<u>COUPLED-MODE EQUATION FOR AXIALLY PERIODIC PERTURBATION</u>:

Consider an optical fiber with RI profile as $n^2(x,y)$ in which there is a periodic z-dependent perturbation in the RI profile as $\Delta \mathrm{n}^2(x,y,z)$

It could be a periodic stress

OR a periodic undulation of this fiber axis.

For a sinusoidal z-perturbation: $\Delta n^2(x,y,z) = \Delta n^2(x,y)\sin kz$

$$K = \frac{2\pi}{\Lambda}; \Lambda = \text{spatial period}$$

If $\psi_1(x,y)$ and $\psi_2(x,y)$ are the two modes of the fiber, then the total field under perturbation may be written as

$$\psi(x,y,z) = A(z)\psi_1 e^{-i\beta_1 z} + B(z)\psi_2 e^{-i\beta_2 z} \tag{1}$$

$\beta_1, \beta_2 \rightarrow$ are mode propagation constants without perturbation.

$A(z), B(z) \rightarrow$ are the amplitudes of the modes.

* Without perturbation A, B are constant.
* Perturbation couples power among modes, hence A, B are z-dependent.

In absence of any perturbation:

$$\begin{aligned} \nabla_{xy}^2\psi_1 + [k_0^2 n^2(x,y) - \beta_1^2]\psi_1 = 0 \\ \nabla_{xy}^2\psi_2 + [k_0^2 n(x,y) - \beta_2^2]\psi_2 = 0 \end{aligned} \tag{2}$$

Since modes are orthogonal,

$$\iint_{-\infty}^{+\infty} \psi_1^*\psi_2 dxdy = 0 \tag{3}$$

Under perturbation, the wave equation to be satisfied by $\psi(x,y,z)$ is then

$$\nabla_{xy}^2\psi + \frac{\partial^2\psi}{\partial z^2} + [k_0^2 n^2(x,y) + \Delta n^2(x,y,z)]\psi = 0 \tag{4}$$

Substituting (1) in (4):

<u>1st term:</u>

$$\nabla_{xy}^2\psi = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2}\right]\left\{A(z)\psi_1 e^{-i\beta_1 z} + B(z)\psi_2 e^{-i\beta_2 z}\right\}$$

$$= \left[\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2}\right]A(z)e^{-i\beta_1 z} + \left[\frac{\partial^2\psi_z}{\partial x^2} + \frac{\partial^2\psi_2}{\partial y^2}\right]B(z)e^{-i\beta_2 z}$$

$$= A(z)e^{-i\beta_1 z} \cdot \frac{\partial^2\psi_1}{\partial x^2} + B(z)e^{-i\beta_2 z} \cdot \frac{\partial^2\psi_2}{\partial x^2} + A(z)e^{-i\beta_1 z} \cdot \frac{\partial^2\psi_1}{\partial y^2} + B(z)e^{-i\beta_2 z}\frac{\partial^2\psi_2}{\partial y^2}$$

$$= \nabla_{xy}^2\psi_1 \cdot Ae^{-i\beta_1 z} + \nabla_{xy}^2\psi_2 Be^{-i\beta_1 z}$$

2nd term:

$$\frac{\partial\psi}{\partial z} = \frac{\partial A(z)}{\partial z} \cdot \psi_1 e^{-i\beta_1 z} - i\beta_1 A\psi_1 e^{-i\beta_1 z} + \frac{\partial B}{\partial z}\psi_2 e^{-i\beta_2 z} - i\beta_2 B\psi_2 e^{-i\beta_1 z}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial z^2} = {} & \frac{\partial^2 A}{\partial z^2}\psi_1 e^{-i\beta_1 z} - i\beta_1\frac{\partial A}{\partial z}\psi_1 e^{-i\beta_1 z} - i\beta_1\frac{\partial A}{\partial z}\psi_1 e^{-i\beta_1 z} - \beta_1^2 A\psi_1 e^{-i\beta_1 z} \\ & + \frac{\partial^2 B}{\partial z^2}\psi_2 e^{-i\beta_2 z} - i\beta_2\frac{\partial B}{\partial z} \cdot \psi_2 e^{-i\beta_2 z} - i\beta_2\frac{\partial B}{\partial z}\psi_2 e^{-i\beta_2 z} - \beta_2^2 B\psi_2 e^{-i\beta_2 z} \\ = {} & \frac{\partial^2 A}{\partial z^2}\psi_1 e^{-i\beta_1 z} - 2i\beta_1\frac{\partial A}{\partial z}\psi_1 e^{-i\beta_1 z} - \beta_1^2 A\psi_1 e^{-i\beta_1 z} \\ & + \frac{\partial^2 B}{\partial z^2}\psi_2 e^{-i\beta_2 z} - 2i\beta_2\frac{\partial \beta}{\partial z}\psi_2 e^{-i\beta_2 z} - \beta_2^2 B\psi_2 e^{-i\beta_2 z}\end{aligned}$$

(= 0; slowly varying approx.)

