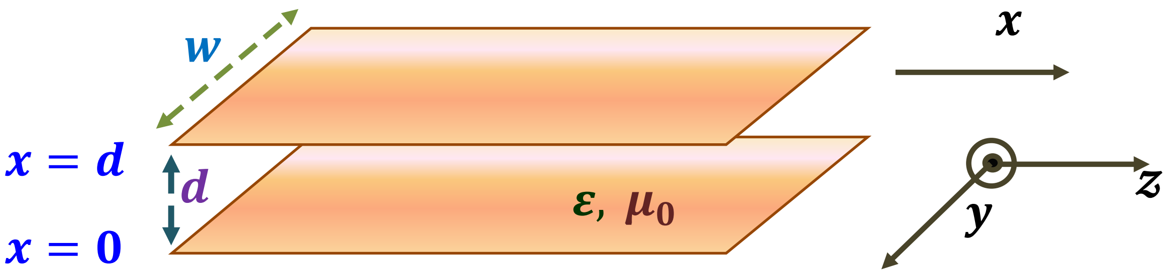
Parallel plate metal waveguide

Broad Topics

- ✓ Parallel plate metal waveguide
- ✓ $\boldsymbol{TE}-$ and $\boldsymbol{TM}-$ modes, $\boldsymbol{E}$ and $\boldsymbol{H}$ fields distribution
- ✓ Dispersion relations, cut-off properties
- ✓ Modes of rectangular metal waveguide

1. Parallel plate metal waveguide



The wave equations:

$\nabla^2 \boldsymbol{E} = -\omega^2 \mu_0 \varepsilon \boldsymbol{E}$and

$\nabla^2 \boldsymbol{H} = -\omega^2 \mu_0 \varepsilon \boldsymbol{H}$

We look for the solutions of the E-field equations: $\nabla^2 \boldsymbol{E} = -\omega^2 \mu_0 \varepsilon \boldsymbol{E}$

The structure is invariant along z

The z- dependence is that of the wave going in z- direction

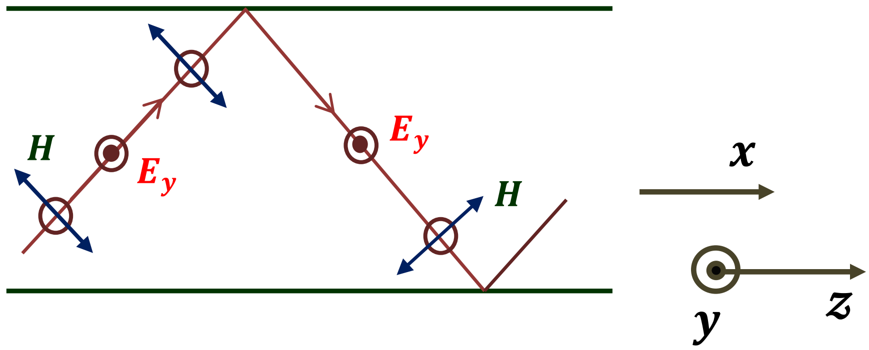
The z- dependence of the solution will be $e^{-ik_z z} \approx e^{-i\beta z}$ $[k_z = \beta]$

Wave equations for TE- and TM modes

$$\frac{\partial^2 E_y}{\partial x^2} + (k_0^2 n^2 - \beta^2) E_y = 0 \qquad \textit{wave equation in } E_y$$

$$\frac{\partial^2 H_y}{\partial x^2} + (k_0^2 n^2 - \beta^2) H_y = 0 \qquad \textit{wave equation in } H_y$$

Transverse Modes



TE-mode

For a TE-guided wave

- ✓ The electric field is transverse to the direction of propagation of wave
- ✓ The field will be represented by

$$E_y = \hat{y}E_y(x)e^{-i\beta z}$$

The $\vec{E}$ field vector is pointing along y –direction

Using this solution in the wave equation: $\nabla^2 E_y = -\omega^2\mu_0\varepsilon E_y$

$$\Rightarrow \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)E_y = -\omega^2\mu_0\epsilon E_y \quad : \quad \frac{\partial^2}{\partial y^2} = no\ dependence$$

$$\Rightarrow \frac{\partial^2 E_y}{\partial x^2} + (\omega^2\mu_0\varepsilon - \beta^2)E_y = 0$$

