Taylor's Series Expansion for a function of two variable:-

Let $f(x,y)$ be a function of two variables x, y in the neighbourhood of (a,b)

$$f(x,y) = f(a,b) + \frac{1}{\lfloor 1}\left[f_x(a,b)(x-a) + f_y(a,b)(y-b)\right] + \frac{1}{\lfloor 2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots \quad (1)$$

Note: Putting $a=0, b=0$ in (1) we have

$$f(x,y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{\lfloor 2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

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is known as Maclaurin's Series for two variables.

① Expand e^{xy} at $(1,1)$ as the Taylors series.

Sol: $f(x,y) = e^{xy}$ and $(a,b) = (1,1)$

$f(x,y) = e^{xy}$, $f(a,b) = f(1,1) = e$

$f_x(x,y) = e^{xy} y$, $f_x(a,b) = f_x(1,1) = e$

$f_{xx}(x,y) = e^{xy} y^2$, $f_{xx}(a,b) = f_{xx}(1,1) = e$

$f_{xy} = e^{xy}(1) + y e^{xy}(x)$; $f_{xy}(a,b) = f_{xy}(1,1)$
$= 2e$

$f_y(x,y) = e^{xy} x$, $f_y(a,b) = f_y(1,1) = e$

$f_{yy}(x,y) = e^{xy} x^2$, $f_{yy}(a,b) = f_{yy}(1,1) = e$

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The Taylors Series is

$$f(x,y) = f(a,b) + \frac{1}{\lfloor 1}\left[f_x(a,b)(x-a) + f_y(a,b)(y-b)\right] + \frac{1}{\lfloor 2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] + \cdots$$

$$e^{xy} = e\left[1 + \left((x-1)+(y-1)\right) + \frac{1}{2}\left((x-1)^2 + 4(x-1)(y-1) + (y-1)^2\right)\right] + \cdots$$

(2) Expand $e^x \cos y$ as the Taylors series

Sol: If the point is not given, Consider the point as $(0,0)$.

$f(x,y) = e^x \cos y$, $(a,b) = (0,0)$

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$f(x,y) = e^x \cos y \,, \quad f(a,b) = f(0,0) = 1$

$f_x(x,y) = e^x \cos y \,, \quad f_x(a,b) = f_x(0,0) = 1$

$f_{xx}(x,y) = e^x \cos y \,, \quad f_{xx}(a,b) = f_{xx}(0,0) = 1$

$f_{xy}(x,y) = e^x(-\sin y) \,, \quad f_{xy}(a,b) = f_{xy}(0,0) = 0$

$f_y(x,y) = e^x(-\sin y) \,, \quad f_y(a,b) = f_y(0,0) = 0$

$f_{yy}(x,y) = e^x(-\cos y) \,, \quad f_{yy}(a,b) = f_{yy}(0,0) = -1$

The Taylor's Series is

$$f(x,y) = 1 + \frac{1}{\underline{1}}\left[(x-0)(1) + (y-0)(0)\right] + \frac{1}{\underline{2}}\left[(x-0)^2(1) + 2(x-0)(y-0)0 + (y-0)^2(-1)\right] + \ldots$$

$$f(x,y) = 1 + \frac{x}{\underline{1}} + \frac{1}{2}(x^2 - y^2) + \ldots$$

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0)=1

Expand

① $\tan^{-1}(y/x)$ at $(1,1)$

Sol:-

$$f(x,y) = \tan^{-1}(y/x) \;\&\; (a,b) = (1,1)$$

$$f(x,y) = \tan^{-1}(y/x) \;,\; f(a,b) = \tan^{-1}(1) = \pi/4$$

$$f_x(x,y) = -\frac{y}{x^2+y^2} \;,\; f_x(a,b) = f_x(1,1) = -1/2$$

$$f_{xx}(x,y) = \frac{2xy}{x^2+y^2} \;,\; f_{xx}(a,b) = f_{xx}(1,1) = 2/4 = 1/2$$

$$f_{xy}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} \;,\; f_{xy}(a,b) = f_{xy}(1,1) = 0$$

$$f_y(x,y) = \frac{x}{x^2+y^2} \;,\; f_y(a,b) = f_y(1,1) = 1/2$$

$$f_{yy}(x,y) = -\frac{2xy}{(x^2+y^2)^2} \;,\; f_{yy}(a,b) = f_{yy}(1,1) = \frac{-2}{4} = -$$

$\therefore$ The Taylor's series is

$$f(x,y) = \pi/4 + \left[(x-1)(-1/2) + (y-1)(1/2)\right] + 1/2\left[(x-1)^2(1/2) + 2(x-1)(y-1)(0) + (y-1)^2(-1/2)\right] + \cdots$$

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$$f(x,y) = \pi/4 + 1/2\,(y-x) + 1/4\left[(x-1)^2 - (y-1)^2\right] + \cdots$$

(2) Expand $e^x \log(1+y)$ as the Taylors Series in the neighbourhood of $(0,0)$.

Sol:-

$$f(x,y) = e^x \log(1+y) \;\&\; (a,b) = (0,0)$$

$$f(x,y) = e^x \log(1+y), \quad f(a,b) = f(0,0) = 0$$

$$f_x(x,y) = e^x \log(1+y), \quad f_x(a,b) = f_x(0,0) = 0$$

$$f_{xx}(x,y) = e^x \log(1+y), \quad f_{xx}(a,b) = f_{xx}(0,0) = 0$$

$$f_{xy}(x,y) = e^x \frac{1}{1+y}, \quad f_{xy}(a,b) = f_{xy}(0,0) = 1$$

$$f_y(x,y) = e^x \frac{1}{1+y}, \quad f_y(a,b) = f_y(0,0) = 1$$

$$f_{yy}(x,y) = -\frac{e^x}{(1+y)^2}, \quad f_{yy}(a,b) = f_{yy}(0,0) = -1$$

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Taylor's series is

$$f(x,y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \cdots$$

③ obtain the Expansion of $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem.

Sol:- Taylor's series of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] +$$

$$\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

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Here $a=1, b=-2$.

$f(x,y) = x^2y + 3y - 2$; $(a,b) = (1,-2)$

$f(a,b) = f(1,-2) = -10$

$f_x(x,y) = 2xy$, $f_x(a,b) = f_x(1,-2) = -4$

$f_y = x^2 + 3$, $f_y(1,-2) = 4$

$f_{xx} = 2y$, $f_{xx}(1,-2) = -4$

$f_{xy} = 2x$, $f_{xy}(1,-2) = 2$

$f_{yy} = 0$, $f_{yy}(1,-2) = 0$

$f_{xxx} = 0$, $f_{xxx}(1,-2) = 0$

$f_{xxy} = 2$, $f_{xxy}(1,-2) = 2$

$f_{xyy} = 0$, $f_{xyy}(1,-2) = 0$

$f_{yyy} = 0$, $f_{yyy}(1,-2) = 0$

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The Taylor's Series is

$$x^2y+3y-2 = -10+\left[(x-1)(-4)+(y+2)(4)\right]+$$

$$\frac{1}{2}\left[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)\right]+$$

$$\frac{1}{6}\left[(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)\right]$$

$$=-10+\left[-4(x-1)+4(y+2)\right]+$$

$$\left[-2(x-1)^2+2(x-1)(y+2)\right]+(x-1)^2(y+2).$$

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