3rd term:

$$k_0^2 n^2(x,y)A\psi_1 e^{-i\beta_1 z} + k_0^2 n^2(xy)B\psi_2 e^{-i\beta_2 z} + k_0^2\Delta n^2(x,yz)A\psi_1 e^{-i\beta_1 z}$$
$$+ k_0^2\Delta n^2(x,yz)B\psi_2 e^{-i\beta_2 z}$$

$$\mathrm{I}: \ \underline{\nabla_{xy}^2\psi} = \underbrace{\nabla_{xy}^2\Psi_1} \cdot Ae^{-i\beta_1 z} + \nabla_{xy}^2\psi_2 \cdot Be^{-i\beta_2 z}$$

$$\mathrm{II}: \ \frac{\partial^2\psi}{\partial z^2} = -2i\beta_1\frac{\partial A}{\partial z}\psi_1 e^{-i\beta_1 z} \underbrace{-\beta_1^2\psi_1} \cdot Ae^{-i\beta_1 z} - 2i\beta_2\frac{\partial\beta}{\partial z}\psi_2 e^{-i\beta_2 z} - \beta_2^2\psi_2 Be^{-i\beta_2 z}$$

$$\underbrace{k_0^2 n^2(xy)\psi_1} \cdot Ae^{-i\beta_1 z} + \underbrace{k_0^2 n^2(xy)\psi_2} \cdot Be^{-i\beta_2 z} \qquad \text{by (2)}$$
$$+ k_0^2\Delta^2(xu_2)\psi_1 \cdot Ae^{-i\beta_1 z} + k_0^2\Delta^2(xyz)\psi_2 \cdot Be^{-i\beta_2 z}$$

So,

$$-2i\beta_1\frac{\partial A}{\partial z}\psi_1 e^{-i\beta_1 z} 2i\beta_2\frac{\partial B}{\partial z}\psi_2 e^{-i\beta_2 z} + k_0^2\Delta^2(xyz)\left[A\psi_1 e^{-i\beta_1 z} + B\psi_2 e^{-i\beta_2 z}\right] = 0$$

Dividing by $e^{-i\beta_1 z}$ all-throughout: $\Delta\beta = \beta_1 - \beta_2$

$$-2i\beta_1\frac{\partial A}{\partial z}\psi_1 - 2i\beta_2\frac{\partial B}{\partial z}\psi_2 e^{i\Delta\beta z} + k_0^2\Delta \mathrm{n}^2(xyz)\left[A\psi_1 + \beta\psi_2 e^{i\Delta\beta z}\right] = 0 \qquad (5)$$

Multiply eqn (5) by ψ^* from left and integrating over the whole space across the fiber cross-section:

$$-2i\beta_1\frac{\partial A}{\partial z}\int \psi_1^*\psi_1 dxdy - 2i\beta_2\frac{\partial B}{\partial z} \cdot \int \psi_1^*\psi_2 dxdy + k_0^2 A\int \psi_1^*\Delta n^2\psi_1 dxdy$$
$$+ k_0^2 B\int \psi_1^*\Delta n^2\psi_2 dxdy \cdot e^{i\Delta\beta z} = 0.$$

using (3)

Hence, $\frac{dA}{dz} = -ic_{11}A - ic_{12}Be^{i\Delta\beta z} - (6.1)$

where we have used

$$\frac{k_0^2}{2\beta_1}\frac{\int \psi_1^* \Delta n^2 \psi_1 dxdy}{\int \psi_1^* \psi_1 dxdy} = c_{11}$$
$$\text{and } \frac{k_0^2}{2\beta_1}\frac{\int \psi_1^* \Delta n^2 \psi_2 dxdy}{\int dxdy} = c_{12}$$

Similarly, multiply equation (5) from the right and integrating, we shall obtain

$$\frac{d\beta}{dz} = -ic_{22}B - ic_2Ae^{-i\Delta\beta z} \tag{6.2}$$

Eqn (6) are the coupled mode eqns. describing z-dependents of amplitudes A, B.

