Module-3

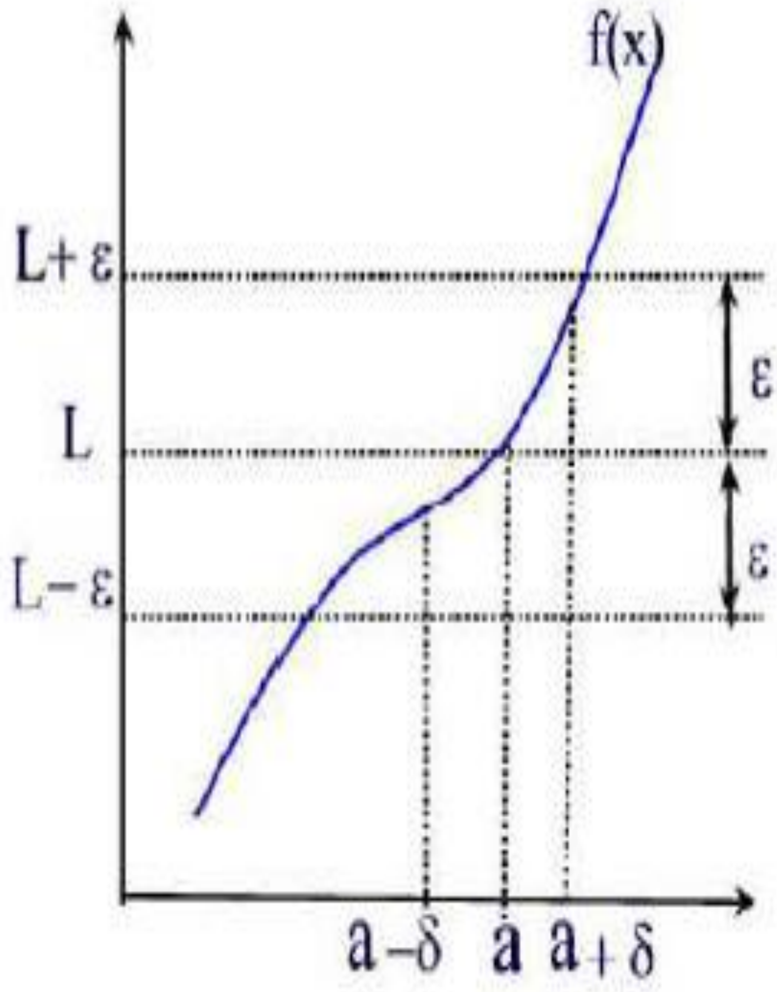
(Multi Variable Calculus)

Limit of a function

Let f be a function and a be a point in its domain. We say that $f(x)$ has a limit L at a if and only if for every $\epsilon > 0$ there exists a positive number δ depending on ϵ such that for any x in the domain of f with the property $0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$. In symbol, we write

$$\lim_{x \to a} f(x) = L$$

or $f(x) \to L$ as $x \to a$.
Geometrically, the definition says that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that any point inside the interval $(a - \delta, a + \delta)$ is mapped to a point inside the interval $(L - \epsilon, L + \epsilon)$ as shown in the figure below.


f(x)
L+ ε
L
L− ε
ε
ε
a−δ
a
a+δ

Functions of Several variables:

We do come across quantities which depend on two or more variables.

For example the area of a triangle A is given by the relation A= xh/2 where 'x' is the length of the base and 'h' is the altitutde.

The change in the value of 'x' or value of 'h' or both affects the area. For a given pair of values of 'x' and 'h', A has a definite value.

We say that A is a function of 'x' and 'h' that is A=f(x,h), 'x' and 'h' are independent variables and 'A' is called the dependent variable.

Similarly the volume 'V' of a rectangular parallelepiped is given by V=xyz, where 'x' is the length, 'y' is the breadth and 'z' is its height.

In this case 'V' is a function of three variables 'x', 'y' and 'z'.

Symbolically, we write it as V = f(x,y,z)

The problems in

(1) computer science,
(2) statistics,
(3) fluid dynamics,
(4) economics etc

deals with functions of two or more independent variables.

Limit of a function of two variables:

A function $f(x,y)$ is said to tend to a limit l as the point (x,y) tends to the point (a,b), if to every arbitrary small positive number ε, there is a corresponding positive δ such that

$$|f(x,y)-l|<\varepsilon$$

whenever $|x-a|<\delta$, $|y-b|<\delta$

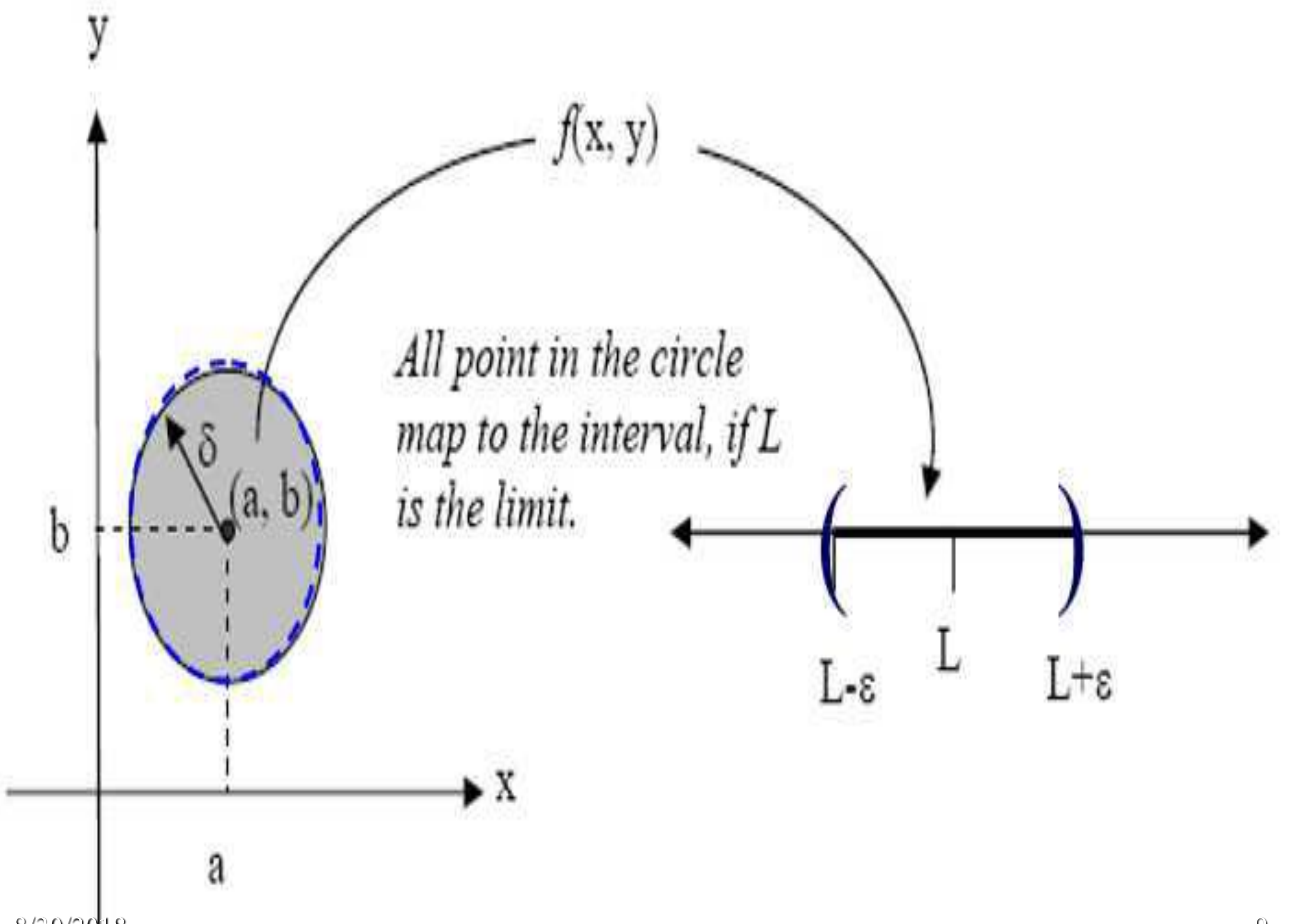
In which case we say $\lim\limits_{\substack{x\to a\\ y\to b}}[f(x,y)]=l$

