Increasing and decreasing functions:

Theorem: 2

definition-2

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a,b)$ then f is strictly increasing on $[a,b]$.

If $f'(x) < 0$ at each point $x \in (a,b)$, then f is strictly decreasing on $[a,b]$.

(converse is not true)

(*) proof

Theorem - 1:

definition-1

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

Then

(i) f is increasing if and only if $f'(x) \geq 0$

(ii) f is decreasing if and only if $f'(x) \leq 0$.

A function that is completely increasing or completely decreasing on I is called monotonic on I.

Every constant function is an increasing function

Every identity function is an increasing function

Ex:

Show that $f(x) = \tan^{-1}(\sin x + \cos x), x > 0$ is a strictly increasing function in the interval $(0, \pi/4)$.

Soln. $f(x) = \tan^{-1}(\sin x + \cos x)$

$$f'(x) = \frac{1}{1+(\sin x+\cos x)^2}(\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{2 + \sin 2x} > 0.$$

Since $\cos x - \sin x > 0$ in the interval $(0, \pi/4)$ and $2 + \sin 2x > 0$.

$\therefore$ $f(x)$ is strictly increasing function of x in the interval $(0, \pi/4)$.

(1) Find the intervals on which f is increasing or decreasing.

(i) $f(x) = 20 - x - x^2$

(ii) $f(x) = x^3 - 3x + 1$

(iii) $f(x) = x^3 + x + 1$

(iv) $f(x) = x - 2\sin x$, $[0, 2\pi]$

(v) $f(x) = x + \cos x$ in $[0, \pi]$

(vi) $f(x) = \sin^4 x + \cos^4 x$ in $[0, \pi/2]$

Ans

(i) increasing in $(-\infty, -\frac{1}{2}]$ and decreasing in $[-\frac{1}{2}, \infty)$

(ii) increasing in $(-\infty, -1] \cup [1, \infty)$ and decreasing in $[-1, 1]$

(iii) strictly increasing on R

(iv) decreasing in $[0, \frac{\pi}{3}] \cup [\frac{5\pi}{3}, 2\pi]$ increasing in $[\frac{\pi}{3}, \frac{5\pi}{3}]$

(v) increasing in $[0, \pi]$

(vi) increasing in $[\frac{\pi}{4}, \frac{\pi}{2}]$ and decreasing in $[0, \frac{\pi}{4}]$

(2) Which of the following functions are increasing or decreasing on the interval given?

(i) $x^2 - 1$ on $[0, 2]$

(ii) $2x^2 + 3x$ on $[-\frac{1}{2}, \frac{1}{2}]$

(iii) e^{-x} on $[0, 1]$

(iv) $x(x-1)(x+1)$ on $[-2, -1]$

(v) $x \sin x$ on $[0, \pi/4]$

Ans.

(i) increasing (ii) st. increasing

(iii) st. decreasing (iv) str. increasing

(v) increasing

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* proof

Define $f : R \to R$ by $f(x) = x^3$

Suppose $x_1 < x_2$, Then $x_2 - x_1 > 0$

and $x_1^2 + x_2^2 > 0$

This implies $x_2^3 - x_1^3 = (x_2 - x_1)(x_2^2 + x_1^2 + x_1 x_2)$

$= (x_2 - x_1)\frac{1}{2}\left[(x_1^2 + x_2^2) + (x_1 + x_2)^2\right] > 0$

$\Rightarrow x_1^3 < x_2^3$

Thus whenever $x_1 < x_2$, $f(x_1) < f(x_2)$

Hence $f(x) = x^3$ is strictly increasing.

But its derivative $f'(x) = 3x^2$ and $f'(0) = 0$

Hence its derivate f' is not strictly positive.

Definition 2:

$f : I \to R$ is said to be strictly increasing if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$. We can similarly say that a function defined on I is strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

$I = [a, b]$

Definition 1: A function f is called increasing on an interval I if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ in I. It is called decreasing on I if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$ in I.

A function that is Completely increasing or Completely decreasing on I is Called monotonic on I.

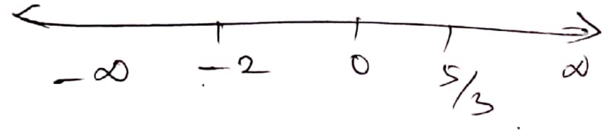
① Find the interval in which $f(x) = 2x^3 + x^2 - 20x$ is increasing and decreasing.

