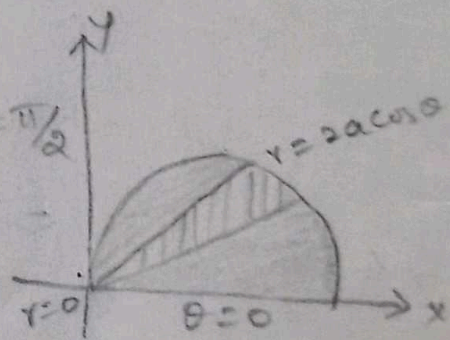
Change of variables :-

(1) By changing to polar co-ordinates, find the value of the integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$

(AU Dec 1999)

Sol: The Region of integration is bounded by $x=0$, $x=2a$, $y=0$ and

$y = \sqrt{2ax-x^2}$



$y^2 = 2ax - x^2$

$x^2+y^2 = 2ax$

$x^2+y^2-2ax = 0$

put $x = r\cos\theta$, $y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta - 2ar\cos\theta = 0$

$r^2 - 2ar\cos\theta = 0$

$r(r-2a\cos\theta) = 0$

$r=0$, $r = 2a\cos\theta$

Now it is evident that the region of integration is $\{(r,\theta) \mid 0 \le \theta \le \pi/2 , 0 \le r \le 2a\cos\theta\}$

$$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\cancel{16}^{\,4} a^4 \cos^4\theta}{\cancel{4}}\, d\theta$$

$$= 4a^4 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \cancel{4}a^4 \left(\frac{3}{\cancel{4}} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)$$

$$= \frac{3\pi a^4}{4}$$

By changing into polar coordinates show that $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \pi/4$.

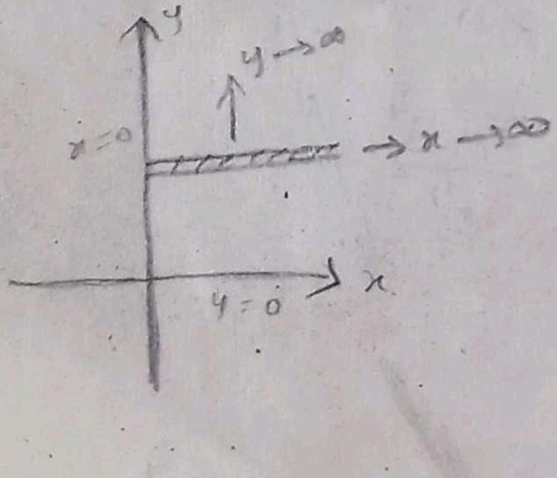
Hence evaluate $\int_0^\infty e^{-t^2}\, dt$.

Sol: Let us transform this integral in polar coordinates by taking

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$x=0,\ x=\infty,\ y=0,\ y=\infty$



put $r^2 = t$

$2r\,dr = dt$

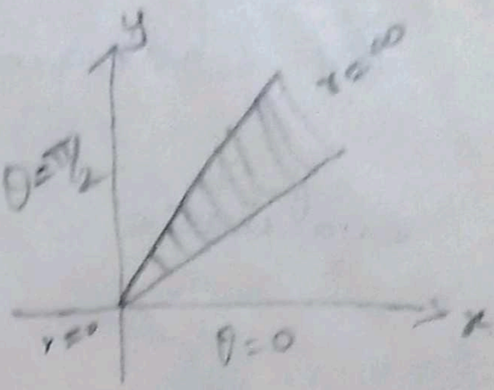
when $r=0$, $t=0$

$r=\infty$, $t=\infty$

$$= \int_0^{\pi/2} \int_0^\infty e^{-t}\, \frac{dt}{2}\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{-t}\, dt\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{e^{-t}}{-1} \right]_0^{\infty} d\theta$$



