Course Title: ***Engineering Physics***

Course Code: **BPHY101L** Slot: Class No.:

Module:2	Electromagnetic waves	7 hours
Physics of divergence - gradient and curl - Qualitative understanding of surface and volume integral - Maxwell Equations (Qualitative) - Displacement current - Electromagnetic wave equation in free space - Plane electromagnetic waves in free space - Hertz's experiment.		

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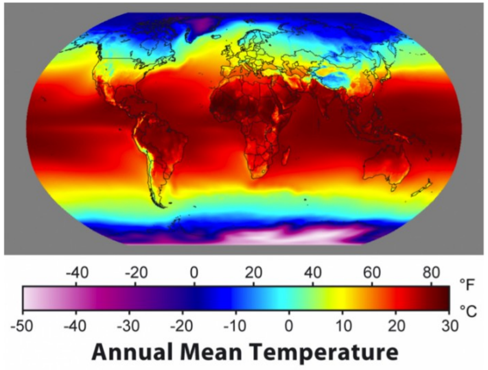
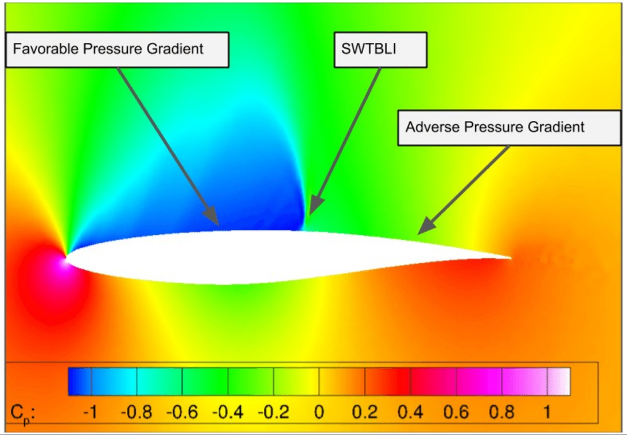
Scalar and Vector Fields

In mathematics and physics, a **scalar field** *is a function associating a single number to every point in a space – possibly physical space.* The scalar may either **be a pure mathematical number (dimensionless)** or **a scalar physical quantity (with units).**

Examples: Temperature distribution throughout space, the pressure distribution in a fluid.

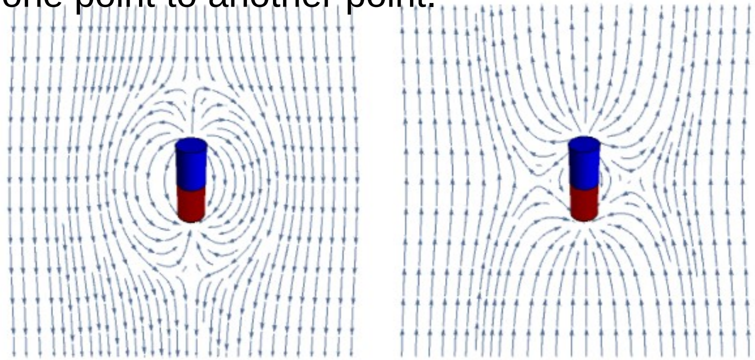




Vector field

In vector calculus and **physics**, **a vector field** is an *assignment of a vector to each point in a region of space.*

- **A vector field** in the plane can be ***visualized as a collection of arrows with a given magnitude and direction, each attached to a point in the plane.***
- Often **used to model**, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as *the magnetic or gravitational force*, as it changes from one point to another point.



Vector fields are commonly used to create patterns in computer graphics. Here: abstract composition of curves following a vector field generated with OpenSimplex noise.

Physics of **Gradient**, **Divergence** & **Curl**

Discussion on **partial derivative** (mathematical form and physical meaning)**-Del operator.**

- The **del symbol (**or **nabla** ∇**)** can be **interpreted as a vector of partial derivative operators.**

- *When* ***applied*** *to a* ***field*** (a function defined on a multi-dimensional domain), it *may denote* any **one of three operators** *depending on the way it is applied*: **the gradient** or (locally) ***steepest slope of a scalar field***; the **divergence of a vector field**; or the **curl (*rotation*) of a vector field**.

Del (∇) Operator

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Partial derivative (∂) vs Total derivative (D)

- **A partial derivative** *of a function of several variables* **is its derivative with respect to one of those variables, with the others held constant** (as opposed to the **total derivative**, in which **all variables are allowed to vary**)**.**

- Unlike partial derivatives, **the total derivative approximates the function with respect to all of its arguments, not just a single one.**

- In many situations, this is the **same as considering all partial derivatives simultaneously.**

- The term **"total derivative" is** primarily **used when f is a function of several variables**

For example, suppose

$$f(x,y) = xy.$$

The rate of change of *f* with respect to *x* is usually the partial derivative of *f* with respect to *x*; in this case,

$$\frac{\partial f}{\partial x} = y.$$

However, if *y* depends on *x*, the partial derivative does not give the true rate of change of *f* as *x* changes because the partial derivative assumes that *y* is fixed. Suppose we are constrained to the line

$$y = x.$$

Then

$$f(x,y) = f(x,x) = x^2,$$

and the total derivative of *f* with respect to *x* is

$$\frac{df}{dx} = 2x,$$

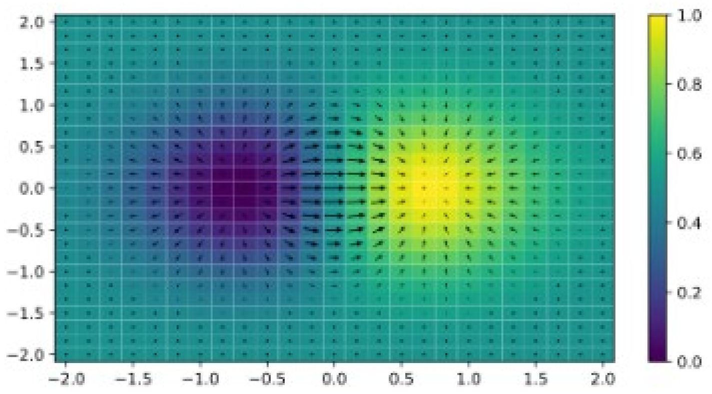
which we see is not equal to the partial derivative $\partial f/\partial x$. Instead of immediately substituting for *y* in terms of *x*, however, we can also use the chain rule as above:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = y + x \cdot 1 = x + y = 2x.$$

Gradient (∇f)

The gradient of a *scalar-valued differentiable function f* of several variables **is the vector field** (or vector-valued function) **∇f** whose value at a point "p" is the "**direction and rate of fastest increase**".

- The **gradient ∇f** points in the direction of **maximum increase of the function f.**
- **The magnitude |∇f| gives the slope** (rate of increase) along this maximal direction.



Gradient of the 2D function $f(x, y) = xe^{-(x^2+y^2)}$ is plotted as arrows over the pseudocolor plot of the function.

Gradient of a function in different coordinate systems

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$ In Cartesian Coordinate System

$$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$ In Spherical Coordinate System

$$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$ In Cylindrical Coordinate System

Divergence of a Vector Field ($\nabla . \vec{F}$)

Let consider a vector field, $\overrightarrow{F}(x, y, z)$, then its divergence is calculated as $div\ \overrightarrow{F}(x, y, z) = \nabla . \overrightarrow{F}$

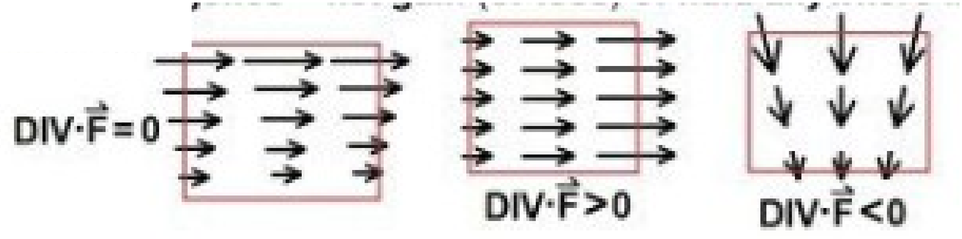
$$\nabla . \overrightarrow{F}(x, y, z) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

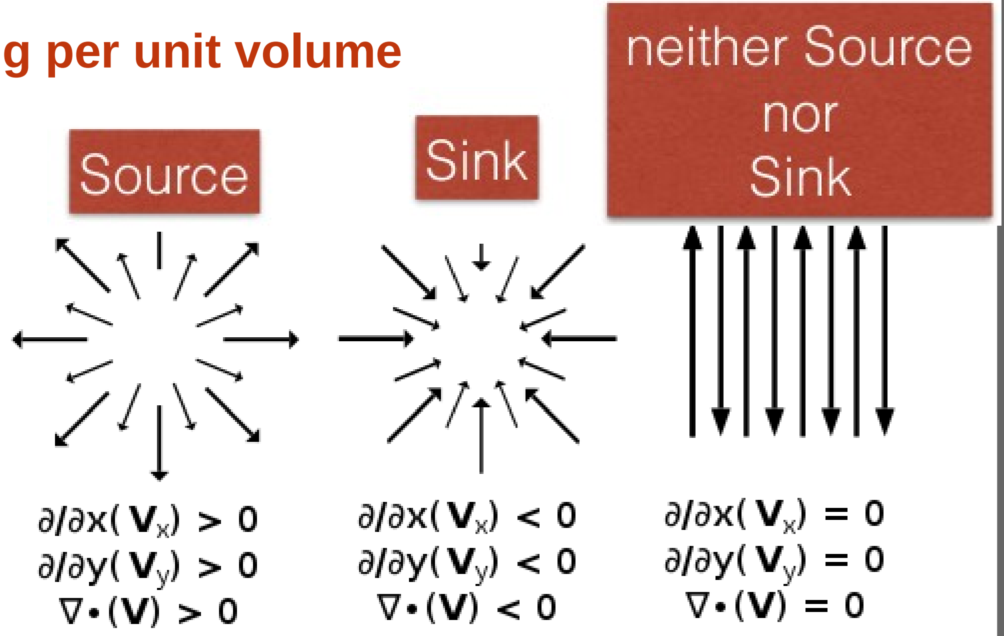
$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

A key point: F is a vector and the divergence of F is a scalar.

