13.6.1 Fractional modal power in the core

One of the important parameters associated with a fibre optic waveguide is the fractional power carried in the core. Now, the power in the core of the fibre is given by

$$P_{\text{core}} = \text{const.} \int_0^a \int_0^{2\pi} |\psi|^2 r \, \mathrm{d}r \, \mathrm{d}\phi$$

$$= \frac{2C}{J_l^2(U)} \int_0^a J_l^2\left(\frac{Ur}{a}\right) r \, \mathrm{d}r \int_0^{2\pi} \cos^2 l\phi \, \mathrm{d}\phi$$

or

$$P_{\text{core}} = C \frac{\pi a^2}{U^2} \frac{2}{J_l^2(U)} \int_0^U J_l^2(x) x \, \mathrm{d}x$$

$$= C\pi a^2 \left[1 - \frac{J_{l-1}(U) J_{l+1}(U)}{J_l^2(U)} \right] \tag{13.55}$$

where C is a constant and use has been made of the result derived in Problem 13.14. Similarly the power in the cladding is given by

$$P_{\text{clad}} = \text{const.} \int_a^\infty \int_0^{2\pi} |\psi|^2 r \, \mathrm{d}r \, \mathrm{d}\phi$$

$$= C\pi a^2 \left[\frac{K_{l-1}(W) K_{l+1}(W)}{K_l^2(W)} - 1 \right] \tag{13.56}$$

The total power is

$$P_{\text{tot}} = P_{\text{core}} + P_{\text{clad}}$$

$$= C\pi a^2 \frac{V^2}{U^2} \frac{K_{l+1}(W) K_{l-1}(W)}{K_l^2(W)} \tag{13.57}$$

where use has been made of the equation

$$\frac{U^2 J_{l+1}(U) J_{l-1}(U)}{J_l^2(U)} = -W^2 \frac{K_{l+1}(W) K_{l-1}(W)}{K_l^2(W)} \tag{13.58}$$

which follows from the eigenvalue equations. The fractional power propagating in the core is thus given by

$$\eta = \frac{P_{\text{core}}}{P_{\text{tot}}} = \left[\frac{W^2}{V^2} + \frac{U^2}{V^2} \frac{K_l^2(W)}{K_{l+1}(W) K_{l-1}(W)} \right] \tag{13.59}$$

Thus as the mode approaches cutoff

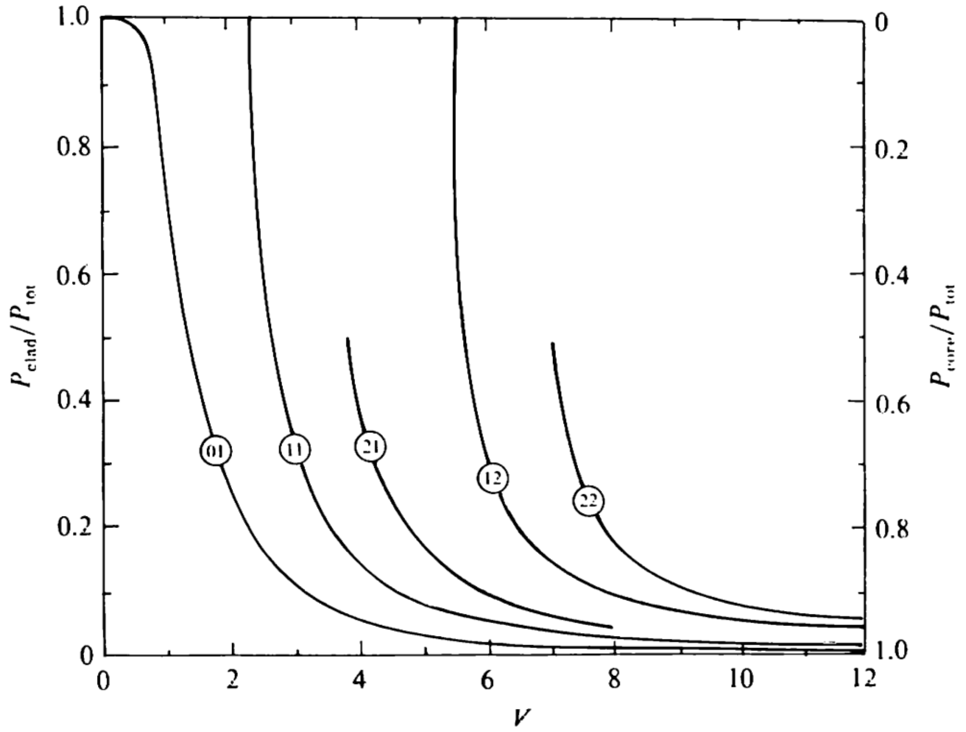
$$V \to V_c, \quad W \to 0, \quad U \to V_c \tag{13.60}$$