$$= \frac{1}{2} \int_0^{\pi/2} d\theta$$

$$= \frac{1}{2} \left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

Let $I = \int_0^{\infty} e^{-x^2} dx \longrightarrow (1)$

also $I = \int_0^{\infty} e^{-y^2} dy \longrightarrow (2)$

Multiply (1) & (2) we have

$$I^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

$$I^2 = \pi/4$$

$$I = \sqrt{\pi/4} = \frac{\sqrt{\pi}}{2}$$

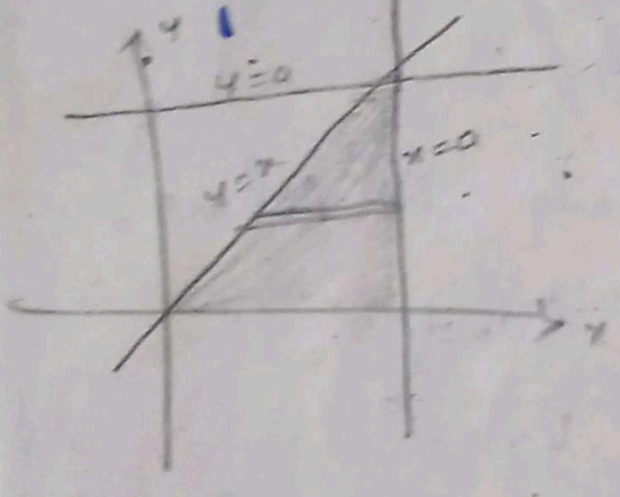
ie $\int_0^{\infty} e^{-t^2} dt = \sqrt{\pi}/2$

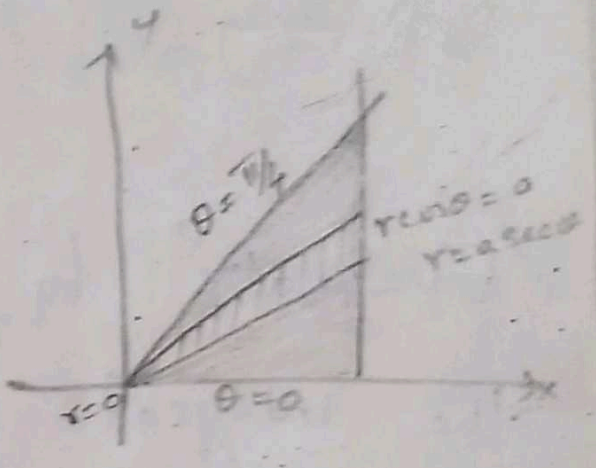
Evaluate by changing to polars, the integral $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$.

Sol: The region of integration is bounded by $y=0$, $y=a$, $x=y$, $x=a$.

Let us transform this integral in polar coordinates by taking

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$





$$\therefore \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{\cancel{r}\cos\theta}{r^{\cancel{2}}}\, \cancel{r}\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta \,(r)_0^{a\sec\theta}\, d\theta$$

$$= \int_0^{\pi/4} \cos\theta \; a\sec\theta \, d\theta$$

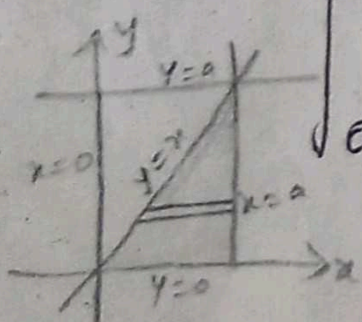
$$= \int_0^{\pi/4} \cancel{\cos\theta}\; a \frac{1}{\cancel{\cos\theta}}\, d\theta$$

$$= a\int_0^{\pi/4} d\theta = a(\pi/4)$$

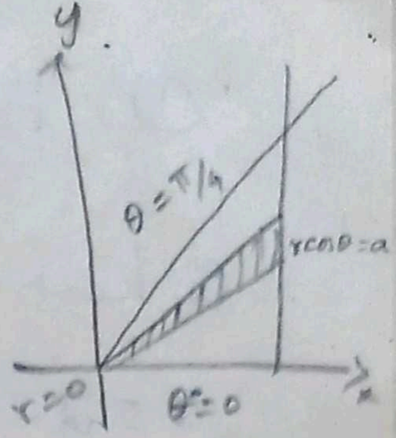
$$= \pi a/4 \,//$$

/ Evaluate by changing to polars, the integral $\int_0^a \int_y^a \frac{x^2\,dx\,dy}{\sqrt{x^2+y^2}}$.

