13.5 Scalar wave equation and the modes of a fibre

In Sec. 11.9 we have shown that for an inhomogeneous medium the electric field $\mathscr{E}$ satisfies the equation

$$\nabla^2 \mathscr{E} + \nabla\left(\frac{\nabla n^2}{n^2}\cdot\mathscr{E}\right) - \epsilon_0\mu_0 n^2 \frac{\partial^2 \mathscr{E}}{\partial t^2} = 0 \tag{13.13}$$

and a similar equation holds for the magnetic field $\mathscr{H}$ (see Eq. (11.137)). The above equation tells us that the different components of the electric field are coupled.

Now, for an infinitely extended homogeneous medium, the second term on the LHS is zero everywhere and each Cartesian component of the electric field satisfies the scalar wave equation:

$$\nabla^2 \Psi = \epsilon_0\mu_0 n^2(\partial^2\Psi/\partial t^2) \tag{13.14}$$

The solution of the above equation can be written in the form of plane waves and using Maxwell's equations one can easily show that the waves are transverse (see Chapter 1). For a medium having weak inhomogeneity (i.e., the variation of n is small in a region $\sim \lambda$) the second term on the LHS of Eq. (13.13) can be assumed to be negligible and the waves can be assumed to be nearly transverse with the transverse component of the electric field satisfying Eq. (13.14). The above equation is obtained after neglecting the term depending on ∇n^2 in Eq. (13.13). This is known as the scalar wave approximation and in this approximation the modes have been assumed to be nearly transverse and can have an arbitrary state of polarization. Thus the two independent sets of modes can be assumed to be x-polarized and y-polarized and in the scalar approximation they have the same propagation constants. These linearly polarized modes are usually referred to as LP modes. We may compare this with the discussion in Sec. 11.5 where we mentioned that when $n_1 \approx n_2$, the modes are nearly transverse and the propagation constants of the TE and TM modes are almost equal.

For n^2 depending only on the transverse coordinates (r, ϕ), we may write

$$\Psi(r, \phi, z, t) = \psi(r, \phi) e^{i(\omega t - \beta z)} \tag{13.15}$$

where ω is the angular frequency and β is known as the propagation constant. The above equation represents the modes of the system. Substituting in Eq. (13.14), we readily obtain

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\psi + \left[\frac{\omega^2}{c^2} n^2(r, \phi) - \beta^2\right]\psi = 0$$

In most practical fibres n^2 depends only on the cylindrical coordinate r and therefore it is convenient to use the cylindrical system of coordinates to obtain†

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + [k_0^2 n^2(r) - \beta^2]\psi = 0 \tag{13.16}$$

where

$$k_0 = \omega / c = 2\pi / \lambda_0 \tag{13.17}$$

is the free space wave number. Since the medium has cylindrical symmetry, we can solve Eq. (13.16) by the method of separation of variables:

$$\psi(r, \phi) = R(r)\Phi(\phi) \tag{13.18}$$

On substitution and dividing by ψ, we obtain

$$\frac{r^2}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + r^2[n^2(r)k_0^2 - \beta^2] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = +l^2 \tag{13.19}$$

where l is a constant. The ϕ dependence will be of the form $\cos l\phi$ or $\sin l\phi$ and for the function to be single valued (i.e., $\Phi(\phi + 2\pi) = \Phi(\phi)$) we must have

$$l = 0, 1, 2, \ldots, \text{etc.} \tag{13.20}$$

(Negative values of l correspond to the same field distribution.) Since for each value of l, there can be two independent states of polarization, modes with $l \geqslant 1$ are four-fold degenerate; modes with $l = 0$ are ϕ independent and have two fold degeneracy. The radial part of the equation gives us

$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} + \{[n^2(r)k_0^2 - \beta^2]r^2 - l^2\}R = 0 \tag{13.21}$$

The solution of the above equation for a step index profile will be given in Sec. 13.6. However, we can make some general comments about the solutions of Eq. (13.21) for an arbitrary cylindrically symmetric profile having a refractive index which decreases monotonically from a value n_1 on the axis to a constant value n_2 beyond the core–cladding interface $r = a$. The solutions of Eq. (13.21) can be divided into two distinct classes (cf. Sec. 11.3):