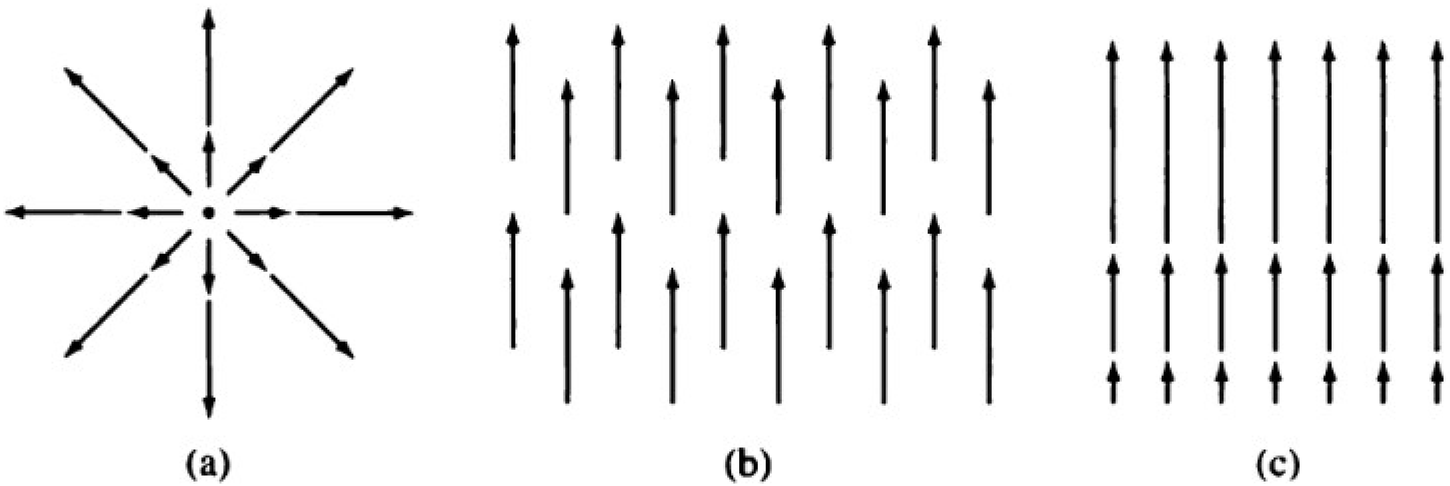
Physical significance of a Divergence ($\nabla . \vec{F}$)

- It is a operation **performed on a vector**, **results into a scalar**
- It **tells**, **how much flux is entering/leaving per unit volume**

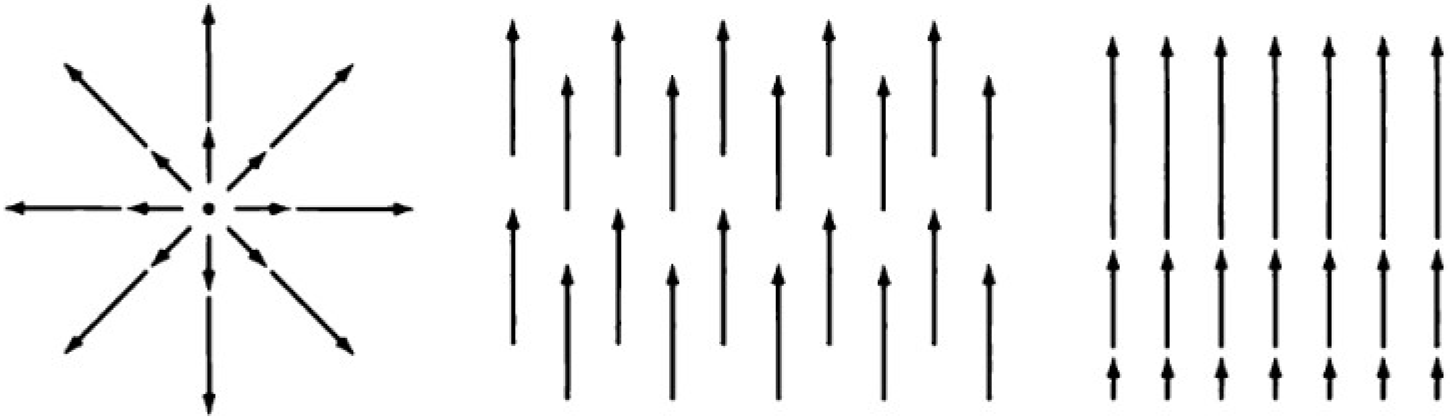




Divergence ?



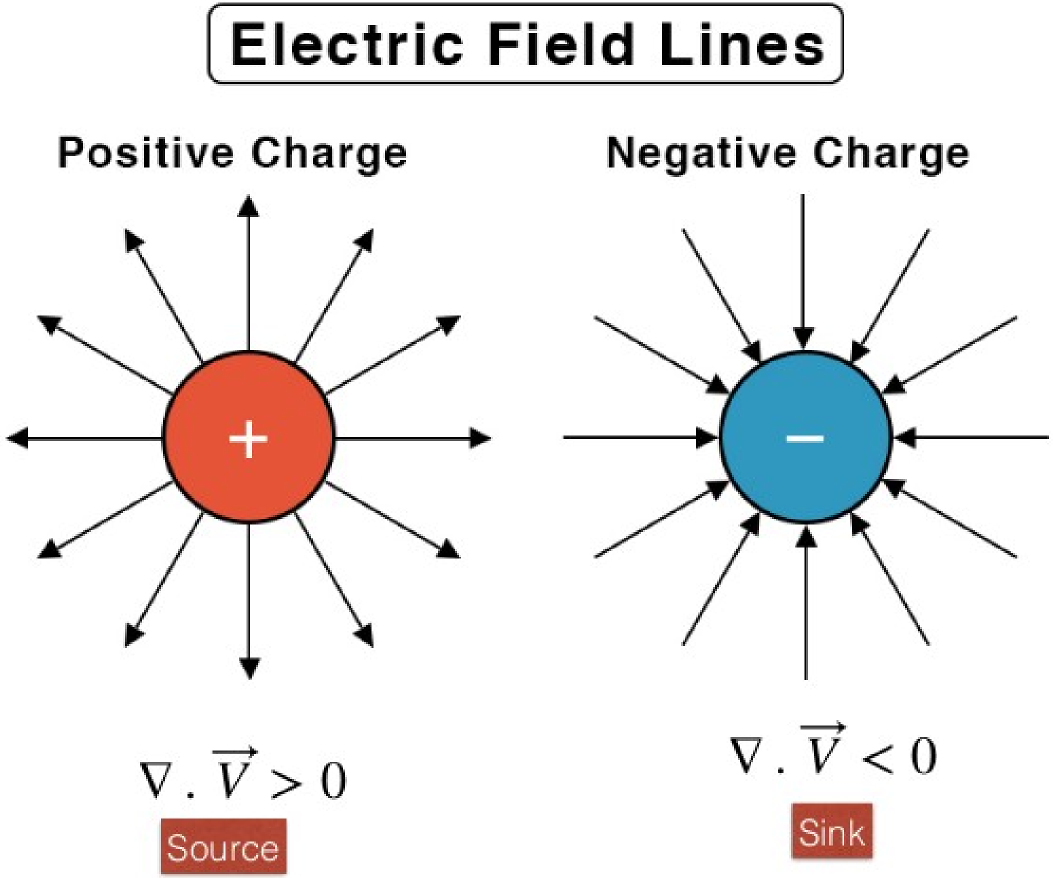
Divergence ?



(a) (b) (c)

Positive **Zero** **Positive**

Divergence of a Vector Field ($\nabla . F$)



Curl of a Vector field ($\nabla \times \vec{F}$)

- Consider the vector fields

$$\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$$

- The curl of **F** is another vector field defined as:

$$\boxed{\textbf{curl } \vec{\mathbf{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}}$$

- In terms of the differential operator ∇, the curl of **F**

$$\boxed{\textbf{Curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

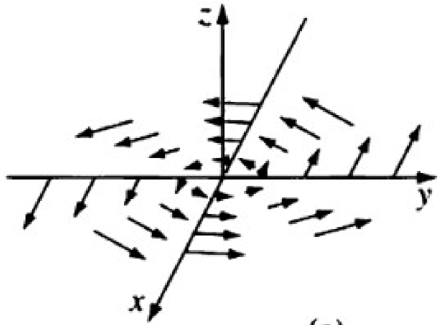
- A key point: **F** is a vector and the **curl** of **F** is a **vector**.

Direction of curl is perpendicular to the plane of circulation

Physical significance of a Curl (∇x$\vec{v_a}$)

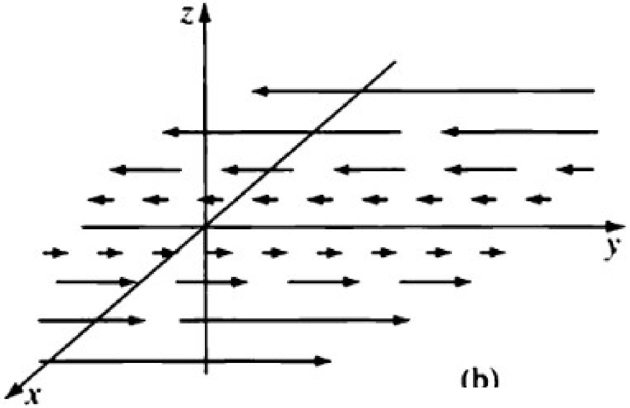
It is a measure of how much the vector field ***F*** *"curls around"* the point in question.

$$\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$$



(a)

$$\mathbf{v}_b = x\hat{\mathbf{y}}$$



(b)

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{\mathbf{z}}.$$

Divergence and Curl in different coordinate systems

In Cartesian Coordinate System

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

In Spherical Coordinate System

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Divergence and Curl in different coordinate systems

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

In Cylindrical Coordinate System

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Second Derivative

The gradient ∇T is a *vector*, so we can take the *divergence* and *curl* of it:

(1) Divergence of gradient: $\nabla \cdot (\nabla T)$.