and if we now use the limiting forms of $K_l(W)$ given in the footnote on page 379, we obtain

$$\eta \to \begin{cases} 0 & \text{for} \quad l = 0 \quad \text{and} \quad 1 \\ (l-1)/l & \text{for} \quad l \geqslant 2 \end{cases} \tag{13.61}$$

In Fig. 13.11 we have plotted the fractional power contained in the core and in the cladding as a function of V for various modes of a step index fibre. Notice that the power associated with a particular mode is concentrated in the core for large values of V, i.e., far from cutoff.

Fig. 13.11 Variation of the fractional power contained in the cladding with V for some low order modes in a step index fibre. (Adapted from Gloge (1971).)



13.13 Splice loss

One of the great advantages of the Gaussian approximation is the fact that it gives us simple analytical expressions for losses at fibre splices. At the joint there could be three types of misalignments: (*a*) longitudinal offset (*b*) transverse offset, (*c*) angular misalignment (see Fig. 13.22). As a specific example, we consider transverse misalignment (Fig. 13.22(*b*)). The two single mode fibres are represented by Gaussian fundamental modes with spot sizes w_1 and w_2. Let us consider the direction of misalignment to be along the x-direction. With respect to the coordinate axes fixed on the fibre the normalized Gaussian modes can be represented by

$$\psi_1(x, y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{w_1} \mathrm{e}^{-(x^2+y^2)/w_1^2} \tag{13.156}$$

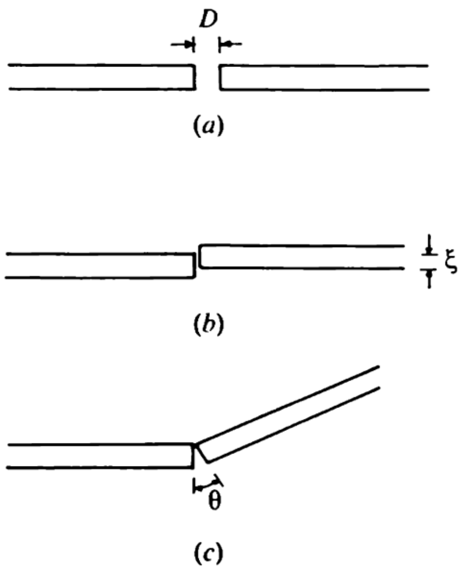
$$\psi_2(x, y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{w_2} \mathrm{e}^{-[(x-\xi)^2+y^2]/w_2^2} \tag{13.157}$$

where ξ is the transverse misalignment and the factors are such that

$$\iint_{-\infty}^{+\infty} \psi_{1,2}^2 \,\mathrm{d}x\,\mathrm{d}y = 1$$

The fractional power that is coupled to the fundamental mode of the second

Fig. 13.22 Various misalignments at a splice between two fibres: (*a*) Longitudinal misalignment, (*b*) transverse misalignment, (*c*) angular misalignment.



fibre is given by

$$T = \left| \iint_{-\infty}^{+\infty} \psi_1 \psi_2^* \, dx \, dy \right|^2 \tag{13.158}$$

Thus

$$T = \left| \frac{2}{\pi w_1 w_2} \iint_{-\infty}^{+\infty} \exp\left[-x^2\left(\frac{1}{w_1^2} + \frac{1}{w_2^2}\right) + \frac{2x\xi}{w_2^2} - y^2\left(\frac{1}{w_1^2} + \frac{1}{w_2^2}\right) - \frac{\xi^2}{w_2^2} \right] dx \, dy \right|^2$$

$$= \left(\frac{2w_1 w_2}{w_1^2 + w_2^2}\right)^2 \exp\left[-\frac{2\xi^2}{(w_1^2 + w_2^2)} \right] \tag{13.159}$$

