Coupled mode equations

In this appendix we will derive the coupled mode equations which describe the variation in the amplitude of the waves propagating in each individual waveguide of a directional coupler. Let $n_1(x, y)$ and $n_2(x, y)$ represent the refractive index variation in the transverse plane of waveguide 1 in the absence of waveguide 2 and that of waveguide 2 in the absence of waveguide 1. Let $n(x, y)$ represent the refractive index variation of the directional coupler consisting of the waveguides 1 and 2. For example, for a directional coupler consisting of two step-index planer waveguides, $n_1(x)$, $n_2(x)$ and $n(x)$ are shown in Fig. G1.

If β_1 and β_2 represent the propagation constants of the modes of

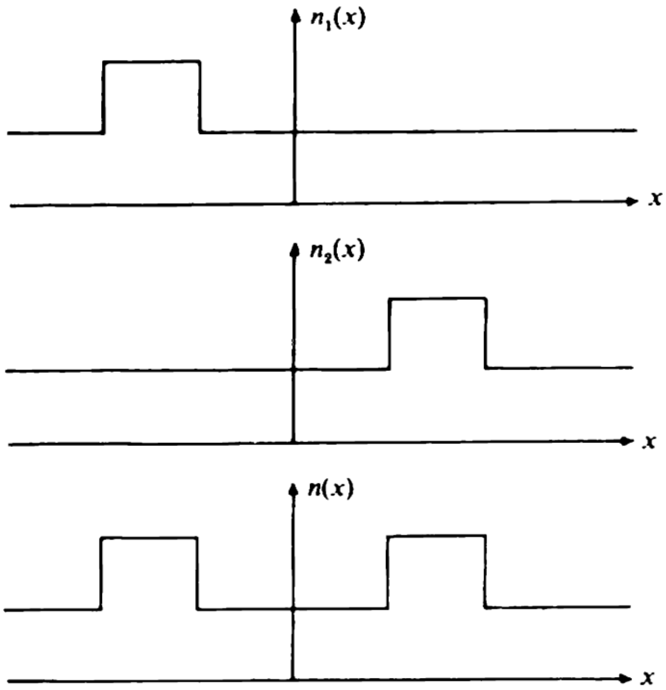

Fig. G1. (*a*) and (*b*) The refractive index profiles corresponding to two isolated step index planar waveguides and (*c*) The refractive index profile corresponding to a directional coupler formed by the two waveguides.

waveguides 1 and 2 in the absence of the other then we may write

$$\nabla_t^2 \psi_1 + [k_0^2 n_1^2(x, y) - \beta_1^2]\psi_1 = 0 \tag{G1}$$

$$\nabla_t^2 \psi_2 + [k_0^2 n_2^2(x, y) - \beta_2^2]\psi_2 = 0 \tag{G2}$$

where

$$\nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{G3}$$

and $\psi_1(x, y)$ and $\psi_2(x, y)$ represent the transverse modc field patterns of waveguides 1 and 2 respectively in the absence of the other.

If $\Psi(x, y, z)$ represents the total field of the directional coupler structure then we have

$$\nabla_t^2 \Psi + \frac{\partial^2 \Psi}{\partial z^2} + k_0^2 n^2(x, y)\Psi = 0 \tag{G4}$$

We now approximate Ψ as follows

$$\Psi(x, y, z) = A(z)\psi_1(x, y)\mathrm{e}^{-\mathrm{i}\beta_1 z} + B(z)\psi_2(x, y)\mathrm{e}^{-\mathrm{i}\beta_2 z} \tag{G5}$$

which is valid when the two waveguides are not very strongly interacting. In Eq. (G5) we have written the total field as a superposition of the fields in the first and second waveguides with amplitudes $A(z)$ and $B(z)$ which are functions of z. For infinite separation between the two waveguides, obviously the waveguides are noninteracting and A and B would then be *independent of z. The coupling between the two waveguides leads to z* dependent amplitudes. Substituting for Ψ in Eq. (G4) we obtain

$$\begin{aligned} &A\mathrm{e}^{-\mathrm{i}\beta_1 z}(\nabla_t^2\psi_1 - \beta_1^2\psi_1 + k_0^2 n^2\psi_1) + B\mathrm{e}^{-\mathrm{i}\beta_2 z}(\nabla_t^2\psi_2 - \beta_2^2\psi_2 + k_0^2 n^2\psi_2) \\ &\quad - 2\mathrm{i}\beta_1(\mathrm{d}A/\mathrm{d}z)\psi_1\mathrm{e}^{-\mathrm{i}\beta_1 z} - 2\mathrm{i}\beta_2(\mathrm{d}B/\mathrm{d}z)\psi_2\mathrm{e}^{-\mathrm{i}\beta_2 z} = 0 \end{aligned} \tag{G6}$$