(2) Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\nabla \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient:*

(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$.

The curl $\nabla \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl:*

(4) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$.

(5) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

Second Derivative cont'd...

(1) $$\nabla \cdot (\nabla T) = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}.$$

Laplacian

(2) The curl of a gradient is always zero:

Important $$\nabla \times (\nabla T) = \mathbf{0}.$$

(3) $\nabla(\nabla \cdot \mathbf{v})$ seldom occurs in physical applications, and it has not been given any special name of its own—it's just **the gradient of the divergence.**

$\nabla(\nabla \cdot \mathbf{v})$ is *not* the same as the Laplacian of a vector: $\nabla^2\mathbf{v} = (\nabla \cdot \nabla)\mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$.

Second Derivative cont'd...

(4) The divergence of a curl, like the curl of a gradient, is always zero:

Important $\nabla \cdot (\nabla \times \mathbf{v}) = 0.$ Using $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}.$)

(5) As you can check from the definition of ∇:

Important $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}.$

Using **Vector triple product:** $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ so-called **BAC-CAB** rule:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

Vector Integral calculus

Calculus is a branch that deals with the study of the rate of change of a function. Calculus plays an integral role in many fields such as Science, Engineering, Navigation, and so on. Generally, calculus is used to develop a Mathematical model to get an optimal solution

- Differential Calculus
- Integral Calculus

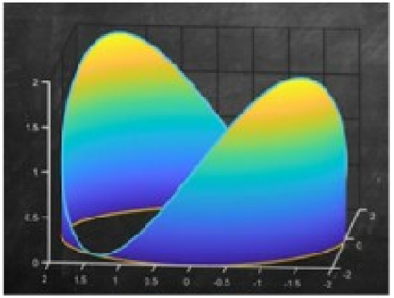
Vector Calculus, also known as vector analysis, deals with the differentiation and integration of vector fields, especially in three-dimensional space. Vector analysis is an analysis that deals with quantities that have both magnitude and direction. Vector calculus deals with three types:

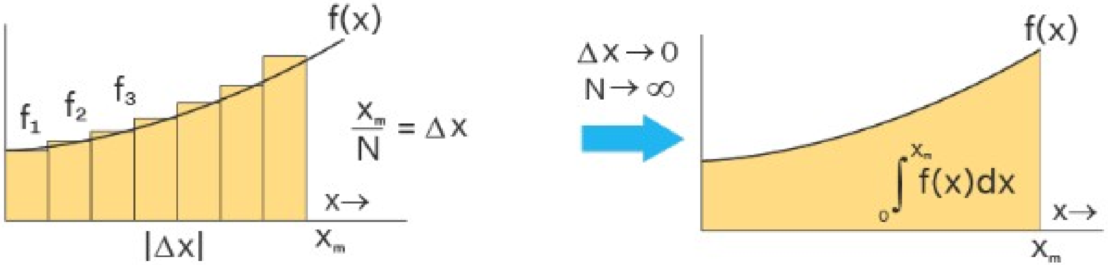
- Line Integral
- Surface Integral
- Volume Integral

Integration

For a function, f(x), the integration can be defined for a point a—to—b is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$





$$\text{Area} = \int_0^{X_m} f(x)dx = \lim_{\Delta x\to 0}\sum_{i=1}^{N} f_i(x)\Delta x$$

The integration denotes the summation of discrete data

Line Integral (of a Scalar Field/ function)

A line integral is integral in which the function to be integrated is determined along a curve in the coordinate system. The function which is to be integrated may be either a scalar field or a vector field. We can integrate a scalar-valued function or vector-valued function along a curve. The value of the line integral can be evaluated by adding all the values of points on the vector field.

Consider a scalar function $\phi(x, y)$. In this case, the line-integral is defined as

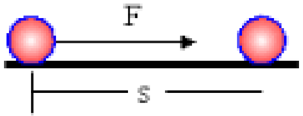
$$\int_A^B \phi(x,y)dr$$

where dr is the infinitesimal line segment along a specific path between A and B

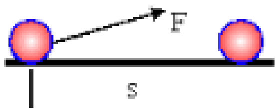
If the path is closed, in which case A = B, we write the integral as

$$\oint_C \phi(x,y)dr$$

Line Integral (of a Vector Field)



Work done by a uniform force in the displacement direction: W = Fs



Work done by a uniform force out of the displacement direction: W = Fs cos Θ

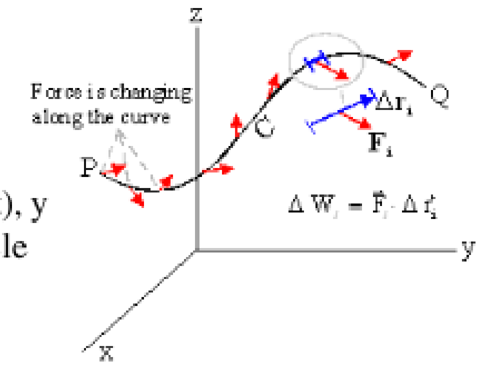
If the function is a vector the line-integral then defined as

$$\int_A^B \vec{F} \cdot d\vec{r}$$

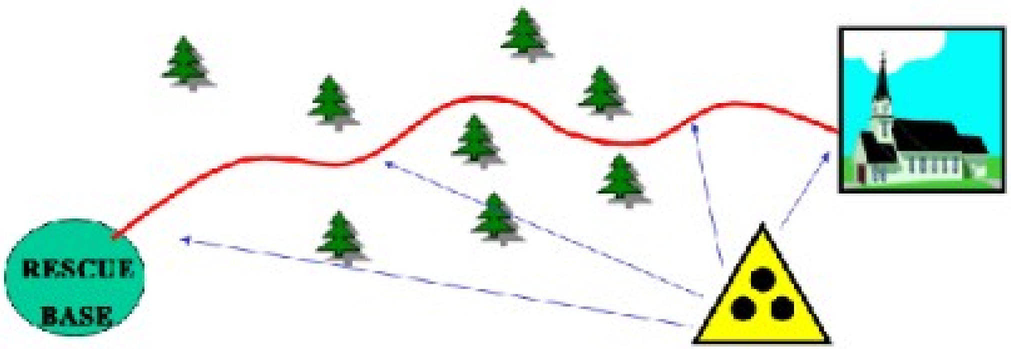
Note that this time angle between the line segment dr and the vector filed F becomes relevant. If the path is closed, like before, A = B, we write the integral is written as:

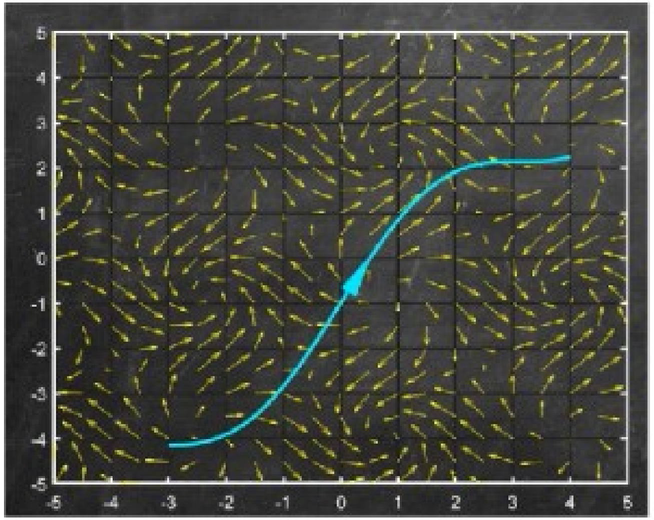
$$\oint_C \vec{F} \cdot d\vec{r}$$

Suppose F = F (x, y, z) is the force acting on a particle which moves along the curve C given by r = [x (t), y (t), z (t)], and with its initial point at P and terminal point Q: What is the work done in moving the particle form P to Q



A rescue team follows a path in a danger area where for each position the degree of radiation is defined. Compute the total amount of radiation gathered by the rescue team along the path.





charge particle in a electric filed

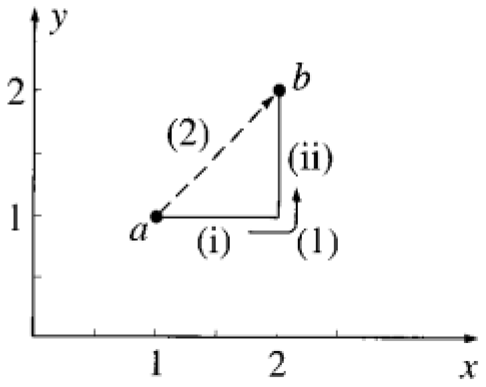
Example of Line Integral

Lets calculate the line integral of $\overrightarrow{V} = y^2\hat{\imath} + 2x(y+1)\hat{\imath}$ from point a(1, 1,0) to point b(2, 2, 0)

$$\int_A^B \vec{F} \cdot d\vec{r}$$

P: path of integral

$$\overrightarrow{dl} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$



along path (i); dy=0 $\Rightarrow$ $\int_a^b \vec{v} \cdot \overrightarrow{dl} = 1$

along path (ii); dx=0 $\Rightarrow$ $\int_a^b \vec{v} \cdot \overrightarrow{dl} = 10$

along path (2);x=y, dx=dy

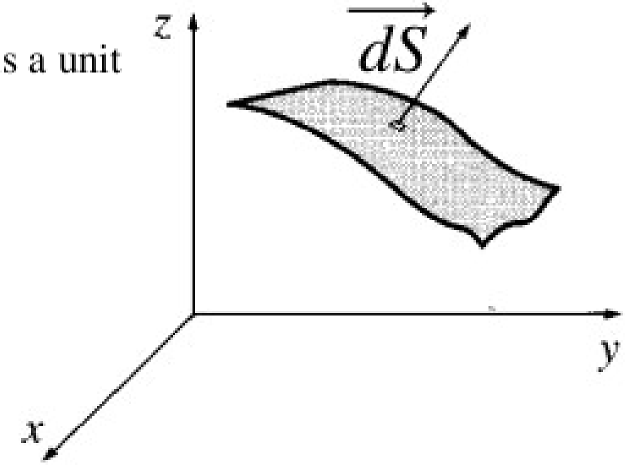
$$\Rightarrow \int_a^b \vec{v} \cdot \overrightarrow{dl} = 10$$

