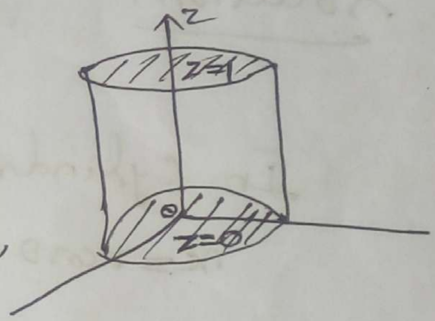
Change to cylindrical Co-ordinates:

① By changing into cylindrical coordinates, evaluate the integral $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the region of space defined by $x^2+y^2 \le 1$ and $0 \le z \le 1$.

Solution:

Here the region of space is enclosed by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$.



In cylindrical coordinates,

$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\iiint (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_0^1 (r^2+z^2)\,r\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\int_0^1 (r^3+rz^2)\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\left(\frac{r^4}{4}+\frac{r^2}{2}z^2\right)_0^1 d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\left[\frac{1}{4}+\frac{z^2}{2}\right] d\theta\,dz$$

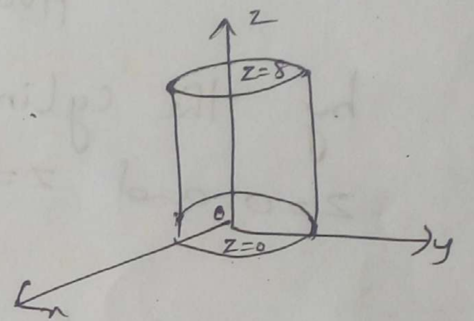
$$= \int_0^1 \left(\frac{\theta}{4} + \frac{\theta z^2}{2}\right)_0^{2\pi} dz$$

$$= \int_0^1 \left(\pi/2 + \pi z^2\right) dz$$

$$= \left(\frac{\pi}{2} z + \pi \frac{z^3}{3}\right)_0^1 = \pi/2 + \pi/3 = 5\pi/6.$$

(2) Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^8 2yz \, dz\, dy\, dx$ by changing into cylindrical polar coordinates.

<u>Solution:</u>



In cylindrical coordinates

$x = r\cos\theta, \; y = r\sin\theta, \; z = z$

r varies from 0 to 2.

θ varies from 0 to $\pi/2$.

z varies from 0 to 8.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^8 2yz \, dz\, dy\, dx = \int_0^{\pi/2}\int_0^2\int_0^8 2r\sin\theta \, z\, r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^2 2r^2\sin\theta \left(z^2/2\right)_0^8 dr\, d\theta$$

$$= 32\int_0^{\pi/2} \sin\theta \left(2r^3/3\right)_0^2 d\theta$$

$$= 32 \times \frac{2\times 8}{3}\left(-\cos\theta\right)_0^{\pi/2} = \frac{512}{3}(1) = \frac{512}{3}$$

③ Evaluate $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x \, dx\,dy\,dz$ by transforming into cylindrical Coordinates.

Solution:

The region of integration is the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Transforming into cylindrical Coordinates

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$

$dx\,dy\,dz = r\,dr\,d\theta\,dz.$

Now the sphere becomes $r^2+z^2=a^2$

$z^2 = a^2-r^2$

$z = \pm\sqrt{a^2-r^2}$

$z = \sqrt{a^2-r^2}$

$$\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x\,dx\,dy\,dz = \iiint r\cos\theta \; r\,dr\,d\theta\,dz$$

$$= \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} r^2\cos\theta \, dz\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^a r^2\cos\theta \, (z)_0^{\sqrt{a^2-r^2}} \, dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^a r^2\cos\theta \sqrt{a^2-r^2}\, dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \int_0^a r^2\sqrt{a^2-r^2}\, dr\,d\theta$$

put $r = a\sin t$

$dr = a\cos t\,dt$

when $r=0,\ t=0$

when $r=a,\ t=\pi/2$

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$$= \int_0^{\pi/2} \cos\theta\, d\theta \int_0^{\pi/2} a^2 \sin^2 t \sqrt{a^2 - a^2 \sin^2 t}\; a\cos t\, dt$$

$$= \int_0^{\pi/2} \cos\theta\, d\theta \int_0^{\pi/2} a^4 \sin^2 t \cos^2 t\, dt$$

$$= a^4 \int_0^{\pi/2} \cos\theta\, d\theta \int_0^{\pi/2} (1-\cos^2 t) \cos^2 t\, dt$$

$$= a^4 \left(\sin\theta\right)_0^{\pi/2} \left(\int_0^{\pi/2} \cos^2 t\, dt - \int_0^{\pi/2} \cos^4 t\, dt \right)$$

$$= a^4 (1-0) \left(\tfrac{1}{2} \tfrac{\pi}{2} - \tfrac{3}{4} \tfrac{1}{2} \tfrac{\pi}{2} \right)$$

$$= a^4 \frac{\pi}{16}$$

$$= \frac{\pi a^4}{16}.$$

(4) Evaluate $\iiint\limits_V xyz\, dx\, dy\, dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Solution:

Transform to cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Now, $x^2+y^2+z^2=a^2 \Rightarrow r^2+z^2=a^2 \Rightarrow z^2=a^2-r^2$

$$z = \sqrt{a^2-r^2}$$

The limits are z : $z=0$ to $z=\sqrt{a^2-r^2}$

r : $r=0$ to $r=a$

θ : $\theta=0$ to $\theta=\pi/2$

$$\iiint_V xyz\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} r^3\cos\theta\sin\theta\, z\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^3 \frac{\sin 2\theta}{2}\left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-r^2}} dr\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\int_0^a r^3\sin 2\theta\,(a^2-r^2)\,dr\,d\theta.$$

$$= \frac{1}{4}\int_0^{\pi/2}\sin 2\theta\int_0^a (r^3a^2 - r^5)\,dr\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\sin 2\theta\left[\frac{r^4}{4}a^2 - \frac{r^6}{6}\right]_0^a d\theta$$

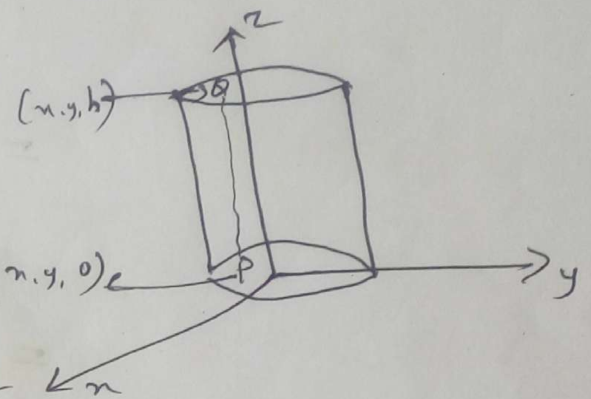
$$= \frac{1}{4}\int_0^{\pi/2}\sin 2\theta\left(\frac{a^6}{4} - \frac{a^6}{6}\right)d\theta$$

$$= \frac{1}{4}\frac{a^6}{12}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^6}{48}\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{a^6}{48}.$$

(5) Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$ where V is the region of space inside the cylinder $x^2+y^2=a^2$ that is bounded by the planes $z=0$ and $z=h$.



$$\iiint_V (x+y+z)\,dx\,dy\,dz = \int_0^h\int_0^{2\pi}\int_0^a (r\cos\theta + r\sin\theta + z)\,r\,dr\,d\theta\,dz$$

$$= \int_0^h\int_0^{2\pi}\left(\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta + z\frac{r^2}{2}\right)_0^a d\theta\,dz$$

$$= \int_0^h\left(\frac{a^3}{3}(\sin\theta) + \frac{a^3}{3}(-\cos\theta) + \frac{a^2}{2}z\theta\right)_0^{2\pi} dz$$

$$= \int_0^h\left[\frac{a^3}{3}(0+0) + \frac{a^3}{3}(-1+1) + \frac{a^2}{\cancel{2}}z(\cancel{2}\pi)\right]dz$$

$$= \pi a^2\left(\frac{z^2}{2}\right)_0^h$$

$$= \frac{\pi a^2 h^2}{2}$$

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(b) Find the volume of the portion of the cylinder $x^2+y^2=1$ intercepted between the plane $z=0$ and the paraboloid $x^2+y^2=4-z$.

Solution:

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$V = \iiint r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^{1}\int_0^{4-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r\,(z)_0^{4-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r\,(4-r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(4\frac{r^2}{2}-\frac{r^4}{4}\right)_0^1 d\theta$$

$$= \int_0^{2\pi}\left(2-\frac{1}{4}\right)d\theta = \frac{7}{4}(2\pi) = \frac{7\pi}{2}.$$

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Evaluate

(7) $\iiint dx\,dy\,dz$, taken throughout the volume of the cylinder $x^2+y^2=4$ bounded by the planes $z=0$ and $y+z=3$.

$z=3-y$
$=3-r\sin\theta$

$x=r\cos\theta$
$y=r\sin\theta$
$z=z$
$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\iiint dx\,dy\,dz = \iiint r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r\,(z)_0^{3-r\sin\theta}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r(3-r\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[3\frac{r^2}{2} - \frac{r^3}{3}\sin\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[6 - \frac{8}{3}\sin\theta\right] d\theta$$

$$= \int_0^{2\pi}\left(6 - \frac{8}{3}\sin\theta\right) d\theta$$

$$= \left(6\theta - \frac{8}{3}(-\cos\theta)\right)_0^{2\pi}$$

$$= 12\pi - \frac{8}{3}\left(\cancel{-1+1}\right) = 12\pi$$

0.

(8) $\iiint dx\,dy\,dz$ taken throughout the volume of the cylinder $x^2+y^2=4$ bounded by the plane $z=0$ and the surface $z=x^2+y^2+2$.

$z = r^2+2$

$$\iiint dx\,dy\,dz = \int_0^{2\pi}\int_0^{2}\int_0^{2+r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r(2+r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\cancel{2}\frac{r^2}{\cancel{2}} + \frac{r^4}{4}\right)_0^2 d\theta$$

$$= 8(2\pi) = 16\pi.$$

For full sphere $x^2+y^2+z^2=a^2$

r varies from 0 to a

θ varies from 0 to π

ϕ varies from 0 to 2π.

For hemisphere

r varies from 0 to a

θ " " 0 to $\pi/2$

ϕ " " 0 to 2π.

quadrant sphere (octant sphere)

r varies from 0 to a

θ varies from 0 to $\pi/2$

ϕ " " 0 to $\pi/2$.