Evaluate $\iiint_R (x-y+z)\,dx\,dy\,dz$ where R is given by $1 \le x \le 2$; $2 \le y \le 3$, $1 \le z \le 3$.

Sol: $\iiint_R (x-y+z)\,dx\,dy\,dz$

$$= \int_1^3 \int_2^3 \int_1^2 (x-y+z)\,dx\,dy\,dz$$

$$= \int_1^3 \int_2^3 \left[\frac{x^2}{2} - yx + zx \right]_1^2 dy\,dz$$

$$= \int_1^3 \int_2^3 \left(\frac{\cancel{4}^{2}}{\cancel{2}} - 2y + 2z - \frac{1}{2} + y - z \right) dy\,dz$$

$$= \int_1^3 \int_2^3 \left(\frac{3}{2} - y + z \right) dy\,dz$$

$$= \int_1^3 \left[\frac{3}{2}y - \frac{y^2}{2} + zy \right]_2^3 dz$$

$$= \int_1^3 \left[\frac{3}{\cancel{2}}(3) - \cancel{\frac{9}{2}} + 3z - \frac{3}{\cancel{2}}(\cancel{2}) + \frac{\cancel{4}^{2}}{\cancel{2}} - 2z \right] dz$$

$$= \int_1^3 (z-1)\,dz$$

$$= \left(\frac{z^2}{2} - z\right)_1^3$$

$$= 9/2 - 3 - 1/2 + 1$$

$$= 8/2 - 2$$

$$= 4 - 2$$

$$= 2 //.$$

Evaluate $\int_0^{2\pi} \int_0^{\pi/4} \int_0^a r^2 \sin\theta \, dr\, d\theta\, d\phi$

Sol:

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^a r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left(\frac{r^3}{3}\right)_0^a \sin\theta \, d\theta\, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{a^3}{3} \sin\theta \, d\theta\, d\phi$$

$$= a^3/3 \int_0^{2\pi} (-\cos\theta)_0^{\pi/4} \, d\phi$$

$$= a^3/3 \int_0^{2\pi} (-1/\sqrt{2} + 1) \, d\phi$$

$$= a^3/3 \left[-1/\sqrt{2}\, \phi + \phi \right]_0^{2\pi}$$

$$= a^3/3 \left[-\frac{1}{\sqrt{2}}(2\pi) + 2\pi \right]$$

$$= \frac{2\pi a^3}{3} \left[-1/\sqrt{2} + 1 \right]$$

$$= \frac{\pi a^3}{3} \left[2 - \sqrt{2} \right) //.$$

Evaluate $\iiint_R (x+y+z) \, dx\, dy\, dz$ where R is the region bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$ and $z=0$, $z=1$.

Sol: $$\iiint_R (x+y+z)\, dx\, dy\, dz = \int_0^1 \int_0^1 \int_0^1 (x+y+z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \left(\frac{x^2}{2} + yx + zx \right)_0^1 dy\,dz$$

$$= \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z \right) dy\,dz$$

$$= \int_0^1 \left(\frac{1}{2}y + \frac{y^2}{2} + zy \right)_0^1 dz$$

$$= \int_0^1 \left(\frac{1}{2} + \frac{1}{2} + z \right) dz$$

$$= \int_0^1 (1+z)\,dz = \left(z + \frac{z^2}{2} \right)_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$$

Evaluate $\iiint_D (x+y+z)\,dx\,dy\,dz$ where

$D: \; 1 \le x \le 2.\; 2 \le y \le 3.\; 1 \le z \le 3.$

Sol: $\iiint_D (x+y+z)\,dz\,dy\,dx$

$$= \int_1^2 \int_2^3 \int_1^3 (x+y+z)\,dz\,dy\,dx$$

$$= \int_1^2 \int_2^3 \left(xz + yz + \frac{z^2}{2} \right)_1^3 dy\, dx$$

$$= \int_1^2 \int_2^3 \left(3x + 3y + \frac{9}{2} - x - y - \frac{1}{2} \right) dy\, dx$$

