6: Waveguides

6.1: Introduction

Rectangular waveguides are used to route millimeter-wave signals and high power microwave signals. The rectangular waveguide, often called just waveguide, shown in Figure $6.1.1$ has metal walls forming a rectangular pipe. Charges and currents induced in the conductive walls guide propagating EM fields in the $+z$ and $-z$ directions. The rectangular waveguide has very little loss compared to a coaxial line because the EM field is away from the walls and there is little current in the walls, and what is there is spread out resulting in low current density. All of the waveguide loss, as with the loss of most transmission systems, is resistive loss so minimizing current density minimizes loss.

This chapter begins with Section 6.2 where symmetries and restricting

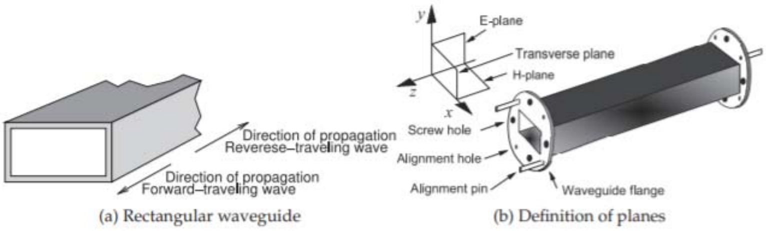

Figure $6.1.1$: Rectangular waveguide.

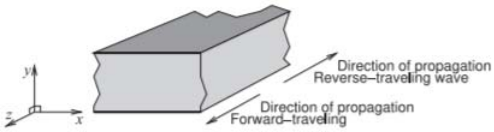

Figure $6.1.2$: Parallel-plate waveguide.

propagation to only the $\pm z$ direction are applied to Maxwell's equations to yield the rectangular wave equation. These are then used in Section 6.3 to describe propagation between two metal planes forming what is called a parallel-plate waveguide, see Figure $6.1.2$. Then the rectangular wave equations are applied to a rectangular waveguide in Section 6.4 to derive the field distribution inside a rectangular waveguide.

6.2: The Rectangular Wave Equation

Maxwell's equations will be put in a form that can be used in establishing the field descriptions in parallel-plate and rectangular waveguides. The EM fields in these structures vary sinusoidally with respect to both position and time so the first simplification of Maxwell's equations is to use phasors. Boundary conditions are established by the metal walls, and these walls match the Cartesian coordinate system. So Maxwell's equations are put in Cartesian coordinate form. Simplifications of the fields can be made that relate to the positions of the metal walls. Another simplification is made by assuming that there can only be propagation in the $\pm z$ direction. When the wave propagates in the $+z$ direction it is called the forward-traveling wave, and when it propagates in the $-z$ direction it is called the reverse-traveling wave.

The development begins with Maxwell's equations (Equations (1.5.1)–(1.5.4)) in a source-free region ($\rho = 0$ and $J = 0$). A simplification comes from assuming a linear, isotropic, and homogeneous medium so that ε and μ are independent of signal level and are independent of the field direction and of position, thus

$$\nabla \times \overline{\mathcal{E}} = -\frac{\partial \overline{\mathcal{B}}}{\partial t} = -\mu \frac{\partial \overline{\mathcal{H}}}{\partial t} \tag{6.2.1}$$

$$\nabla \cdot \overline{\mathcal{D}} = 0 = \nabla \cdot \overline{\mathcal{E}} \tag{6.2.2}$$

$$\nabla \times \overline{\mathcal{H}} = \frac{\partial \overline{\mathcal{D}}}{\partial t} = \varepsilon \frac{\partial \overline{\mathcal{E}}}{\partial t} \tag{6.2.3}$$

$$\nabla \cdot \overline{\mathcal{B}} = 0 = \nabla \cdot \overline{\mathcal{H}} \tag{6.2.4}$$

Taking the curl of Equation (6.2.1) leads to

$$\nabla \times \nabla \times \overline{\mathcal{E}} = -\nabla \times \mu \frac{\partial \overline{\mathcal{H}}}{\partial t} = -\mu \frac{\partial (\nabla \times \overline{\mathcal{H}})}{\partial t} \tag{6.2.5}$$

Applying the identity $\nabla \times \nabla \times \overline{\mathcal{A}} = \nabla(\nabla \cdot \overline{\mathcal{A}}) - \nabla^2 \overline{\mathcal{A}}$ to the left-hand side of Equation (6.2.5), and replacing $\nabla \times \overline{\mathcal{H}}$ with the right-hand side of Equation (6.2.3), the equation above becomes

$$-\nabla^2 \overline{\mathcal{E}} + \nabla(\nabla \cdot \overline{\mathcal{E}}) = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \overline{\mathcal{E}}}{\partial t}\right) = -\mu\varepsilon \frac{\partial^2 \overline{\mathcal{E}}}{\partial t^2} \tag{6.2.6}$$

