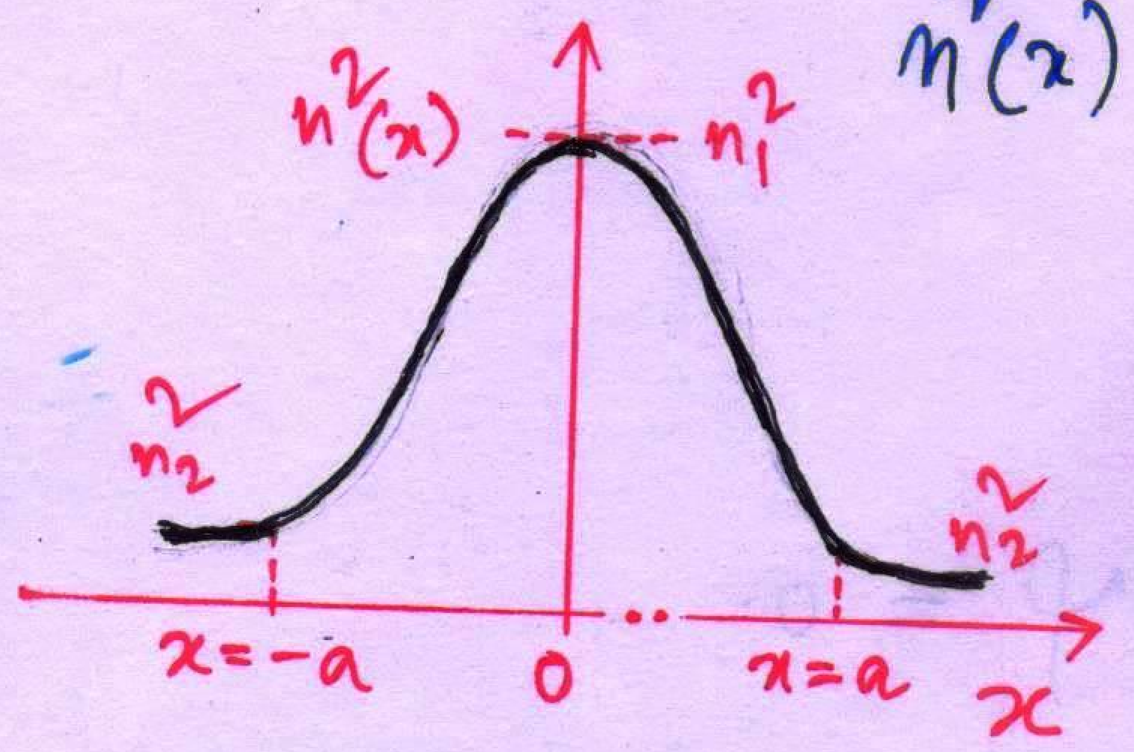
Modes of Parabolic index planar waveguide

For a parabolic R.I. profile planar slab waveguide we write

$$n^2(x) = n_1^2\left[1 - 2\Delta\left(\frac{x}{a}\right)^2\right]$$



$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \simeq \frac{n_1 - n_2}{2n_1}$$

= Relative core-cladding index diff.

Such that at $x = 0$, $n^2(0) = n_1^2$: center of core

and for $x = a$, $n^2(a) = n_2^2$: cladding

Therefore the wave equation becomes

$$\frac{d^2\psi}{dx^2} + \left\{k_0^2 n_1^2\left[1 - 2\Delta\left(\frac{x}{a}\right)^2\right] - \beta^2\right\}\psi = 0$$

or $$\frac{d^2\psi}{dx^2} + \left[k_0^2 n_1^2 - \beta^2 - \frac{k_0^2 n_1^2 2\Delta}{a^2}\cdot x^2\right]\psi = 0$$

Now substitute $\gamma = \left[\frac{k_0^2 n_1^2 2\Delta}{a^2}\right]^{\frac{1}{4}}$ and $\xi = \gamma x$

Then $$\frac{d^2\psi}{dx^2} + \left[(k_0^2 n_1^2 - \beta^2) - \gamma^4 x^2\right]\psi = 0$$

We may write —

$$\frac{d^2\psi}{dx^2} + \left[\frac{k_0^2 n_1^2 - \beta^2}{\gamma^2} - \gamma^2 x^2\right]\gamma^2\psi = 0.$$

