Quantum Mechanics cont'd…

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SAS

Wave Function

- A wave function is a mathematical description of the state of a system.
- It is a complex function and is generally represented as ψ.
- All measurable quantities, such as energy, momentum, position, etc of the system can be deduced from the wave function.
- Wave function ψ is continuous and single-valued everywhere.
- Derivatives $\partial\psi/\partial x$, $\partial\psi/\partial y$, $\partial\psi/\partial z$ is a continuous and single-valued everywhere.
- It follows the **principle of superposition**. $\psi = \psi_1 + \psi_2$ like waves we have studied so far.
- The wave function is not a physical quantity (can not be measured), but the square of the wave function, $|\psi|^2$ is real and has physical meaning.
- Wave function can be obtained by solving the Schrödinger equation for the system

Properties of Wave Function

1. $\Psi(\vec{r}, t)$ is complex. It can be written in the form

$$\Psi(\vec{r}, t) = A(\vec{r}, t) + i\, B(\vec{r}, t)$$

   where A and B are real functions.

2. Complex conjugate of Ψ is defined as

$$\Psi^* = A - iB$$

3. $|\Psi|^2 = \Psi^*\Psi = A^2+B^2$

   Therefore $|\Psi|^2 = \Psi^*\Psi$ is always positive and real.

4. While Ψ itself has no physical interpretation, $|\Psi|^2$ evaluated at a particular place at a particular time equals to the probability of finding the body there at that time.

Properties of Wave Function

$|\psi|^2 dx$ = Probability of finding the particle in a region having length dx

$\int_{-x_1}^{+x_2} |\psi|^2 dx$ = Probability of finding the particle in a region between x_1 and x_2

$\int_{-\infty}^{+\infty} |\psi|^2 dx$ = Total probability of finding the particle in the entire space = 1

In 3-Dimensional

$|\psi|^2 \mathrm{dxdydz} = |\psi|^2 \mathrm{dV}$ is the probability of finding the body in a box of volume dV at time t

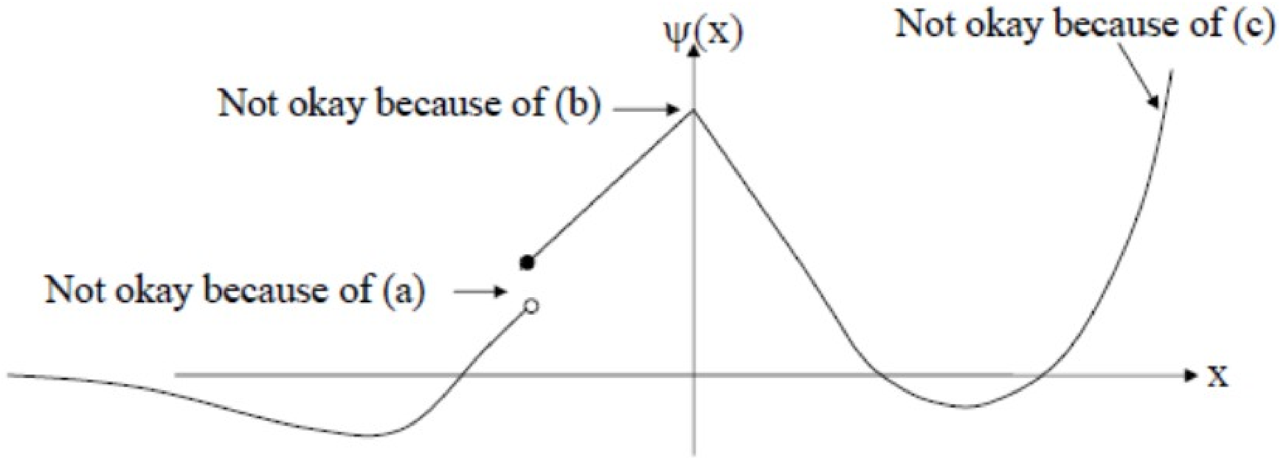
Properties of Wave Function

5. Mathematical properties of Ψ:

a. Ψ must be continuous and single-valued everywhere.

b. $\partial\Psi/\partial x$, $\partial\Psi/\partial y$, $\partial\Psi/\partial z$ must be continuous and single-valued everywhere. (There may be exception in some special situations, we will discuss this later.)