$$= \int_1^2 \int_2^3 (2x + 2y + 4)\, dy\, dx$$

$$= \int_1^2 \left(2xy + \cancel{2}\frac{y^2}{\cancel{2}} + 4y \right)_2^3 dx$$

$$= \int_1^2 (6x + 9 + \cancel{12} - 4x - \cancel{4} - \cancel{8})\, dx$$

$$= \int_1^2 (2x + 9)\, dx = \left(\cancel{2}\frac{x^2}{\cancel{2}} + 9x \right)_1^2 = 4 + 18 - 1 - 9$$

$$= 12$$

Evaluate $\iiint_V (x + y + z)\, dx\, dy\, dz$, where the region V is bounded by $x + y + z = a$ $(a > 0)$, $x = 0$, $y = 0$, $z = 0$.

Sol:

$$\iiint_V (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left(xz + yz + z^2/2 \right)_0^{a-x-y} dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left[x(a-x-y) + y(a-x-y) + \frac{(a-x-y)^2}{2} \right] dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left[x(a-x) - xy + y(a-x) - y^2 + \frac{(a-x-y)^2}{2} \right] dy\,dx$$

$$= \int_0^a \left[x(a-x)y - xy^2/2 + (a-x)y^2/2 - y^3/3 + \frac{(a-x-y)^3}{-6} \right.$$

$$= \int_0^a x(a-x)^2 - \frac{x(a-x)^2}{2} + \frac{(a-x)^3}{2} - \frac{(a-x)^3}{3} - \frac{(a-x-[illegible])}{6} + \frac{(a-x)^3}{6} \Big]$$

$$= \int_0^a \left(\frac{x(a-x)^2}{2} + \frac{(a-x)^3}{3} \right) dx$$

$$= \int_0^a \left(\frac{x(a^2+x^2-2ax)}{2} + \frac{(a-x)^3}{3} \right) dx$$

$$= \int_0^a \left(\frac{a^2x + x^3 - 2ax^2}{2} + \frac{(a-x)^3}{3} \right) dx$$

$$= \left[\frac{1}{2} \left(\frac{a^2x^2}{2} + \frac{x^4}{4} - \frac{2ax^3}{3} \right) - \frac{(a-x)^4}{12} \right]_0^a$$

$$= \frac{1}{2} \left(\frac{a^4}{2} + \frac{a^4}{4} - \frac{2a^4}{3} \right) - \cancel{\frac{(a-a)^4}{12}}_{0} + \frac{a^4}{12}$$

$$= \frac{1}{2} \left(\frac{6a^4 + 3a^4 - 8a^4}{12} \right) + \frac{a^4}{12}$$

$$= \frac{1}{2} \left(\frac{a^4}{12} \right) + \frac{a^4}{12}$$

$$= \frac{a^4}{12} \left(\frac{1}{2} + 1 \right) = \frac{a^4}{12} \left(\frac{3}{2} \right) = \frac{3a^4}{24} \,//$$

$$= \frac{a^4}{8} \,//$$

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

Sol:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx.$$

$$\left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1} x/a\right.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1} 1 - \sin^{-1} 0\right] dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} (\pi/2 - 0)\,dy\,dx$$

$$= \pi/2 \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$$

$$= \pi/2 \int_0^a \left(\sqrt{a^2-x^2}\right) dx$$

$$= \pi/2 \left[a^2/2 \sin^{-1} x/a + x/2 \sqrt{a^2-x^2} \right]_0^a$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = x/2 \sqrt{a^2-x^2} + a^2/2 \sin^{-1} x/a \right]$$

$$= \pi/2 \left[\left(a^2/2 \sin^{-1} 1 + 0\right) - (0+0) \right]$$

$$= \pi/2 \left(a^2/2 \, \pi/2 \right) = \pi^2 a^2 / 8 \;//$$

Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$.