Now, changing variable from x to ξ —

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi}\frac{d\xi}{dx} = \gamma\frac{d\psi}{d\xi} \qquad \xi = \gamma x \;\Big|\; \frac{d\xi}{dx} = \gamma$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\gamma\frac{d\psi}{d\xi}\right) = \gamma\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\right)\frac{d\xi}{dx}$$

$$= \gamma^2\frac{d^2\psi}{d\xi^2}$$

So the wave equation becomes

$$\gamma^2 \frac{d^2\psi}{d\xi^2} + \left[\frac{k_0^2 n_1^2 - \beta^2}{\gamma^2} - \xi^2\right]\gamma^2\psi = 0$$

Define $\frac{k_0^2 n_1^2 - \beta^2}{\gamma^2} = \Lambda$, then

$$\frac{d^2\psi}{d\xi^2} + (\Lambda - \xi^2)\psi = 0.$$

This equation corresponds to 1d Schrödinger's eqn for a Linear Harmonic Oscillator (LHO).

$$\psi(\xi) \longrightarrow 0 \text{ as } \xi \to \pm\infty$$

and $\Lambda = 2m+1$ with $m = 0, 1, 2, 3, \ldots$

this yields allowed solutions for β:

$$\beta_m = k_0 n_1 \left[1 - (2m+1)\frac{\sqrt{2\Delta}}{k_0 n_1 a}\right]^{1/2}$$

Since, $k_0^2 n_1^2 - \beta^2 = \gamma^2 \Lambda$

$$\beta^2 = k_0^2 n_1^2 - (2m+1)\sqrt{\frac{k_0^2 n_1^2 2\Delta}{a^2}}$$

$$= k_0^2 n_1^2 \left[1 - (2m+1)\frac{\sqrt{2\Delta}}{k_0 n_1 a}\right]$$

The complete solution for the wavefunction is

$$\psi_m(\xi) = N_m H_m(\xi)\, e^{-\xi^2/2}$$

with $N_m = \sqrt{\frac{\gamma}{2^m m! \sqrt{\pi}}}$. Hermite Gauss solution

m designates the order of modes

(order of eigen states in LHO problem)