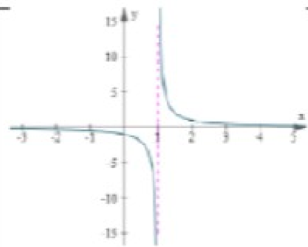
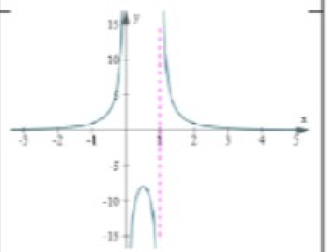
c. Ψ must be normalizable. $|\Psi|^2$ must go 0 fast enough as x, y, or z $\rightarrow \pm\infty$ so that $\int |\Psi|^2\, dV$ remains finite.



Properties of Wave Function

- ψ must be finite for all values of x Ψ can never be infinite
- Ex: $\psi = \psi_0 \tan x$ cannot be an acceptable – x=90° → tan x=∞ 。
- $\psi = \psi_o \sin(1/x)$ is also not an acceptable wave function – x=0→ sin (1/x)=∞
- ψ must be single-valued
  →there is no multiple probabilities of finding the particle at the same point
  Ex: $\psi = \psi_o \sin\sqrt{x}$- not an acceptable wavefunction (for given x, √x has (±) two values)
- ψ should be continuous
  i.e. probability of finding particle at all the points in the region of interest can be specific

Examples of discontinuous ψ

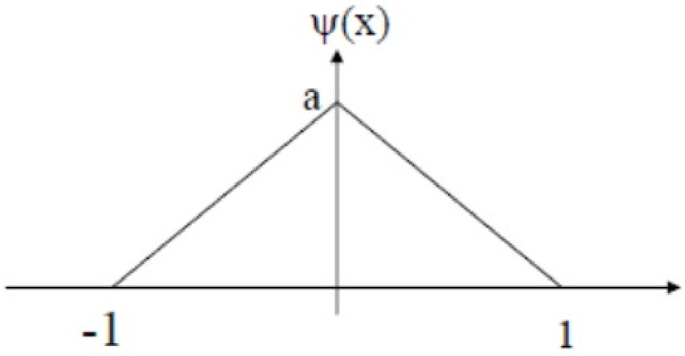
Formula	Graph	Formula	Graph
$\psi(x) = \frac{1}{x-1}$		$\psi(x) = \frac{2}{x^2 - x} = \frac{2}{x(x-1)}$	

Normalisation of Wave Function ψ

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dV = 1$$

If a wavefunction is not normalized, we can make it so by dividing it with a normalization constant. E.g.

$$f(x) = \begin{cases} a(1-x) & x \geq 0 \\ a(1+x) & x < 0 \end{cases}$$



$$\therefore \int_{-\infty}^{\infty} |f(x)|^2 \, dx = 2\int_0^1 [a(1-x)]^2 dx$$

$$= 2a^2 \left[-\frac{(1-x)^3}{3} \right]_0^1$$

$$= \frac{2}{3}a^2 \neq 1$$

$$\therefore \text{ f(x) is not normalixed, but } \psi(x) = \frac{f(x)}{\sqrt{\frac{2}{3}}a} \text{ is!}$$

Wave Nature of Electron: Invention of Electron Microscope

With a visible light microscope, we are limited to being able to resolve objects which are at least about $\underline{0.5 \times 10^{-6}\ m} = \underline{0.5\ \mu m} = \underline{500\ nm}$ in size.

This is because visible light, with a wavelength of ~500 nm cannot resolve objects whose size is smaller than it's wavelength.

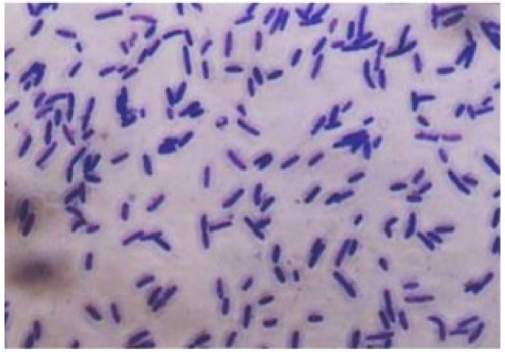
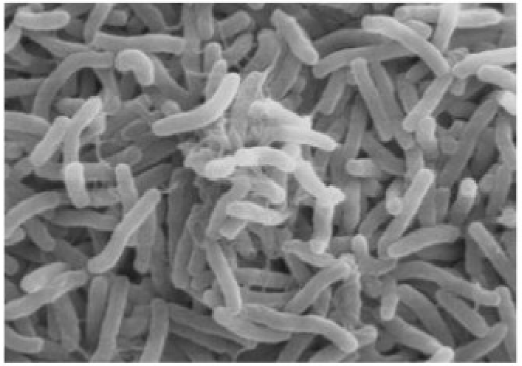