A similar definition extends to functions in two variables: We say that L is the **limit** of a function f at the point (a, b), written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if $f(x, y)$ is as close to L as we please whenever the distance from the point (x, y) to the point (a, b) is sufficiently small, but not zero.

Using $\epsilon\delta$ definition we say that L is the limit of $f(x,y)$ as (x,y) approaches (a,b) if and only if for every given $\epsilon > 0$ we can find a $\delta > 0$ such that for any point (x,y) where $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ we have $|f(x,y) - L| < \epsilon$. What does this mean in words? To say that L is the limit of $f(x,y)$ as $(x,y) \to (a,b)$ means that for any given $\epsilon > 0$, we can find an open punctured disk (i.e. without the center and the boundary) centered at (a,b) such that for any point (x,y) inside the disk the difference $f(x,y) - L$ is within ϵ, i.e., $L - \epsilon < f(x,y) < L + \epsilon$.


y
f(x, y)
All point in the circle
map to the interval, if L
is the limit.
δ
(a, b)
b
L-ε
L
L+ε
x
a

Continuity of a function of two variables:

A function $f(x,y)$ is said to be continuous at the point (a,b) if

$$\lim_{\substack{x\to a\\ y\to b}}[f(x,y)] = f(a,b)$$

in any manner x and y approach a and b

It should not take for granted that the path along which the point (x,y) tends to (a,b) is immaterial. Since $\lim_{y\to b}\left\{\lim_{x\to a}[f(x,y)]\right\}$ need not always be equal to $\lim_{x\to a}\left\{\lim_{y\to b}[f(x,y)]\right\}$

Example (1): **Evaluate** $\lim\limits_{\substack{x\to 1\\ y\to 2}}\left[\dfrac{2x^2y}{x^2+y^2+1}\right]$

Sol: $$\lim_{\substack{x\to 1\\ y\to 2}}\left[\frac{2x^2y}{x^2+y^2+1}\right]=\lim_{x\to 1}\left\{\lim_{y\to 2}\left[\frac{2x^2y}{x^2+y^2+1}\right]\right\}$$

$$=\lim_{x\to 1}\left[\frac{4x^2}{x^2+5}\right]=\frac{4}{6}=\frac{2}{3}$$

Example (2): Evaluate $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left[\dfrac{xy}{x^2+y^2}\right]$

Sol: let $y = mx$ so that as $x \to 0$, $y \to 0$

$$\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{xy}{x^2+y^2}\right]=\lim_{x\to 0}\left[\frac{x.mx}{x^2+m^2x^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{m}{1+m^2}\right]=\frac{m}{1+m^2}$$

Which depends up on the value of m. Hence the limit is different for different lines.

The limit does not exist.

Example (3): **Evaluate** $\lim\limits_{\substack{x\to\infty \\ y\to 2}}\left[\dfrac{xy}{x^2+2y^2}\right]$

Sol: $\lim\limits_{\substack{x\to\infty \\ y\to 2}}\left[\dfrac{xy}{x^2+2y^2}\right]=\lim\limits_{x\to\infty}\left\{\lim\limits_{y\to 2}\left[\dfrac{xy}{x^2+2y^2}\right]\right\}$

$$=\lim_{x\to\infty}\left\{\frac{2x}{x^2+8}\right\}=\lim_{x\to\infty}\left\{\frac{(2/x)}{1+\dfrac{8}{x^2}}\right\}=0$$

Example (4): Evaluate $\lim\limits_{\substack{x\to 1\\ y\to 1}}\left[\dfrac{x(y-1)}{y(x-1)}\right]$

Sol: $\lim\limits_{\substack{x\to 1\\ y\to 1}}\left[\dfrac{x(y-1)}{y(x-1)}\right]=\lim\limits_{x\to 1}\left\{\lim\limits_{y\to 1}\left[\dfrac{x(y-1)}{y(x-1)}\right]\right\}=\lim\limits_{x\to 1}[0]=0$ **(i)**

Again $\lim\limits_{\substack{x\to 1\\ y\to 1}}\left[\dfrac{x(y-1)}{y(x-1)}\right]=\lim\limits_{y\to 1}\left\{\lim\limits_{x\to 1}\left[\dfrac{x(y-1)}{y(x-1)}\right]\right\}=\lim\limits_{y\to 1}\left[\dfrac{1}{0}\right]=\infty$ **(ii)**

Since (i) $\neq$ (ii), therefore the limit does not exist.

Example (5): If $f(x,y)=\dfrac{x-y}{2x+y}$**, show that** $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0}\{f(x,y)\}\right]\neq\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0}\{f(x,y)\}\right]$

Sol: $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0}\{f(x,y)\}\right]=\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0}\dfrac{x-y}{2x+y}\right]=\lim\limits_{x\to 0}\left[\dfrac{1}{2}\right]=\dfrac{1}{2}$ **(i)**

Again $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0}\{f(x,y)\}\right]=\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0}\dfrac{x-y}{2x+y}\right]=\lim\limits_{y\to 0}[-1]=-1$ **(ii)**

Therefore **(i)** $\neq$ **(ii)**

Example (6): Show that the function $f(x,y)=\begin{cases} x^2+2y & ;(x,y)\neq(1,2) \\ 0 & ;(x,y)=(1,2) \end{cases}$

is discontinuous at $(1,2)$

Sol: $\lim\limits_{\substack{x\to1\\y\to2}}\left[f(x,y)\right]=\lim\limits_{x\to1}\left\{\lim\limits_{y\to2}\left[x^2+2y\right]\right\}$ **=** $\lim\limits_{x\to1}\left\{x^2+4\right\}=5$

But $f(1,2)=0$ **(given)**

$$\lim_{\substack{x\to1\\y\to2}}\left[f(x,y)\right]\neq f(1,2).$$

Therefore, $f(x,y)$ **is discontinuous at the point** $(1,2)$.

