \tag{7}$$

Now if we study the subspace of any 2-states only, as from the semi-classical description we saw that in the lossless limit, we have sharp enough resonance to excite at best only one excitation for a given frequency of applied field. Let the states be $|g\rangle$ and $|e\rangle$, then the manifold of all the atomic states in this subspace can be generated from an SU(2) group. So we introduce analogous Pauli matrices:

$$\sigma_x = |e\rangle \langle g| + |e\rangle \langle g| \tag{8a}$$

$$\sigma_y = -\iota |e\rangle \langle g| + \iota |e\rangle \langle g| \tag{8b}$$

$$\sigma_z = |e\rangle \langle e| + |g\rangle \langle g| \tag{8c}$$

These clearly obey the Lie-Algebra of SU(2) i.e $[\sigma_i, \sigma_j] = \iota\epsilon_{ijk}\sigma_k$ and we can also define creation and annihilation operators as $\sigma_\pm = \sigma_x \pm \sigma_y$ and subsequently $\sigma_+ = |e\rangle \langle g|$ and $\sigma_- = |g\rangle \langle e|$. Now due to parity symmetry of atomic states, $\rho_{gg} = \rho_{ee} = 0$ and let $\rho_{eg} = \rho_{ge} = \rho$. The total hamiltonian now reduced to the subspace of $|e\rangle$ and $|g\rangle$

atomic states tensored with the only electromagnetic field mode $|k\rangle$ that resonates with the transition is given by:

$$H = \sum_i E_i \sigma_{ii} + \hbar\omega_k (a_k^\dagger a_k + 1/2) - g_k(\sigma_+ + \sigma_-)(a_k + a_k^\dagger) \tag{9}$$

Now in order extract the essential physics, we look at the interaction term H_I which we can see as a perturbation to the decoupled hamiltonian $H_0 = H_A + H_F$. So we go to the interaction picture in order to see its effect. In this picture the different terms of $V_I = e^{-\iota H_0 t} H_I e^{\iota H_0 t}$ evolve in time as follows:

$$\sigma_+ a_k \to \sigma_+ a_k e^{\iota(\omega_0 - \omega_k)t} \tag{10a}$$

$$\sigma_- a_k \to \sigma_- a_k e^{-\iota(\omega_0 + \omega_k)t} \tag{10b}$$

$$\sigma_+ a_k^\dagger \to \sigma_+ a_k^\dagger e^{\iota(-\omega_0 - \omega_k)t} \tag{10c}$$

$$\sigma_+ a_k^\dagger \to \sigma_+ a_k^\dagger e^{\iota(\omega_0 + \omega_k)t} \tag{10d}$$

Here ω_0 is $(E_e - E_g)/\hbar$ the resonating frequency of transition. Since the time evolution of the state in interaction picture is given as:

$$|\psi(t)\rangle_I = |\psi(0)\rangle_I - \frac{\iota}{\hbar} \int_0^t V_I(\lambda) \, |\psi(\lambda)\rangle \, d\lambda \tag{11}$$

At this point we introduce the assumption that the applied frequency is very close to the resonating frequency. Then $\omega_0 + \omega_k \gg |\omega_0 - \omega_k|$ implying the over any timescale, $e^{\iota(\omega_0 + \omega_k)t}$ rotates much faster in Argand plane than $e^{\pm\iota(\omega_0 - \omega_k)t}$, so when integrated in Eq (11), the $e^{\iota(\omega_0 + \omega_k)t}$ terms average out to 0. So we can effectively neglect these terms in the total Hamiltonian. This is called the **Rotating Wave Approximation**. An important phenomenology ignored in this approximation is the Lamb shift, which corresponds to the term $\sigma_+ a_k^\dagger$ that practically just shifts the entire spectrum upwards by the photon energy + excitation energy. Lamb shift is not relevant for the energy scales we would be probing in microcavities so we can safely ignore it. So finally the Jaynes-Cummings hamiltonian is:

$$H = \sum_i E_i \sigma_{ii} + \hbar\omega_k (a_k^\dagger a_k + 1/2) - g_k(\sigma_+ a_k + \sigma_- a_k^\dagger) \tag{12}$$

5. Polaritons

We can apply the Jaynes-Cummings hamiltonian our considered solid-state system to where we have em-modes resonating in a microcavity and the matter excitations are excitonic. The only addition is that the hamiltonian will be integrated over all the values of in-plane momentum of the exciton and the photon. Then the total hamiltonian becomes:

$$H = \sum_k E_C(\mathbf{k}) a_k^\dagger a_k + \sum_k E_X(\mathbf{k}) b_k^\dagger b_k + \sum_k g(\mathbf{k})(b_k^\dagger a_k + b_k a_k^\dagger) \tag{13}$$

Where the individual dispersion of exciton and photon along the in-plane momentum are:

$$E_C(\mathbf{k}) = \hbar c k_z + \frac{\hbar^2 k_\parallel^2}{2(\hbar k_z / c)} \tag{14a}$$