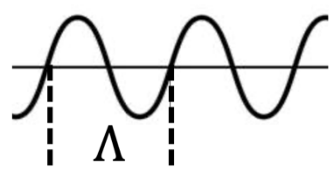
$$C_{21} = \frac{k_0^2}{2\beta_2^2}\cdot\frac{\int 2\Delta\mathrm{n}^2 1}{\int 22}$$

$$C_{22} = \frac{k_0^2}{2\beta_2^2}\cdot\frac{\int 2\Delta\mathrm{n}^2 1}{\int 22}$$

So far, we have considered perturbation $\Delta\mathrm{n}^2(x,y,z)$ which is general and weak.

For a <u>periodic</u> & <u>z-dependent</u> perturbation such as in fiber Bragg grating, we may write

$$\Delta\mathrm{n}^2(x,y,z) = \Delta\mathrm{n}^2(x,y)\cdot\sin kz \text{ where } k = \frac{2\pi}{\Lambda}$$



And then we have,

$$c_{11} = \frac{k_0^2}{2\beta_1}\cdot\frac{\int 1\,\Delta n^2 1}{\int 11}\cdot\sin kz = 2\kappa_{11}\sin k_z \cdot.$$

$$\kappa_{11} = \frac{k_0^2}{4\beta_1}\cdot\frac{\int 1\,\Delta n^2 1}{\int 11}$$

and similarly,

$$c_{12} = 2\kappa_{12}\sin kz \quad c_{22} = 2\kappa_{22}\sin kz$$
$$c_{21} = 2\kappa_{21}\sin kz$$

So, coupled-mode equations take the form:

$$\frac{dA}{dz} = -2i\kappa_{11}A\sin kz - \kappa_{12}Be^{i(\Delta\beta+k)z} + \kappa_{12}Be^{i(\Delta\beta-k)z}$$

Integrating above eq. over a length L, which is small compared with the length over which A and B change appreciably,

$$\begin{aligned}
&A\left(z+\frac{L}{2}\right)-A\left(z-\frac{L}{2}\right) \\
&= +\,4i\kappa_{11}A\cos K_z\,\frac{\sin KL/2}{K} \\
&-2i\kappa_{12}Be^{i(\Delta\beta+K)z}\left\{\frac{\sin(\Delta\beta+K)L/2}{\Delta\beta+K}\right\} \\
&+2i\kappa_{12}Be^{i(\Delta\beta-K)z}\left\{\frac{\sin(\Delta\beta-K)L/2}{\Delta\beta-K}\right\}
\end{aligned}$$

Since $\Delta\beta = \frac{2\pi}{\lambda_0}.\,\Delta n_{\text{eff}}$

$\Delta h_{\text{eff}} \approx$ index difference between core-cladding.

$$\approx 0.005 \text{ for } \lambda_0 = 1.0 \text{ mm.}$$

$$\Delta\beta \approx 3\times 10^4\ \text{m}^{-1}$$

If we choose $K \approx \Delta\beta$ and $L \approx 2\times 10^{-3}$ m (typical values)

then

$$\begin{aligned}
&\left|\frac{\sin(\Delta\beta-K)L/2}{(\Delta\beta-K)}\right| \approx \frac{L}{2} = 10^{-3}\ \text{m} \\
&\left|\frac{\sin(\Delta\beta+K)L/2}{\Delta\beta+K}\right| \leqslant \frac{1}{\Delta\beta+K} \approx \frac{1}{2\Delta\beta} \approx 1.7\times 10^{-5}\ \text{m} \\
&\left|\frac{\sin KL/2}{K}\right| \leqslant \frac{1}{K} \approx \frac{1}{\Delta\beta} \approx 3\times 10^{-5}\ \text{m.}
\end{aligned}$$

Thus, for $k \approx \Delta\beta$, the 1^{st} & 2^{nd} terms are negligible.

And for $\Delta\beta = -K$, 2^{nd} term would have made significant contribution.

1^{st} & 3^{rd} terms are negligible.

So, coupling takes place if $\Delta\beta \approx K\ or - K$.

Thus, if we choose $K = \frac{2\pi}{\Lambda} \simeq \Delta\beta = \beta_1 - \beta_2$: but $\Delta\beta - K = \Gamma$

$$\left.\begin{aligned}
&\frac{dA}{dz} = \kappa_{12}Be^{i\Gamma z} \\
&and\ \frac{dB}{dz} = -\kappa_{21}Ae^{-i\Gamma z}
\end{aligned}\right\} \text{----------(7.1) \& (7.2)}$$

$\xrightarrow{\beta_1}$

$\xrightarrow{\beta_2}$

Under weakly guiding approximation, the modes ψ_1, ψ_2 can be normalized as

$$\left.\begin{aligned} &\frac{\beta_1}{2\omega\mu_0}\iint \psi_1^*\psi_1 dxdy = 1 \\ \text{and } &\frac{\beta_2}{2\omega\mu_0}\iint \psi_2^*\psi_2 dxdy = 1 \end{aligned}\right\} - - - - - - - - - - - -(8)$$

