(1) Evaluate $\iiint \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\,dy\,dz$ over the region bounded by the sphere $x^2+y^2+z^2=1$.

Solution:

Let us transform this integral in spherical polar coordinates by taking

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\varphi$

Here φ varies from 0 to 2π

θ varies from 0 to π

r varies from 0 to 1.

$$\iiint \frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\, r^2\sin\theta\, dr\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta\left[\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr\right] d\theta\,d\varphi$$

Put $r = \sin t$

$dr = \cos t\,dt$

when $r=0,\ t=0$

when $r=1,\ t=\pi/2$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cos t\,dt\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \int_0^{\pi/2} \frac{\sin^2 t}{\cancel{\cos t}}\,\cancel{\cos t}\,dt\,d\theta\,d\varphi$$

Scanned by CamScanner

$$= \int_0^{2\pi} \int_0^{\pi} \sin\theta \ \tfrac{1}{2} \tfrac{\pi}{2} \, d\theta \, d\varphi$$

$$= \tfrac{\pi}{4} \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} d\varphi$$

$$= \tfrac{\pi}{4} \int_0^{2\pi} \left[1+1\right] d\varphi$$

$$= \tfrac{2\pi}{4} \left(\varphi\right)_0^{2\pi} = \frac{\cancel{2}\pi}{\cancel{4}} \cancel{2}\pi = \pi^2$$

(2) Evaluate the integral $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$.

Solution:

Let us Convert the given integral into spherical polar coordinates.

$x = r\sin\theta\cos\varphi \qquad dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\varphi$

$y = r\sin\theta\sin\varphi \qquad x^2+y^2+z^2 = r^2$

$z = r\cos\theta$

$$\iiint x^2+y^2+z^2\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2 \cdot r^2 \sin\theta\,dr\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \left(\frac{r^5}{5}\right)_0^1 d\theta\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \,(1/5)\, d\theta\, d\varphi$$

$$= 1/5 \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta\, d\varphi$$

$\cos\pi = -1$
$\cos 0 = 1.$

$$= 1/5 \int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\varphi$$

$$= 1/5 \int_0^{2\pi} [1+1]\, d\varphi$$

$$= 2/5 \,(\varphi)_0^{2\pi} = 2/5(2\pi) = 4\pi/5$$

3 Evaluate $\iiint (x^2+y^2+z^2)\, dx\,dy\,dz$ taken over the volume closed by the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical coordinates.

Solution:

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

$$dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$\iiint (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \left(\frac{r^5}{5}\right)_0^a d\theta\, d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi} (-\cos\theta)_0^{\pi}\, d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi} (1+1)\, d\phi$$

$$= \frac{2a^5}{5}(\phi)_0^{2\pi} = \frac{4\pi a^5}{5}.$$

(4) Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar Coordinates.

Solution:

$x = r\sin\theta\cos\phi$ $\qquad x^2+y^2+z^2 = r^2$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$.

r varies from 0 to 1.

θ varies from 0 to $\pi/2$.

ϕ varies from 0 to $\pi/2$.

In spherical polar Coordinates system we have $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$ $\quad dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\varphi$

$z = r\cos\theta.$

The region is bounded by

(i) $z = 0,$ $\quad z = \sqrt{a^2 - x^2 - y^2}$

$r\cos\theta = 0$ $\quad r\cos\theta = \sqrt{a^2 - r^2\sin^2\theta\cos^2\varphi - r^2\sin^2\theta\sin^2\varphi}$

ie $\theta = \pi/2$ $\quad r\cos\theta = \sqrt{a^2 - r^2\sin^2\theta}$

$r^2\cos^2\theta = a^2 - r^2\sin^2\theta$

$r^2 = a^2$

$r = \pm a.$

(ii) $y = 0,$ $\quad y = \sqrt{a^2 - x^2} \Rightarrow y^2 = a^2 - x^2 \Rightarrow x^2 + y^2 = a^2$

$r\sin\theta\sin\varphi = 0$ $\quad r\sin\theta\sin\varphi = \sqrt{a^2 - x^2}$

$\Rightarrow \varphi = 0$ $\quad r\sin\theta\sin\varphi = \sqrt{a^2 - r^2\sin^2\theta\cos^2\varphi}$

$r^2\sin^2\theta\sin^2\varphi = a^2 - r^2\sin^2\theta\cos^2\varphi$

$r^2\sin^2\theta = a^2$

$r = \pm a \quad (\because \theta = \pi/2)$

φ Varies from 0 to $\pi/2$.

