1. *Plane harmonic waves*

   Consider the following scalar harmonic plane waves. Here $\psi_0 \in \mathbb{R}$ is a constant amplitude, $\boldsymbol{k} \in \mathbb{R}^3$ and $\omega > 0$ are the wave-vector and frequency associated with the waves:

   (a) $\psi(\boldsymbol{r}, t) = \psi_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$,

   (b) $\psi(\boldsymbol{r}, t) = \psi_0 \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$,

   (c) $\psi(\boldsymbol{r}, t) = \psi_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} + \omega t)}$,

   (d) $\psi(\boldsymbol{r}, t) = \psi_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} + \psi_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} + \omega t)}$,

   (e) $\psi(\boldsymbol{r}, t) = \psi_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} + \dfrac{1}{2} \psi_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} + \omega t)}$.

   Using the conventions for complex notation of time-harmonic fields, evaluate the real physical field, simplify as far as reasonable. Clarify the notions of a *forward traveling wave*, *backward traveling wave*, *standing wave*, and *partly traveling, partly standing wave*.

2. *Time-averaged Poynting vector and energy density*

   (a) Suppose that $a(\boldsymbol{r}, t)$ and $b(\boldsymbol{r}, t)$ are scalar fields (or components of vector fields) of the form $a(\boldsymbol{r}, t) = \tilde{a}(\boldsymbol{r}) \, \mathrm{e}^{\mathrm{i}\omega t}$, $b(\boldsymbol{r}, t) = \tilde{b}(\boldsymbol{r}) \, \mathrm{e}^{\mathrm{i}\omega t}$. Using the conventions for the complex notation of time-harmonic fields, here for angular frequency $\omega = 2\pi/T$, with time period T, write out the (real, physical) product $a\,b$, and show that the time-average

   $$\bar{f}(t) = \frac{1}{T} \int_t^{t+T} f(t') \, \mathrm{d}t'$$

   of that product evaluates to $\overline{a\,b} = \dfrac{1}{2} \mathrm{Re}(\tilde{a}^* \, \tilde{b})$, where * denotes complex conjugation.

   (b) Show that, for time-harmonic electromagnetic fields of the form $\boldsymbol{E}(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}(\boldsymbol{r}) \, \mathrm{e}^{\mathrm{i}\omega t}$ ($\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ analogously), the time-averaged Poynting vector $\overline{\boldsymbol{S}}$ and the time averaged energy density $\overline{w}$ take the forms

   $$\overline{\boldsymbol{S}} = \frac{1}{2} \mathrm{Re} \left(\tilde{\boldsymbol{E}}^* \times \tilde{\boldsymbol{H}} \right), \quad \text{and} \quad \overline{w} = \frac{1}{4} \mathrm{Re} \left(\tilde{\boldsymbol{E}}^* \cdot \tilde{\boldsymbol{D}} + \tilde{\boldsymbol{H}}^* \cdot \tilde{\boldsymbol{B}} \right).$$

3. *Plane harmonic electromagnetic waves*

   Consider an electromagnetic wave with an electric field of the form

   $$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)},$$

   with frequency ω, wave vector $\boldsymbol{k}$, and amplitude vector $\boldsymbol{E}_0$ (remember the conventions for complex notation of time harmonic fields). The wave is assumed to propagate through a lossless isotropic homogeneous linear medium without free charges or currents, characterized by the relative permittivity ϵ, relative permeability μ, and refractive index $n = \sqrt{\epsilon\mu}$.

   (a) Write out the remaining parts $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ of the electromagnetic field in terms of the quantities introduced above. State conditions such that all Maxwell equations are satisfied.

   (b) Show that, for this wave, the time averaged energy density $\overline{w}$ and the time-averaged Poynting vector $\overline{\boldsymbol{S}}$ take the following alternative forms:

   $$\overline{w} = \frac{1}{2} \epsilon_0 \epsilon \, |\boldsymbol{E}_0|^2 = \frac{1}{2} \mu_0 \mu \, |\boldsymbol{H}_0|^2,$$

   $$\overline{\boldsymbol{S}} = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon}{\mu_0 \mu}} \, |\boldsymbol{E}_0|^2 \, \frac{\boldsymbol{k}}{k} = \frac{1}{2} \sqrt{\frac{\mu_0 \mu}{\epsilon_0 \epsilon}} \, |\boldsymbol{H}_0|^2 \, \frac{\boldsymbol{k}}{k} = \overline{w} \, \mathrm{c_m} \, \frac{\boldsymbol{k}}{k}.$$

   Here $\mathrm{c_m} = (\sqrt{\epsilon_0 \epsilon \mu_0 \mu})^{-1}$ is the phase velocity in the medium, and $\boldsymbol{H}_0$ is the amplitude vector associated with the magnetic part of the wave field. *(→)*

(3., continued)

Now consider a lossy medium, with the attenuation given by the imaginary part of the complex permttivity $\epsilon = \epsilon' - \mathrm{i}\epsilon''$, with $\epsilon'' > 0$.

