Constrained maxima and minima

Lagrangean Multiplier Method:

This method is to determine the maximum or minimum value of a function of three or more variables, the variables are not independent.

Suppose we are given a function $f(x, y, z) = 0$ Subject to $\varphi(x, y, z) = 0$.
To find the maxima and minima of $f(x, y, z) = 0$ use $\varphi(x, y, z) = 0$.

Consider $g(x, y, z) = f(x, y, z) + \lambda \varphi(x, y, z)$

Where λ is a parameter which is independent of x, y, z and is Called the Lagrangean multiplier.

Form the equations $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = \frac{\partial g}{\partial \lambda} = 0$

Solving all these equations we obtain the values of x, y, z.

Example 15: Determine the maximum of the function u given by $u = (x + 1)(y + 1)(z + 1)$ subject to the condition $a^x b^y c^z = k$

Solution: We have $u = (x + 1)(y + 1)(z + 1)$. Then

$$\log u = \log(x + 1) + \log(y + 1) + \log(z + 1) \qquad \text{....(1)}$$

and $a^x b^y c^z = k$ implies that

$$x \log a + y \log b + z \log c = \log k. \qquad \text{....(2)}$$

Now when u is maximum logu will also be maximum.

Let $F(x,y,z) = \log(x+1) + \log(y+1) + \log(z+1) + \lambda(x\log a + y\log b + z\log c - \log k]$

$$\frac{\partial F}{\partial x} = \frac{1}{x+1} + \lambda \log a; \quad \frac{\partial F}{\partial y} = \frac{1}{y+1} + \lambda \log b; \quad \frac{\partial F}{\partial z} = \frac{1}{z+1} + \lambda \log c$$

Stationary point of F are obtained from $\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0;$

i.e $\quad 1+(x+1)\lambda \log a = 0 \Rightarrow 1+\lambda x\log a + \lambda \log a = 0 \quad (3)$

$1+(y+1)\lambda \log b = 0 \Rightarrow 1+\lambda y\log b + \lambda \log b = 0 \quad (4)$

$1+(z+1)\lambda \log c = 0 \Rightarrow 1+\lambda z\log c + \lambda \log c = 0 \quad (5)$

adding these equations we get

$$3+\lambda(x\log a + y\log b + z\log c) + \lambda(\log a + \log b + \log c) = 0$$

(or) $\quad 3 + \lambda \log k + \lambda \log(abc) = 0$ from (2)

$$\Rightarrow \quad \lambda = -\frac{3}{\log(kabc)}$$

Substituting this value of λ in (3) (4) and (5) we get

$$(x+1) = -\frac{1}{\lambda \log a} = -\frac{1}{\frac{-3\log a}{\log(kabc)}} = \frac{\log(kabc)}{3\log a}$$

Similarly $(y+1) = \frac{\log(kabc)}{3\log b}; \quad (z+1) = \frac{\log(kabc)}{3\log c}$

$$\therefore \left(\frac{\log\left(k\left(\frac{bc}{a^2}\right)\right)}{3\log a}, \frac{\log\left(k\left(\frac{ca}{b^2}\right)\right)}{3\log b}, \frac{\log\left(k\left(\frac{ab}{c^2}\right)\right)}{3\log c} \right)$$ is a stationary point at which

u has maximum.

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Example 16: The temperature T at any point x,y,z in space is $T = 400\,xyz^2$. Find the highest temperature on the surface of unit sphere $x^2+y^2+z^2=1$

Solution: We have to maximize $T = 400\,xyz^2$ ....(1)

subject to $x^2+y^2+z^2-1=0$ ....(2)

Let $F(x,y,z) = 400\,xyz^2 + \lambda(x^2+y^2+z^2-1)$ ....(3)

where λ is Lagrangian Multiplier. Then

$$\frac{\partial F}{\partial x} = 400yz^2 + \lambda(2x) = 400\,yz^2 + 2\lambda x$$

$$\frac{\partial F}{\partial y} = 400xz^2 + \lambda(2y) = 400xz^2 + 2\lambda y$$

$$\frac{\partial F}{\partial z} = 800(xyz) + 2z\lambda$$

Then stationary points of F are obtained from $\frac{\partial F}{\partial x} = 0;\ \frac{\partial F}{\partial y} = 0;\ \frac{\partial F}{\partial z} = 0$