(iii) $x = 0,\ x = a$ $\quad r\sin\theta\cos\varphi = a$

$r\sin\theta\cos\varphi = 0$ $\quad r = a$

$r = 0$

r varies from 0 to a.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1}{\sqrt{1-r^2}}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

put $r = \sin t$

$dr = \cos t\, dt$

when $r=0,\ t=0$

when $r=1,\ t=\pi/2$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}} \cos t\, dt\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \int_0^{\pi/2} \sin^2 t\, dt\, d\theta\, d\phi.$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\ \tfrac{1}{2}\tfrac{\pi}{2}\, d\theta\, d\phi.$$

$$= \frac{\pi}{4}\int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\, d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} (0+1)\, d\phi$$

$$= \frac{\pi}{4}\,\frac{\pi}{2} = \frac{\pi^2}{8}.$$

(5) Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\, dz\, dy\, dx$ by transforming to spherical polar coordinates.

Hence the region is bounded by $\rho=0, \rho=a$;

$0 \le \theta \le \pi/2$; $0 \le \varphi \le \pi/2$.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r\sin\theta\cos\varphi\, r\sin\theta\sin\varphi\, r\cos\theta\; r^2\sin\theta\, dr\,d\theta\,d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^5 \sin^3\theta\cos\theta\sin\varphi\cos\varphi\, dr\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin^3\theta\cos\theta\sin\varphi\cos\varphi \left(\frac{r^6}{6}\right)_0^a d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin^3\theta\cos\theta\sin\varphi\cos\varphi\, \frac{a^6}{6}\, d\theta\, d\varphi$$

$$= a^6/6 \int_0^{\pi/2} \sin^3\theta\cos\theta \left(\int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi\right) d\theta$$

$$= \frac{a^6}{6}\int_0^{\pi/2} \sin^3\theta\cos\theta \left(\int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\varphi\right) d\theta$$

$$= \frac{a^6}{6}\int_0^{\pi/2} \sin^3\theta\cos\theta\, \frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi/2} d\theta$$

$$= \frac{a^6}{6}\int_0^{\pi/2} \sin^3\theta\cos\theta\, 1/2\,(1/2 + 1/2)\, d\theta$$

$t = \sin\theta$

$dt = \cos\theta\, d\theta$

when $\theta=0$, $t=0$

when $\theta=\pi/2$, $t=1$

$$= \frac{a^6}{12}\int_0^{\pi/2} \sin^3\theta\cos\theta\, d\theta$$

$$= \frac{a^6}{12}\int_0^1 t^3\, dt = \frac{a^6}{12}\left(\frac{t^4}{4}\right)_0^1$$

$$= \frac{a^6}{12}\left(\frac{1}{4}\right)$$

$$= \frac{a^6}{48}.$$

$$= \frac{a^6}{48}$$ //.

(6) Evaluate $\iiint_V \sqrt{1-x^2-y^2-z^2}\, dx\, dy\, dz$ where V is the volume of the sphere $x^2+y^2+z^2=1$ by transforming to spherical polar coordinates.