(*a*) $$k_0^2 n_1^2 > \beta^2 > k_0^2 n_2^2 \tag{13.22}$$

For β^2 lying in the above range, the fields $R(r)$ are oscillatory in the core and decay in the cladding and β^2 assumes only discrete values; these are known as the *guided modes* of the system. For a given value of l, there will be several guided modes which are designated LP_{lm} modes ($m = 1, 2, 3, \ldots$); LP stands for linearly polarized.† Further, since the modes are solutions of the scalar wave equation, they can be assumed to satisfy the orthonormality condition

$$\int_0^\infty \int_0^{2\pi} \psi_{lm}^*(r, \phi)\psi_{l'm'}(r, \phi) r \, dr \, d\phi = \delta_{ll'}\delta_{mm'} \tag{13.23}$$

(*b*) $$\beta^2 < k_0^2 n_2^2 \tag{13.24}$$

For such β values, the fields are oscillatory even in the cladding and β can assume a continuum of values. These are known as the *radiation modes.*

The guided and radiation modes form a complete set of modes in the sense that an arbitrary field distribution can be expanded in terms of these modes, i.e.,

$$\psi(x, y, z) = \sum_\nu a_\nu \psi_\nu(x, y) e^{-i\beta_\nu z} + \int a(\beta)\psi(\beta, x, y) e^{-i\beta z} \, d\beta \tag{13.25}$$

where the first term represents a sum over discrete modes and the second term an integral over the continuum of modes‡. The quantity $|a_\nu|^2$ is proportional to the power carried by the ν^{th} mode; the constants a_ν can be determined by knowing the incident field at $z = 0$ and using the orthonormality condition.

The calculation of the modal field distributions and the corresponding propagation constants are of extreme importance in the study of waveguides. For example, knowing the frequency dependence of the propagation constant one can calculate the temporal broadening of a pulse (see Sec. 13.8) which determines the information-carrying capacity. Knowledge of the modal field distribution is essential for the calculation of excitation efficiencies, splice losses at joints and in the development of new fibre optic devices like directional couplers etc. We will now present a detailed modal analysis for step index and parabolic refractive index distributions.

13.6 Modal analysis for a step index fibre

In this section, we will obtain the modal fields and the corresponding propagation constants for a step index fibre for which the refractive index variation is given by Eq. (13.1). For such a fibre it is possible to obtain rigorous solutions of the vector equations (see, e.g., Sodha and Ghatak (1977)). However, most practical fibres used in communication are weakly guiding i.e., relative refractive index difference $(n_1 - n_2)/n_1 \ll 1$ and in such a case the radial part of the transverse component of the electric field satisfies the following equation (see Eq. (13.21)):

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + \{[k_0^2n^2(r) - \beta^2]r^2 - l^2\}R = 0 \tag{13.26}$$

and the complete transverse field is given by

$$\Psi(r, \phi, z, t) = R(r)\,\mathrm{e}^{\mathrm{i}(\omega t - \beta z)}\begin{Bmatrix}\cos l\phi\\ \sin l\phi\end{Bmatrix} \tag{13.27}$$

If we substitute in Eq. (13.26) for $n^2(r)$, we obtain

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + \left(U^2\frac{r^2}{a^2} - l^2\right)R = 0; \quad r < a \tag{13.28}$$

and

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} - \left(W^2\frac{r^2}{a^2} + l^2\right)R = 0; \quad r > a \tag{13.29}$$

where

$$U = a(k_0^2n_1^2 - \beta^2)^{\frac{1}{2}} \tag{13.30}$$

$$W = a(\beta^2 - k_0^2n_2^2)^{\frac{1}{2}} \tag{13.31}$$

and the normalized waveguide parameter V is defined by

$$V = (U^2 + W^2)^{\frac{1}{2}} = k_0a(n_1^2 - n_2^2)^{\frac{1}{2}} \tag{13.32}$$

Guided modes correspond to $n_2^2k_0^2 < \beta^2 < n_1^2k_0^2$ and therefore for guided modes both U and W are real.

