Powerflow with a mode

Powerflow with a mode can be determined from the time average of the Poynting vector:

$$\langle S \rangle = \frac{1}{2} \mathrm{Re}\left(\mathcal{E} \times H^*\right)$$

We calculate the power associated with the TE mode of a step-index symmetric slab waveguide.

For TE-mode of a slab, we have

$$E_y = E_y(x)\, e^{i(\omega t - \beta z)}$$

$$H_x = -\frac{\beta}{\omega\mu_0} E_y = -\frac{\beta}{\omega\mu_0} E_y\, e^{i(\omega t - \beta z)}$$

$$H_z = \frac{i}{\omega\mu_0} \frac{\partial E_y}{\partial x} = \frac{i}{\omega\mu_0} \frac{dE_y}{dx} e^{i(\omega t - \beta z)}$$

$E_y(x) = A\cos \chi x$: core

$= C e^{-\gamma|x|}$: cladding

Using these relations, we readily get

$$\langle S_y \rangle = \frac{1}{2}\mathrm{Re}\left(E_z \cdot H_x\right) = 0$$

ie, no power flow along y-direction

$E_z = 0$

$$\langle S_x \rangle = \frac{1}{2} \mathrm{Re}\left(E_y H_z\right)$$

$E_y = A\cos\chi x$; $\frac{dE_y}{dx} = A\chi \sin\chi x$ (for core)

Hence, $\int_{-d/2}^{d/2} E_y H_z\, dx = \frac{iA^2}{2\omega\mu_0} \int_{-d/2}^{+d/2} \sin 2\chi x\, dx = 0$

In cladding: $\int_{-\infty}^{-d/2} E_y H_z\, dx + \int_{d/2}^{\infty} E_y H_z\, dx = \int_{-\infty}^{-d/2} c^2 \gamma e^{+2\gamma x} dx - \int_{d/2}^{\infty} c^2 \gamma e^{-2\gamma x} dx$

$$= c^2 \frac{e^{-\gamma d}}{2} - c^2 \frac{e^{-\gamma d}}{2} = 0 \text{ : no power flow along } y.$$

Now $\langle S_z \rangle = -\frac{1}{2} \mathrm{Re}(\mathcal{E}_y . H_x) = \frac{\beta}{2\omega\mu_0} |E_y|^2$

So power associated with mode per unit length in the y-direction is

$$P = \frac{1}{2} \frac{\beta}{\omega\mu_0} \int_{-\infty}^{+\infty} |E_y|^2 dx .$$

Using this expression one can evaluate the power associated with any of the symmetric and antisymmetric modes of a slab waveguide.

Considering a symmetric TE-mode, we write

$$P = \frac{\beta}{2\omega\mu_0} \cdot 2 \times \left[A^2 \int_0^{d/2} \cos^2 \kappa x \, dx + C^2 \int_{d/2}^{\infty} e^{-2\gamma x} dx \right]$$

$$= \frac{\beta}{2\omega\mu_0} \times 2 \left[\frac{A^2}{2} \int_0^{d/2} (1 + \cos 2\kappa x) \, dx + C^2 \left. \frac{e^{-2\gamma x}}{-2\gamma} \right|_{d/2}^{\infty} \right]$$

$$= \frac{\beta}{2\omega\mu_0} \left[A^2 \left(\frac{d}{2} + \frac{\sin \kappa d}{2\kappa} \right) + C^2 \frac{e^{-\gamma d}}{\gamma} \right]$$

Now $\quad C e^{-\gamma d/2} = A \cos \kappa d/2 \quad$ at $d/2$

$$C^2 = A^2 \cos^2 \kappa d/2 \, . \, e^{\gamma d/2}$$

$$P = \frac{\beta A^2}{2\omega\mu_0} \left[\frac{d}{2} + \frac{\sin \kappa d}{2\kappa} + \frac{\cos^2 \kappa d/2}{\gamma} \right] = \frac{\beta A^2}{4\mu\omega} \left[d + \frac{\sin \kappa d}{\kappa} + \frac{2\cos^2 \kappa d/2}{\gamma} \right]$$

$$= \frac{\beta A^2}{4\mu\omega_0} \cdot \left[d + \frac{2 \sin \kappa d/2 \cdot \cos \kappa d/2}{\kappa} + \frac{2 \cos^2 \kappa d/2}{\gamma} \right]$$

$$= \frac{\beta A^2}{4\mu\omega_0} \left[d + 2 \cos^2 \kappa d/2 \left(\frac{\tan \kappa d/2}{\kappa} + \frac{1}{\gamma} \right) \right]$$

$$\boxed{P = \frac{\beta A^2}{4\mu\omega_0} \cdot \left(d + \frac{2}{\gamma} \right)} \qquad \left\{ \text{using } \tan \kappa d/2 = \frac{\gamma}{\kappa} \right\}$$

