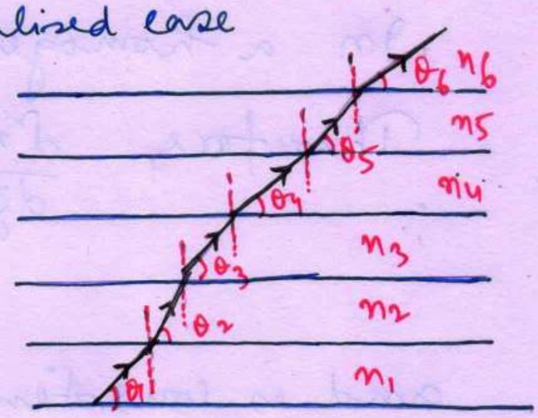
# Ray Equation in a planar slab structure

A. Graded RI profile slab: generalised case



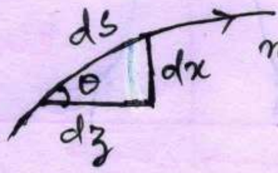

Consider a graded medium where RI depends only on $x$.

A medium with continuously graded RI can be assumed as layers of infinitesimal thickness. Such situation can be described by Snell's law:

$$n_1 \sin\phi_1 = n_2 \sin\phi_2 = \cdots$$

ie, $$n_1 \cos\theta_1 = n_2 \cos\theta_2 = \cdots \; n(x)\cos\theta(x) = \tilde{\beta}$$

say a constant.




Next consider the ray path as shown.

Here $ds^2 = dx^2 + dz^2 \Rightarrow \left(\frac{ds}{dz}\right)^2 = \left(\frac{dx}{dz}\right)^2 + 1$

But $\frac{ds}{dz} = \frac{1}{\cos\theta(x)} = \frac{n(x)}{\tilde{\beta}}$ $\left[\text{using } n(x)\cos\theta(x) = \tilde{\beta}\right]$

Therefore, $$\left(\frac{dx}{dz}\right)^2 = \frac{n^2(x)}{\tilde{\beta}^2} - 1$$

Differentiating –

$$2\frac{dx}{dz}\cdot\frac{d^2x}{dz^2} = \frac{1}{\tilde{\beta}^2}\cdot\frac{d}{dz}n^2(x)$$

ie, $$\frac{d^2x}{dz^2} = \frac{1}{2\tilde{\beta}^2}\cdot\frac{d}{dz}n^2(x)\cdot\frac{dz}{dx}$$

$$= \frac{1}{2\tilde{\beta}^2}\cdot\frac{d}{dx}n^2(x)$$

Ray path equation: $$\boxed{\frac{d^2x}{dz^2} = \frac{1}{2\tilde{\beta}^2}\cdot\frac{dn^2(x)}{dx}}$$

A. Ray path in homogeneous medium:

In a homogeneous medium $n^2(x) = $ constant

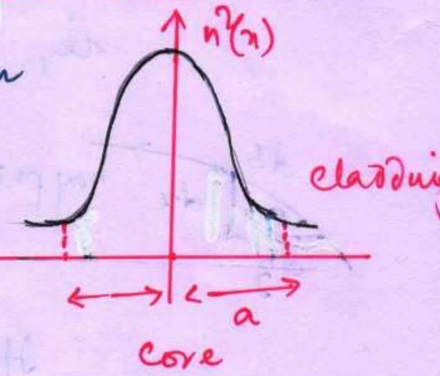
Therefore, $\frac{d^2x}{dz^2} = 0$. The solution is

$$x(z) = A + Bz$$

This is a straight line, and is consistent with the concept that rays travel in a straight line in a homogeneous medium.

B. Ray path in a square-law/parabolic medium.

The index profile in such a medium is




$$n^2(x) = n_1^2\left[1 - 2\Delta\left(\frac{x}{a}\right)^2\right] : \text{core} \quad |x|<a$$

$$= n_2^2 : \text{cladding} \quad |x|>a$$

Hence, $\frac{d\, n^2(x)}{dx} = -\frac{n_1^2 2\Delta}{a^2}\cdot 2x$

Ray equation: $\frac{d^2x}{dz^2} + \frac{n_1^2 2\Delta}{a^2 \tilde{\beta}^2}\cdot x = 0$

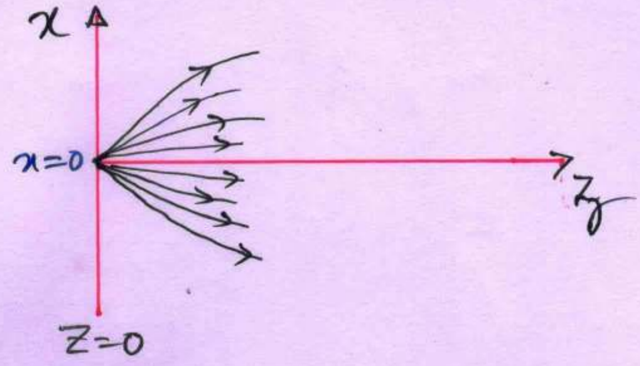
ie, $\frac{d^2x}{dz^2} + \Gamma^2 x = 0$

where $\Gamma = \frac{n_1\sqrt{2\Delta}}{a\tilde{\beta}}$

The solution is

$$x(z) = A\sin(\Gamma z) + B\cos(\Gamma z)$$

Consider launching of light at the input of the slab guide.




