Module 1 : Introduction to waves

a) Waves on a string - (wave equation derivation)

b) Harmonic waves

c) Reflection & transmission of waves at a boundary (Qualitative)

d) Standing waves & their eigenfrequencies

Book :- The physics of vibration & Waves
by H J. Pain & P. Rankin

Role of physics in Computer Engineering

* Computer games (Newton's laws, friction, motion, hydrodynamics)
* ~~ph~~ Face recognition
* 3D modelling
* Computer graphics (reflection of light, how light interact with atmosphere)

  higher order interaction of reflection, diffraction & polarization
* Rate of heat generation in CPU.

* Classification of waves based on their broad properties

a) Mechanical waves, etc, example

b) Matter waves

* Discussion based on other criteria

a) longitudinal & transverse waves

b) Mixed waves

* Classification based on dimensionality

1-D, 2-D & 3-D waves

* Derivation of 1-D transverse wave on a string (representative figure - assumption of forces on a string)

derivation of wave equation $v = \sqrt{\frac{T}{\rho}}$

* Discussion on general solutions $y = (x \mp vt)$ & its linear combination.

Wave?

Is a disturbance which propagates in a medium/vaccum & transforms energy from one place to another without a net transfer of medium particle.

Characteristic of a wave

Amplitude :- maximum displacement of a pt. on a wave from its undisturbed position.

wave length :- distance from a pt. on the wave to the equivalent pt. on the adjacent wave (i.e. Crest, trough)

Period :- Time taken by two successive (crests or troughs) to pass a specific point.

Phase :- The position of a wave pt. at a particular instant of time on a wave form.

velocity :- The velocity of a wave relates period & wavelength of oscillations of the wave. If T is the time period, λ is the wavelength the velocity is $v = \frac{\lambda}{T} \Rightarrow v = \frac{\omega}{K}$

Classification of Waves :

Mechanical waves (Requires a medium)

Transeverse waves — Longitudinal wave

water waves — Sound wave

Electromagnetic waves (Don't require a medium)

↓

Transverse waves – Microwave, X-ray, Visible light, radio wave etc

Matter waves

Associated with subatomic particles mainly

Arises because of dual nature of particle.

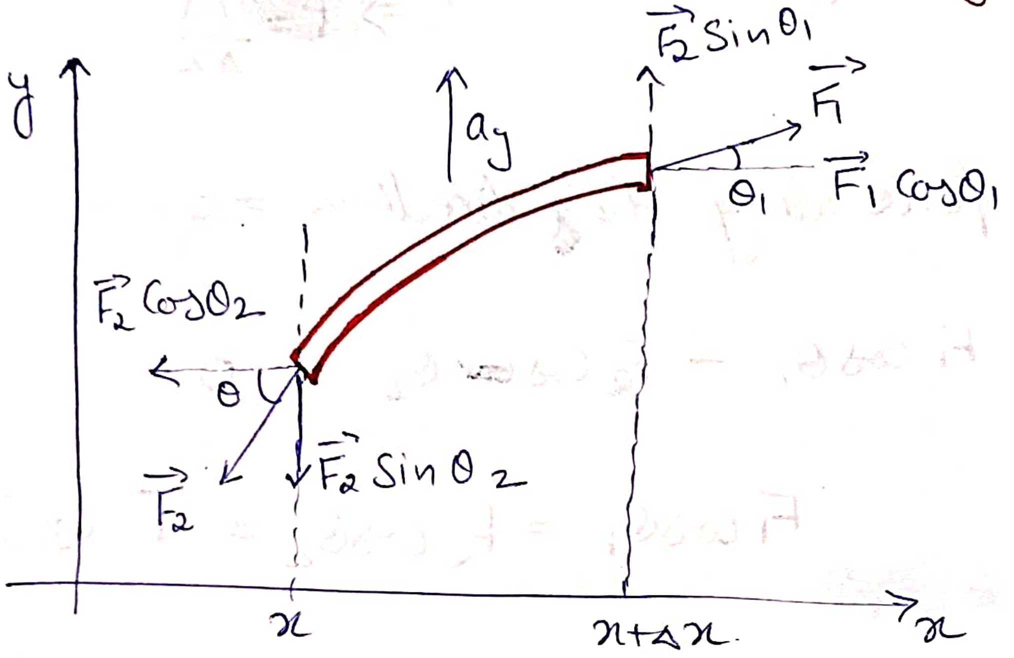
Classification based on dimensionality

1D waves :- Waves on a string

2D waves – Waves in water

3D waves – Waves in space (i.e. sound waves)