Image is in the public domain

Bacteria, as viewed using visible light



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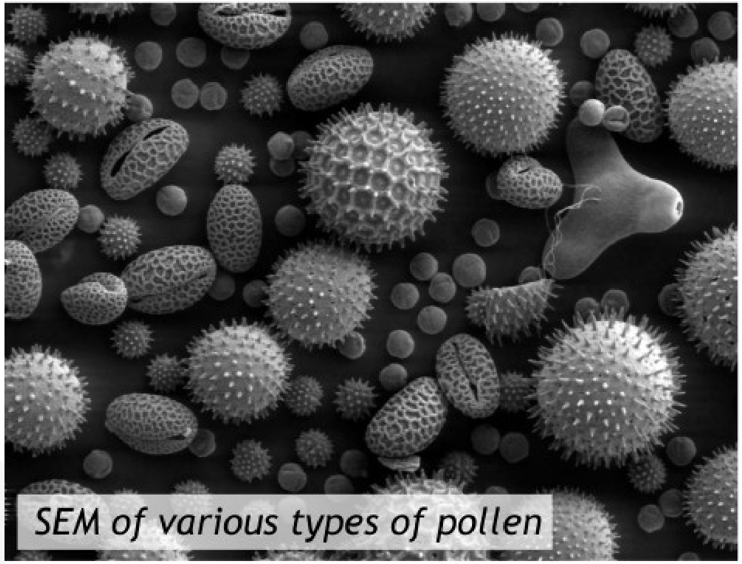
Bacteria, as viewed using electrons!

Wave Nature of Electron: Invention of Electron Microscope



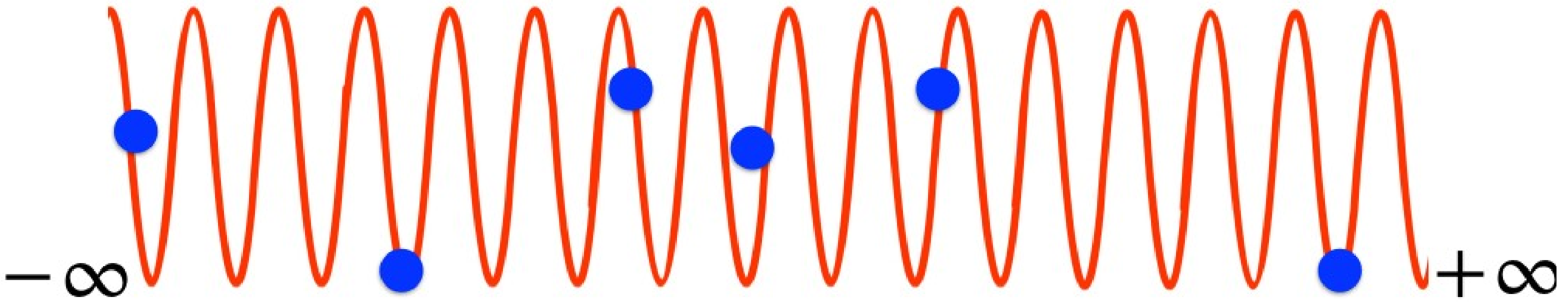
SEM of various types of pollen

SEM of an ant head

Heisenberg Realised that…

According to De Broglie, for matter wave of definite momentum, the wavelength can be defined as:

$$\boxed{\lambda_B = \frac{h}{p}}$$



Heisenberg Uncertainty Principle

It is impossible to determine simultaneously with unlimited precision the position and momentum of a particle