Perfect metallic boundary

Boundary Conditions: $E_y(x=0) = E_y(x=d) = 0$

Now $\omega^2\mu_0\varepsilon - k_z^2 = \omega^2\mu_0\varepsilon_0\varepsilon_r - \beta^2$

$$= \frac{\omega^2 n^2}{c^2} - \beta^2 = k^2 - \beta^2 = k_x^2$$

Also $k_y = 0$ therefore $k^2 = k_x^2 + k_z^2$;

Therefore, $\frac{\partial^2 E_y}{\partial x^2} + (\omega^2\mu_0\varepsilon - \beta^2)E_y = 0 \rightarrow \frac{\partial^2 E_y(x)}{\partial x^2} + k_x^2 E_y(x) = 0$

This has the solution:

$$E_y(x) = A_0 e^{-ik_x x} + B_0 e^{+ik_x x} = E_0 \sin k_x x + E_0' \cos k_x x$$

Complete field solutions: TE Modes

Boundary Condition: $E_y(x=0) = 0, \Rightarrow E_y = E_0 sin\, k_x x$

But k_x cannot be arbitrary ------

Second Boundary Condition: $E_y(x=d)=0$ restricts $E_y = E_0 sin\, k_x d = 0$

$$\Rightarrow \sin(k_x d) = \ sin(m\pi) \Rightarrow \ k_x = \frac{m\pi}{d} \quad where\, m = 1, 2, 3, \dots$$

So, the solution becomes: $E_y(x) = E_0 \sin(\frac{m\pi}{d}x)$

The complete solution is: $E_y(x,z) = \hat{y}E_0 \sin(\frac{m\pi}{d}x)e^{-i\beta z}$

Electric field: TE Modes: from field configuration also

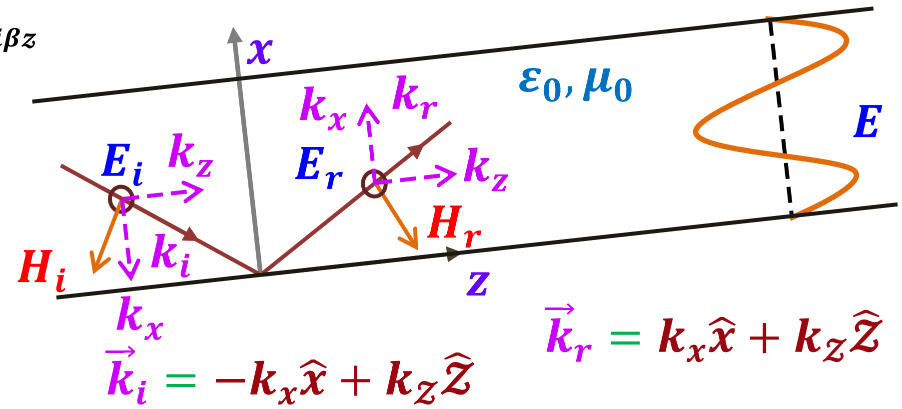
$\vec{E} = \hat{y}\, E_i e^{-i(-k_x x + k_z Z)} + \hat{y}\Gamma_{TE} E_i e^{-i(k_x x + k_z Z)} where\, \Gamma_{TE} = -1$

$= \hat{y}\, E_i e^{i(k_x x - k_z Z)} - \hat{y}\, E_i e^{-i(k_x x + k_z Z)}$

$= \hat{y}\, E_i \left(e^{ik_x x} - e^{-ik_x x}\right) e^{-ik_z Z}$

$= \hat{y}\, E_i\, 2i \sin(k_x x)\, e^{-ik_z Z}$

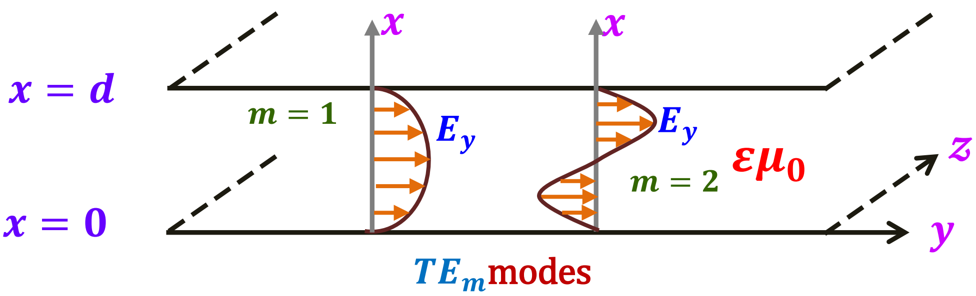
$= \hat{y}\, E_0\, sin(\frac{m\pi}{d}x)e^{-i\beta z}$