1D transverse Wave on a string



Assumptions:-

1) String is perfectly flexible & offers no resistance to bending.

* The tension in the string is tangential to the curve of the string

2) Points on the string move only in the verticle direction, there is no motion in the horizontal direction (Longitudinal) direction.

* No Net force along horizontal direction
= Sum of the forces in the horizontal direction is zero.

3) Gravitational force on the string is negligible
$\Rightarrow$ Net force resultant = mass x acceleration

Mass density of the string $= \rho$

$$\Rightarrow \rho = \frac{M}{\Delta x}$$

Now

* Net force along the x-direction $= 0$

$$F_1 \cos\theta_1 - F_2 \cos\theta_2 = 0$$

$$F_1 \cos\theta_1 = F_2 \cos\theta_2 = T \text{ (const.)}$$

* Net force along y-axis

$$\Delta F = F_1 \sin\theta_1 - F_2 \sin\theta_2 = m\, a_y$$

$$\Delta F = F_1 \sin\theta_1 - F_2 \sin\theta_2 = \rho \Delta x\, a_y$$

$$\Rightarrow T \tan\theta_1 - T \tan\theta_2 = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow T \left[\tan\theta_1 - \tan\theta_2\right] = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\because \frac{\partial y}{\partial x} = \frac{y(x+\Delta x) - y(x)}{\Delta x} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{y'(x+\Delta x) - y'(x)}{\Delta x}}$$

$$\Rightarrow T \left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x} \right] = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\Delta F \Rightarrow T \frac{\partial^2 y}{\partial x^2} \Delta x = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} = \frac{\partial^2 y}{\partial t^2}$$

Compairing with

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

$$v = \sqrt{\frac{T}{\rho}}$$

Dimension

$$T = Ma = MLT^{-2}$$

$$\rho = ML^{-1}$$

$$\frac{\rho}{T} = \frac{1}{L^2 T^{-2}} = \frac{1}{v^2}$$

Example

A cooper wire is pulled by using an external tension, $T = 0.98\ N$. The density of a cooper wire is $\rho = 9.86\ g\,cm^{-1}$. Compute the speed of wave supported by the string.

$$v = \sqrt{\frac{T}{\rho}}$$

General solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

must be fn. of x & t

$\therefore$ $f_1(x-vt)$ & $f_2(x+vt)$

& also superposition of f_1 & f_2

$$y = f_1(x-vt) + f_2(x+vt)$$

Problem

1) Show that $f(x,t) = A \sin[B(x-vt)]$ satisfies the wave equation.

2) Show that $f(x,t) = A e^{-B(x-vt)^2}$ satisfies the wave equation.

Simple Harmonic Motion

A Simple Harmonic Motion (S.H.M.) is that in which restoring force is directly proportional to the displacement of the particle from its mean position.

$$\boxed{F = -ky} \Rightarrow m\frac{d^2y}{dt^2} + ky = 0$$

$$(D^2+\omega) = 0$$

Sol:

$$y = A \sin \omega t \quad \Big| \text{ where } \omega = \sqrt{\frac{k}{m}}$$

* Link b/w $\underbrace{\text{oscillation}}$ & wave

is a periodic motion which can further leads to a wave.

* The collective oscillations of a number of coupled oscillators form waves

- Oscillations of E & M fields produce EM waves
- Oscillation of particles in a medium produce Sound / water waves.