$$\frac{\partial F}{\partial x} = 400yz^2 + \lambda(2x) = 0 \Rightarrow \lambda = \frac{-400yz^2}{2x} = \frac{-200yz^2}{x} \qquad ....(4)$$

$$\frac{\partial F}{\partial y} = 400xz^2 + \lambda(2y) = 0 \Rightarrow \lambda = \frac{-400yz^2}{2y} = \frac{-200xz^2}{y} \qquad ....(5)$$

$$\frac{\partial F}{\partial z} = 800(xyz) + 2z\lambda \Rightarrow \lambda = \frac{-800xyz}{2z} = -400xy \qquad ....(6)$$

From (4) and (5) we have $\frac{-200yz^2}{x} = \frac{-200xz^2}{y}$

$\Rightarrow x^2 = y^2 \Rightarrow x = y$ ($\because$ we have to find the maximum temperature)

From (4) and (6) $\dfrac{-200yz^2}{x} = -400xy$

$$z^2 = 2x^2 \Rightarrow x = \frac{z}{\sqrt{2}} \Rightarrow z = \sqrt{2}x$$

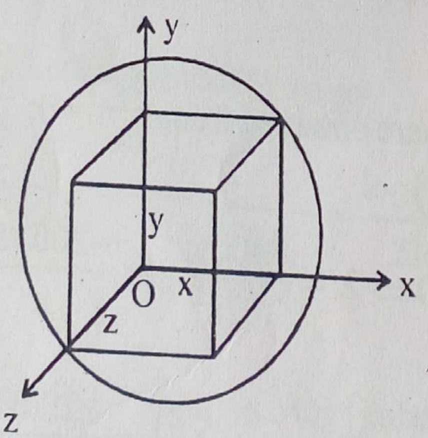
Substitute these in (2) we get $x^2 + x^2 + 2x^2 = 1 \Rightarrow 4x^2 = 1 \Rightarrow x = \dfrac{1}{4}$

$$x = \frac{1}{2}, y = \frac{1}{2}, z = \frac{1}{\sqrt{2}}$$

$\therefore \left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right)$ is a stationary point.

$\therefore$ The maximum temperature is $T = 400\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = 50.$

Example 17: Prove that rectangular solid of maximum volume which can be inscribed in a sphere is a cube.



Solution: : Let 2x,2y,2z be the edges of the rectangular solid with faces parallel to the coordinate planes inscribed in a sphere

Then its volume $V = 2x.2y.2z = 8xyz$

....(1)

We have to maximize V subject to the condition that

(Equation of sphere with center at (0,0) and radius a.)

Let $F = 8xyz + \lambda(x^2+y^2+z^2-a^2)$ then

$$\frac{\partial F}{\partial x} = 8yz + \lambda(2x), \quad \frac{\partial F}{\partial y} = 8xz+\lambda(2y), \quad \frac{\partial F}{\partial z} = 8xy+\lambda(2z)$$

The stationary points of F are obtained from $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$

i.e $$\frac{\partial F}{\partial x} = 8yz+\lambda(2x) = 0 \Rightarrow \lambda = -\frac{8yz}{2x} \quad ....(2)$$

$$\frac{\partial F}{\partial y} = 8xz+\lambda(2y) = 0 \Rightarrow \lambda = -\frac{8xz}{2y} \quad ....(3)$$

$$\frac{\partial F}{\partial z} = 8xy+\lambda(2z) = 0 \Rightarrow \lambda = -\frac{8xy}{2z} \quad ....(4)$$

From (1) and (2) $-\frac{8yz}{2x} = -\frac{8xz}{2y} \Rightarrow x^2 = y^2 \Rightarrow x = y$

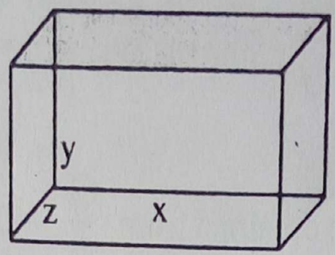
From (2) and (3) $-\frac{8xz}{2y} = -\frac{8xy}{2z} \Rightarrow y^2 = z^2 \Rightarrow y = z$

Substituting x=y=z in $F = x^2+y^2+z^2 = a^2$, we get $x^2+x^2+x^2 = a^2$. So $3x^2 = a^2$.