Eq. (13.28) and (13.29) are of the standard form of Bessel's equation (see, e.g., Irving and Mullineux (1959)). The solutions of Eq. (13.28) are $J_l(x)$ and $Y_l(x)$ where $x = Ur/a$. The solution $Y_l(x)$ has to be rejected since it diverges as $x \to 0$. The solutions of Eq. (13.29) are the modified Bessel functions $K_l(\tilde{x})$ and $I_l(\tilde{x})$ with the asymptotic forms

$$K_l(\tilde{x}) \xrightarrow[\tilde{x}\to\infty]{} \left(\frac{\pi}{2\tilde{x}}\right)^{\frac{1}{2}} e^{-\tilde{x}} \tag{13.33}$$

$$I_l(\tilde{x}) \xrightarrow[\tilde{x}\to\infty]{} \frac{1}{(2\pi\tilde{x})^{\frac{1}{2}}} e^{\tilde{x}} \tag{13.34}$$

where $\tilde{x} = Wr/a$. Obviously the solution $I_l(\tilde{x})$ which diverges as $\tilde{x} \to \infty$ has to be rejected. Thus the transverse dependence of the modal field is given by

$$\psi(r,\phi) = \begin{cases} \dfrac{A}{J_l(U)} J_l\left(\dfrac{Ur}{a}\right)\begin{bmatrix}\cos l\phi \\ \sin l\phi\end{bmatrix}; & r < a \\ \dfrac{A}{K_l(W)} K_l\left(\dfrac{Wr}{a}\right)\begin{bmatrix}\cos l\phi \\ \sin l\phi\end{bmatrix}; & r > a \end{cases} \tag{13.35}$$

where we have assumed the continuity of ψ at the core–cladding interface. Continuity of $\partial\psi/\partial r$ at $r = a$ leads to[†]

$$\frac{UJ_l'(U)}{J_l(U)} = \frac{WK_l'(W)}{K_l(W)} \tag{13.36}$$

Using the identities

$$\pm UJ_l'(U) = lJ_l(U) - UJ_{l\pm 1}(U) \tag{13.37}$$

$$\pm WK_l'(W) = lK_l(W) \mp WK_{l\pm 1}(W) \tag{13.38}$$

$$J_{l+1}(U) = (2l/U)J_l(U) - J_{l-1}(U) \tag{13.39}$$

and

$$K_{l+1}(W) = (2l/W)K_l(W) + K_{l-1}(W) \tag{13.40}$$

Eq. (13.36) can be written in either of the following two forms

$$U\frac{J_{l+1}(U)}{J_l(U)} = W\frac{K_{l+1}(W)}{K_l(W)} \tag{13.41}$$

[†] It should be mentioned that as long as ψ is assumed to satisfy the scalar wave equation (Eq. (13.16)), both ψ and $d\psi/dr$ have to be continuous at any refractive index discontinuity. This follows from the fact that if $d\psi/dr$ does not happen to be continuous then $d^2\psi/dr^2$ will be a Dirac delta function which would therefore be inconsistent with Eq. (13.16) unless, of course, there is an infinite discontinuity in n^2 which indeed happens at the interface between a dielectric and a perfect conductor.

or

$$U\frac{J_{l-1}(U)}{J_l(U)} = -W\frac{K_{l-1}(W)}{K_l(W)} \tag{13.42}$$

However using the proper limiting forms of $K_l(W)$ as $W \to 0$, one can show that†

$$\underset{W\to 0}{\mathrm{Lt}}\; W\frac{K_{l-1}(W)}{K_l(W)} \to 0; \quad l = 0, 1, 2, \ldots \tag{13.43}$$

and therefore we use Eq. (13.42) for studying the modes. For $l = 0$ we get

$$U\frac{J_1(U)}{J_0(U)} = W\frac{K_1(W)}{K_0(W)} \tag{13.44}$$

where we have used the relations $J_{-1}(U) = -J_1(U)$ and $K_{-1}(W) = K_1(W)$.

