## Wave equation in

## INHOMOGENEOUS medium : $\Delta\epsilon \neq 0$

Maxwell's Equations :

$$\left.\begin{array}{l}\nabla\cdot D = 0 \\ \nabla\cdot B = 0\end{array}\right\}$$ No excess charge: Charge-free region

$$\nabla\times E = -\frac{\partial B}{\partial t}$$

$$\nabla\times H = \frac{\partial D}{\partial t}$$

Now, $\nabla\cdot D = \nabla\cdot(\epsilon E)$

$$= \nabla\cdot(\epsilon_0 \epsilon_r E)$$

$$= \epsilon_0 \nabla\cdot(n^2 E)$$

$$= \epsilon_0 \left(\nabla n^2\cdot E + n^2 \nabla\cdot E\right) = 0 \quad \nabla\cdot D = 0$$

So, $\nabla\times(\nabla\times E)$

$$= \nabla\cdot E = -\frac{\nabla n^2\cdot E}{n^2}$$

$$= -\frac{\nabla\epsilon\cdot E}{\epsilon}$$

$$= \nabla(\nabla\cdot E) - E(\nabla\cdot\nabla)$$

$$= -\nabla\left[\frac{\nabla\epsilon\cdot E}{\epsilon}\right] - \nabla^2 E = -\frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2}$$

or $$\nabla^2 E + \nabla\left[\frac{\nabla\epsilon\cdot E}{\epsilon}\right] = \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2}$$ in terms of permittivity $\epsilon$.

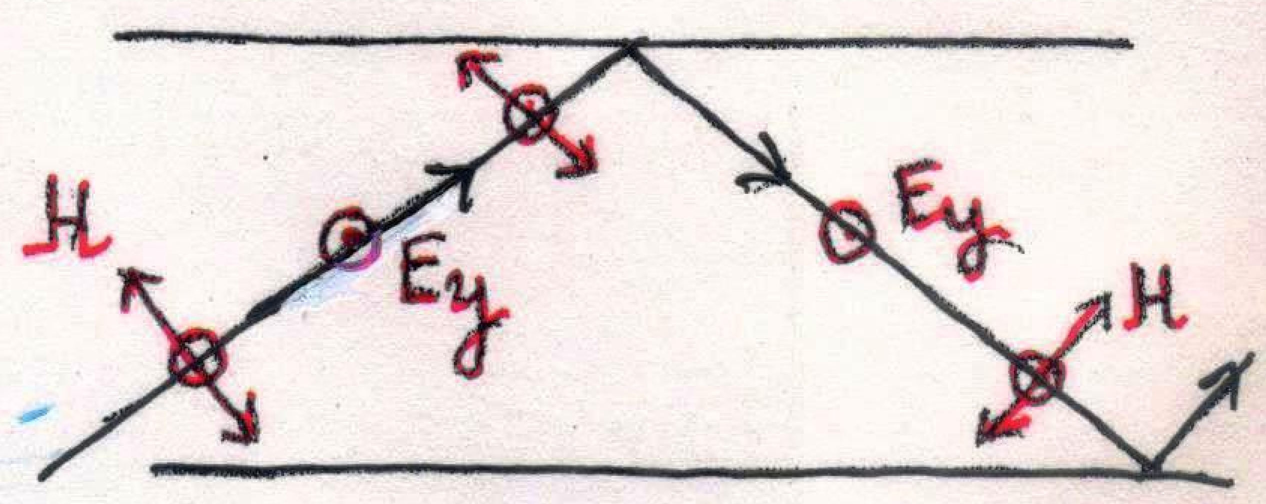
or $$\boxed{\nabla^2 E + \nabla\left[\frac{\nabla n^2\cdot E}{n^2}\right] = \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2}}$$

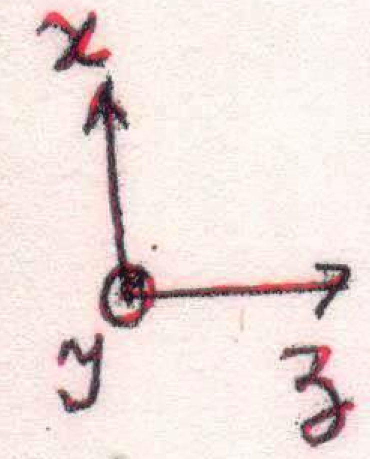
Similarly for H-field, the wave equation is