Surface Integral

Surface integral of a vector field F (x, y, z) along the surface S is defined as

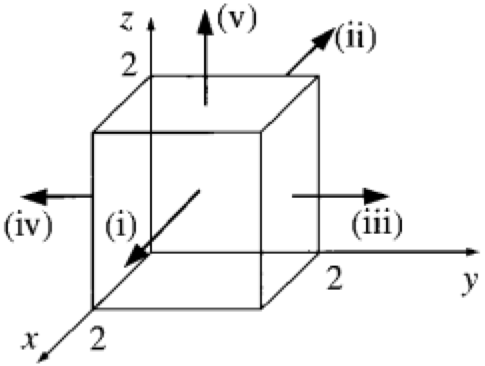
$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \left(\vec{F}\cdot\hat{n}\right) dS$$

where $\overrightarrow{dS}$ is an infinitesimal vector perpendicular to the surface element and $\hat{n}$ is a unit vector in the direction of $\overrightarrow{dS}$.



Surface Integral cont'd...

If $\vec{\mathbf{F}}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{\mathbf{F}} \cdot \overrightarrow{dS}$ represents total mass per unit time passing through the surface.



$(i)\ \overrightarrow{dS} = \hat{x}\ dydz$

$(ii)\ \overrightarrow{dS} = -\hat{x}\ dydz$

$(iii)\ \overrightarrow{dS} = \hat{y}\ dxdz$

$(iv)\ \overrightarrow{dS} = -\hat{y}\ dxdz$

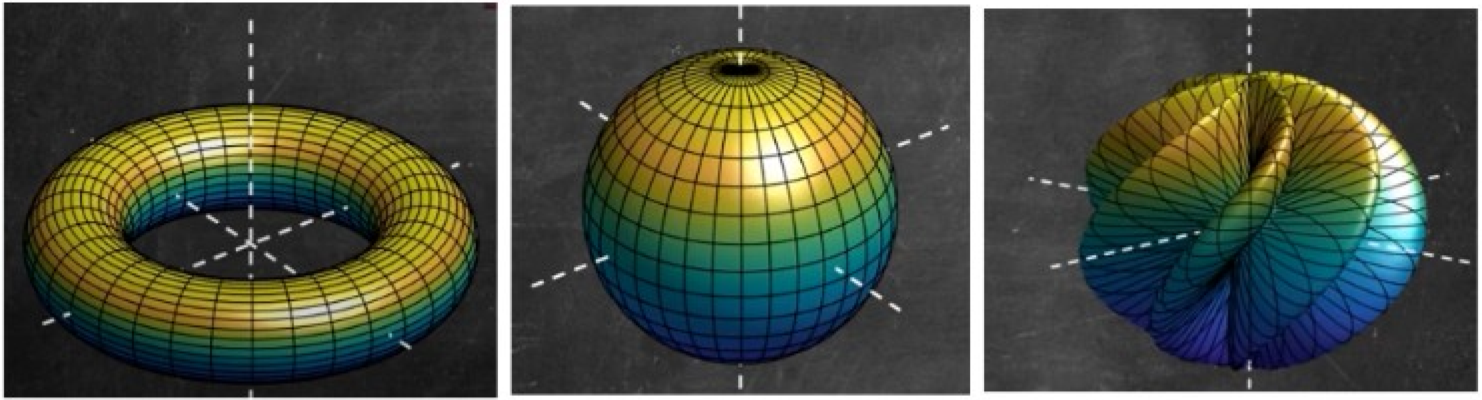
$(v)\ \overrightarrow{dS} = \hat{z}\ dxdy$

$(vi)\ \overrightarrow{dS} = -\hat{z}\ dxdy$

Surface Integral over closed surface cont'd...

Surface integral of a vector field $\vec{F}$ (x, y, z) along the surface S, and the **surface is closed**, then it is defined as

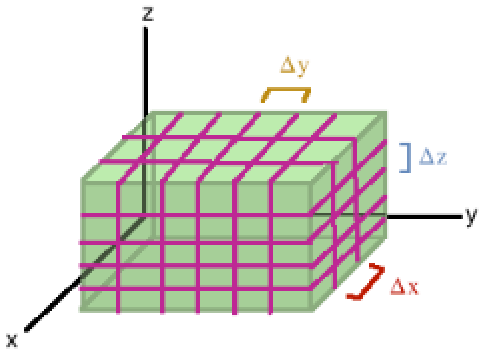
$$\oiint \vec{F}(x, y) \cdot d\vec{S}$$

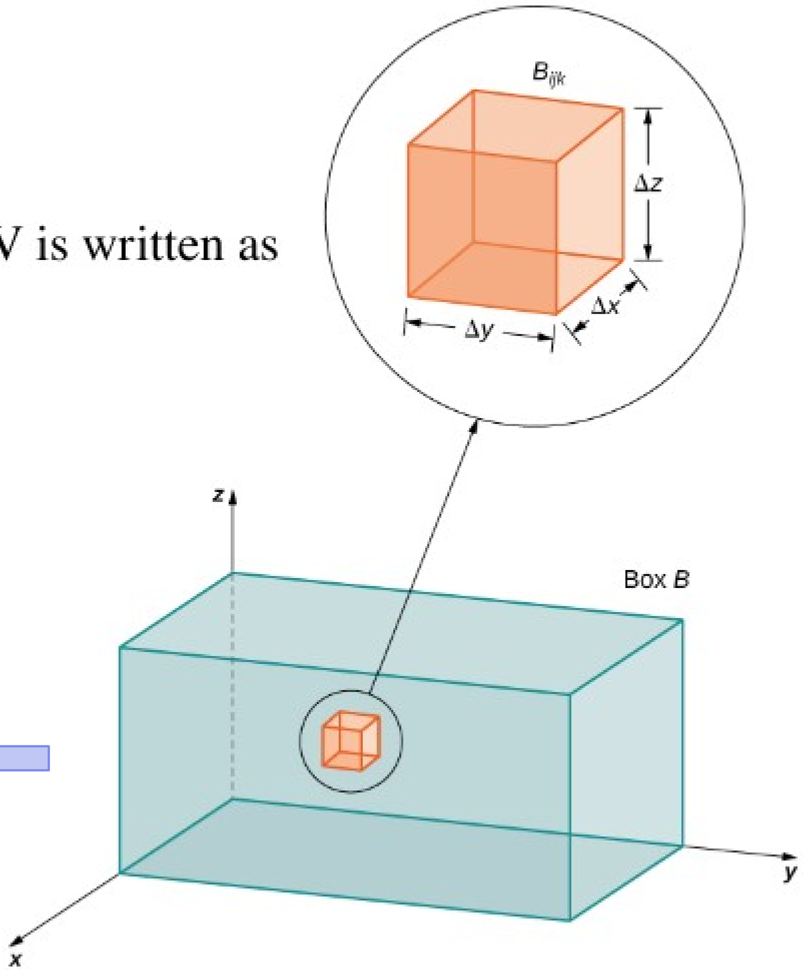


Volume Integral

Volume integral of a scalar field f(x,y) within a region V is written as

$$\iiint_{\mathcal{V}} f(x,y,z)dV = \iiint_{\mathcal{V}} f(x,y,z)dxdydz$$





Sample Problem:

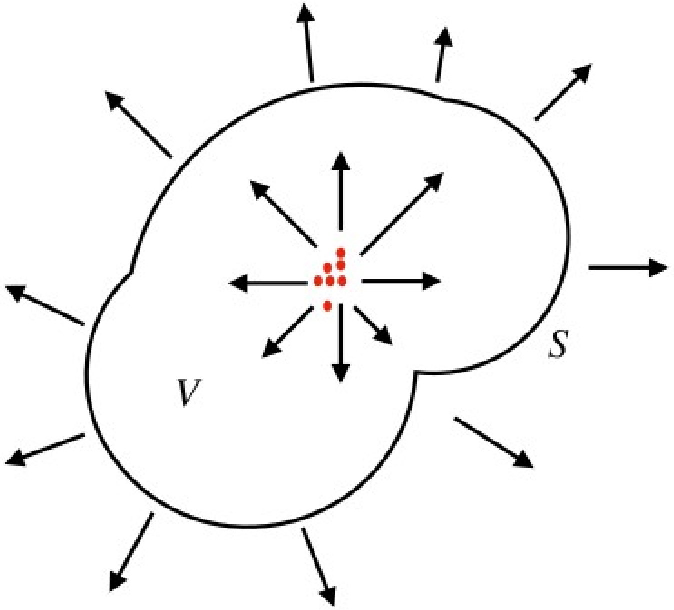
Prove that the following vector field $\overrightarrow{F}$ is conservative:

$$\overrightarrow{F} = \left(4y^2 + \frac{3x^2y}{z^2}\right)\hat{i} + \left(8xy + \frac{x^3}{z^2}\right)\hat{j} + \left(11 - \frac{2x^3y}{z^3}\right)\hat{k}$$

Gauss Divergence Theorem **or** Divergence Theorem

The normal surface integral of a vector function $\vec{F}(x, y, z)$ over the boundary of a closed region is equal to the volume integral of the divergence of the vector function, $\vec{\nabla} \cdot \vec{F}(x, y, z)$, taken throughout that region

$$\oint_S \vec{F} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{F}) dV$$



Statement:

Total outward flux of $\vec{F}$ through a closed surface S is equal to the volume integral of the divergence of $\vec{F}$ over volume V enclosed by S.