Modal field profiles

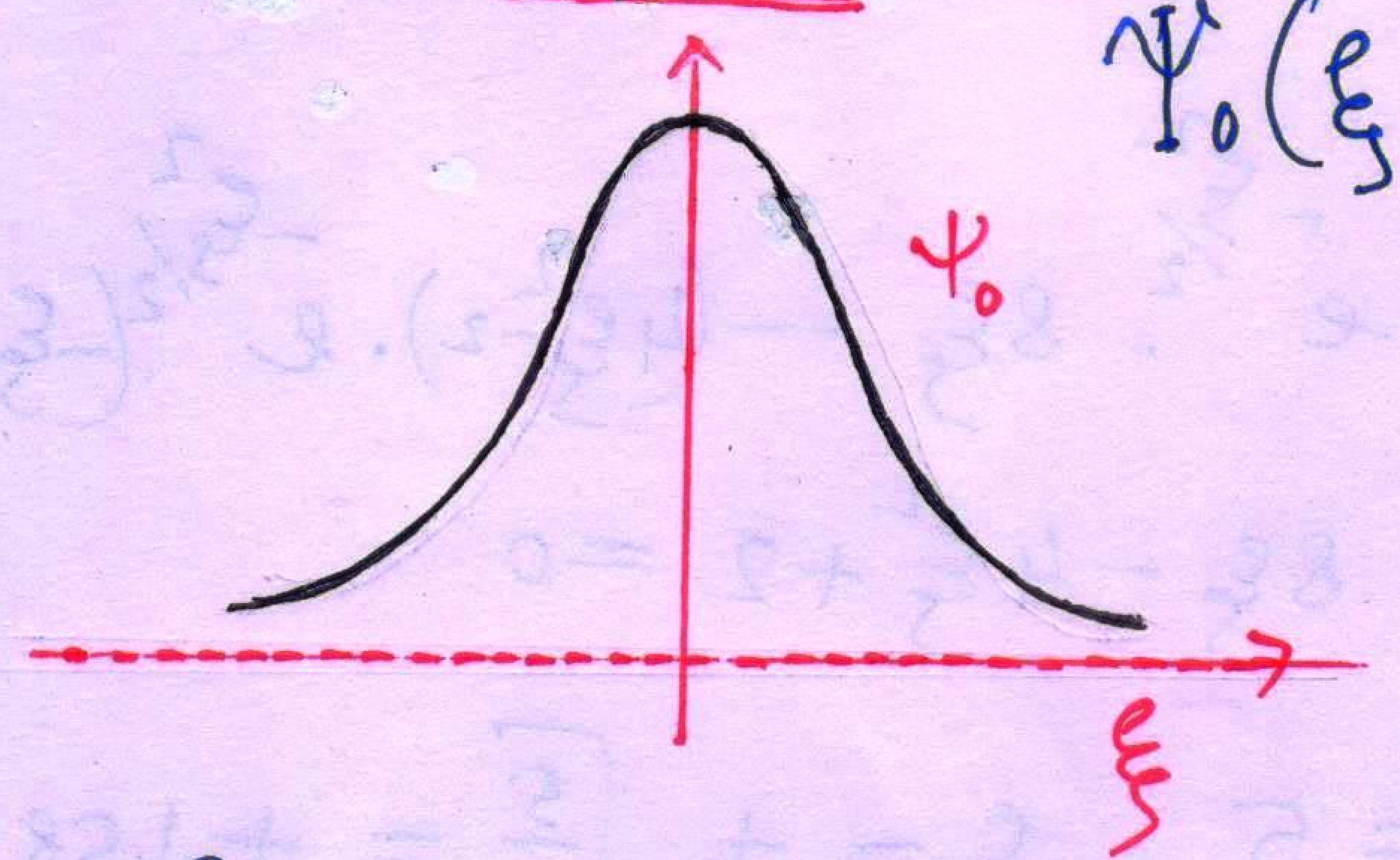
$$\psi_m(\xi) = C_m H_m(\xi)\, e^{-\xi^2/2} \quad ; \quad C_m = \left[\frac{\gamma}{2^m\, m!\sqrt{\pi}}\right]^{1/2}$$

Hermite polynomials (a few lower orders):

$$H_0(\xi) = 1 \qquad H_2(\xi) = 4\xi^2 - 2$$

$$H_1(\xi) = 2\xi \qquad H_3(\xi) = 8\xi^3 - 12\xi$$

For $\underline{m = 0}$:



