$$A = \cancel{4}\frac{9a^2}{\cancel{4}}\int_0^1\int_0^{1-Y} X^{1/2}\,Y^{1/2}\,dX\,dY$$

$$= 9a^2\int_0^1\left[\frac{X^{3/2}}{3/2}\right]_0^{1-Y} Y^{1/2}\,dY$$

$$= 6a^2\int_0^1 (1-Y)^{3/2}\,Y^{1/2}\,dY$$

$$= 6a^2\int_0^1 Y^{1/2}(1-Y)^{3/2}\,dY$$

$$= 6a^2\,\beta\left(\tfrac{3}{2},\tfrac{5}{2}\right)$$

$$= 6a^2\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)}$$

$$= \cancel{6}a^2\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\cancel{3}!} = \frac{3\pi a^2}{8}.$$

PROBLEMS FOR PRACTICE

1. Evaluate the following:

(i) $\int_0^a\int_0^b x^2y\,dy\,dx$ **Answer:** $\frac{a^3b^2}{6}$

(ii) $\int_0^2\int_0^1 (x^2+y^2)\,dx\,dy$ **Answer:** 10/3

(iii) $\int_0^3\int_1^2 xy(x+y)\,dx\,dy$ **Answer:** 21

Answer: 2/9

(iv) $\int_0^{\pi/2}\int_0^{a\cos\theta} r^2\,dr\,d\theta$

Answer: $(a-1)^2/2$

(v) $\int_0^{\log a}\int_0^{y} e^{x+y}\,dxdy$

(vi) Show that $\int_0^1\int_0^{1-x} xydxdy = 1/24$

(vii) Show that $\int_0^2\int_0^y \frac{1}{x^2+y^2}\,dxdy = (\pi/4)l$

(viii) Show that $\int_0^a\int_0^{\sqrt{a^2-y^2}} \frac{dxdy}{\sqrt{a^2-y^2}}$

Answer: (e^2-1)

(ix) $\int_0^2\int_0^{x^2} e^{y/x}\,dxdy$

Answer: $(\log 2)^2$

(x) $\int_{r=0}^{2a\sin\theta}\int_1^2 \frac{1}{xy}\,dxdy$

2 (i) $\iint (x+y)dydx$ over the positive quadrant of the Ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Answer: $\frac{1}{3}ab(a+b)$

(ii) Evaluate $\iint (x^2+y^2)dxdy$ over the region for which $x=0$, $y=0$ and $x+y=1$.

Answer: 1/6

(iii) Evaluate $\iint xydxdy$ over region bounded by $x=0, y=0, x=a, y=b$

Answer: $\frac{1}{4}a^2b^2$

(iv) Evaluate $\iint e^{2x+y}\,dx\,dy$ over the triangle bounded by $x=0, y=0$ and $x+y=2$.

Answer: $\frac{1}{2}(e^2-1)^2$

(v) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$. **Answer:** $(1/6)\pi a^3$

(vi) Show that $\int_0^1 \int_0^{x^2} e^{y/x}\,dx$

Answer: (1/2)

Change the order of integration and hence evaluate the following:

3. $\int_0^{\pi/2} \int_0^{x} \frac{\cos y}{\sqrt{(\pi/2-x)(x-y)}}\,dy\,dx$ **Answer:** $\pi/2$

4. $\int_0^{a} \int_0^{bx/a} x\,dy\,dx$ **Answer:** $a^2b/3$

5. $\int_0^{1} \int_{x^2}^{2-x} xy\,dy\,dx$ **Answer:** 3/8

6. $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$ **Answer:** $16a^2/3$

7. $\int_0^{2} \int_{y^2/4}^{3-y} (x^2+y^2)\,dx\,dy$ **Answer:** $\frac{314}{35}$

8. $\int_0^{\infty} \int_0^{x} x \exp\left(-\frac{x^2}{y}\right) dy\,dx$ **Answer:** 1/2

9. $\int_0^{a} \int_{\sqrt{ax}}^{a} \frac{4}{\sqrt{y^4-a^2x^2}}\,dx\,dy$ **Answer:** 2π

