Mean value theorems and their applications:

In this section our main objective is to prove that between any two points of a smooth curve there is a point at which the tangent is parallel to the chord joining two points. To do this we need the following theorem due to Michael Rolle. (1652–1719)

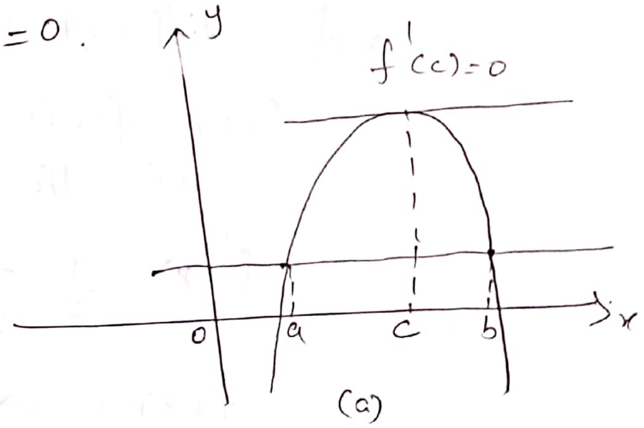
Rolle's Theorem: As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero. Let f be a real value function that satisfies the following three conditions.

(i) f is defined and continuous on the closed interval $[a, b]$

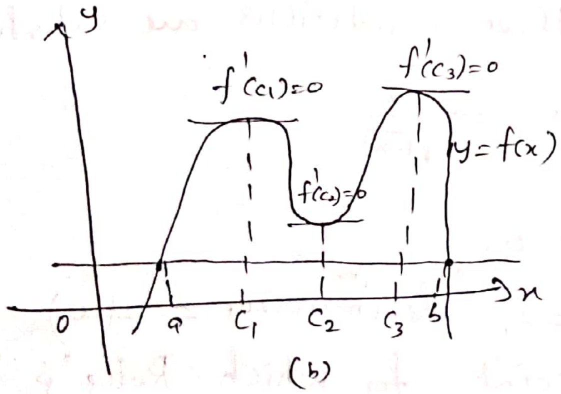
(ii) f is differentiable on the open interval (a, b)

(iii) $f(a) = f(b)$

Then there exists atleast one point $c \in (a, b)$ such that $f'(c) = 0$.



(a)



(b)

Rolle's theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a) or it may have more (b).

(1) Using Rolle's theorem find the values of c for the following

(i) $f(x) = 2x^3 - 5x^2 - 4x + 3, \ 1/2 \le x \le 3.$

Ans: f is continuous on $[1/2, 3]$ and differentiable in $(1/2, 3)$

$f(1/2) = f(3) = 0.$

All the conditions are satisfied.

$f'(x) = 6x^2 - 10x - 4$

$f'(x) = 0 \Rightarrow 6x^2 - 10x - 4 = 0 \Rightarrow 3x^2 - 5x - 2 = 0$

$\Rightarrow (3x+1)(x-2) = 0$

$\Rightarrow x = -1/3 \text{ (or) } x = 2$

$x = -1/3$ does not lie in $(1/2, 3)$.

$\therefore x = 2$ is the suitable 'c' of Rolle's theorem

(ii) $f(x) = \sqrt{1-x^2}, \ -1 \le x \le 1.$

Ans: The function is continuous in $[-1, 1]$ and differentiable in $(-1, 1)$.

$f(1) = f(-1) = 0$

All the three conditions are satisfied.

$$f'(x) = \frac{1}{2} \frac{-2x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$f'(x) = 0 \Rightarrow x = 0.$

(Note that for $x = 0$, denominator $= 1 \neq 0$)

Thus the suitable point for which Rolle's theorem holds is $c = 0$.

(iii) $f(x) = (x-a)(b-x)$, $a \le x \le b$, $a \ne b$

$f(x)$ is continuous on $[a, b]$ and

$f'(x)$ exists at every point of (a, b).

$f(a) = f(b) = 0$

All the conditions are satisfied.

$\therefore f'(x) = (b-x) - (x-a)$

$f'(x) = 0 \Rightarrow -2x = -b-a \Rightarrow x = \frac{a+b}{2}$

The suitable point 'c' of Rolle's theorem is $c = \frac{a+b}{2}$

Note: ② Rolle's theorem cannot be applied if any one of the conditions does not hold.

Verify Rolle's theorem for the following:

① $f(x) = x^3 - 3x + 3$, $0 \le x \le 1$.

f is continuous on $[0,1]$ and differentiable in $(0,1)$.

$f(0) = 3$ and $f(1) = 1$

$\therefore f(a) \ne f(b)$

$\therefore$ Rolle's theorem, does not hold, since $f(a) = f(b)$

is not satisfied.

Also note that $f'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

There exists no point $c \in (0,1)$ satisfying $f'(c) = 0$.

(ii) $f(x) = \tan x$, $0 \le x \le \pi$

$f'(x)$ is not continuous in $[0, \pi]$ as $\tan x$ tends

to $+\infty$ at $x = \pi/2$,

$\therefore$ Rolles theorem is not applicable.

(vi) $f(x) = x(x-1)(x-2)$, $0 \le x \le 2$.

f is continuous in $[0,2]$ and differentiable in $(0,2)$

$f(0) = 0 = f(2)$, satisfying hypothesis of Rolle's theorem.

Now $f'(x) = (x-1)(x-2) + x(x-2) + x(x-1) = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{\sqrt{3}}$

The required c in Rolle's theorem is $1 \pm \frac{1}{\sqrt{3}} \in (0,2)$.

Note: There could exist more than one such c appearing in the statement of Rolle's theorem

(iii) $f(x) = |x|, \quad -1 \le x \le 1.$

f is Continuous in $[-1,1]$ but not differentiable in $(-1,1)$ since $f'(0)$ does not exists

$\therefore$ Rolles theorem is not applicable.

(iv) $f(x) = \sin^2 x, \quad 0 \le x \le \pi.$

f is continuous in $[0,\pi]$ and differentiable in $(0,\pi)$.

$$f(0) = f(\pi) = 0$$

ie f satisfies hypothesis of Rolle's theorem

$$f'(x) = 2\sin x \cos x = \sin 2x$$

$$f'(c) = 0 \Rightarrow \sin 2c = 0 \Rightarrow 2c = 0, \pi, 2\pi, 3\pi$$

$$\Rightarrow c = 0, \pi/2, \pi, 3\pi/2, \ldots$$

Since $c = \pi/2 \in (0,\pi)$, the suitable c of Rolle's theorem is $c = \pi/2$.

(v) $f(x) = e^x \sin x, \quad 0 \le x \le \pi$

e^x and $\sin x$ are Continuous for all x, therefore the product $e^x \sin x$ is Continuous in $0 \le x \le \pi$.

$$f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

exist in $0 < x < \pi \Rightarrow f'(x)$ is differentiable in $(0,\pi)$.

$$f(0) = e^0 \sin 0 = 0$$
$$f(\pi) = e^\pi \sin \pi = 0.$$

$\therefore$ f satisfies hypothesis of Rolle's theorem.

Thus there exists $c \in (0,\pi)$ satisfying $f'(c) = 0 \Rightarrow$

$e^c(\sin c + \cos c) = 0 \Rightarrow e^c = 0$ or $\sin c + \cos c = 0$

$e^c = 0 \Rightarrow c = -\infty$ which is not meaningful here.

$\Rightarrow \sin c = -\cos c \Rightarrow \frac{\sin c}{\cos c} = -1 \Rightarrow \tan c = -1 = \tan \frac{3\pi}{4}$

$\Rightarrow c = 3\pi/4$ is the required point.