Orthogonality of modes

Let us rewrite the wave equation determining the TE-modes as an eigen value problem

$$\frac{d^2\psi_m}{dx^2} + \left[k_0^2 n^2(x) - \beta_m^2\right]\psi_m = 0$$

i.e.,

$$\frac{d^2\psi_m}{dx^2} + k_0^2 n^2(x)\psi_m = \lambda_m \psi_m \quad \text{——(1)}$$

Here $\lambda_m = \beta_m^2$ the eigenvalue of the operator $\left[\frac{d^2}{dx^2} + k_0^2 n^2(x)\right]$ corresponding to eigenstate ψ_m.

The complex conjugate of the eigenvalue equation corresponding to λ_k is

$$\frac{d^2\psi_k^*}{dx^2} + k_0^2 n^2(x)\psi_k^* = \lambda_k^* \psi_k^* \quad \text{——(2)}$$

Now, $\psi_k^* \times (1) - (2) \times \psi_m$ yields —

$$\psi_k^* \cdot \frac{d^2\psi_m}{dx^2} - \frac{d^2\psi_k^*}{dx^2}\cdot\psi_m = (\lambda_m - \lambda_k^*)\psi_m\psi_k^*$$

i.e.,

$$\frac{d}{dx}\left(\psi_k^* \frac{d\psi_m}{dx} - \psi_m \frac{d\psi_k^*}{dx}\right) = (\lambda_m - \lambda_k^*)\psi_m\psi_k^*$$

Integrating from $-\infty$ to $+\infty$

$$\underbrace{(\lambda_m - \lambda_k^*)}_{=0} \underbrace{\int \psi_m\psi_k^* \, dx}_{=0} = \left.\psi_k^* \frac{d\psi_m}{dx} - \psi_m \frac{d\psi_k^*}{dx}\right|_{-\infty}^{+\infty} = 0.$$

possible :

<u>for $m = k$</u> : $\int_{-\infty}^{+\infty} \psi_m\psi_k^* \, dx = \int_{-\infty}^{+\infty} |\psi_m|^2 dx$ = +ve & definitive

$\therefore \lambda_m = \lambda_k^* \Rightarrow \lambda_m = \lambda_m^*$ all λ's are <u>real</u>.

<u>for $m \neq k$</u> : $\lambda_m \neq \lambda_k^*$ $\quad \therefore \int_{-\infty}^{+\infty} \psi_k^* \psi_m \, dx = 0$

<u>orthogonal</u>.

Symmetric R.I. Profile : symmetric & antisymmetric modes.

Consider symmetric R.I. Profile of the slab waveguide:

$n^2(-x) = n^2(x)$.

for TE-modes : $\frac{d^2 E_y}{dx^2} + k_0^2 n^2(x) E_y(x) = \beta^2 E_y(x)$.

Transform $x \to -x$, then

$\frac{d^2 E_y(-x)}{dx^2} + k_0^2 n^2(x) E_y(-x) = \beta^2 E_y(-x)$ | since $n^2(-x) = n^2(x)$

We see that both $E_y(x)$ and $E_y(-x)$ satisfy the same eigenvalue equation. Hence these $E_y(x)$ and $E_y(-x)$ are eigenfunctions belonging to the same eigenvalue β^2.

So, if the mode is non-degenerate, then $E_y(-x)$ must be a multiple of $E_y(x)$.

ie, $E_y(-x) = \lambda E_y(x)$

Transforming $x \longrightarrow -x$, $E_y(x) = \lambda E_y(-x) = \lambda^2 E_y(x)$

ie, $\lambda^2 = 1$ or, $\lambda = \pm 1$.

Hence $E_y(-x) = \pm E_y(x)$. So modes are symmetric or antisymmetric.