$\Rightarrow x = \frac{a}{\sqrt{3}}$, so $y = \frac{a}{\sqrt{3}}$ and $z = \frac{a}{\sqrt{3}}$. Since all the dimensions are equal in solid so formed is a cube.

Example 18: Find, using Lagrange's Multipliers the dimensions of a rectangular box open at the top of the maximum capacity with surface area 432 Square meters.

Solution: Let x,y,z be the dimensions of the box.



Let Volume $V = xyz$ ....(1)

Given surface area $S = xy+2yz+2zx = 432$ ....(2)

We have to maximize V such that $xy+2yz+2zx-432 = 0$.

Let $F = xyz + \lambda(xy+2yz+2zx-432) = 0$

Now $\frac{\partial F}{\partial x} = yz+\lambda(y+2z)$; $\frac{\partial F}{\partial y} = xz+\lambda(x+2z)$; $\frac{\partial F}{\partial z} = xy+\lambda(2y+2x)$

The stationary values of F can be obtained from $\frac{\partial F}{\partial x} = 0;\ \frac{\partial F}{\partial y} = 0;\ \frac{\partial F}{\partial z} = 0;$

$$\frac{\partial F}{\partial x} = yz+\lambda(y+2z) = 0 \quad \Rightarrow \quad \lambda = \frac{-yz}{y+2z} \qquad ....(3)$$

$$\frac{\partial F}{\partial y} = xz+\lambda(x+2z) = 0 \quad \Rightarrow \quad \lambda = \frac{-xz}{x+2z} \qquad ....(4)$$

$$\frac{\partial F}{\partial z} = xy+\lambda(2x+2y) = 0 \quad \Rightarrow \quad \lambda = \frac{-xy}{2x+2y} \qquad ....(5)$$

From (3) and (4) we get $\frac{-xz}{x+2z} = \frac{-yz}{y+2z}$

$$\Rightarrow \quad -xyz - 2yz^2 = -xyz - 2xz^2 \Rightarrow y = x$$

From (4) and (5) we get $\frac{-xz}{x+2z} = \frac{-xy}{2x+2y}$

$$\Rightarrow -2x^2z - 2xyz = -x^2y - 2xyz \quad \Rightarrow 2z = y$$

substituting $x = y = 2z$ in $xy + 2yz + 2zx = 432$, we get

$$x^2 + (2x)(x/2) + 2x(x/2) = 432$$

$$\Rightarrow 3x^2 = 432$$

$$\Rightarrow x^2 = 144 \Rightarrow x = 12$$

$$\therefore x = y = 12, z = x/2 = 6.$$

Therefore the required dimension of the box are respectively 12, 12, 6m.

Example 19: In a plane triangle ABC find the maximum value of $u = \cos A \cos B \cos C$

Solution: Given $u = \cos A \cos B \cos C$ ....(1)

where $A+B+C = \pi$ ....(2)

u is maximum when log u is maximum, so

$$\log u = \log \cos A + \log \cos B + \log \cos C$$

Let $F = (\log \cos A + \log \cos B + \log \cos C) + \lambda(A+B+C-\pi)$

$$\frac{\partial F}{\partial A} = 1/\cos A - \sin A + \lambda \,;\ \frac{\partial F}{\partial B} = -\frac{\sin B}{\cos B} + \lambda,\ \frac{\partial F}{\partial C} = \frac{-\sin C}{\cos C} + \lambda$$

The stationary points of F are obtained from $\frac{\partial F}{\partial A} = 0;\ \frac{\partial F}{\partial B} = 0;\ \frac{\partial F}{\partial C} = 0$

$$\frac{\partial F}{\partial A} = -\tan A + \lambda = 0 \Rightarrow \lambda = \tan A \quad ....(3)$$

$$\frac{\partial F}{\partial B} = -\tan B + \lambda = 0 \Rightarrow \lambda = \tan B \quad ....(4)$$

$$\frac{\partial F}{\partial C} = -\tan C + \lambda = 0 \Rightarrow \lambda = \tan C. \quad ....(5)$$

Solving these we get A=B=C, substituting in (1) we get, A+A+A=π

$$\Rightarrow \qquad A=\frac{\pi}{3}$$

$$B=A=\frac{\pi}{3}, C=A=\frac{\pi}{3}$$

$\therefore$ The stationary point is $\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)$

The maximum value of $u=\cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3}$

Example 20: Show that if the perimeter of a triangle is constant, its area is a maximum when it is equilateral.