Example (7):

Investigate the continuity of the function $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

at the origin.

Sol: **consider** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left[\dfrac{xy}{x^2+y^2}\right]$

let $y=mx$ **so that as** $x\to 0$ **,** $y\to 0$

$$\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{xy}{x^2+y^2}\right]=\lim_{x\to 0}\left[\frac{x.mx}{x^2+m^2x^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{m}{1+m^2}\right]=\frac{m}{1+m^2}$$

which depends up on the value of m **and therefore not unique. Hence the limit does not exist and as such that function is discontinuous at the origin.**

PARTIAL DIFFERENTIATION

PARTIAL DIFFERENTIAL COEFFICIENTS

The partial derivative of a function of several variables is the ordinary derivative with respect to any one of the variables whenever, all the remaining variables are held constant. The difference between partial and ordinary differentiation is that while differentiating (partially) with respect to one variable, all other variables are treated (temporarily) as constants and in ordinary differentiation no variable taken as constant,

Definition: Let $z = f(x, y)$

Keeping y constant and varying only x, the partial derivative of z w.r.t. 'x' is denoted by $\frac{\partial z}{\partial x}$ and is defined as the limit

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Partial derivative of z, w.r.t. y is denoted by $\frac{\partial z}{\partial y}$ and is defined as

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}.$$

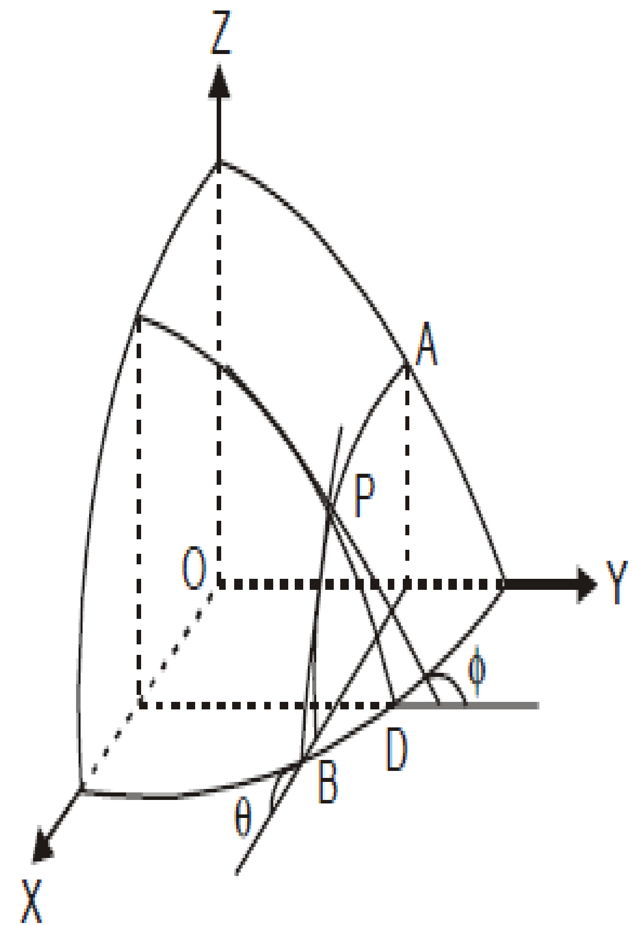
Notation: The partial derivative $\frac{\partial z}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x}$ or f_x similarly $\frac{\partial z}{\partial y}$ is denoted by $\frac{\partial f}{\partial y}$ or f_y. The partial derivatives for higher order are calculated by successive differentiation.

Thus, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \text{ and so on.}$$

Geometrical interpretation of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

Let $z = f(x, y)$ represents the equation of a surface in xyz-coordinate system. Suppose APB is the curve which a plane through any point P on the surface $\parallel$ to the xz-plane, cuts. As point P moves along this curve APB, its coordinates z and x vary while y remains constant. The slope of the tangent line at P to APB represents the rate at which z-changes w.r.t. x.



Hence, $\frac{\partial z}{\partial x} = \tan\theta$ (slope of the curve APB at the point P)

and $\frac{\partial z}{\partial y} = \tan\phi$ (slope of the curve CPD at point P)

Example 1. Verify that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ where $u(x, y) = \log_e \left(\dfrac{x^2 + y^2}{xy} \right)$

Sol. We have $$u(x, y) = \log_e \left(\frac{x^2 + y^2}{xy}\right)$$

$$\Rightarrow \quad u(x, y) = \log (x^2 + y^2) - \log x - \log y \quad \text{...}(i)$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x}$$

Now differentiating partially w.r.t. y.

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{4xy}{\left(x^2 + y^2\right)^2} \quad \text{...}(A)$$

Again differentiate (i) partially w.r.t. y, we obtain

$$\frac{\partial u}{\partial y} = \frac{2y}{\left(x^2 + y^2\right)} - \frac{1}{y}$$

Next, we differentiate above equation w.r.t. x.

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{\left(x^2 + y^2\right)^2} \quad \text{...}(B)$$

Thus, from (A) and (B), we find

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$ **Hence proved.**

Example 2. If $f = \tan^{-1}\left(\frac{y}{x}\right)$, verify that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Sol. We have $$f = \tan^{-1}\left(\frac{y}{x}\right) \qquad \ldots(i)$$

Differentiating (*i*) partially with respect to *x*, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \left(\frac{-y}{x^2+y^2}\right) \qquad \ldots(ii)$$

Differentiating (*i*) partially with respect to *y*, we get

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2} \qquad \ldots(iii)$$

Differentiating (*ii*) partially with respect to *y*, we get

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right)(-1)-(-y)(2y)}{\left(x^2+y^2\right)^2}$$

$$= \frac{y^2-x^2}{\left(x^2+y^2\right)^2} \qquad \ldots(iv)$$

Differentiating (*iii*) partially with respect to x, we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right)(1)-x(2x)}{\left(x^3+y^2\right)^2}$$

$$= \frac{y^2-x^2}{\left(x^2+y^2\right)^2} \qquad \ldots(v)$$

$\therefore$ From eqns. (*iv*) and (*v*), we get $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$. **Hence proved.**

Example 3. If $u(x+y) = x^2 + y^2$, prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Sol. Given $$u = \frac{x^2+y^2}{x+y}$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{(x+y)(2x)-(x^2+y^2)(1)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(x+y)(2y)-(x^2+y^2)(1)}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

$\therefore$ $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{4xy}{(x+y)^2}$$

or $$1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y} = 1-\frac{4xy}{(x+y)^2}=\frac{(x-y)^2}{(x+y)^2} \quad ...(i)$$

and $$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y} = \frac{\left(x^2+2xy-y^2\right)-\left(y^2+2xy-x^2\right)}{(x+y)^2}$$

$$= \frac{2\left(x^2-y^2\right)}{(x+y)^2}=\frac{2(x-y)}{(x+y)} \quad ...(ii)$$

$\therefore$ From (*ii*), we get

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2}=4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)$$, from (*i*). **Hence proved.**

