1. *Guided 2-D TE / TM modes, orthogonality properties*

   Reconsider the orthogonality properties of guided modes, as discussed in the lecture, now for a set of non-degenerate TE- and TM-polarized modes supported by a 2-D waveguide configuration with permittivity $\epsilon(x)$. These are electromagnetic fields of the form

$$\begin{pmatrix} \boldsymbol{E}_m \\ \boldsymbol{H}_m \end{pmatrix}(x,z) = \begin{pmatrix} \bar{\boldsymbol{E}}_m \\ \bar{\boldsymbol{H}}_m \end{pmatrix}(x)\, \mathrm{e}^{-\mathrm{i}\beta_m z} \tag{1}$$

   with mode profiles $\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m$ and pairwise different propagation constants β_m. We introduce an abbreviation $\boldsymbol{\psi}_m^{\mathrm{p}} = (\bar{\boldsymbol{E}}_m, \bar{\boldsymbol{H}}_m)$ for the mode profile; superscripts $^{\mathrm{p\,=\,TE,\,TM}}$ denote the polarization. Similar to the 3-D case, a product

$$(\boldsymbol{E}_1, \boldsymbol{H}_1; \boldsymbol{E}_2, \boldsymbol{H}_2) := \frac{1}{4}\int \left(E_{1x}^* H_{2y} - E_{1y}^* H_{2x} + H_{1y}^* E_{2x} - H_{1x}^* E_{2y}\right) \mathrm{d}x \tag{2}$$

   of two electromagnetic fields $(\boldsymbol{E}_j, \boldsymbol{H}_j)$, $j = 1, 2$, will be used.

   (a) Recall the specific modal properties of the TE / TM modes supported by straight 2-D waveguides.

   (b) Show that the time averaged power P_m per lateral (y) unit length carried by mode $\boldsymbol{\psi}_m^{\mathrm{p}}$, β_m can be given the form

$$P_m := \int S_z\,\mathrm{d}x = (\boldsymbol{\psi}_m^{\mathrm{p}}; \boldsymbol{\psi}_m^{\mathrm{p}}) = \begin{cases} \dfrac{\beta_m}{2\omega\mu_0}\displaystyle\int |E_{m,y}|^2\,\mathrm{d}x, & \text{if p = TE,} \\[2ex] \dfrac{\beta_m}{2\omega\epsilon_0}\displaystyle\int \frac{1}{\epsilon}\,|H_{m,y}|^2\,\mathrm{d}x, & \text{if p = TM.} \end{cases} \tag{3}$$

   (c) Starting from the identity

$$\boldsymbol{\nabla}\cdot(\boldsymbol{E}_l^* \times \boldsymbol{H}_m + \boldsymbol{E}_m \times \boldsymbol{H}_l^*) = 0 \quad \text{for all } l, m, \tag{4}$$

   and taking into account the vanishing y-derivatives of all fields in the 2-D setting, conclude that

$$(\beta_l - \beta_m)\int \left(\bar{\boldsymbol{E}}_l^* \times \bar{\boldsymbol{H}}_m + \bar{\boldsymbol{E}}_m \times \bar{\boldsymbol{H}}_l^*\right)_z \mathrm{d}x = 0\,. \tag{5}$$

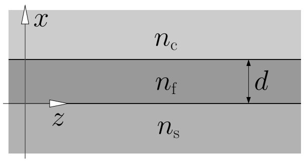
   (d) Verify that the orthogonality properties of the modal set in question can be stated in the form

$$\begin{aligned} &(\boldsymbol{\psi}_l^{\mathrm{TE}}; \boldsymbol{\psi}_m^{\mathrm{TM}}) = 0\,, \qquad (\boldsymbol{\psi}_l^{\mathrm{TM}}; \boldsymbol{\psi}_m^{\mathrm{TE}}) = 0\,, \\ &(\boldsymbol{\psi}_l^{\mathrm{TE}}; \boldsymbol{\psi}_m^{\mathrm{TE}}) = \frac{\beta_m}{2\omega\mu_0}\int E_{l,y}^*\,E_{m,y}\,\mathrm{d}x = \delta_{lm}P_m\,, \\ &(\boldsymbol{\psi}_l^{\mathrm{TM}}; \boldsymbol{\psi}_m^{\mathrm{TM}}) = \frac{\beta_m}{2\omega\epsilon_0}\int \frac{1}{\epsilon}\,H_{l,y}^*\,H_{m,y}\,\mathrm{d}x = \delta_{lm}P_m\,, \end{aligned} \tag{6}$$