Gauss Divergence Theorem cont'd...

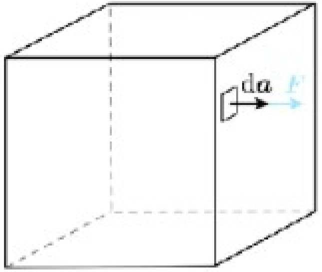
$$\oint_S \vec{F} \cdot \vec{dS}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\vec{dS} = dS_x\hat{i} + dS_y\hat{j} + dS_z\hat{k}$$

$$\vec{F} \cdot \vec{dS} = F_x dS_x + F_y dS_y + F_z dS_z$$

$$= Flux = \Phi$$

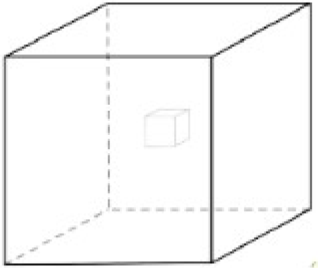


$$\int_V (\vec{\nabla} \cdot \vec{F})dV$$

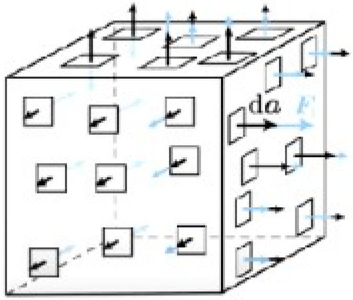
$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$dV = dxdydz$$

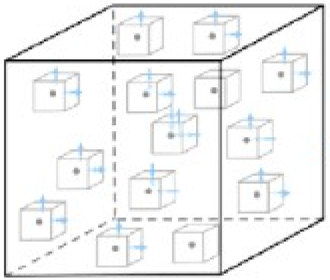
$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$



$$\oint_S \vec{F} \cdot \vec{dS} =$$



$$=$$



$$= \int_V (\vec{\nabla} \cdot \vec{F})dV$$

Stokes' theorem

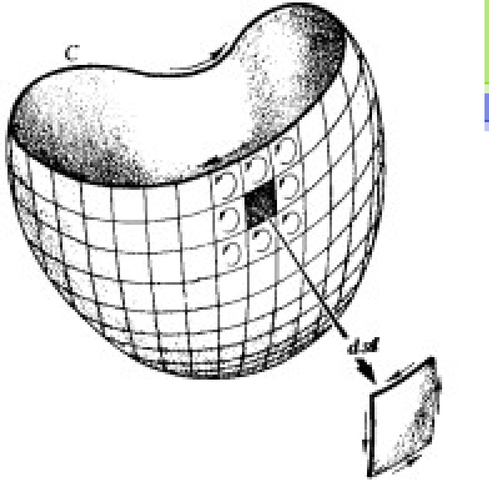
Circulation of a vector field function, $\vec{F}$ around a closed path L, is equal to the surface integral of the curl $\vec{F}$, $(\vec{\nabla} \times \vec{F})$ over the surface surface S bounded by L.

$$\boxed{\oint_L \vec{F} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{dS}}$$

Statement:

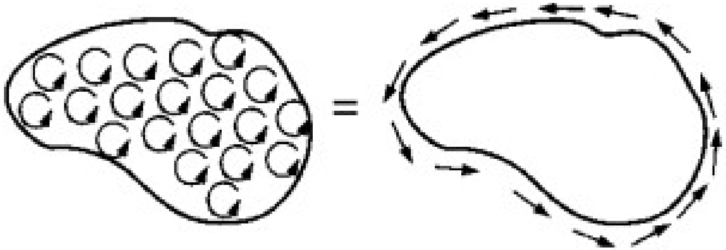
The surface integral of the curl of a vector function over a surface bounded by a closed surface is equal to the line integral of the particular vector function around that surface.

Stokes' theorem cont'd...



RHS: Flux of the curl through the surface S →total amount of swirls

LHS: amount of flow following the boundary.



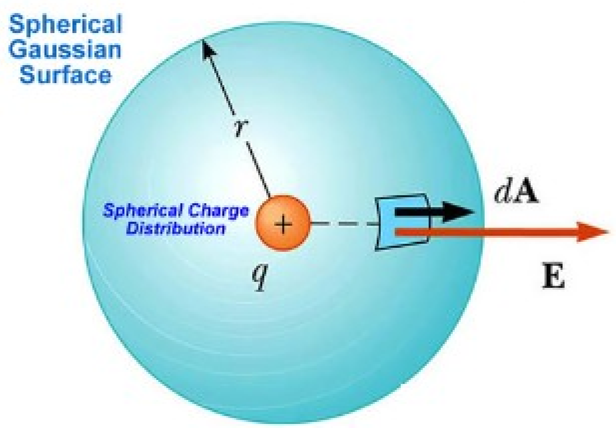
It relates the microscopic circulation of a vector field with the macroscopic circulations

Integral forms of Electricity and Magnetism (Before Maxwell)

1. Gauss's Law

The net electric flux through a closed surface (3D) is $\dfrac{1}{\epsilon_0}$ times the net charge enclosed by the surface

$$\boxed{\oint \vec{E} \cdot \vec{da} = \frac{Q_{encl}}{\epsilon_0}}$$

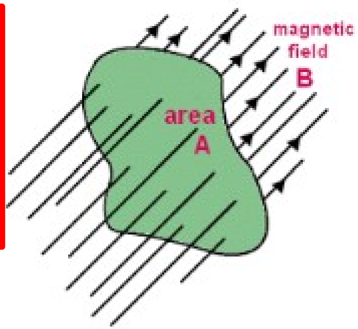


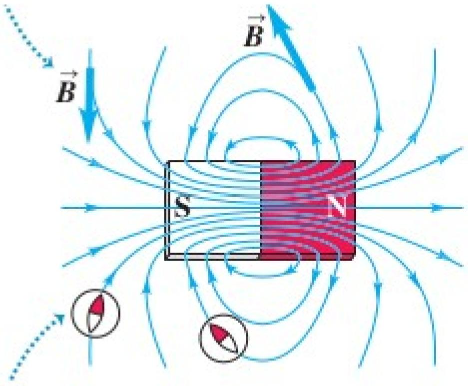
Integral forms of Electricity and Magnetism (Before Maxwell) cont'd...

2. Gauss's Law of magnetism

The net magnetic flux through any closed surface is Zero

$$\boxed{\oint \vec{B} \cdot \vec{da} = 0}$$





Means

Magnetic monopoles does not exist

Integral forms of Electricity and Magnetism (Before Maxwell) cont'd...

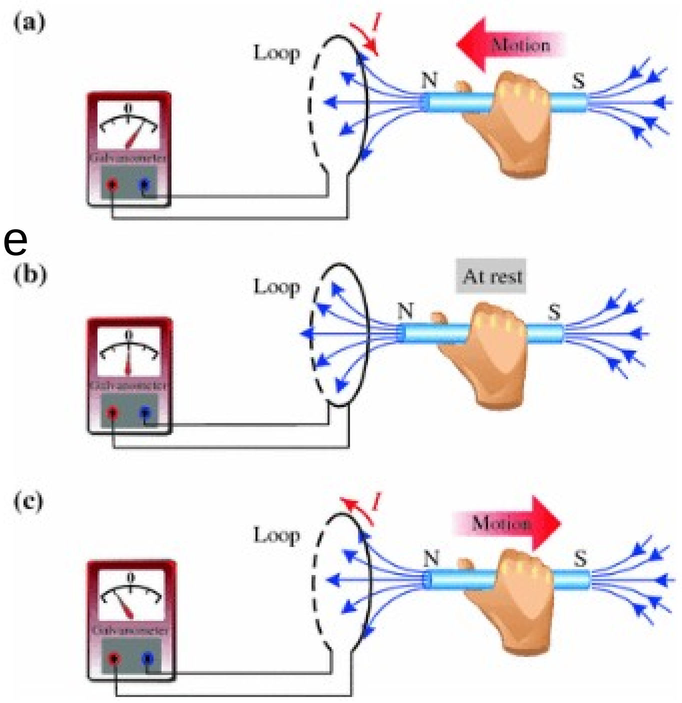
3. Faradays' Law

Whenever a conductor is placed in a varying magnetic field, an electromotive force is induced. If the conductor circuit is closed, a current is induced, which is called induced current." the induced emf in a coil is equal to the rate of change of flux linkage

$$\varepsilon = -\frac{\partial \Phi_B}{\partial t}$$

$$\boxed{\oint \vec{E} \cdot \vec{dl} = -\frac{\partial \Phi_B}{\partial t}}$$

Change in Magnetic field create a current

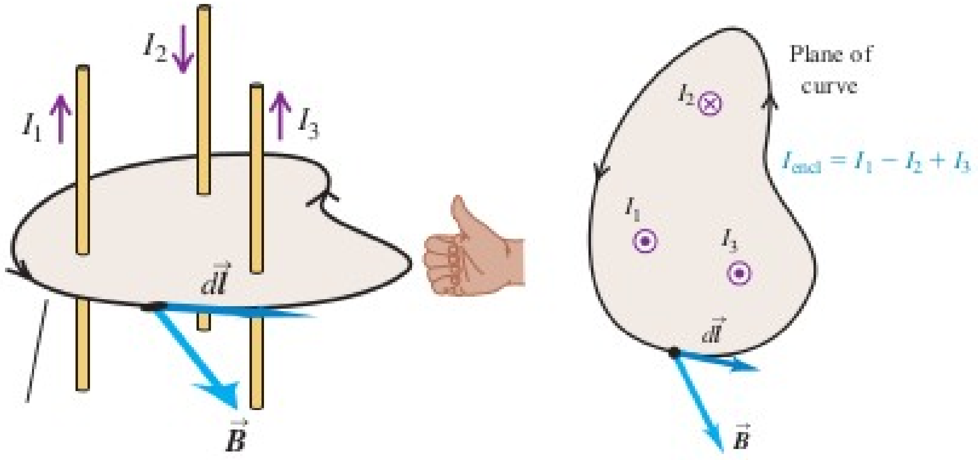


Integral forms of Electricity and Magnetism (Before Maxwell) cont'd...