Using Equation (6.2.2) this reduces to

$$\nabla^2 \overline{\mathcal{E}} = \mu\varepsilon \frac{\partial^2 (\overline{\mathcal{E}})}{\partial t^2} \tag{6.2.7}$$

where

$$\nabla^2 \overline{\mathcal{E}} = \frac{\partial^2 \overline{\mathcal{E}}}{\partial x^2} + \frac{\partial^2 \overline{\mathcal{E}}}{\partial y^2} + \frac{\partial^2 \overline{\mathcal{E}}}{\partial z^2} = \nabla_t^2 \overline{\mathcal{E}} + \frac{\partial^2 \overline{\mathcal{E}}}{\partial z^2} \tag{6.2.8}$$

and

$$\nabla_t^2 \overline{\mathcal{E}} = \frac{\partial^2 \overline{\mathcal{E}}}{\partial x^2} + \frac{\partial^2 \overline{\mathcal{E}}}{\partial y^2} \tag{6.2.9}$$

is used for fields propagating in the $\pm z$ direction and the subscript t indicates the transverse plane (the $x-y$ plane here). Equation (6.2.9) can be put into the form of its components. Since

$$\overline{\mathcal{E}} = \mathcal{E}_x \hat{\mathbf{x}} + \mathcal{E}_y \hat{\mathbf{y}} + \mathcal{E}_z \hat{\mathbf{z}} \tag{6.2.10}$$

then

$$\nabla^2\overline{\mathcal{E}} = \left(\frac{\partial^2 \mathcal{E}_x}{\partial x^2}\hat{\mathbf{x}} + \frac{\partial^2 \mathcal{E}_y}{\partial x^2}\hat{\mathbf{y}} + \frac{\partial^2 \mathcal{E}_z}{\partial x^2}\hat{\mathbf{z}}\right) + \left(\frac{\partial^2 E_x}{\partial y^2}\hat{\mathbf{x}} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2}\hat{\mathbf{y}} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2}\hat{\mathbf{z}}\right)$$

$$+ \left(\frac{\partial^2 E_x}{\partial z^2}\hat{\mathbf{x}} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2}\hat{\mathbf{y}} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2}\hat{\mathbf{z}}\right) \tag{6.2.11}$$

$$= \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} + \frac{\partial^2 \mathcal{E}_x}{\partial z^2}\right)\hat{\mathbf{x}} + \left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2}\right)\hat{\mathbf{y}} \tag{6.2.12}$$

$$+ \left(\frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2}\right)\hat{\mathbf{z}} \tag{6.2.13}$$

and

$$\nabla_t^2\overline{\mathcal{E}} = \left(\frac{\partial^2 \mathcal{E}_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2}\right)\hat{\mathbf{x}} + \left(\frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2}\right)\hat{\mathbf{y}} + \left(\frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2}\right)\hat{\mathbf{z}} \tag{6.2.14}$$

Invoking the phasor form, $\partial/\partial t$ is replaced by $\jmath\omega$, and with propagation only in the $\pm z$ direction there is an assumed $\mathrm{e}^{(\jmath\omega t - \gamma z)}$ dependence of the fields. Development is now simplified by introducing the phasor $\overline{\mathcal{E}}$ defined so that

$$\overline{\mathcal{E}} = \overline{E}\mathrm{e}^{-\gamma z} \tag{6.2.15}$$

Now Equation (6.2.7) further reduces to

$$\nabla^2\overline{E} = \left(\nabla_t^2\overline{E} + \frac{\partial^2\overline{E}}{\partial z^2}\right) = \nabla_t^2\overline{E} + \gamma^2\overline{E} = (\jmath\omega)^2\mu\varepsilon\overline{E} = -k^2\overline{E} \tag{6.2.16}$$

where $k = \omega\sqrt{\mu\varepsilon}$ is the **wavenumber** (with SI units of m^{-1}). Rearranging Equation (6.2.16) yields

$$\nabla_t^2\overline{E} = -(\gamma^2 + k^2)\overline{E} \tag{6.2.17}$$

A similar expression can be derived for the magnetic field:

$$\nabla_t^2\overline{H} = -(\gamma^2 + k^2)\overline{H} \tag{6.2.18}$$

Equations (6.2.17) and (6.2.18) are called wave equations, or Helmholtz equations, for phasor fields propagating in the z direction. Equations (6.2.17) and (6.2.18) are usually written as

$$\nabla_t^2\overline{E} = -k_c^2\overline{E} \tag{6.2.19}$$

$$\nabla_t^2\overline{H} = -k_c^2\overline{H} \tag{6.2.20}$$

where the **cutoff wavenumber** is

$$k_c^2 = \gamma^2 + k^2 \tag{6.2.21}$$

Equations (6.2.19) and (6.2.20) describe the transverse fields (the fields in the x–y plane) between the conducting plates of the parallel-plate as well as within the walls of the rectangular waveguide having a $\mathrm{e}^{(\jmath\omega t - \gamma z)}$ dependence. The general form of the solution of these equations is a sinusoidal wave moving in the z direction. For propagating waves in a lossless medium, $\gamma = \jmath\beta$, where β is the phase constant:

$$\beta = \pm\sqrt{k^2 - k_c^2} \tag{6.2.22}$$

If β is not real, which occurs when $|k_c| < |k|$, then an EM wave cannot propagate and such modes are called evanescent modes. These are like fringing fields. If they are generated, say at a discontinuity, they will store reactive energy locally.