   for pairs of modes with indices l, m, with $\delta_{lm} = 1$, if $l = m$, and $\delta_{lm} = 0$, otherwise. Note that the statements still hold (?) in case of a degeneracy between modes of different polarization.

2. *Three layer slab waveguide, transverse resonance condition*

   Specialize the model of 2-D of dielectric multilayer slab waveguides, as discussed in the lecture, to a three-layer structure as shown in the figure.



A not necessarily symmetric three layer slab waveguide, with a core thickness d, and refractive indices n_s, n_f, and n_c for the subtrate, film, and cover layers, described in Cartesian coordinates x and z.

   We use the notation as introduced in the lecture. Interest is in guided TE / TM modes with effective indices n_{eff} from the interval $n_s, n_c < n_{\text{eff}} < n_f$, and with propagation constants $\beta = kn_{\text{eff}}$.

(a) Verify that the profile of the principal field component ϕ can be stated in the form

$$\phi(x) = \begin{cases} B_\mathrm{c}\,\mathrm{e}^{-\kappa_\mathrm{c}\,x}, & \text{for } d < x, \\ A_\mathrm{f}\sin(\kappa_\mathrm{f}\,x) + B_\mathrm{f}\cos(\kappa_\mathrm{f}\,x), & \text{for } 0 < x < d, \\ A_\mathrm{s}\,\mathrm{e}^{\kappa_\mathrm{s}\,x}, & \text{for } x < 0, \end{cases} \tag{7}$$

where the local wavenumbers in x-direction are

$\kappa_\mathrm{s} = \sqrt{\beta^2 - k^2 n_\mathrm{s}^2}\,,\ \kappa_\mathrm{f} = \sqrt{k^2 n_\mathrm{f}^2 - \beta^2}\,,\ \kappa_\mathrm{c} = \sqrt{\beta^2 - k^2 n_\mathrm{c}^2}\,.$

(b) Use the conditons of continuity of ϕ and continuity of $\eta\partial_x\phi$ at $x = 0$ and at $x = d$ to establish a system of equations for the local coefficients A_s, A_f, B_f, and B_c. Here η distinguishes the polarizations as $\eta(x) = 1$ (TE) and $\eta(x) = 1/n^2(x)$ (TM); introduce quantities η_s, η_f, η_c accordingly.

(c) By eliminating successively first A_f and B_f, then B_c, and finally A_s (requiring a nonzero solution, i.e. a nonzero A_s), show that the condition for the existence of guided modes can be written as a transcendental equation

$$\tan(\kappa_\mathrm{f} d) = \frac{(\eta_\mathrm{s}\kappa_\mathrm{s} + \eta_\mathrm{c}\kappa_\mathrm{c})\eta_\mathrm{f}\kappa_\mathrm{f}}{\eta_\mathrm{f}^2\kappa_\mathrm{f}^2 - \eta_\mathrm{s}\kappa_\mathrm{s}\eta_\mathrm{c}\kappa_\mathrm{c}}\,, \tag{8}$$

the so-called *transverse resonance condition* for the waveguide.

(d) Check whether the condition (8) is satisfied for the effective index values as given in the lecture for the example with waveguide parameters $n_\mathrm{s} = 1.45$, $n_\mathrm{f} = 1.99$, $n_\mathrm{c} = 1.0$, $d = 1.5\,\mu$m, $\lambda = 1.55\,\mu$m.

