Evaluation of double integrals by changing cartesian to polar coordinates (change of variables).

Let the variables x, y in the double integral $\iint_R f(x,y)\,dx\,dy$ be changed to u, v by means of the relations $x = \phi(u,v)$, $y = \psi(u,v)$, then the double integral is transformed into $\iint_{R'} f\{\phi(u,v), \psi(u,v)\}\,|J|\,du\,dv$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the Jacobian of transformation from (x, y) to (u, v) coordinates and R' is the region in the uv-plane which corresponds to the region R in the xy-plane.

(i) To change Cartesian Coordinates (x, y) to polar co-ordinates (r, θ).

Here we have $x = r\cos\theta$, $y = r\sin\theta$

So that $x^2 + y^2 = r^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

$$\therefore \iint_R f(x, y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

ie replace x by $r\cos\theta$, y by $r\sin\theta$ and $dx\,dy$ by $r\,dr\,d\theta$.

(ii) To change Cartesian Coordinates (x, y, z) to spherical polar Coordinates (r, θ, φ).

Here, we have $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

So that $x^2+y^2+z^2 = r^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\therefore \iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\varphi$$

(iii) To change Cartesian Co-ordinates (x, y, z) to cylindrical Co-ordinates (r, φ, z).

Here we have $x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\varphi + \sin^2\varphi) = r$$

$$\therefore \iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V f(r\cos\varphi, r\sin\varphi, z)\, r\, dr\, d\varphi\, dz$$