Sol: $\int_0^a \int_y^a \frac{x^2\,dx\,dy}{\sqrt{x^2+y^2}} = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\cancel{r}}\, \cancel{r}\,dr\,d\theta$



$$= \int_0^{\pi/4} \cos^2\theta \left(\frac{r^3}{3} \right]_0^{a\sec\theta} d\theta$$



$$= \int_0^{\pi/4} \cos^2\theta \, \frac{a^3 \sec^3\theta}{3} \, d\theta$$

$$= a^3/3 \int_0^{\pi/4} \cancel{\cos^2\theta} \, \frac{1}{\cos^3\theta} \, d\theta$$

$$= a^3/3 \int_0^{\pi/4} \sec\theta \, d\theta$$

$$= a^3/3 \left[\log(\sec\theta + \tan\theta) \right]_0^{\pi/4}$$

$$= a^3/3 \left[\log(\sqrt{2}+1) - \log(1+0) \right]$$

$$= a^3/3 \left[\log(\sqrt{2}+1) - 0 \right]$$

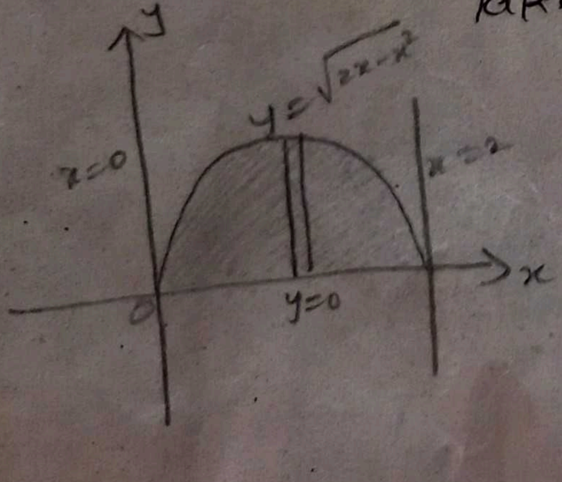
$$= a^3/3 \log(\sqrt{2}+1) \,//.$$

By changing into polar coordinates, evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$ (AU, NOV. 2001)

Sol: Given $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dx\,dy$

$$= \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx \text{ (correct form)}$$

The region of integration is bounded by $x=0$, $x=2$, $y=0$ and $y=\sqrt{2x-x^2}$



Take $y=\sqrt{2x-x^2}$

$$y^2 = 2x-x^2$$

$$x^2+y^2 = 2x$$

$$x^2+y^2-2x = 0$$

In this, the polar equation of the circle is $(r\cos\theta)^2 + (r\sin\theta)^2 - 2r\cos\theta = 0$

(or)

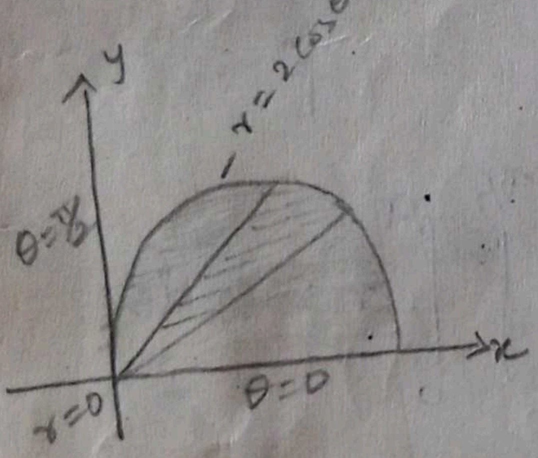
$$r = 2\cos\theta$$

$$\left[\because\ x = r\cos\theta,\quad y = r\sin\theta,\quad dx\,dy = r\,dr\,d\theta\right]$$

Now it is evident that the region of integration is

$$\left\{(r,\theta)\ \middle|\ 0 \le \theta \le \pi/2\ ;\ 0 \le r \le 2\cos\theta\right\}$$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{\cancel{r}\cos\theta}{\cancel{r^2}}\,\cancel{r}\,dr\,d\theta$$



$$= \int_0^{\pi/2}\int_0^{2\cos\theta} \cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta\,(r)_0^{2\cos\theta}\,d\theta$$

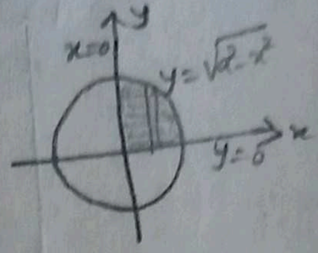
$$= \int_0^{\pi/2} 2\cos^2\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta\,d\theta = \cancel{2}\left(\frac{1}{\cancel{2}}\,\frac{\pi}{2}\right)$$

$$= \pi/2$$

Transform the integral into polar Co-ordinates and hence evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$