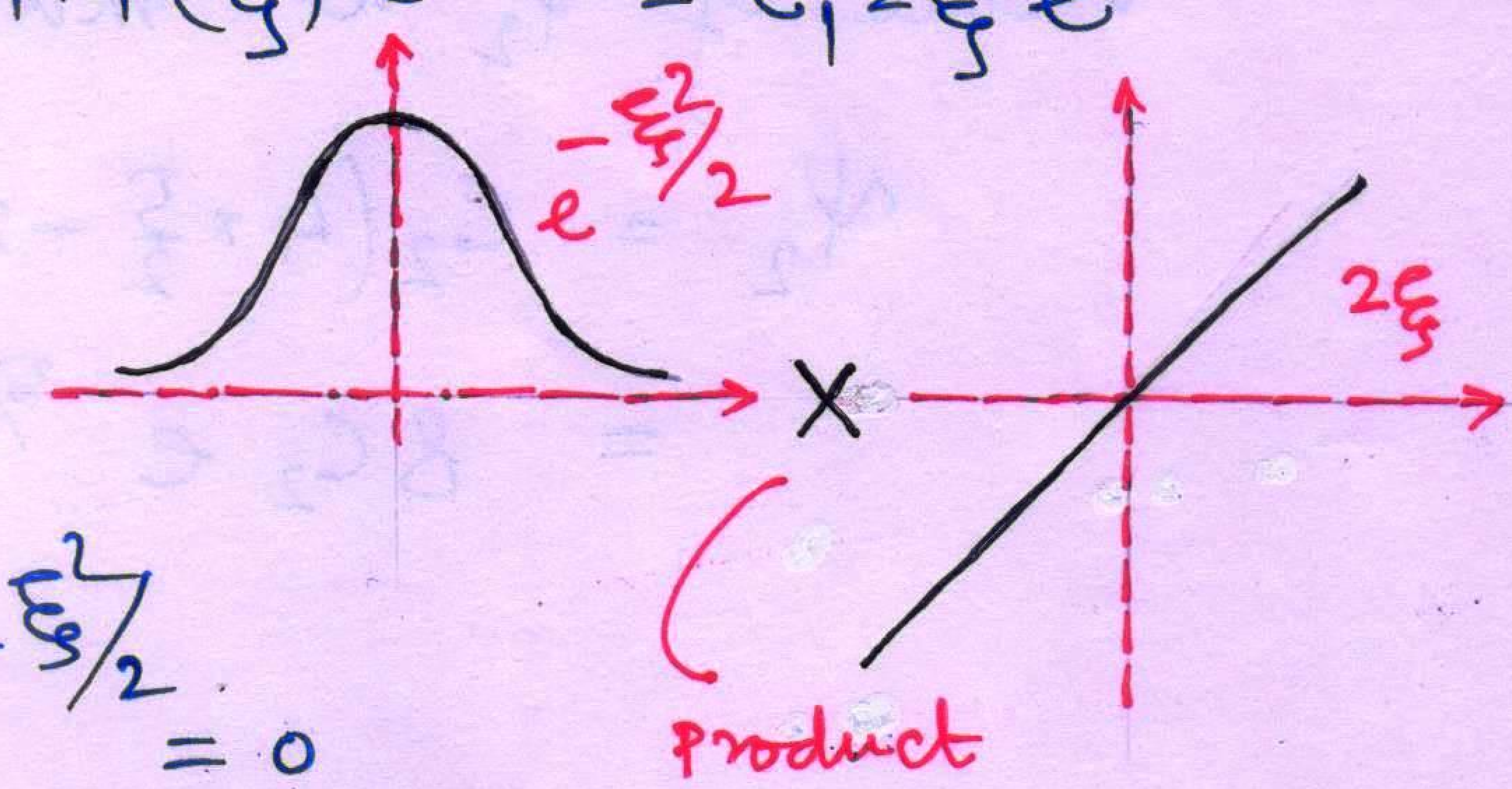
$$\psi_0(\xi) = C_0 H_0(\xi)\, e^{-\xi^2/2} = C_0\, e^{-\xi^2/2}$$

$\Rightarrow$ a simple Gaussian with amplitude $\psi_0 = C_0$

For $\underline{m = 1}$: $\quad \psi_1(\xi) = C_1 H_1(\xi)\, e^{-\xi^2/2} = C_1\, 2\xi\, e^{-\xi^2/2}$