Boundary conditions, resulting from the charges and current on the plates, further constrain the solutions. Equation (6.2.1) with Equation (1.A.39) becomes

$$\nabla \times \overline{E} = \jmath\omega\mu\overline{H} \tag{6.2.23}$$

In rectangular coordinates, $\overline{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ and $\overline{H} = H_x\hat{\mathbf{x}} + H_y\hat{\mathbf{y}} + H_z\hat{\mathbf{z}}$, and Equation (6.2.23) becomes

$$\left.\begin{array}{ll}\frac{\partial E_z}{\partial y}+\gamma E_y=-\jmath\omega\mu H_x & -\frac{\partial E_z}{\partial x}-\gamma E_x=-\jmath\omega\mu H_y \\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\jmath\omega\mu H_z & \end{array}\right\} \tag{6.2.24}$$

Similarly for $\nabla\times\overline{H}\jmath\omega\varepsilon\overline{E}$:

$$\left.\begin{array}{ll}\frac{\partial H_z}{\partial y}+\gamma H_y=\jmath\omega\varepsilon E_x & -\frac{\partial H_z}{\partial x}-\gamma H_x=\jmath\omega\varepsilon E_y \\ \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=\jmath\omega\varepsilon E_z & \end{array}\right\} \tag{6.2.25}$$

Solving Equations (6.2.24) and (6.2.25) yields the rectangular wave equations for $k_c\neq 0(k_c^2=k^2+\gamma^2$ and if there is no loss $k_c^2=\omega^2\mu\varepsilon-\beta^2$):

$$\left.\begin{array}{ll}E_x=\frac{-1}{k_c^2}\left(\gamma\frac{\partial E_z}{\partial x}+\jmath\omega\mu\frac{\partial H_z}{\partial y}\right) & E_y=\frac{1}{k_c^2}\left(-\gamma\frac{\partial E_z}{\partial y}+\jmath\omega\mu\frac{\partial H_z}{\partial x}\right) \\ H_x=\frac{1}{k_c^2}\left(-\gamma\frac{\partial H_z}{\partial x}+\jmath\omega\varepsilon\frac{\partial E_z}{\partial y}\right) & H_y=\frac{-1}{k_c^2}\left(\gamma\frac{\partial H_z}{\partial y}+\jmath\omega\varepsilon\frac{\partial E_z}{\partial z}\right) \\ E_z=\frac{-\jmath}{\omega\varepsilon}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right) & H_z=\frac{\jmath}{\omega\mu}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\end{array}\right\} \tag{6.2.26}$$

The solution for $k_c=0$ is arrived at separately. Since $k_c=0$ there is no loss. Also propagation at DC is a solution and the phasor fields at $\omega=0$ will also be the field descriptions at any frequency. At $\omega=0,\ \gamma=\jmath\beta=0$ and $k=0$. Equations (6.2.24)–(6.2.25) are now written as

$$\left.\begin{array}{ll}\frac{\partial E_z}{\partial y}+0\cdot E_y=0\cdot H_x=0 & -\frac{\partial E_z}{\partial x}-0\cdot E_x=0\cdot H_y=0 \\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-0\cdot H_z=0 & \frac{\partial H_z}{\partial y}+0\cdot H_y=0\cdot E_x=0 \\ -\frac{\partial H_z}{\partial x}-0\cdot H_x=0\cdot E_y=0 & \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=0\cdot E_z=0\end{array}\right\} \tag{6.2.27}$$

The only solutions to these with $\partial/\partial x=0$ that also satisfies boundary conditions are that $E_z=0=H_z=E_x=H_y$, and H_y and E_x are constants.

Now that the fields are in the appropriate forms, classification of possible solutions (i.e. modes) can be developed for the parallel-plate and rectangular waveguides. At this stage the following simplifications have been made to Maxwell's equations to get them into the form of Equation (6.2.26)):

- Using phasors
- Restriction of propagation to the $+z$ and $-z$ directions
- Assuming that ε and μ are constants
- Putting the wave equations in rectangular form so that boundary conditions established by the metal walls can be easily applied.

6.3: Parallel-Plate Waveguide

This section derives the propagating EM fields for the parallel-plate waveguide shown in Figure $6.3.1$. The parallel-plate waveguide shown in Figure $6.3.1$(a) has conducting planes at the top and bottom that (as an approximation) extend infinitely in the x direction. Electromagnetic fields introduced between the plates, say by a sinusoidally varying voltage generator across the plates, will be guided by the charges and currents induced in the conductors.

The parallel-plate waveguide structure occurs in many planar circuits, such as between the ground and power planes of circuit boards. Understanding the EM propagation supported by parallel-plate waveguides enables design choices to be made that suppress unwanted propagation modes.