We should mention here that the boundary conditions used in deriving the eigenvalue equation (Eq. (13.42)) are consistent with the approximation involved in using the scalar wave equation. For example, if ψ is assumed to represent E_y then, rigorously speaking, E_y and $\partial E_y/\partial r$ are *not* continuous at $r = a$ for all ϕ; indeed, one must make E_ϕ, E_z and $n^2 E_r$ continuous at the interface $r = a$. However, if $n_1 \approx n_2$ the error involved is negligible (see, e.g., Sodha and Ghatak (1977)).

Since $V^2 = U^2 + W^2$ the solution of the transcendental equation (for given values of l and V) will give us universal curves describing the dependence of U or W on V. It is, however, more convenient to define the normalized propagation constant

$$b = \frac{\beta^2/k_0^2 - n_2^2}{n_1^2 - n_2^2} = \frac{W^2}{V^2} \tag{13.45}$$

using which Eqs. (13.42) and (13.44) reduce to

$$V(1-b)^{\frac{1}{2}}\frac{J_{l-1}(V(1-b)^{\frac{1}{2}})}{J_l(V(1-b)^{\frac{1}{2}})} = -Vb^{\frac{1}{2}}\frac{K_{l-1}(Vb^{\frac{1}{2}})}{K_l(Vb^{\frac{1}{2}})}; \quad l \geqslant 1 \tag{13.46}$$

and

$$V(1-b)^{\frac{1}{2}}\frac{J_1(V(1-b)^{\frac{1}{2}})}{J_0(V(1-b)^{\frac{1}{2}})} = Vb^{\frac{1}{2}}\frac{K_1(Vb^{\frac{1}{2}})}{K_0(Vb^{\frac{1}{2}})}; \quad l = 0 \tag{13.47}$$

† The limiting forms are

$$K_0(W) \xrightarrow[W\to 0]{} -\ln(W/2)$$

and

$$K_l(W) \xrightarrow[W\to 0]{} \tfrac{1}{2}\Gamma(l)(2/W)^l; \quad l > 0$$

Table 13.1. *Cutoff frequencies of various LP_{lm} modes in a step index fibre*

$l = 0$ modes	$(J_1(V_c) = 0)$	$l = 1$ modes	$(J_0(V_c) = 0)$
Mode	V_c	Mode	V_c
LP_{01}	0	LP_{11}	2.4048
LP_{02}	3.8317	LP_{12}	5.5201
LP_{03}	7.0156	LP_{13}	8.6537
LP_{04}	10.1735	LP_{14}	11.7915

$l = 2$ modes	$(J_1(V_c) = 0;\ V_c \neq 0)$	$l = 3$ modes	$(J_2(V_c) = 0;\ V_c \neq 0)$
Mode	V_c	Mode	V_c
LP_{21}	3.8317	LP_{31}	5.1356
LP_{22}	7.0156	LP_{32}	8.4172
LP_{23}	10.1735	LP_{33}	11.6198
LP_{24}	13.3237	LP_{34}	14.7960

respectively. Since for guided modes $k_0^2 n_2^2 < \beta^2 < k_0^2 n_1^2$, we must have $0 < b < 1$. For a given value of l, there will be a finite number of solutions and the m^{th} solution ($m = 1, 2, 3, \ldots$) is referred to as the LP_{lm} mode. In Fig. 13.7 we have plotted the LHS and RHS of the above equations for the $l = 0$ and $l = 1$ modes corresponding to $V = 8$. The points of intersection give the allowed values of b for different modes. By studying the zeroes of Bessel functions (see

Fig. 13.7 Variation of the LHS (solid curves) and the RHS (dashed curves) of Eq. (13.46) for (*a*) $l = 0$ and (*b*) $l = 1$ for $V = 8$. The points of intersection represent the discrete modes of the waveguide.