Sol:- $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$.

$$= \int_1^e \int_1^{\log y} \left(z\log z - z \right)_1^{e^x} dx\, dy$$

$$\left[\because \int \underset{u}{\log z}\, \underset{dv}{dz} = z\log z - z \right]$$

$$= \int_1^e \int_1^{\log y} \left(e^x \log e^x - e^x - 1\log 1 + 1 \right) dx\, dy$$

$$\left[\because \log 1 = 0,\ \log e^x = x \right]$$

$$= \int_1^e \int_1^{\log y} (\underset{u\,dv}{x e^x} - e^x + 1)\, dx\, dy$$

$$= \int_1^e \left[x e^x - e^x - e^x + x \right]_1^{\log y} dy$$

$$= \int_1^e \left[\log y\, e^{\log y} - 2 e^{\log y} + \log y - (e - 2e + 1)\right]$$

$$= \int_1^e (y \log y - 2y + \log y + e - 1)\, dy \longrightarrow (y)$$

$$\left[\because e^{\log y} = y\right]$$

Now $\int y \log y \, dy$

$u = \log y \quad dv = y\,dy \qquad \int u\,dv = \left[uv - \int v\,du\right]$

$du = 1/y\, dy \quad v = y^2/2$

$$= y^2/2 \log y - \int y^{\cancel{2}}/2 \cdot 1/\cancel{y}\, dy$$

$$= y^2/2 \log y - \left[y^2/4\right] \longrightarrow (4)$$

Substituting (2) in (1) we get

$$= \left[\frac{y^2}{2}\log y - \frac{y^2}{4} - \cancel{2}\frac{y^2}{\cancel{2}} + y\log y - y + (e-1)y \right]_1^e$$

$$= \frac{e^2}{2}\log e - \frac{e^2}{4} - e^2 + e\log e - e + (e-1)e - \frac{1}{2}\underbrace{\cancel{\log 1}}_{0} + \frac{1}{4} + 1 - 1\underbrace{\cancel{\log 1}}_{0} + 1 - (e-1)$$

$$= \frac{e^2}{2}(1) - \frac{e^2}{4} - e^2 + e(1) + e(e-1) - e + \frac{1}{4} + 1 + 1 - e + 1 \quad \left[\because \log 1 = 0, \ \log_e e = 1\right]$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - \cancel{e^2} + \cancel{e} + \cancel{e^2} - \cancel{e} - e + \frac{1}{4} + 1 + 1 - e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{1}{4} + 3 = \frac{e^2}{4} - 2e + \frac{13}{4}$$

$$= \frac{e^2 - 8e + 13}{4}$$

Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$.

Sol:

$$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

$$= \int_0^a \int_0^x \left(e^{x+y+z}\right)_0^{x+y} dy\,dx$$

$$= \int_0^a \int_0^x \left[\left(e^{x+y+x+y}\right) - \left(e^{x+y+0}\right)\right] dy\,dx$$

$$= \int_0^a \int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dy\,dx$$

$$= \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx$$

$$= \int_0^a \left[\left(\frac{e^{2(x+x)}}{2} - e^{x+x}\right) - \left(\frac{e^{2(x+0)}}{2} - e^{x+0}\right)\right] dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x} \right]_0^a$$

$$= \frac{e^{4a}}{8} - \frac{e^{2a}}{2} - \frac{e^{2a}}{4} + e^{a} - 1/8 + 1/2 + 1/4 -$$

$$= e^{4a}/8 - \frac{3e^{2a}}{4} + e^{a} - 3/8$$

$$= \frac{e^{4a} - 6e^{2a} + 8e^{a} - 3}{8}$$

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^{z} \, dx \, dy \, dz$

Sol:

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} e^{z} \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \left[e^{z} \right]_0^{x+y} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[e^{x+y} - e^{0} \right] dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left(e^{x+y} - 1 \right) dy \, dx$$

$$= \int_0^1 \left(e^{x+y} - y \right)_0^{1-x} dx$$

$$= \int_0^1 \left[e^{x+1-x} - (1-x) - e^{x} \right] dx$$

$$= \int_0^1 (e - 1 + x - e^{x})\, dx$$

$$= \left(ex - x + x^2/2 - e^{x} \right)_0^1$$

$$= \left[(\cancel{e} - 1 + 1/2 - \cancel{e}) - (0 - 0 + 0 - 1) \right]$$

$$= -1/2 + 1$$

$$= 1/2$$

Evaluate $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xy \, dz\, dy\, dx$ AU 2001. No

