Reflection and refraction of electromagnetic waves

2.1 Introduction

In this chapter we will study the reflection and refraction of electromagnetic waves from an interface separating two media and from a stack of films. Such studies are very important in understanding many practical optical devices such as Fabry–Perot etalons, interference filters, special optical coatings etc. Furthermore, by studying the state of polarization of a light beam reflected from a medium, one can obtain its optical characteristics; this forms the basis of the field of ellipsometry.

In deriving the reflection and transmission coefficients we will use the following continuity conditions† at the interface:

(*a*) continuity of the tangential components of the electric vector $\mathscr{E}$;
(*b*) continuity of the normal components of the displacement vector $\mathscr{D}$;
(*c*) continuity of the tangential components of the magnetic field vector $\mathscr{H}$; and
(*d*) continuity of the normal components of the magnetic induction vector $\mathscr{B}$.

We will find that the equations determining the reflection and transmission coefficients fall into two groups: one of the groups contains only the components of $\mathscr{E}$ parallel to the plane of incidence (and $\mathscr{H}$ perpendicular to the plane) and the other group contains only the components of $\mathscr{E}$ perpendicular to the plane of incidence (and $\mathscr{H}$ parallel to the plane). Therefore the two cases (being independent of each other) will be considered separately and using them we can study the reflection (and refraction) of electromagnetic waves which have arbitrary states of polarization. We will, for example, show that a circularly polarized wave on reflection from a dielectric surface can become elliptically polarized. Similarly, a linearly polarized wave reflected from a metal surface may be linearly polarized or circularly polarized or elliptically polarized depending on the angle of incidence and the direction of the electric vector associated with the incident wave.

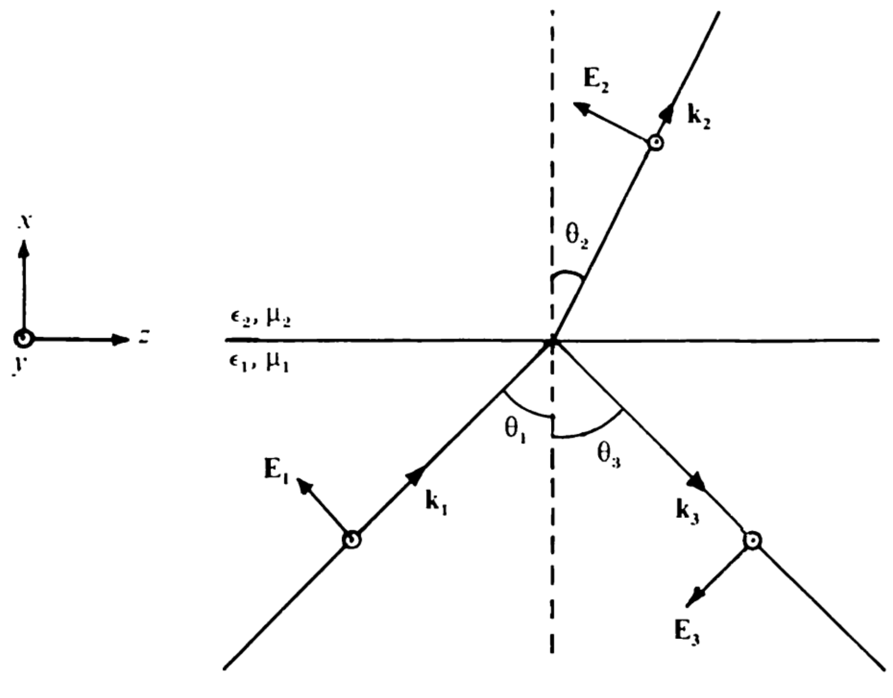
2.2 Reflection and refraction at the interface of two homogeneous nonabsorbing dielectrics

We first consider the reflection and refraction of a plane electromagnetic wave incident at the interface of two dielectrics characterized by (ϵ_1, μ_1) and (ϵ_2, μ_2) (see Fig. 2.1). We assume the media to be nonabsorbing, isotropic and homogeneous. Let

$$\left.\begin{aligned} \mathscr{E}_1 &= \mathbf{E}_1 e^{i(\omega t - \mathbf{k}_1 \cdot \mathbf{r})} \\ \mathscr{E}_2 &= \mathbf{E}_2 e^{i(\omega_2 - \mathbf{k}_2 \cdot \mathbf{r})} \\ \text{and} \quad \mathscr{E}_3 &= \mathbf{E}_3 e^{i(\omega_3 t - \mathbf{k}_3 \cdot \mathbf{r})} \end{aligned}\right\} \tag{2.1}$$

represent the electric fields associated with the incident, refracted and reflected waves respectively; the vectors $\mathbf{E}_1, \mathbf{E}_2$ and $\mathbf{E}_3$ are independent of space and time. Let θ_1, θ_2 and θ_3 represent the (acute) angles that the vectors $\mathbf{k}_1, \mathbf{k}_2$, and $\mathbf{k}_3$ make with the normal to the interface (see Fig. 2.1). Fig. 2.1 also shows the direction of the Cartesian axes; the plane $x = 0$ represents the interface of the two dielectrics, and the direction of the y-axis

Fig. 2.1 The reflection and refraction of a plane wave incident at the interface of two dielectrics; the electric vector is assumed to be in the plane of incidence (i.e., p-polarized).



is such that

$$k_{1y} = 0 \tag{2.2}$$

i.e., the propagation vector associated with the incident wave is parallel to the x–z plane.

Since the fields satisfy the wave equation we must have

$$k_1^2 = \omega^2 \epsilon_1 \mu_1 \tag{2.3a}$$

$$k_2^2 = \omega_2^2 \epsilon_2 \mu_2 \tag{2.3b}$$

$$k_3^2 = \omega_3^2 \epsilon_1 \mu_1 \tag{2.3c}$$

We consider two special cases: one in which the electric vector lies in the plane of incidence and the other in which the electric vector lies perpendicular to the plane of incidence. By using appropriate boundary conditions it can easily be shown that if the electric vector associated with the incident plane wave lies in the plane of incidence then the electric vectors associated with the reflected and refracted waves will also lie in the plane of incidence (see Problem 2.1) and the same is true for the perpendicular case.

