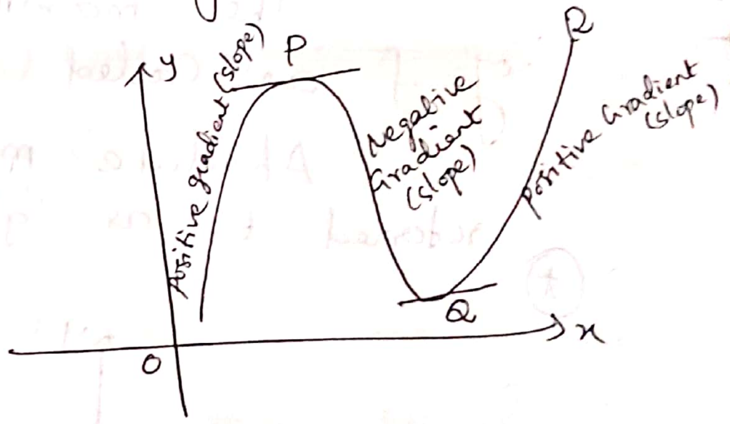
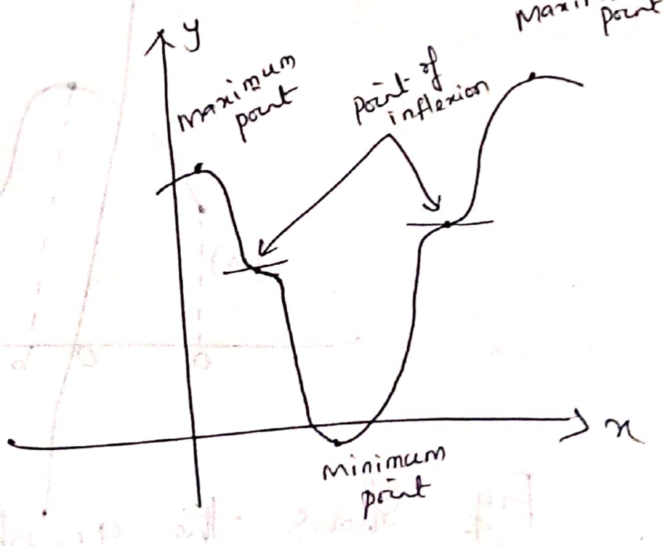
Module-2

Applications of Single variable Differentiation.

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. In many case these problems can be reduced to finding the maximum or minimum values of a function. Many problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation.





Definition:

A function f has an absolute maximum at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called maximum value of f on D.

Similarly f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called minimum value of f on D.

The maximum and minimum values of f are called extreme values of f.

Absolute maxima and minima are also referred to as global maxima or minima.

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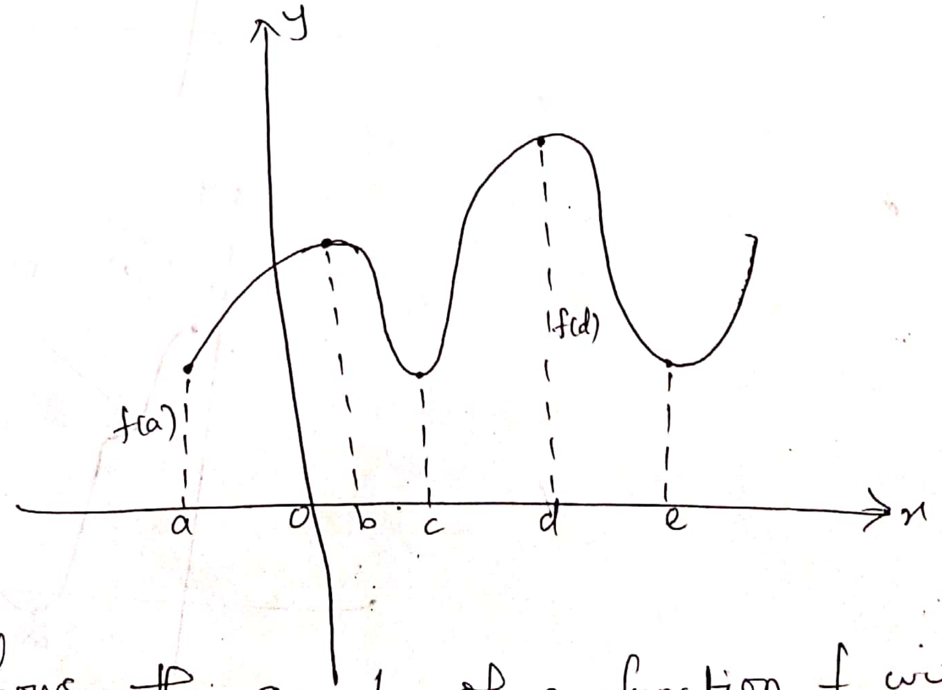
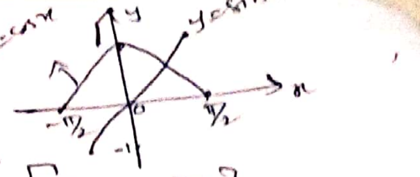