The maximum power coupling appears at $\xi = 0$ and the transmitted fractional power is

$$T_{\max} = \left(\frac{2w_1 w_2}{w_1^2 + w_2^2}\right)^2 \tag{13.160}$$

which is unity for two fibres having identical Gaussian fundamental modes. Also the transmitted energy decreases by a factor of 1/e for a misalignment of

$$\xi_e = (w_1^2 + w_2^2)^{\frac{1}{2}}/\sqrt{2} \tag{13.161}$$

For two identical fibres, the transverse offset loss varies as

$$T = e^{-\xi^2/w_1^2} \tag{13.162}$$

Hence by measuring the transmission loss across a splice as a function of ξ one can experimentally obtain the spot size w_1.

The corresponding expressions for tilt loss and longitudinal separations for a pair of identical fibres are given by (see Problems 13.11 and 13.13)

$$T_t = e^{-(k_0 n_2 \theta w)^2/4} \tag{13.163}$$

$$T_l = \frac{1 + 4\tilde{D}^2}{\tilde{D}^2 + (1 + 2\tilde{D}^2)^2}; \quad \tilde{D} = \frac{4D}{n_2 k_0 w^2} \tag{13.164}$$

where θ is the angle between the two fibre axes and D the separation between the two fibre end faces; n_2 is the refractive index of the medium between the fibre ends.

Calculation of Angular Misalignment

Problem 13.11: Consider two single mode fibres and assume that the fundamental modes of the two fibres can be represented by Gaussian distributions. Calculate the power transmission loss as a function of angular misalignment between the two fibres. Assume that the fibre ends are placed in a liquid of refractive index n_2.

Solution: We describe the fundamental modes of the two fibres by Eq. (13.156) and a similar equation with w_1 replaced by w_2. Let us assume that there is a small angular misalignment of θ between the two fibres (see Fig. 13.22(c)). In order to calculate the power transmission loss, we must transform the Gaussian beam of the first fibre into the coordinate system of the second fibre. If (x, y, z) and (x', y', z') represent the coordinate systems of the first and second fibres respectively then we have

$$\left.\begin{aligned} x &= x'\cos\theta + z'\sin\theta \\ y &= y' \\ z &= -x'\sin\theta + z'\cos\theta \end{aligned}\right\} \tag{13.183}$$

where we assume an angular misalignment in the x–z plane. Since the medium between the two fibres is of refractive index n_2, the Gaussian mode of the first fibre, as it emerges, will propagate aproximately as

$$\psi_1(x, y, z) \approx \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{w_1} e^{-(x^2+y^2)/w_1^2} e^{-ik_0 n_2 z} \tag{13.184}$$

for small values of z so that we can neglect diffraction effects. We transform Eq. (13.184) into the (x', y', z') coordinate system and obtain for the incident beam at the input plane $z' = 0$ of the second fibre

$$\psi_1(x', y, z') = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{w_1} e^{-(x'^2+y^2)/w_1^2} e^{ik_0 n_2 x'\theta} \tag{13.185}$$

where we have assumed $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Calculation of the overlap of this field with the field of the second fibre and taking modulus squared gives us the power transmission coefficient as

$$T(\theta) = \left(\frac{2w_1 w_2}{w_1^2 + w_2^2}\right)^2 \exp\left[-\frac{k_0^2 n_2^2 \theta^2 w_1^2 w_2^2}{2(w_1^2 + w_2^2)}\right] \tag{13.186}$$

Thus the coupled power decreases to e^{-1} of its maximum value (corresponding to $\theta = 0$) in an angle

$$\theta_e = \sqrt{2}(w_1^2 + w_2^2)^{\frac{1}{2}}/w_1 w_2 k_0 n_2 \tag{13.187}$$

It is interesting to note that for two identical fibres with $w_1 = w_2 = w$ and using Eqs. (13.161) and (13.187) we get

$$\xi_e \theta_e = 2/k_0 n_2 = \lambda_0/\pi n_2 \tag{13.188}$$

which is independent of w_1. Eq. (13.188) is similar to the uncertainty principle in quantum mechanics (see e.g., Ghatak and Lokanathan (1984)). Eq. (13.188) implies that increasing the mode spot size w_1 would increase ξ_e, i.e., tolerance towards transverse misalignment but this would correspondingly decrease θ_e, i.e., the tolerance towards angular misalignment.