Case 1: $\mathscr{E}$ lying in the plane of incidence. We will first consider the case when the electric field associated with the incident wave lies in the plane of incidence (see Fig. 2.1). We resolve the electric vector $\mathscr{E}$ along the x and z-axis and since the z-component is tangential to the surface we must have $\mathscr{E}_z$ continuous across the interface; thus

$$\mathscr{E}_{1z} + \mathscr{E}_{3z} = \mathscr{E}_{2z} \tag{2.4}$$

or

$$\begin{aligned}[-E_1 e^{i(\omega t - \mathbf{k}_1\cdot\mathbf{r})}\cos\theta_1 - E_3 e^{i(\omega_3 t - \mathbf{k}_3\cdot\mathbf{r})}\cos\theta_3]_{x=0} \\ = [-E_2 e^{i(\omega_2 t - \mathbf{k}_2\cdot\mathbf{r})}\cos\theta_2]_{x=0}\end{aligned}$$

where all expressions are to be evaluated at the interface $x = 0$. Now

$$\begin{aligned}\mathbf{k}\cdot\mathbf{r} &= k_x x + k_y y + k_z z \\ &= k_y y + k_z z \qquad \text{(at the interface } x = 0)\end{aligned}$$

Thus, we must have

$$\begin{aligned}-E_1 e^{i(\omega t - k_{1y}y - k_{1z}z)}\cos\theta_1 - E_3 e^{i(\omega_3 t - k_{3y}y - k_{3z}z)}\cos\theta_3 \\ = -E_2 e^{i(\omega_2 t - k_{2y}y - k_{2z}z)}\cos\theta_2\end{aligned} \tag{2.5}$$

The above equation has to be valid at *all* times and for *all* values of y and

z (on the plane $x = 0$) and therefore we must have

$$\omega = \omega_2 = \omega_3 \tag{2.6}$$

$$k_{1y} = k_{2y} = k_{3y} \tag{2.7}$$

$$k_{1z} = k_{2z} = k_{3z} \tag{2.8}$$

Thus the frequencies associated with the reflected and refracted waves must be the same as that of the incident wave – which is also physically obvious. Thus

$$k_1 = \omega(\epsilon_1 \mu_1)^{\frac{1}{2}} = k_3 \tag{2.9}$$

and

$$k_2 = \omega(\epsilon_2 \mu_2)^{\frac{1}{2}} \tag{2.10}$$

Now, since $k_{1y} = 0$ (see Eq. (2.2)) we must have

$$k_{2y} = k_{3y} = 0 \tag{2.11}$$

i.e., $\mathbf{k}_1, \mathbf{k}_2$ and $\mathbf{k}_3$ will all be parallel to the x–z plane. Eq. (2.8) gives

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 = k_1 \sin\theta_3 \tag{2.12}$$

where we have used Eq. (2.9). Thus

$$\theta_1 = \theta_3 \tag{2.13}$$

which says that the angle of incidence equals the angle of reflection. Further

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{k_2}{k_1} = \left(\frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1}\right)^{\frac{1}{2}} = \frac{n_2}{n_1} \tag{2.14}$$

where

$$n_1 = \frac{c}{v_1} = \left(\frac{\epsilon_1 \mu_1}{\epsilon_0 \mu_0}\right)^{\frac{1}{2}} \tag{2.15}$$

and

$$n_2 = \frac{c}{v_2} = \left(\frac{\epsilon_2 \mu_2}{\epsilon_0 \mu_0}\right)^{\frac{1}{2}} \tag{2.16}$$

represent the refractive indices of media 1 and 2 respectively and $c = (\epsilon_0 \mu_0)^{-\frac{1}{2}}$ represents the velocity of light in free space. Eq. (2.14) gives us

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{2.17}$$

which is Snell's law of refraction.

We should mention here that in the derivation of the laws of reflection and refraction (*viz.* Eqs. (2.6)–(2.17)) we have only used the fact that *a* particular continuity condition should be valid at all times for all values of y and z on the plane $x=0$. Since this argument is valid for *all* continuity conditions, Eqs (2.6)–(2.17) will be valid for any *arbitrary* state of polarization associated with the incident wave.

From now on we will assume that both media are non-magnetic, i.e.,

$$\mu_1 \approx \mu_2 \approx \mu_0 \tag{2.18}$$

which is indeed true for all dielectrics. Thus

$$n_1 = \left(\frac{\epsilon_1}{\epsilon_0}\right)^{\frac{1}{2}} = K_1^{\frac{1}{2}} \tag{2.19}$$

and

$$n_2 = \left(\frac{\epsilon_2}{\epsilon_0}\right)^{\frac{1}{2}} = K_2^{\frac{1}{2}} \tag{2.20}$$

K_1 and K_2 being the dielectric constants of media 1 and 2 respectively.

Now, using Eqs. (2.6)–(2.8), Eq. (2.5) becomes

$$-E_1\cos\theta_1 - E_3\cos\theta_1 = -E_2\cos\theta_2 \tag{2.21}$$

where we ~~are~~ have also used Eq. (2.12). Similarly, since the normal component of $\mathscr{D}(=\mathscr{D}_x)$ is also continuous across the interface we must have

$$\mathscr{D}_{1x} + \mathscr{D}_{3x} = \mathscr{D}_{2x} \tag{2.22}$$

or

$$\epsilon_1 E_1 \sin\theta_1 - \epsilon_1 E_3 \sin\theta_3 = \epsilon_2 E_2 \sin\theta_2 \tag{2.23}$$

where we have used the relation $\mathscr{D} = \epsilon\mathscr{E}$ and Eq. (2.1). Substituting for E_2 from Eq. (2.23) in Eq. (2.21) and carrying out elementary manipulations we get

$$r_p \equiv \frac{E_3}{E_1} = \frac{\epsilon_1\sin\theta_1\cos\theta_2 - \epsilon_2\sin\theta_2\cos\theta_1}{\epsilon_2\sin\theta_2\cos\theta_1 + \epsilon_1\sin\theta_1\cos\theta_2} \tag{2.24}$$

where r_p denotes the *amplitude reflection coefficient* for parallel polarization.† If we now substitute for E_3 from Eq. (2.24) in Eq. (2.21) and carry out elementry simplifications, we get

$$t_p \equiv \frac{E_2}{E_1} = \frac{2\epsilon_1\sin\theta_1\cos\theta_1}{\epsilon_2\cos\theta_1\sin\theta_2 + \epsilon_1\sin\theta_1\cos\theta_2} \tag{2.25}$$

where t_p denotes the *amplitude transmission coefficient* for parallel polarization.