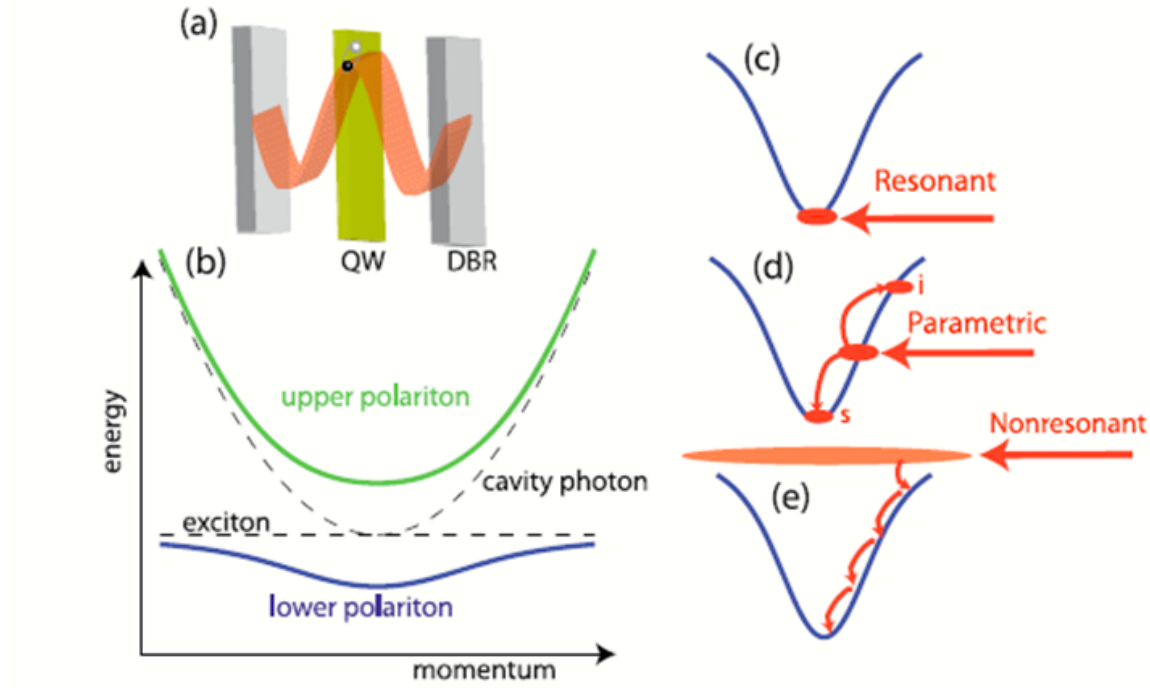

Figure 2: (a)Pictorial representation of microcavity, (b) Complete dispersion picture for exciton, photon and polariton, (c) Resonant Condensation, (d) Parametric Oscillation, (e) Nonresonant/Bose-Einstein Condensation[BM13]

$$E_X(\mathbf{k}) = \frac{\hbar^2 k^2}{2M} \tag{14b}$$

And the momentum dependent coupling is given by:

$$g(\mathbf{k}) = \mu\sqrt{\frac{E_C(k)}{2\epsilon_0 V}} \tag{15}$$

Where μ is the dipole element of the em-exciton coupling. If we diagonalize the hamiltonian for every momentum state(since they commute), the eigenenergies we get are:

$$E_\pm(k) = (E_X(k) + E_C(k))/2 \pm \sqrt{\Delta_k^2 + 4g^2(k)} \tag{16}$$

Here $\Delta_k = E_C(k) - E_X(k)$ is the detuning which is an important control parameter. So we get 2 modes for each momentum state, and each of these eigenmodes of coupled electromagnetics and excitonic vibration is called the **upper and lower polaritons**.

6. Condensation of Polaritons

Condensation would be characterized by a macroscopic number of polaritons occupying the lowest energy state, i.e the $k = 0$ in-plane momentum mode. There are 3 ways to achieve this. The common step in all 3 is to use a laser pump beam with oblique incidence to induce a particular in-plane momentum of the cavity photon and subsequently that of the polariton. The first way is to pump at $k = 0$ mode, i.e at perpendicular

incidence. This way a large number of polaritons are excited in the ground state. This is called resonant excitation or even less accurately resonant condensation though the polaritons didn't spontaneously occupy the ground state.

The second way is to pump at the inflection point of the lower-polariton branch by tuning the wavelength and angle of incidence carefully. Due to a particular symmetry of the dispersion curve of the lower polariton branch, this kind of pumping would produce 2-different populations of polaritons in the ground state(signal) and in an excited state(idler) further from the inflection point, such that the energy and in-plane momentum is conserved. This kind of scatter is experimentally shown to be quantum coherent scattering in the sense that the particular phase relationship between the pump, signal and idler polaritons is obeyed which is $2\theta_p = \theta_s + \theta_i$. Note that here we have a symmetry in $\theta_s - \theta_i$ and this allows *Goldstone modes* which would be coupled phase fluctuations of both the signal and idler polaritons(hence it is different from superfluids in this sense).