Sol: The region of integration is

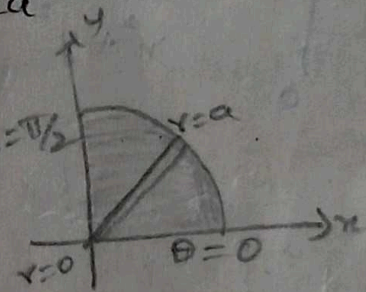
$y=0, \quad y=\sqrt{a^2-x^2}, \quad x=0$ and $x=a$.

ie $y=0, \quad x^2+y^2=a^2, \quad x=0$ and $x=a$.

$$x^2+y^2=a^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$$

$$\Rightarrow r^2=a^2$$

$$\Rightarrow r=a.$$



$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^a r\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^a d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} d\theta = \frac{a^3}{3}\left(\frac{\pi}{2}\right)$$

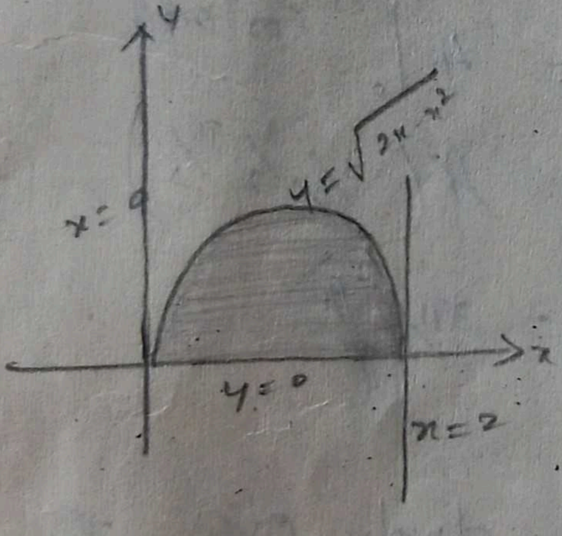
$$= \frac{\pi a^3}{6}$$

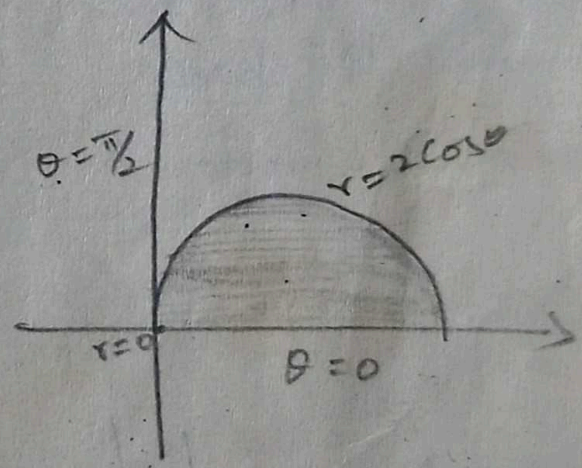
Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ by Changing into polar Co-ordinates.

Sol: The region of integration is

$y=0, \quad y=\sqrt{2x-x^2}, \quad x=0, \quad x=2.$

ie $y=0, \quad x^2+y^2=2x, \quad x=0, \quad x=2 \quad \rightarrow (1)$





put $x=r\cos\theta, \quad y=r\sin\theta$ in (1) we get

$y=0 \Rightarrow r\sin\theta=0 \Rightarrow \theta=0$

$[\because r\neq 0]$,

$x=0 \Rightarrow r\cos\theta=0 \Rightarrow \theta=\pi/2 \quad [\because r\neq 0]$

$y=\sqrt{2x-x^2} \Rightarrow x^2+y^2=2x$

$\Rightarrow r^2=2r\cos\theta$

$\Rightarrow r=2\cos\theta$

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(r^4/4\right)_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\overset{4}{\cancel{16}}\cos^4\theta}{\cancel{4}}\, d\theta$$

$$= 4\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \cancel{4}\left(\frac{3}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= 3\pi/4$$

Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$ by transforming into polar coordinates.