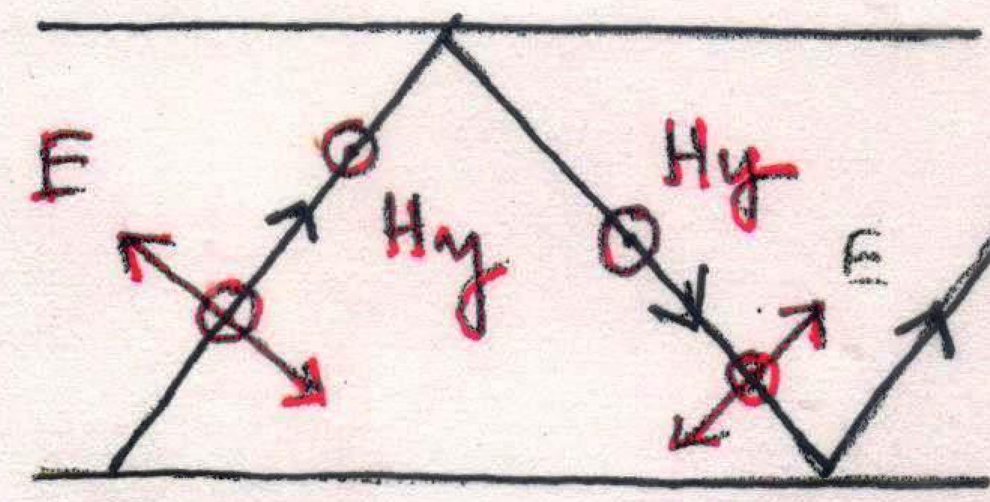
$$\boxed{\nabla^2 H + \frac{\nabla n^2}{n^2}\times(\nabla\times H) = \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2}}$$

Transverse Modes:

Waves propagating in the xz-plane.




TE-mode

TM-mode

$H_x, H_z, E_y$ - field components

$E_x, E_z, H_y$ - field components

Electric and magnetic fields are of the form:

$$E = E_0 e^{i(\omega t - \beta z)} \qquad H = H_0 e^{i(\omega t - \beta z)}$$

Now Maxwell's curl eqns:

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} = \epsilon_0 n^2 \frac{\partial E}{\partial t} \qquad \nabla \times E = -\mu_0 \frac{\partial H}{\partial t}$$

x - Compt

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \epsilon\, i\omega E_x$$

propagating in xz-plane

no variation along y

$$\frac{\partial}{\partial y} \equiv 0$$

$$\underset{\times}{\parallel} \quad - \frac{\partial H_y}{\partial z} = i\omega\, \epsilon_0 n^2 E_x$$

$$\cancel{i}\beta H_y = \cancel{i}\omega\, \epsilon_0 n^2 E_x$$

$$\boxed{H_y = \frac{\omega \epsilon_0 n^2}{\beta} E_x}$$ TM

y - compt

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i\omega\epsilon E_y$$

$$\boxed{-i\beta H_x - \frac{\partial H_z}{\partial x} = i\omega\epsilon_0 n^2 E_y}$$ TE

z - compt

$$\frac{\partial H_y}{\partial x} - \underset{\underset{\times}{\parallel}}{\frac{\partial H_x}{\partial y}} = i\omega\epsilon_0 n^2 E_z$$

TM

$$\boxed{\frac{\partial H_y}{\partial x} = i\omega\epsilon_0 n^2 E_z}$$

From the $\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}$

$$\boxed{H_x = -\frac{\beta}{\mu_0 \omega} E_y} \quad \text{TE}$$

$$\boxed{i\beta E_x + \frac{\partial E_z}{\partial x} = \mu_0 i \omega H_y} \quad \text{TM}$$

$$\boxed{H_z = \frac{i}{\mu_0 \omega} \frac{\partial E_y}{\partial x}} \quad \text{TE}$$

$$\downarrow$$

$$\frac{\partial}{\partial x}\left[\frac{i}{\omega \mu_0} \frac{\partial E_y}{\partial x}\right] = \frac{\partial H_z}{\partial x}$$