6.3.1 TEM Mode

In the transverse EM (TEM) mode, all of the E and H field components are in the plane transverse to the direction of propagation, that is, $E_z = 0 = H_z$. Thus Equation (6.2.26) requires that $E_x,\ E_y,\ H_x$, and H_y cannot vary with position in the transverse plane (i.e., with respect to x and y). Thus $E_x,\ E_y,\ H_x$, and H_y must be constant between the plates. Furthermore, boundary conditions require that $H_y = 0$ and $E_x = 0$ at the conductors. So in the TEM parallel-plate mode, only E_y and H_x exist, and they are constant. Equation

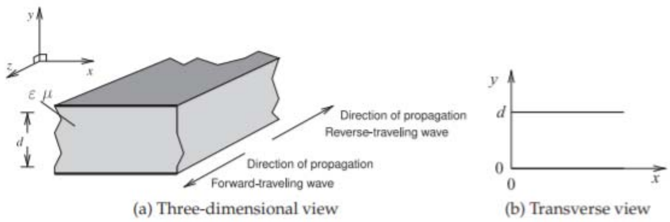

Figure $6.3.1$: Parallel-plate waveguide.

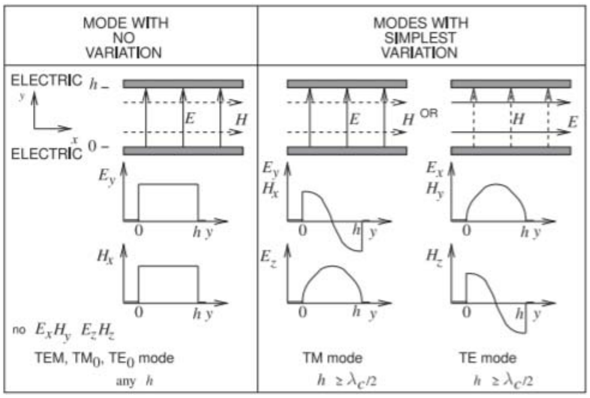

Figure $6.3.2$: Lowest-order modes supported by combinations of electric and magnetic walls for the TEM $(= \mathrm{TM}_0 = \mathrm{TE}_0)$, TM_1, and TE_1 modes.

(6.2.25) leads to

$$H_x = \frac{\gamma E_y}{\jmath\omega\mu} = \pm\frac{\jmath\omega\sqrt{\mu\varepsilon}}{\jmath\omega\mu}E_y = \pm\sqrt{\frac{\varepsilon}{\mu}}E_y = \pm\frac{1}{\eta}E_y \tag{6.3.1}$$

where the plus sign describes forward-traveling fields (propagating in the $+z$ direction) and the minus sign describes backward-traveling fields (propagating in the $-z$ direction). The quantity $\eta = \sqrt{\mu/\varepsilon}$ is called the **wave impedance**, it is the **intrinsic impedance** of the medium between the parallel plates. This field variation is shown on the left in Figure $6.3.2$(a). The TEM mode exists down to DC.

To determine the characteristic impedance of the parallel-plate waveguide first calculate the voltage of the top plate with respect to the bottom plate. This voltage is the integral of the electric field between the plates:

$$V = -\int_{y=0}^{d} E_y \mathrm{e}^{-\gamma z} dy = E_y d \mathrm{e}^{-\gamma z} \tag{6.3.2}$$

since E_y is a constant. The current on the top plate in the z direction is obtained by integrating the surface current density in the x direction. Assuming that the plates have a width W in the x direction then the current on the top plate is

$$I = -\int_{x=0}^{W} J_s \cdot \hat{z} dx = H_x W \mathrm{e}^{-\jmath z} \tag{6.3.3}$$

since E_y is a constant. In terms of voltage and current (and hence treating the parallel-plate waveguide as a transmission line) the characteristic impedance of the TEM mode is

$$Z_0 = \frac{V}{I} = \frac{E_y d}{H_x W} = \frac{\eta d}{W} \tag{6.3.4}$$

Here η is the intrinsic impedance of a TEM mode in the medium. Since we are considering a TEM mode, the wave impedance of the TEM mode is just the intrinsic impedance, that is,

$$Z_{\text{TEM}} = E_y / H_x = Z_0|_{\text{free-space}} = \eta \tag{6.3.5}$$

With $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ being the free-space impedance, the characteristic impedance can be written

$$Z_0 = \frac{\eta_0 d}{W}\sqrt{\frac{\mu_r}{\varepsilon_r}} \quad \text{and} \quad Z_{\text{TEM}} = \eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} \tag{6.3.6}$$

The phase velocity $(= \omega/\beta)$ is just the speed of light in the medium:

$$v_p = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \tag{6.3.7}$$

Formulas for attenuation are developed in [1] and the conductor attenuation

$$\alpha_c = \frac{R_s}{\eta d} \quad \text{(with SI units of Np/m)} \tag{6.3.8}$$

where $R_s = 1/(\sigma\delta_s)$ is the surface resistance of the conductor, σ is the conductivity of the conductor, and δ_s is the skin depth in the conductor. The attenuation due to dielectric loss is

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} \quad \text{(with SI units of Np/m)} \tag{6.3.9}$$

6.3.2 TM Mode

The **Transverse Magnetic Mode** (**TM**) is characterized by $H_z = 0$. Another restriction that will be used here in developing the field equations is that there is no variation of the fields in the x direction. Examining Equation (6.2.25) the only components of the field that could exist are E_y, E_z, and H_x. Everywhere E_y is perpendicular, and H_y is parallel, to the electrical walls so boundary conditions are satisfied for E_y and H_x. E_z will be parallel to the electrical walls at the walls so boundary conditions need to be applied in deriving E_z.