Fig. Shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

when $x = -\pi/2$, $y = \cos x = 0$
when $x = \pi/2$, $y = \cos x = 0$
$x = 0$, $y = 1$.

The points are $(-\pi/2, 0)$
$(0, 1)$
$(\pi/2, 0)$.



For example,

on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice).

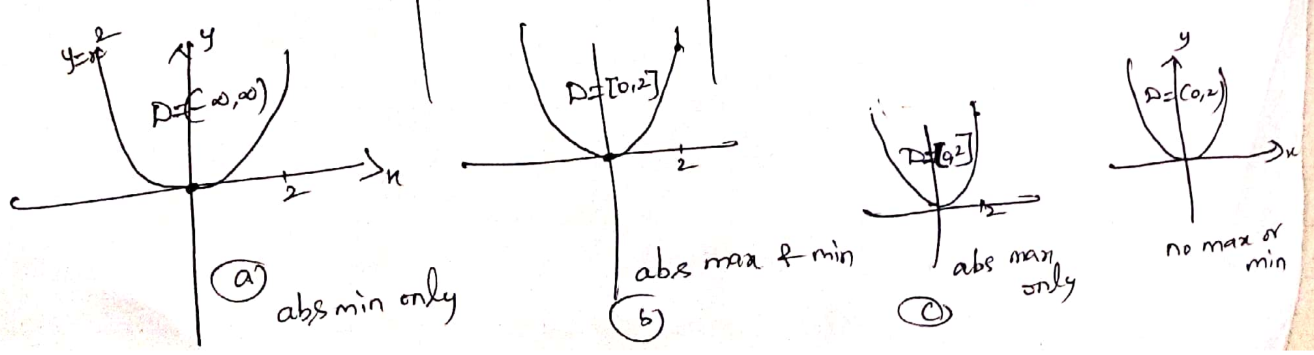
On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1.

Functions with the same defining Rule or formula can have different extrema (maximum or minimum values), depending on the domain.

when $x = -\pi/2$, $y = \sin x = -1$
when $x = \pi/2$, $y = \sin x = 1$
when $x = 0$, $y = \sin x = 0$
The points are $(-\pi/2, -1)$
$(0, 0)$, $(\pi/2, 1)$

Ex:

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute max of 4 at $x = 2$. " Min of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute max of 4 at $x = 2$. No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema.



In the above figure if we consider only values of x near b, for instance, if we restrict our attention to the interval (a, c) then $f(b)$ is the largest of those value of $f(x)$ and is called a local maximum value of f.

Likewise $f(c)$ is called a local minimum value of f because $f(c) \leq f(x)$ for x near c, in the interval (b, d).

In general we have the following definition.

Definition:

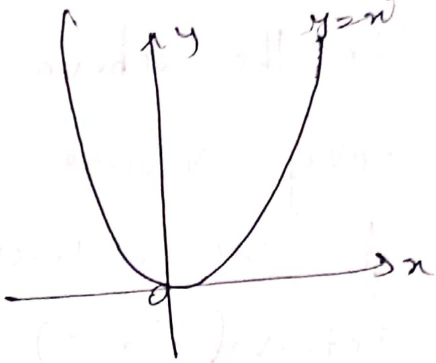
A function f has a local maximum (or relative maximum) at c if there is an open interval I containing c such that $f(c) \geq f(x)$ for all x in I.

Similarly, f has a local minimum at c if there is an open interval I containing c such that $f(c) \leq f(x)$ for all x in I.

Ex:

(1) $y = x^2$

$f(0) = 0$ is the absolute (and local) mininum value of f.

