Chapter 5

Vector Calculus

In the previous sections, we have studied **real-valued** multivariable functions, that is functions of the type

$$\begin{aligned} f: \quad & \mathbb{R}^2 \to \mathbb{R} \\ & (x,y) \mapsto f(x,y) \end{aligned}$$

or

$$\begin{aligned} f: \quad & \mathbb{R}^3 \to \mathbb{R} \\ & (x,y,z) \mapsto f(x,y,z) \end{aligned}$$

or in general

$$\begin{aligned} f: \quad & \mathbb{R}^n \to \mathbb{R} \\ & (x_1,x_2,...,x_n) \mapsto f(x_1,x_2,...,x_n) \end{aligned}$$

In each case, the domain is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ or in general $\mathbb{R}^n$. The range is a subset of $\mathbb{R}$, this is why these functions are called real-valued.

We have also studied **vector-valued** functions, that is functions of the type

$$\begin{aligned} f: \quad & \mathbb{R} \to \mathbb{R}^2 \\ & t \mapsto \langle x(t), y(t) \rangle \end{aligned}$$

or

$$\begin{aligned} f: \quad & \mathbb{R} \to \mathbb{R}^3 \\ & t \mapsto \langle x(t), y(t), z(t) \rangle \end{aligned}$$

or in general

$$\begin{aligned} f: \quad & \mathbb{R} \to \mathbb{R}^n \\ & t \mapsto \langle x_1(t), x_2(t), ..., x_n(t) \rangle \end{aligned}$$

In each case, the domain is a subset of $\mathbb{R}$, the range is a subset of $\mathbb{R}^n$ for some integer n. This is why these functions are called vector-valued.

In this chapter, we will concentrate our attention on functions of the type

$$\begin{aligned} f: \quad & \mathbb{R}^2 \to \mathbb{R}^2 \\ & (x,y) \mapsto \langle P(x,y), Q(x,y) \rangle \end{aligned}$$

for some real-valued multivariable functions P and Q. We will also study functions of the type

$$\begin{array}{rl} f: & \mathbb{R}^3 \to \mathbb{R}^3 \\ & (x,y,z) \mapsto \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \end{array}$$

These are multivariable functions, their domain is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. They are vector-valued functions, their range is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. Such functions will be called vector fields.

5.1 Vector Fields

5.1.1 Definitions and Examples

There are situations in which it is natural to attach a vector to each point in a given region. For example, we may want to know the velocity of the wind at each point in space. In this case, the vector at each point would represent the speed and direction of the wind at that point. Another example might be to know how gravity acts on each point in space. In this case, the vector at each point would represent the intensity and direction of the force due to gravity. A region in which each point is attached a vector is called a vector field. A more precise mathematical definition follows.

Definition 366 *Let D be a set in $\mathbb{R}^2$. A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x,y) of D a two dimensional vector $\mathbf{F}(x,y)$. In other words, it is a function of the form*

$$\begin{array}{rl} \mathbf{F}: & \mathbb{R}^2 \to \mathbb{R}^2 \\ & (x,y) \mapsto \mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle \end{array}$$

*where P and Q are real-valued functions. P and Q are called the **component functions**.*

Instead of writing $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, we can also write $\mathbf{F}(x,y) = P(x,y)\overrightarrow{i} + Q(x,y)\overrightarrow{j}$.

Definition 367 *Let D be a set in $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x,y,z) of D a three dimensional vector $\mathbf{F}(x,y,z)$. In other words, it is a function of the form*

$$\begin{array}{rl} \mathbf{F}: & \mathbb{R}^3 \to \mathbb{R}^3 \\ & (x,y,z) \mapsto \mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \end{array}$$

*where P, Q and R are real-valued functions. P, Q and R are called the **component functions**.*

Instead of writing $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$, we can also write $\mathbf{F}(x,y,z) = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k}$.

Vector fields are well known in physics. Some of the best examples include electrostatic fields, magnetic fields, and gravitational fields.