† We use the standard convention of using the subscripts p and s for parallel and perpendicular polarizations.

The reflectivity R and the transmittivity T are defined through the following equations

$$R = \frac{\text{Amount of energy reflected (per unit time) from an area } da \text{ of the interface}}{\text{Amount of energy incident (per unit time) on the same area } da \text{ of the interface}}$$

$$= \frac{|\langle \mathbf{S}_{\text{ref}} \cdot \mathbf{da} \rangle|}{|\langle \mathbf{S}_{\text{inc}} \cdot \mathbf{da} \rangle|} = \frac{|\langle (\mathbf{S}_{\text{ref}})_x \rangle|}{|\langle (\mathbf{S}_{\text{inc}})_x \rangle|} \tag{2.26}$$

$$T = \frac{\text{Amount of energy transmitted (per unit time) from an area } da \text{ of the interface}}{\text{Amount of energy incident (per unit time) on the same area } da \text{ of the interface}}$$

$$= \frac{|\langle \mathbf{S}_{\text{tr}} \cdot \mathbf{da} \rangle|}{|\langle \mathbf{S}_{\text{inc}} \cdot \mathbf{da} \rangle|} = \frac{|\langle (\mathbf{S}_{\text{tr}})_x \rangle|}{|\langle (\mathbf{S}_{\text{inc}})_x \rangle|} \tag{2.27}$$

where $\mathbf{S}_{\text{inc}}$, $\mathbf{S}_{\text{ref}}$ and $\mathbf{S}_{\text{tr}}$ represent the Poynting vectors (see Sec. 1.4) associated with the incident wave, reflected wave and transmitted wave respectively and $\langle \cdots \rangle$ represents the time average of the quantity inside the angular bracket. Recalling Eq. (1.76), we have

$$\begin{aligned} |\langle S_x \rangle| &= |\langle (\mathscr{E} \times \mathscr{H})_x \rangle| = \tfrac{1}{2}(k/\omega\mu)|E|^2 \cos\theta \\ &= \tfrac{1}{2}(\epsilon/\mu)^{\frac{1}{2}} |E|^2 \cos\theta \end{aligned} \tag{2.28}$$

where θ is the (acute) angle that $\mathscr{E} \times \mathscr{H}$ (i.e., the propagation vector $\mathbf{k}$) makes with the x-axis. Thus

$$R_p = \frac{\frac{1}{2}(\epsilon_1/\mu_1)^{\frac{1}{2}} |E_3|^2 \cos\theta_3}{\frac{1}{2}(\epsilon_1/\mu_1)^{\frac{1}{2}} |E_1|^2 \cos\theta_1} = \left|\frac{E_3}{E_1}\right|^2 = |r_p|^2 \tag{2.29a}$$

or

$$R_p = |r_p|^2 = \left(\frac{\epsilon_2 \cos\theta_1 \sin\theta_2 - \epsilon_1 \sin\theta_1 \cos\theta_2}{\epsilon_2 \cos\theta_1 \sin\theta_2 + \epsilon_1 \sin\theta_1 \cos\theta_2} \right)^2 \tag{2.29b}$$

Similarly

$$\begin{aligned} T_p &= \frac{\frac{1}{2}(\epsilon_2/\mu_2)^{\frac{1}{2}} |E_2|^2 \cos\theta_2}{\frac{1}{2}(\epsilon_1/\mu_1)^{\frac{1}{2}} |E_1|^2 \cos\theta_1} \\ &= \left(\frac{\epsilon_2}{\epsilon_1}\right)^{\frac{1}{2}} \left[\left(\frac{\epsilon_2}{\epsilon_1}\right)^{\frac{1}{2}} \frac{\sin\theta_2}{\sin\theta_1} \right] \\ &\quad \times \left(\frac{2\epsilon_1 \sin\theta_1 \cos\theta_1}{\epsilon_2 \cos\theta_1 \sin\theta_2 + \epsilon_1 \sin\theta_1 \cos\theta_2} \right)^2 \frac{\cos\theta_2}{\cos\theta_1} \end{aligned} \tag{2.30a}$$

where we have use Eq. (2.14) to substitute for $(\mu_1/\mu_2)^{\frac{1}{2}}$. Simplifying, we get

$$T_p = \frac{4\epsilon_1\epsilon_2 \sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2}{(\epsilon_2 \cos\theta_1 \sin\theta_2 + \epsilon_1 \sin\theta_1 \cos\theta_2)^2} \tag{2.30b}$$

It can readily be verified that

$$R_p + T_p = 1 \tag{2.31}$$

as it indeed should be.

We return to Eq. (2.24) and use Eqs. (2.19) and (2.20) to write it in the form

$$r_p = \frac{n_1^2 \sin\theta_1 \cos\theta_2 - n_2^2 \cos\theta_1 \sin\theta_2}{n_1^2 \sin\theta_1 \cos\theta_2 + n_2^2 \cos\theta_1 \sin\theta_2} \tag{2.32}$$

From the above equation we can make the following observations.

(*a*) When $n_2 = n_1$, $\theta_2 = \theta_1$ and

$$r_p = 0$$

implying that there is no reflected wave. Thus if we put a transparent substance (like glass) inside a liquid which has the same refractive index as glass then the glass will not be visible from outside!