So, at $z=0$, $x=0$

The ray equation gives: $x(z) = A\sin(\Gamma z)$ as $B = 0$.

Now at $z=0$, the ray makes an angle $\theta_1$ with $z$. Then

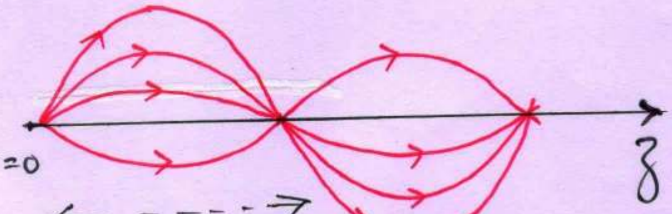
$$\tan\theta_1 = \frac{dx}{dz} = (A\cos\Gamma z).\Gamma\Big|_{z=0}$$
$$= A\Gamma$$

or, $$A = \frac{\tan\theta_1}{\Gamma} = \frac{\sin\theta_1}{\cos\theta_1}\cdot\frac{\tilde{\beta}a}{n_1\sqrt{2\Delta}}$$
$$= \frac{a\sin\theta_1}{\sqrt{2\Delta}}$$

$$\therefore\ x(z) = \frac{a\sin\theta_1}{\sqrt{2\Delta}}\cdot\sin\left(\frac{n_1\sqrt{2\Delta}}{\tilde{\beta}a}\cdot z\right).$$

This is the ray path in the graded slab waveguide.

Note: The period doesnot depend on the launching angle $\theta_1$.




Hence all rays having different angle meet the axis with same period.

Period $\Lambda = \frac{2\pi\tilde{\beta}a}{n_1\sqrt{2\Delta}}\ \left(\frac{2\pi}{\Gamma}\right)$

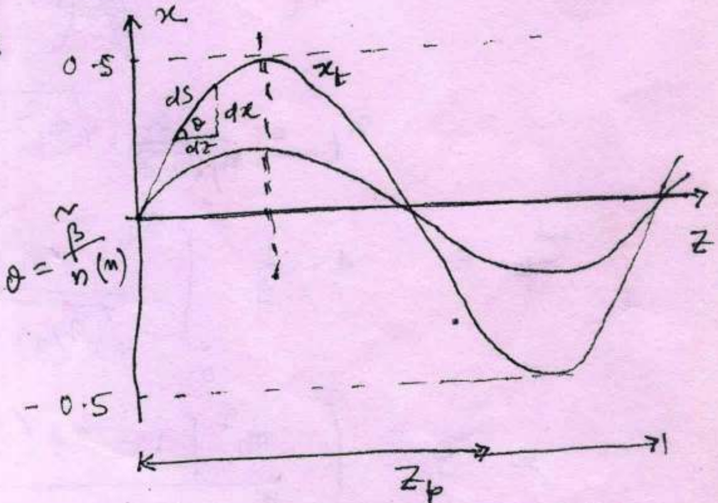
The maximum travel of a ray along $x$ (amplitude) depends on the launch angle $\theta_1$.

All rays periodically focus at the axis.

# Graded Index slab waveguides

## Time of travel:

In a graded-index medium, the time taken by different rays are, in general different. Therefore, in graded slab waveguide, if all rays are launched simultaneously, they will appear at the output at different times. This leads to broadening of a time pulse and is known as temporal dispersion of the pulse.




The time taken to travel an arc ds is $d\tau = \frac{ds}{v(x)} = \frac{1}{c}\, n(x)\, ds$