Solution:

In spherical polar coordinates

we have $x = r\sin\theta\cos\phi$ $\quad dxdydz = r^2\sin\theta\, dr d\theta d\phi$

$y = r\sin\theta\sin\phi$ $\quad x^2+y^2+z^2 = r^2$

$z = r\cos\theta$

$$\iiint_V \sqrt{1-x^2-y^2-z^2}\, dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \sqrt{1-r^2}\, r^2\sin\theta\, dr d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\int_0^1 \sqrt{1-r^2}\, r^2\, dr\right)\sin\theta\, d\theta\, d\phi$$

put $r = \sin t$ $\quad$ when $r=0,\ t=0$

$dr = \cos t\, dt$ $\quad r=1,\ t=\pi/2$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\int_0^{\pi/2} \sqrt{1-\sin^2 t}\ \sin^2 t\cos t\, dt\right)\sin\theta\, d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\int_0^{\pi/2} \sin^2 t\cos^2 t\, dt\right)\sin\theta\, d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\int_0^{\pi/2} \sin^2 t(1-\sin^2 t)\, dt\right)\sin\theta\, d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\int_0^{\pi/2} \sin^2 t\,dt - \int_0^{\pi/2} \sin^4 t\,dt\right) \sin\theta\, d\theta\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{2}\frac{\pi}{2} - \frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right] \sin\theta\, d\theta\, d\varphi$$

$$= \pi/16 \int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\varphi$$

$$= \pi/16 \int_0^{2\pi} [1+1]\, d\varphi$$

$$= \pi/8\, (\varphi)_0^{2\pi} = \pi/8\,(2\pi) = \pi^2/4.$$

(9) Find the volume of the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical polar coordinates.

Solution:

$$\text{Required volume} = \iiint_V dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \left(\frac{r^3}{3}\right)_0^a d\theta\, d\varphi$$

$$= \frac{a^3}{3}\int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\varphi$$

$$= \frac{a^3}{3}\int_0^{2\pi} (1+1)\, d\varphi$$

$$= \frac{2a^3}{3} \int_0^{2\pi} d\phi$$

$$= \frac{2a^3}{3}(2\pi)$$

$$= \frac{4\pi a^3}{3}.$$

(8) Evaluate $\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ throughout the volume of the sphere $x^2+y^2+z^2=a^2$.

Solution:

$$\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\cancel{r^2}\sin\theta\,dr\,d\theta\,d\phi}{\cancel{r^2}}$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta\,(r)_0^a\,d\theta\,d\phi$$

$$= a\int_0^{2\pi} (-\cos\theta)_0^{\pi}\,d\phi$$

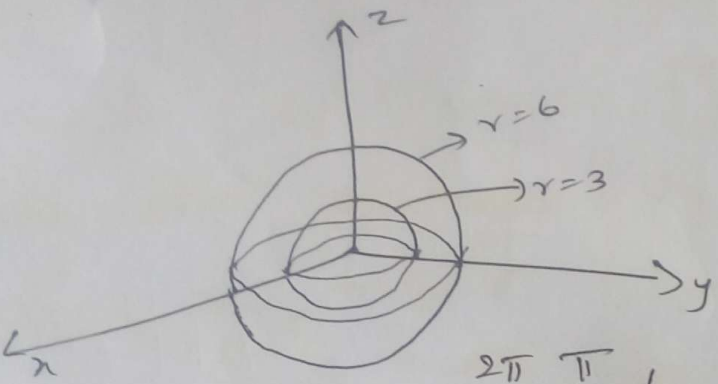
$$= a\int_0^{2\pi} (1+1)\,d\phi$$

$$= 2a\,(\phi)_0^{2\pi}$$

$$= 4\pi a.$$

Scanned by CamScanner

(9) Evaluate $\iiint_D \frac{1}{x^2+y^2+z^2}\, dv$ with D is the region between the sphere $x^2+y^2+z^2=9$ & $x^2+y^2+z^2=36$.



$$\iiint_D \frac{1}{x^2+y^2+z^2}\, dv = \int_0^{2\pi}\int_0^{\pi}\int_3^{6} \frac{1}{\cancel{r^2}}\, \cancel{r^2} \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta\, (r)_3^6\, d\theta\, d\phi$$

$$= 3\int_0^{2\pi} (-\cos\theta)_0^{\pi}\, d\phi$$

$$= 2(3)\int_0^{2\pi} d\phi = 2\pi(6)$$

$$= 12\pi.$$