Example 4. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Sol. Given $$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \qquad ...(i)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\sqrt{\left\{1-\left(\frac{x}{y}\right)^2\right\}}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{(y^2-x^2)}} - \frac{yx}{x^2+y^2} \qquad ...(ii)$$

from (i), $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{\left\{1-\left(\frac{x}{y}\right)^2\right\}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{(y^2-x^2)}} + \frac{xy}{x^2+y^2} \qquad ...(iii)$$

Adding (ii) and (iii), we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$ **Hence proved.**

Example 5. If $f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ then prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Sol. $$\frac{\partial f}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right)$$

$$\frac{\partial f}{\partial x} = 2x \cdot \tan^{-1}\left(\frac{y}{x}\right) - \frac{yx^2}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y$$

Differentiating both sides with respect to y, we get

$$\frac{\partial^2 f}{\partial y \partial x} = 2x \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) -1 = \frac{2x^2}{x^2+y^2} -1 = \frac{x^2-y^2}{x^2+y^2} \qquad \text{...}(i)$$

Again $$\frac{\partial f}{\partial y} = x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - 2y \tan^{-1}\left(\frac{x}{y}\right) - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right)$$

$$\frac{\partial f}{\partial y} = \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x\left(x^2+y^2\right)}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) = x - 2y\tan^{-1}\left(\frac{x}{y}\right).$$

Differentiating both sides with respect to x, we get

$$\frac{\partial^2 f}{\partial x\,\partial y} = 1 - 2y\,\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \qquad \ldots(ii)$$

Thus, from (i) and (ii), we get

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}.$$ **Hence proved.**

Example 6. If $V = (x^2 + y^2 + z^2)^{-1/2}$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Sol. Given $$V = (x^2 + y^2 + z^2)^{-1/2}. \quad \text{...}(i)$$

$$\therefore \quad \frac{\partial V}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}\, 2x = -x\,(x^2 + y^2 + z^2)^{-3/2}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = -\left[x\left\{-\frac{3}{2}\left(x^2+y^2+z^2\right)^{-5/2}\cdot 2x\right\}+\left(x^2+y^2+z^2\right)^{-3/2.}\cdot 1\right]$$

$$= 3x^2\,(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

$$= (x^2 + y^2 + z^2)^{-5/2}\,[3x^2 - (x^2 + y^2 + z^2)]$$

$$\frac{\partial^2 V}{\partial x^2} = (x^2 + y^2 + z^2)^{-5/2}\,(2x^2 - y^2 - z^2) \quad \text{...}(ii)$$

Similarly from (*i*), we can find

$$\frac{\partial^2 V}{\partial y^2} = (x^2 + y^2 + z^2)^{-5/2} (2y^2 - x^2 - z^2) \quad ...(iii)$$

$$\frac{\partial^2 V}{\partial z^2} = (x^2 + y^2 + z^2)^{-5/2} (2z^2 - x^2 - y^2) \quad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = (x^2 + y^2 + z^2)^{-5/2} [(2x^2 - y^2 - z^2) + (2y^2 - x^2 - z^2) + (2z^2 - x^2 - y^2)]$$

$$= (x^2 + y^2 + z^2)^{-5/2} [0] = 0.$$ **Hence proved.**

Example 7. If $u = f(r)$, where $r^2 = x^2 + y^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Sol. Given $$r^2 = x^2 + y^2 \quad ...(i)$$

Differentiating both sides partially with respect to x, we have

$$2r\frac{\partial r}{\partial x} = 2x \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \quad ...(ii)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} \quad ...(iii)$$

Now, $$u = f(r)$$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x} = f'(r)\cdot\frac{x}{r}, \text{ from } (ii)$$

Again differentiating partially w.r.to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{f'(r)x}{r}\right] = \frac{r\left[f'(r).1 + xf''(r)(\partial r/\partial x)\right] - xf'(r)(\partial r/\partial x)}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r^2}\left[rf'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)\right], \text{ from } (ii).$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[rf'(r) + y^2 f''(r) - \frac{y^2}{r} f'(r)\right]$$

Adding, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2rf'(r) + \left(x^2 + y^2\right) f''(r) - \frac{\left(x^2 + y^2\right)}{r} f'(r)\right]$$

$$= \frac{1}{r^2}\,[2rf'(r) + r^2 f''(r) - rf'(r)], \text{ from } (i)$$

$$= \frac{1}{r} f'(r) + f''(r).$$ **Hence proved.**

Example 8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$; show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Sol. Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \qquad \ldots(i)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \qquad \ldots(ii)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}. \qquad \ldots(iii)$$

Adding eqns. (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)}$$

$$\left| \text{As } a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \right.$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}. \qquad \ldots(iv)$$

Now, $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ from } (iv)$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2}.$$

Hence proved.

Example 9. If $u = \tan^{-1} \dfrac{xy}{\sqrt{(1+x^2+y^2)}}$, show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}.$$

Sol. Given $$u = \tan^{-1} \frac{xy}{\sqrt{(1+x^2+y^2)}}.$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{1}{\left[1+\left\{x^2y^2/\left(1+x^2+y^2\right)\right\}\right]}$$

$$\times\, x \left[\frac{\sqrt{\left(1+x^2+y^2\right)}.1 - y\frac{1}{2}\left(1+x^2+y^2\right)^{-1/2} 2y}{\left(1+x^2+y^2\right)}\right]$$

$$= \frac{x}{1+x^2+y^2+x^2y^2} \cdot \frac{\left(1+x^2+y^2\right)-y^2}{\sqrt{\left(1+x^2+y^2\right)}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{(1+x^2)(1+y^2)} \cdot \frac{1+x^2}{\sqrt{(1+x^2+y^2)}} = \frac{x}{(1+y^2)\sqrt{(1+x^2+y^2)}}$$

Again differentiating partially w.r.to x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{x}{(1+y^2)\sqrt{(1+x^2+y^2)}}\right] = \frac{1}{(1+y^2)}\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{(1+x^2+y^2)}}\right]$$

$$= \frac{1}{(1+y^2)}\left[\frac{\sqrt{(1+x^2+y^2)}1 - x\frac{1}{2}(1+x^2+y^2)^{-1/2}2x}{(1+x^2+y^2)}\right]$$

$$= \frac{1}{(1+y^2)} \cdot \frac{(1+x^2+y^2)-x^2}{(1+x^2+y^2)^{3/2}} = \frac{1}{(1+x^2+y^2)^{3/2}}.$$ Hence proved.