If a measurement of position is made with precision Δx and a simultaneous measurement of momentum in the x direction is made with precision Δp, then the product of the two uncertainties can never be smaller than $h/(4\pi)$. That is,

$$\boxed{\Delta x \Delta p \geq \frac{h}{4\pi}}$$

Heisenberg Realised that…

- In the world of very small particles, one cannot measure any property of a particle without interacting with it in some way

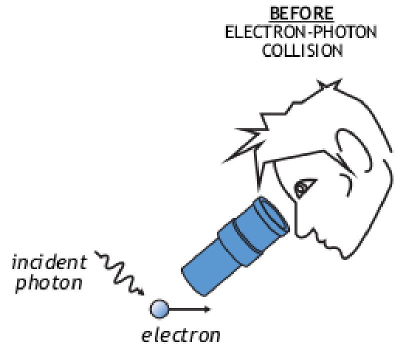
- This introduces an unavoidable uncertainty into the result

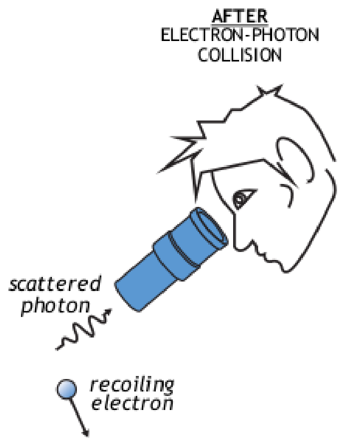
- One can never measure all the properties exactly

Werner **Heisenberg**
Nobel Prize for Physics for 1932.

Position-Momentum of an electron Measurement

- Shine light on electron and detect reflected light using a microscope
- Minimum uncertainty in position is given by the wavelength of the light
- So to determine the position accurately, it is necessary to use light with a short wavelength





- By Planck's law $E = hc/\lambda$, a photon with a short wavelength has a large energy
- Thus, it would impart a large 'kick' to the electron
- But to determine its momentum accurately, electron must only be given a small kick
- This means using light of long wavelength !

Implications

- It is impossible to know *both* the position and momentum exactly, i.e., $\Delta x=0$ and $\Delta p=0$
- These uncertainties are inherent in the physical world and have nothing to do with the skill of the observer
- Because *h* is so small, these uncertainties are not observable in normal everyday situations

$$\boxed{\Delta x \Delta p \geq \frac{h}{4\pi}}$$

Heisenberg Uncertainty Principle: Examples

Cricket ball
Macroscopic Object

- A pitcher throws a 0.1-kg baseball at 40 m/s
- So momentum is 0.1 x 40 = 4 kg m/s
- Suppose the momentum is measured to an accuracy of 1 %, i.e.,

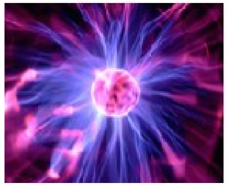
$$\Delta p = 0.01\, p = 4 \times 10^{-2} \text{ kg m/s}$$

- The uncertainty in position is then

$$\Delta x \geq \frac{h}{4\pi \Delta p} = 1.3 \times 10^{-33} \mathrm{m}$$

- No wonder one does not observe the effects of the uncertainty principle in everyday life!

Heisenberg Uncertainty Principle: Examples



Electron
(microscopic
object)

Same situation, but baseball replaced by an electron which has mass 9.11 x 10^{-31} kg traveling at 40 m/s

So momentum = 3.6 x 10^{-29} kg m/s
and its uncertainty = 3.6 x 10^{-31} kg m/s

$\Delta p = 0.01\, p = 4 \times 10^{-2}$ kg m/s

Uncertainty in momentum for a ball

The uncertainty in position is then

$$\Delta x \geq \frac{h}{4\pi \Delta p} = 1.4 \times 10^{-4} \mathrm{m}$$

Heisenberg Uncertainty Principle: Energy and time

$$v = \frac{\Delta x}{\Delta t}$$

$$\Delta x = v.\Delta t$$

Change in kinetic energy of the particle is

$$\Delta E = \Delta p.v$$

=> $$\Delta p = \frac{\Delta E}{v}$$

Hence, $$\Delta x.\Delta p = v.\Delta t \times \frac{\Delta E}{v}$$

=> $$\Delta x.\Delta p = \Delta E.\Delta t$$

Then from uncertainty principle

$$\boxed{\Delta E.\Delta t = \hbar / 2}$$

The time-dependent wavefunction of a particle confined to a region between 0 and **L** is

$$\psi(x,t) = A\,e^{-i\omega t}\sin\left(\pi x/L\right)$$

where ω is angular frequency and E is the energy of the particle. (**Note:** The function varies as a sine because of the limits (0 to **L**). When $x = 0$, the sine factor is zero and the wavefunction is zero, consistent with the boundary conditions.