5.1.2 Plotting Vector Fields

Let us first look at some examples. We visualize vector fields by plotting the arrow representing $\mathbf{F}(x,y)$ at the point (x,y) or $\mathbf{F}(x,y,z)$ at each point (x,y,z). We obviously cannot do this for every point, so we do it for some representative points. We can also use Mathematical software to plot a vector field. The command to plot a vector field depends on whether it is a vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. **To plot a vector field with Scientific Workplace or Notebook:**

   (a) To plot $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, simply highlight the vector to plot (remember in Scientific Notebook, vector are written using parentheses or square brackets), then click on **Compute** then select **Plot 2D** then **Vector Field**.

   (b) To plot $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$, simply highlight the vector to plot (remember in Scientific Notebook, vector are written using parentheses or square brackets), then click on **Compute** then select **Plot 3D** then **Vector Field**.

2. **To plot a vector field with Maple:**

   (a) To plot $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, use the command

   $$\begin{gathered} fieldplot([P,Q], x = x_{\min}..x_{\max}, y = y_{\min}..y_{\max}, \\ ARROW = option, grid = [numx, numy]) \end{gathered}$$

   where *option* is one of LINE, THIN, SLIM, THICK, $numx, numy$ are the number of sample points to use in the x and $y-$directions.

   (b) To plot $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$, use the command

   $$\begin{aligned} fieldplot3d([P,Q,R], x &= x_{\min}..x_{\max}, y = y_{\min}..y_{\max}, z = z_{\min}..z_{\max}, \\ ARROW &= option, grid = numx, numy, numz) \end{aligned}$$

Example 368 *Describe the vector field given by*

$$\mathbf{F}(x,y) = \langle -y, x\rangle$$

We can first plot some of the vectors in the vector field. For this, we take some sample points (x,y), compute the vector $\mathbf{F}(x,y)$, then draw the vector $\mathbf{F}(x,y)$, starting at the point (x,y) (not origin like we usually draw vectors). Let's do it for a few points.

- *At $(1,0)$: $F(1,0) = \langle 0,1\rangle$, So, from the point $(1,0)$ we draw the vector $\langle 0,1\rangle = \overrightarrow{j}$.*

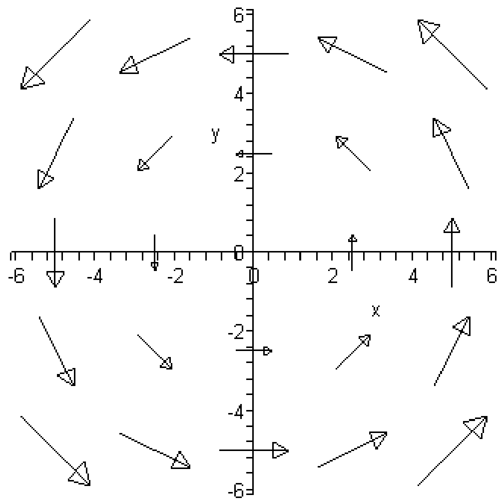


Figure 5.1: Vector field $F(x,y) = \langle -y, x \rangle$

- *At* $(0,1) : F(0,1) = \langle -1, 0 \rangle = -\overrightarrow{i}$. *So, starting at* $(0,1)$, *we draw the vector* $-\overrightarrow{i}$.
- *At* $(-1,0) : F(-1,0) = \langle 0, -1 \rangle = -\overrightarrow{j}$.
- *At* $(1,1) : F(1,1) = \langle -1, 1 \rangle$
- *By picking up enough points, we get a picture of the vector field which looks like figure 5.1. This picture was generated by Maple.The picture seems to indicate a circular motion, each arrow being tangent to a circle centered at the origin, with speed increasing as we move away from the center of the circle. This can be easily verified with simple computations. The position vector of a sample point is* $\langle x, y \rangle$. *The vector field at that point is* $\langle -y, x \rangle$. *The dot product of the two is*

$$\begin{aligned} \langle x, y \rangle \cdot \langle -y, x \rangle &= -xy + xy \\ &= 0 \end{aligned}$$

This indicates that $F(x,y)$ *is perpendicular to the position vector* $\langle x, y \rangle$ *and is therefore tangent to a circle centered at the origin, of radius* $\|\langle x, y \rangle\| = \sqrt{x^2 + y^2}$. *Also, note that* $\|F(x,y)\| = \sqrt{x^2 + y^2}$. *So, we see that the magnitude of* F *increases as the point moves away from the center of the circle.*