But $ds = \sqrt{(dx)^2 + (dz)^2}$

$$= \boxed{\sqrt{1 + \left(\frac{dz}{dx}\right)^2}} \cdot dx \quad = \operatorname{cosec}\theta \cdot dx \quad \& \ \cos\theta = \frac{\tilde{\beta}}{n(x)}$$

$$= \frac{n(x)}{\sqrt{n^2(x) - \tilde{\beta}^2}}\, dx.$$

$$\frac{dz}{ds} = \cos\theta = \frac{\tilde{\beta}}{n(x)}$$

$$\frac{dz}{dx} = \cot\theta = \frac{\cos\theta}{\sqrt{1-\cos^2\theta}} = \cancel{\frac{\tilde{\beta}^2}{n(x)\sqrt{1-\frac{\tilde{\beta}^2}{n^2(x)}}}}$$

$$= \frac{\tilde{\beta} \times n(x)}{n(x) \times \sqrt{n^2(x) - \tilde{\beta}^2}}.$$

$$1 + \left(\frac{dz}{dx}\right)^2 = 1 + \frac{\tilde{\beta}^2}{n^2(x) - \tilde{\beta}^2} = \frac{n^2(x)}{n^2(x) - \tilde{\beta}^2}$$

Alt. $\frac{ds}{dx} = \operatorname{cosec}\theta = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{1 - \frac{\tilde{\beta}^2}{n^2(x)}}} = \frac{n(x)}{\sqrt{n^2(x) - \tilde{\beta}^2}}$

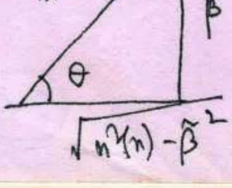
The time taken to travel a quarter of period is

$$\frac{\tau_p}{4} = \int_0^{x_t} \frac{n(x)}{c} \cdot ds$$

$$= \frac{1}{c} \int_0^{x_t} \frac{n^2(x)}{\sqrt{n^2(x) - \tilde{\beta}^2}}\, dx$$

$x_t$ is the turning point of a ray (the distance from the axis).

$$\sqrt{1 + \left(\frac{dz}{dx}\right)^2} = \sqrt{1 + \cot^2\theta} = \sqrt{\operatorname{cosec}^2\theta} = \operatorname{cosec}\theta = \frac{n(x)}{\sqrt{n^2(x) - \tilde{\beta}^2}}$$




$$\cos\theta = \frac{\tilde{\beta}}{n(x)}$$

Pulse dispersion in a parabolic medium:

At the turning point; $n(x_t) = \tilde{\beta}$, since $n(x_t)\cos 0^\circ = \tilde{\beta}$.

Now, we have $n^2(x) = n_1^2\left[1 - 2\Delta\left(\frac{x_t}{a}\right)^2\right]$ for a parabolic medium.

so, at $x_t$ : $n^2(x_t) = n_1^2\left[1 - 2\Delta\left(\frac{x_t}{a}\right)^2\right] = \tilde{\beta}^2$

Hence $$x_t^2 = \cancel{\frac{n_1^2}{\tilde{\beta}^2}}\left(1 - \frac{\tilde{\beta}^2}{n_1^2}\right)\Big/\frac{2\Delta}{a^2} = \frac{n_1^2 - \tilde{\beta}^2}{n_1^2}\cdot\frac{a^2}{2\Delta}.$$

$$x_t = \frac{a}{n_1\sqrt{2\Delta}}\cdot\sqrt{n_1^2 - \tilde{\beta}^2}.$$

$$\tau_p = 4\cdot\frac{1}{c}\int_0^{x_t}\frac{n^2(x)}{\sqrt{n^2(x) - \tilde{\beta}^2}}\,dx$$

$$= \frac{4}{c}\int\frac{n_1^2\left[1 - 2\Delta\left(\frac{x}{a}\right)^2\right]}{\left[n_1^2\left[1 - 2\Delta\left(\frac{x}{a}\right)^2\right] - \tilde{\beta}^2\right]^{1/2}}\,dx$$

$$= \frac{4}{c}\int\frac{n_1^2 - \frac{2\Delta n_1^2}{a^2}\cdot x^2 - \tilde{\beta}^2 + \tilde{\beta}^2}{\sqrt{n_1^2 - \frac{2\Delta n_1^2}{a^2}\cdot x^2 - \tilde{\beta}^2}}\cdot dx \qquad n_1^2 - \tilde{\beta}^2 = x_t^2\cdot\frac{n_1^2(2\Delta)}{a^2}$$

$$= \frac{4}{c}\left[\int_0^{x_t}\frac{(x_t^2 - x^2)\cdot\frac{n_1^2\cdot 2\Delta}{a^2}}{(x_t^2 - x^2)^{1/2}\cdot\frac{n_1\sqrt{2\Delta}}{a}}\cdot dx + \int_0^{x_t}\frac{\tilde{\beta}^2}{\sqrt{x_t^2 - x^2}\cdot\frac{n_1\sqrt{2\Delta}}{a}}\,dx\right]$$

Carry out the elementary integration

$$\tau_p = \frac{\pi}{c}\cdot\frac{a}{n_1\sqrt{2\Delta}}\cdot\left(n_1^2 + \tilde{\beta}^2\right).$$