Sol:

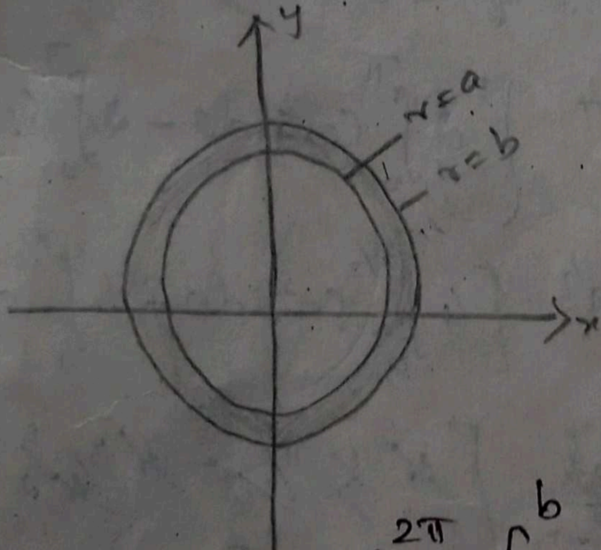
Putting $x = r\cos\theta$, $y = r\sin\theta$ the given two circles becomes

$$x^2+y^2=a^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$$

$$\Rightarrow r = a$$

$$x^2+y^2=b^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = b^2$$

$$\Rightarrow r = b$$



and θ varies from 0 to 2π.

$$\iint \frac{x^2y^2\,dx\,dy}{x^2+y^2} = \int_0^{2\pi}\int_a^b \frac{r^2\cos^2\theta\,\cancel{r^2}\sin^2\theta}{\cancel{r^2}}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b r^3\cos^2\theta\,\sin^2\theta\,d\theta$$

$$= \int_0^{2\pi}\cos^2\theta\,\sin^2\theta\,d\theta\left(\frac{r^4}{4}\right)_a^b$$

$$= \int_0^{2\pi} \cos^2\theta \sin^2\theta \left(\frac{b^4-a^4}{4}\right) d\theta$$

$$= \frac{b^4-a^4}{4} \int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta$$

$$= \frac{b^4-a^4}{4} \int_0^{2\pi} \cos^2\theta (1-\cos^2\theta) \, d\theta$$

$$= \frac{b^4-a^4}{4} \left[\int_0^{2\pi} \cos^2\theta \, d\theta - \int_0^{2\pi} \cos^4\theta \, d\theta \right]$$

$$= \frac{b^4-a^4}{4} \left[2\int_0^{\pi} \cos^2\theta \, d\theta - 2\int_0^{\pi} \cos^4\theta \, d\theta \right]$$

$$= \frac{b^4-a^4}{4} \left[4\int_0^{\pi/2} \cos^2\theta \, d\theta - 4\int_0^{\pi/2} \cos^4\theta \, d\theta \right]$$

$$= \frac{b^4-a^4}{4} \left[\cancel{4} \, \frac{1}{\cancel{2}} \, \frac{\pi}{\cancel{2}} - \cancel{4} \, \frac{3}{\cancel{4}} \, \frac{1}{2} \, \frac{\pi}{2} \right]$$

$$= \frac{b^4-a^4}{4} \left[\pi - \frac{3\pi}{4} \right]$$

$$= \frac{b^4-a^4}{4} \left[\frac{\pi}{4} \right]$$

$$= \frac{\pi (b^4-a^4)}{16}$$

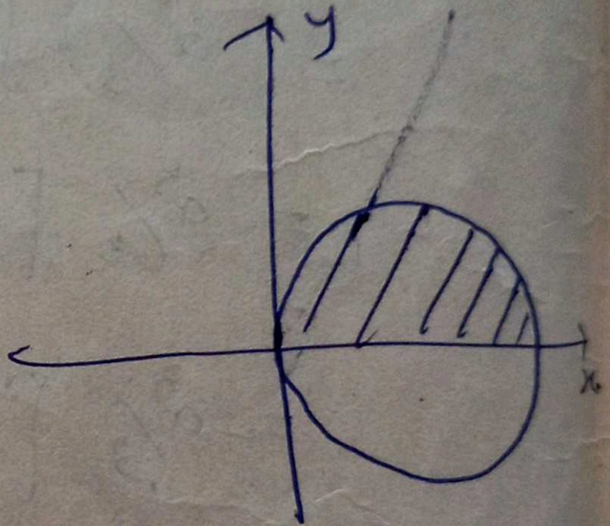
① Evaluate $\iint r^2 \sin\theta \, dr \, d\theta$ where R is the semi circle $r = 2a\cos\theta$ about the initial line.