The boundary conditions are that the E field parallel to the conducting walls is zero. Considering E_z only, Equation (6.2.18) becomes

$$\frac{d^2 E_z}{dy^2} = -k_c^2 E_z \tag{6.3.10}$$

The solution to Equation (6.3.10) is

$$E_z = [E_0 \sin(k_c y) + E_1 \cos(k_c y)]\mathrm{e}^{-\gamma z} \tag{6.3.11}$$

To find the coefficients E_0 and E_1 boundary conditions are applied so that E_z is zero at $y = 0$ and $y = d$ (since the E field parallel to the conductors must be zero), that is,

$$E_z|_{y=0} = 0 = E_1 \quad \text{and} \quad E_z|_{y=d} = 0 = E_0 \sin(k_c d) \tag{6.3.12}$$

This requires that $\sin(k_c d) = 0$, and thus requiring that there are discrete values of k_c:

$$k_c = m\pi/d \quad m = 1, 2, 3, \ldots \tag{6.3.13}$$

(Note that $m = 0$ is also a solution but requires a separate derivation, see the summary for this section.) Each value of k_c identifies a different mode and m is the mode index. The mth mode is the TM_m mode and m indicates the number of half-sinusoidal variations of the fields in the y direction. The TM_m mode propagates if the wavelength of the signal is such that $\lambda \geq \lambda_c$, where λ_c is the critical wavelength of the mth mode

Substituting the above results and assumptions (e.g., $\partial/\partial x = 0$) in Equation (6.2.25),

$$H_z = 0 \; E_x = 0 \; H_y = 0 \tag{6.3.14}$$

$$E_z = E_0 \sin(k_c y) \mathrm{e}^{-\gamma z} \tag{6.3.15}$$

$$E_y = -\frac{\gamma}{k_c^2} \frac{dE_z}{dy} = -\frac{\gamma}{k_c} E_0 \cos(k_c y) \mathrm{e}^{-\gamma z} \tag{6.3.16}$$

$$H_x = \frac{\jmath\omega\varepsilon}{k_c^2} \frac{dE_z}{dy} = \frac{\jmath\omega\varepsilon}{k_c} E_0 \cos(k_c y) \mathrm{e}^{-\gamma z} \tag{6.3.17}$$

These are the complete field descriptions of the TM parallel-plate waveguide modes with zero variation in the x direction. Recall that the wavenumber $k = \omega\sqrt{\mu\varepsilon}$.

There are an infinite number of TM modes identified by the index m, which determines the cutoff wavenumber, k_c, of the particular mode. The propagation constant of the mth mode, i.e. the TM_m mode, is

$$\gamma = \sqrt{k_c^2 - k^2} = \sqrt{(m\pi/d)^2 - \omega^2\mu\varepsilon} \tag{6.3.18}$$

Propagation is only possible if γ has an imaginary component. Thus in a lossless medium $\gamma = \jmath\beta$ and $\beta = \sqrt{k^2 - k_c^2}$. The cutoff frequency below which propagation is not possible is

$$f_c = \frac{1}{2\pi} \frac{k_c}{\sqrt{\mu\varepsilon}} = \frac{1}{2\pi} \frac{m\pi}{d\sqrt{\mu\varepsilon}} = \frac{m\nu}{2d} \tag{6.3.19}$$

where $\nu = 1/\sqrt{\mu\varepsilon}$ is the velocity of a TEM mode in the medium. The cutoff wavelength can also be defined as

$$\lambda_c = \frac{\nu}{f_c} = \frac{2d}{m} \tag{6.3.20}$$

where ν is the speed of light in the medium. The wavelength of the TM_m mode, at a particular frequency, is the guide wavelength

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{6.3.21}$$

where λ is the wavelength of a TEM mode in the medium: $\lambda = \nu/f$ (so $\lambda_g = \lambda$ when $k_c = 0$). The phase velocity of the modes is dependent on the mode index m through the cutoff frequency:

$$v_p = \frac{\omega}{\beta} = \frac{\nu}{\sqrt{1 - (f_c/f)^2}} \tag{6.3.22}$$

and the group velocity is

$$v_g = \frac{d\omega}{d\beta} = \nu\sqrt{1 - (f_c/f)^2} \tag{6.3.23}$$

The phase velocity, v_p, of a TM mode is greater than ν while the group velocity, v_g, is slower than ν. The group velocity is the velocity at which energy is transmitted and thus can never be faster than the speed of light, c. The phase velocity, however, can be greater than c. The TM mode field variation is shown on the right in Figure $6.3.2$.