Using this

$$\kappa_{12} = \frac{\omega\epsilon_0}{8}\iint \psi_1^*\Delta \mathrm{n}^2\psi_2 dxdy \ \&\ \kappa_{21} = \frac{\omega\epsilon_0}{8}\iint \psi_2^*\Delta n^2\psi_1 dxdy$$

yielding that $\kappa_{12} = \kappa_{21} = \kappa$ (say).
Hence

$$\left.\begin{aligned} &\frac{dA}{dz} = \kappa B e^{i\Gamma z} \\ \text{and } &\frac{dB}{dz} = -\kappa A e^{-i\Gamma z} \end{aligned}\right\} - - - - - - - - - - - -(9)$$

Equations (9) describe the coupling between two modes propagating along the same direction is (β_1 and β_2 are along +z direction) <u>CODIRECTIONAL COUPLING</u>.

For <u>CONTRADIRECTIONAL COUPLING</u>, coupling occurs between the modes traveling in the opposite direction.
There we start form $\psi(x,y,z) = A(z)\psi_1(x,y)e^{-i\beta_1 z} + B(z)\psi_2(x,y)e^{i\beta_2 z}$.
Thus, following a same procedure we can obtain the CME as

$$\left.\begin{aligned} &\frac{dA}{dz} = \kappa B e^{i\Gamma z} \\ \text{and } &\frac{dB}{dz} = +\kappa A e^{-i\Gamma z} \end{aligned}\right\} - - - - - - - - - - - -(10)$$

where $\Gamma = \beta_1 + \beta_2 - K$

$$\overset{\beta_1}{\longleftarrow}\ \overset{\beta_2}{\longrightarrow}$$

Since $\Delta\beta = \beta_1 - (-\beta_2)$

CONTRA-DIRECTIONAL COUPLING: between the same modes

We have the coupled-mode eques for this case as

$$\frac{dA}{dz} = \kappa B e^{i\Gamma z}$$
$$\frac{dB}{dz} = \kappa A e^{-i\Gamma z} \text{ and } \Gamma = \beta_1 + \beta_2 - K, K = \frac{2\pi}{\Lambda}.$$

If the coupling between the two identical modes traveling in opposite direction, then $\beta_1 = \beta_2 = \frac{2\pi}{\lambda_0} n_{eff}$, $n_{\text{eff}} =$ mode-index

So, $\Lambda = \frac{\lambda_0}{2n_{\text{eff}}}$ Compare this with the case of codirectional case, see the periodicity required here is much smaller.

When the modes are phase-matched, *i.e.*, $\Gamma = 0$, we obtain the equations as

$$\frac{d^2B}{dz^2} = \kappa^2 B$$

Whose solution is $\boxed{B(z) = b_1 e^{\kappa z} + b_2 e^{-\kappa z}}$

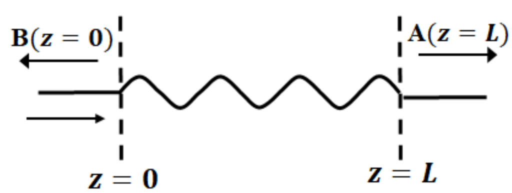
(the solutions are not oscillatory)

And then $\boxed{A(z) = b_1 e^{\kappa z} - b_2 e^{-\kappa z}}$

Boundary condition: A unit power is incident in mode A propagating through a periodic wavelength of length L.

i.e., $A(z = 0) = 1$

Since there is no back-coupled wave beyond $z = L$, $B(z = L) = 0$



Thus, $b_1 e^{\kappa L} + b_2 e^{-\kappa L} = 0$; $b_1 - b_2 = 1$

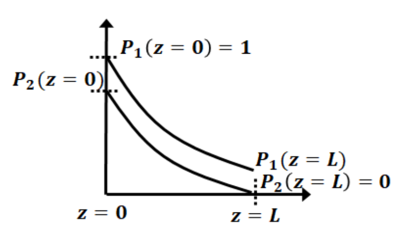
This gives

$$b_1 = \frac{e^{-\kappa L}}{2\cosh \kappa L} \quad \| \quad b_2 = \frac{-e^{\kappa L}}{2\cosh \kappa L}$$

$$\therefore B(z) = \frac{\sinh \kappa(z - L)}{\cosh \kappa L}, A(z) = \frac{\cosh \kappa(z - L)}{\cosh \kappa L}$$

Note that $|A(z)|^2 - |B(z)|^2 = (\cosh^2 \kappa L)^{-1} =$ const.