4. Amperes' Law

The magnetic field in space around an electric current is proportional to the electric current which serves as its source

$$\boxed{\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{encl}}$$



It relates the net magnetic field along a closed loop to the electric current passing through the loop

Integral and Differential form Maxwell equations

Integral form		Differential form
Gauss's Law $\oint \vec{E} \cdot \vec{da} = \frac{Q_{encl}}{\epsilon_0}$	**Using Gauss divergences theorem** →	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Gauss's Law for magnetism $\oint \vec{B} \cdot \vec{da} = 0$	**Using Gauss divergences theorem** →	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday's Law $\oint \vec{E} \cdot \vec{dl} = -\frac{\partial \Phi_B}{\partial t}$	**Using Stoke's theorem** →	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampere's Law $\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{encl}$	**Using Stoke's theorem** →	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Transformation of Maxwell eqns. from Integral to Differential form

Gauss's Law

$$\boxed{\oint \vec{E} \cdot \vec{da} = \frac{Q_{encl}}{\epsilon_0}}$$

Integral form

---(1)

Using Gauss divergences theorem, the L.H.S can be written as

$$\Rightarrow \oint_S \vec{E} \cdot \vec{da} = \int_V (\vec{\nabla} \cdot \vec{E})dV$$

---(2)

Substituting (2) in (1), we get

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{E})dV = \frac{Q_{encl}}{\epsilon_0}$$

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{E})dV = \frac{1}{\epsilon_0}\int_V \rho dV$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$

Differential form

Transformation of Maxwell eqns. from Integral to Differential form

Gauss's Law of Magnetism

$$\oint \vec{B} \cdot \vec{da} = 0$$

Integral form

---(1)

Using Gauss divergences theorem, the L.H.S can be written as

$$\Rightarrow \oint_S \vec{B} \cdot \vec{da} = \int_V (\vec{\nabla} \cdot \vec{B})dV \quad \text{---(2)}$$

Substituting (2) in (1), we get

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{B})dV = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

Differential form

Transformation of Maxwell eqns. from Integral to Differential form

Faraday's Law

$$\varepsilon = -\frac{\partial \Phi_B}{\partial t}$$

$$\Rightarrow \oint \vec{E} \cdot \vec{dl} = -\frac{\partial \Phi_B}{\partial t}$$

Integral form

---(1)

Using Stoke's theorem, the L.H.S can be written as

$$\Rightarrow \oint_L \vec{E} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS}$$

---(2)

Substituting (2) in (1), we get

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS} = -\frac{\partial \Phi_B}{\partial t}$$

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \vec{dS}$$

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{dS}$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Differential form

Transformation of Maxwell eqns. from Integral to Differential form

Ampere's Law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{encl}$$

Integral form

---(1)

Using Stoke's theorem, the L.H.S can be written as

$$\Rightarrow \oint_L \vec{B} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS}$$

---(2)

Substituting (2) in (1), we get

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} = \mu_0 I_{encl}$$

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} = \mu_0 \int_S \vec{J} \cdot \vec{dS}$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

Differential form

Physical Significance of Maxwell's Equations

Gauss's Law $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

According to this total electric flux through any closed surface is $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the closed surfaces, representing Gauss's law of electrostatics. As this does not depend on time, it is a steady state equation. Here for positive ρ, divergence of electric field is positive and for negative ρ divergence is negative. It indicates that ρ is scalar quantity.

Gauss's Law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$

magnetic monopoles cannot exist

Faraday's Law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Time-varying magnetic flux produce electric field

Ampere's Law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

A magnetic field is produced due to conduction current density

Physical Significance of Maxwell's Equations

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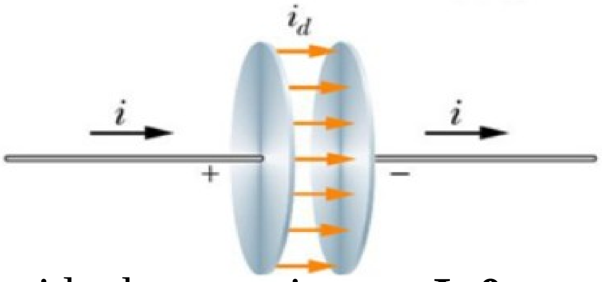
A magnetic field is produced due to conduction current density

Anomalies with this equation

Discussion on anomalies in $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Assume a capacitor of radius, **r**, is connected to a circuit with a flowing current of **I**

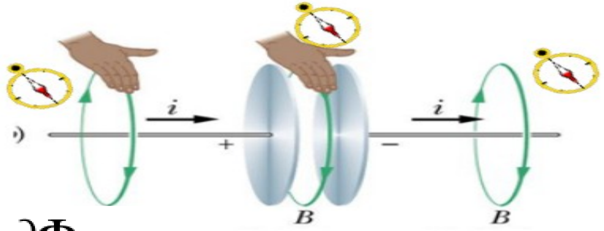


$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

According to Ampere's law, there must be no magnetic field inside the capacitor, as **I=0**

However, Maxwell found the same magnetic field with same deflection inside the capacitor, so he predict that:

The change in Electric Field ⇨ Creates the Magnetic Field



$$\oint \vec{B} \cdot \vec{dS} \propto \frac{\partial \Phi_E}{\partial t} \Rightarrow \oint \vec{B} \cdot \vec{dS} = \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} \Rightarrow \quad \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} = i_d$$

displacement current

Corrected form Ampere's law

Maxwell said that not only current produces a magnetic field **but a changing electric field** in vacuum/free space also produces a magnetic field

Ampere's Law $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

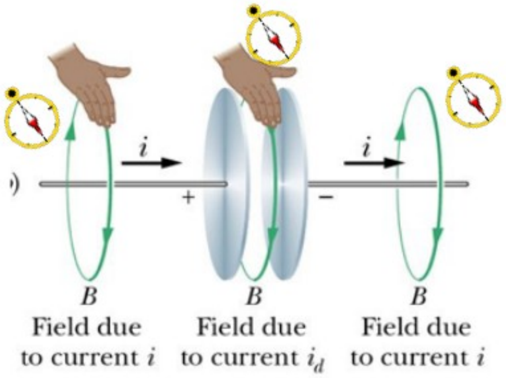
Corrected to

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$$

Ampere's-Maxwell Law

Conduction current

Displacement current



Corrected (Modified) form of Maxwell's equations

Gauss's Law $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Gauss's Law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$

Faraday's Law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Same as Earlier

Ampere's-Maxwell Law $\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$

modified

A magnetic field is produced due to conduction current density and varying electric field

Maxwell Equations in free space

A **free space** means there is **no source of current or Charge**

$$\rho = 0, \quad J = 0$$

General form

Gauss’s Law $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$

Gauss’s Law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$

Faradav’s Law $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

Ampere’s-Maxwell Law $\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}\right]$

$\rho = 0, \quad J = 0$

Free Space

$\vec{\nabla} \cdot \vec{E} = 0$

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

Derivation of Electromagnetic (EM) wave equation in free space

A free space means there is no source of current or Charge; ρ = 0, J = 0, considering the Maxwell equation:

$$\vec{\nabla} \cdot \vec{E} = 0 \;\ldots\ldots (1) \qquad \vec{\nabla} \cdot \vec{B} = 0 \;\ldots\ldots (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \;\ldots\ldots (3) \qquad \vec{\nabla} \times \vec{B} = \mu_0\, \epsilon_0 \frac{\partial \vec{E}}{\partial t} \;\ldots\ldots (4)$$

Lets operate Curl on eqⁿ-3

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} . \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

using Maxwell equation-1 & 4

$$\Rightarrow \vec{\nabla}(0) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\mu_0\, \epsilon_0 \frac{\partial \vec{E}}{\partial t})$$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0\, \epsilon_0 \frac{\partial}{\partial t}(\frac{\partial \vec{E}}{\partial t}) \qquad \boxed{c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0\, \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow \frac{\partial^2 \vec{E}}{\partial t^2} = c^2\, \nabla^2 \vec{E}$$

Lets operate Curl on eqⁿ-4

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\mu_0\, \epsilon_0 \frac{\partial \vec{E}}{\partial t})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} . \vec{B}) - \nabla^2 \vec{B} = \mu_0\, \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

using Maxwell equation-2 & 3

$$\Rightarrow \vec{\nabla}(0) - \nabla^2 \vec{B} = \mu_0\, \epsilon_0 \frac{\partial}{\partial t}(\frac{-\partial \vec{B}}{\partial t})$$

$$\Rightarrow -\nabla^2 \vec{B} = -\mu_0\, \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad \boxed{c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

$$\Rightarrow \frac{\partial^2 \vec{B}}{\partial t^2} = c^2\, \nabla^2 \vec{B}$$

EM wave equation in free space

$$\Rightarrow \quad \frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \nabla^2 \vec{E} \qquad \Rightarrow \quad \frac{\partial^2 \vec{B}}{\partial t^2} = c^2 \nabla^2 \vec{B}$$

$$\boxed{c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

c, is the speed of EM wave. If we substitute the value of

$$\epsilon_0 = 8.854 \times 10^{-12}\ Fm^{-1},\ and\ \mu_0 = 4\pi \times 10^{-7}$$

$$\Rightarrow \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.99776 \times 10^8\ m/s$$