(*b*) Polarization by reflection (Brewster's law): the amplitude reflection coefficient r_p vanishes when

$$n_1^2 \sin\theta_1 \cos\theta_2 = n_2^2 \cos\theta_1 \sin\theta_2$$

If we multiply both sides by $\sin\theta_2$ and use Snell's law (i.e., $n_1 \sin\theta_1 = n_2 \sin\theta_2$) we get

$$n_1^2 \sin\theta_1 \cos\theta_2 \sin\theta_2 = n_1^2 \sin^2\theta_1 \cos\theta_1$$

or

$$\sin 2\theta_1 = \sin 2\theta_2$$

Thus either $\theta_1 = \theta_2$ (which corresponds to the case (*a*) above) or

$$\theta_1 = \tfrac{1}{2}\pi - \theta_2$$

Thus when

$$\theta_1 + \theta_2 = \tfrac{1}{2}\pi \tag{2.33}$$

(i.e., when the reflected and transmitted rays are at right angles to each other) there is no reflection for the parallel component of $\mathscr{E}$. The corresponding angle of incidence can readily be found:

$$\cos\theta_1 = \sin\theta_2 = n_1 \sin\theta_1 / n_2$$

where we have used Eq. (2.17). Thus when

$$\theta_1 = \theta_p \equiv \tan^{-1}(n_2/n_1) \tag{2.34}$$

$r_p = 0$ and if the incident wave is unpolarized then only the perpendicular component of $\mathscr{E}$ will be reflected and the (reflected) light will be linearly polarized (see Fig. 2.2). This is Brewster's law and the angle θ_p is usually referred to as the *polarizing angle* or the *Brewster angle*. For the air–glass interface $n_1 = 1.0$, $n_2 = 1.5$ and

$$\theta_p = \tan^{-1} 1.5 \approx 56.3^\circ \tag{2.35}$$

(*c*) Reflection at normal and grazing incidence: in order to study reflection at normal incidence we use Snell's law ($n_2 \sin\theta_2 = n_1 \sin\theta_1$) to put Eq. (2.32) in the following form:

$$r_p = \frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} \tag{2.36}$$

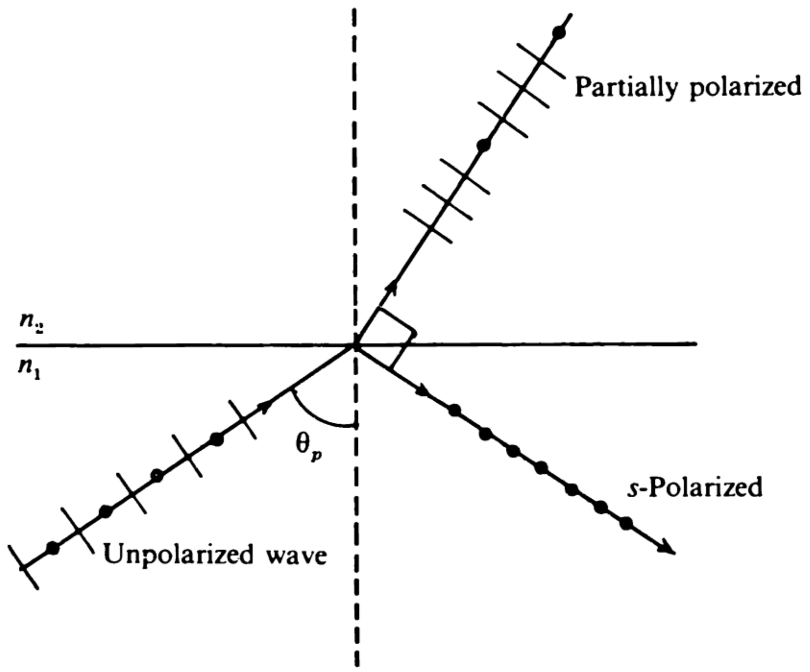
At normal incidence, $\theta_1 = \theta_2 = 0$ and we have

$$r_p = \frac{n_1 - n_2}{n_1 + n_2} \tag{2.37}$$

For the air–glass interface $n_1 = 1.0$, $n_2 = 1.5$ giving

$$r_p = -0.2 \quad \text{and} \quad R = 0.04 \tag{2.38}$$

Fig. 2.2 When an electromagnetic wave is incident at the Brewster angle ($\theta_1 = \theta_p = \tan^{-1}(n_2/n_1)$), the reflected wave is linearly polarized with its electric vector oscillating at right angles to the plane of incidence; thus an incident unpolarized wave becomes *s*-polarized on reflection.



implying that only 4% of the light intensity is reflected.

On the other hand, for grazing incidence (i.e., $\theta_1 \to \frac{1}{2}\pi$) we have the following limiting formula (see Problem 2.2)

$$r_p \approx 1 - \frac{2(n_2/n_1)^2 \alpha_1}{[(n_2/n_1)^2 - 1]^{\frac{1}{2}}} \tag{2.39}$$

where $\alpha_1 \equiv \frac{1}{2}\pi - \theta_1$, and is measured in radians. (Obviously, we are assuming $n_2 > n_1$, otherwise total internal reflection will occur – see Sec. 2.3.) Eq. (2.39) tells us that at grazing incidence, $|r_p|$ tends to unity and reflection is almost complete. Thus if we hold a glass plate at the level of the eye, it will almost act as a mirror!

(*d*) Complete expressions for the electric and magnetic fields: using the definitions for r_p and t_p and Eq. (1.32) we can write down the complete expressions for the electric and magnetic fields associated with the incident, transmitted and reflected waves:

$$\mathscr{E}_\mathrm{i} = \mathscr{E}_1 = (\hat{\mathbf{x}} \sin\theta_1 - \hat{\mathbf{z}} \cos\theta_1) E_1 \mathrm{e}^{\mathrm{i}[\omega t - k_1(x\cos\theta_1 + z\sin\theta_1)]} \tag{2.40}$$

$$\mathscr{H}_\mathrm{i} = \mathscr{H}_1 = \hat{\mathbf{y}}(E_1/Z_1)\mathrm{e}^{\mathrm{i}[\omega t - k_1(x\cos\theta_1 + z\sin\theta_1)]} \tag{2.41}$$

$$\mathscr{E}_\mathrm{t} = \mathscr{E}_2 = (\hat{\mathbf{x}} \sin\theta_2 - \hat{\mathbf{z}} \cos\theta_2) t_p E_1 \mathrm{e}^{\mathrm{i}[\omega t - k_2(x\cos\theta_2 + z\sin\theta_2)]} \tag{2.42}$$