Sol:- $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xy \, dz\, dy\, dx$

$$= \int_1^3 \int_{1/x}^1 \left[xyz \right]_0^{\sqrt{xy}} dy\, dx$$

$$= \int_1^3 \int_{1/x}^1 xy\sqrt{xy}\, dy\, dx$$

$$= \int_1^3 \int_{1/x}^1 x^{3/2} y^{3/2} \, dy \, dx$$

$$= \int_1^3 \left[x^{3/2} \frac{y^{5/2}}{5/2} \right]_{1/x}^1 dx$$

$$= \int_1^3 \frac{2x^{3/2}}{5} \left[1 - \left(\frac{1}{x}\right)^{5/2} \right] dx$$

$$= 2/5 \int_1^3 \left(x^{3/2} - \frac{x^{3/2}}{x^{5/2}} \right) dx$$

$$= 2/5 \int_1^3 \left(x^{3/2} - x^{-1} \right) dx$$

$$= 2/5 \int_1^3 \left(x^{3/2} - 1/x \right) dx$$

$$= 2/5 \left[\frac{x^{5/2}}{5/2} - \log x \right]_1^3$$

$$= 2/5 \left[\left(2/5 \, 3^{5/2} - \log 3 \right) - \left(2/5 - \log 1 \right) \right]$$

$$= \frac{4}{25} 3^{5/2} - 2/5 \log 3 - 4/25 + \cancelto{0}{\log 1}$$

$$= \frac{4}{25} (3)^{5/2} - 4/25 - 2/5 \log 3$$

$$= \frac{2}{25} \left[2(3)^{5/2} - 2 - 5 \log 3 \right] \quad //$$

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{(x+y)^2} x \, dz \, dy \, dx$. AU 1996

Sol:

$$\int_0^1 \int_0^{1-x} \int_0^{(x+y)^2} x \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left(xz \right)_0^{(x+y)^2} dy \cdot dx$$

$$= \int_0^1 \int_0^{1-x} x(x+y)^2 \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} x(x^2+y^2+2xy) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} (x^3+xy^2+2x^2y) \, dy \, dx$$

$$= \int_0^1 \left[x^3y + xy^3/3 + \cancel{2}x^2y^2/\cancel{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^3(1-x) + x/3 \, (1-x)^3 + x^2(1-x)^2 \right] dx$$

$$= \int_0^1 x(1-x) \left[x^2 + \frac{(1-x)^2}{3} + x(1-x) \right] dx$$

$$= \int_0^1 (x-x^2)\left(\cancel{x^2}+\tfrac{1}{3}(1+x^2-2x)+x-\cancel{x^2}\right]dx$$

$$= \int_0^1 (x-x^2)\left[\tfrac{1}{3}+\tfrac{1}{3}x^2-\tfrac{2}{3}x+x\right]dx$$

$$= \int_0^1 (x-x^2)\left[\tfrac{1}{3}x^2+\tfrac{1}{3}x+\tfrac{1}{3}\right]dx$$

$$= \tfrac{1}{3}\int_0^1 (x-x^2)(x^2+x+1)\,dx$$

$$= \tfrac{1}{3}\int_0^1 (\cancel{x^3}+\cancel{x^2}+x-x^4-\cancel{x^3}-\cancel{x^2})\,dx$$

$$= \tfrac{1}{3}\int_0^1 (-x^4+x)\,dx = \tfrac{1}{3}\left[-x^5/5+x^2/2\right]_0^1$$

$$= \tfrac{1}{3}\left[-1/5+1/2\right] = \tfrac{1}{\cancel{3}}\left[\tfrac{\cancel{3}}{10}\right] = 1/10 \,\|.$$

Evaluate $\iiint_V \dfrac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integeation bounded by the planes $x=0$, $y=0$, $z=0$. $x+y+z=1$.