Discussion about the general solution of a wave equation.

$$y(x,t) = A \sin\left(\frac{2\pi}{\lambda} x \mp \frac{2\pi}{T} t\right)$$

$$\frac{2\pi}{T} = \omega$$

$$\therefore \quad y(x,t) = A \sin(kx \mp \omega t)$$

or

$$\omega = 2\pi \nu \Rightarrow \frac{2\pi v}{\lambda}$$

$$\therefore \quad y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x \mp vt)\right]$$

Dispersion Relation

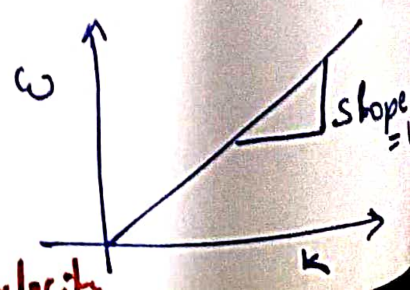
It relates the wavelength or wavenum of a wave to its frequency.

$$y(x,t) = A \sin(kx \mp \omega t)$$

$$\Rightarrow \quad kx - \omega t = 0$$

$$\Rightarrow \quad \frac{\omega}{k} = \frac{x}{t} = v_p$$

Phase velocity



Impedance (in the context of a wave propagates in a medium)

It is resistance offered by the medium when a wave propagates throught it.

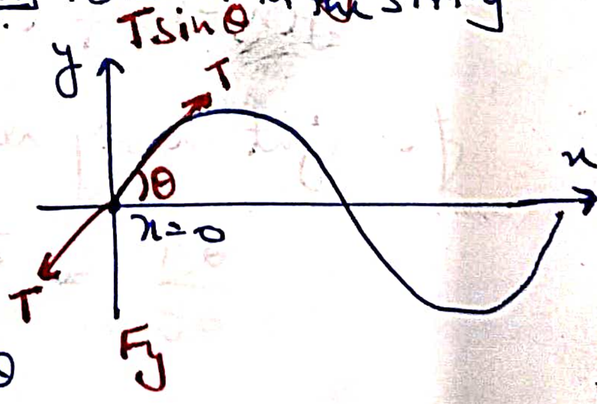
* The impedance offered by a string to the transverse wave travelling through it is known as the <u>characteristic impedance</u>.
* Denoted by (Z)
* $$Z = \frac{\text{Transverse force}}{\text{Transverse velocity}} = \frac{F_y}{v_y} = \frac{F_y}{\frac{\partial y}{\partial t}}$$

Characteristic Impedance of a String

at $x=0$, the vertical component of ~~string~~ tension in the string must balance the driving force.

magnitude of transverse force

$$F_y = -T\sin\theta$$



for small θ

$$\Rightarrow F_y = -T\sin\theta \approx \cancel{t} -T\tan\theta$$

$$\Rightarrow F_y = -T\left(\frac{\partial y}{\partial x}\right)$$

We know that a ~~wave~~ wave moving in right direction $\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$

previous expression coming from using the relation

particle velocity = direction × wave velocity × slope

$$Z = \frac{F_y}{v_y} = \frac{F_y}{\frac{\partial y}{\partial t}} = \frac{-T\frac{\partial y}{\partial x}}{-v\frac{\partial y}{\partial x}} = \frac{T}{v}$$

$$Z = \frac{T}{v}$$

As $v^2 = \frac{T}{\rho}$

$T = v^2 \rho$

$$\Rightarrow Z = \frac{v^2 \rho}{v} = \rho v$$

$$\Rightarrow \boxed{Z = \rho v}$$

$$y(x,t) = f(x \mp vt)$$

for right moving wave

$$\frac{\partial y}{\partial t} = \mp v \frac{\partial y}{\partial x}$$

for left moving moving wave

$$\frac{\partial y}{\partial t} = v \frac{\partial y}{\partial x}$$

particle velocity = direction × wave Velocity × slope

Simple Harmonic Waves ctd

$$y(x,t) = A\sin(kx \mp \omega t)$$

Ex $$y(x,t) = 0.02 \sin\left(\frac{x}{0.01} + \frac{t}{0.05}\right)$$

Find :→

a) Amplitude (0.02)

b) Wavelength ($\pi/50$)

c) Frequency ($10/\pi$)

d) Velocity ($v = 0.02$ m/s)

Characteristics of Harmonic Waves

1) For harmonic motion, amplitud best represents by a 'Sine fn.'