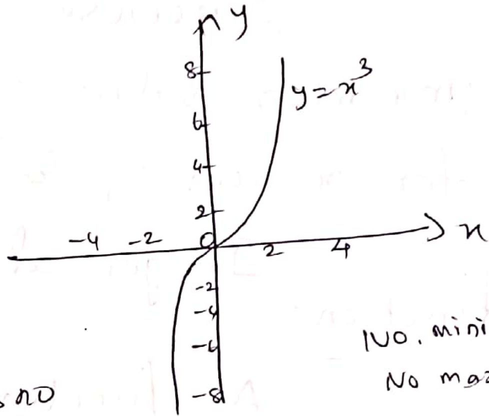


Min. value $= 0$

No Max.

(2) $y = x^3$

This function has neither an absolute maximum value nor an absolute minimum value. In fact it has no local extreme values either.

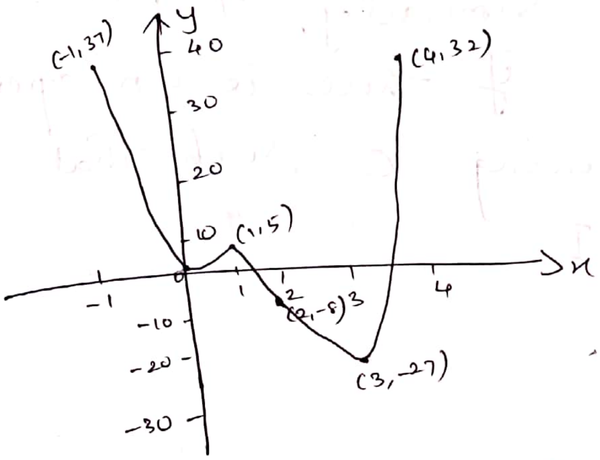


No. minimum

No maximum

(3) Consider the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \quad -1 \le x \le 4.$$



In this figure we can see that $f(1) = 5$ is local maximum, whereas the absolute maximum is $f(-1) = 37$.

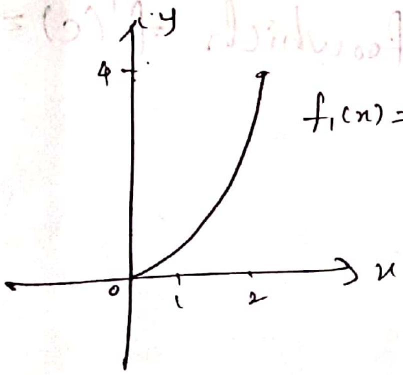
Also $f(0) = 0$ is a local minimum and $f(3) = -27$ is both local and absolute minimum.

We have seen that some function have extreme values, while others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

The Extreme Value theorem:

If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some number c and d in $[a, b]$.

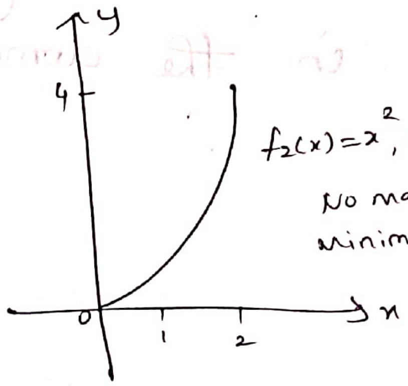
Ex:



$f_1(x) = x^2, \; 0 < x \leq 2$

Maximum $f_1(2) = 4$

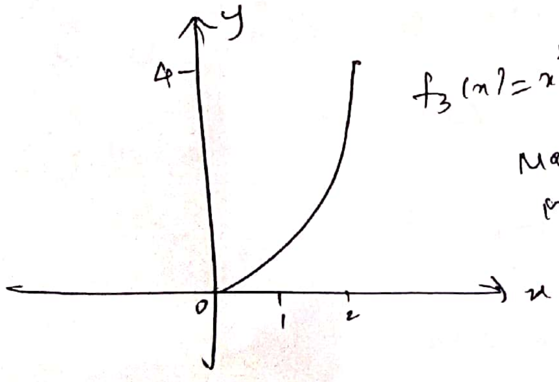
No minimum



$f_2(x) = x^2, \; 0 \leq x < 2$

No maximum

minimum $f_2(0) = 0$



$f_3(x) = x^2, \; 0 \leq x \leq 2$

Maximum $f_3(2) = 4$

Minimum $f_3(0) = 0$.

Fermat's Theorem:

The First Derivative Theorem for local Extreme Values:

If f has a local extremum (maximum or minimum) at c and if $f'(c)$ exists then $f'(c) = 0$.

Definition: A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Stationary points are critical number c in the domain of f, for which $f'(c) = 0$.

To find the absolute maximum and absolute minimum values of a continuous function f on a closed interval $[a,b]$:

① Find the values of f at the critical numbers, of f in (a,b).