Solution: Let a,b,c denote the sides of the triangle, then its perimeter is

$$2s=a+b+c=\text{constant}=k\ (\text{say})$$

Area of the triangle is u where $u^2=s(s-a)(s-b)(s-c)$

$$2\log u=\log s+\log(s-a)+\log(s-b)+\log(s-c)$$

and $\quad a+b+c=k$ ....(1)

Let $F=\log s+\log(s-a)+\log(s-b)+\log(s-c)+\lambda(a+b+c-k)$. Then ....(2)

$$\frac{\partial F}{\partial a}=-\frac{1}{s-a}+\lambda;\ \frac{\partial F}{\partial b}=-\frac{1}{s-b}+\lambda,\ \frac{\partial F}{\partial c}=-\frac{1}{s-c}+\lambda$$

The stationary points are obtained from $\frac{\partial F}{\partial a}=0;\ \frac{\partial F}{\partial b}=0;\ \frac{\partial F}{\partial c}=0;$

i.e $\quad \frac{\partial F}{\partial a}=-\frac{1}{s-a}+\lambda=0 \Rightarrow \lambda=\frac{1}{s-a}$ ....(3)

$\frac{\partial F}{\partial b}=-\frac{1}{s-b}+\lambda=0 \Rightarrow \lambda=\frac{1}{s-b}$ ....(4)

$\frac{\partial F}{\partial c}=-\frac{1}{s-c}+\lambda=0 \Rightarrow \lambda=\frac{1}{s-c}$ ....(5)

From (3) and (4), $\frac{1}{s-a}=\frac{1}{s-b} \Rightarrow a=b$

From (4) and (5), $\frac{1}{s-b}=\frac{1}{s-c} \Rightarrow b=c$

Therefore a=b=c

i.e u is maximum when a=b=c . Then the area of the triangle is maximum when it is equilateral.

Example 21: Find the points on the surface $z^2=xy+1$, nearest to the origin.

Solution: Let P(x,y,z) be any point on $z^2= xy+1$ then

$$OP^2 = (x-0)^2+(y-0)^2 +(z-0)^2= x^2+y^2+z^2 \quad ....(1)$$

We have to find P such that OP is minimum and OP is minimum when OP^2

$$z^2 = xy+1 \quad ....(2)$$

Let $F = x^2+y^2+z^2+\lambda(z^2-xy-1)$. Then

$$\frac{\partial F}{\partial x} = 2x+\lambda(-y), \quad \frac{\partial F}{\partial y} = 2y+\lambda(-x), \quad \frac{\partial F}{\partial z} = 2z+\lambda(2z).$$

The stationary points are obtained from $\frac{\partial F}{\partial x}=0; \frac{\partial F}{\partial y}=0; \frac{\partial F}{\partial z}=0$; So

$$\frac{\partial F}{\partial x}=2x+\lambda(-y) = 0 \Rightarrow \lambda y = x \quad \text{or } \lambda = x/y \quad ....(3)$$

$$\frac{\partial F}{\partial y} = 2y+\lambda(-x) = 0 \Rightarrow \lambda x=y \quad \text{or } \lambda = y/x \quad ....(4)$$

$$\frac{\partial F}{\partial z} = 2z+\lambda(2z) = 0 \Rightarrow \lambda = -1 \quad ....(5)$$

Substitute $\lambda = -1$ in (3) $\Rightarrow -1 = x/y \quad x = -y$

and from (3) and (4) we have $x/y = y/x \quad x=y$.

But $x=-y$, $x=y$ possible only when $x = 0 = y$.

Substitute, $x=y=0$ in (2) we get $z^2 = 0+1 = z^2 =1 \quad z=\pm 1$

The points $(0,0,1)$, $(0,0,-1)$ are nearest to origin on the given surface.