At $\xi = 0 \qquad \psi_1 = 0$

and maxima are obtained by



$$\frac{d}{d\xi}\left(\xi\, e^{-\xi^2/2}\right) = 0$$

ie, $\quad 1 \cdot e^{-\xi^2/2} + \xi(-\xi)\, e^{-\xi^2/2} = 0$

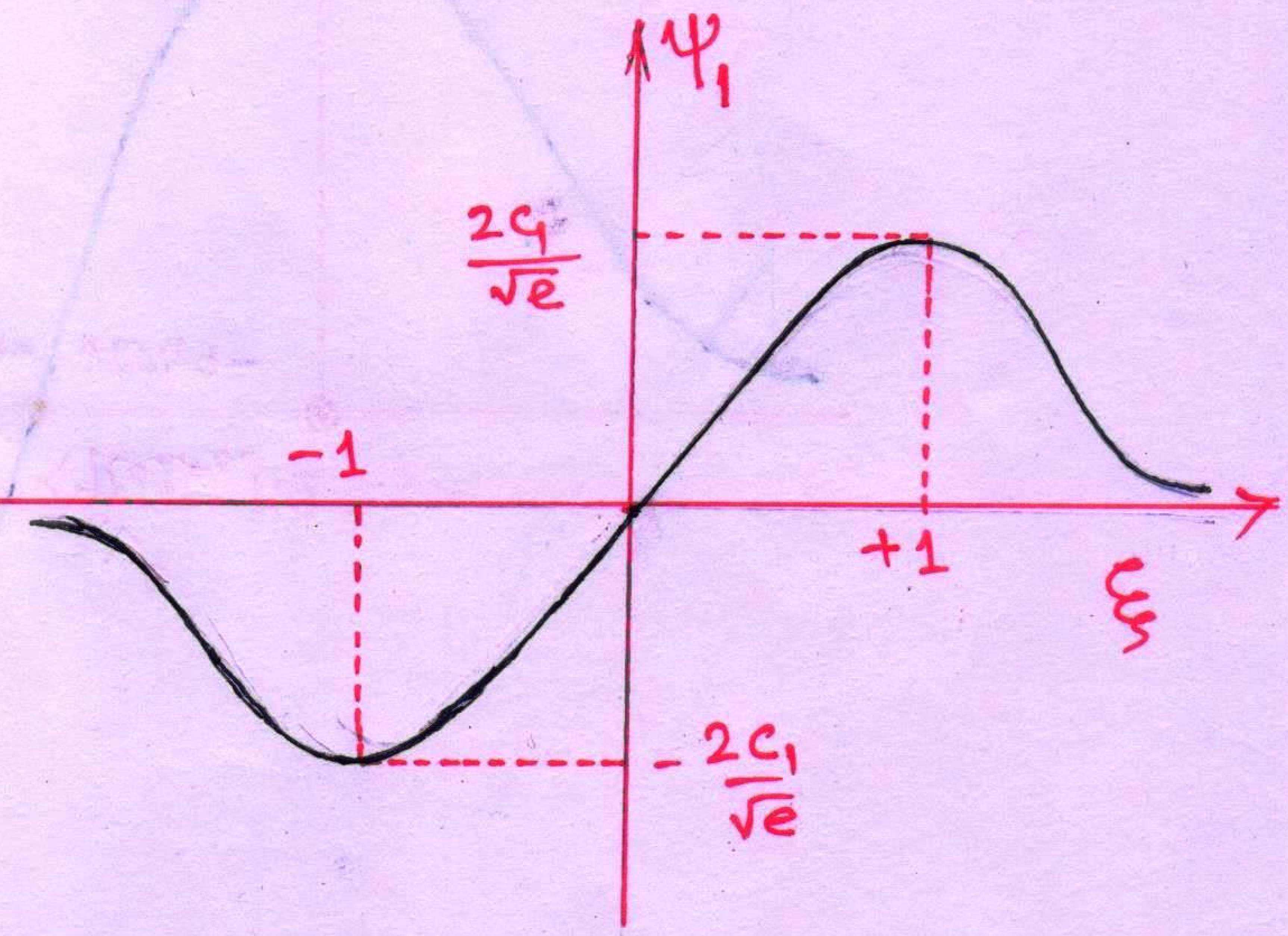
or, $\quad (1 - \xi^2)\, e^{-\xi^2/2} = 0$

Then $\xi^2 = 1 \qquad \xi = \pm 1$

with these values

$\psi_1 = \dfrac{2C_1}{\sqrt{e}}$ for $\xi = +1$

and $\psi_2 = -\dfrac{2C_1}{\sqrt{e}}$ for $\xi = -1$



For $\underline{m=2}$:

$$\psi_2(\xi) = C_2 H_2(\xi)\, e^{-\xi^2/2}$$

$$= c_2(4\xi^2-2)\, e^{-\xi^2/2}$$

At $\xi = 0$ $\quad \psi_2 = -2c_2$

ψ_2 has zeros at $4\xi^2-2=0$ ie, $\xi = \pm\frac{1}{\sqrt{2}}$

$= \pm 0.7071$

The maxima (peaks) are obtained from:

$$\frac{d}{d\xi}(4\xi^2-2)e^{-\xi^2/2} = e^{-\xi^2/2}\cdot 8\xi - (4\xi^2-2)\cdot e^{-\xi^2/2}\cdot(-\xi)$$

$$= 8\xi - 4\xi^2 + 2 = 0$$

$$\therefore \quad 2\xi^2 = 5, \quad \xi = \pm\sqrt{\frac{5}{2}} = \pm 1.5811$$

Value of ψ_2 at maxima:

$$\psi_2 = c_2\left(4\times\frac{5}{2}-2\right) e^{-\frac{5}{2}/2}$$

$$= 8c_2\, e^{-5/4}$$