Field distribution: TE Modes

The $\vec{E}$ field solution: $E_y(x,z) = \hat{y}E_0 \sin(\frac{m\pi}{d}x)e^{-i\beta z}$

Mode field ($\vec{E}$ field) distributions of two lowest order modes $m=1,\ m=2$



Magnetic field components: TE Modes

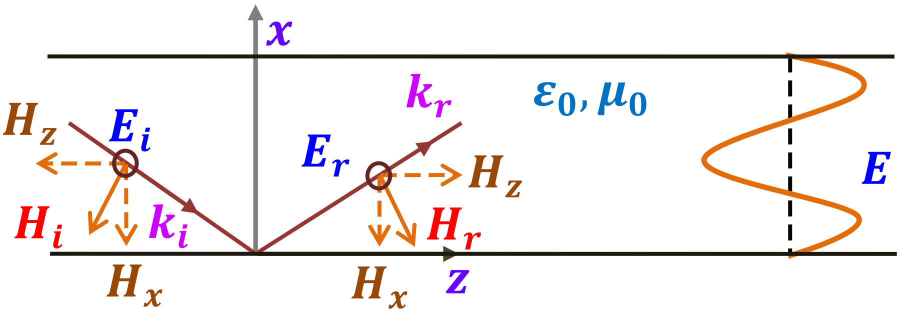
Electric field: $E_y(x,z) = \hat{y}E_0 \sin(\frac{m\pi}{d}x)e^{-i\beta z}$

Magnetic field is determined from: $\nabla \times \vec{E} = -i\omega\mu_0\vec{H}$

$$\vec{H} = \frac{iE_0}{\omega\mu_0}\left[\hat{z}\left(\frac{m\pi}{d}\right)\cos\left(\frac{m\pi}{d}x\right) + \hat{x}\, i\beta \sin\left(\frac{m\pi}{d}x\right)\right]e^{-i\beta z}$$

Perfect metallic boundary condition: $H_x(x=0) = H_x(x=d) = 0$ is automatically satisfied

Magnetic field: TE Modes from field configuration also



$$\vec{H}_i = -H_x\hat{x} - H_z\hat{z} \qquad \vec{H}_r = -H_x\hat{x} + H_z\hat{z}$$

$$\vec{k}_i = -k_x\hat{x} + k_z\hat{z} \qquad \vec{k}_r = k_x\hat{x} + k_z\hat{z}$$

$\hat{z}H_z = -H_i\hat{z} + \Gamma H_i\hat{z}$ where $\Gamma = -1$

$$= -\hat{z}\, H_i e^{-i(-k_x x + k_z z)} - \hat{z} H_i e^{-i(k_x x + k_z z)}$$

$$= -\hat{z}\, H_i\left(e^{ik_x x} + e^{-ik_x x}\right) e^{-ik_z z}$$

$$= -\hat{z}\, H_i\, 2\cos(k_x x)\, e^{-ik_z z}$$

$$= -\hat{z}\, H_0\, \cos\left(\frac{m\pi}{d}x\right) e^{-i\beta z}$$

Similarly for $\hat{x}H_x$ field and the complete magnetic field is

$$\vec{H} = \frac{iE_0}{\omega\mu_0}\left[\hat{z}\left(\frac{m\pi}{d}\right)\cos\left(\frac{m\pi}{d}x\right) + \hat{x}\, i\beta \sin\left(\frac{m\pi}{d}x\right)\right]e^{-i\beta z}$$

Complete field solutions: TE mode

$$E_y = \hat{y}\, E_0 \sin\left(\frac{m\pi}{d}x\right) e^{-i\beta z}$$

$$H_x = -\hat{x}\,\frac{\beta E_0}{\omega\mu_0}\sin\left(\frac{m\pi}{d}x\right) e^{-i\beta z}$$

$$H_z = \hat{z}\,\frac{iE_0}{\omega\mu_0}\frac{m\pi}{d}\cos\left(\frac{m\pi}{d}x\right) e^{-i\beta z}$$

Each value of m refers to a particular field configuration or mode. It denotes the number of half period variations of electric field along the x-direction of the parallel planes. A particular mode is written as TE_{m0} where subscript 0 refers to y-direction and will have some integral value in a waveguide where planes in y-direction will be considered for restricting/guiding the wave.