Sol:

$$f'(x) = 6x^2 + 2x - 20 = 2(3x^2 + x - 10)$$
$$= 2(x+2)(3x-5)$$

$$f'(x) = 0 \Rightarrow x = -2,\ x = 5/3$$

The values -2 and $5/3$ divide the real line into intervals $(-\infty, -2)$, $(-2, 5/3)$ & $(5/3, \infty)$.



$f'(x) \geq 0$ in $(-\infty, -2]$

$\therefore f$ is increasing on $(-\infty, -2]$

$f'(x) \leq 0$ in $[-2, 5/3]$

$\therefore f$ is decreasing on $[-2, 5/3]$

$f'(x) \geq 0$ in $[5/3, \infty)$

$\therefore$ f is increasing on $[5/3, \infty)$.

Note: If a function changes its signs at different points of a region (interval) then the function is not monotonic in that region. So to prove the non-monotonicity of a function, it is enough to prove that f' has different signs at different points

Ex: Prove that the function $f(x) = \sin x + \cos 2x$ is not monotonic on the interval $[0, \pi/4]$

Sol: Let $f(x) = \sin x + \cos 2x$

Then $f'(x) = \cos x - 2 \sin 2x$

Now $f'(0) = \cos 0 - 2 \sin 0 = 1 - 0 = 1 > 0$

and $f'(\pi/4) = \cos(\pi/4) - 2\sin 2(\pi/4)$

$= \frac{1}{\sqrt{2}} - 2 \times 1 < 0$.

Thus f' is of different signs at 0 and $\pi/4$.

Therefore f is not monotonic on $[0, \pi/4]$.

Concavity (Convexity) and points of inflection:

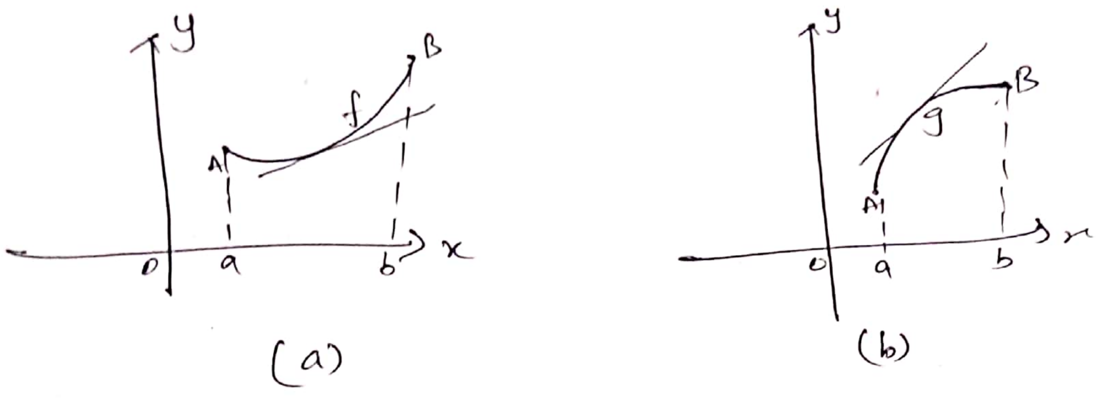
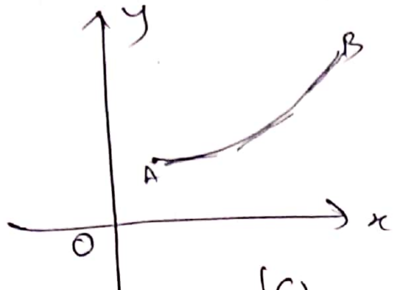


Figure (a) & (b) shows the graph of two increasing functions on [a, b]. Both graphs join point A to point B but they look different because they bend in different directions.

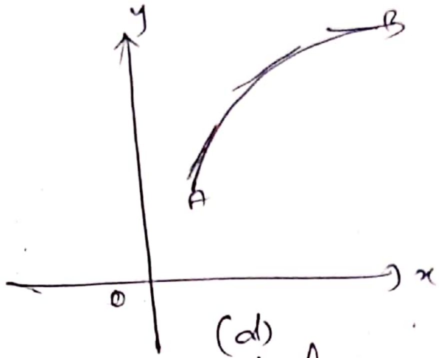
How can we distinguish between these two types of behaviour?

In (a) the curve lies above the tangents and f is called concave upward (convex downward) on [a, b]

In (b) the curve lies below the tangents and g is called concave downward (convex upward) on [a, b].



(c)



(d)

Let us see how the second derivative helps to determine the interval of concavity (convexity).

Looking at (c), you can see that, going from left to right, the slope of the tangent increases. This means that the derivative $f'(x)$ is an increasing function and therefore its derivative $f''(x)$ is positive.