2) Harmonic waves propagates down the string one wavelength (λ) along x-axis during one time-period (T) along x-axis.

3) Speed of propagation $v = \frac{\lambda}{T}$

4) Value of 'Sin' argument θ varies b/w −1 & +1.

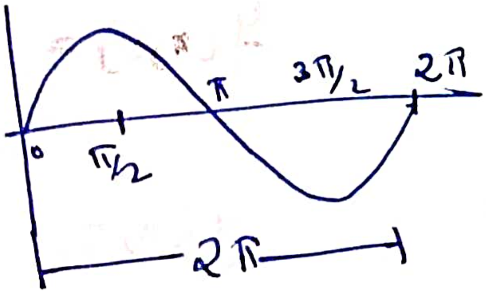
5) Repeats every 2π radian.

6) Repeats after wave length (λ) along x-axis.

$\theta = \frac{x}{r}$ $\quad \left[\because r = 1\right]$

$\theta = x$

$\frac{\theta}{x} = 1$



$2\pi = \lambda$

$\frac{2\pi}{\lambda} = 1$

$\therefore \frac{\theta}{x} = \frac{2\pi}{\lambda} \Rightarrow \theta = \frac{2\pi}{\lambda} x$

$\therefore y = A\sin\theta = A\sin\left(\frac{2\pi}{\lambda} x\right)$

$\Rightarrow y = A\sin\left(\frac{2\pi}{\lambda} x\right)$

$\Rightarrow y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x \pm vt)\right)$

$\Rightarrow y(x,t) = A\sin\left(\frac{2\pi}{x} \pm \frac{2\pi}{\lambda} vt\right)$

Impedance on a string

Considering a sinusoidal wave motion

$$y(x,t) = A \sin(kx - \omega t + \phi).$$

The impedance offered by the string if the tension in the string is T & density ρ

We have,

$$y(x,t) = A \sin(kx - \omega t + \phi)$$

$$\Rightarrow \quad F_y = -T \frac{\partial y}{\partial x} = -TAk \cos(kx - \omega t + \phi)$$

$$\Rightarrow \quad \frac{\partial y}{\partial t} = -A\omega \cos(kx - \omega t + \phi)$$

$$\Rightarrow \quad Z = \frac{F_y}{v} = \frac{-Tk}{-\omega} = \frac{Tk}{\omega} = \frac{T 2\pi}{\lambda} \cdot \frac{1}{2\pi\nu}$$