$$\mathscr{H}_\mathrm{t} = \mathscr{H}_2 = \hat{\mathbf{y}}(t_p E_1/Z_2)\mathrm{e}^{\mathrm{i}[\omega t - k_2(x\cos\theta_2 + z\sin\theta_2)]} \tag{2.43}$$

$$\mathscr{E}_\mathrm{r} = \mathscr{E}_3 = (-\hat{\mathbf{x}} \sin\theta_1 - \hat{\mathbf{z}} \cos\theta_1) r_p E_1 \mathrm{e}^{\mathrm{i}[\omega t - k_1(-x\cos\theta_1 + z\sin\theta_1)]} \tag{2.44}$$

$$\mathscr{H}_\mathrm{r} = \mathscr{H}_3 = -\hat{\mathbf{y}}(r_p E_1/Z_1)\mathrm{e}^{\mathrm{i}[\omega t - k_1(-x\cos\theta_1 + z\sin\theta_1)]} \tag{2.45}$$

where

$$Z_1 = (\mu_1/\epsilon_1)^{\frac{1}{2}} \quad \text{and} \quad Z_2 = (\mu_2/\epsilon_2)^{\frac{1}{2}} \tag{2.46}$$

are known as the characteristic impedances of media 1 and 2 respectively. Notice that the direction of the magnetic field is along the y-axis which is tangential to the surface. Thus, continuity of the tangential component of $\mathscr{H}$ at the interface leads to

$$\frac{t_p E_1}{Z_2} = \frac{E_1}{Z_1} - \frac{r_p E_1}{Z_1}$$

or

$$t_p = \left(\frac{\mu_2 \epsilon_1}{\mu_1 \epsilon_2}\right)^{\frac{1}{2}} (1 - r_p)$$

$$= \frac{\epsilon_1}{\epsilon_2} \frac{\sin\theta_1}{\sin\theta_2} (1 - r_p) \tag{2.47}$$

If we substitute for r_p from Eq. (2.24) we obtain Eq. (2.25). Thus the continuity condition for $\mathscr{H}_y$ does not give any additional result. For nonmagnetic media (see Eqs. (2.19) and (2.20)), Eq. (2.47) becomes

$$t_p = (n_1/n_2)(1 - r_p) \tag{2.48}$$

Case 2: $\mathscr{E}$ lying perpendicular to the plane of incidence. We next consider the case in which the electric vector associated with the incident wave is perpendicular to the plane of incidence, i.e., along the y-axis (see Fig. 2.3). The electric vectors associated with the reflected and transmitted waves will also be perpendicular to the plane of incidence (see Problem 2.1). Thus, the electric fields associated with the incident, transmitted and reflected waves will be given by (cf. Eq. (2.1))

$$\mathscr{E}_1 = \hat{\mathbf{y}} E_1 e^{i(\omega t - \mathbf{k}_1 \cdot \mathbf{r})} \tag{2.49a}$$

$$\mathscr{E}_2 = \hat{\mathbf{y}} E_2 e^{i(\omega t - \mathbf{k}_2 \cdot \mathbf{r})} \tag{2.49b}$$

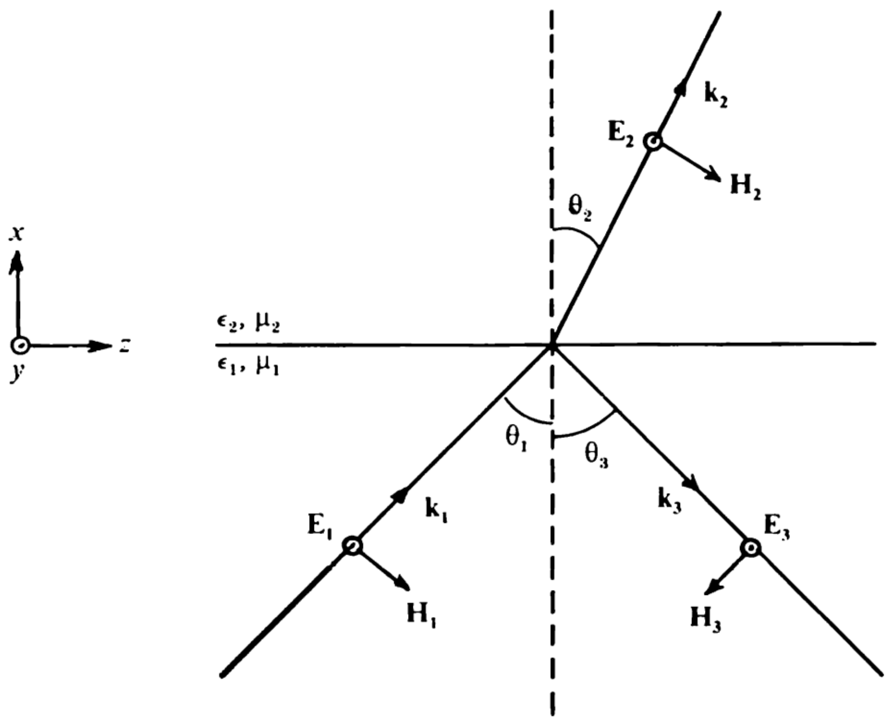
and

$$\mathscr{E}_3 = \hat{\mathbf{y}} E_3 e^{i(\omega t - \mathbf{k}_3 \cdot \mathbf{r})} \tag{2.49c}$$

The magnetic field is given by

$$\mathscr{H} = \frac{\mathbf{k} \times \mathscr{E}}{\omega \mu} = \frac{1}{\omega \mu} [\hat{\mathbf{x}}(-k_z \mathscr{E}_y) + \hat{\mathbf{z}}(k_x \mathscr{E}_y)] \tag{2.50}$$

Fig. 2.3 The reflection and refraction of a plane wave incident at the interface of two dielectrics; the electric vector is assumed to be perpendicular to the plane of incidence (i.e., s-polarized).



where we have used the fact that $k_y = 0$ (see Eqs. (2.2) and (2.11)). Thus if we write

$$\mathscr{H}_1 = \mathbf{H}_1 e^{i(\omega t - \mathbf{k}_i \cdot \mathbf{r})} \tag{2.51}$$

