If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ such that $|\vec{r}| = r$,

Prove that (i) $\nabla r = \frac{\vec{r}}{r} = \hat{r}$ (ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$

(iii) $\nabla r^n = n r^{n-2}\vec{r}$ (iv) $\nabla f(r) = f'(r)\nabla r$.

Solution:

Given that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = x/r$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = y/r$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \partial r/\partial z = z/r$$

(i) $$\nabla r = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) r$$

$$= \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\,\partial r/\partial z$$

$$\nabla r = \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\,z/r$$

$$\nabla r = \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\vec{r}}{r} = \hat{r}$$

(ii) $$\nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)(1/r)$$

$$= -1/r^2\,\partial r/\partial x\,\vec{i} + \left(-1/r^2\,\partial r/\partial y\right)\vec{j} + \left(-1/r^2\,\partial r/\partial z\right)\vec{k}$$

$$= -1/r^2\left(\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right)$$

Scanned by CamScanner

$$= -\frac{1}{r^3}(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= -\frac{\vec{r}}{r^3} = -\frac{1}{r^2}\left(\frac{\vec{r}}{r}\right)$$

$$\nabla(1/r) = -\frac{1}{r^2}(\hat{r}) = \frac{-\hat{r}}{r^2}.$$

(iii) $$\nabla r^n = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) r^n$$

$$= \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\,n r^{n-1}\frac{\partial r}{\partial x} + \vec{j}\,n r^{n-1}\frac{\partial r}{\partial y} + \vec{k}\,n r^{n-1}\partial r/\partial z$$

$$= n r^{n-1}\left[\vec{i}\,\frac{x}{r} + \vec{j}\,\frac{y}{r} + \vec{k}\,z/r\right]$$

$$= \frac{n r^{n-1}}{r}\left[x\vec{i}+y\vec{j}+z\vec{k}\right]$$

$$\nabla r^n = n r^{n-2}\,\vec{r}.$$

(iv) $$\nabla f(r) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) f(r)$$

$$= \vec{i}\,\frac{\partial f(r)}{\partial x} + \vec{j}\frac{\partial}{\partial y}f(r) + \vec{k}\frac{\partial}{\partial z}f(r)$$

$$= \vec{i}\,f'(r)\frac{\partial r}{\partial x} + \vec{j}\,f'(r)\frac{\partial r}{\partial y} + \vec{k}\,f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\vec{i}\,\frac{x}{r} + \vec{j}\,\frac{y}{r} + \vec{k}\,\frac{z}{r}\right] = \frac{f'(r)}{r}\vec{r}$$

$$\nabla f(r) = f'(r)\frac{\vec{r}}{r} = f'(r)\,\hat{r} = f'(r)\,\nabla r.$$

Divergence of a Vector point Function:

Let $\vec{F}$ be any given Continuously differentiable vector point function then the divergence of $\vec{F}$ is defined as

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \vec{F}$$

$$\nabla \cdot \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j}\frac{\partial F}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

Note: $\nabla \cdot \vec{F}$ is a Scalar point function.

Solenoidal vector:

A vector $\vec{F}$ is said to be Solenoidal vector if $\operatorname{div} \vec{F} = 0$ (or) $\nabla \cdot \vec{F} = 0$

Note: If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a Continuously differentiable vector point function then $\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\left[\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)\right.$$

$$\left. \operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right]$$

Curl of vector point Function:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any given continuously differentiable vector point function, the curl or rotation of $\vec{F}$ is defined as

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\times\vec{F}$$

$$= \vec{i}\times\frac{\partial\vec{F}}{\partial x} + \vec{j}\times\frac{\partial\vec{F}}{\partial y} + \vec{k}\times\frac{\partial\vec{F}}{\partial z}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\left[\nabla\times\vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\times\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)\right.$$

$$= \vec{k}\frac{\partial F_2}{\partial x} - \vec{j}\frac{\partial F_3}{\partial x} - \vec{k}\frac{\partial F_1}{\partial y} + \vec{i}\frac{\partial F_3}{\partial y} + \vec{j}\frac{\partial F_1}{\partial z} - \vec{i}\frac{\partial F_2}{\partial z}$$

$$\left. = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

Note: $\nabla \times \vec{F}$ is a vector point function.

irrotational vector:

A vector $\vec{F}$ is said to be irrotational if $\nabla \times \vec{F} = 0$.

ie $\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = 0.$

Laplacian operator ∇^2 :-

∇^2 is defined as $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

[Since $\nabla^2 = \nabla \cdot \nabla$

$= \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) \cdot \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)$

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$]

(1) prove that $\text{curl}(\nabla\phi) = 0$ (or) $\nabla \times \nabla\phi = 0$

Solution:

$$\nabla = \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z$$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\nabla \times \nabla \phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \partial\phi/\partial y & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \Sigma \vec{i} \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z} \right)$$

$$= 0$$

(2) prove that $\operatorname{div}(\operatorname{Curl} \vec{F}) = 0$ (or) $\nabla \cdot (\nabla \times \vec{F}) = 0$

(A.U. June 2005)

Solution:

$$\operatorname{Curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$ if $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \partial/\partial y\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \partial/\partial z\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

(5) Find div F and curl F where F = grad$(x^3+y^3+z^3-3xyz)$

(AU May 2001)

Solution:

Given $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

$\vec{F} = (3x^2-3yz)\vec{i} + (3y^2-3xz)\vec{j} + (3z^2-3xy)\vec{k}$

$\nabla\cdot\vec{F} = \partial/\partial x\,(3x^2-3yz) + \partial/\partial y\,(3y^2-3xz) + \partial/\partial z\,(3z^2-3xy)$

$= 6x+6y+6z$

$\nabla\cdot\vec{F} = 6(x+y+z)$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$= \vec{i}(-3x+3x) - \vec{j}(-3y+3y) + \vec{k}(-3z+3z)$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$

$\nabla\times\vec{F} = 0$ //

Prove $\text{div}(u\,\text{grad}\,v) = u\nabla^2 v + (\text{grad}\,u)\cdot(\text{grad}\,v)$

Solution:

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u\cdot\nabla v$$

$$u\nabla v = u\left[\vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}\right]$$

$$= \vec{i}\,u\frac{\partial v}{\partial x} + \vec{j}\,u\frac{\partial v}{\partial y} + \vec{k}\,u\frac{\partial v}{\partial z}$$

$$\nabla\cdot(u\nabla v) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left(\vec{i}\,u\frac{\partial v}{\partial x} + \vec{j}\,u\frac{\partial v}{\partial y} + \vec{k}\,u\frac{\partial v}{\partial z}\right)$$

$$= \partial/\partial x\left(u\frac{\partial v}{\partial x}\right) + \partial/\partial y\left(u\frac{\partial v}{\partial y}\right) + \partial/\partial z\left(u\frac{\partial v}{\partial z}\right)$$

$$= u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

$$= u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

$$\nabla u = \vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}$$

$$\nabla v = \vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}$$

$$\nabla u\cdot\nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u\cdot\nabla v.$$

Scanned by CamScanner

prove that curl (curl $\vec{F}$) = grad (div $\vec{F}$) $-$ $\nabla^2 \vec{F}$

(2006, 2004)

ution :

$$\nabla \times (\nabla \times \vec{F}) = (\nabla \cdot \vec{F}) \nabla - (\nabla \cdot \nabla) \vec{F}$$

$$\left[\because \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} \right]$$

$$= \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

$$= \text{grad} (\text{div} \vec{F}) - \nabla^2 \vec{F}.$$