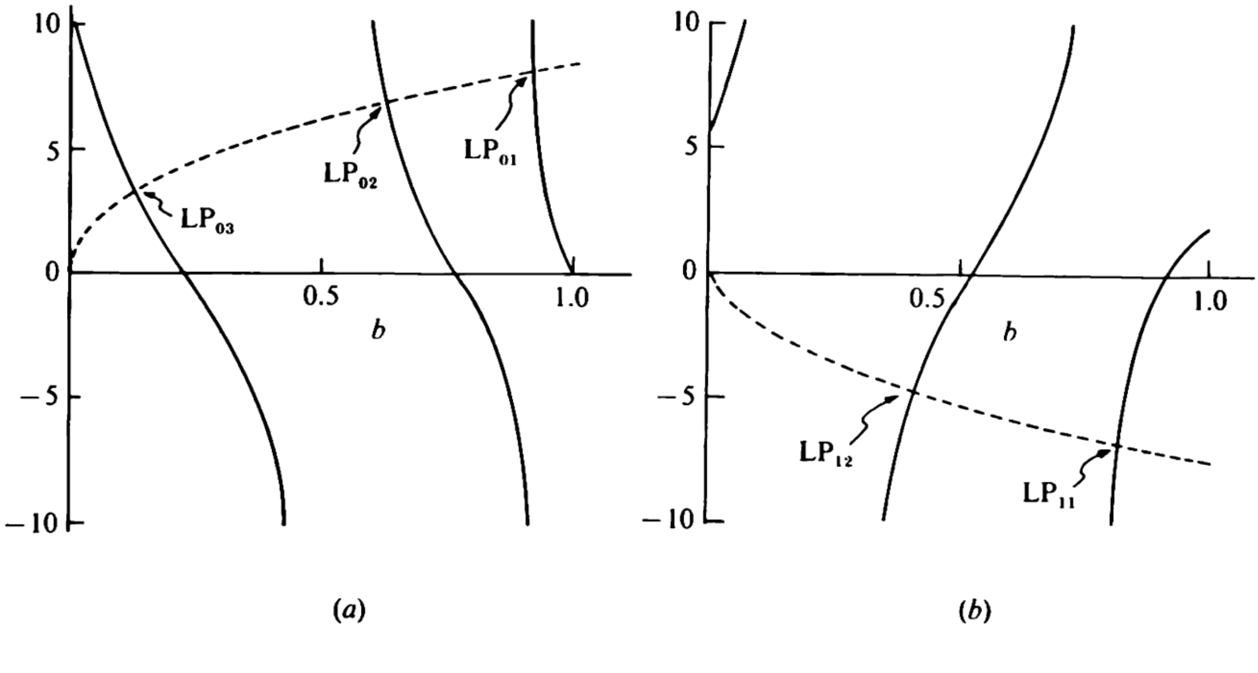


Table 13.1) one can immediately see that since $0 < b < 1$, there will be only a finite number of guided modes. The guided modes which are given by the points of intersection in Figs. 13.7(*a*) and (*b*) are designated in decreasing values of b as LP_{l1}, LP_{l2}, LP_{l3} etc. The variation of b with V forms a set of universal curves, which are plotted in Fig. 13.8. As can be seen, at a particular V value there are only a finite number of modes.

The condition $b = 0$ (i.e., $\beta^2 = k_0^2 n_2^2$) corresponds to what is known as the *cutoff* of the mode. For $b < 0$, $\beta^2 < k_0^2 n_2^2$ and the fields are oscillatory even in the cladding and we have what are known as radiation modes. Obviously at cut off $\beta = k_0 n_2$ implying