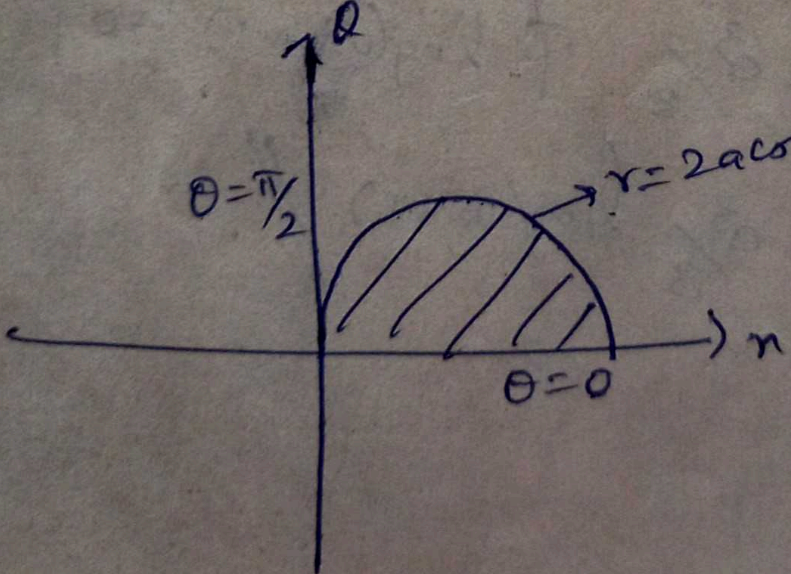
$$\iint r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta$$

centre ← $x^2 + y^2 = 2ax$
is (a,0)

$$r^2 = 2ar\cos\theta$$

$$r = 2a\cos\theta$$





$$= \int_0^{\pi/2} \sin\theta \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \sin\theta \, \frac{8a^3\cos^3\theta}{3} \, d\theta$$

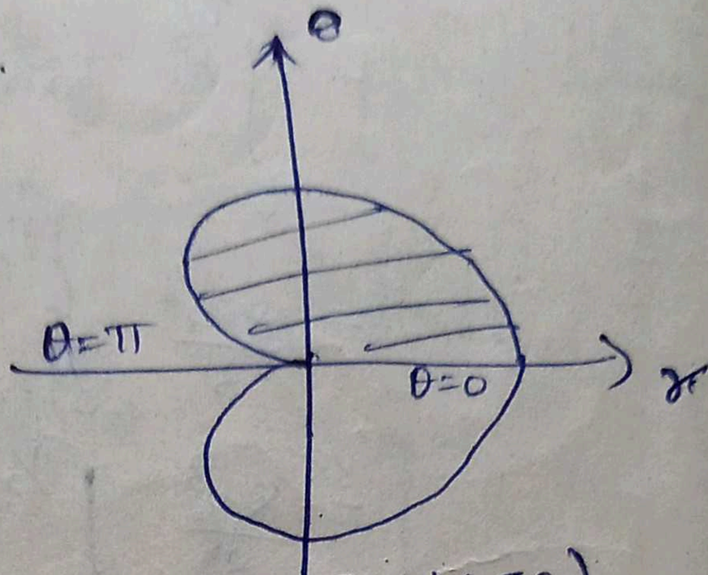
$$= \frac{8a^3}{3} \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta$$

$$= \frac{8a^3}{3} \int_0^{\pi/2} \cos^3\theta \, d(\cos\theta)$$

$$= \frac{8a^3}{3} \left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{8a^3}{3} \left[0 - 1/4\right] = -\frac{2a^3}{3}$$

Find, using a double integral, the area of the Cardioid $r = a(1+\cos\theta)$



Required area $= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$

$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \cancel{2}\int_0^{\pi}\frac{a^2(1+\cos\theta)^2}{\cancel{2}}\,d\theta$$

$$= a^2\int_0^{\pi}(1+\cos^2\theta+2\cos\theta)\,d\theta$$

$$= a^2 \int_0^{\pi} \left[1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + \tfrac{1}{2} + \frac{\cos 2\theta}{2} + 2\cos\theta\right) d\theta$$

$$= a^2 \int_0^{\pi} \left(\tfrac{3}{2} + \tfrac{1}{2}\cos 2\theta + 2\cos\theta\right) d\theta$$