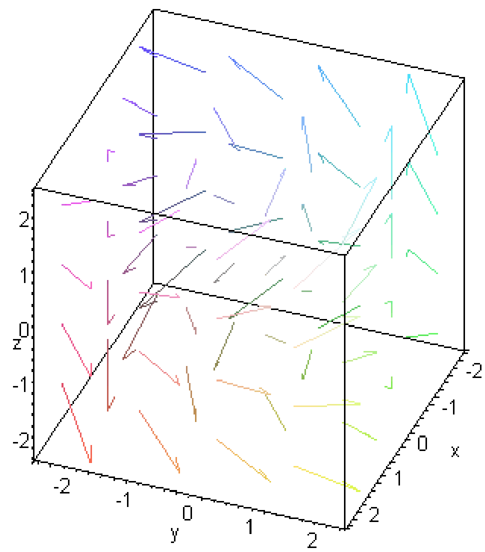


Figure 5.2: Vector field $F(x, y, z) = \langle -y, x, z \rangle$

Example 369 *Same question with* $F(x, y, z) = \langle -y, x, z \rangle$.
A Maple plot of this vector field is shown in figure 5.2.This vector field seems to represent a circular motion around the z-axis along with a linear motion in the z direction. Which is easy to understand. From the previous example, the circular motion is caused by the first two coordinates of the vector field. The last one causes the motion in the z direction.

Vector fields abound in nature. Here are some examples of vector fields you go through everyday.

Example 370 *Recall that an example of a vector is a force (it has both direction and magnitude). The earth gravity subjects every point not too far from the earth to a force, called the force of gravity. As such it is a vector field. If you have taken a physics class, you even know the formula for this vector field. It is given by*

$$\begin{aligned} F(x, y, z) &= -\frac{mMG}{\|\overrightarrow{x}\|^3}\overrightarrow{x} \\ &= \left\langle -\frac{mMGx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{mMGy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{mMGz}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle \end{aligned}$$

where $\overrightarrow{x} = \langle x, y, z \rangle$*, the position vector of a point* (x, y, z)*.* G *is the gravitational constant,* m *is the mass of an object at* (x, y, z) *and* M *is the mass of the earth. A picture of this vector field is shown in figure 5.3.*

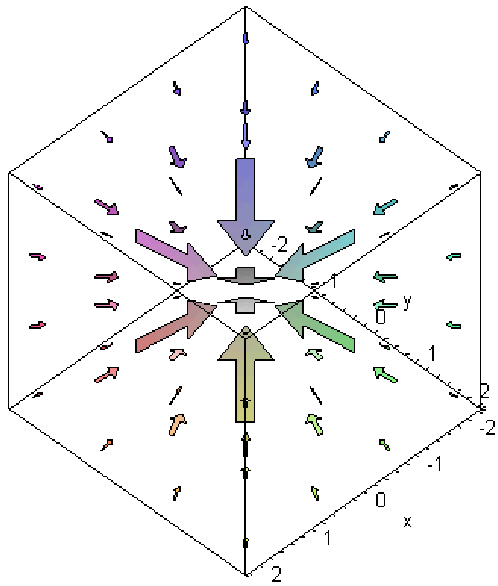


Figure 5.3: Gravitational vector field

Example 371 *A river flowing is another example of a vector field. As the water flows, each molecule of the water is moving in a certain direction, at a certain speed. At each molecule, we could draw an arrow representing the speed and direction of motion. This is an example of a vector field. In fact, it is a difficult mathematical problem to derive the vector field which describes the flow of a moving liquid (or gas).*

Example 372 *Similar to the previous example is the air which surrounds us. When there is wind, each molecule in the air is moving in a certain direction, at a certain speed. The vectors representing this motion at each point form a vector field. You may remember in the movie "twister", when the team studying the storm throws little balls which are carried by the funnel of the tornado. Each ball is equipped with a device which allows it to be tracked. Thus, as it is taken away by the tornado, we can capture its speed and direction. Thus, we can capture the vector field which describes the tornado.*

5.1.3 Gradient Vector Fields

If $f(x, y)$ is a real-valued function of two variables, then you will recall that

$$\nabla f(x, y) = \left\langle \frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right\rangle$$

Similarly, if $f(x,y,z)$ is a real-valued function of three variables, then

$$\nabla f(x,y,z) = \left\langle \frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z} \right\rangle$$

These are vector fields. They are called gradient vector fields.