$\Rightarrow$ energy Conservation.



The reflection coefficient

$$r = \frac{B(z = 0)}{A(z = 0)} = -\tanh \kappa L$$

So, the energy reflection coefficient is

$$\boxed{R = \tanh^2\ \kappa L}$$

for a medium of index variation as $n(z) = n_0 + \Delta n \sin \kappa z$, the coupling coefficient can be shown to be

$$\kappa = \frac{\pi \Delta n}{\lambda_0}$$

For fiber then assuming a similar expression,

$$R = \tanh^2\left(\frac{\pi \Delta n L}{\lambda_0}\right)$$

⇒ Thus, if we wish a reflection centered around 1550 mm, then the required period is

$$\Lambda = \frac{\lambda_0}{2n_{\mathrm{eff}}} = \frac{1550}{2 \times 1.46} = 513 \text{ nm}$$
$$\approx \frac{1}{2}\mu\text{m}$$

⇒ Typical UV written gratings have $\Delta n = 0.4 \times 10^{-3}$.

For a grating length of $L = 2$ mm $= 2 \times 10^{-3}$ m, the reflectivity is

$$R = tanh^2\left(\frac{\pi \times 0.4 \times 10^{-3} \times 2 \times 10^{-3}}{1.55 \times 10^{-3}}\right) = 0.85$$

⇒ The corresponding BW of reflection is

$$\Delta\lambda_0 = \frac{\lambda_B^2}{\pi \eta_{eff}} \sqrt{\kappa^2 L^2 + \pi^2} = 0.8\ nm$$

CO-DIRECTIONAL COUPLING: Phase-Matched

We have coupled mode equation for this case as

$$\left.\begin{aligned}\frac{dA}{dz} &= \kappa B e^{i\Gamma z}\\ \frac{dB}{dz} &= -\kappa A e^{-i\Gamma z}\end{aligned}\right\}$$

where $\Gamma = \beta_1 - \beta_2 - K$ is the phase mismatch parameter and β_1, β_2 are the propagation constants of the modes between which the coupling is to take place.

We first consider the coupling under a phase matching condition i.e., $\Gamma = 0$ ie, the periodic pshmbation has a spatial period

$$\Lambda = \frac{2\pi}{\beta_1 - \beta_2} = \frac{\lambda_0}{n_{\text{eff }1} - n_{\text{eff }2}}.$$

Under this condition,

$$\frac{dA}{dz} = \kappa B \text{ and } \frac{dB}{dz} = -\kappa A$$

which yields on differentiation

$$\frac{d^2B}{dz^2} = -\kappa^2 B.$$

The Solution of this differential equation is

$$B(z) = b_1 \cos \kappa z + b_2 \sin \kappa z;$$

And

$$A(z) = -\frac{1}{\kappa}\frac{dB}{dz} \Rightarrow A(z) = b_1 \sin \kappa z - b_2 \cos \kappa z$$

Boundary conditions:

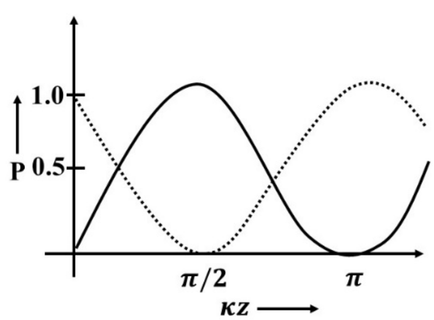
At $z = 0$, mode $1, \{E_1, \beta,\}$ is excited with unit power, $A(z = 0) = 1$ and $B(z = 0) = 0$.

$$\therefore b_1 = 0 \text{ and } b_2 = -1$$

So, $\begin{aligned}A(z) &= \cos \kappa z\\ B(z) &= -\sin \kappa z\end{aligned}$

Hence, the power carried by modes $\{E_1, \beta_1\}$ and $\{E_2, \beta_2\}$ vary with z as

$$\begin{aligned}P_1 &= |A(z)|^2 = \cos^2 \kappa z\\ P_2 &= |B(z)|^2 = \sin^2 \kappa z\end{aligned}$$



Thus, we see that a periodic exchange of power between the modes takes place. Under phase-matching condition, complete transfer of power is possible.

The length of interaction required for complete transfer is $z = L_c = \frac{\pi}{2\kappa}$.

<u>Problem:</u> Consider a planar waveguide: $n_f = 1.51; n_s = 1.50; n_c = 1.0$ and $d = 4\ \mu$m. Solve it for eigen modes at $\lambda_0 = 0.6\mu$m.