The third way is to excite polariton is practically far away and random position in the E, k space, but the incoherent scattering of polaritons via various relaxation mechanisms like exciton-exciton and exciton-phonon coupling, amplified by boson enhancement, we get a macroscopic population in the ground state. The kinetics of the Bose condensation can be studied by considering the thermalization of reservoir excitons as the dominating relaxation mechanism[Por+02]. Only this mechanism can be considered to be utilizing Bose-Einstein Condensation and the temperature threshold to achieve this via this so-called non-resonant excitation is $T_c = 3.3125\frac{\hbar^2 n^{2/3}}{mk_B}$. Because the effective mass of the polaritons is 10^{-5} times smaller that that of the electrons and even smaller that alkali atoms which require nanokelvin scale temperatures to condense, polaritons can effectively condense at cryogenically available temperatures like few kelvins. Because after a series of incoherent scattering, the polaritons occupy the ground state coherently, we have U(1) symmetry breaking giving space for Goldstone modes of phase fluctuations. The first experimental verification[Den+02] of non-resonant polaritonic condensation was justified by long-range order characterized by the first order correlation function $g^{(1)}(x_1, x_2) = \langle \psi(x_1)\psi^*(x_2)\rangle$ measured using a Michelson interferometer, owing to the fact that the emitted light from the polaritons is in-phase with the condensate's wavefunction.

7. Microscopic theory of interacting Bose gases

In the extensive study of interacting Bose gases, almost all kinds of phenomena can be modelled with significant accuracy by only considering 2-body interactions. Some exceptions were found in strongly interacting Unitary Bose gases[Fle+17]. We can solve this many-body problem in the second-quantization framework by writing the Hamiltonian in the following form:

$$H = \int (\frac{-\hbar^2}{2m}\psi^\dagger(\mathbf{r})\boldsymbol{\nabla}^2\psi(\mathbf{r}) + U\mathbf{r})\psi^\dagger(\mathbf{r})\psi(\mathbf{r}))d\mathbf{r} + \frac{1}{2}\int V(|\mathbf{r}' - \mathbf{r}|)\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})d\mathbf{r}d\mathbf{r}' \tag{17}$$

Now representing the same in the Fourier basis by defining $\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} a_p e^{\iota \mathbf{p}\cdot\mathbf{r}}$ where V is the volume of our domain. Then the Hamiltonian becomes:

$$H = \sum_p \frac{p^2}{2m} a_p^\dagger a_p + \frac{1}{2V} \sum_{p_1 p_2 q} V(q) a_{p_1+q}^\dagger a_{p_2-q}^\dagger a_{p_2} a_{p_1} \tag{18}$$

Here, $V(q)$ is the Fourier transform of $V(|\mathbf{r}' - \mathbf{r}|) = V(R)$. Because in the usual Bose gases that are not dipolar in nature, the interaction strength is Van-der-Waals or induced dipole in nature. These are significantly short-ranged compared to the mean-free path of the particles, and this makes $V(R)$ close to the form of $\delta(R)$. So we can approximate $V(q) = V_0$

Now in the case of BECs, a macroscopic number of particles occupy the ground state, so the annihilation and creation operators for $p = 0$ can be ignored to have quantum fluctuations and rather be approximated as $a_p = a_p^\dagger = \sqrt{N_0}$ where N_0 is the number of particles in the ground state. Hence, in the 0^{th}-order, the energy of the system is $N_0^2 V_0 / 2V$. Now suppose we look for terms *linear* in a_p in the Hamiltonian i.e $a_0^\dagger a_0^\dagger a_p a_0$, then in the interaction picture it shall correspond to a scattering process where a particle with momentum p collides with a 0-momentum particle and both scatter to 0-momentum states finally. This process violates momentum conservation, so we move on to the quadratic term next:

$$\sum_{p_1 p_2 q} a_{p_1+q}^\dagger a_{p_2-q}^\dagger a_{p_2} a_{p_1} = \sum_{p \neq 0} (4 a_p^\dagger a_p a_0^\dagger a_0 + a_p^\dagger a_{-p} a_0^\dagger a_0 + a_{-p}^\dagger a_p a_0^\dagger a_0) \tag{19}$$

Further defining $n = N_0/V$, the Hamiltonian up to second order in annihilation operators can be written as:

$$H = \frac{N_0^2 U_0}{2V} + \sum_{p \neq 0} (\frac{p^2}{2m} + 2 n_0 V_0) a_p^\dagger a_p + \frac{n_0 V_0}{2} \sum_{p \neq 0} (a_p^\dagger a_{-p} + a_{-p}^\dagger a_p) \tag{20}$$

Now the total number operator for the entire system shall be given as $N = N_0 + \sum_{p \neq 0} a_p^\dagger a_p$, so upto 2^{nd}-order in a_p we have $N_0^2 = N^2 - 2N \sum_{p \neq 0} a_p^\dagger a_p$. Since we have taken into account the 2^{nd}-order correlations in momentum, we can safely approximate $N_0 \to N$. Then the Hamiltonian becomes:

$$H = \frac{N^2 U_0}{2V} + \sum_{p \neq 0} [(\frac{p^2}{2m} + n V_0)(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}) + n V_0 \sum_{p \neq 0} (a_p^\dagger a_{-p} + a_{-p}^\dagger a_p)] \tag{21}$$