Example 10. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Sol. We have

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u} = 1 \qquad \ldots(i)$$

where u is a function of x, y and z

Differentiating (i) partially with respect to x, we get

$$= \frac{2x}{a^2+u}-\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = 0$$

or
$$\frac{\partial u}{\partial x} = \frac{2x/(a^2+u)}{\left[x^2/(a^2+u)^2+y^2/(b^2+u)^2+z^2/(c^2+u)^2\right]} = \frac{2x/(a^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Similarly,
$$\frac{\partial u}{\partial y} = \frac{2y/(b^2+u)}{\sum\left[x^2/(a^2+u)^2\right]};\ \frac{\partial u}{\partial z} = \frac{2z/(c^2+u)}{\sum\left[x^2/(a^2+u)^2\right]}$$

Adding with square

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \frac{4\left[x^2/(a^2+u)^2+y^2/(b^2+u)^2+z^2/(c^2+u)^2\right]}{\left[\sum\left\{x^2/(a^2+u)^2\right\}\right]^2}$$

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{\sum\left[x^2/\left(a^2+u\right)^2\right]} \qquad \ldots(ii)$$

Also, $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \frac{1}{\sum\left[x^2/\left(a^2+u\right)^2\right]}\left[\frac{2x^2}{\left(a^2+u\right)}+\frac{2y^2}{\left(b^2+u\right)}+\frac{2z^2}{\left(c^2+u\right)}\right]$$

$$= \frac{2}{\sum\left[x^2/\left(a^2+u\right)^2\right]}\ [1],\ \text{from } (i) \qquad \ldots(iii)$$

From (*ii*) and (*iii*), we have

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$ **Hence proved.**

Example 11. If $x^x y^y z^z = c$, show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$ Where z is a function of x and y.

Sol. Given $x^x . y^y . z^z = c$, where z is a function of x and y.

Taking logarithms, $x \log x + y \log y + z \log z = \log c$. ...(*i*)

Differentiating (*i*) partially with respect to x, we get

$$= \left[x\left(\frac{1}{x}\right)+(\log x)1\right]+\left[z\left(\frac{1}{z}\right)+(\log z)1\right]\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \quad ...(ii)$$

Similarly, from (*i*), we have $\dfrac{\partial z}{\partial y} = -\dfrac{(1+\log y)}{(1+\log z)}$ ...(*iii*)

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\left(\frac{1+\log y}{1+\log z}\right)\right], \text{ from } (iii)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(1 + \log y) \cdot \frac{\partial}{\partial x} [(1 + \log z)^{-1}]$$

$$= -(1 + \log y) \cdot \left[-(1+\log z)^{-2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}\right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log y)}{z(1+\log z)^2} \cdot \left\{-\left(\frac{1+\log x}{1+\log z}\right)\right\}, \text{ from } (iii)$$

$\therefore$ At $x = y = z$, we have $\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3}$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log_e e+\log x)} = (\text{As } \log_e e = 1)$$

$$= -\frac{1}{x\log(ex)} = -\{x \log (ex)\}^{-1}.$$ **Hence proved.**

Example 12. If $u = \log(x^3 + y^3 - x^2y - xy^2)$ then show that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Sol. We have $u = \log(x^3 + y^3 - x^2y - xy^2)$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 2xy - y^2}{(x^3 + y^3 - x^2y - xy^2)} \quad ...(i)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - x^2 - 2xy}{(x^3 + y^3 - x^2y - xy^2)} \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{(3x^2 - 2xy - y^2) + (3y^2 - x^2 - 2xy)}{(x^3 + y^3 - x^2y - xy^2)}$$

$$= \frac{2(x-y)^2}{(x+y)(x^2 + y^2 - 2xy)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(x-y)^2}{(x+y)(x-y)^2} = \frac{2}{(x+y)} \quad ...(iii)$$

Now, $\frac{\partial^2 u}{\partial x^2} + 2\,\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \cdot \frac{2}{x+y} \quad \text{(from } iii\text{)}$$

$$= 2\,\frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) + 2\frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)$$

$$= -\frac{2}{(x+y)^2} - \frac{2}{(x+y)^2} = -\frac{4}{(x+y)^2}.$$ **Hence proved.**

Example 13. If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

Sol. We have $$u = e^{xyz} \therefore \frac{\partial u}{\partial z} = e^{xyz}.xy$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}(e^{xyz}.xy) = e^{xyz}\, x^2yz + e^{xyz}.x$$

$$\frac{\partial^2 u}{\partial y \partial z} = (x^2yz + x)\, e^{xyz}$$

Hence $$\frac{\partial^3 u}{\partial x \partial y \partial z} = (2xyz + 1)\, e^{xyz} + (x^2yz + x)\, e^{xyz}.yz$$

$$= (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$ **Hence proved.**

Example 14. If $u = \log r$, where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}.$$

Sol. Given $$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2, \quad \ldots(i)$$

Differentiating partially with respect to x, we get

$$2r\frac{\partial r}{\partial x} = 2(x-a) \text{ or } \frac{\partial r}{\partial x} = \left(\frac{x-a}{r}\right). \quad \ldots(ii)$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{(y-b)}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{(z-c)}{r}$$

Now, $$u = \log r.$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\left(\frac{x-a}{r}\right), \text{ from } (ii)$$

$$\frac{\partial u}{\partial x} = \frac{x-a}{r^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right) = \frac{r^2(1)-(x-a)\,2r(\partial r/\partial x)}{r^4}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r^2-2\,(x-a)^2}{r^4}, \text{ from } (ii)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{r^2-2\,(y-b)^2}{r^4};\ \frac{\partial^2 u}{\partial z^2} = \frac{r^2-2\,(z-c)^2}{r^4}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{3r^2-2\left\{(x-a)^2+(y-b)^2+(z-c)^2\right\}}{r^4}$$

$$= \frac{3r^2-2r^2}{r^4}, \text{ from } (i) = \frac{1}{r^2}.$$ **Hence proved.**

Example 15. If $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, prove that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Sol. Given $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$.

Differentiating partially with respect to x, we get

$$\frac{\partial u}{\partial x} = x^2 \frac{1}{1+(y/x)^2} \cdot \left(-\frac{y}{x^2}\right) + 2x \tan^{-1}\left(\frac{y}{x}\right) - y^2 \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y}$$

$$= -\frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2} + 2x \tan^{-1} \frac{y}{x}$$

$$= -\frac{y\left(x^2+y^2\right)}{\left(x^2+y^2\right)} + 2x \tan^{-1}\left(\frac{y}{x}\right) = -y + 2x \tan^{-1}\left(\frac{y}{x}\right)$$

Again differentiating partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{-y+2x\tan^{-1}\left(\frac{y}{x}\right)\right\} = -1 + 2x \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= -1 + \frac{2x^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$ **Hence proved.**

Example 16. If $z = f(x - by) + \phi(x + by)$, prove that

$$b^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}.$$

Sol. Given $$z = f(x - by) + \phi(x + by) \quad \text{...}(i)$$

$\therefore$ $$\frac{\partial z}{\partial x} = f'(x - by) + \phi'(x + by)$$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x - by) + \phi''(x + by). \quad \text{...}(ii)$$

Again from (*i*), $$\frac{\partial z}{\partial y} = -bf'(x - by) + b\phi'(x + by)$$

and $$\frac{\partial^2 z}{\partial y^2} = b^2 f''(x - by) + b^2\phi''(x + by) = b^2 \frac{\partial^2 z}{\partial x^2},$$ from (*ii*). **Hence proved.**

Example 17. If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$. Prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

Sol.