Example 22: Find the maximum and minimum distances of the point (3,4,12) from the sphere $x^2+y^2+z^2 = 1$

Solution: Let P(x,y,z) be any point on the sphere and A(3,4,12) the given point so that

$$AP^2 = (x-3)^2+(y-4)^2+(z-12)^2 = f(x,y,z) \text{ (say)}$$

We have to find the maximum and minimum values of f(x,y,z) subject to the condition

$$x^2+y^2+z^2 = 1 \qquad \text{....(2)}$$

Let $F(x,y,z) = (x-3)^2 +(y-4)^2 +(z-12)^2 +\lambda(x^2+y^2+z^2 - 1)$

Then $\dfrac{\partial F}{\partial x} = 2(x-3)+\lambda 2x = 0 \Rightarrow \lambda = -\dfrac{x-3}{x}$ ....(3)

$\dfrac{\partial F}{\partial y} = 2(y-4) +\lambda 2y = 0 \Rightarrow \lambda = -\dfrac{y-4}{y}$ ....(4)

$\dfrac{\partial F}{\partial z} = 2(z-12)+\lambda 2z = 0 \Rightarrow \lambda = -\dfrac{z-12}{z}$ ....(5)

we have $\lambda = -\dfrac{x-3}{x} = -\dfrac{y-4}{y} = -\dfrac{z-12}{z}$

$$= \pm\frac{\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}}{\sqrt{x^2+y^2+z^2}} = \pm\sqrt{f}$$

$\therefore \lambda = \pm\sqrt{f}$

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substituting for λ in (3) (4),(5) we get

$$x=\frac{3}{1+\lambda}=\frac{3}{1\pm\sqrt{f}}\;;\;y=\frac{4}{1\pm\sqrt{f}}\;;\;z=\frac{12}{1\pm\sqrt{f}}$$

$$\therefore x^2+y^2+z^2=\frac{9+16+144}{\left(1\pm\sqrt{f}\right)^2}=\frac{169}{\left(1\pm\sqrt{f}\right)^2}$$

Using (2), $\quad 1=\dfrac{169}{\left(1\pm\sqrt{f}\right)^2} \quad 1\pm\sqrt{f}=\pm 13 \;\Rightarrow\; \sqrt{f}=12,\,14$

Therefore, maximum AP is 14, minimum AP is 12.

Example 23: Find the volume of greatest rectangular parallelopiped that can be insecribed in the Ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

Solution: Let the edges of the parallelopiped to 2x, 2y, 2z which are parallel to the coordinate axes (as explained in Example 17).

Now we have to find the maximum value of V subject to the condition.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad ....(1)$$

$$V=2x\,(2y)\,(2z)=8xyz \qquad ....(2)$$

Let $\quad F=8xyz+\lambda\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)$

The stationary points of F are obtained by $\dfrac{\partial F}{\partial x}=0;\ \dfrac{\partial F}{\partial y}=0;\ \dfrac{\partial F}{\partial z}=0;$

$$\frac{\partial F}{\partial x}=8yz+\lambda\left(\frac{2x}{a^2}\right)=0 \;\Rightarrow\; \lambda=-\frac{8yza^2}{2x} \qquad ....(3)$$

$$\frac{\partial F}{\partial y} = 8xz + \lambda\left(\frac{2x}{b^2}\right) = 0 \quad \Rightarrow \quad \lambda = -\frac{8xzb^2}{2y} \qquad \text{....(4)}$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2x}{c^2}\right) = 0 \quad \Rightarrow \quad \lambda = -\frac{8xyc^2}{2z} \qquad \text{....(5)}$$

Equating the value of λ from (3) (4) (5) we get

$$-\frac{8yza^2}{2x} = -\frac{8xzb^2}{2y}$$

$$\Rightarrow \quad b^2x^2 = y^2 a^2 \Rightarrow \quad \frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$-\frac{8xzb^2}{2y} = -\frac{8xyc^2}{2z}$$

$$\Rightarrow \quad c^2 y^2 = b^2 z^2 \Rightarrow \quad \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

substituting these in (1) we get $\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \Rightarrow \frac{x^2}{a^2} = \frac{1}{3}$

i.e. $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3} \quad \Rightarrow x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$ ....(6)

Hence the greatest volume $= 8xyz = \frac{8abc}{3\sqrt{3}}$ (when $x = 0$, the paralleopiped is just a rectangular sheet and as such its volume V=0. As x increases V also increases continuously). Thus V must be greatest at the stage given by (6).