Now at the turning point $z = \frac{z_p}{4}$: ($z_p$ corresponds to $\tau_p$)

$$\frac{1}{4}\Gamma z_p = \frac{\pi}{2} \Rightarrow z_p = \frac{2\pi a\tilde{\beta}}{n_1\sqrt{2\Delta}}.$$

Thus if $\tau(z)$ represents the time taken by a ray to travel a distance $z$,

$$\frac{\tau(z)}{z} = \frac{\tau_p}{z_p} = \frac{\frac{\pi}{c}\cdot\frac{a}{n_1\sqrt{2\Delta}}\left(n_1^2+\tilde{\beta}^2\right)}{\frac{2\pi a\tilde{\beta}}{n_1\sqrt{2\Delta}}} = \frac{1}{2c}\cdot\left(\tilde{\beta}+\frac{n_1^2}{\tilde{\beta}}\right)$$

$$\therefore\ \tau(z) = \frac{1}{2c}\cdot\left(\tilde{\beta}+\frac{n_1^2}{\tilde{\beta}}\right)$$

The ray which travels along the axis takes the minimum time and for this ray

$$\tilde{\beta} = n_1 \quad (\text{paraxial ray})$$

$$\therefore\ \tau_{min}(z) = \frac{1}{2c}\cdot\left(n_1+\frac{n_1^2}{n_1}\right) = \frac{n_1}{c}\cdot z.$$

But the ray which goes to highest turning point (at the periphery) has $\tilde{\beta} = n_2$ (meridonial ray)

$$\therefore\ \tau_{max}(z) = \frac{1}{2c}\left(n_2+\frac{n_1^2}{n_2}\right)\cdot z$$

Therefore, broadening of the pulse $\Delta\tau$ is

$$\tau_{max}-\tau_{min} = \frac{1}{2cn_2}\cdot(n_1-n_2)^2\cdot z$$

$$= \frac{1}{2c}\cdot\frac{n_1}{n_2 n_1}(n_1-n_2)^2\cdot z = \frac{n_1}{2c}\cdot\left(\frac{n_1-n_2}{n_2}\right)^2\cdot z$$

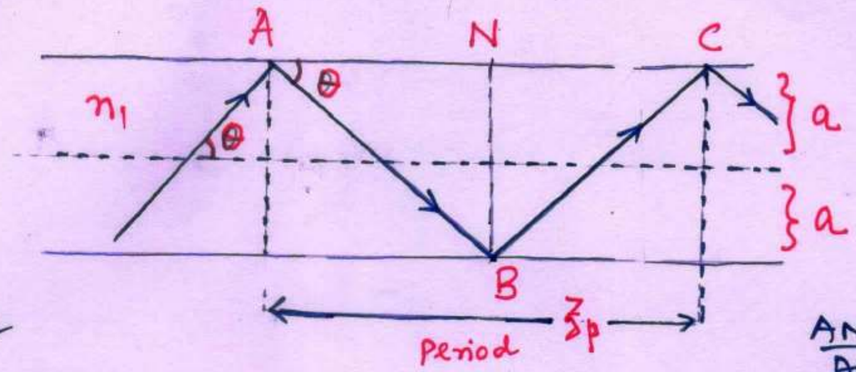
$n_2^2$

$$\approx \frac{n_1}{2c}\Delta^2\cdot z.$$

i.e., $\Delta\tau \propto \Delta^2$

Example: for $n_1 = 1.46$ $\Delta = 0.01 \rightarrow \Delta\tau = 250$ ps/km.

# Pulse Dispersion in Step-Index Waveguide




$\frac{AN}{AB} = \cos\theta$

Time taken to travel one period $z_p$ is

$$\tau_p = \frac{2n_1}{c}\cdot AB = \frac{n_1}{c}\cdot 2AB = \frac{n_1}{c}\cdot\frac{2AN}{\cos\theta} = \frac{n_1 z_p}{c\cos\theta}$$

Now, $n_1\cos\theta = n_2$ is for the maximum value of $\theta$
ie minimum value of $\cos\theta$

So, the rays are from $\theta = 0$ to $\theta = \cos^{-1}\frac{n_2}{n_1}$ are guided.

the extreme rays

Therefore, the minimum time $\tau_{min} = \frac{n_1 z_p}{c}$

and maximum time $\tau_{max} = \frac{n_1 z_p}{c\cos\theta} = \frac{n_1^2}{c\,n_2} z_p$

$$\therefore\ \Delta\tau = \frac{n_1}{c} z_p\left[1 - \frac{n_2}{n_1}\right] = \frac{n_1\Delta}{c}\cdot z_p$$

ie, $\Delta\tau \propto \Delta$.

<u>Example</u> for $n_1 = 1.46$ $\Delta = 0.01$ $\Delta\tau = 50$ ps/km

200 <u>times</u> <u>more than that of Graded</u> w/g.