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)}$$

$= 2.$ **Hence proved.**

CHECK YOUR PROGRESS

1. Find $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ if $u = e^{x^2+y^2+z^2}$. **[Ans.** $8xyzu$**]**

2. Find the first order derivatives of

(i) $u = x^{xy}$. $\left[\textbf{Ans. } \dfrac{\partial u}{\partial x} = x^{xy}\,(y \log x + y);\ \dfrac{\partial u}{\partial y} = x^{xy+1} \log x\right]$

(ii) $u = \log\left(x + \sqrt{x^2 - y^2}\right)$ $\left[\textbf{Ans. } \dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{x^2 - y^2}}\ \dfrac{\partial u}{\partial y} = -y\left(x^2 - y^2\right)^{\frac{-1}{2}}\left(x + \sqrt{x^2 - y^2}\right)^{-1}\right]$

3. If $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

4. If $u = e^x\,(x \cos y - y \sin y)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

5. If $u = a \log(x^2 + y^2) + b \tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

6. If $z = \tan(y - ax) + (y + ax)^{3/2}$, prove that $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2} = 0$.

7. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

8. If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$. **[Ans.** 0]

9. If $u = \log(x^2 + y^2 + z^2)$, find the value of $\dfrac{\partial^2 u}{\partial y \partial z}$. $\left[\textbf{Ans. } \dfrac{-4yz}{\left(x^2 + y^2 + z^2\right)^2}\right]$

10. If $u = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$, then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{u}$.

11. If $z = f(x + ay) + \phi(x - ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$.

12. If $u = \cos^{-1}\left[(x - y)/(x + y)\right]$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

13. If $u = \log(x^2 + y^2)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

14. If $u = x^2y + y^2z + z^2x$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x + y + z)^2.$

15. Prove that $f(x, t) = a \sin bx . \cos bt$ satisfies $\dfrac{\partial^2 f}{\partial x^2} = b^2 \dfrac{\partial^2 f}{\partial t^2}$.

16. If $u = r^m$, where $r = \sqrt{x^2 + y^2 + z^2}$ find $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$. [**Ans.** $m(m + 1)r^{m-2}$]

17. If $u = (x^2 + y^2 + z^2)^{-1}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 2(x^2 + y^2 + z^2)^{-2}.$

18. If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n, when $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$. $\left[\textbf{Ans. } n = -\dfrac{3}{2}\right]$

19. For $n = 2$ or -3 show that $u = r^n(3\cos^2\theta - 1)$ satisfies the differential equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0.$$

20. If $u = e^{a\theta} \cos(a \log r)$, show that $\left(\frac{\partial^2 u}{\partial r^2}\right) + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

21. If $e^{-\frac{z}{(x^2-y^2)}} = (x-y)$, show that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2 - y^2$.

[**Hint:** Solve for $z = (y^2 - x^2) \log(x - y)$].

22. If $u = \frac{1}{r}$ and $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

TOTAL DIFFERENTIAL COEFFICIENT

Let $$z = f(x, y) \qquad ...(i)$$

where $x = \phi(t)$ and $y = \psi(t)$, then z can be expressed as a function of t alone by substituting the values of x and y in terms of t from the last two equations in equation (i).

And we can find the ordinary differential coefficient $\frac{dz}{dt}$, which is called total differential coefficient of z with respect to t. Since it is very difficult sometimes to express z in terms of t alone by eliminating x and y. So we are now to find $\frac{dz}{dt}$ without actually substituting the values of x and y in terms of t in $z = f(x, y)$.

Let δx, δy and δz be the increaments in x, y and z corresponding to a small increament δt in the value of t.

then $\qquad z + \delta z = f(x + \delta x, y + \delta y) \qquad \ldots(ii)$

where $\qquad x + \delta x = \phi(t + \delta t),\ y + \delta y = \psi(t + \delta t)$

Now,
$$\frac{dz}{dt} = \lim_{\delta t \to 0} \frac{\delta z}{\delta t} = \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x,y)}{\delta t} \quad \text{(from } ii\text{)}$$

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y) + f(x+\delta x, y) - f(x,y)}{\delta t}$$

{Adding and subtracting $f(x + \delta x, y)$}

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta t} + \lim_{\delta t \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta t}$$

$$= \lim_{\delta t \to 0} \left[\frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}\right] + \lim_{\delta t \to 0} \left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x} \cdot \frac{\delta x}{\delta t}\right]$$

Also, as $\delta t \to 0, \delta x \to 0, \delta y \to 0$

$$\therefore \quad \frac{dz}{dt} = \lim_{\delta t \to 0}\left[\frac{\partial f(x,y)}{\partial y}\frac{\delta y}{\delta t}\right] + \lim_{\delta t \to 0}\left[\frac{\partial f(x,y)}{\partial x}\frac{\delta x}{\delta t}\right]$$

$$= \frac{\partial f(x,y)}{\partial y}\frac{dy}{dt} + \frac{\partial f(x,y)}{\partial x}\cdot\frac{dx}{dt}$$

$$\therefore \quad \boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}} \qquad \text{...(iii) (As } z = f(x, y))$$

In general $$\frac{dz}{dt} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial z}{\partial x_2}\frac{dx_2}{dt} + \dots + \frac{\partial z}{\partial x_n}\frac{dx_n}{dt}$$

The above relation can be also written as

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$, which is called total differential of z.

Corollary: If $z = f(x, y)$ and suppose y is the function of x, then f is a function of one independent variable x. Here y is intermediate variable. Identifying t with x in (*iii*), we get

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \Rightarrow \boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}}.$$

Change of Variables

Let $z = f(x, y)$ where $x = \phi(s, t)$ and $y = \psi(s, t)$ then z is considered as function of s and t.

Now the derivative of z with respect s is partial but not total. Keeping t constant the equation (iii) modified as

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \qquad \ldots(A)$$

In a similar way, we get

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \cdot \qquad \ldots(B)$$

The equations (A) and (B) are known as chain rule for partial differentiation.

Example

Find the total differential coefficient of $x^2 y$ w.r.t. x when x, y are connected by

$$x^2 + xy + y^2 = 1.$$

Sol. Let $$z = x^2 y \qquad \text{...}(i)$$

Then the total differential coefficient of z

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial x} \cdot \frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad \text{...}(ii)$$

From (i) $\dfrac{\partial z}{\partial x} = 2xy$, $\dfrac{\partial z}{\partial y} = x^2$ and we have

$$x^2 + xy + y^2 = 1 \qquad \text{...}(iii)$$

Differentiating w.r.t. x, we get

$$2x + \frac{dy}{dx}\, x + y + 2y\,\frac{dy}{dx} = 0$$

$$(2x + y) + (x + 2y)\,\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\,\frac{2x+y}{x+2y}$$

Putting these values in equation (ii), we get

$$\frac{dz}{dx} = 2xy + x^2\left(-\frac{2x+y}{x+2y}\right) = 2xy - \frac{x^2(2x+y)}{(x+2y)}.$$

Example

If $u = f(y - z, z - x, x - y)$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Sol. Let $r = y - z, s = z - x, t = x - y$