Likewise in (d), the slope of the tangent decreases from left to right, so $f'(x)$ decreases and therefore $f''(x)$ is negative. This reasoning can be reversed and suggests that the following theorem is true.

Test for Concavity (convexity)

Suppose f is twice differentiable on an interval I.

(i) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward (convex downward) on I.

(ii) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward (convex upward) on I.

Definition: A point P on a curve is called a point of inflection if the curve changes from concave upward (convex downward) to concave downward (convex upward) or from concave downward (convex upward) to concave upward (convex downward) at P.

That is the point that separate the convex part of a continuous curve from the concave part is called the point of inflection of the curve.

Note: points of inflections need not be critical points and critical points need not be points of inflections.

Note: For points of inflections x_0, $f''(x_0) = 0$ and in the immediate neighbourhood (a, b) of x_0, $f''(a)$ and $f''(b)$ must differe in sign.

Ex:-

① Determine the domain of Concavity (Convexity) of the curve $y = 2 - x^2$.

Sol: $y = 2 - x^2$

$y' = -2x$ and $y'' = -2 < 0$, $x \in R$.

Here the curve is everywhere Concave downwards (convex upwards).

② Determine the domain of convexity of the function $y = e^x$.

Sol: $y = e^x$, $y'' = e^x > 0$ for x.

Hence the curve is everywhere Convex downward (Concave upward).

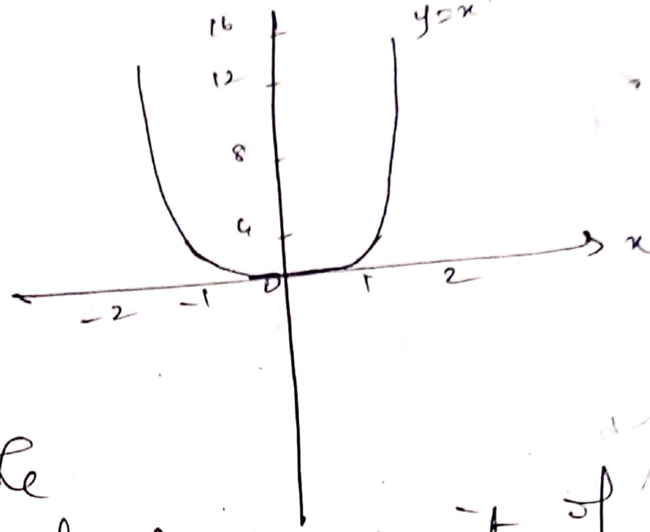
③ Test the curve $y = x^4$ for points of inflection:

Sol: $y = x^4$, $y' = 4x^3$

$y'' = 12x^2 = 0$ for $x = 0$

and $y'' > 0$ for $x < 0$ and $x > 0$

Therefore the curve is Concave upward and y'' does not change sign as $y(x)$ passes through $x=0$. Thus the Curve does not admit any point of inflection.



Note: The Curve is Concave upward in $(-\infty, 0)$ and $(0, \infty)$.

4) Determine where the curve $y = x^3 - 3x + 1$ is Concave upward, and where it is Concave downward. Also find the inflection points.

Sol:

$-\infty$ $x=0$ ∞

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

$$f''(x) = 6x$$

Thus $f''(x) > 0$ when $x > 0$ and

$f''(x) < 0$ when $x < 0$.

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The test for Concavity tell us that the curve is Concave downward on $(-\infty, 0)$ and Concave upward on $(0, \infty)$. Since the curve changes from Concave downward to Concave upward when $x=0$, the point $(0, f(0))$, ie $(0, 1)$ is a point of inflection. Note that $f''(0)=0$.

(5) Discuss the curve $y = x^4 - 4x^3$ with respect to Concavity and points of inflection.

Sol:

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x = 12x(x-2).$$

Since $f''(x) = 0$ when $x = 0$ or $x = 2$, we divide the real line into three intervals $(-\infty, 0), (0, 2), (2, \infty)$

$-\infty \quad 0 \quad 2 \quad \infty$

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In the interval $(-\infty, 0)$,

$$f''(x) > 0$$

$\therefore$ The curve is Concave upward.

In the interval $(0, 2)$,

$$f''(x) < 0.$$

$\therefore$ The curve is Concave downward

In the interval $(2, \infty)$.

$$f''(x) > 0$$

$\therefore$ The curve is Concave upward.

The point $(0, f(0))$ ie $(0, 0)$ is an inflection point since the curve changes from Concave upward to Concave downward there.

Also $(2, f(2))$ ie $(2, -16)$ is an inflection point since the curve changes from Concave downward to Concave upward there.

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