In this 2^{nd}-order Hamiltonian, if we have to find the normal modes of excitations represented in the momentum space, we shall diagonalize the Hamiltonian using the Bogoliubov transformation:

$$\alpha = u a_p + v a_{-p}^\dagger \tag{22a}$$

$$\beta = u a_{-p} + v a_p^\dagger \tag{22b}$$

The constraint leading from Bosonic commutation relations suggests that the normal modes can be parameterized by a number λ such that $u = \cosh\lambda$ and $v = \sinh\lambda$. In this space of normal modes which diagonalize the Hamiltonian, the final form becomes:

$$H = \frac{N^2 U_0}{2V} + \sum_{p \neq 0} \epsilon_p \alpha_p^\dagger \alpha_p - \frac{1}{2} \sum_{p \neq 0} (\epsilon_p^0 + n V_0 - \epsilon_p) \tag{23}$$

Here $\epsilon_p^0 = \frac{p^2}{2m}$ and the Bogoliubov excitation spectrum is given by $\epsilon_p = \sqrt{(\epsilon_p^0)^2 + 2\epsilon_p^0 n V_0}$. There are many characteristics that can be extracted from the functional form of the excitation spectrum, while there are two energy scales, ϵ_p^0 and nV_0, govern them. In the low kinetic energy regime where $\epsilon_p^0/nV_0 \ll 1$, we have a linear dispersion $\epsilon_p = p\sqrt{nV_0/m}$ suggesting the existence of **sound like modes**. On the other end where $\epsilon_p^0/nV_0 \gg 1$, we have free particle like spectrum i.e $\epsilon_p = \frac{p^2}{2m}$. Another important conclusion is that a finite speed of sound can only exist in the case of interacting Bose gases. This poses a constraint that only interacting Bose gases can exhibit superfluidity against external perturbations according to the Landau criterion.

8. Mean Field Dynamics

In order to find the time evolution and corresponding spatial density and phase profile of the condensate, we can neglect the quantum fluctuations about the mean-field order parameter $\psi(\mathbf{x}, t)$. For a condensate in a closed equilibrated system, the time evolution in the weakly interacting regime is governed by the Gross-Pitaevskii equation:

$$\iota\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + g|\psi|^2\psi + V(\mathbf{x})\psi \tag{24}$$

Where the self-interaction coefficient $g = \frac{4\pi\hbar^2 a_s}{m}$ corresponds to s-wave scattering length a_s as we are considering only weak interactions. But in the case of exciton-polaritons, since we continuously pump the system, the total number changes with time. In this case, unitary time evolution is no longer possible and hence we introduce non-hermitian terms in the hamiltonian. Let us now consider the case of non-resonant condensation:

The mechanism which pumps up the number of polaritons in the ground state(i.e near $k = 0$ mode) is the stimulated scattering down of the reservoir pool of polaritons which are resonantly being pumped by the incident laser. Now as discussed in previous sections, the kinetics of these relaxation mechanisms are complex and mean-field description shall fail to model it accurately because of the strongly momentum-dependent interactions of the polaritons with phonons or the thermalized excitons. Thus adhering to developments made in the work [WC07], we adopt a phenomenological model for the scattering of polaritons into the condensate. Now at any point in real space, the scattering for polaritons from the entire spectrum of high-momentum modes(the reservoir) to the condensate shall be directly proportional to the density of both particles. While the condensate density is $|\psi(r)|^2$, let the density profile of reservoir be $n(r)$. Then we can model the scattering rate as $Rn(r)|\psi(r)|^2$.

Further, the mechanism for loss in the number of polaritons is the escape of photons from the microcavity owing to their finite lifetime. Since the loss of multiple photons is not correlated, and the condensate density corresponds to photon density as well, we can adopt a single loss rate γ_p for the photons. So the modified mean-field equation for the time evolution of the condensate is:

$$\iota\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + g|\psi|^2\psi + \iota\Big(\frac{nR}{2} - \frac{\gamma_p}{2}\Big)\psi \tag{25}$$

Where n is the number density of reservoir polaritons, R is the amplification factor for scattering from reservoir to condensate, and $1/\gamma_p$ is the photon lifetime. Now because the number density of the reservoir polariton is reduced due to Bose-Condensation but also revived back from the pump laser, we have a dynamic evolution for $n(\mathbf{r})$ as:

$$\frac{\partial n}{\partial t} = P(\mathbf{r}) - n(\mathbf{r})R|\psi|^2 - \gamma_R n \tag{26}$$

Here $P(\mathbf{r})$ is proportional to the pump intensity profile and $1/\gamma_R$ is the lifetime of the reservoir polariton. Thus in this case of non-equilibrium Bose condensation of the polaritons is modelled by coupled mean-field equations.