Definition 373 *Let f be a real-valued function.*

1. *If f is a function of two variables, then ∇f is a vector field on $\mathbb{R}^2$.*
2. *If f is a function of three variables, then ∇f is a vector field on $\mathbb{R}^3$.*
3. *In both cases, ∇f is called a **gradient vector field**.*

Example 374 *Find the gradient vector field associated with $f(x,y) = e^{3x}\cos 4y$*
Since $\frac{\partial f(x,y)}{\partial x} = 3e^{3x}\cos 4y$ and $\frac{\partial f(x,y)}{\partial y} = -4e^{3x}\sin 4y$, it follows that

$$\nabla f(x,y) = \left\langle 3e^{3x}\cos 4y, -4e^{3x}\sin 4y \right\rangle$$

Definition 375 *A vector field F is called a **conservative vector field** if it is the gradient of some real-valued function in other words if there exists some real-valued function f such that*

$$F = \nabla f$$

*In this case, f is called a **potential function** for F.*

Remark 376 *We can draw an analogy with functions of one variables. If we replace ∇f by f' (the derivative of f), then saying that f is a potential function for F amounts to saying that f is an antiderivative of F.*

Not every vector field is conservative. However, many of the vector fields which arise in physics are. An example is shown next.

Example 377 *Let $f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$. Then we have*

$$\begin{aligned}
\frac{\partial f(x,y,z)}{\partial x} &= \frac{-mMGx}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\frac{\partial f(x,y,z)}{\partial y} &= \frac{-mMGy}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\frac{\partial f(x,y,z)}{\partial z} &= \frac{-mMGz}{(x^2+y^2+z^2)^{\frac{3}{2}}}
\end{aligned}$$

So,

$$\begin{aligned}
\nabla f &= \left\langle -\frac{mMGx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{mMGy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{mMGz}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle \\
&= F(x,y,z)
\end{aligned}$$

where F is the gravitational field of example 370 on page 253. This means that the gravitational field is a conservative vector field. Its potential function is $f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$.

The following theorem gives a necessary and sufficient condition for a vector field in $\mathbb{R}^2$ to be a conservative vector field. A similar condition for $\mathbb{R}^3$ vector fields will be given in the next subsection.

Theorem 378 (Test for Conservative Vector Fields in the Plane) *Suppose that $P(x,y)$ and $Q(x,y)$ have continuous first partial derivatives on an open disk D. The vector field $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is conservative if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Proof. *Why the condition is sufficient is more difficult to prove and will not be done here. It is easy to see why it is necessary. For F to be conservative, there must exist a function f such that $F = \nabla f$. Therefore we must have*

$$\langle f_x, f_y\rangle = \langle P, Q\rangle$$

Under the conditions of the theorem, Clairaut's theorem applies, thus the mixed partials must be equal that is $f_{xy} = f_{yx}$. Since $f_x = P$, $f_{xy} = \dfrac{\partial P}{\partial y}$. Similarly since $f_y = Q$, $f_{yx} = \dfrac{\partial Q}{\partial x}$. ■

Example 379 *Prove that $F(x,y) = \langle 2xy, x^2 - y\rangle$ is a conservative vector field and find a potential function for it.*
We use the theorem above. For this, we compute

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial (2xy)}{\partial y} \\ &= 2x \end{aligned}$$

and

$$\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{\partial (x^2 - y)}{\partial x} \\ &= 2x \end{aligned}$$

Since the two partials are equal, by the theorem, we know our vector field is conservative. We can now find its potential function. We are looking for a function f such that $F = \nabla f$ that is

$$\langle f_x, f_y\rangle = \langle 2xy, x^2 - y\rangle$$

Thus we need to solve

$$\begin{cases} f_x(x,y) = 2xy \\ f_y(x,y) = x^2 - y \end{cases}$$

that is

$$\begin{cases} f(x,y) = \int f_x(x,y)\,dx = \int 2xydx \\ f(x,y) = \int f_y(x,y)\,dy = \int \left(x^2 - y\right) dy \end{cases}$$