The wave impedance of the TM_m mode is

$$Z_{\mathrm{TM}} = -E_y/H_x = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k} \tag{6.3.24}$$

Formulas for attenuation are developed in [1] and the conductor attenuation

$$\alpha_c = \frac{2kR_s}{\beta\eta d} \quad \text{(with SI units of Np/m)} \tag{6.3.25}$$

where $R_s = 1/(\sigma\delta_s)$ is the surface resistance of the conductor, σ is the conductivity of the conductor, and δ_s is the skin depth in the conductor. The attenuation due to dielectric loss is

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} \quad \text{(with SI units of Np/m)} \tag{6.3.26}$$

6.3.3 TE Mode

The **transverse electric (TE) mode** is characterized by $E_z = 0$. Following the same development as for the TM modes, the TE_n mode fields with n variations of the H_z are

$$E_z = 0, \; E_y = 0, \; H_x = 0 \tag{6.3.27}$$

$$H_z = H_0 \cos(k_c y) \mathrm{e}^{-\gamma z} \tag{6.3.28}$$

$$E_x = \frac{\jmath\omega\mu}{k_c} H_0 \sin(k_c y) \mathrm{e}^{-\gamma z} \tag{6.3.29}$$

$$H_y = \frac{\gamma}{k_c} H_0 \sin(k_c y) \mathrm{e}^{-\gamma z} \tag{6.3.30}$$

The equations for k_c, v_p, v_g, f_c, λ_c, and λ_g of the TE_n mode are the same as for the TM_m mode considered in the previous section with the replacement of the mode index m by n, $n = 1, 2, 3, \ldots$. (Note that $n = 0$ is also a solution but requires a separate derivation, see the summary for this section.) The TE mode field variation is shown on the right in Figure $6.3.2$.

The wave impedance of the TE mode is

$$Z_{\mathrm{TE}} = \frac{k\eta}{\beta} \tag{6.3.31}$$

Formulas for attenuation are developed in [1] and the conductor attenuation

$$\alpha_c = \frac{2k_c^2 R_s}{k\beta\eta d} \quad \text{(with SI units of Np/m)} \tag{6.3.32}$$

where $R_s = 1/(\sigma\delta_s)$ is the **surface resistance**, η, of the conductor, σ is the conductivity of the conductor, and δ_s is the skin depth pf the conductor. The attenuation due to dielectric loss is

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} \quad \text{(with SI units of Np/m)} \tag{6.3.33}$$

6.3.4 Summary

The TEM mode (where $k_c = 0$) is the same as the TM_0 mode and the TE_0 mode. Here derivations of the TE and TM modes began with from Equation (6.2.25) and were only solutions for $k_c \neq 0$. Development of the $0th$ order TE and TM modes requires derivation from Equation (6.2.26) but this was done for the TEM mode and so was not repeated for the TE_0 and TM_0 modes.

6.4: Rectangular Waveguide

A rectangular waveguide is shown in Figure $6.4.1$(a). Rectangular waveguides guide EM energy between four connected electrical walls, and there is little current created on the walls. As a result, resistive losses are quite low, much lower than can be achieved using coaxial lines for example. One of the major uses of a rectangular waveguide is when losses must be kept to a minimum, so that a rectangular waveguide is used in very high-power situations such as radar, and at a few tens of gigahertz and above. At higher frequencies the loss of coaxial lines becomes very large, and it also becomes difficult to build small-diameter coaxial lines at 100 GHz and above. As a result, a rectangular waveguide is nearly always used above 100 GHz. There are many low- to medium-power legacy systems that use rectangular waveguides down to 1 GHz.

A rectangular waveguide supports many different modes, but it does not support the TEM mode. The modes are categorized as being either TM or TE, denoting whether all of the magnetic fields are perpendicular to the direction of propagation (these are the transverse magnetic fields) or whether all of the electric fields are perpendicular to the direction of propagation (these are the transverse electric fields). Dimensions of the waveguide can be chosen so that only one mode can propagate for a range of frequencies. With more than one mode propagating, the different components of a signal would travel at different speeds and thus combine at a load incoherently, since the ratio of the energy in the modes would vary (usually) randomly.

The TE and TM field descriptions are derived from the solution of differential equations—Maxwell's equations—subject to boundary conditions. The general solutions for rectangular systems are sinewaves and there are possibly many discrete solutions. The nomenclature that has developed over

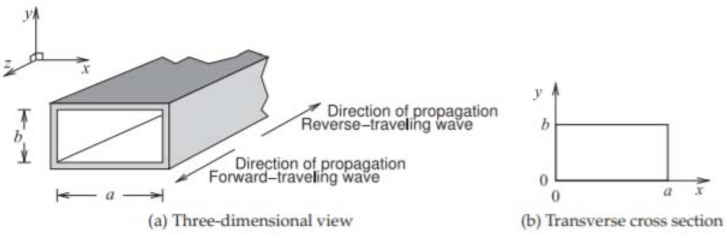

Figure $6.4.1$: Rectangular waveguide with internal dimensions of a and b.

the years to classify modes references the number of variations in the x direction, using the index m, and the number of variations in the y direction, using the index n. So there are TE_{mn} and TM_{mn} modes, and dimensions are usually selected so that only the TE_{10} mode can propagate.