Then $u = f(r, s, t)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\times 0 + \frac{\partial f}{\partial s}(-1) + \frac{\partial f}{\partial t}(1) \quad \left| \text{As } \frac{\partial r}{\partial x} = 0, \frac{\partial s}{\partial x} = -1, \frac{\partial t}{\partial x} = 1 \right.$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \qquad \ldots(i)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-1) \quad \left| \text{As } \frac{\partial r}{\partial y} = 1, \frac{\partial s}{\partial y} = 0, \frac{\partial t}{\partial y} = -1 \right.$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \qquad \ldots(ii)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial f}{\partial r}(-1) + \frac{\partial f}{\partial s}(1) + \frac{\partial f}{\partial t}(0) \quad \Bigg| \text{ As } \frac{\partial r}{\partial z} = -1, \frac{\partial s}{\partial z} = 1, \frac{\partial t}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial z} = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \qquad \ldots(iii)$$

Adding equations (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0. \quad \text{Hence proved.}$$

Example

If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$, find $\dfrac{du}{dx}$.

Sol. By total differentiation, we know that

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad ...(i)$$

we have $u = x \log xy$

$$\frac{\partial u}{\partial x} = \log xy + \frac{1}{y}\cdot y = \log xy + 1 \quad ...(ii)$$

$$\frac{\partial u}{\partial y} = \frac{x}{xy}(x) = \frac{x}{y} \quad ...(iii)$$

Also, given that

$$x^3 + y^3 + 3xy = 1$$

Differentiating w.r.t. 'x', we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x^2 + y) + (x + y^2)\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{\left(x^2+y\right)}{\left(x+y^2\right)} \quad ...(iv)$$

Using (*ii*), (*iii*) and (*iv*) in (*i*), we get

$$\frac{du}{dx} = (1 + \log xy) - \frac{x}{y}\left(\frac{x^2+y}{x+y^2}\right)$$

Example

If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0.$

Sol. We have

$$x^2 + y^2 + z^2 - 2xyz = 1$$

$$x^2 - 2xyz + y^2 z^2 = 1 - y^2 - z^2 + y^2 z^2$$

$$(x - yz)^2 = (1 - y^2)(1 - z^2)$$

$$(x - yz) = \sqrt{(1-y^2)}\ \sqrt{(1-z^2)} \qquad \text{...}(i)$$

Again $\quad y^2 - 2xyz + z^2 x^2 = 1 - x^2 - z^2 + z^2 x^2$

$$(y - zx)^2 = (1 - x^2)(1 - z^2)$$

$$(y - zx) = \sqrt{(1-x^2)}\ \sqrt{(1-z^2)} \qquad \text{...}(ii)$$

Similarly $\quad (z - xy) = \sqrt{(1-x^2)} \cdot \sqrt{(1-y^2)} \qquad \text{...}(iii)$

Let $\quad u \equiv x^2 + y^2 + z^2 - 2xyz - 1 = 0$

By total differentiation, we get

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$(2x - 2yz)dx + (2y - 2zx)dy + (2z - 2xy)dz = 0$$

As $\frac{\partial u}{\partial x} = 2x - 2yz$

$\frac{\partial u}{\partial y} = 2y - 2zx$

$\frac{\partial u}{\partial z} = 2z - 2xy$

$$(x - yz)dx + (y - zx)dy + (z - xy)dz = 0 \qquad ...(iv)$$

Putting (*i*), (*ii*) and (*iii*) in equation (*iv*), we get

$$\sqrt{(1-y^2)}\ \sqrt{(1-z^2)}\ .dx + \sqrt{(1-x^2)}\ \sqrt{(1-z^2)}\ dy + \sqrt{(1-x^2)}\ \sqrt{(1-y^2)}\ dz = 0$$

Dividing by $\sqrt{(1-x^2)}\ \sqrt{(1-y^2)}\ \sqrt{(1-z^2)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$ Hence proved.

CHECK YOUR PROGRESS

1. Find $\dfrac{dy}{dx}$ if $x^y + y^x = c$. $\left[\textbf{Ans } \dfrac{dy}{dx} = -\left[\dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]\right]$

2. If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$, find $\dfrac{du}{dx}$. $\left[\textbf{Ans } \dfrac{du}{dx} = (1+\log xy) + \dfrac{x}{y}\left[-\dfrac{(x^2+y)}{(y^2+x)}\right]\right]$

3. If $u = x^2 y$, where $x^2 + xy + y^2 = 1$, find $\dfrac{du}{dx}$. $\left[\textbf{Ans } \dfrac{du}{dx} = 2xy - x^2\left[\dfrac{(2x+y)}{(x+2y)}\right]\right]$

4. If V is a function of u, v where $u = x - y$ and $v = x - y$, prove that

$$x\,\frac{\partial^2 V}{\partial x^2} + y\,\frac{\partial^2 V}{\partial y^2} = (x+y)\left(\frac{\partial^2 V}{\partial u^2} + xy\frac{\partial^2 V}{\partial v^2}\right).$$

5. Transform the Laplacian equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ by change of variables from x, y to r, θ

when $x = e^r \cos\theta$, $y = e^r \sin\theta$. $\left[\textbf{Ans } e^{-2r}\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\partial^2 u}{\partial \theta^2}\right) = 0\right]$

6. Find $\dfrac{du}{dx}$, if $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$. $\left[\textbf{Ans } \dfrac{du}{dx} = 1 + \log xy - \dfrac{x}{y}\cdot\dfrac{x^2+y}{x+y^2}\right]$

JACOBIAN

If $u = u(x, y)$ and $v = v(x, y)$ where x and y are independent, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is known as the Jacobian of u, v with respect to x, y and is denoted by

$$\frac{\partial (u, v)}{\partial (x, y)} \text{ or } J(u, v)$$

Similarly, the Jacobian of three functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ is defined as

$$J(u, v, w) = \frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobians

1. If $u = u\ (x, y)$ and $v = v\ (x, y)$, then

$$\frac{\partial\,(u, v)}{\partial\,(x, y)} \times \frac{\partial\,(x, y)}{\partial\,(u, v)} = 1 \text{ or } JJ' = 1$$

2. Chain rule: If u, v, are function of r, s and r, s are themselves functions of x, y *i.e.*, $u = u\,(r, s)$, $v = v\,(r, s)$ and $r = r\,(x, y)$, $s = s\,(x, y)$

then
$$\frac{\partial\,(u,v)}{\partial\,(x,y)} = \frac{\partial\,(u,v)}{\partial\,(r,s)} \cdot \frac{\partial\,(r,s)}{\partial\,(x,y)}$$

Example

Find $\dfrac{\partial(u,v)}{\partial(x,y)}$, when $u = 3x + 5y$, $v = 4x - 3y$.