⇒ Two TE -mode will be supposed:

$$n_{\text{eff }1} = 1.50862$$

$$n_{\text{eff }2} = 1.50460$$

For a phase-matching condition to achieve complete transfer of power, we need the pitch

$$\Lambda = \frac{2\pi}{\Delta\beta} = \frac{\lambda_0}{\Delta n_{\text{eff}}} = \frac{\lambda_0}{n_{\text{eff }1} - n_{\text{eff }2}} = 149.3\mu\text{m}$$

<u>For a planar waveguide: sinusoidal perturbation:</u>

Coupling coefficient:

$$\kappa \simeq \frac{\pi}{\lambda_0} \cdot \frac{h}{\sqrt{d_1 d_2}} \cdot \sqrt{\frac{\left(n_f^2 - n_{eff_1}^2\right)\left(n_f^2 - n_{eff_2}^2\right)}{n_{eff_1} \cdot n_{eff_2}}}$$

.

Here h = amplitude of periodic thickness variation.

$$d_1 = d + \frac{1}{k_0\sqrt{n_{\text{eff1}}^2 - n_s^2}} + \frac{1}{k_0\sqrt{n_{\text{eff1}}^2 - n_c^2}}.$$

$$d_2 = d + \frac{1}{k_0\sqrt{n_{\text{eff2}}^2 - n_s^2}} + \frac{1}{k_0\sqrt{n_{\text{eff2}}^2 - n_c^2}}.$$

$d_1, d_2 \rightarrow$ effective waveguide thickness for the two modes and $n_f, n_c, n_s \rightarrow$ are the indices.

Here $d_1 = 4.678\ \mu\text{m}\ and\ d_2 = 4.897$" " µm. Assume $h = 0.01$ µm ⇒ $\kappa = 0.598\ \text{cm}^{-1}$.

So, the coupling length $L_C = \frac{\pi}{2\kappa} = 2.63$ cm

CO-DIRECTIONAL COOPLING: Phase Mismatched

Here

$$\Gamma = \beta_1 - \beta_2 - K \neq 0$$

So, from

$$\frac{dA}{dz} = \kappa B e^{i\Gamma z}$$
$$\frac{dB}{dz} = -\kappa A e^{-i\Gamma z}$$

give together

$$\frac{d^2B}{dz^2} = -\kappa \frac{dA}{dz} e^{-i\Gamma z} + i\kappa\Gamma e^{-i\Gamma z}$$

i.e.,

$$\frac{d^2B}{dz^2} + \kappa^2 B + i\Gamma \frac{dB}{dz} = 0$$

General sols:

$$B(z) = e^{-i\frac{\Gamma}{2}\cdot z}\left(b_1 e^{i\gamma z} + b_2 e^{-i\gamma z}\right)$$

.

$$r^2 = \kappa^2 + \frac{\Gamma^2}{4}$$

Thus,

$$A(z) = \frac{i}{\kappa} e^{i\Gamma/2\cdot z}\left[\left(\frac{\Gamma}{2} - \gamma\right) b_1 e^{i\gamma z} + \left(\frac{\Gamma}{2} + \gamma\right) b_2 e^{-i\gamma z}\right]$$

Boundary Conditions:

$$A(z=0) = 1 \text{ and } B(z=0) = 0.$$

So,

$$b_1 + b_2 = 0 \;\Rightarrow\; b_1 = -b_2$$

and

$$\frac{i}{\kappa}\left[\left(\frac{\Gamma}{2} - \gamma\right) b_1 + \left(\frac{\Gamma}{2} + \gamma\right) b_2\right] = 1$$

Solving

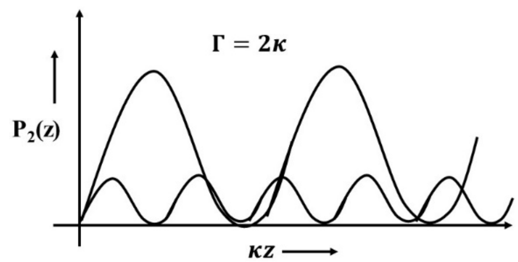
$$b_1 = \frac{i\kappa}{2\gamma} = -b_2$$

So,

$$B(z) = -\frac{\kappa}{\gamma} e^{-\frac{i}{2}\cdot z} \sin \gamma z$$
$$A(z) = e^{i\frac{\Gamma}{2}\cdot z} \left[\cos \gamma z - i \frac{\Gamma}{2\gamma} \sin \gamma z\right]$$

