TM-modes of symmetric slab

TM-modes of a planar waveguide are described by the field components H_y, E_x and E_z. These are related as the following:

$$E_x = \frac{\beta}{\omega \epsilon_0 n^2} H_y \qquad \text{and} \qquad i\beta E_x + \frac{\partial E_z}{\partial x} = i\omega \mu_0 H_y$$

$$E_z = \frac{1}{i\omega \epsilon_0 n^2} \frac{\partial H_y}{dx}$$

Eliminating E_x and E_z from the above eqns, we get

$$n^2(x) \frac{d}{dx}\left(\frac{1}{n^2(x)} \frac{dH_y}{dx}\right) + \left(k_0^2 n^2(x) - \beta^2\right) H_y = 0$$

This can be written as

$$\frac{d^2 H_y}{dx^2} - \left[\frac{1}{n^2(x)} \frac{dn^2}{dx}\right] \frac{dH_y}{dx} + \left[k_0^2 n^2(x) - \beta^2\right] H_y = 0$$

This equation is different from the TE-modes' equation we discussed earlier (the E_y equation). However for a step-index waveguide, the R.I. is constant in each region; therefore x-dependence of $n^2\left[\frac{d}{dx} n^2\right]$ does not exist. So, we may write the H_y-equation for the TM-modes as

$$\frac{d^2 H_y}{dx^2} + \left(k_0^2 n_1^2 - \beta^2\right) H_y = 0 \qquad |x| < d/2$$

$$\frac{d^2 H_y}{dx^2} - \left(\beta^2 - k_0^2 n_2^2\right) H_y = 0 \qquad |x| > \frac{d}{2}$$

In this case, the boundary conditions require that tangential field components, ie, H_y and E_z to be continuous across the interfaces i.e, at $x = \pm \frac{d}{2}$

But for TM-mode, $E_z = \frac{1}{i\omega\epsilon_0 n^2(x)} \frac{\partial H_y}{\partial x}$

Hence the required conditions are

H_y and $\frac{1}{n^2}\frac{\partial H_y}{\partial x}$ be continuous at $x = \pm \frac{d}{2}$.

The above equations yield the solutions for the symmetric modes as

$$H_y(x) = A\cos \kappa x \quad |x| < d/2 : \text{core}$$
$$B e^{-\gamma|x|} \quad |x| > d/2 : \text{cladding}$$

The boundary conditions give

$$A\cos(\kappa d/2) = B e^{-\gamma d/2}$$

and $$\frac{1}{n_1^2}\left[-A\kappa \sin(\kappa d/2)\right] = \frac{1}{n_2^2}\left(-B\gamma e^{-\gamma d/2}\right).$$

Dividing, the above two eqns constitute the eigenvalue equation as

$$\kappa \tan(\kappa d/2) = \gamma \frac{n_1^2}{n_2^2}$$

This can be written as $x \tan x = y$ in the same way as the TE-case. | where $x = \kappa d/2$, $y = \gamma d/2 \cdot \frac{n_1^2}{n_2^2}$

Considering the antisymmetric mode solutions

$$H_y(x) = C\sin \kappa x \quad |x| < d/2 : \text{core}$$
$$= D e^{-\gamma|x|} \quad |x| > d/2 : \text{cladding}$$

We arrive at the eigenvalue equation, on applying the boundary conditions, as

$$\tan(\kappa d/2) = -\frac{\kappa}{\gamma} \cdot \frac{n_2^2}{n_1^2}$$

In the same way as the TE-case, this can be written as

$$x \cot x = -y$$

The V-number can be defined in the same way as before

$$V = k_0 \cdot \frac{d}{2} \cdot \sqrt{n_1^2 - n_2^2} \quad \text{with} \quad x^2 + \frac{y^2}{\left(\frac{n_1^2}{n_2^2}\right)^2} = V^2.$$

The solutions of the eigenvalue equations can be discussed in the same way as the TE-case, except the difference that the RHS of the equations now describe an ellipse whose semimajor axis is $\frac{n_1^2}{n_2^2} V$ and semi minor axis is V.

All the quantities related to the TM-modes, namely, the cut-off frequency, no. of zeros etc. are similar to the ones discussed in TE-mode.

A waveguide having V lying between $0 - \pi/2$, is referred to as a single mode. But actually for this case there will be two modes one TE- and the other TM which are having slightly different β-values: β_{TE}, β_{TM}. However, if the incident field is polarized to excite only TE or only TM-mode, the no. of modes will be due to the respective conditions.

Note: $K\frac{d}{2} = x \qquad \left(\frac{\gamma d}{2}\right)\frac{n_1^2}{n_2^2} = y$

$$\text{ie,} \quad \frac{\gamma d}{2} = y \cdot \frac{n_2^2}{n_1^2}$$

$$V^2 = \left(K\frac{d}{2}\right)^2 + \left(\frac{\gamma d}{2}\right)^2 = x^2 + y^2\left(\frac{n_2^2}{n_1^2}\right)^2$$

$$\Rightarrow \quad \frac{x^2}{V^2} + \frac{y^2}{V^2\left(\frac{n_1^2}{n_2^2}\right)^2} = 1 \qquad \underbrace{A = V}_{\text{Semi-major}}, \quad \underbrace{B = V \cdot \frac{n_2^2}{n_2^2}}_{\text{semi minor}}$$