Now to find $\int 2xydx$*, we remember that we are looking for a function of* x *and* y*. From differential calculus, you remember that* $\int 2xydx = yx^2 + C$*. What is* C*? It is a constant. But when we integrate with respect to* x*, a function of* y *is also a constant. So, we see that*

$$f(x,y) = \int 2xydx = x^2y + g(y)$$

Similarly

$$f(x,y) = \int \left(x^2 - y\right) dy = x^2y - \frac{y^2}{2} + h(x)$$

Thus, we see that $g(y) = -\frac{y^2}{2} + C$ *and* $h(x) = C$*. It follows that*

$$f(x,y) = x^2y - \frac{y^2}{2} + C$$

5.1.4 Curl and Divergence of a Vector Field

Definition 380 *Consider the vector field in space* $F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$*. The **curl** of* F*, denoted* $\operatorname{curl} F$ *is defined to be*

$$\begin{aligned} \operatorname{curl} F &= \nabla \times F \\ &= \left\langle \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right), \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right), \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \right\rangle \end{aligned}$$

Definition 381 *If* $\operatorname{curl} F = \overrightarrow{0}$*,* F *is said to be **irrotational**.*

Remark 382 *It is easy to derive the formula for* $\operatorname{curl} F$ *by using the determinant form of the cross product.*

$$\nabla \times F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Theorem 383 (Test for Conservative Vector Fields in Space) *Suppose that* $P(x,y,z)$*,* $Q(x,y,z)$*, and* $R(x,y,z)$ *have continuous first partial derivatives in an open sphere* S *in space. The vector field* $F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ *is conservative if and only if*

$$\operatorname{curl} F = \overrightarrow{0}$$

that is if and only if

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Definition 384 *1. The **divergence** of a plane vector field $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is*

$$\begin{aligned}\operatorname{div} F(x,y) &= \nabla \cdot F(x,y)\\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\end{aligned}$$

*2. The **divergence** of a space vector field $F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is*

$$\begin{aligned}\operatorname{div} F(x,y,z) &= \nabla \cdot F(x,y,z)\\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\end{aligned}$$

Remark 385 (Interpretation of curl and div) *Think of a vector field v as the velocity field of some fluid. Then:*

1. $\operatorname{div} v(x,y,z)$ *gives us an indication of whether the fluid tends to accumulate near the point (x,y,z) (negative divergence) or tends to move away from the point (x,y,z) (positive divergence).*

2. $\operatorname{curl} v(x,y,z)$ *measure the rotational tendency of the fluid at the point (x,y,z).*

Example 386 *Let α be a constant and set $v(x,y,z) = \langle \alpha x, \alpha y, \alpha z\rangle$. Find $\operatorname{div} v$ and $\operatorname{curl} v$.*

$$\begin{aligned}\operatorname{div} v &= \nabla \cdot v\\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle \alpha x, \alpha y, \alpha z\rangle\\ &= \alpha\frac{\partial x}{\partial x} + \alpha\frac{\partial y}{\partial y} + \alpha\frac{\partial z}{\partial z}\\ &= 3\alpha\end{aligned}$$

$$\begin{aligned}\operatorname{curl} v &= \nabla \times v\\ &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \alpha x & \alpha y & \alpha z\end{vmatrix}\\ &= 0\end{aligned}$$

Figure 5.4 shows a plot of this vector field when $\alpha = -1$. We see that all the arrows converge toward the origin. Also, this vector field has no rotational tendency.