$$= \frac{T}{\lambda \nu}$$

$$Z = \frac{T}{v}$$

$$= \frac{v^2 \rho}{v}$$

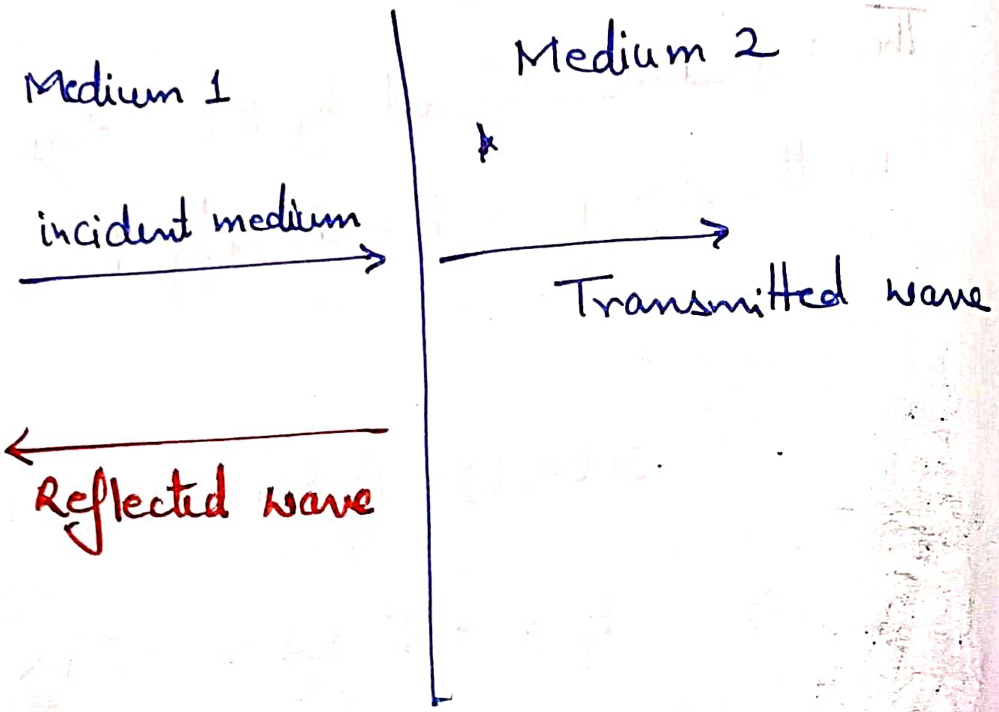
$$\boxed{Z = v\rho}$$

$$\nu = \frac{v}{\lambda} \Rightarrow \lambda\nu = v$$

$$v^2 = \frac{T}{\rho}$$

Wave travelling in different medium

Studying the properties of wave in terms of impedance



Properties of waves when moving from one medium to another

Properties	Reflection	Transmission
velocity (v)	Same	change
frequency (ν or f) $\omega = 2\pi\nu$	Same	same
wavelength (λ) $k = \frac{2\pi}{\lambda}$	Same	Change
Phase (ϕ)	$\phi = 0$; denser ⇒ rarer $\phi = \pi$; rarer ⇒ denser	$\phi = 0$; denser ⇒ rarer $\phi = 0$; rarer ⇒ [illegible]

Wave travel in different medium

M1

M2

$y_i(x,t) = A \sin(k_1 x - \omega t)$

$y_t(x,t) = C \sin(k_2 x - \omega t)$

$y_r(x,t) = B \sin(-k_1 x - \omega t)$

$x = 0$

At the boundary $(x=0)$ it must satisfied the boundary conditions

- Displacement must be the same immediately to the left & right of $x=0$ at all the times.

- There is no discontinuity of displacement at the boundary (i.e. Transverse force F_y must be continous at $x=0$)

$$= -T \frac{\partial y}{\partial x}$$

$\therefore$ at the boundary $x=0$, satisfy the boundary conditions:

$$y_i(x,t) + y_r(x,t) = y_t(x,t)$$

$$A \sin(k_1 x - \omega t) + B \sin(-k_1 x - \omega t) = C \sin(k_2 x - \omega t)$$

putting $x=0$, we get

$$A \sin(-\omega t) + B \sin(-\omega t) = C \sin(-\omega t)$$

$$\Rightarrow \quad \boxed{A + B = C} \quad \text{—} (1)$$

from second boundary conditions

$$(F_y)_i + (F_y)_r = (F_y)_t$$

$$-T \frac{\partial y_i}{\partial x} - T \frac{\partial y_r}{\partial x} = -T \frac{\partial y_t}{\partial x}$$

$$k_1 A \cos(k_1 x - \omega t) - k_1 B \cos(k_1 x - \omega t)$$
$$= k_2 C \cos(k_2 x - \omega t)$$

putting $x=0$

$$K_1 A \cos(-\omega t) - K_1 B \cos(-\omega t) = K_2 C \cos(-\omega t)$$

$$K_1 (A - B) = K_2 C$$

$$\boxed{A - B = \frac{K_2}{K_1} C} \quad \text{—} (2)$$

adding (1) & (2)

$$2A = \left(\frac{K_1 + K_2}{K_1}\right) C$$

transmission coefficient
$$\boxed{\frac{C}{A} = \frac{2K_1}{K_1 + K_2}} \quad \text{—} (3)$$