where we have neglected terms proportional to $\mathrm{d}^2A/\mathrm{d}z^2$ and $\mathrm{d}^2B/\mathrm{d}z^2$ which is justified when $A(z)$ and $B(z)$ are slowly varying functions of z. Using Eqs. (G1) and (G2), Eq. (G6) becomes

$$\begin{aligned} &k_0^2\Delta n_1^2 A\psi_1 + k_0^2\Delta n_2^2 B\psi_2\mathrm{e}^{\mathrm{i}\Delta\beta z} - 2\mathrm{i}\beta_1(\mathrm{d}A/\mathrm{d}z)\psi_1 \\ &\quad - 2\mathrm{i}\beta_2(\mathrm{d}B/\mathrm{d}z)\psi_2\mathrm{e}^{\mathrm{i}\Delta\beta z} = 0 \end{aligned} \tag{G7}$$

where

$$\Delta n_1^2 = n^2(x, y) - n_1^2(x, y) \tag{G8}$$

$$\Delta n_2^2 = n^2(x, y) - n_2^2(x, y) \tag{G9}$$

$$\Delta\beta = \beta_1 - \beta_2 \tag{G10}$$

Multiplying Eq. (G7) by ψ_1^* and integrating over the whole cross section, we obtain

$$dA/dz = -i\kappa_{11} A(z) - i\kappa_{12} B e^{i\Delta\beta z} \tag{G11}$$

where

$$\kappa_{11} = \frac{k_0^2}{2\beta_1} \frac{\iint_{-\infty}^{\infty} \psi_1^* \Delta n_1^2 \psi_1 \, dx dy}{\iint_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx dy} \tag{G12}$$

$$\kappa_{12} = \frac{k_0^2}{2\beta_1} \frac{\iint_{-\infty}^{\infty} \psi_1^* \Delta n_2^2 \psi_2 \, dx dy}{\iint_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx dy} \tag{G13}$$

In writing Eq. (G11) we have neglected the overlap integral of the modes i.e. we assume

$$\iint_{-\infty}^{\infty} \psi_1^* \psi_2 \, dx dy \ll \iint_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx dy$$

which is valid for weak coupling between the waveguides.

Similarly if we multiply Eq. (G7) by ψ_2^* and integrate we would obtain

$$dB/dz = -i\kappa_{22} B - i\kappa_{21} A e^{-i\Delta\beta z} \tag{G14}$$

where

$$\kappa_{22} = \frac{k_0^2}{2\beta_2} \frac{\iint_{-\infty}^{\infty} \psi_2^* \Delta n_2^2 \psi_2 \, dx dy}{\iint_{-\infty}^{\infty} \psi_2^* \psi_2 \, dx dy} \tag{G15}$$

$$\kappa_{21} = \frac{k_0^2}{2\beta_2} \frac{\iint_{-\infty}^{\infty} \psi_2^* \Delta n_1^2 \psi_1 \, dx dy}{\iint_{-\infty}^{\infty} \psi_2^* \psi_2 \, dx dy} \tag{G16}$$

We can write Eqs. (G11) and (G14) in a different form if we define

$$a(z) = A(z) e^{-i\beta_1 z} \tag{G17}$$

$$b(z) = B(z) e^{-i\beta_2 z} \tag{G18}$$

Substituting from Eqs. (G17) and (G18) in Eqs. (G11) and (G14), we have

$$da/dz = -\mathrm{i}(\beta_1 + \kappa_{11})a - \mathrm{i}\kappa_{12}b \tag{G19}$$

$$db/dz = -\mathrm{i}(\beta_2 + \kappa_{22})b - \mathrm{i}\kappa_{21}a \tag{G20}$$

The above two equations represent the coupled mode equations. It follows from Eqs. (G19) and (G20) that κ_{11} and κ_{22} represent the corrections to the propagation constants of each individual waveguide mode due to the presence of the other waveguide. These correction factors are normally neglected in the analysis although one can very easily incorporate them. Thus the coupled equations may be written as

$$da/dz = -\mathrm{i}\beta_1 a - \mathrm{i}\kappa_{12}b \tag{G21}$$

$$db/dz = -\mathrm{i}\beta_2 b - \mathrm{i}\kappa_{21}a \tag{G22}$$

These are the coupled mode equations which have been used in Chapter 14.