Sol. We have

$$u = 3x + 5y$$
$$v = 4x - 3y$$

$$\therefore \quad \frac{\partial u}{\partial x} = 3,\ \frac{\partial u}{\partial y} = 5,\ \frac{\partial v}{\partial x} = 4 \text{ and } \frac{\partial v}{\partial y} = -3$$

Thus,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix} = -9 - 20 = -29.$$

Example Calculate the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ of the following:

$$u = x + 2y + z,\ v = x + 2y + 3z,\ w = 2x + 3y + 5z.$$

Sol. We have

$$u = x + 2y + z$$
$$v = x + 2y + 3z$$
$$w = 2x + 3y + 5z$$

$$\therefore \quad \frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 2,\ \frac{\partial u}{\partial z} = 1,\ \frac{\partial v}{\partial x} = 1,\ \frac{\partial v}{\partial y} = 2,\ \frac{\partial v}{\partial z} = 3,$$

$$\frac{\partial w}{\partial x} = 2,\ \frac{\partial w}{\partial y} = 3 \text{ and } \frac{\partial w}{\partial z} = 5.$$

Now,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 1\ (10 - 9) - 2\ (5 - 6) + 1\ (3 - 4) = 2.$$

Example

Calculate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ if $u = \dfrac{2yz}{x}$, $v = \dfrac{3zx}{y}$, $w = \dfrac{4xy}{z}$.

Sol. Given $\quad u = \dfrac{2yz}{x}, v = \dfrac{3zx}{y}, w = \dfrac{4xy}{z}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{-2yz}{x^2}, \frac{\partial u}{\partial y} = \frac{2z}{x}, \frac{\partial u}{\partial z} = \frac{2y}{x}, \frac{\partial v}{\partial x} = \frac{3z}{y}, \frac{\partial v}{\partial y} = -\frac{3zx}{y^2}, \frac{\partial v}{\partial z} = \frac{3x}{y},$$

$$\frac{\partial w}{\partial x} = \frac{4y}{z}, \frac{\partial w}{\partial y} = \frac{4x}{z} \text{ and } \frac{\partial w}{\partial z} = -\frac{4xy}{z^2}$$

$$\text{Now,} \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & \frac{-3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & \frac{-4xy}{z^2} \end{vmatrix}$$

$$= -\frac{2yz}{x^2}\left[\frac{12x^2yz}{y^2z^2}-\frac{12x^2}{yz}\right] - \frac{2z}{x}\left[\frac{-12xyz}{yz^2}-\frac{12xy}{yz}\right] + \frac{2y}{x}\left[\frac{12xz}{yz}+\frac{12xyz}{zy^2}\right]$$

$$\Rightarrow \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 + 48 + 48 = 96.$$

But, we have $\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$ (Property 1)

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{96}.$$

Example

If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r \sin\theta \cos\phi$,

$v = r\sin\theta\sin\phi$, $w = r\cos\theta$, calculate $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

Sol. Here x, y, z are functions of u, v, w and u, v, w are functions of r, θ, ϕ so we apply IInd property.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} \qquad \ldots(i)$$

Consider
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\left[\sqrt{\frac{w}{v}\frac{v}{u}\frac{u}{w}} + \sqrt{\frac{v}{w}\frac{w}{u}\frac{u}{v}}\right] = \frac{1}{8}\left[\sqrt{1}+\sqrt{1}\right] = \frac{2}{8} = \frac{1}{4}$$

$$\text{Next} \quad \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta \cos\phi\, (r^2 \sin^2\theta \cos\phi) - r\cos\theta\cos\phi\,(-r^2 \sin\theta\cos\theta\cos\phi) + r^2 \sin\theta\sin\phi\,(\sin^2\theta\sin\phi + \cos^2\theta\sin\phi)$$

$$= r^2 \sin\theta\cos^2\phi\,(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$

$$\Rightarrow \quad \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = r^2 \sin\theta\cos^2\phi + r^2\sin\theta\sin^2\phi = r^2\sin\theta \qquad ...(iii)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{1}{4} \times r^2 \sin\theta = \frac{r^2\sin\theta}{4}.$$

Functional Dependence

Let $u = f_1(x, y)$, $v = f_2(x, y)$ be two functions. Suppose u and v are connected by the relation $f(u, v) = 0$, where f is differentiable. Then u and v are called functionally dependent on one another (*i.e.*, one function say u is a function of the second function v) if the $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are not all zero simultaneously.

Necessary and sufficient condition for functional dependence (Jacobian for functional dependence functions):

Let u and v are functionally dependent then

$$f(u, v) = 0 \qquad \text{...}(i)$$

Differentiate partially equation (i) w.r.t. x and y, we get

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 0 \qquad \text{...}(ii)$$

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \qquad \text{...}(iii)$$

There must be a non-trivial solution for $\frac{\partial f}{\partial u} \neq 0, \frac{\partial f}{\partial v} \neq 0$ to this system exists.

Thus, $$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$ | For non-trivial solution $|A| = 0$

or $$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$ | Changing all rows in columns

or $$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

Hence, two functions u and v are "functionally dependent" if their Jacobian is equal to zero.

Note: The functions u and v are said to be "functionally independent" if their Jacobian is not equal to zero *i.e.*, $J(u, v) \neq 0$

Similarly for three functionally dependent functions say u, v and w.

$$J(u, v, w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

Example

Show that the functions $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Also find the relation between them.

Sol. Here $u = x + y - z$, $v = x - y + z$ and $w = x^2 + y^2 + z^2 - 2yz$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2y-2z & 0 \end{vmatrix} \quad (C_3 \to C_3 + C_2)$$

$= 0$. Hence u, v, w are not independent.

$$
\begin{aligned}
\text{Again} \qquad u + v &= x + y - z + x - y + z = 2x \\
u - v &= x + y - z - x + y - z = 2\,(y - z) \\
\therefore \quad (u + v)^2 + (u - v)^2 &= 4x^2 + 4\,(y - z)^2 \\
&= 4(x^2 + y^2 + z^2 - 2yz) = 4w \\
\Rightarrow \quad (u + v)^2 + (u - v)^2 &= 4w \\
\text{or} \qquad 2(u^2 + v^2) &= 4w \text{ or } u^2 + v^2 = 2w.
\end{aligned}
$$

Example Find Jacobian of $u = \sin^{-1} x + \sin^{-1} y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$. Also find relation between u and v.

Sol. We have $\quad u = \sin^{-1} x + \sin^{-1} y,\ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

Now,
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{\sqrt{1-x^2}} & \dfrac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \dfrac{xy}{\sqrt{1-x^2}} & -\dfrac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= -\frac{xy}{\sqrt{1-x^2}\,\sqrt{1-y^2}} + 1 - 1 + \frac{xy}{\sqrt{1-x^2}\,\sqrt{1-y^2}} = 0.$$ Hence u and v are dependent.

Next, $\quad u = \sin^{-1} x + \sin^{-1} y \Rightarrow u = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

$$\left| \text{As } \sin^{-1} A + \sin^{-1} B = \sin^{-1}\left\{A\sqrt{1-B^2} + B\sqrt{1-A^2}\right\}\right.$$

$\Rightarrow \quad \sin u = x\sqrt{1-y^2} + y\sqrt{1-x^2} = v$

$v = \sin u.$