This conclude that the EM wave travels in speed of light in free space

General Solution to EM wave Equation

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2\, \nabla^2 \vec{E} \quad \Rightarrow \quad \frac{\partial^2 \vec{B}}{\partial t^2} = c^2\, \nabla^2 \vec{B}$$

Let **x** be the direction of propagation of waves, in that case: $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2}$ *and* $\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial x^2}$

$$\Rightarrow \frac{\partial^2 \vec{E}}{\partial t^2} = c^2\, \frac{\partial^2 \vec{E}}{\partial x^2} \quad \Rightarrow \quad \frac{\partial^2 \vec{B}}{\partial t^2} = c^2\, \frac{\partial^2 \vec{B}}{\partial x^2}$$

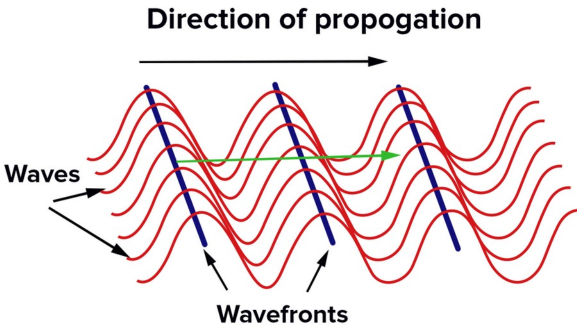
The most familiar form of solution to Maxwell's equation is of the form:

$$\vec{E}(x,t) = \vec{E}_{max}\, Sin(kx - \omega t + \phi) \qquad \vec{B}(x,t) = \vec{B}_{max}\, Sin(kx - \omega t + \phi)$$

Plane EM waves in free space

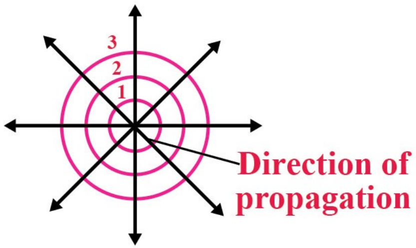
Wavefronts and its Types

Wave Fronts are the parallel surfaces connecting equivalent points on adjacent waves

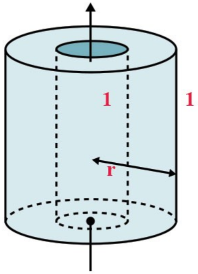


Types

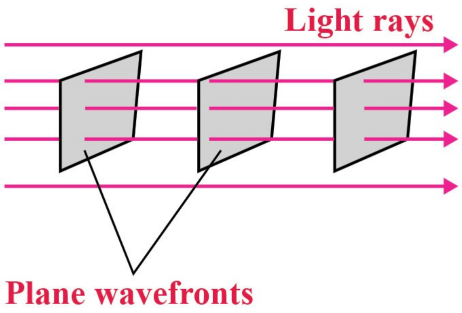
1.Spherical Wave Front



1.Cylindrical Wave Front

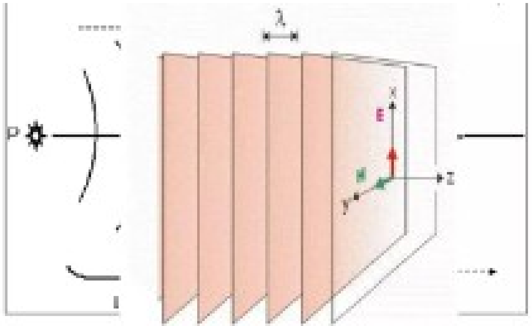


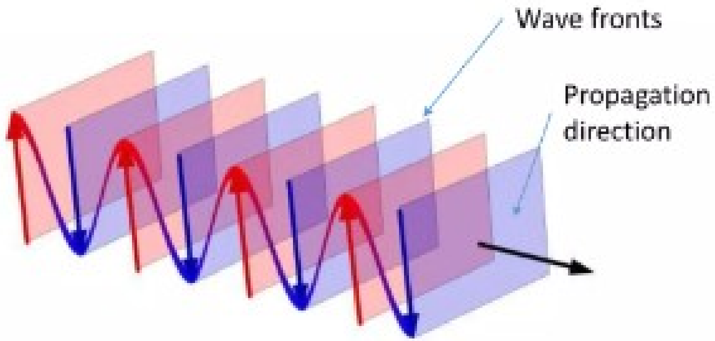
3.Plane Wave Front



Plane Wave

A **plane wave** is defined as a wave **whose value remains constant (i.e., whose wave fronts are parallel planes) throughout a plane** (constant phase on surface) and **is transverse to the direction of the propagation of the wave.**



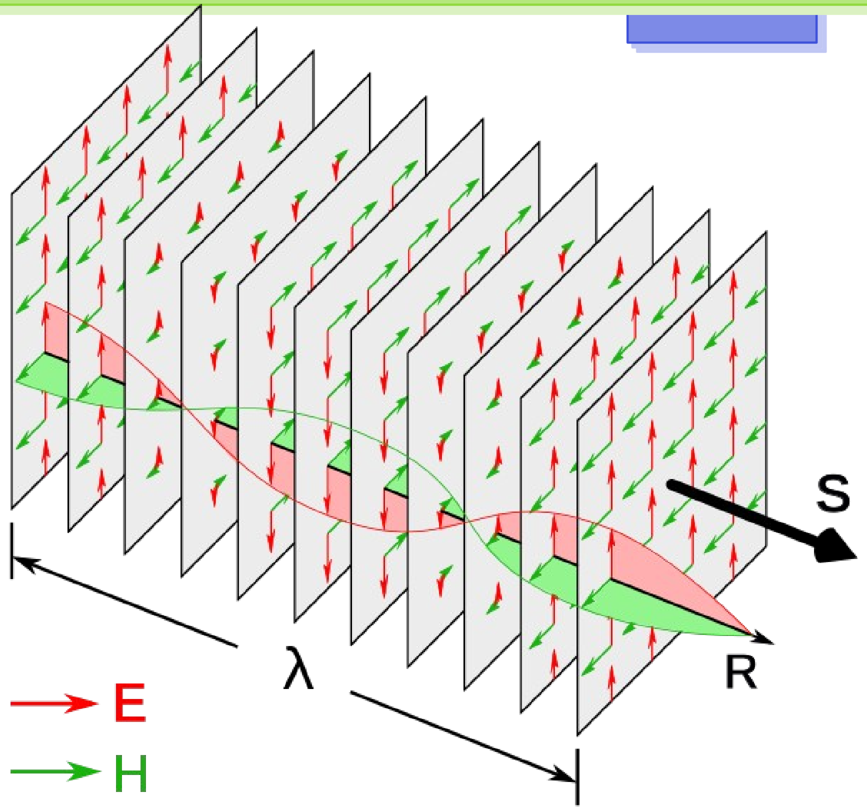


Properties of a plane wave:

1) A plane wave's **wave front are equally spaced** (i.e., a wavelength apart).
2) EM plane **wave fronts propagates at the speed of light.**
3) Plane waves **can be represented by the solution of "wave equation".**

Plane EM wave cont'd...

A **plane EM wave** is a wave in which the ***E and B fields have same magnitude in a plane which is perpendicular to the direction of the propagation***. The plane waves magnitude, phase, and orientation depends only on the variables associated with the direction of propagation.



Solution to EM Wave in 1D

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2\ \nabla^2 \vec{E}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = c^2\ \nabla^2 \vec{B}$$

Let "***x***" be the direction of propagation of waves, in that case:

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2}$$

$$\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial x^2}$$

The most familiar form of solution to Maxwell's equation is of the form:

$$\vec{E}(x,t) = \vec{E}_0\ Sin(kx - \omega t + \phi)$$

$$\vec{B}(x,t) = \vec{B}_0\ Sin(kx - \omega t + \phi)$$

$\vec{E}_0$ and $\vec{B}_0$ are the constant vectors, only the phase term is varying along z direction.

Transverse nature of EM Wave

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \nabla^2 \vec{E}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = c^2 \nabla^2 \vec{B}$$

The most familiar form of solution to Maxwell's equation is of the form:

$$\vec{E}(x,t) = \vec{E}_0 \, Sin(kx - \omega t + \phi)$$

$$\vec{B}(x,t) = \vec{B}_0 \, Sin(kx - \omega t + \phi)$$

$$\vec{E}_0 = E_{0x}\hat{i} + E_{0x}\hat{j} + E_{0x}\hat{k}$$

$$\vec{B}_0 = B_{0x}\hat{i} + B_{0x}\hat{j} + B_{0x}\hat{k}$$

Transverse nature of EM Wave cont'd...

using Maxwell equation in free space

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{E}\perp\vec{k}$$
$$\Rightarrow \qquad\qquad \Rightarrow \vec{E}\perp\vec{B}$$
$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{B}\perp\vec{k}$$

If $\vec{E}$ oscillates in the y-direction, $\vec{B}$ will oscillates on the z-direction and vice-versa....i.e mutually orthogonal to each other.

using Maxwell equation (Faraday's law) in free space

lets assume, the Electric field in y-direction

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{B}_0=-\hat{k}\left(\frac{k}{\omega}\right)\vec{E}_y$$

$$\Rightarrow \vec{B}_0=-\hat{k}\left(\frac{1}{c}\right)\vec{E}_y$$

similarly, If the Electric field in z-direction

$$\Rightarrow \vec{B}_0=-\hat{j}\left(\frac{1}{c}\right)\vec{E}_z$$

Concept: Accelerating Charge Particle

Lets consider a charge particle:

Rest, v=0

Electric Field

$E \neq 0, \quad B = 0$

Charge in motion $(v \neq 0)$

V= Constant

$(E \neq 0, \quad B \neq 0)$

E, B are constant

$\neq$

Accelerating

$(E \neq 0, \quad B \neq 0)$

E, B are time varying

EM Waves

Accelerating charge particles/oscillating charge particles create EM waves. Hence, the LC circuit is can be used as the source of the EM wave.