② Find the values of $f(a)$ and $f(b)$.

③ The largest of the values from steps 1 and 2 is the absolute maximum value, and the smallest of these values is the absolute minimum value.

① Find the absolute maximum and minimum values of the function.

$$f(x) = x^3 - 3x^2 + 1, \quad -\tfrac{1}{2} \le x \le 4.$$

Note that f is continuous on $[-\tfrac{1}{2}, 4]$.

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Since $f'(x)$ exists for all x, the only critical number of f are $x=0$ and $x=2$.

Both of these critical numbers lie in the interval $[-\tfrac{1}{2}, 4]$.

Value of f at these critical numbers are $f(0)=1$ and $f(2)=-3$.

The values of f at the end points of the interval are

$$f(-\tfrac{1}{2}) = (-\tfrac{1}{2})^3 - 3(-\tfrac{1}{2})^2 + 1 = \tfrac{1}{8}$$

$$f(4) = 4^3 - 3\times 4^2 + 1 = 17.$$

Comparing these four numbers, we see that the absolute maximum value is $f(4)=17$ and the absolute minimum value is $f(2)=-3$.

Note that in this example the absolute maximum occurs at an end point, whereas the absolute minimum occurs at a critical number.

② Find the absolute maximum and absolute minimum values of

$$f(x) = x - 2\sin x, \quad 0 \le x \le 2\pi.$$

Sol: $f(x) = x - 2\sin x$ is continuous in $[0, 2\pi]$

$f'(x) = 1 - 2\cos x$

$f'(x) = 0 \Rightarrow \cos x = \tfrac{1}{2}$ $\qquad \cos 60 = \tfrac{1}{2}$

$\Rightarrow x = \pi/3$ or $5\pi/3$ $\qquad \cos 300 = \tfrac{1}{2}$

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The value of f at these critical points are

$$f(\pi/3) = \pi/3 - 2\sin\pi/3$$
$$= \pi/3 - \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)$$
$$f(\pi/3) = \pi/3 - \sqrt{3} = 1.0472 - 1.7321$$
$$= -0.6849$$
$$f(5\pi/3) = 5\pi/3 - 2\sin 5\pi/3$$
$$= 5\pi/3 - \cancel{2}\left(-\frac{\sqrt{3}}{\cancel{2}}\right)$$
$$= 5\pi/3 + \sqrt{3}$$
$$f(5\pi/3) = 6.968039$$

The values of f at the end points are $f(0) = 0$ and $f(2\pi) = 2\pi \approx 6.28$

Comparing these four number, the absolute minimum is $f(\pi/3) = \pi/3 - \sqrt{3}$ and the absolute maximum is $f(5\pi/3) = 5\pi/3 + \sqrt{3}$.

In this example both absolute minimum and absolute maximum occurs at the critical numbers.

Find the absolute maximum and minimum values of $f(x) = x^2$.

$$f'(x) = 2x = 0$$
$$\Rightarrow x = 0.$$

Critical point value $f(0) = 0$

Endpoint values $f(-2) = 4$

$f(1) = 4$

The function has a absolute maximum value of 4 at $x=-2$ and an absolute minimum value of 0 at $x=0$.

H.W

Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

$$g'(t) = 8 - 4t^3$$
$$g'(t) = 0 \Rightarrow 8 - 4t^3 = 0$$
$$t = \sqrt[3]{2} > 1.$$

$t = \sqrt[3]{2} > 1$, a point not in the given domain.

$g(-2) = -32$ (absolute mini)

$g(1) = 7$ (absolute max).

Ex: Consider the function

$f(x) = 3x^4 - 16x^3 + 18x^2$; $-1 \le x \le 4$.

$f(x) = 3x^4 - 16x^3 + 18x^2$

$f'(x) = 12x^3 - 48x^2 + 36x$

$f''(x) = 36x^2 - 96x + 36$

$f'(x) = 0 \Rightarrow 12x^3 - 48x^2 + 36x = 0$

$12x(x^2 - 4x + 3) = 0$

$\Rightarrow x = 0, (x-1)(x-3) = 0$

$\Rightarrow x = 0, 1, 3.$

$f(0) = 0$

$f(1) = 5$

$f(3) = -27$

$f(-1) = 37$

$f(4) = 32$

Absolute Max. value $= 37$, $x = -1$

" Min. value $= -27$, $x = 3$.