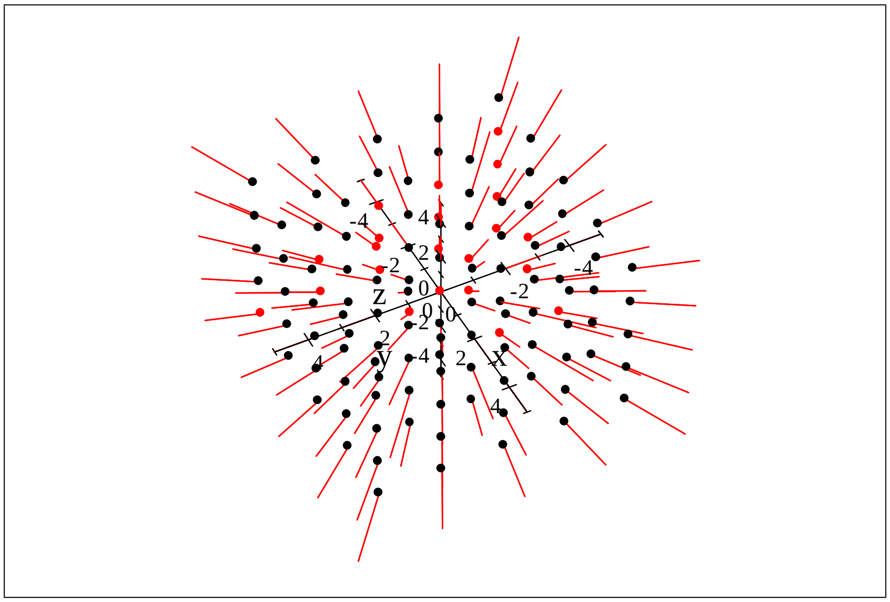


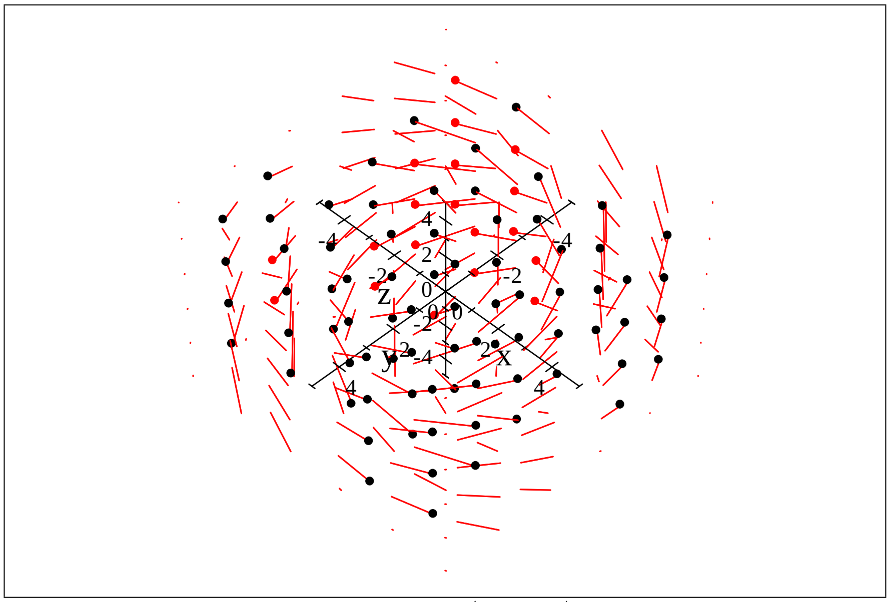
Figure 5.4: Vector field $\langle -x - y - z \rangle$

Example 387 *Let* $F(x,y,z) = \langle -y, x, z \rangle$. *Find* $\operatorname{div} F$ *and* $\operatorname{curl} F$.

$$\begin{aligned}
\operatorname{div} F &= \nabla \cdot F \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle -y, x, z \rangle \\
&= -\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial z}{\partial z} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} F &= \nabla \times F \\
&= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} \\
&= \left(\frac{\partial z}{\partial y} - \frac{\partial x}{\partial z} \right) \overrightarrow{i} - \left(\frac{\partial z}{\partial x} + \frac{\partial y}{\partial z} \right) \overrightarrow{j} + \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) \overrightarrow{k} \\
&= 2\overrightarrow{k} \\
&= \langle 0, 0, 2 \rangle
\end{aligned}$$

A plot of this vector field is shown in figure 387.



Vector Field $\langle -y, x, z\rangle$

5.1.5 Assignment

1. Sketch the vector fields below:

   (a) $F(x,y) = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$.

   (b) $F(x,y) = \left\langle y, \frac{1}{2} \right\rangle$.

   (c) $F(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$.