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\omega \mu_0}{i} \cdot \frac{\partial H_z}{\partial x} = \frac{\omega \mu_0}{\cancel{i}} \left(-\cancel{i}\beta H_x - \cancel{i}\omega \epsilon_0 n^2 E_y\right)$$

$$= -\omega \mu_0 \beta H_x - \omega^2 \mu_0 \epsilon_0 n^2 E_y$$

$$= -\cancel{\omega \mu_0} \beta \left(-\frac{\beta}{\cancel{\omega \mu_0}}\right) E_y - \omega^2 \mu_0 \epsilon_0 n^2 E_y$$

$$= +\beta^2 E_y - k_0^2 n^2 E_y$$

$$\boxed{\frac{\partial^2}{\partial x^2} E_y + \left(k_0^2 n^2 - \beta^2\right) E_y = 0}$$

$\Rightarrow$ wave eqn for $E_y$.

The same can be obtained for other field components: $E_x$, $E_z$, $H_y$, $H_x$, $H_z$ .......

**TE**

$$H_x = -\frac{\beta}{\mu_0 \omega} E_y$$

$$H_z = \frac{i}{\mu_0 \omega} \frac{\partial E_y}{\partial x}$$

$$E_y = -\frac{\beta}{\omega \epsilon_0 n^2} H_x + \frac{i}{\omega \epsilon_0 n^2} \frac{\partial H_z}{\partial x}$$

**TM**

$$E_x = \frac{\beta}{\omega \mu_0} H_y$$

$$E_z = -\frac{i}{\omega \epsilon_0 n^2} \frac{\partial H_y}{\partial x}$$

$$H_y = \frac{\beta}{\mu_0 \omega} E_x - \frac{i}{\mu_0 \omega} \frac{\partial E_z}{\partial x}$$

- Transverse Electric (TE), or
- Transverse Magnetic (TM).

Proceeding from the Maxwell curl equations:

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\bar{H}$$

or

$$\hat{x}: \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\hat{y}: \quad -\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = -j\omega\mu H_y$$

$$\hat{z}: \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

However, the spatial variation in $z$ is known so that

$$\frac{\partial\left(e^{-j\beta z}\right)}{\partial z} = -j\beta\left(e^{-j\beta z}\right)$$

Consequently, these curl equations simplify to

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \qquad (3.3a),(1)$$

$$-\frac{\partial E_z}{\partial x} - j\beta E_x = -j\omega\mu H_y \qquad (3.3b),(2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad (3.3c),(3)$$

We can perform a similar expansion of Ampère's equation $\nabla \times \bar{H} = j\omega\varepsilon\bar{E}$ to obtain

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\varepsilon E_x \qquad (3.4a),(4)$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \qquad (3.4b),(5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \qquad (3.5c),(6)$$

Now, (1)-(6) can be manipulated to produce simple algebraic equations for the transverse ($x$ and $y$) components of $\bar{E}$ and $\bar{H}$. For example, from (1):

$$H_x = \frac{j}{\omega\mu}\left(\frac{\partial E_z}{\partial y} + j\beta E_y\right)$$

Substituting for $E_y$ from (5) we find

$$H_x = \frac{j}{\omega\mu}\left[\frac{\partial E_z}{\partial y} + j\beta\frac{1}{j\omega\varepsilon}\left(-j\beta H_x - \frac{\partial H_z}{\partial x}\right)\right]$$

$$= \frac{j}{\omega\mu}\frac{\partial E_z}{\partial y} + \frac{\beta^2}{\omega^2\mu\varepsilon}H_x - \frac{j\beta}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x}$$

or, $$\boxed{H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)} \qquad (3.5a),(7)$$

where $\boxed{k_c^2 \equiv k^2 - \beta^2}$ and $k^2 = \omega^2\mu\varepsilon$. (3.6)

Similarly, we can show that

$$\boxed{H_y = -\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)} \qquad (3.5b),(8)$$

$$\boxed{E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)} \qquad (3.5c),(9)$$

$$\boxed{E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)} \qquad (3.5d),(10)$$

Most important point: From (7)-(10), we can see that **all transverse** components of $\bar{E}$ and $\bar{H}$ can be determined from **only the axial** components $E_z$ and $H_z$. It is this fact that allows the mode designations TEM, TE, and TM.

Furthermore, we can use superposition to reduce the complexity of the solution by considering each of these mode types separately, then adding the fields together at the end.