9. Hydrodynamics of exciton-polaritons

One of the foremost advantages of the mean-field description of Bosonic many-body is that it allows for hydrodynamic analysis very similar to that of classical fluids. To begin with, we first perform the Madelung transform of our field $\psi(r) = \sqrt{\rho(r,t)}e^{\iota\theta(r,t)}$. Plugging this into equations 25 and 26, we get:

$$\frac{\partial \rho}{\partial t} + \frac{\hbar}{m}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\theta) = (n(r)R - \gamma_p)\rho \tag{27a}$$

$$\hbar\frac{\partial \theta}{\partial t} + \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\theta)^2 + g\rho - \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}^2(\sqrt{\rho}) = 0 \tag{27b}$$

The effect of driving and dissipation can be readily seen at this stage. In equation 27a, if we isolate the effect of γ_p and nR, this evolves polariton field as $\rho(r,t) = \rho(r,0)e^{-(\gamma_p - n(r)R)t/2}$, showing that driving and dissipation exponentially changes the polariton density with time. Now from continuity conditions on Schrodinger's equation, we define an analogous velocity field in quantum fluids as $\mathbf{v}(r,t) = \frac{\hbar}{m}\boldsymbol{\nabla}\theta(r,t)$. To get a dynamical equation for the velocity field, we take the gradient of Equation 27b to get:

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}v^2 = -\frac{1}{m}\boldsymbol{\nabla}\left(g\rho - \frac{\hbar^2}{2m}(\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}^2\sqrt{\rho})\right) \tag{28}$$

Now $\frac{1}{2}\boldsymbol{\nabla}v^2 = \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})$ and because $\mathbf{v}$ is computed as a gradient, so it is obviously irrotational. So we obtain Euler's equation for the velocity field:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\frac{1}{m}\boldsymbol{\nabla}\left(g\rho - \frac{\hbar^2}{2m}(\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}^2\sqrt{\rho})\right) \tag{29}$$

This is basically the Navier-Stokes equation written for compressible, barotropic and inviscid fluid. Here the pressure can be divided into the classical part $g\rho$ and the quantum pressure part $-\frac{\hbar^2}{2m}(\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}^2\sqrt{\rho})$. The quantum pressure term is often neglected in the Thomas-Fermi limit when the field is slowly varying in space.

At first, it seems that the driven dissipative nature of the system is isolated from the final hydrodynamic equation. But the fact that pressure is dependent on ρ which

evolves according to the coupled equation 27a, **we can conclude that the effects of driving and dissipation on the velocity field are 2^{nd}-order in time**.

Now although our velocity field is irrotational by definition, it is instable to topological defects where the condensate depletes and loses coherence with the rest of the field. When a region in space is no longer coherent with the phase of the wavefunction around it, the velocity field can no longer be defined there as $\frac{\hbar}{m}\boldsymbol{\nabla}\theta$. But around the region both phase and hence the velocity too is well defined. And for the wavefunction to be single-valued, the following criterion for the loop integral must hold:

$$\int_{\mathcal{C}} \boldsymbol{\nabla}\theta \cdot dl = 2n\pi \tag{30}$$

Where $\mathcal{C}$ is any loop that touches only coherent parts of the wavefunction. Now if this loop surrounds an incoherent region, the loop can't be shrunk to a point, yet the criterion has to be satisfied. Only in this particular scenario non-zero values of n, i.e non-zero winding numbers are allowed. And these are treated as quantized vortices. Usually only $+1$ or -1 vortices are stable while vortices with higher winding numbers segregate into smaller ones. The length scale corresponding to the size of these vortices can also be treated as the coherence length, exactly because of the fact that the core of vortices are incoherent with the bulk of the condensate. Because the condensate density depletes in the core, it would happen at a distance ζ, then the kinetic energy shall be converted to interaction energy for the particles to loose momentum before reaching the core. This amounts to:

$$\frac{\hbar^2}{2m}\frac{\psi}{\zeta^2} = g|\psi|^2\psi \implies \zeta = \frac{\hbar}{\sqrt{2mg|\psi|^2}} \tag{31}$$

The total energy of the system shall still be defined as:

$$E = \int (\frac{\hbar^2}{2m}(\boldsymbol{\nabla}\psi)^2 + \frac{1}{2}g|\psi|^4)dr \tag{32}$$

Plugging in the Madelung transform and then the definition of $\mathbf{v}$, we get:

$$E = \int (\frac{1}{4}\rho v^2 + \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\sqrt{\rho}|^2 + \frac{g\rho^2}{4})dr \tag{33}$$

The three terms in the energy functional are Kinetic Energy, Quantum Pressure Energy and Interaction Energy respectively.