$$= a^2 \left[\tfrac{3}{2}\theta + \tfrac{1}{2}\frac{\sin 2\theta}{2} + 2\sin\theta\right]_0^{\pi}$$

$$= a^2 \left[\left(\tfrac{3}{2}\pi + 0 + 0\right) - (0 + 0 + 0)\right]$$

$$= \frac{3\pi a^2}{2}$$

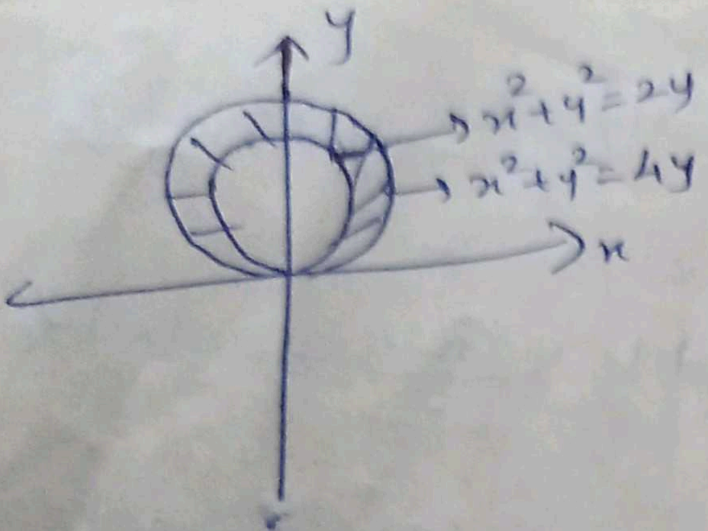
Calculate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Centre is (0,1)

$$x^2+y^2=2y$$

$$r^2=2r\sin\theta$$

$$r=2\sin\theta$$

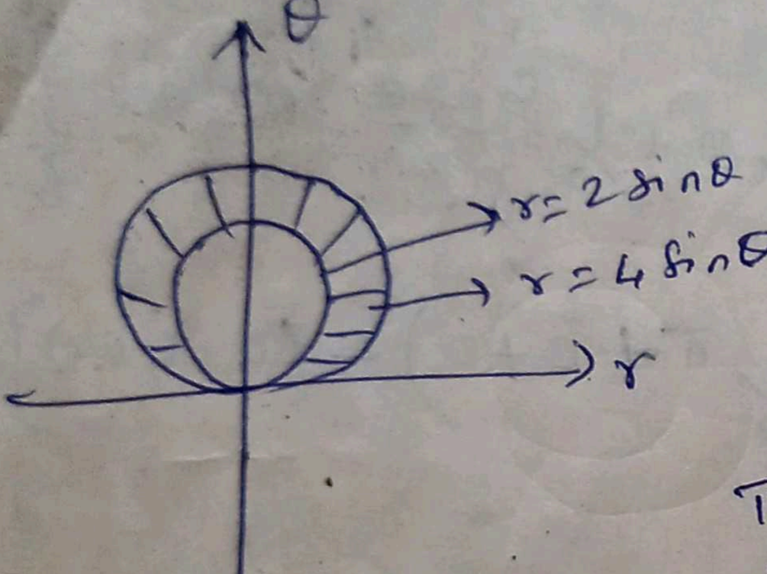


$$x^2+y^2=4y$$

Centre is (0,2)

$$r^2=4r\sin\theta$$

$$r=4\sin\theta$$



$$\iint r^3\,dr\,d\theta=\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$=\int_0^{\pi}\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$=\int_0^{\pi}\left(64\sin^4\theta-4\sin^4\theta\right)d\theta$$

$$= \int_0^{\pi} 60 \sin^4\theta \, d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 120 \, \frac{3}{4} \, \frac{1}{2} \, \frac{\pi}{2}$$

$$= \frac{45}{2}\pi.$$

$\left[\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \right.$ if $f(2a-x) = f(x)$

otherwise zero

$\int_0^{2a} f(x)\,dx = 0$ if $f(2a-x) = -f(x)$ $\Big]$

Find the area of the region outside the inner circle $r = 2\cos\theta$ and inside the outer circle $r = 4\cos\theta$ by double integration.