$$b = 0, \quad W = 0, \quad U = V = V_c$$

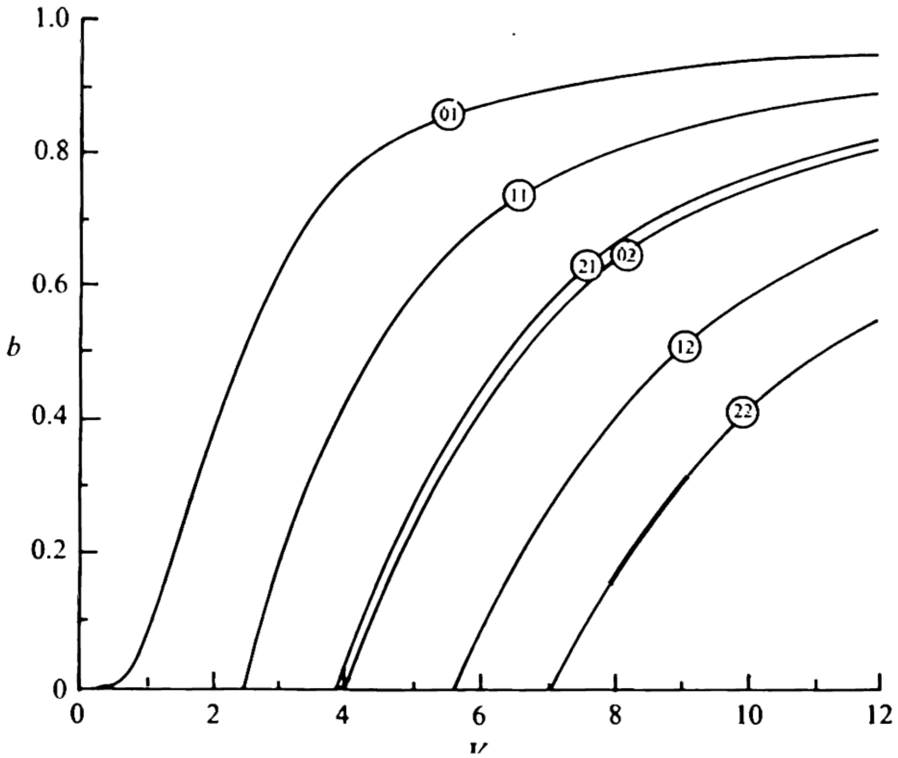
The cutoffs of various modes are determined from the following equations

$$\begin{aligned} l &= 0 \quad \text{modes:} \quad J_1(V_c) = 0 \\ l &= 1 \quad \text{modes:} \quad J_0(V_c) = 0 \\ l &\geqslant 2 \quad \text{modes:} \quad J_{l-1}(V_c) = 0; \quad V_c \neq 0 \end{aligned} \tag{13.48}$$

It must be noted that for $l \geqslant 2$, the root $V_c = 0$ must not be included since

$$\operatorname*{Lt}_{V \to 0} V \frac{J_{l-1}(V)}{J_l(V)} \neq 0 \quad \text{for } l \geqslant 2 \tag{13.49}$$

Fig. 13.8 Variation of the normalized propagation constant b with normalized frequency V for a step index fibre corresponding to some low order modes. The cutoff frequencies of LP_{2m} and $LP_{0,m+1}$ modes are the same. (Adapted from Gloge (1971).)



Thus the cutoff V values (also known as normalized cutoff frequencies) occur at the zeroes of Bessel functions and are tabulated in Table 13.1.

As is obvious from the above analysis and also from Fig. 13.8 for a step index fibre with

$$0 < V < 2.4048 \tag{13.50}$$

we will have only one guided mode namely the LP_{01} mode. Such a fibre is referred to as a single mode fibre and is of tremendous importance in optical communication systems. As will be discussed later, the dispersion curves can be used to calculate the group velocity of various modes of a step index fibre.

The radial field distributions and the schematic intensity patterns of some

Fig. 13.9 Radial intensity distributions (normalized to the same power) of some low order modes in a step index fibre for $V = 8$. Notice that the higher order modes have a greater fraction of power in the cladding.