Writing (2) as, $\frac{K_1}{K_2}(A - B) = C$ — (4)

Subtracting ~~adding~~ (1) & (4)

$$A\left(1 - \frac{K_1}{K_2}\right) + B\left(1 + \frac{K_1}{K_2}\right) = 0$$

$$A\left(1 - \frac{K_1}{K_2}\right) = -B\left(1 + \frac{K_1}{K_2}\right)$$

Reflection coefficient
$$\boxed{\frac{B}{A} = \frac{\left(\frac{K_1}{K_2} - 1\right)}{\left(\frac{K_1}{K_2} + 1\right)} = \frac{K_1 - K_2}{K_1 + K_2}} \quad \text{—} (5)$$

$$\nu = \frac{\mathcal{Q}\,v}{d}$$

$$2\pi\nu = \frac{v\,2\pi}{d}$$

$$\boxed{\omega = \frac{vK}{\mathcal{Q}}} \Rightarrow K = \left(\frac{\omega}{v\mathcal{Q}}\right)$$

$$\boxed{\frac{C}{A} = \frac{2v_2}{v_1+v_2}} \qquad \boxed{\frac{B}{A} = \frac{v_2-v_1}{v_1+v_2}}$$

$$\boxed{\begin{gathered} Z = T/v \\ v = \frac{T}{Z} \end{gathered}}$$

Hence,

$$\frac{C}{A} = \frac{2\,T/Z_2}{T/Z_1 + T/Z_2} = \frac{2\,\frac{1}{Z_2}}{\frac{Z_2+Z_1}{Z_1Z_2}} = \frac{2Z_1}{Z_1+Z_2}$$

$$\boxed{\frac{C}{A} = \frac{2Z_1}{Z_1+Z_2}}$$

$$\frac{B}{A} = \frac{\frac{T}{Z_2} - \frac{T}{Z_1}}{\frac{T}{Z_1}+\frac{T}{Z_2}} = \frac{Z_1-Z_2}{Z_1+Z_2} \Rightarrow \boxed{\frac{B}{A} = \frac{Z_1-Z_2}{Z_1+Z_2}}$$

Cases:

① If the boundary has infinite Impedance
$Z_2 \longrightarrow \infty$ (fixed end at the boundary)

transmission coefficient $\boxed{\frac{C}{A} = 0}$

Reflection coefficient $\frac{B}{A} = \frac{Z_2\left(\frac{Z_1}{Z_2} - 1\right)}{Z_2\left(\frac{Z_1}{Z_2} + 1\right)}$

$\boxed{\frac{B}{A} = -1}$

② If the boundary has low impedance (Open end at the boundary)
$Z_2 \longrightarrow 0$

⇒ transmission coefficient $\boxed{\frac{C}{A} = 2}$

Reflection coefficient $\boxed{\frac{B}{A} = 1}$

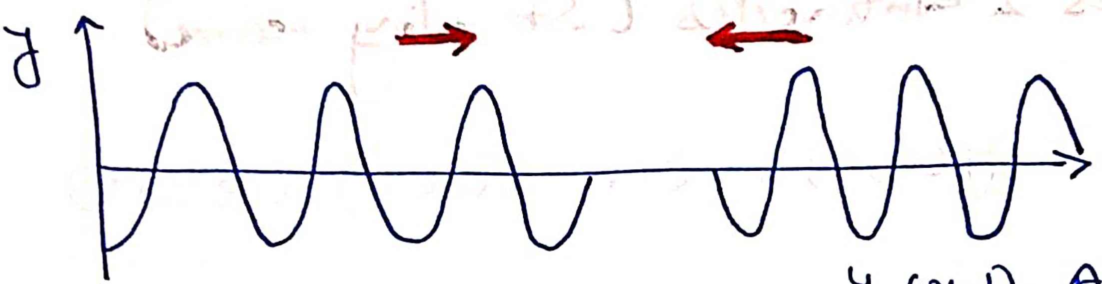
Superposition (Principle) of waves

* When two or more waves with constant phase difference, same intensity & same amplitude travelling ~~travelling~~ through a medium, each wave produces its own displacement irrespective of each other. The resultant of these waves is the vector sum of the amplitude of each wave.