10. Numerical Scheme

Before moving towards the numerical scheme, I shall introduce the units for computation that would non-dimensionalize the quantities. The primary scales for length, time and condensate wavefunction are L_0,L_0/v_0 and $\sqrt{n_0}$. After scaling in these units, Eq (18) becomes:

$$\iota\frac{\partial\psi}{\partial t} = -\alpha\boldsymbol{\nabla}^2\psi + \beta|\psi|^2\psi + \iota\Big(\frac{nR}{2} - \frac{\gamma_p}{2}\Big)\psi \tag{34}$$

Here $\alpha = \frac{1}{\sqrt{2}}\frac{\zeta}{L_0}\frac{c}{v_0}$ and $\beta = \frac{1}{\sqrt{2}}\frac{L_0}{\zeta}\frac{c}{v_0}$ with $\zeta = \hbar/\sqrt{2mgn_0}$ and $c = \sqrt{gn_0/m}$ being the healing length and the speed of sound respectively. The non-hermitian quantities are also scaled but it doesn't matter by how much, because these are phenomenological parameters that will be intractable to be derived from the microscopics. They will have to be fitted on experimental data. Now the length of the computational domain can be defined by $\mathcal{L} = L/L_0$, so the computational grid size is $\delta x = \mathcal{L}/N$. For simplicity we choose the $\alpha = 1$ and in order to tune the nonlinearity in the system we keep β as a computational parameter and then the equation becomes:

$$\iota\frac{\partial\psi}{\partial t} = -\boldsymbol{\nabla}^2\psi + \beta|\psi|^2\psi + \iota\Big(\frac{nR}{2} - \frac{\gamma_p}{2}\Big)\psi \tag{35}$$

Now coming to the numerical scheme, I use the spectral decomposition and time-splitting of the hamiltonian in the following way. Because the hamiltonian can be split into $H_1 + H_2$ where choose H_1 as the Kinetic Energy part, then can increment the state through each hamiltonian sequentially because $\mathcal{U}(\Delta t) = e^{-\iota(H_1+H_2)\Delta t/\hbar} = e^{-\iota(H_2)\Delta t/\hbar}e^{-\iota(H_1)\Delta t/\hbar} + \mathcal{O}(\Delta t^3)[H_2, H_1]$. So first evaluating the Kinetic energy part:

$$\iota\frac{\partial\psi}{\partial t} = -\boldsymbol{\nabla}^2\psi \tag{36}$$

We use the FFT algorithm of both sides which shall give:

$$\iota\frac{\partial\psi_k}{\partial t} = -k^2\psi_k \tag{37}$$

And this is *exactly solvable*. So after incrementing my Δt we do an inverse FFT going back to position space. Now considering the other part of the hamiltonian:

$$\iota\frac{\partial\psi}{\partial t} = \beta|\psi|^2\psi + \iota\Big(\frac{nR}{2} - \frac{\gamma_p}{2}\Big)\psi \tag{38}$$

Conjugating and cross multiplying:

$$\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t} = \frac{\partial|\psi|^2}{\partial t} = (nR - \gamma_p)|\psi|^2 \tag{39}$$

This is also exactly solvable and by substituting the solution in Eq(24), we can solve it exactly too, giving:

$$\psi(x, \Delta t) = \psi(x)e^{-W(x)\Delta t + \iota\beta|\psi|^2(1-e^{-2W(x)\Delta t}/2W(x))} \tag{40}$$

Where $W(x) = (\gamma_p - nR)/2$. Thus each sub-hamiltonian is exactly solvable in time so the only source of error is the non-commutatibility of those leading to third-order error in time difference, hence very small.

11. Turbulence

Turbulence if attempted can be defined as aperiodically evolving states that have a broad spatial spectrum of excitation. If the initial state is well localized in the spectrum but evolves to become a turbulent one, then only one key component of the governing equation can allow for this spreading and that is nonlinearity. Turbulence has been predicted and observed in a wide variety of systems where the governing equation had a nonlinear component that would allow for wave-mixing. In the case of the Navier-Stokes equation, the inertial propagation term $\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}$ and in our case the nonlinear interaction term $g|\psi|^2\psi$ is responsible for the phenomena.

Nonlinearity basically couples the kinetics of excitation modes at any particular wavevector with those spread across the entire spectrum of wavevectors. This results in the exchange of energy between different length scales. So if we keep injecting energy to the system at some length scale, say L_f, then this energy shall be transferred in steps to all other scales. It is apt to say in step rather than directly to all other scales because the energy transfer rate depletes as the difference between the corresponding wavenumbers increase. This *cascaded* transfer of energy from larger to smaller scales happens in the *inertial range* of wavenumbers. In the case of classical fluids, beyond this range, dissipation is strong enough to cease the trend. As shown by Kolmogorov [Kol+91], the kinetic-energy spectrum for incompressible classical fluids in this range scales as $k^{-5/3}$. In the specific case of 2D classical fluids, enstrophy is another conserved quantity apart from energy. In these systems, smaller vortices cluster together to form large scaled structures, thus transferring the energy from smaller to larger length scales. Effectively we have an inverse energy cascade here[BE12].

In the case of 2D Quantum turbulence, a similar argument was given by Onsager [Ons49] were the inverse cascade shall correspond to quasi-equilibrated coherent structures of point vortices clustered to large length scales. We shall study quantum turbulence in context of exciton-polaritons adopting a similar framework to the recent study [Pan+23] but the difference being that we shall consider fully non-resonant excitation of polaritons which essential relax into the BEC rather than pumping polaritons directly into the ground state itself.