(and similar expressions for $\mathscr{H}_2$ and $\mathscr{H}_3$), then

$$\mathbf{H}_1 = \left(\frac{\epsilon_1}{\mu_1}\right)^{\frac{1}{2}} (-\hat{\mathbf{x}} \sin\theta_1 + \hat{\mathbf{z}} \cos\theta_1) E_1 \tag{2.52}$$

$$\mathbf{H}_2 = (\epsilon_2/\mu_2)^{\frac{1}{2}} [-\hat{\mathbf{x}} \sin\theta_2 + \hat{\mathbf{z}} \cos\theta_2] E_2 \tag{2.53}$$

and

$$\mathbf{H}_3 = (\epsilon_1/\mu_1)^{\frac{1}{2}} [-\hat{\mathbf{x}} \sin\theta_1 - \hat{\mathbf{z}} \cos\theta_1) E_3 \tag{2.54}$$

will represent the space and time independent part of the magnetic field associated with the incident, transmitted and reflected waves respectively. Since H_z represents a tangential component, its continuity across the interface leads to

$$(\epsilon_1/\mu_1)^{\frac{1}{2}} (\cos\theta_1) E_1 - (\epsilon_1/\mu_1)^{\frac{1}{2}} (\cos\theta_1) E_3 = (\epsilon_2/\mu_2)^{\frac{1}{2}} (\cos\theta_2) E_2 \tag{2.55}$$

Also, since $\mathscr{E}$ has only a y-component (which is tangential to the interface) we have

$$E_1 + E_3 = E_2 \tag{2.56}$$

Substituting for E_2 from Eq. (2.56) in Eq. (2.55) we get after simple manipulations

$$r_s \equiv \frac{E_3}{E_1} = \frac{(\epsilon_1/\mu_1)^{\frac{1}{2}} \cos\theta_1 - (\epsilon_2/\mu_2)^{\frac{1}{2}} \cos\theta_2}{(\epsilon_1/\mu_1)^{\frac{1}{2}} \cos\theta_1 + (\epsilon_2/\mu_2)^{\frac{1}{2}} \cos\theta_2} \tag{2.57}$$

where the subscript s refers to perpendicular polarization. For nonmagnetic media

$$r_s = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{2.58}$$

Further

$$t_s \equiv \frac{E_2}{E_1} = 1 + r_s = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{2.59}$$

The variation of $|r_p|$ and $|r_s|$ with the angle of incidence θ_1 is plotted in Fig. 2.4 for the air–glass interface $n_1 = 1.0$, $n_2 = 1.5$. Notice that $|r_p|$ vanishes at the Brewster angle. Also $|r_p|$ and $|r_s|$ tend to the same value at normal and grazing incidence.

As for the case of parallel polarization we write the complete expressions for the electric and magnetic fields associated with the incident, transmitted and reflected waves:

$$\mathscr{E}_{\mathrm{i}} = \mathscr{E}_1 = \hat{\mathbf{y}} E_1 \exp\{\mathrm{i}[\omega t - k_1(x\cos\theta_1 + z\sin\theta_1)]\} \tag{2.60}$$

$$\begin{aligned}\mathscr{H}_{\mathrm{i}} = \mathscr{H}_1 &= (-\hat{\mathbf{x}}\sin\theta_1 + \hat{\mathbf{z}}\cos\theta_1)(E_1/Z_1)\\ &\quad\times \exp\{\mathrm{i}[\omega t - k_1(x\cos\theta_1 + z\sin\theta_1)]\}\end{aligned} \tag{2.61}$$

$$\mathscr{E}_{\mathrm{t}} = \mathscr{E}_2 = \hat{\mathbf{y}} t_s E_1 \exp\{\mathrm{i}[\omega t - k_2(x\cos\theta_2 + z\sin\theta_2)]\} \tag{2.62}$$

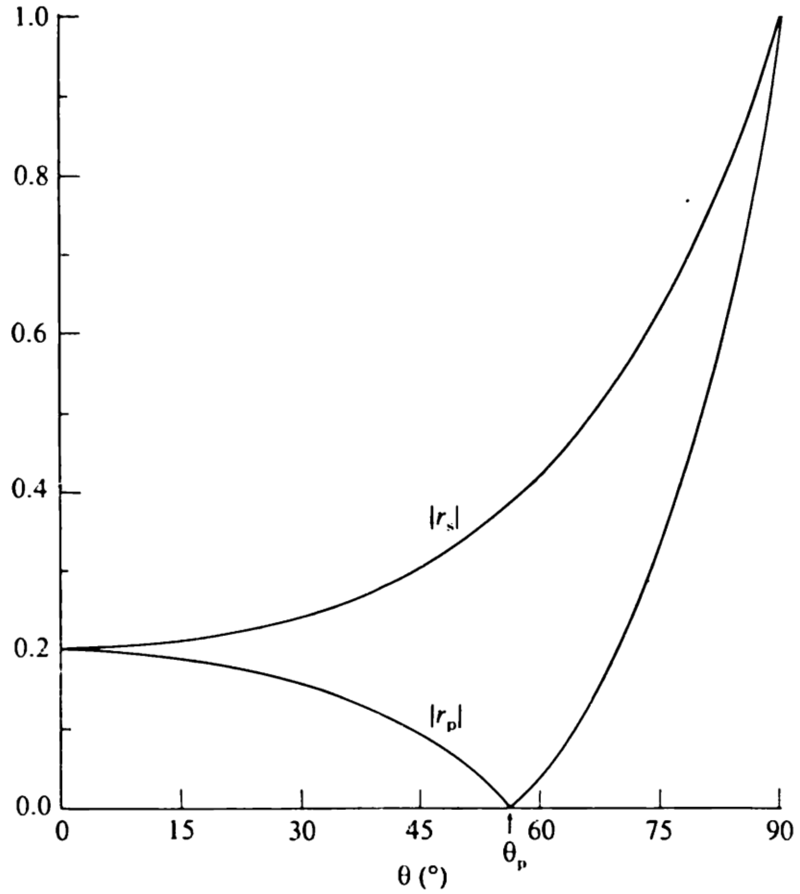
$$\begin{aligned}\mathscr{H}_{\mathrm{t}} = \mathscr{H}_2 &= (-\hat{\mathbf{x}}\sin\theta_2 + \hat{\mathbf{z}}\cos\theta_2)(t_s E_1/Z_2)\\ &\quad\times \exp\{\mathrm{i}[\omega t - k_2(x\cos\theta_2 + z\sin\theta_2)]\}\end{aligned} \tag{2.63}$$