3. *Functional for guided modes of 3-D dielectric waveguides*

Consider a nonmagnetic, lossless, and isotropic 3-D dielectric waveguide with a cross section spanned by the x- and y-plane, with a real scalar permittivity $\epsilon(x, y)$ that is constant along the axis z of the waveguide. Guided modes supported by that waveguide are electromagnetic fields of the form

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}(x, y, z) = \begin{pmatrix} \bar{\boldsymbol{E}} \\ \bar{\boldsymbol{H}} \end{pmatrix}(x, y)\ \mathrm{e}^{-\mathrm{i}\beta z}, \tag{9}$$

with a real propagation constant β and with mode profile components $\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{H}}$ that vanish for $x, y \to \pm\infty$. The Maxwell curl equations in the frequency domain, for a time dependence $\sim\ \mathrm{e}^{\mathrm{i}\omega t}$ with angular frequency ω, need to be satisfied. For a field in the modal form of Eq. (9), these can be stated as:

$$\begin{aligned} (\mathsf{C} + \mathrm{i}\beta\mathsf{R})\bar{\boldsymbol{E}} &= -\mathrm{i}\omega\mu_0\bar{\boldsymbol{H}}, \\ (\mathsf{C} + \mathrm{i}\beta\mathsf{R})\bar{\boldsymbol{H}} &= \mathrm{i}\omega\epsilon_0\epsilon\bar{\boldsymbol{E}}, \end{aligned} \quad \text{with} \quad \mathsf{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}. \tag{10}$$

For simplicity we restrict things to a smooth refractive index profile, such that the permittivity and all fields can be assumed to be continuous, with continuous derivatives. Now consider the functional

$$\mathcal{B}(\boldsymbol{E}, \boldsymbol{H}) = \frac{\omega\varepsilon_0\langle\boldsymbol{E}, \varepsilon\boldsymbol{E}\rangle + \omega\mu_0\langle\boldsymbol{H}, \boldsymbol{H}\rangle + \mathrm{i}\langle\boldsymbol{E}, \mathsf{C}\boldsymbol{H}\rangle - \mathrm{i}\langle\boldsymbol{H}, \mathsf{C}\boldsymbol{E}\rangle}{\langle\boldsymbol{E}, \mathsf{R}\boldsymbol{H}\rangle - \langle\boldsymbol{H}, \mathsf{R}\boldsymbol{E}\rangle}, \tag{11}$$

for an inner product $\langle\boldsymbol{F}, \boldsymbol{G}\rangle = \iint \boldsymbol{F}^* \cdot \boldsymbol{G}\,\mathrm{d}x\,\mathrm{d}y$.

(a) Show that the functional evaluates to the propagation constant, if a valid mode profile is inserted, i.e. show that $\mathcal{B}(\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}) = \beta$ for fields $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$ that satisfy Eq. (10).

(b) Show that the functional becomes stationary at a valid mode profile, i.e. show that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{B}(\bar{\boldsymbol{E}} + s\,\delta\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}} + s\,\delta\bar{\boldsymbol{H}})\bigg|_{s=0} = 0 \tag{12}$$

for fields $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$ that satisfy Eq. (10), and for arbitrary variations $\delta\bar{\boldsymbol{E}}$, $\delta\bar{\boldsymbol{H}}$ (with real s). This implies: If the functional becomes stationary at some field $\boldsymbol{E}, \boldsymbol{H}$, i.e. if Eq. (12) is satisfied for arbitrary $\delta\bar{\boldsymbol{E}}, \delta\bar{\boldsymbol{H}}$, then $\boldsymbol{E}, \boldsymbol{H}$ satisfy Eq. (10).

Hints: Observe the "cc-bilinearity" of $\langle\ \cdot\ ,\ \cdot\ \rangle$. You might wish to show first that, for fields $\boldsymbol{F}, \boldsymbol{G}$ where this is applicable, $\langle\boldsymbol{F}, \mathsf{R}\boldsymbol{G}\rangle = -\langle\mathsf{R}\boldsymbol{F}, \boldsymbol{G}\rangle$, and $\langle\boldsymbol{F}, \mathsf{C}\boldsymbol{G}\rangle = \langle\mathsf{C}\boldsymbol{F}, \boldsymbol{G}\rangle$. Try to work, as far as possible, on the level of the inner products; introduce abbreviations, where appropriate, e.g. for the numerator and the denominator of $\mathcal{B}$.

Hand in your solutions until Wednesday, June 18, 09:15. Good luck!