6.4.1 TM Modes

The development of the field descriptions for the TM modes begins with the rectangular wave equations derived in Section 6.2. Transverse magnetic waves have zero H_z, but nonzero E_z. The differential equation governing E_z is, in rectangular coordinates (from Equations (6.2.13) and (6.2.16)),

$$\nabla_t^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -k_c^2 E_z \tag{6.4.1}$$

Using a separation of variables procedure, this equation has the solution

$$E_z = [A'\sin(k_x x) + B'\cos(k_x x)][C'\sin(k_y y) + D'\cos(k_y y)]\text{e}^{-\gamma z} \tag{6.4.2}$$

where

$$k_x^2 + k_y^2 = k_c^2 \tag{6.4.3}$$

The perfectly conducting boundary at $x = 0$ requires $B' = 0$ to produce $E_z = 0$ there. Similarly the ideal boundary at $y = 0$ requires $D' = 0$. Replacing $A'C'$ by a new constant A, then

$$E_z = A\sin(k_x x)\sin(k_y y)\text{e}^{-\gamma z} \tag{6.4.4}$$

The axial electric field, E_z, must also be zero at $x = a$ and $y = b$. This can only be so (except for the trivial solution $A = 0$) if $k_x a$ is an integral multiple of π so that $\sin(k_x a) = 0$:

$$k_x a = m\pi, \quad m = 1, 2, 3, \ldots \tag{6.4.5}$$

Similarly, for E_z to be zero at $y = b$, $\sin(k_y b) = 0$ and $k_y b$ must also be a multiple of π:

$$k_y b = n\pi, \quad n = 1, 2, 3, \ldots \tag{6.4.6}$$

So the cutoff frequency of the TM wave with m variations in x and with n variations in y (i.e., the TM_{mn} mode) is, from Equation (6.4.3),

$$f_{c_{m,n}} = \frac{k_{c_{m,n}}}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(n\frac{\pi}{b}\right)^2\right]^{1/2} \tag{6.4.7}$$

The remaining field components of the TM_{mn} wave are found with $H_z = 0$ and E_z from Equation (6.4.4) and Equation (6.2.25)):

$$E_x = -\frac{\gamma k_x}{k^2_{c_{m,n}}} A\cos(k_x x)\sin(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.8}$$

$$E_y = -\frac{\gamma k_y}{k^2_{c_{m,n}}} A\sin(k_x x)\cos(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.9}$$

$$H_x = \frac{\jmath\omega\varepsilon k_y}{k^2_{c_{m,n}}} A\sin(k_x x)\cos(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.10}$$

$$H_y = -\frac{\jmath\omega\varepsilon k_x}{k^2_{c_{m,n}}} A\cos(k_x x)\sin(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.11}$$

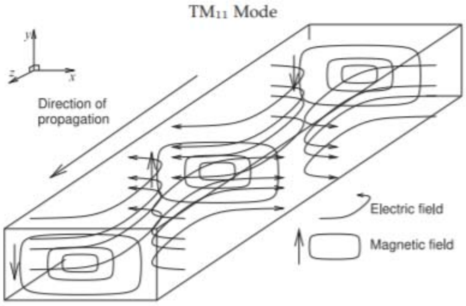

Figure 6.4.2: Electric and magnetic field distribution for the lowest-order TM mode, the TM_{11} mode.

The spatial field variations depend on the x and y cutoff wavenumbers, k_x and k_y, which in turn depend on the mode indexes and the cross-sectional dimensions of the waveguide. The cutoff wavenumber, k_c, is a function of the m and n indexes, and so $k_{c_{m,n}}$ is often used for the cutoff wavenumber with

$$k^2_{c_{m,n}} = k^2_{x,m} + k^2_{y,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{6.4.12}$$

The lowest-order TM mode is the TM_{11} mode, with $m = 1$ and $n = 1$, and this has the minimum variation of the fields (of any TM mode); these are shown in Figure 6.4.2.

In summary, a mode can propagate only at frequencies above the cutoff frequency. Another quantity that defines when cutoff occurs is the cutoff wavelength, defined as

$$\lambda_c = \frac{\nu}{f_{c_{m,n}}} \tag{6.4.13}$$

where $\nu = 1/\sqrt{\mu\varepsilon}$ is the velocity of a TEM mode in the medium (and of course this is not a rectangular waveguide mode). The cutoff wavelength is the wavelength in the medium (without the waveguide walls) at which cutoff occurs. Since k_c^2 is $k^2 - \beta^2$, the attenuation constant of a given mode for frequencies below the cutoff frequency is

$$\alpha = \sqrt{k^2_{c_{m,n}} - k^2} = k_{c_{m,n}}\sqrt{1 - \left(\frac{f}{f_{c_{m,n}}}\right)^2}, \quad f < f_{c_{m,n}} \tag{6.4.14}$$

The phase constant for frequencies above the cutoff frequency is

$$\beta = \sqrt{k^2 - k_{c_{m,n}}^2} = k\sqrt{1 - \left(\frac{f_{c_{m,n}}}{f}\right)^2}, \quad f > f_{c_{m,n}} \tag{6.4.15}$$