Sol:

$$\iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz\,dy\,dx$$

$$= \frac{1}{16}\left[8\log 2 - 5\right]$$

(Already done)

find the volume bounded by the Cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Sol:-

$$\iiint_V dz\,dy\,dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[z\right]_0^{4-y} dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx$$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[\left(4\sqrt{4-x^2} - \frac{4-x^2}{2} \right) - \left(-4\sqrt{4-x^2} - \frac{4-x^2}{2} \right) \right] d$$

$$= \int_{-2}^{2} \left(4\sqrt{4-x^2} - \cancel{\frac{4-x^2}{2}} + 4\sqrt{4-x^2} + \cancel{\frac{4-x^2}{2}} \right) dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx$$

$$= 8 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 16 \int_{0}^{2} \sqrt{4-x^2}\, dx$$

$$= 16 \left[\frac{4}{2} \sin^{-1} \frac{x}{2} + \frac{x}{2} \sqrt{4-x^2} \right]_0^2$$

$$= 16 \left[\left(\cancel{2}\frac{\pi}{\cancel{2}} + 0 \right) - (0 + 0) \right]$$

$$= 16\pi$$

Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Sol:

$V = 8 \times$ Volume in an Octant.

$$\therefore V = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\,dx$$

$$= 8 \int_0^a \left[\frac{a^2-x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} + \frac{y}{2}\sqrt{a^2-x^2-y^2} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \int_0^a \frac{a^2-x^2}{2} \pi/2 \, dx$$

$$= 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi \left[a^2 x - x^3/3 \right]_0^a$$

$$= 2\pi \left[a^3 - a^3/3 \right]$$

$$= 2\pi \left(\frac{2a^3}{3} \right)$$

$$= \frac{4\pi a^3}{3} \,//.$$

Evaluate $\displaystyle\int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} x^2 z \, dz\, dy\, dx$ (AU 1996)

Sol:

$$\int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} x^2 z \, dz\, dy\, dx$$

$$= \int_0^a \int_0^{b(1-x/a)} x^2 \left(z^2/2 \right)_0^{c(1-x/a-y/b)} dy\, dx$$

$$= \int_0^a \int_0^{b(1-x/a)} x^2 \, \frac{c^2 (1-x/a-y/b)^2}{2} \, dy\, dx$$

$$= \frac{c^2}{2}\int_0^a \int_0^{b(1-x/a)} x^2(1-x/a-y/b)^2\,dy\,dx$$

put $t = 1 - x/a$

$dt = -\frac{1}{a}dx$

$dx = -a\,dt$

when $x=0$, $t=1$

when $x=a$, $t=0$

$1 - x/a = t$

$x/a = 1-t$

$x = a(1-t)$

$$= \frac{c^2}{2}\int_1^0 \int_0^{bt} a^2(1-t)^2(t-y/b)^2(-a)\,dy\,dt$$

$$= -\frac{a^3c^2}{2}\int_1^0 \int_0^{bt} (1-t)^2(t-y/b)^2\,dy\,dt$$

$$= -\frac{a^3c^2}{2}\int_1^0 \left[(1-t)^2\frac{(t-y/b)^3}{-3/b}\right]_0^{bt} dt$$

$$= -\frac{a^3c^2}{2}\int_1^0 \left[0 + \frac{b}{3}(1-t)^2 t^3\right] dt$$

$$= -\frac{a^3c^2b}{6}\int_1^0 (1+t^2-2t)t^3\,dt$$

$$= -\frac{a^3c^2b}{6}\int_1^0 (t^3+t^5-2t^4)\,dt$$

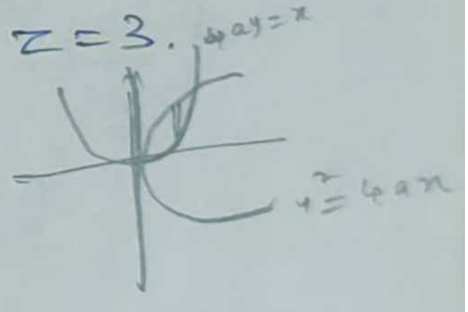
$$= -\frac{a^3c^2b}{6}\left[\frac{t^4}{4}+\frac{t^6}{6}-\frac{2t^5}{5}\right]_1^0$$

$$= -\frac{a^3c^2b}{6}\left[0-\left(\frac{1}{4}+\frac{1}{6}-\frac{2}{5}\right)\right]$$

$$= -\frac{a^3c^2b}{6}\left[-\left(\frac{30+20-48}{120}\right)\right]$$

$$= \frac{a^3c^2b}{6}\left(\frac{2}{120}\right) = \frac{a^3c^2b}{6}\left(\frac{1}{60}\right) = \frac{a^3bc^2}{360}$$

Find the volume of the region bounded by the surfaces $y^2 = 4ax$ and $x^2 = 4ay$ and the plane $z = 0$ and $z = 3$.