Solution

Computation of the normalization constant:

$$
\begin{aligned}
1 &= \int_0^L dx\, \psi^*(x)\psi(x) \\
&= \int_0^L dx\, \left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(Ae^{-i\omega t}\sin\frac{\pi x}{L}\right) \\
&= A^2\int_0^L dx\, \sin^2\frac{\pi x}{L} \\
&= A^2\frac{L}{2} \\
\Rightarrow A &= \sqrt{\frac{2}{L}}.
\end{aligned}
$$

5. The wavefunction for a quantum particle of mass m confined to move in the domain $0 \le x \le L$ is given by

$$\psi(x) = N \sin(4\pi x/L)$$

where N is the normalization factor.

(a) Normalize the wavefunction.

Answer:

$$N^2 \int_0^L \sin^2\left(\frac{4\pi x}{L}\right) dx = 1$$

$$y = \frac{4\pi x}{L} \qquad x = \frac{L}{4\pi} y \qquad dx = \frac{L}{4\pi} dy$$

$$\frac{LN^2}{4\pi} \int_0^{4\pi} \sin^2 y dy = N^2 \frac{L}{4\pi}\frac{4\pi}{2} = N^2 \frac{L}{2} = 1$$

$$N = \left(\frac{2}{L}\right)^{1/2}$$

$$\psi(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{4\pi x}{L}\right)$$

6. The state of a one-dimensional quantum system is represented by the wavefunction

$$\psi(x) = N \sin(3\pi x)$$

for $0 < x < 1$ with N being the normalization factor. Calculate the probability that a measurement of the position of the particle will give a result in the range $2/3 \leq x < 1$.

Answer:

$$N^2 \int_0^1 \sin^2 3\pi x \; dx = 1$$

$$y = 3\pi x \qquad dx = \frac{1}{3\pi}\, dy$$

$$\frac{N^2}{3\pi} \int_0^{3\pi} \sin^2 y \; dy = \frac{N^2}{3\pi} \left[\frac{y}{2} - \frac{1}{4} \sin 2y \right]_0^{3\pi} = \frac{N^2}{2} = 1$$

or

$$N = \sqrt{2}$$

13. A one-dimensional particle of mass m occupies the interval $0 \leq x < \infty$ in a state defined by the wavefunction

$$\psi(x) = Nxe^{-ax}$$

where N is the normalization constant and a is a constant having units of inverse length. Normalize the wavefunction, and use the normalized wavefunction to calculate the expectation value of the kinetic energy $\langle T \rangle$ of the particle.

Answer:

$$N^2 \int_0^\infty x^2 e^{-2ax} dx = N^2 \frac{2!}{(2a)^3} = \frac{N^2}{4a^3} = 1 \qquad N = 2a^{3/2}$$

14. The unnormalized wavefunction for a quantum particle on the domain $0 \leq x < \infty$ is given by

$$\psi(x) = Nxe^{-ax^2}$$

where N is the normalization and a is a constant having units of the square of the inverse length. Calculate the expectation value of x^2 for the particle.

Answer:

$$N^2 \int_0^\infty x^2 \, e^{-2ax^2} \, dx = N^2 \frac{1}{4} \frac{\pi^{1/2}}{(2a)^{3/2}} = 1 \qquad N = 2 \left[\frac{(2a)^{3/2}}{\pi^{1/2}} \right]^{1/2}$$

$$\Gamma(z) = \int_0^\infty x^{z-1} \cdot e^{-x}\, dx$$

or you can write ...

$$\Gamma(z+1) = \int_0^\infty x^{z} \cdot e^{-x}\, dx$$

5) $\Gamma(n) = (n - 1)!$

Because the gamma function reduces in this special case to (n - 1)! it is often convenient to view it as a **generalized factorial function**.