   (d) $F(x,y,z) = (\sin x, y, z)$.

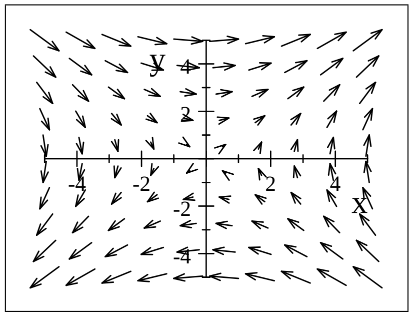
2. Match the vector fields in $\mathbb{R}^2$ with the given plots.

   (a) $F(x,y) = \langle y, x\rangle$.

   (b) $F(x,y) = \langle x-2, x+1\rangle$.

   (c) $F(x,y) = \langle 1, \sin y\rangle$.

   (d) $F(x,y) = \left\langle y. \frac{1}{x} \right\rangle$.



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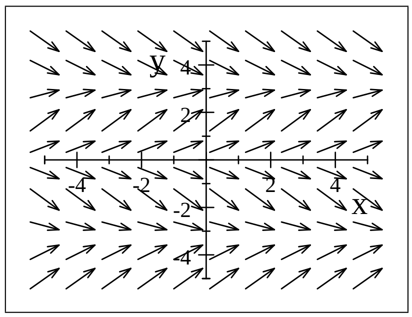

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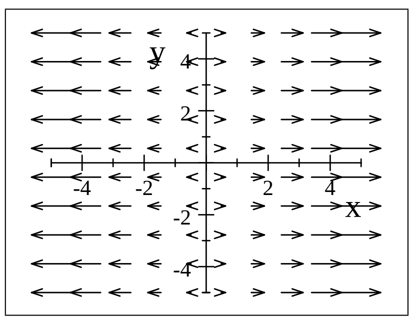

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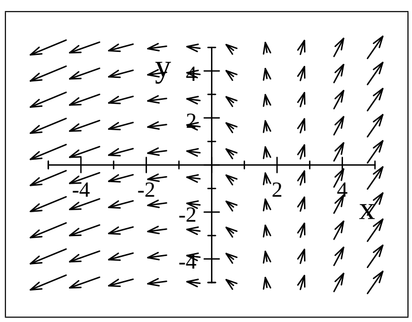


(h)

3. Match the vector fields in $\mathbb{R}^3$ with the given plots.

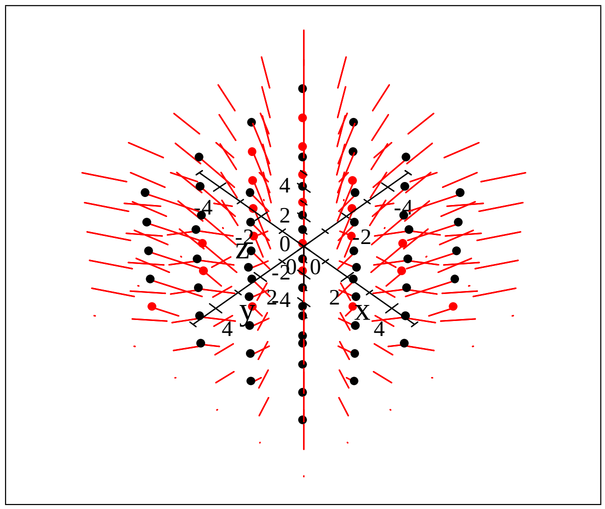
   (a) $F(x, y, z) = \langle 1, 2, 3 \rangle$.

   (b) $F(x, y, z) = \langle 1, 2, z \rangle$.

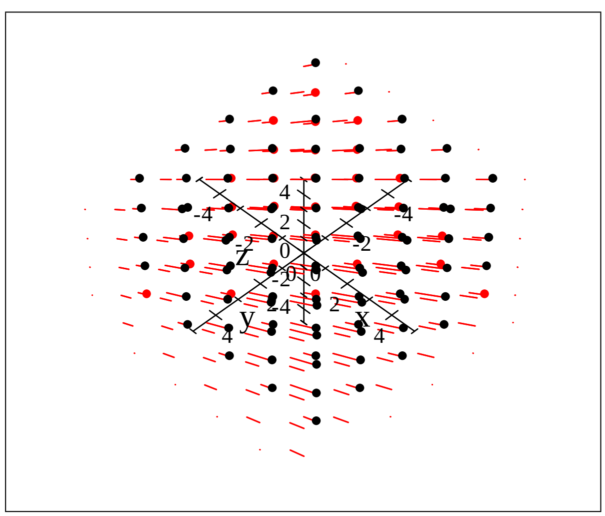
   (c) $F(x, y, z) = \langle x, y, 3 \rangle$.