* The modification or the redistribution of intensity of resultant wave due to superposition of two or more waves is known as interference

Standing Wave

When two identical progressive waves (i.e. waves having same amplitude, wavelength & speed) travelling through medium along same path in opposite direction, interfere with each other, by superposition of waves, the resultant wave is obtained in the form of loops, is called a stationary wave or standing wave.



$y_l(x,t) = A\sin(kx+\omega t)$

$y_r(x,t) = A\sin(kx-\omega t)$

Resultant wave is

$$y(x,t) = y_r(x,t) + y_l(x,t)$$

$$= A\sin(kx-\omega t) + A\sin(kx+\omega t)$$

$$= A\left[\sin(kx-\omega t) + \sin(kx+\omega t)\right]$$

$$= A\left[2\sin\left(\frac{kx-\omega t+kx+\omega t}{2}\right)\cos\left(\frac{kx-\omega t-kx-\omega t}{2}\right)\right]$$

$$= A\left[2\sin(kx)\cos(\omega t)\right]$$

$$\boxed{y(x,t) = 2A\sin(kx)\cos(\omega t)}$$

* This is not in the form $f(x-vt)$ / $f(x+vt)$
* But satisfies the wave eq. $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$
* $\therefore$ it is a wave but not a traveling wave/ progressive wave, hence a standing wave.

Nodes & Antinodes (Standing wave)

$$y(x,t) = 2A\,\mathrm{Sin}(kx)\,\mathrm{Cos}(\omega t)$$

Nodes = pt. of least Amplitudes

$$\mathrm{Sin}(kx) = 0$$

$$\mathrm{Sin}(kx) = \mathrm{Sin}(n\pi)$$

$$kx = n\pi$$

$$\frac{2\pi}{\lambda}x = n\pi$$

$$\boxed{x = \frac{n\lambda}{2}}$$

$$\therefore x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, 2\lambda, \ldots$$

Antinodes = pt. of maximum amplitude

$$\mathrm{Sin}(kx) = \pm 1$$

$$\mathrm{Sin}(kx) = \left(n + \frac{1}{2}\right)\pi$$

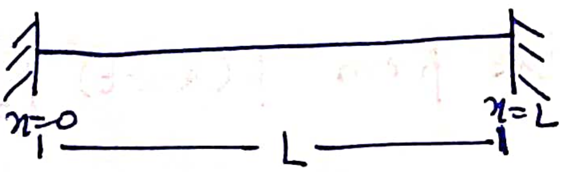
$$\frac{2\pi}{\lambda}x = \left(n + \frac{1}{2}\right)\pi$$

$$x = \left(n + \frac{1}{2}\right)\lambda/2$$

$$\Rightarrow \boxed{x = \frac{(2n+1)\lambda}{4}}$$

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \frac{7\lambda}{4}, \ldots$$

Standing wave in a string (when the end are fixed)



Consider a string of length $= L$

Tension $= T$

linear mass density $\left(\frac{M}{\ell}\right) = \rho$

* end pts. are fixed

$\therefore$ at $x=0$ & $x=L$

displacement in the string at the end pts. is zero

That is

at $x=0$, $y(0,t)=0$

$\Downarrow$

$2A\sin(0)\cos(\omega t) = 0$

at $x=L$, $y(L,t)=0$

$2A \sin(kL)\cos(\omega t) = 0$

$\sin(kL) = \sin(n\pi)$

$kL = n\pi$

$$\boxed{k_n = \frac{n\pi}{L}}$$

or $\frac{2\pi L}{\lambda} = n\pi$

$$\boxed{\lambda = \frac{2L}{n}}$$

$f = \frac{v}{\lambda} = \frac{nv}{2L} \Rightarrow \boxed{f = \frac{nv}{2L}}$ ← Eigen frequency ($n = 1, 2, 3, \ldots$)

$\Rightarrow 2\pi f = \frac{2\pi n v}{2L} \Rightarrow \boxed{\omega = n v k}$

Properties of Standing waves

* for a given harmonic ($n > 1$), there will be $(n-1)$ pts on the string which are always at rest.