Special values

1) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2) $\Gamma\left(m+\frac{1}{2}\right) = \dfrac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m}\sqrt{\pi} \qquad m = 1, 2, 3, \cdots$

3) $\Gamma\left(-m+\frac{1}{2}\right) = \dfrac{(-1)^m 2^m \sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2m-1)} \qquad m = 1, 2, 3, \cdots$

Quantum Mechanics cont’d…

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Time-dependent Schrödinger Wave Equation

Schrödinger Equation is a mathematical expression that describes the change of a physical quantity over time in which the quantum effects like wave-particle duality are significant…

- In other words, we can say that …..It is a differential equation for the de Broglie waves associated with particles and describes the motion of particles.
- The Schrodinger equation is the fundamental equation of wave mechanics in the same sense as Newton's second law of motion of classical mechanics

The Schrödinger Equation has two forms:

- Time-dependent Schrödinger Equation
- Time-independent Schrödinger Equation

$$i\hbar\frac{\partial\psi(r,t)}{\partial t} = H\psi(r,t)$$

$Where,\ H = -\frac{\hbar^2}{2m}\nabla^2 + V$ $\qquad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$

$V = time\ dependant\ Potential\ energy$

Derivation: Time-dependent Schrödinger Wave Equation

Consider a particle of having mass **m**, moving in the **x**-direction; having total energy **E** and momentum, **p**

⇒Then, according to classical mechanics, the total energy associated with the particle is:

$$\Rightarrow\ E = KE + PE$$

$$\Rightarrow\ E = \frac{\mathrm{p}^2}{2\mathrm{m}} + \mathrm{V}$$

⇒If the particle is associated with a matter wave, then it can be represented as a wave function, ψ:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

k, is the propagation vector and ω, angular frequency

Derivation: Time-dependent Schrödinger Wave Equation

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

we know that,

$$\Rightarrow E = h\nu$$

$$\Rightarrow E = h\frac{\omega}{2\pi} = \hbar\omega$$

$$\Rightarrow \omega = \frac{E}{\hbar}\,.\,.\,.\,.\,.\,.\,.\,(a)$$

Also, we know that,

$$\lambda = \frac{h}{p} \Rightarrow p = \frac{h}{\lambda}; \quad k = \frac{2\pi}{\lambda}$$

$$\Rightarrow p = \frac{hk}{2\pi} = \hbar k \;\; \Rightarrow k = \frac{p}{\hbar}\,.\,.\,.\,.\,(b)$$

on substituting the eq. “a & b” in wave function $\psi(x,t)$

$$\psi(x,t) = Ae^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)}$$

$$\Rightarrow\; = Ae^{\frac{i}{\hbar}(px-Et)}\,.\,.\,.\,.\,.\,.\,.\,(1)$$

Taking the partial derivative w.r.t. to position of the wave function $\psi(x,t)$:

$$\Rightarrow \quad \frac{\partial\psi(\mathrm{x,t})}{\partial \mathrm{x}} = = Ae^{\frac{i}{\hbar}(px-Et)}\left(\frac{ip}{\hbar}\right)$$

$$\Rightarrow \frac{\partial^2\psi(\mathrm{x,t})}{\partial \mathrm{x}^2} = Ae^{\frac{i}{\hbar}(px-Et)}\left(\frac{ip}{\hbar}\right)^2$$

$$\Rightarrow \frac{\partial^2\psi(\mathrm{x,t})}{\partial \mathrm{x}^2} = \left(\frac{ip}{\hbar}\right)^2\psi(x,t)$$

$$\Rightarrow p^2\psi(x,t) = -\,\hbar^2\,\frac{\partial^2\psi(\mathrm{x,t})}{\partial \mathrm{x}^2}\,.\,.\,.\,.\,.\,(2)$$

Derivation: Time-dependent Schrödinger Wave Equation

$$\psi(x,t) = Ae^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)}$$
$$= Ae^{\frac{i}{\hbar}(px - Et)} \ldots\ldots.(1)$$