Thus, power in modes 1 and 2 at any value of z will be,

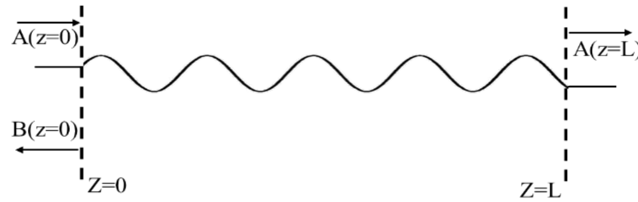
$$P_1 = |A(z)|^2 = \cos^2 \kappa z + \frac{\Gamma^2}{4\gamma^2} \sin^2 \kappa z$$
$$P_2 = |B(z)|^2 = \frac{\kappa^2}{\gamma^2} \sin^2 \kappa z$$



CONTRA-DIRECTIONAL COUPLING: phase-mismatched ($\Gamma \neq 0$)

We have the coupled mode equation for this case as

$$\left.\begin{aligned} \frac{dA}{dz} &= \kappa B e^{i\Gamma z} \\ \frac{dB}{dz} &= -\kappa A e^{-i\Gamma z} \end{aligned}\right\} \ldots\ldots (1)$$



where $\Gamma = \beta_1 + \beta_2 - K$ and $K = \frac{2\pi}{\lambda}$

Differentiating equations (1), we get

$$\frac{d^2A}{dz^2} - i\Gamma \frac{dA}{dz} - \kappa^2 A = 0 \quad \ldots\ldots (2)$$

General solution of equation (2) is $A(z) = e^{\frac{i\Gamma}{2}z}\left[Pe^{gz} + Qe^{-gz}\right]$

where $g^2 = \kappa^2 - \frac{\Gamma^2}{4}$

From (1), $B(z) = e^{-\frac{i\Gamma}{2}z}\left[\frac{(g+i\frac{\Gamma}{2})}{\kappa} Pe^{gz} - \frac{(g-i\frac{\Gamma}{2})}{\kappa} Qe^{-gz}\right]$

Now, use the boundary conditions, $A(z = 0) = 1$ (Unit power launched at input)

& B (z =L) = 0 (no coupling beyond z =L)

We obtain

$$\left.\begin{aligned} P &= \frac{(g - i\frac{\Gamma}{2})\, e^{-gL}}{2\{g\cosh(gL) \ + \ i\frac{\Gamma}{2}\sinh(gL)\}} \\ Q &= \frac{(g - i\frac{\Gamma}{2})\, e^{-gL}}{2\{g\cosh(gL) \ + \ i\frac{\Gamma}{2}\sinh(gL)\}} \end{aligned}\right\} \ldots\ldots (3)$$

So, the reflectivity of the periodic structure is

$$\left.\begin{aligned} R &= \frac{|B(0)|^2}{|A(0)|^2} = \frac{\kappa^2 \, sinh^2(gL)}{g^2 \ cosh^2(gL) \ + \ \frac{\Gamma^2}{4} \ sinh^2(gL)} \\ \& \ \ T &= \frac{|A(L)|^2}{|B(L)|^2} = \frac{g^2}{g^2 \ cosh^2(gL) \ + \ \frac{\Gamma^2}{4} \ sinh^2(gL)} \end{aligned}\right\} \ldots\ldots (4)$$

$\Delta\lambda$: wavelength spacing between the minima

$g^2 = +ve$ if $\kappa^2 > \frac{\Gamma^2}{4}$ (center wavelength λ_B for which $\Gamma = 0$, $g = \kappa$)

As we deviate from λ_B, Γ increases.

And when $\Gamma^2 > 4\kappa^2$, g^2 becomes negative.

i.e., $g^2 = -ve$ when $\Gamma^2 > 4\kappa^2$

when $g^2 = -ve$, hyperbolic functions in R & T become ordinary sin & cosine functions.

Thus, for $\Gamma^2 > 4\kappa^2$,

$$R = \frac{\kappa^2 \sin^2(\tilde{g}L)}{\tilde{g}^2 \cos^2(\tilde{g}L) + \frac{\Gamma^2}{4}\sin^2(\tilde{g}L)} \ldots\ldots (1)$$

(where $\tilde{g}^2 = -g^2$)

The reflectivity $'R'$ besoms zero when-

$\sin(\tilde{g}L) = 0 \quad \Rightarrow \tilde{g}L = m\pi\,; m = 1,2,3\ldots.$ (Zeros in reflected spectrum)