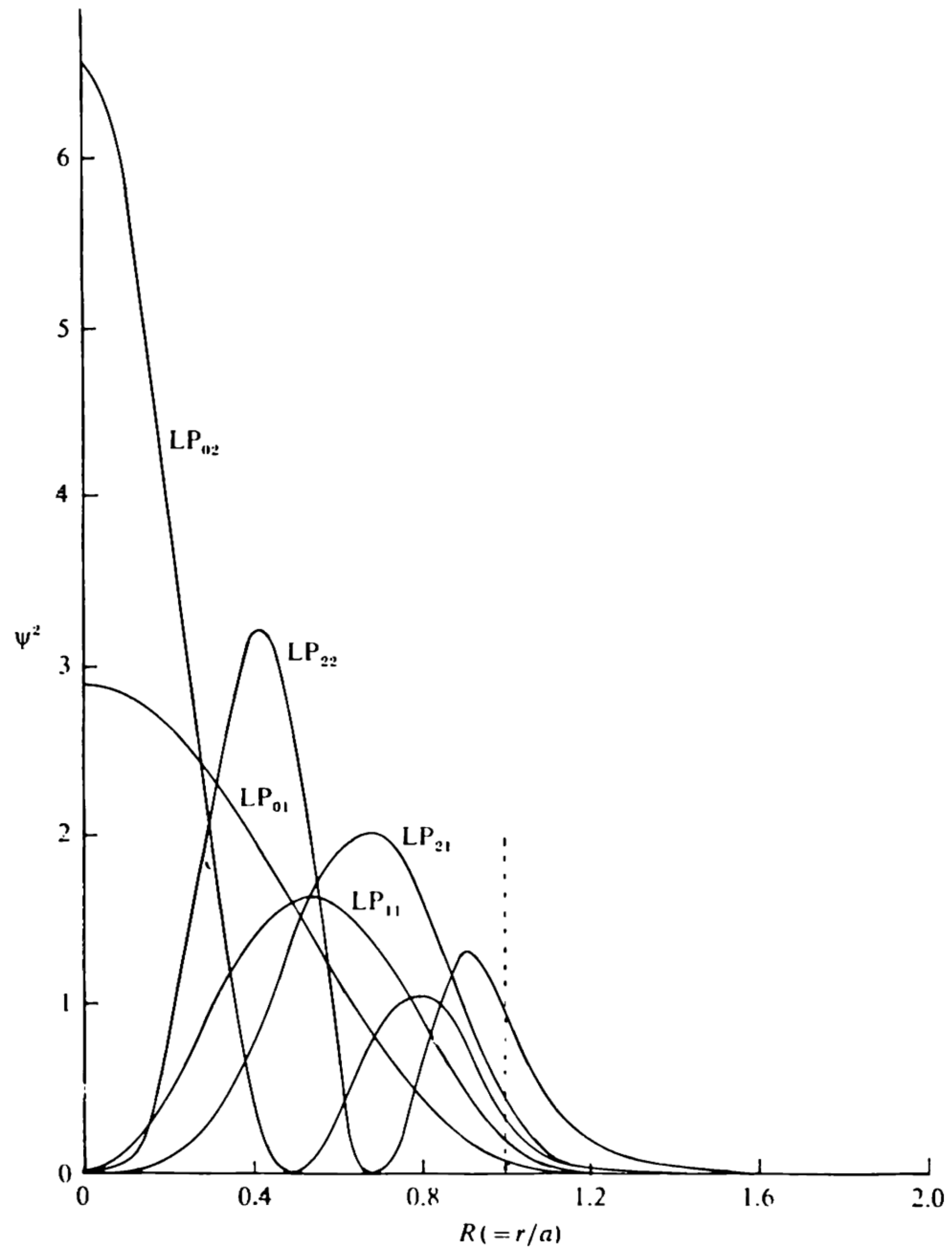
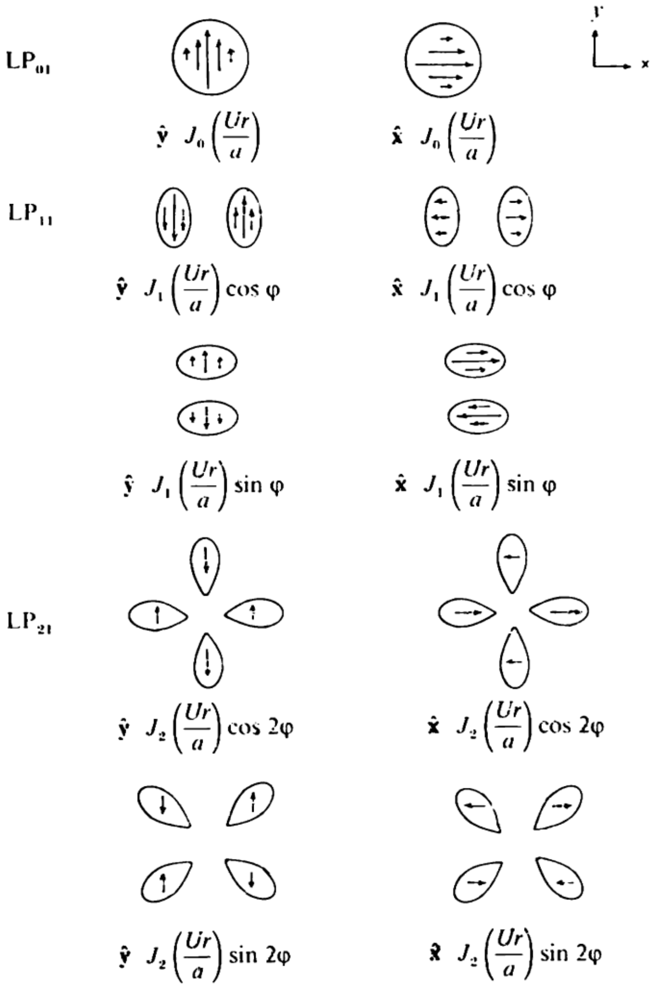
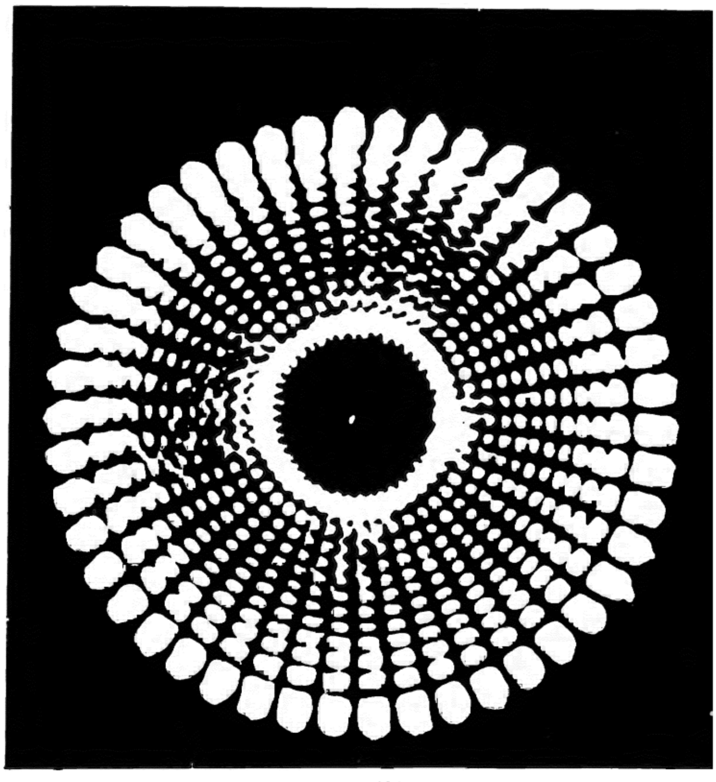