$x^2 + y^2 = 2x$

centre $(1,0)$

$r^2 = 2r\cos\theta$

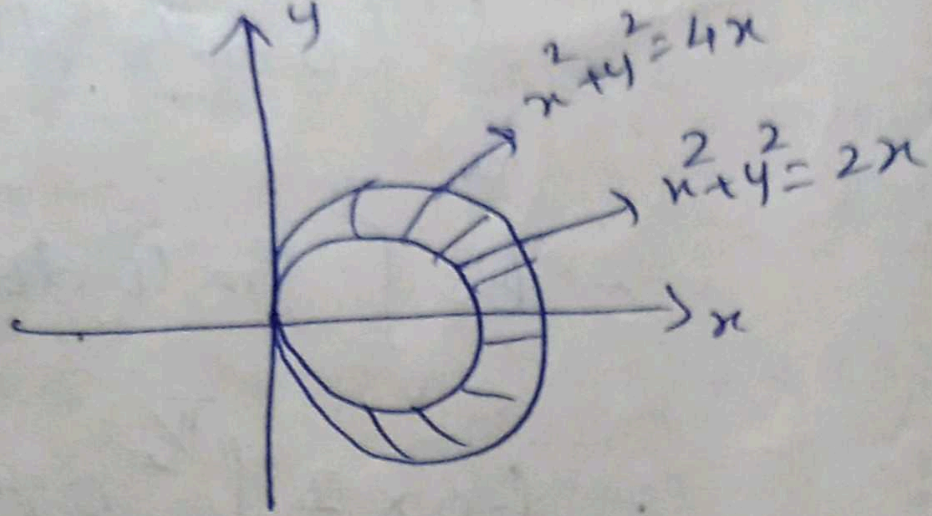
$r = 2\cos\theta$

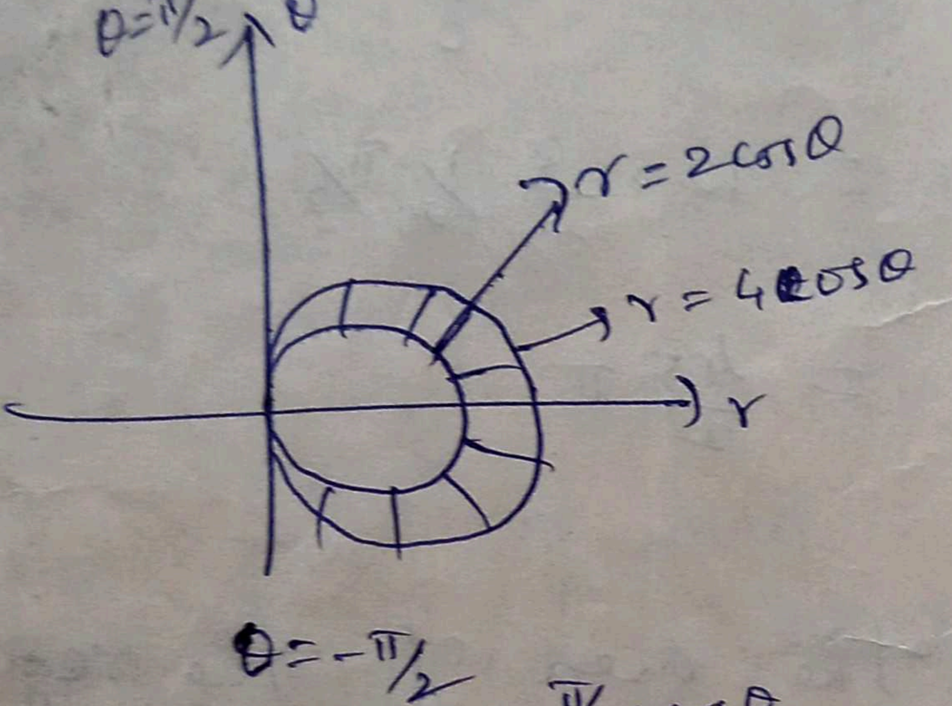
centre $(2,0)$

$x^2 + y^2 = 4x$

$r^2 = 4r\cos\theta$

$r = 4\cos\theta$





$$\iint dx\,dy = 2\int_0^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= \cancel{2}\int_0^{\pi/2}\left(\frac{r^2}{\cancel{2}}\right)_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left(16\cos^2\theta - 4\cos^2\theta\right) d\theta$$

$$= 12 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= 12 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 3\pi$$