To start with, we need to first pump polaritons into the reservoir and we can choose a standard gaussian pump profile for the same. This corresponds to choosing the function $P(r)$ in Equation 26. And we can give a small initial value to the fields $n(r)$ and $\psi(r)$. We also add noise to the pump profile with a signal-to-noise ratio of 0.95. This would add to triggering the instability of the system's dynamics. Other parameters for simulating the coupled equations 35 and 26 are given in the Parameters section.

Because we have a Gaussian pump, the reservoir field and eventually the condensate field also evolve into a Gaussian profile after a given time. Due to the repulsive nonlinear-interaction, the gaussian is forced to spread outwards and eventually reflect back. Near the boundaries, the incoming and reflected waves collide and the instability induced by pump beam's noise as well as machine precision, forms vortices. From this point onwards turbulence shall be dictated by interaction between the vortices.

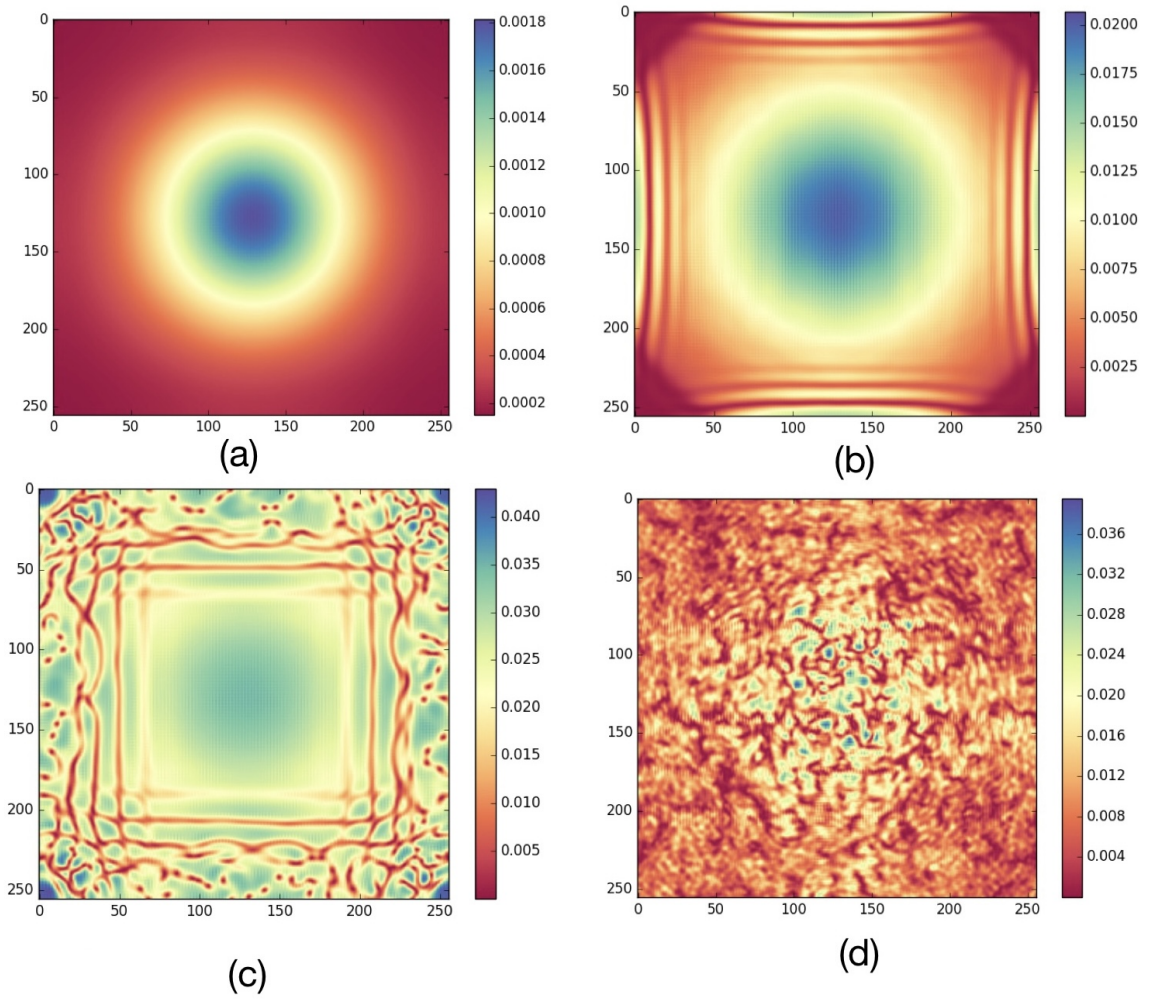

Figure 3: (a) At the 5000th iteration, the condensate mimics the Gaussian beam profile that we use as a pump. (b) The waves are reflected from the boundaries of the domain and interfere with the outgoing waves. (c) Vortices(dark red dots) are created near the boundaries. (d) Vortices are clustered around the center of the domain

Initially, we see only a few vortices as more and more vortices are created, they start to cluster together and move towards the centre of the domain. But they are also pushed outwards by the waves that are emanated due to the pump. The oscillatory movement of the clusters has been seen in the simulation.