When $x = 0$, $f''(0) = 36 > 0$

when $x = 1$, $f''(1) = 36 - 96 + 36 = -24 < 0$

when $x = 3$, $f''(3) = 36 \times 9 - 96 \times 3 + 36 = 72 > 0$

local Max $= 5$ when $x = 1$

Local mini $= -27$ when $x = 3$

Local mini $= 0$ when $x = 0$.

Let us now see how the second derivatives of functions help determining the turning nature (of graphs of functions) and in optimization problems.

The second derivative test:

Suppose f is continuous on an open interval that contains c.

(a) if $f'(c)=0$ and $f''(c)>0$ then f has a local minimum at c.

(b) if $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.

(c) $f''(c)=0$ the point cannot be an extremum (minimum or maximum).

Ex 1) Find the local minimum and maximum values of $f(x)=x^4-3x^3+3x^2-x$.

Sol:

$$f(x)=x^4-3x^3+3x^2-x$$
$$f'(x)=4x^3-9x^2+6x-1$$
$$f''(x)=12x^2-18x+6$$

$$f'(x)=0 \Rightarrow 4x^3-9x^2+6x-1=0$$
$$4x^3-4x^2-5x^2+5x+x-1=0$$
$$4x^2(x-1)-5x(x-1)+(x-1)=0$$
$$(x-1)(4x^2-5x+1)=0$$
$$(x-1)(4x^2-4x-x+1)=0$$
$$(x-1)(4x(x-1)-(x-1))=0$$
$$(x-1)(x-1)(4x-1)=0$$
$$(x-1)^2(4x-1)=0.$$

$x=1,1,\frac{1}{4}.$

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$f(1) = 0$

$f(1/4) = -\dfrac{27}{256}$.

Stationary points are $(1, 0), \left(\frac{1}{4}, \frac{-27}{256}\right)$

When $x = 1$, $f''(1) = 0$. Thus the second derivative test gives no information about the extremum nature of f at $x = 1$.

When $x = 1/4$, $f''\left(\frac{1}{4}\right) = \frac{9}{4} > 0$

hence $\left(\frac{1}{4}, \frac{-27}{256}\right)$ is a minimum point.

H.W

① Find the Critical number and stationary points of each of the following functions.

		Ans Critical Number	Stationary points
①	$f(x) = 2x - 3x^2$	$x = 1/3$	$(1/3, 1/3)$
②	$f(x) = x^3 - 3x + 1$	$x = \pm 1$	$(1, -1), (-1, 3)$
③	$f(x) = \dfrac{x+1}{x^2+x+1}$	$x = 0, -2$	$(0, 1), (-2, -1/3)$
④	$f(\theta) = \theta + \sin\theta$ in $[0, 2\pi]$	$\theta = \pi$	(π, π)
⑤	$f(\theta) = \sin^2 2\theta$ in $[0, \pi]$	$\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi$	$(0, 0), (\pi/4, 1)$ $(\pi/2, 0)$ $(3\pi/4, 1)$
⑥	$f(x) = x^{4/5}(x-4)^2$	$x = 0, 4, 8/7$	$(0,0)$ $(4, 0), \left(\frac{8}{7}, \left(\frac{8}{7}\right)^{4/5}\left(\frac{20}{7}\right)^2\right)$

② Find the absolute maximum and absolute minimum values of f on the given interval.

		abs maxi	abs Mini
①	$f(x) = x^2 - 2x + 2$, $[0, 3]$	5	1
②	$f(x) = 1 - 2x - x^2$, $[-4, 1]$	2	-7
③	$f(x) = x^3 - 12x + 1$, $[-3, 5]$	66	-15
④	$f(x) = \sin x + \cos x$, $[0, \pi/3]$	$\sqrt{2}$	1
⑤	$f(x) = x - 2\cos x$, $[-\pi, \pi]$	$\pi + 2$	$-\pi/6 - \sqrt{3}$
⑥	$f(x) = \sqrt{9 - x^2}$, $[-1, 2]$	3	$\sqrt{5}$
⑦	$f(x) = \dfrac{x}{x+1}$, $[1, 2]$	$2/3$	$1/2$

Find the local maximum and minimum values of the following.

		local max	local mini
1	$x^3 - x$	$\frac{2}{3\sqrt{3}}$	$-\frac{2}{3\sqrt{3}}$
2	$2x^3+5x^2-4x$	12	$-\frac{19}{27}$
3	x^4-6x^2	0	-9
4	$\sin^2\theta$, $[0,\pi]$	1	Nil
5	$t+\cos t$	No maximum and No minimum.	
6	$(x^2-1)^3$	NIL	-1

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