For a propagating mode (i.e., $f > f_{c_{m,n}}$) the wavelength of the mode, called the guide wavelength, is

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{6.4.16}$$

where λ is the wavelength of a TEM mode in the medium (but of course not in the waveguide): $\lambda = \nu/f$. The phase velocity of the mode is also dependent on the mode indexes m and n through the cutoff frequency,

$$v_p = \frac{\omega}{\beta} = \frac{\nu}{\sqrt{1 - (f_{c_{m,n}}/f)^2}} \tag{6.4.17}$$

and the group velocity is

$$v_g = \frac{d\omega}{d\beta} = \nu\sqrt{1 - (f_{c_{m,n}}/f)^2} \tag{6.4.18}$$

6.4.2 TE Modes

Transverse electric waves have zero E_z and nonzero H_z so that, in rectangular coordinates,

$$\nabla_t^2 H_z = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = -k_c^2 H_z \tag{6.4.19}$$

Solving using the separation of variables technique gives

$$H_z = [A''\sin(k_x x) + B''\cos(k_x x)][C''\sin(k_y y) + D''\cos(k_y y)]\mathrm{e}^{-\gamma z} \tag{6.4.20}$$

where

$$k_x^2 + k_y^2 = k_c^2 \tag{6.4.21}$$

Imposition of a boundary condition in this case is a little less direct, but the electric field components are

$$\begin{aligned}
E_x &= -\frac{\jmath\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} && (6.4.22)\\
&= -\frac{\jmath\omega\mu k_y}{k_c^2}[A''\sin(k_x x) + B''\cos(k_x x)][C''\cos(k_y y) - D''\sin(k_y y)]\mathrm{e}^{-\gamma z} && (6.4.23)\\
E_y &= \frac{\jmath\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} && (6.4.24)\\
&= \frac{\jmath\omega\mu k_x}{k_c^2}[A''\cos(k_x x) - B''\sin(k_x x)][C''\sin(k_y y) + D''\cos(k_y y)]\mathrm{e}^{-\gamma z} && (6.4.25)
\end{aligned}$$

For E_x to be zero at $y = 0$ for all x, $C'' = 0$; and for $E_y = 0$ at $x = 0$ for all y, $A'' = 0$. Defining $B''D'' = B$, then

$$\begin{aligned}
H_z &= B\cos(k_x x)\cos(k_y y) && (6.4.26)\\
E_x &= \frac{\jmath\omega\mu k_y}{k_c^2}B\cos(k_x x)\sin(k_y y)\mathrm{e}^{-\gamma z} && (6.4.27)\\
E_y &= -\frac{\jmath\omega\mu k_x}{k_c^2}B\sin(k_x x)\cos(k_y y)\mathrm{e}^{-\gamma z} && (6.4.28)
\end{aligned}$$

E_y is zero at $x = a$, that is, $\sin(k_x a) = 0$, so that $k_x a$ must be a multiple of π:

$$k_x a = m\pi, \quad m = 1, 2, 3, \ldots \tag{6.4.29}$$

Also, E_x is zero at $y = b$, that is, $\sin(k_y b) = 0$, so that $k_y b$ must be zero (so that E_x is always zero) or that it is a multiple of π. Therefore

$$k_y b = n\pi \quad n = 0, 1, 2, 3, \ldots \tag{6.4.30}$$

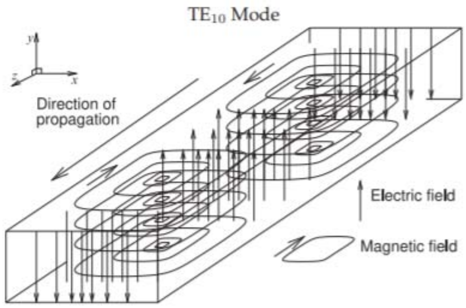

Figure 6.4.3: Electric and magnetic field distribution for the lowest-order TE mode.

The forms of the transverse electric field are then

$$E_x = \frac{\jmath\omega\mu k_y}{k^2_{c_{m,n}}} B\cos(k_x x)\sin(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.31}$$

$$E_y = -\frac{\jmath\omega\mu k_x}{k^2_{c_{m,n}}} B\sin(k_x x)\cos(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.32}$$

and the corresponding transverse magnetic field components are

$$H_x = \frac{\gamma k_x}{k^2_{c_{m,n}}} B\sin(k_x x)\cos(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.33}$$

$$H_y = \frac{\gamma k_y}{k^2_{c_{m,n}}} B\cos(k_x x)\sin(k_y y)\mathrm{e}^{-\gamma z} \tag{6.4.34}$$

Here the use of $k_{c_{m,n}}$ emphasizes that the cutoff wavenumber is a function of the m and n indexes:

$$k^2_{c_{m,n}} = k^2_{x,m} + k^2_{y,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{6.4.35}$$

The lowest-order TE mode is the TE_{10} mode (with $m = 1$ and $n = 0$) and this has the minimum variation of the fields; these are shown in Figure 6.4.3.