Properities of divergence:

① prove that $\nabla\cdot(\varphi\vec{A}) = \nabla\varphi\cdot\vec{A} + \varphi\,\nabla\cdot\vec{A}$

proof: $\nabla\cdot(\varphi\vec{A}) = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot(\varphi\vec{A})$

$$= \Sigma\,\vec{i}\cdot\frac{\partial}{\partial x}(\varphi\vec{A})$$

$$= \Sigma\,\vec{i}\cdot\left[\varphi\frac{\partial\vec{A}}{\partial x} + \frac{\partial\varphi}{\partial x}\vec{A}\right]$$

$$= \Sigma\left[\vec{i}\cdot\varphi\frac{\partial\vec{A}}{\partial x} + \vec{i}\cdot\frac{\partial\varphi}{\partial x}\vec{A}\right]$$

$$= \Sigma\left[\varphi\left(\vec{i}\cdot\frac{\partial\vec{A}}{\partial x}\right) + \vec{i}\frac{\partial\varphi}{\partial x}\cdot\vec{A}\right]$$

$$= \Sigma\left[\varphi\left(\vec{i}\frac{\partial}{\partial x}\cdot\vec{A}\right) + \vec{i}\frac{\partial\varphi}{\partial x}\cdot\vec{A}\right]$$

$$= \varphi(\nabla\cdot\vec{A}) = \nabla\varphi\cdot\vec{A}$$

properities of curl:

② Prove that $\nabla\times(\varphi\vec{u}) = \nabla\varphi\times\vec{u} + \varphi(\nabla\times\vec{u})$

proof: $\nabla\times(\varphi\vec{u}) = \Sigma\,\vec{i}\times\frac{\partial}{\partial x}(\varphi\vec{u})$

$$= \Sigma\,\vec{i}\times\left[\varphi\frac{\partial\vec{u}}{\partial x} + \frac{\partial\varphi}{\partial x}u\right]$$

$$= \Sigma\left[\vec{i}\times\varphi\frac{\partial\vec{u}}{\partial x} + \vec{i}\times\frac{\partial\varphi}{\partial x}\vec{u}\right]$$

$$= \Sigma\left[\varphi\,\vec{i}\times\frac{\partial\vec{u}}{\partial x} + \vec{i}\frac{\partial\varphi}{\partial x}\times\vec{u}\right]$$

$$= \Sigma\left[\varphi\left(\vec{i}\frac{\partial}{\partial x}\times\vec{u}\right) + \vec{i}\frac{\partial\varphi}{\partial x}\times\vec{u}\right]$$

$$= \varphi(\nabla\times\vec{u}) + \nabla\varphi\times\vec{u}.$$

Property of divergence:

(b) Prove that $\nabla\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v})$

Proof: $\nabla\cdot(\vec{u}\times\vec{v}) = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot(\vec{u}\times\vec{v})$

$$= \sum \vec{i}\cdot\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$

$$= \sum \vec{i}\cdot\left[\vec{u}\times\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{u}}{\partial x}\times\vec{v}\right]$$

$$= \sum\left[\vec{i}\cdot\left(\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right) + \vec{i}\cdot\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right)\right]$$

$$= \sum\left[-\vec{i}\cdot\left(\frac{\partial\vec{v}}{\partial x}\times\vec{u}\right) + \vec{i}\cdot\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right)\right]$$

$$= \sum\left[-\left(\vec{i}\times\frac{\partial\vec{v}}{\partial x}\right)\cdot\vec{u} + \left(\vec{i}\,\frac{\partial}{\partial x}\times\vec{u}\right)\cdot\vec{v}\right]$$

$$= \sum\left[-\left(\vec{i}\,\frac{\partial}{\partial x}\times\vec{v}\right)\cdot\vec{u}\right] + \left(\vec{i}\,\frac{\partial}{\partial x}\times\vec{u}\right)\cdot\vec{v}$$

$$= -(\nabla\times\vec{u})\cdot\vec{u} + (\nabla\times\vec{u})\cdot\vec{v}$$

$$= (\nabla\times\vec{u})\cdot\vec{v} - \vec{u}\cdot(\nabla\times\vec{v})$$

Note: $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$.