Since in this case, $m = 0$ reduces the whole field of TE mode to zero (all components), the lowest order mode that can exist is TE_{10}.

Dispersion relation: TE Modes

The propagation constant is given by $\beta^2 + k_x^2 = \omega^2\mu_0\varepsilon$

$$\Rightarrow \quad \beta^2 = \omega^2\mu_0\varepsilon - k_x^2 = \omega^2\mu_0\varepsilon - \left(\frac{m\pi}{d}\right)^2$$

$$\text{so} \quad \beta_m = \sqrt{\omega^2\mu_0\varepsilon - \left(\frac{m\pi}{d}\right)^2}$$

The propagation is possible only when $\beta_m > 0$

That means $\omega^2\mu_0\varepsilon > \left(\frac{m\pi}{d}\right)^2$ which decides the frequency above which the propagation is possible.

Cut-off frequency: TE Modes

- ✓ Dispersion relation for TE_mmode: $\beta = \sqrt{\omega^2\mu_0\varepsilon - \left(\frac{m\pi}{d}\right)^2}$
- ✓ For this TE_mmode, if the frequency ω is less than $\frac{1}{\sqrt{\mu_0\varepsilon}}\left(\frac{m\pi}{d}\right)$

- ✓ Then βbecomes entirely imaginary
- ✓ The mode does not propagate but decays exponentially with distance
- ✓ Then the cut-off frequency is determined by the condition: $\omega_c^2\mu_0\varepsilon = \left(\frac{m\pi}{d}\right)^2$

So, the cut-off frequency: $\omega_c = \frac{1}{\sqrt{\mu_0\varepsilon}}\left(\frac{m\pi}{d}\right) = \frac{v_0 m\pi}{d}$ and the frequency, $f_c = \frac{v_0 m}{2d}$.

Also, the cut-off wavelength is $\lambda_c = \frac{v_0 2d}{f_c m}$ implying that higher the separation between the plates, lower is the cut-off frequency. So propagation is possible if $\omega > \omega_c$or $\lambda < \lambda_c$.

Phase velocity: TE Modes

The phase velocity/speed with which the wave travels within the guide is $v_{ph} = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu_0\varepsilon - \left(\frac{m\pi}{d}\right)^2}} = \frac{\omega}{\sqrt{\omega^2\mu_0\varepsilon - \omega_c{}^2\mu_0\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon}\sqrt{1-\omega_c{}^2/\omega^2}} = \frac{v_0}{\sqrt{1-\lambda_0^2/\lambda_c^2}}$

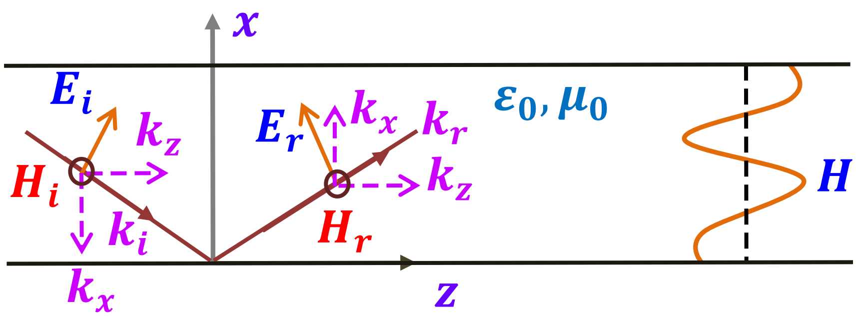
TM-mode

For a TM-guided wave

- ✓ The magnetic field is transverse to the direction of propagation of the wave
- ✓ The field will be represented by