Hertz Experiment setup:

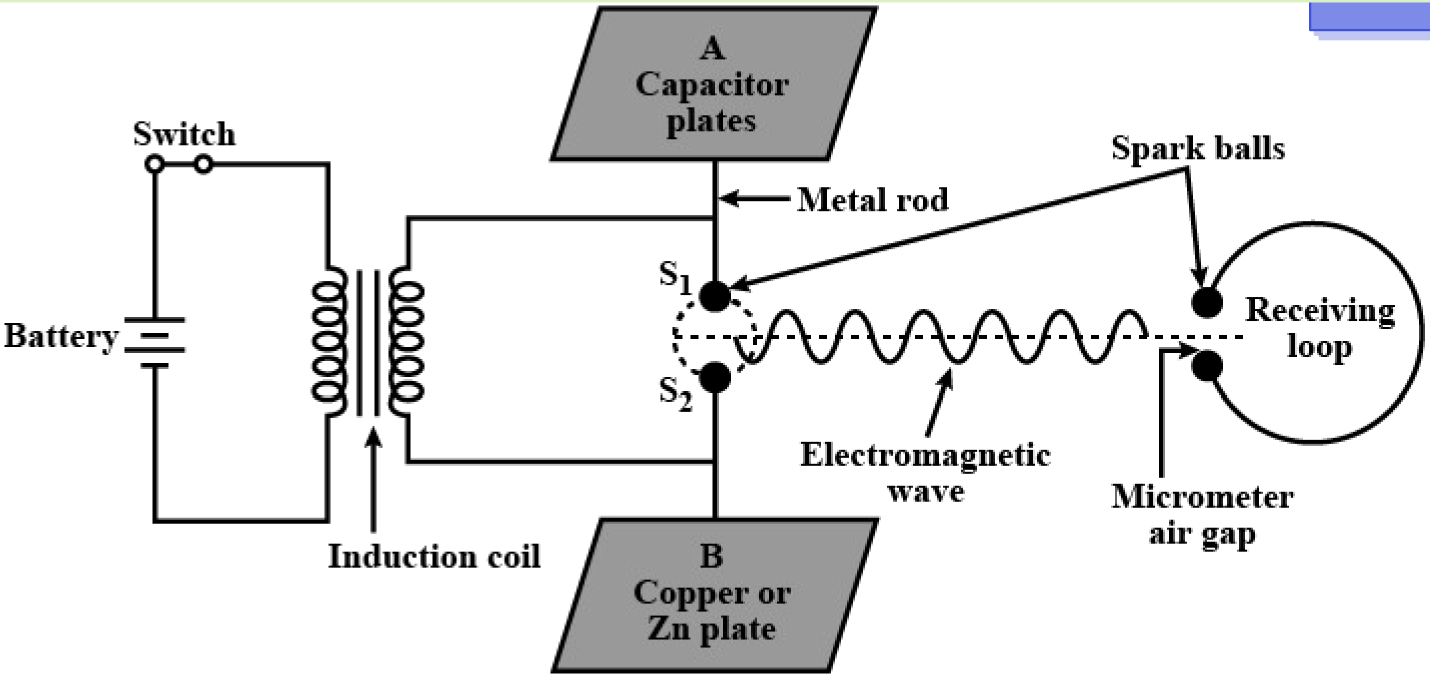
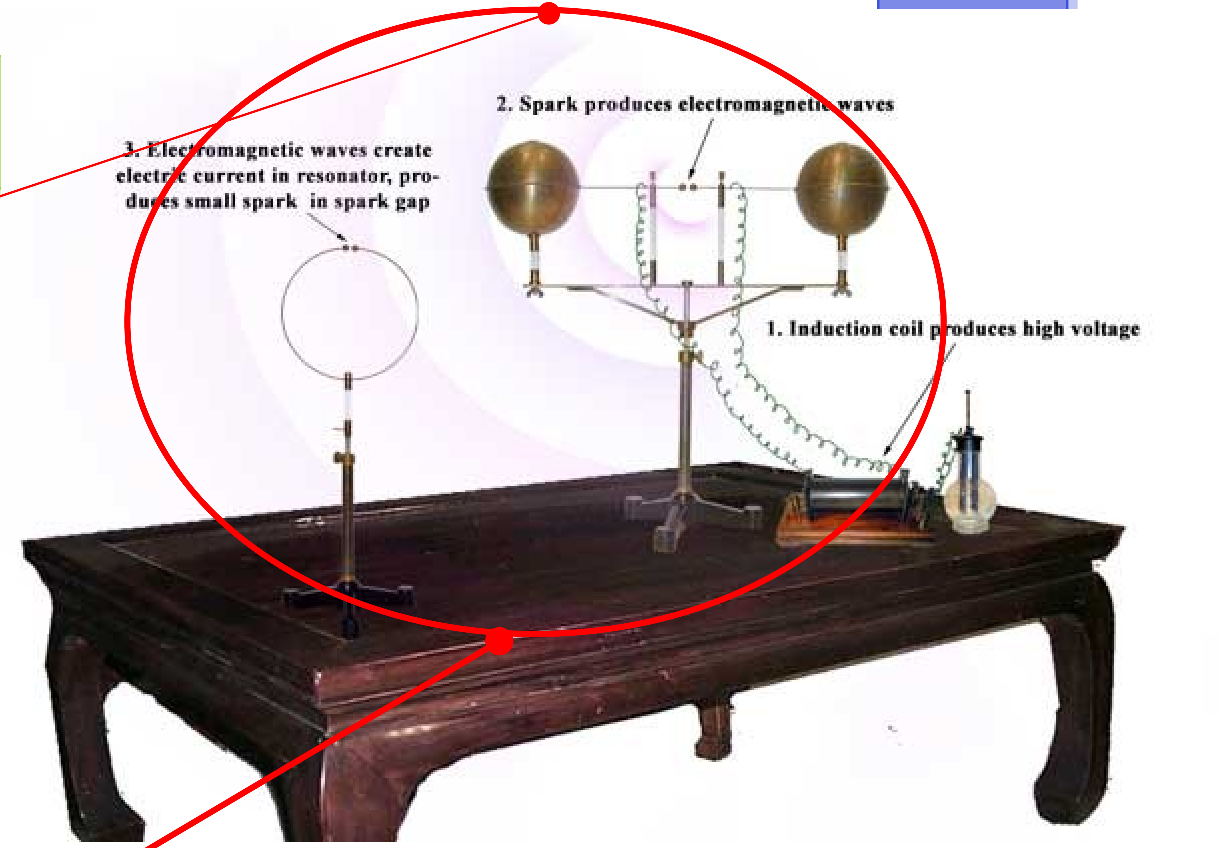
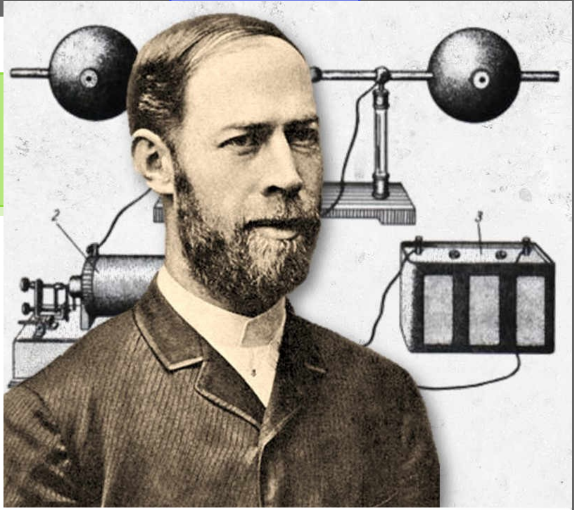


Fig : Sketch of the apparatus used by Hertz for producingand detecting radiowaves

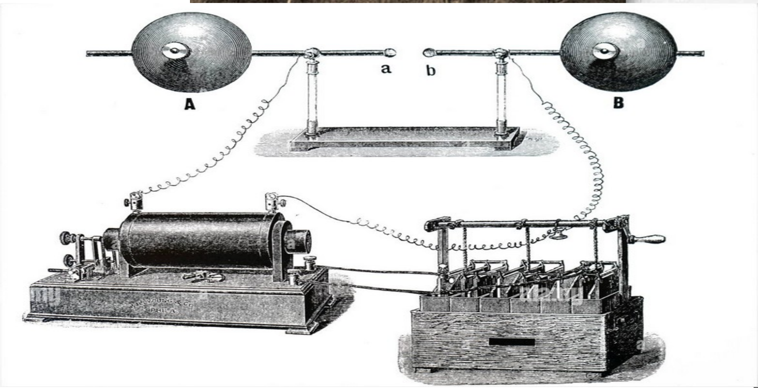
Picture of Hertz Experiment setup:



Importance of Hertz Experiment



- ➔Heinrich Hertz between 1885 and 1889 had an exceptional influence on the subsequent development of science and technology.
- ➔In 1888, hertz was the first to experimentally generate and detect electromagnetic waves and proved the theory predicted by Maxwell in 1865.
- ➔He used an oscillator made of polished brass knobs, connected to an induction coil and separated by a tiny gap over which sparks could leap.
- ➔If Maxwell's predictions were correct, electromagnetic waves would be transmitted during each series of sparks.



Numericals:

Q1. A plane progressive wave is given by $y = 2\cos 6.284(330tx)$. What is period of the wave?

The correct option is **A** $\frac{1}{330}$ s

Solution:

We have the progressive wave given by:

$y = 2\cos 6.284(330t - x)$

$y = 2\cos 2\pi(330t - x)$

$y = 2\cos(2\pi \times 330t - 2\pi x)$

On comparing the given progressive equation with $y = A\cos(wt - kx)$,

we get, $\omega = 2\pi \times 330$ and time period $T = \frac{2\pi}{\omega} = \frac{1}{330}$ s