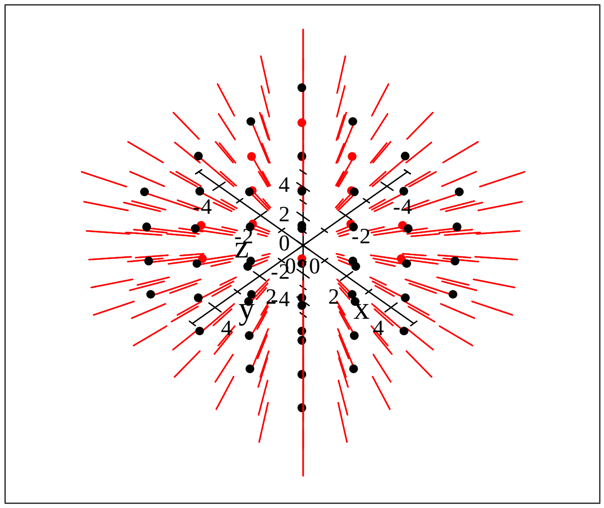
(d) $F(x, y, z) = \langle x, y, z \rangle$.

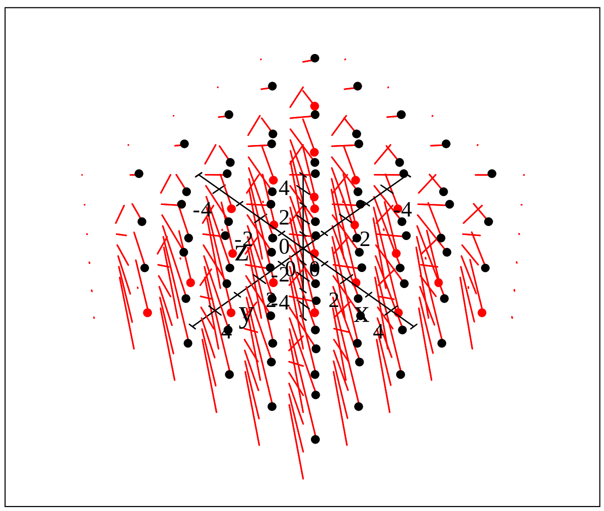
(e)



(f)



(g) 

(h) 

4. Find the gradient vector fiels of f for the functions below.

   (a) $f(x,y) = \ln(x+2y)$.

   (b) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$.

   (c) $f(x,y) = x^{\alpha}e^{-\beta x}$.

5. Find the curl and the divergence of the vector fields below.

   (a) $F(x,y,z) = \langle xyz, 0, -x^2y\rangle$.

   (b) $F(x,y,z) = \langle 1, x+yz, xy-\sqrt{z}\rangle$.

   (c) $F(x,y,z) = \langle e^x \sin y, e^x \cos y, z\rangle$.

   (d) $F(x,y,z) = \langle e^{-x}\sin y, e^{-y}\sin z, e^{-z}\sin x\rangle$.

6. Determine if the given vector fields F are conservative. If they are, find a potential function for them, that is a function f such that $F = \nabla f$.

    (a) $F(x, y) = \left\langle (1 + xy) e^{xy}, e^{y} + x^{2} e^{xy} \right\rangle$.

    (b) $F(x, y, z) = \langle e^{x} \cos y + yz, xz - e^{x} \sin y, xy + z \rangle$.

    (c) $F(x, y, z) = \langle yz, xz, xy \rangle$.

    (d) $F(x, y, z) = \left\langle 2xy, x^{2} + 2yz, y^{2} \right\rangle$.

    (e) $F(x, y, z) = \langle ye^{-x}, e^{-x}, 2z \rangle$.