$$\mathscr{E}_{\mathrm{r}} = \mathscr{E}_3 = \hat{\mathbf{y}} r_s E_1 \exp\{\mathrm{i}[\omega t - k_1(-x\cos\theta_1 + z\sin\theta_1)]\} \tag{2.64}$$

$$\begin{aligned}\mathscr{H}_{\mathrm{r}} = \mathscr{H}_3 &= (-\hat{\mathbf{x}}\sin\theta_1 - \hat{\mathbf{z}}\cos\theta_1)(r_s E_1/Z_1)\\ &\quad\times \exp\{\mathrm{i}[\omega t - k_1(-x\cos\theta_1 + z\sin\theta_1)]\}\end{aligned} \tag{2.65}$$

Problem 2.1: Consider the case in which $\mathscr{E}$ associated with the incident wave lies in the plane of incidence. Show that reflected or transmitted wave will not have any y-component of $\mathscr{E}$. Similarly for perpendicular polarization.

Fig. 2.4 Variation of the reflection coefficients $|r_p|$ and $|r_s|$ as a function of the angle of incidence for an air–glass interface.



2.3 Total internal reflection and evanescent waves

According to Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \tag{2.79}$$

and thus if the wave is incident on a rarer medium (i.e., if $n_2 < n_1$) then $\sin \theta_2$ will be greater than unity if

$$\theta_1 > \theta_c \tag{2.80}$$

where

$$\theta_c = \sin^{-1}(n_2/n_1) \tag{2.81}$$

This angle is known as the critical angle. For $\theta_1 > \theta_c$ we have the phenomenon known as total internal reflection. Since Eq. (2.79) is derived by applying the continuity conditions, it remains valid even for $\theta_1 > \theta_c$; however, for $\theta_1 > \theta_c$, we have a wave for which $\sin \theta_2$ exceeds unity (i.e., $k_{2z} = k_2 \sin \theta_2 > k_2$) and therefore $\cos \theta_2$ is a purely imaginary quantity (i.e., $k_{2x} = k_2 \cos \theta_2$ is purely imaginary). Now

$$\cos^2 \theta_2 = 1 - \sin^2 \theta_2 = -(n_1^2 \sin^2 \theta_1 / n_2^2 - 1)$$

where the quantity inside the brackets is positive when $\theta_1 > \theta_c$. Thus

$$\cos \theta_2 = -\mathrm{i}(n_1^2 \sin^2 \theta_1 / n_2^2 - 1)^{\frac{1}{2}} \tag{2.82}$$

(the reason for choosing the minus sign will soon become obvious). For definiteness we consider perpendicular polarization for which the amplitude transmission coefficient is given by Eq. (2.59). Since $\cos \theta_2$ is now an imaginary quantity, t_s is complex and we may write

$$t_s = |t_s| \mathrm{e}^{\mathrm{i}\alpha} \tag{2.83}$$

Thus the transmitted electric field can be written in the form

$$\mathscr{E}_t = \hat{\mathbf{y}} |t_s| E_1 \exp\{\mathrm{i}[\omega t - k_2(x \cos \theta_2 + z \sin \theta_2) + \alpha]\} \tag{2.84}$$

If now substitute for $\sin \theta_2$ and $\cos \theta_2$ from Eqs. (2.79), and (2.82) respectively and use $k_2 = (\omega/c)n_2$ we obtain†

$$\mathscr{E}_t = \hat{\mathbf{y}}|t_s|E_1 \exp(-\gamma x)\exp\{i[\omega t - \omega(n_1/c)z\sin\theta_1 + \alpha]\} \tag{2.85}$$

where

$$\gamma = (\omega/c)(n_1^2 \sin^2\theta_1 - n_2^2)^{\frac{1}{2}} \tag{2.86}$$

Eq. (2.85) represents a wave which decays exponentially in the x-direction and propagates in the z-direction; such a wave is known as an *evanescent wave*. The corresponding magnetic field is given by (see Eq. (2.63)):

$$\mathscr{H}_t = \left[-\hat{\mathbf{x}}\frac{n_1\sin\theta_1}{n_2} - \hat{\mathbf{z}}i\left(\frac{n_1^2\sin^2\theta_1}{n_2^2} - 1\right)^{\frac{1}{2}}\right]$$

$$\times \frac{|t_s|E_1}{Z_2}\exp(-\gamma x)\exp\left[i\left(\omega t - \frac{\omega n_1}{c}z\sin\theta_1 + \alpha\right)\right] \tag{2.87}$$

Since $-i = e^{-\frac{1}{2}i\pi}$, the z-component of $\mathscr{H}$ is $\frac{1}{2}\pi$ out of phase with respect to $\mathscr{H}_x$ and $\mathscr{E}_y$. The actual electric and magnetic fields (which would be the real parts of Eqs. (2.85) and (2.87)) would be given by

$$\mathscr{E}_y = |t_s|E_1 e^{-\gamma x}\cos\left(\omega t - \frac{\omega n_1}{c}z\sin\theta_1 + \alpha\right) \tag{2.88}$$

$$\mathscr{H}_x = -\frac{n_1\sin\theta_1}{n_2}|t_s|\frac{E_1}{Z_2}e^{-\gamma x}\cos\left(\omega t - \frac{\omega n_1}{c}z\sin\theta_1 + \alpha\right) \tag{2.89}$$

and

$$\mathscr{H}_z = \left(\frac{n_1^2\sin^2\theta_1}{n_2^2} - 1\right)^{\frac{1}{2}}\frac{|t_s|E_1}{Z_2}e^{-\gamma x}\sin\left(\omega t - \frac{\omega n_1}{c}z\sin\theta_1 + \alpha\right) \tag{2.90}$$

Thus

$$\langle S_x\rangle = \langle \mathscr{E}_y\mathscr{H}_z\rangle = 0$$

implying that there is *no* power flow along the x-direction and therefore

† We should mention here that if, instead of Eq. (2.82) we had taken

$$\cos\theta_2 = +i(n_1^2\sin^2\theta_1/n_2^2 - 1)^{\frac{1}{2}}$$

we would have obtained a wave whose amplitude would have *increased* exponentially with x; clearly this would be physically impossible.