In order to mathematically diagnose the evolution through different stages, we decompose the velocity field into incompressible part v_{inc} and compressible part v_{comp}, thus $\boldsymbol{\nabla \cdot \mathbf{v_{inc}}} = 0$ and $\boldsymbol{\nabla \times \mathbf{v_{comp}}} = 0$. Then we compute the corresponding kinetic energies $E_i = \frac{m}{4}\int |v_{inc}|^2 dr$ and $E_c = \frac{m}{4}\int |v_{comp}|^2 dr$. The spectra of compressible kinetic energy shall correspond to sound waves in the polaritonic fluid since those are density perturbations essentially. While the incompressible kinetic energy contains the signature of the vortex configuration in the fluid. At the initial stages when the polariton field is smooth enough, the E_q i.e. quantum pressure energy is minimal. Now the appearance of vortices corresponds to an upsurge in the compressible part of kinetic energy as well as quantum pressure energy, both adhering to density depletion

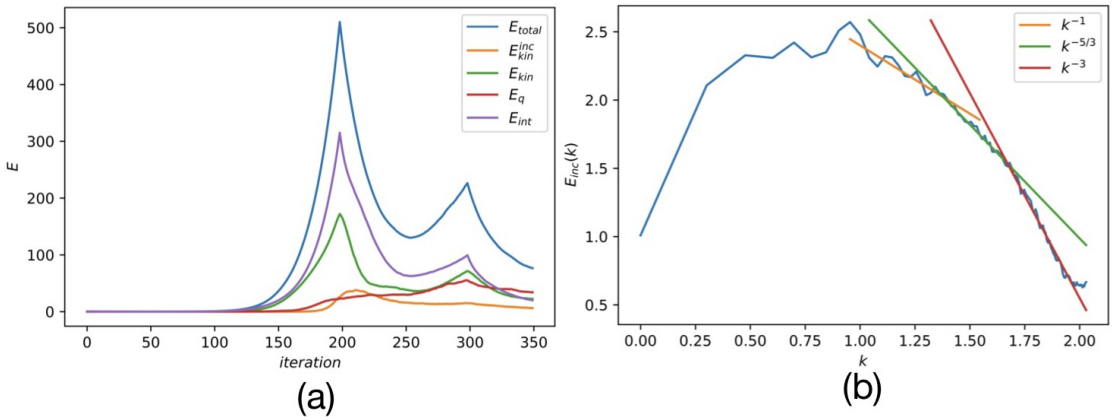

Figure 4: (a) Evolution of different energy components with time. Observe that the incompressible kinetic energy doesn't increase even though total kinetic energy increases. This corresponds to upsurge in compressible kinetic energy due to formation of vortices. (b) The IR, mid-range and UV cascades can be seen in the spectrum of incompressible kinetic energy

near the vortices.

Now in the turbulent state, which can be visually identified by looking at the density profile with a lot many clusters of vortices, we obtain the spectrum of incompressible kinetic energy. The spectrum is computed as:

$$E(k) = \frac{1}{4}\int_{|\mathbf{k}|=k} \left|\widehat{(\rho^{1/2}\mathbf{v})^i}\right|^2 d^2k \tag{41}$$

This is computed basically by summing up Fourier modes in a finitely thick shell in k-space. In the spectrum, we clearly see three cascades. The low wavenumber or the IR-cascade has k^{-1} dependence. In the range, the lengths $1/k$ are much more than the intervortex separation where the interaction is strong enough to produce clusters.

For the highest wavenumbers, i.e in the UV range, where we have k^{-3}, the length scales $1/k$ are shorter than the healing length and hence this cascade corresponds to *intravortex* excitations. Because there is no forcing given at such short length scales, by the very construction of our simulation, **these excitations are being driven by those at longer length scales**. Hence this part corresponds to what we can call **direct energy cascade**.

For the mid-range wavenumbers, the lengths are more than the size of vortex cores and the intervortex length is also small enough to produce intervortex interactions strong enough to produce clusters. In this range, the energy from the length scale of vortices, i.e. healing length is propagated towards larger lengths via the formation of clusters. Hence this range is referred to as **inverse cascade**. A typical or an average size of clusters can be estimated by the lower limit of this range because, at wavenumbers lower than that, clusters fail to form. In our case, we see that this range is less than half an order of magnitude long. Increasing the nonlinearity or number of vortices shall change this range.

12. Future Work

Turbulence is one of the standard far out-of-equilibrium processes in the context of fluids, classical or quantum. But there are other non-equilibrium or even near-equilibrium processes as well, which can be modelled again in the mean-field regime. But an even broader class of phenomenology can be studied beyond the mean field, condensation kinetics in the presence of thermalized polaritons, and BKT transition with quantum noise. Even if we add a stochastic component to the mean-field description itself, after averaging over-density fluctuations, we get KPZ dynamics for the phase profile of the polariton fluid. So if given an opportunity, I would like to work in the space of problems modelled either by stochastic mean-field or even beyond the mean-field approach.

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