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Property of curl :-

(4) Prove that $\text{curl}\,(\vec{u}\times\vec{v}) = (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v} + \vec{u}\,\text{div}\,\vec{v} - \vec{v}\,\text{div}\,\vec{u}$

Solution:

$$\text{curl}\,(\vec{u}\times\vec{v}) = (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v} + \vec{u}\,\text{div}\,\vec{v} - \vec{v}\,\text{div}\,\vec{u}$$

$$\text{Curl}\,(\vec{u}\times\vec{v}) = \Sigma\,\vec{i}\times \partial/\partial x\,(\vec{u}\times\vec{v})$$

$$= \Sigma\,\vec{i}\times\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v} + \vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$= \Sigma\,\vec{i}\times\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right) + \Sigma\,\vec{i}\times\left(\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$= \Sigma(\vec{i}\cdot\vec{v})\frac{\partial\vec{u}}{\partial x} - \Sigma\left(\vec{i}\cdot\frac{\partial\vec{u}}{\partial x}\right)\vec{v} + \Sigma\left(\vec{i}\cdot\frac{\partial\vec{v}}{\partial x}\right)\vec{u} - \Sigma(\vec{i}\cdot\vec{u})\frac{\partial\vec{v}}{\partial x}$$

$$= \left(\Sigma\,\vec{i}\cdot\frac{\partial\vec{v}}{\partial x}\right)\vec{u} - \left(\Sigma\,\vec{i}\cdot\frac{\partial\vec{u}}{\partial x}\right)\vec{v} + \vec{v}\cdot\left(\Sigma\,\vec{i}\,\partial/\partial x\right)\vec{u} - \vec{u}\cdot\left(\Sigma\,\vec{i}\,\partial/\partial x\right)\vec{v}$$

$$\text{Curl}\,(\vec{u}\times\vec{v}) = \vec{u}\,\text{div}\,\vec{v} - \vec{v}\,\text{div}\,\vec{u} + (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v}.$$

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① If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A} \times \vec{B}$ is solenoidal.

proof: Given $\vec{A}$ and $\vec{B}$ are irrotational.

$$\nabla \times \vec{A} = 0 \quad \text{and} \quad \nabla \times \vec{B} = 0$$

ie

$$\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$$

$$= 0 - 0 = 0$$

Hence $\vec{A} \times \vec{B}$ is solenoidal.

② If $\vec{A}$ is a constant vector, prove that $\operatorname{div} \vec{A} = 0$

Solution: Given : $\vec{A}$ is a constant vector.

$$\text{Let } \vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$$

$$\frac{\partial A_1}{\partial x} = 0, \quad \frac{\partial A_2}{\partial y} = 0, \quad \frac{\partial A_3}{\partial z} = 0$$

$$\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = 0 + 0 + 0 = 0.$$

③ If $\vec{A}$ is a constant vector, prove that Curl $\vec{A} = 0$

$$\text{Let } \vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$\nabla \times \vec{A} = 0$$

Hence Curl $\vec{A} = 0$.

(4) If $\vec{a}$ is a constant vector, show that $\nabla \cdot (\vec{a} \times \vec{r}) = 0$.

Solution: Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(a_2 z - a_3 y) - \vec{j}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x)$$

$$\nabla \cdot (\vec{a} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) - \frac{\partial}{\partial y}(a_1 z - a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 - 0 + 0 = 0.$$

(5) If $\nabla^2 \varphi = 0$ prove that $\nabla \varphi$ is both solenoidal and irrotational.

Solution: $\nabla^2 \varphi = 0$ (given)

(i) To prove that $\nabla \varphi$ is solenoidal,

we have to prove that $\operatorname{div} \nabla \varphi = 0$

$$\therefore \operatorname{div}(\nabla \varphi) = \nabla \cdot \nabla \varphi$$
$$= \nabla^2 \varphi$$
$$= 0$$

$\therefore \nabla \varphi$ is solenoidal.

(ii) To prove that $\nabla\phi$ is irrotational, we have to prove that curl $\nabla\phi = 0$

$$\text{Curl } \nabla\phi = \nabla \times \nabla\phi$$

$$= \Sigma \vec{i} \times \partial/\partial x \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \right)$$

$$= \Sigma \left[\vec{k}\frac{\partial^2\phi}{\partial x\partial y} - \vec{j}\frac{\partial^2\phi}{\partial x\partial z} \right]$$

$$= \vec{k}\frac{\partial^2\phi}{\partial x\partial y} - \vec{j}\frac{\partial^2\phi}{\partial x\partial z} + \vec{i}\frac{\partial^2\phi}{\partial y\partial z} - \vec{k}\frac{\partial^2\phi}{\partial y\partial z} + \vec{j}\frac{\partial^2\phi}{\partial z\partial x} - \vec{i}\frac{\partial^2\phi}{\partial z\partial y}$$

$$= 0$$

$$\text{Curl } \nabla\phi = 0$$

$\therefore \nabla\phi$ is irrotational.