  $H_y = \hat{y}H_y(x)e^{-i\beta z}$

  The$\vec{H}$ field vector is pointing along y −direction



Using this solution in the wave equation becomes

$$\nabla^2 H_y = -\,\omega^2 \mu_0 \varepsilon H_y$$

$$\Rightarrow \qquad \frac{\partial^2 H_y}{\partial x^2} + (\omega^2 \mu_0 \varepsilon - \beta^2) H_y = 0$$

This has the solution: $H_y(x) = H_0 \cos k_x x$

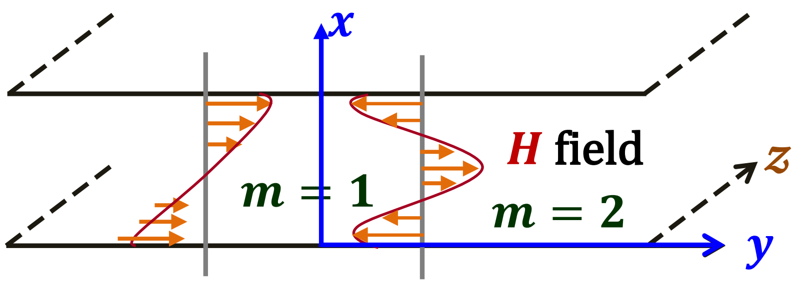
$here\ \ k_x = \frac{m\pi}{d}$; where $m = 1, 2, 3, \dots$

Using appropriate boundary conditions, the solution becomes:

$$H_y(x) = H_0 \cos\left(\frac{m\pi}{d} x\right)$$

And the complete solution is

$H_y(x, z) = \hat{y} H_0 \cos\left(\frac{m\pi}{d} x\right) e^{-i\beta z}$



Electric field components: TM Modes

Magnetic field: $H_y(x, z) = \hat{y} H_0 \cos\left(\frac{m\pi}{d} x\right) e^{-i\beta z}$

Electric field is determined from the relation: $\nabla \times \vec{H} = i\omega\varepsilon\vec{E}$

And therefore the electric fields

$$\vec{E} = -\frac{iH_0}{\omega\varepsilon}\left[-\hat{z}\left(\frac{m\pi}{d}\right) sin\left(\frac{m\pi}{d} x\right) + \hat{x}\, i\beta\, cos\left(\frac{m\pi}{d} x\right)\right] e^{-i\beta z}$$

Perfect metallic boundary condition: $E_z(x = 0) = E_z(x = d) = 0$ is automatically satisfied

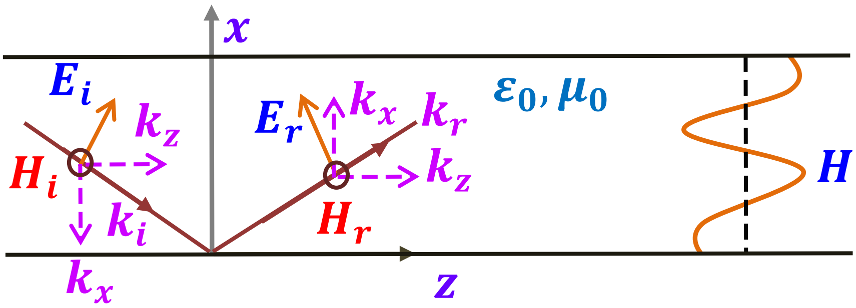
Magnetic field: TM Modes: from field configuration also

$$H_y = \hat{y}\, H_i e^{-i(-k_x x + k_z z)} + \hat{y} \Gamma_{TM} H_i e^{-i(k_x x + k_z z)}$$

for this case $\Gamma_{TM} = +1$

$$H_y = \hat{y}\, H_i e^{i(k_x x - k_z z)} + \hat{y} H_i e^{-i(k_x x + k_z z)}$$

$$H_y(x,z) = \hat{y}\, 2H_i \cos(k_x x) e^{-i\beta z}$$



$$\vec{k}_i = -k_x \hat{x} + k_z \hat{z} \qquad \vec{k}_r = k_x \hat{x} + k_z \hat{z}$$