Substituting for $\tilde{g}$,

$$\frac{\Gamma^2}{4} - \kappa^2 = \frac{m^2\pi^2}{L^2}$$

And $\quad \Gamma = \beta_1 + \beta_2 - K = \pm 2\sqrt{\kappa^2 + \frac{m^2\pi^2}{L^2}} \ldots\ldots\ldots (2)$

Now, for contra directional coupling between same modes

$$\beta_1 = \beta_2 = \frac{2\pi}{\lambda_0} n_{eff}$$

Thus,

$$\left(\frac{4\pi}{\lambda_0} n_{\mathrm{eff}} - \frac{2\pi}{\Lambda}\right) = \pm 2\sqrt{\kappa^2 + \frac{m^2\pi^2}{L^2}} \ldots\ldots (3)$$

$m = 1 \Rightarrow$ first zero on either side of λ_B.

$\lambda_B =$ center wavelength $= 2n_{\mathrm{eff}}\Lambda$.

So, substitute $\lambda_0 = \lambda_B \pm \frac{\Delta\lambda}{2}$ where $\Delta\lambda =$ deviation from λ_B to minima when $R = 0$

Putting λ_0 in equation (3) and using $\Delta\lambda \ll \lambda_B$ and $\lambda_B = 2n_{eff}\Lambda$

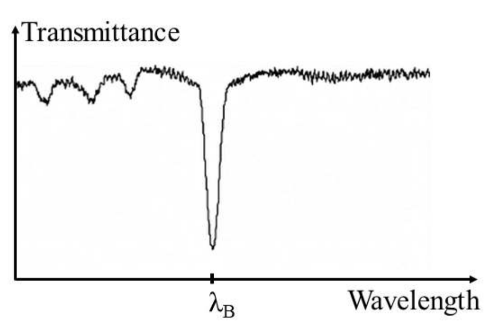
$$\left(\frac{4\pi}{\lambda_0} n_{\mathrm{eff}} - \frac{2\pi}{\Lambda}\right) = \left\{\frac{4\pi n_{\mathrm{eff}}}{\lambda_B}\left(1 \pm \frac{\Delta\lambda}{2\lambda_B}\right) - \frac{2\pi}{\Lambda}\right\} = \pm \frac{2\pi n_{\mathrm{eff}}\,\Delta\lambda}{\lambda_B^2} \ldots\ldots (4)$$

From equations (3) and (4):

$$\Delta\lambda = \frac{\lambda_B^2}{\pi n_{\mathrm{eff}} L} \cdot (\kappa^2 L^2 + \pi^2)^{\frac{1}{2}}$$

In other form:

$$\frac{\Delta\lambda}{\lambda_B} = \frac{2}{\pi}\left(\frac{\Lambda}{L}\right)\sqrt{\kappa^2 L^2 + \pi^2}$$



FIBER BRAGG GRATING:

$\beta_g - (-\beta_g) = K = \frac{2\pi}{\Lambda}$ (Λ = spatial frequency)

$2 \times \frac{2\pi}{\lambda_B} n_{eff} = \frac{2\pi}{\Lambda}$

$\lambda_B = 2\Lambda\, n_{eff}$

Consider a SIF: n_2=1.45, $a = 3\mu$m, $NA = 0.1$, λ_c for $LP_{11} = 0.784$ μm. ($\lambda_B = 2\Lambda \cdot n_{\text{eff}}$)

⇒At 850 mm n_{eff} of $LP_{01} = 1.4517$

$$\Lambda = \frac{\lambda_B}{2n_{\text{eff}}} = 0.293\ \mu\text{m}.$$

Reflation coefficient:

$$R = \tanh^2 \kappa L \quad , L \rightarrow \text{Length of the grating}$$

$$\kappa = \frac{\pi \Delta n}{\lambda_B} \cdot I \quad ,$$

$I \rightarrow$ Overlap Integral of the modes distribution over the regain of the grating.

$= 1$ for a plane wave in uniform grating

$$\Delta\lambda = \frac{\lambda_B^2}{\pi\, n_{\text{eff}}\, L} \cdot (\kappa^2 L^2 + \pi^2)^{\frac{1}{2}}$$

We need an FBG at 800 mm, with $R = 90\%$ with $L = 25$ mm. Calculate the bandwidth. Assume $I = 0.5$

$$\Rightarrow \quad \tanh \kappa L = \sqrt{0.9} \qquad \kappa = \frac{1}{2L} \cdot \ln\left(\frac{1+\sqrt{0.9}}{1-\sqrt{0.9}}\right) \simeq 0.073\ \text{mm}^{-1}.$$

Assuming $I = 0.5$, $\Delta n \simeq 3.72 \times 10^{-5}$

The corresponding $BW = 0.02$ mm.