1. *Vectorial mode equations (transverse-H-variant)*

   Specialize the Maxwell-curl equation in the frequency domain $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu_0\boldsymbol{H}$, $\boldsymbol{\nabla} \times \boldsymbol{H} = \mathrm{i}\omega\epsilon_0\epsilon\boldsymbol{E}$ to a waveguide configuration with its axis parallel to the Cartesian z-axis (use of symbols as in the lecture). Introduce a field in the form of a waveguide mode with mode profile $\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{H}}$ and propagation constant β, and select the transverse magnetic components of the mode profile $\bar{H}_x$, $\bar{H}_y$ as principal fields. Show that these satisfy the vectorial mode equations

$$\partial_x^2\bar{H}_x + \epsilon\partial_y\frac{1}{\epsilon}\partial_y\bar{H}_x + \partial_{xy}\bar{H}_y - \epsilon\partial_y\frac{1}{\epsilon}\partial_x\bar{H}_y + (k^2\epsilon - \beta^2)\bar{H}_x = 0\,, \tag{1}$$

$$\epsilon\partial_x\frac{1}{\epsilon}\partial_x\bar{H}_y + \partial_y^2\bar{H}_y + \partial_{yx}\bar{H}_x - \epsilon\partial_x\frac{1}{\epsilon}\partial_y\bar{H}_x + (k^2\epsilon - \beta^2)\bar{H}_y = 0\,, \tag{2}$$

   and that the electromagnetic profile components can be expressed through the principal fields as

$$\begin{pmatrix}\bar{E}_x\\ \bar{E}_y\\ \bar{E}_z\end{pmatrix} = \frac{1}{\omega\epsilon_0\epsilon}\begin{pmatrix}\beta\bar{H}_y - \beta^{-1}(\partial_{yx}\bar{H}_x + \partial_y^2\bar{H}_y)\\ -\beta\bar{H}_x + \beta^{-1}(\partial_{xy}\bar{H}_y + \partial_x^2\bar{H}_x)\\ -\mathrm{i}(\partial_x\bar{H}_y - \partial_y\bar{H}_x)\end{pmatrix}, \quad \begin{pmatrix}\bar{H}_x\\ \bar{H}_y\\ \bar{H}_z\end{pmatrix} = \begin{pmatrix}\bar{H}_x\\ \bar{H}_y\\ -\mathrm{i}\beta^{-1}(\partial_x\bar{H}_x + \partial_y\bar{H}_y)\end{pmatrix}. \tag{3}$$

2. *2-D mode equations*

   Consider 2-D TE and TM problems in the frequency domain, as discussed in the lecture, for a 2-D waveguide configuration with its axis oriented along the Cartesian z-axis (assume nonmagnetic media, $\mu = 1$). The structure is constant along the y-axis; the permittivity ϵ depends (smoothly, or with discontinuities) on the x-coordinate only. Introduce a field in the form of a waveguide mode with 1-D mode profile $\bar{\boldsymbol{E}}(x)$, $\bar{\boldsymbol{H}}(x)$ and propagation constant β.

   (a) For the 2-D TE setting, choose the transverse electric component $\bar{E}_y$ of the mode profile as the principal field. Show that this component satisfies the scalar 2-D TE mode equation

$$\partial_x^2\bar{E}_y + (k^2\epsilon - \beta^2)\bar{E}_y = 0, \tag{4}$$

   that the other electromagnetic field components can be expressed as

$$\bar{E}_x = 0, \quad \bar{E}_z = 0, \quad \bar{H}_x = \frac{-\beta}{\omega\mu_0}\,\bar{E}_y, \quad \bar{H}_y = 0, \quad \bar{H}_z = \frac{\mathrm{i}}{\omega\mu_0}\,\partial_x\bar{E}_y, \tag{5}$$

   and that continuity of $\bar{E}_y$ and of $\partial_x\bar{E}_y$ is required at any dielectric interfaces.

   (b) Now look at the 2-D TM setting. Choose the transverse magnetic component $\bar{H}_y$ of the mode profile as the principal field. Show that this component satisfies the scalar 2-D TM mode equation

$$\epsilon\partial_x\frac{1}{\epsilon}\partial_x\bar{H}_y + (k^2\epsilon - \beta^2)\bar{H}_y = 0, \tag{6}$$

   that all electromagnetic field components can be expressed as

$$\bar{E}_x = \frac{\beta}{\omega\epsilon_0\epsilon}\,\bar{H}_y, \quad \bar{E}_y = 0, \quad \bar{E}_z = \frac{-\mathrm{i}}{\omega\epsilon_0\epsilon}\,\partial_x\bar{H}_y, \quad \bar{H}_x = 0, \quad \bar{H}_z = 0, \tag{7}$$

   and that continuity of $\bar{H}_y$ and of $\frac{1}{\epsilon}\partial_x\bar{H}_y$ is required at any dielectric interfaces.

3. *Guided modes of dielectric multilayer slab waveguides*

   Consider the solver for the guided modes of dielectric multilayer slab waveguides with 1-D cross sections at `www.siio.eu/oms.html`.

   Imagine that you would need to test the results for reliability. To that end, have the script compute the guided TE and TM modes of a waveguide with one interior core layer with refractive index 2.0 and a thickness of $0.5\,\mu$m, on top of a substrate with refractive index 1.5, covered by air with refractive index 1.0, at a wavelength of $1.55\,\mu$m. The solver output needs to satisfy the relations from problem 2, and the results need to describe guided modes (decaying fields for $x \to \pm\infty$).

   Outline a procedure for numerical assessment of the mode solver results:

   - Check the properties of the profile components (dis- / continuity, signs) that can be verified by inspecting the respective mode profile plots.
   - Describe a numerical procedure for veryifying the differential equations, after exporting adequate mode profile components.

4. *Guided modes of a rectangular dielectric channel, approximate solutions*

   Consider the solver for the guided modes of dielectric multilayer waveguides with rectangular 2-D cross sections at `www.siio.eu/eims.html`. Note that this is an explicitly *approximate* solver.

   Imagine that you would need to quickly check the results for reliability. To that end, have the script compute the guided TE- and TM-like modes of a waveguide with a rectangular core of width $1.0\,\mu$m, thickness $0.4\,\mu$m, with refractive index 1.99, embedded in a material with refractive index 1.45, at a wavelength of $1.55\,\mu$m.

   - State the properties (differential equations (the most appropriate of the many forms), dis- / continuity, symmetry, ...) that need to be satisfied by the modes that are exact solutions of the problem that the solver addresses.
   - Prepare a list of issues where the solver results obviously violate these conditions.

Hand in your solutions until Wednesday, June 04, 09:15. Good luck!