* The pts. which are always at rest known as nodes.

$$v = \sqrt{\frac{T}{\rho}} \quad , \quad \lambda_n = \frac{2L}{n}$$

* for fundamental mode (First harmonic)

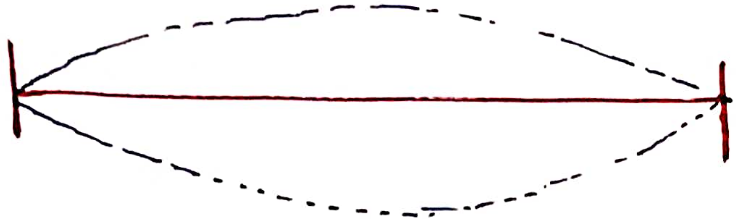
$$\boxed{n = 1} \quad \boxed{\lambda = 2L}$$

$$\boxed{f_1 = \frac{v}{\lambda_1} = \frac{1}{\lambda_1}\sqrt{\frac{T}{\rho}} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}}$$

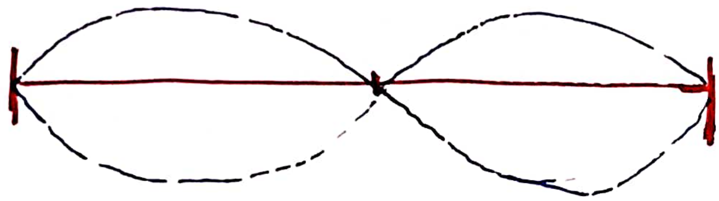
frequency

for $n = 1$

$$\lambda_1 = 2L$$

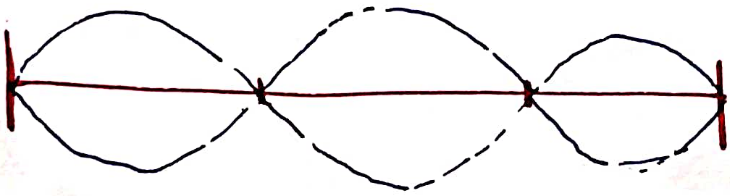


for $n = 2$



$$\lambda_2 = L$$

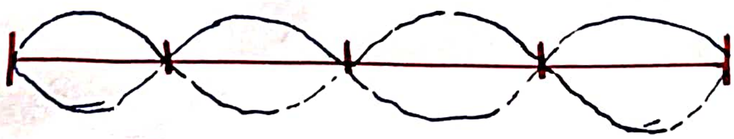
for $n = 3$



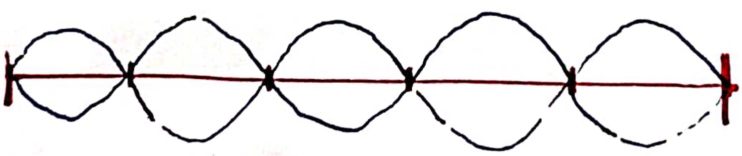
$$\lambda_3 = \frac{2L}{3}$$

for $n = 4$

$$\lambda_4 = \frac{2L}{4} = \frac{L}{2}$$

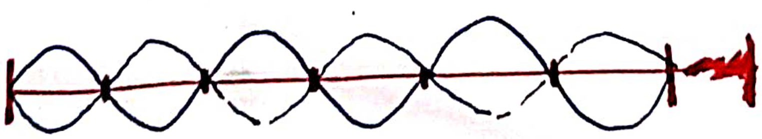


for $n = 5$



$$\lambda_5 = \frac{2L}{5}$$

for $n = 6$



$$\lambda_6 = \frac{2L}{6} = \frac{L}{3}$$