Sol:

$$y^2 = 4ax \longrightarrow (1)$$

$$x^2 = 4ay \longrightarrow (2)$$

$$(2) \Rightarrow \quad y = \frac{x^2}{4a}$$

sub in (1) we get

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\frac{x^4}{16a^2} = 4ax$$

$$x^4 = 64a^3x$$

$$x^4 - 64a^3x = 0$$

$$x(x^3 - 64a^3) = 0$$

$$x = 0, \quad x^3 = 64a^3$$

$$x = 4a$$

Required volume $= \displaystyle\int_0^{4a} \int_{x^2/4a}^{\sqrt{4ax}} \int_0^3 dz\,dy\,dx$

$$= \int_0^{4a} \int_{x^2/4a}^{\sqrt{4ax}} (z)_0^3\,dy\,dx$$

$$= \int_0^{4a} \int_{x^2/4a}^{\sqrt{4ax}} 3\,dy\,dx$$

$$= 3\int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} dy\,dx$$

$$= 3\int_0^{4a} (y)_{x^2/4a}^{\sqrt{4ax}}\, dx$$

$$= 3\int_0^{4a}\left(\sqrt{4ax} - x^2/4a\right)dx$$

$$= 3\int_0^{4a}\left(\sqrt{4a}\, x^{1/2} - x^2/4a\right)dx$$

$$= 3\left[\sqrt{4a}\,\frac{x^{3/2}}{3/2} - x^3/12a\right]_0^{4a}$$

$$= 3\left[\frac{2}{3}\sqrt{4a}\,(4a)^{3/2} - \frac{(4a)^3}{12a}\right]$$

$$= 3\left[2/3\,\sqrt{4}\,\sqrt{a}\,4^{3/2}a^{3/2} - \frac{64a^3}{12a}\right]$$

$$= 3\left[2/3\; 16a^2 - \frac{64a^2}{12}\right]$$

$$= 3\left[\frac{32}{3}a^2 - \frac{16a^2}{3}\right]$$

$$= \cancel{3}\left[\frac{16a^2}{\cancel{3}}\right]$$

$$= 16a^2$$

Evaluate $\iiint\limits_V dx\,dy\,dz$, where V is the Volume of the tetrahedron whose vertices are $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$.

Sol:- Now the plane through the points $(0, 1, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$ is $x+y+z=1$.

$\therefore$ If we first integrate w.r.t x, then its limits are 0 and $1-y-z$.

If the second integration w.r.t. 'y', its limits are 0 and $1-z$.

Finally, the limits of integration for z are 0 and 1.

$$\iiint_V dx\,dy\,dz = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-z} \left[x\right]_0^{1-y-z} dy\,dz$$

$$= \int_0^1 \int_0^{1-z} [1-y-z]\,dy\,dz$$

$$= \int_0^1 \left[y - y^2/2 - yz\right]_0^{1-z} dz$$

$$= \int_0^1 \left[(1-z) - \frac{(1-z)^2}{2} - (1-z)z\right] dz$$

$$= \int_0^1 (1-z)\left[1 - \frac{1-z}{2} - z\right] dz$$

$$= \int_0^1 (1-z)\left[\frac{2-1+z-2z}{2}\right] dz$$

$$= \int_0^1 (1-z)\frac{(1-z)}{2}\,dz$$

$$= \frac{1}{2}\int_0^1 (1-z)^2 dz = \frac{1}{2}\left[\frac{(1-z)^3}{-3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3}\right]$$

$$= \frac{1}{6}$$