Let's take the partial derivative w.r.t. to time of the wave function $\psi(x,t)$:

$$\Rightarrow \frac{\partial\psi(\mathrm{x,t})}{\partial \mathrm{t}} = Ae^{\frac{i}{\hbar}(px-Et)}(\frac{-iE}{\hbar})$$

$$\Rightarrow \frac{\partial\psi(\mathrm{x,t})}{\partial \mathrm{t}} = (\frac{-iE}{\hbar})\ \psi(x,t)$$

$$\Rightarrow E\psi(x,t) = (\frac{\hbar}{-i})\ \frac{\partial\psi(\mathrm{x,t})}{\partial \mathrm{t}}$$

$$\Rightarrow E\psi(x,t) = i\hbar\ \frac{\partial\psi(\mathrm{x,t})}{\partial \mathrm{t}} \ldots\ldots(3)$$

$$E = \frac{\mathrm{p}^2}{2\mathrm{m}} + \mathrm{V}$$

operating over the wave function $\psi(x,t)$:

$$E\psi(x,t) = \frac{\mathrm{p}^2}{2\mathrm{m}}\psi(\mathrm{x,t}) + \mathrm{V}\psi(\mathrm{x,t})$$

using the equation 2 and 3,the above equation can be changes to

$$\boxed{\mathrm{i}\hbar\frac{\partial\psi(\mathrm{x,t})}{\partial \mathrm{t}} = -\frac{\hbar^2}{2\mathrm{m}}\frac{\partial^2\psi(\mathrm{x,t})}{\partial \mathrm{x}^2} + \mathrm{V}\psi(\mathrm{x,t})}$$

$$\Downarrow$$

Time-dependent Schrödinger Wave Equation in 1D

Derivation: Time-dependent Schrödinger Wave Equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V\psi(x,t)$$

1D

$$\Rightarrow i\hbar\frac{\partial\psi(x,y,z,t)}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi(x,y,z,t)}{\partial x^2} + \frac{\partial^2\psi(x,y,z,t)}{\partial y^2} + \frac{\partial^2\psi(x,y,z,t)}{\partial z^2}\right] + V\psi(x,y,z,t)$$

3D

$$\Rightarrow i\hbar\frac{\partial\psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + V\psi(r,t)$$

$$i\hbar\frac{\partial\psi(r,t)}{\partial t} = H\psi(r,t)$$

$$H \equiv -\frac{\hbar^2}{2m}\nabla^2 + V$$

Derivation: Time-dependent Schrödinger Wave Equation

$$i\hbar\frac{\partial\psi(r,t)}{\partial t} = H\psi(r,t) \qquad H \equiv -\frac{\hbar^2}{2m}\nabla^2 + V$$

Any condition imposed on the motion of a particle will affect the potential energy U, which is a function of x & t. By knowing the exact form of U, the equation may be solved for Ψ. The time-dependent Schrodinger equation is used to explain non-stationary phenomena, such as the electronic transition between two states of atom.

Time-dependent Schrödinger Wave Equation

The Schrödinger Equation has two forms:

- Time-dependent Schrödinger Equation
- Time-independent Schrödinger Equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V\psi(x,t)$$

$$i\hbar\frac{\partial\psi(r,t)}{\partial t} = H\psi(r,t)$$

Time-independent Schrödinger Wave Equation

In many atomic phenomena, the **potential energy** of the particle is **independent of time and depends only on the position** of the particle. In such situations, the differential equation for de-Broglie waves associated with particles is called the time-independent (**stationary/steady state**) Schrodinger wave equation.

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

$$\Rightarrow \psi(x,t) = Ae^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)}$$

$$\Rightarrow = Ae^{i(\frac{p}{\hbar}x)}e^{(-i\frac{E}{\hbar}t)} \qquad (\therefore\ e^{m+n} = e^m + e^n)$$

$$\Rightarrow = \psi(x)\ \phi(t)$$

$$\boxed{\psi(x,t) = \psi(x)\ \phi(t)}$$

separation of variables

Time-independent Schrödinger Wave Equation

we know that, the time dependent scrounger wave equation for particle moving in x direction is

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V\psi(x,t)$$

$$\boxed{\psi(x,t) \ = \ \psi(x)\ \phi(t)}$$

$$\Rightarrow \ i\hbar\frac{\partial\psi(x)\phi(t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)\phi(t)}{\partial x^2} + V\psi(x)\phi(t)$$

$$\Rightarrow \ \psi(x)\ \left[i\hbar\frac{\partial\phi(t)}{\partial t}\right] = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x)\right]\phi(t)$$

$$\Rightarrow \ \underbrace{\frac{1}{\phi(t)}\ \left[i\hbar\frac{\partial\phi(t)}{\partial t}\right]}_{\text{Function of time}} = \left[-\frac{\hbar^2}{2m}\underbrace{\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x)}_{\text{Function of position}}\right]\frac{1}{\psi(x)}$$