the transmission coefficient (see Eq. (2.27)) is zero. However,

$$\langle S_z \rangle = \langle -\mathscr{E}_y \mathscr{H}_x \rangle = \frac{1}{2} \frac{n_1 \sin\theta_1}{n_2} |t_s|^2 \frac{E_1^2}{Z_2} e^{-2\gamma x} \tag{2.91}$$

Thus there *is* power flow along the z-axis in the second medium (see Fig. 2.6). We should point out that since $\cos\theta_2$ is purely imaginary[†].

$$|r_p| = |r_s| = 1 \tag{2.92}$$

(see Eqs. (2.36) and (2.57)). Thus although the entire energy is reflected back, there is power flowing in the second medium. Physically, we can understand this by considering the incidence of a spatially bounded beam at the interface (see Fig. 2.7). As shown in the figure, the beam undergoes

Fig. 2.6 When a plane wave undergoes total internal reflection (at the interface of two dielectrics) there is an evanescent wave in the rarer medium which propagates along the z-axis and whose amplitude decreases exponentially along the x-axis.

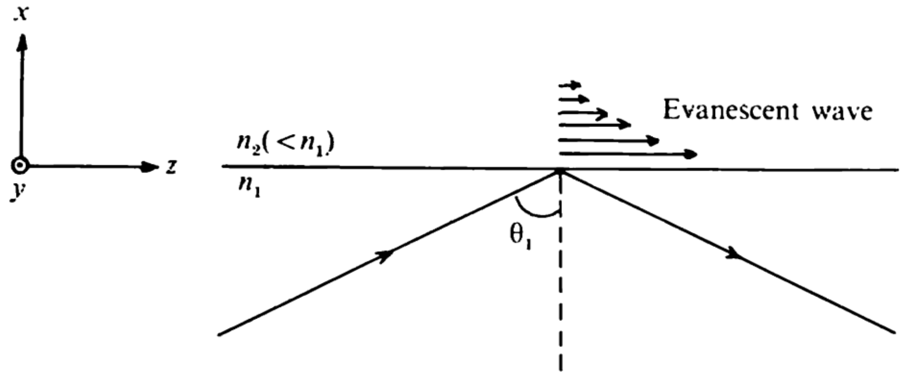
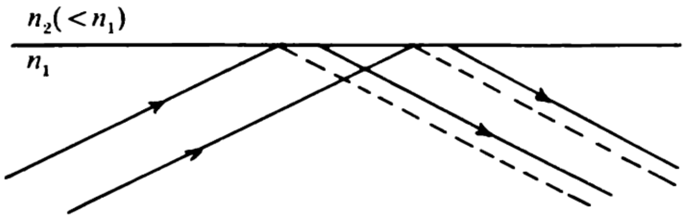


Fig. 2.7 When a spatially bounded beam is incident at an interface making an angle greater than the critical angle, then the beam undergoes a lateral shift which can be interpreted as the beam entering the rarer medium and reemerging (from the rarer medium) after reflection.

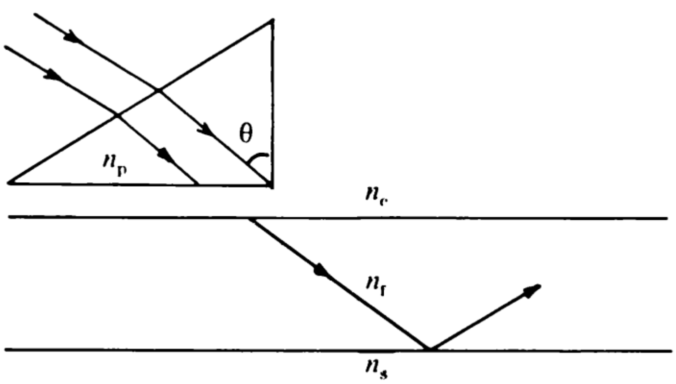


[†] Using Eqs. (2.58) and (2.82), we obtain $r_s = e^{-i\chi}$ where $\chi = -2\tan^{-1}\{[n_1^2 \sin^2\theta_1 - n_2^2]^{1/2}/n_1 \cos\theta_1\}$.

a lateral shift which can be interpreted as the beam entering the rarer medium and reemerging (from the rarer medium) after reflection. This shift is known as the Goos–Hanchen shift (see, e.g., Ghatak and Thyagarajan (1978) for a detailed account of the Goos–Hanchen shift). It is now physically obvious that if, instead of a spatially bounded beam, we have an infinitely extended plane wave incident on the interface then although the reflection is complete, energy will flow along the z-axis in the rarer medium; the magnitude of this energy decays along the x-axis.

A very important application of evanescent waves is the prism–film coupler (see Fig. 2.8) which is extensively used in the coupling of a laser beam into a thin film waveguide used in integrated optics (see Chapter 14). It consists of a high refractive index prism placed in close contact with a thin film waveguide and the laser beam is incident on the base of the prism at an angle greater than the critical angle (i.e., $\theta > \sin^{-1}(n_c/n_p)$; see Fig. 2.8). In the air space between the prism and the film, an evanescent wave is generated which couples energy from the incident beam into the waveguide. Detailed calculation shows that maximum power transfer occurs when the angle of incidence satisfies the following condition: $\theta = \sin^{-1}(\beta/k_0 n_p)$ where k_0 is the free space wave number and β is the propagation constant of the waveguide mode†. Thus different modes can be excited independently by choosing different angles of incidence.

Fig. 2.8 The prism–film coupler arrangement: n_f, n_s and n_c represent the refractive indices of the film, substrate and cover respectively, $n_f > n_s, n_c$.



† In Chapters 11 and 14 we will show that a thin film waveguide can support various modes of propagation and each mode is characterized by a specific value of the propagation constant β. The phase velocity of the mode is given by ω/β which can be written as $c/(\beta/k_0)$ where we have used $\omega = ck_0$. Thus β/k_0 can be interpreted as the effective refractive index of the medium for the mode.