10. $\int_0^2 \int_1^{e^x} dydx$ **Answer:** e^2-3

11. $\int_0^1 \int_x^{\sqrt{x}} xydxdy$ **Answer:** 1/24

12. $\int_0^{\sqrt{1-x^2}} y^2dxdy$ **Answer:** $\pi/16$

13. Find the area of the segment cut off from the parabola $y^2=2x$ by the straight line $y=4x-1$. **Answer:** 9/32

14. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **Answer:** πab

15. Find by double Integration the area of the region enclosed by the curves $x^2+y^2=a^2$, $x+y=a$(in the first quadrant) **Answer:** $(1/4)a^2(\pi-2)$

16. Find the area bounded by the curves $y^2=x^3$, $x^2=y^3$ **Answer:** 1/5

17. Find the area common to the circle $x^2+y^2=4$ and the Ellipse $x^2+4y^2=9$.

Answer: $4\pi+9\sin^{-1}\left(\frac{1}{3}\sqrt{7/3}\right)-2\sin^{-1}\left(\frac{1}{2}\sqrt{7/3}\right)$

18. Find the area of the portion bounded by the curve $x(x^2+y^2)=a(x^2-y^2)$and its asymptote. **Answer:** $\frac{1}{2}a^2(4-\pi)$

19. Find the area bounded by the curve $xy^2=4a^2(2a-x)$ and its asymptote

Answer: $4\pi a^2$

20. Find the area bounded by the curve $y^2(2a-x)=x^3$ and its asymptote.

Answer: $3\pi a^2$

21. Find the area bounded by the parabola $y^2=4ax$ and its latusrectum.

Answer: $\frac{8}{3}$

22. Find the area bounded by the curves $3y^2=25x$ and $5x^2=9y$ **Answer:** 5

23. Find the area bounded by the curves $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and $x+y=$

Answer: $a^2/3$

24. Find the area enclosed by the curves $y=2-x$, $x^2+y^2=4$

Answer: $\pi+2$

25. Find the area of the region bounded by upper half of the circle $x^2+y^2=25$ the x-axis and the ordinates $x=-3, x=4$ **Answer:** 24 : 43

26. Find the area enclosed by the curves $y=2-x$, $y^2=2(2-x)$ **Answer:** 2/3

27. Find the area enclosed by the lines $x=0, y=0$ $\frac{x}{a}+\frac{y}{b}=1$ **Answer:** ab/2

28. Find the area in the first quadrant bounded by the x-axis and the curves $x^2+y^2=10$, $y^2=9x$

29. Find the area outside the circle $r=2a\cos\theta$ and inside the Cardioid $r=a(1+\cos\theta)$ **Answer:** $\pi a^2/2$

30. Find the area of the loop of the curver $r=\dfrac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$

[Hint: Given curve is $x^3+y^3=3axy$) **Answer:** $(3/2)\pi a^2$

31. Find the area of the curve $r=a(1+\sin\theta)$ **Answer:** $(3/2)\pi a^2$

32. Find the area bounded by the curve $r=a\cos5\theta$ **Answer:** $\pi a^2/4$

33. Find the area of the ellipse $\frac{\ell}{r}=1+e\cos\theta$ **Answer:** $\pi l^2/(1-e^2)^{3/2}$

34. Find the area lying inside the circle $r=a\sin\theta$ and outside the cardiod $r=a(1-\cos\theta)$

35. Find the area common to the circles $r=a\sqrt{2}$ and $r=2a\cos\theta$

36. Find the area of the curve $r^2=a^2\cos 2\theta$ **Answer:** a^2

37. Calculate the area included between the curve $r=a(\sec\theta+\cos\theta)$ and its asymptote **Answer:** $\frac{5}{4}\pi a^2$

38. Show that the volume of the solid bounded by the coordinate planes and the surface is $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}$ is $\frac{abc}{90}$.

39. Evaluate $\iint[xy(1-x-y)]^{1/2}\,dxdy$, over the area enclosed by the lines $x=0$, $y=0$ and $x+y=1$ in the positive quadrant. **Answer: $2\pi/105$**

40. Find the value of $\iint x^m y^n\,dxdy$ taken over the area $x\geq 0$, $y\geq 0$, $x+y\leq 1$ interms of Gamma functions, if $m, n>0$. **Answer:** $\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+3)}$

41. Evaluate $\iint x^{m-1} y^{n-1}(1-x-y)^{p-1}\,dxdy$, taken over the area in the first quadrant enclosed by the lines $x=0, y=0, x+y=1$. **Answer:** $\frac{\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(m+n+p)}$.