Time-independent Schrödinger Wave Equation

$$\underbrace{\frac{1}{\phi(t)}\left[i\hbar\frac{\partial\phi(\mathrm{t})}{\partial \mathrm{t}}\right]}_{\text{Function of time}} = \underbrace{\left[-\frac{\hbar^2}{2\mathrm{m}}\frac{\partial^2\psi(\mathrm{x})}{\partial \mathrm{x}^2} + V\psi(\mathrm{x})\right]\frac{1}{\psi(x)}}_{\text{Function of position}}$$

$$\boxed{\begin{aligned} \psi(x,t) &= \psi(x)\ \phi(t) \\ \psi(x) &= Ae^{i(\frac{p}{\hbar}x)} \\ \phi(t) &= e^{(-i\frac{E}{\hbar}t)} \end{aligned}}$$

$\Rightarrow$ LHS= RHS = S (Separation variable constant)

Lets find that constant, by calculating the LHS, as we know that:

$$LHS = \frac{1}{\phi(t)}\left[i\hbar\frac{\partial\phi(\mathrm{t})}{\partial \mathrm{t}}\right] \quad \text{and} \quad \phi(t) = e^{(-i\frac{E}{\hbar}t)}$$

upon substitution, we will have

$$\frac{1}{\phi(t)}\left[i\hbar\frac{\partial}{\partial \mathrm{t}}e^{(-i\frac{E}{\hbar}t)}\right] \Rightarrow \frac{1}{\phi(t)}\left[i\hbar e^{(-i\frac{E}{\hbar}t)}\left(\frac{-iE}{\hbar}\right)\right] \Rightarrow \frac{1}{\phi(t)}\left[i\hbar\ \phi(t)\ \left(\frac{-iE}{\hbar}\right)\right] \Rightarrow E$$

$\Rightarrow$ LHS= RHS = E (total energy of the system)

Time-independent Schrödinger Wave Equation

$$E = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x)\right]\frac{1}{\psi(x)}$$

$$\Rightarrow \frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x)\right] = E$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x) = E\psi(x)}$$

The is differential equation in positio only and can be easily solved to get energy of the system

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\psi(x) = E\psi(x)$$

$$H\psi(x) = E\psi(x)$$

where the above equation is Eigen value equation and H is the hamiltonian, and E is the solution

Time-independent Schrödinger Wave Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x) = E\psi(x)$$

we can also rearrange the above equation as:

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x) - E\psi(x) = 0$$

$$\Rightarrow \frac{\partial^2\psi(x)}{\partial x^2} - \frac{2m}{\hbar^2}(V - E)\,\psi(x) = 0$$

$$\boxed{\frac{\partial^2\psi(x)}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\,\psi(x) = 0}$$

Free particle: Time-independent Schrödinger Wave Equation

For a free particle, $V(x) = 0$

$$\Rightarrow \frac{\partial^2 \psi(\mathrm{x})}{\partial \mathrm{x}^2} + \frac{2\mathrm{m}}{\hbar^2}(\mathrm{E} - \mathrm{V})\ \psi(\mathrm{x}) = 0$$

$$\Rightarrow \frac{\partial^2 \psi(\mathrm{x})}{\partial \mathrm{x}^2} = -\frac{2\mathrm{mE}}{\hbar^2}\ \psi(\mathrm{x})$$

$$\Rightarrow \frac{\partial^2 \psi(\mathrm{x})}{\partial \mathrm{x}^2} = -\mathrm{k}^2\ \psi(\mathrm{x}) \qquad \therefore k = \sqrt{\frac{2mE}{\hbar^2}}$$

So, The solution to time independent Schrodinger Equation is: $\psi(x) = Ae^{ikx}$

and we know that, $\phi(t) = e^{(-i\frac{E}{\hbar}t)}$

So, The final solution is: $\boxed{\psi(x,t) = \psi(x)\ \phi(t) = Ae^{-i(kx - \frac{E}{\hbar}t)}}$