Electric field components: TM Modes: from field configuration also

$$E_x = \hat{x}\, E_i e^{-i(-k_x x + k_z z)} + \hat{x} E_r e^{-i(k_x x + k_z z)}$$

$$= \hat{x}\, E_i e^{-i(-k_x x + k_z z)} + \hat{x} \Gamma_{TM} E_i e^{-i(k_x x + k_z z)}$$

for this case $\Gamma_{TM} = +1$

$$E_x = \hat{x}\, E_i e^{-i(-k_x x + k_z z)} + \hat{x} E_i e^{-i(k_x x + k_z z)}$$

$$E_x(x,z) = \hat{x}\, 2E_i \cos(k_x x) e^{-i\beta z}$$

And for E_z

$$E_z = \hat{z}\, E_i e^{-i(-k_x x + k_z z)} - \hat{z} E_r e^{-i(k_x x + k_z z)}$$

$$= \hat{z}\, E_i e^{-i(-k_x x + k_z z)} - \hat{z} \Gamma_{TM} E_i e^{-i(k_x x + k_z z)}$$

for this case $\Gamma_{TM} = +1$

$$E_z = \hat{z}\, E_i e^{-i(-k_x x + k_z z)} - \hat{z} E_i e^{-i(k_x x + k_z z)}$$

$$E_z(x,z) = \hat{z}\, 2iE_i \sin(k_x x) e^{-i\beta z}$$

So we get

$$\vec{E} = -\frac{iH_0}{\omega\varepsilon}\left[-\hat{z}\left(\frac{m\pi}{d}\right) sin\left(\frac{m\pi}{d}x\right) + \hat{x}\, i\beta\, cos\left(\frac{m\pi}{d}x\right)\right] e^{-i\beta z}$$

Complete field solutions: TM mode

$$H_y = \hat{y}\ H_0 \cos\left(\frac{m\pi}{d}x\right) e^{-i\beta z}$$

$$E_x = \hat{x}\frac{\beta H_0}{\omega\varepsilon}\cos\left(\frac{m\pi}{d}x\right) e^{-i\beta z}$$

$$E_z = \hat{z}\frac{iH_0}{\omega\varepsilon}\frac{m\pi}{d}\sin\left(\frac{m\pi}{d}x\right) e^{-i\beta z}$$

It is obvious that in this case for $m = 0$, all field components are not zero. That means, for the TM case, lowest mode will be TM_{00} containing

$H_y = \hat{y}H_0 e^{-i\beta z}$ and $E_x = \hat{x}\frac{\beta H_0}{\omega\varepsilon}e^{-i\beta z}$ and $E_z = 0$

Thus, in TM_{00} mode the longitudinal components of both electric and magnetic fields are zero. This is electric and magnetic fields both are transverse to the direction of propagation. €therefore this wave is TEM wave.

Dispersion relation: TM_m Modes

$$\beta^2 + k_x^2 = \omega^2\mu_0\varepsilon$$

$$\Rightarrow \quad \beta_m^2 = \omega^2\mu_0\varepsilon - k_x^2 = \omega^2\mu_0\varepsilon - \left(\frac{m\pi}{d}\right)^2$$

$$so \quad \beta_m = \sqrt{\omega^2\mu_0\varepsilon - \left(\frac{m\pi}{d}\right)^2}$$

Cut-off frequency: TM_m Modes

- ✓ Dispersion relation for TM_mmode: $\beta_m = \sqrt{\omega^2\mu_0\varepsilon - \left(\frac{m\pi}{d}\right)^2}$
- ✓ For this TM_mmode, if the frequency ω is less than $\frac{1}{\sqrt{\mu_0\varepsilon}}\left(\frac{m\pi}{d}\right)$
- ✓ Then βbecomes entirely imaginary
- ✓ The mode does not propagate but decays exponentially with distance

- ✓ Then the cut-off frequency is determined by the condition: $\omega_c^2 \mu_0 \varepsilon = \left(\frac{m\pi}{d}\right)^2$

So, the cut-off frequency: $\omega_c = \frac{1}{\sqrt{\mu_0 \varepsilon}}\left(\frac{m\pi}{d}\right)$

Phase velocity: TM Modes

The phase velocity/speed with which the wave travels within the guide is $v_{ph} = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2 \mu_0 \varepsilon - \left(\frac{m\pi}{d}\right)^2}} = \frac{\omega}{\sqrt{\omega^2 \mu_0 \varepsilon - \omega_c{}^2 \mu_0 \varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon}\sqrt{1-\omega_c{}^2/\omega^2}} = \frac{v_0}{\sqrt{1-\lambda_0^2/\lambda_c^2}}$