(c) Show that the the real- and imaginary parts of the complex refractive index $n = n' - \mathrm{i}n''$, defined through the relation $n^2 = \epsilon\mu$, are related to the complex permittivity ϵ and real permeability μ as

$$(n')^2 = \frac{1}{2}\left(|\epsilon\mu| + \epsilon'\mu\right), \qquad (n'')^2 = \frac{1}{2}\left(|\epsilon\mu| - \epsilon'\mu\right).$$

Choose n' to be positive; then select the correct sign for n''.

(d) Modify your expressions from (3a) and (3b) to take into account the attenuation. Write the wavenumber $\boldsymbol{k} = k_0\, n\, \boldsymbol{\kappa}$ as a product of the vacuum wavenumber $k_0 = \omega/\mathrm{c}$, the complex refractive index n, and a direction $\boldsymbol{\kappa} \in \mathbb{R}^3$, with $|\boldsymbol{\kappa}|^2 = 1$. Show that the time-averaged Poynting vector and the time-averaged energy density can be given the form

$$\overline{\boldsymbol{S}}(\boldsymbol{r}) = \frac{k_0 n'}{2\omega\mu_0\mu}\,\mathrm{e}^{-2k_0 n''\boldsymbol{\kappa}\cdot\boldsymbol{r}}|\boldsymbol{E}_0|^2\boldsymbol{\kappa}, \qquad \overline{w}(\boldsymbol{r}) = \frac{\epsilon_0 (n')^2}{2\mu}\,\mathrm{e}^{-2k_0 n''\boldsymbol{\kappa}\cdot\boldsymbol{r}}|\boldsymbol{E}_0|^2,$$

such that the relation $\overline{\boldsymbol{S}}(\boldsymbol{r}) = \overline{w}(\boldsymbol{r})\,(\mathrm{c}/n')\,\boldsymbol{\kappa}$ holds. Verify that all results relate to *damped* waves, for positive ϵ'', n'', μ and the time dependence $\sim \exp(\mathrm{i}\omega t)$ introduced before.

4. *2-D Gaussian beams*

The principal electric field component $E_y(x,z)$ of 2-D TE waves with vacuum wavelength $\lambda = 2\pi/k$ and vacuum wavenumber k in a homogeneous medium with refractive index n satisfies the 2-D Helmholtz equation

$$(\partial_x^2 + \partial_z^2 + k^2n^2)E_y = 0\,. \tag{1}$$

Elementary solutions can be stated in the form

$$E_y(x,z) \sim \mathrm{e}^{-\mathrm{i}(k_x x + k_z z)}, \quad \text{with} \quad k^2n^2 = k_x^2 + k_z^2. \tag{2}$$

We consider a Gaussian superposition of these solutions with wave vectors around the z-axis,

$$E_y(x,z) = E_0 \int \mathrm{e}^{-k_x^2/\kappa^2}\,\mathrm{e}^{-\mathrm{i}(k_x x + k_z(k_x) z)}\mathrm{d}k_x\,, \tag{3}$$

with a spectral width κ. Here E_0 is an arbitrary amplitude; the wavevector components k_x and k_z need to satisfy Eq. (2), which requires a dependence

$$k_z(k_x) = kn\sqrt{1 - \frac{k_x^2}{k^2n^2}}. \tag{4}$$

(a) For a well directed bundle with "narrow" spectral width κ, only waves with $|k_x^2/(k^2n^2)| \ll 1$ contribute significantly. Use a first order approximation $\sqrt{1-x} \approx 1 - x/2$ of the squareroot in (4) to evaluate (3). Show that the 2-D Gaussian wave bundle can then be given the form

$$E_y(x,z) \approx E_0 \frac{2\sqrt{\pi}}{w(z)}\,\mathrm{e}^{-x^2/w^2(z)}\,\mathrm{e}^{-\mathrm{i}knz}, \quad \text{with} \quad w(z) = \frac{2}{\kappa}\sqrt{1 - \mathrm{i}\frac{\kappa^2}{2kn}z}. \tag{5}$$

You might wish to use the integral identity $\displaystyle\int_0^\infty \mathrm{e}^{-at^2}\cos(2bt)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-b^2/a}$, $\mathrm{Re}\,a > 0$.

(b) Split the exponent $-x^2/w^2$ of the first exponential term in Eq. (5) into real and imaginary parts. Interpret the functional dependence of the wave bundle on the coordinates x and z.

(c) Visualize the field behaviour by means of suitable plots, e.g. plots of $|E_y|$ and $\mathrm{Re}E_y$ versus x, z, for parameters $n = 1.0$, $\lambda = 1.0\,\mu$m, and for beams with field-1/e-widths at focus $w(0) = 2/\kappa$ of $0.5\,\mu$m and $4\,\mu$m.

Hand in your solutions until Wednesday, May 07, 09:15. Good luck!