Fig. 13.10 (*a*) Schematic of the modal field patterns for some low order modes in a step index fibre. The arrows represent the direction of the electric field. (*b*) Modal intensity pattern of the $LP_{23,12}$ mode in a multimode fibre. (Photograph courtesy of Dr W. Freude.)



(*a*)



(*b*)

lower order modes are shown in Figs. 13.9 and 13.10(*a*). We may note the following points:

(*a*) The $l = 0$ modes are two fold degenerate corresponding to two independent states of polarization.
(*b*) The $l \geqslant 1$ modes are four fold degenerate because for each polarization, the ϕ dependence could be either $\cos l\phi$ or $\sin l\phi$.

Further

$$\text{number of zeroes in the } \phi \text{ direction} = 2l \tag{13.51}$$

and

$$\text{number of zeroes in the radial direction (excluding } r = 0) = m - 1 \tag{13.52}$$

Fig. 13.10(*b*) shows the mode field distribution of a typical higher order mode in a multimode fibre. The figure corresponds to the $LP_{23,12}$ mode; the values of l and m are obtained using Eqs. (13.51) and (13.52). In Appendix F we have shown that when $V \gg 1$, the total number of modes is given by

$$N \approx V^2/2 \tag{13.53}$$

and such a fibre which supports a large number of guided modes is known as a multimode fibre. For a typical multimode step index fibre

$$n_1 = 1.47, \quad n_2 = 1.46, \quad a = 25\,\mu\text{m} \tag{13.54}$$

and for $\lambda_0 = 0.8\,\mu$m, we obtain

$$V \approx 34$$

which would support approximately 580 modes.