Maxima and minima of functions of two Variables :-

Definition :- (Maximum)

A function $f(x, y)$ is Said to have a relative maximum (or maximum) at (a, b) if $f(a, b) > f(a+h, b+k)$ for

Small values of h and k.

Minimum :- A function $f(x, y)$ is said to have a relative minimum at (a, b) if $f(a, b) < f(a+h, b+k)$ for small values of h and k.

Note :- The maximum and minimum value of a function are called as extreme values.

Rule to find the maximum or minimum value of a function of two variables :-

Let $f(x, y)$ be the given function. To find the maximum and minimum values of $f(x, y)$ we have to follow the following rules.

Step-1:- Solve the equations

$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0 \longrightarrow (1)$$

Let (a,b) be a solution of (1).

Step-2:- Compute $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x \partial y}$

and $C = \frac{\partial^2 f}{\partial y^2}$ at (a,b).

Step-3:-

(a) If $AC - B^2$ is +ve and A is −ve (or B is −ve) then $f(x,y)$ has a maximum at (a,b).

(b) If $AC - B^2$ is +ve and A is +ve (or B is +ve) then $f(x,y)$ has a minimum at (a,b).

(c) If $AC - B^2$ is $-ve$ then $f(x, y)$ has neither a maximum nor a minimum at (a, b). In this case (a, b) is called a saddle point.

(d) No information is obtained if $AC - B^2 = 0$. In such a case further investigation is necessary.

Example 8: Discuss the maximum and minimum of the following function $f=x^3y^3-3x-3y$.

Solution: The given function if $f=x^3y^3-3x-3y$ ....(1)

Differentiating (1) with respect to x partially we get,

$$\frac{\partial f}{\partial x}=3x^2y^3-3=3(x^2y^3-1)$$

Differentiating (1) with respect to y partially

$$\frac{\partial f}{\partial y}=3x^3y^2-3=3(x^3y^2-1)$$

now $\frac{\partial f}{\partial x}=0 \Rightarrow (x^2y^3-1)=0$ i.e., $x^2y^3=1$ ....(2)

and $\frac{\partial f}{\partial y}=0 \Rightarrow (x^3y^2-1)=0$ i.e., $x^3y^2=1$ ....(3)

substituting $x^2=\frac{1}{y^2}$ in (3) we get $\frac{1}{y^{9/2}}y^2=1$ i.e $y=1$

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When y = 1, we have x = 1

$$\frac{\partial^2 f}{\partial x^2} = 6xy^3; \quad \frac{\partial^2 f}{\partial y^2} = 6x^3y \text{ and } \frac{\partial^2 f}{\partial x \partial y} = 9x^2y^2$$

Hence the stationary point is (1,1)

Now $rt-s^2 = 36x^4y^4 - 81x^4y^4 = -45x^4y^4$

At (1,1) $rt-s^2 = -45 < 0$

Hence there is neither maximum nor minimum at (1,1). Therefore (1,1) is saddle point.

Example 9: In a plane triangle ABC, find the maximum values of cos A cos B cos C

Solution: Let $f = \cos A \cos B \cos C$

$$= \cos A \cos B \cos(\pi - (A+B))$$

$$f = -\cos A \cos B \cos(A+B)$$

$$\frac{\partial f}{\partial A} = -\cos B[-\sin A \cos(A+B) + \cos A(-\sin(A+B))]$$

$$= \cos B \sin(A+A+B) = \cos B \sin(2A+B)$$

also $$\frac{\partial f}{\partial B} = -\cos A[-\sin B \cos(A+B) + \cos B(-\sin(A+B)]$$

$$= \cos A \sin(A+2B)$$

For f has minimum or maximum let $\frac{\partial f}{\partial A} = 0, \frac{\partial f}{\partial B} = 0$

$$\Rightarrow \cos B \sin(2A+B) = 0$$

$$\Rightarrow \cos A \sin(2A+B) = 0$$

i.e. $B = \frac{\pi}{2}, 2A + B = \pi$ and $\cos A \sin(A+2B) = 0$

i.e $A = \frac{\pi}{2}, A + B = \pi$ now solving these $2A + B = \pi$ and $A + 2B = \pi$,

we get $A = B = \frac{\pi}{3}$

$$r = \frac{\partial^2 f}{\partial A^2} = 2\cos B \cos(2A + B)$$

$$s = \frac{\partial^2 f}{\partial A \partial B} = \cos B \cos(2A+B) - \sin B \sin(2A + b) = \cos(2A + 2B)$$

$$t = \frac{\partial^2 f}{\partial B} = 2\cos A \cos(A + 2B)$$

at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $r = 2\cos\frac{\pi}{3}\cos\pi = -1$

at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $s = \cos\left(2\frac{\pi}{3} + 2\frac{\pi}{3}\right) = -1/2$

at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $r = 2\cos\frac{\pi}{3}\cos\pi = -1$

$$\therefore\ rt - s^2 = (-1)(-1) - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} > 0$$

$$\therefore\ r = -1 < 0$$

$\therefore$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, f has maximum.

If $A = \frac{\pi}{3}$ and $B = \frac{\pi}{3}$ then $C = \pi - (\pi/3) - (\pi/3) = \pi/3$

$\therefore\ f_{max} = \cos\pi/3 \cos\pi/3 \cos\pi/3 = 1/8.$

Example 10: Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Solution: Let $(x_r, y_r), r = 1, 2, 3$ be the vertices of the triangle and let (x,y) be any point inside the triangle. Then let

$$v = \sum_{r=1}^{3}\left[(x-x_r)^2+(y-y_r)^2\right]$$

$$\frac{\partial v}{\partial x} = \sum_{r=1}^{3} 2(x-x_r) = 2\left[(x-x_1)+(x-x_2)+(x-x_3)\right]$$

Similarly $\quad \dfrac{\partial v}{\partial y} = 2\,[(y-y_1)+(y-y_2)+(y-y_3)]$

For stationary point put $\dfrac{\partial v}{\partial x} = 0 \quad \dfrac{\partial v}{\partial y} = 0$

$$\frac{\partial v}{\partial x} = 0 \Rightarrow 3x-(x_1+x_2+x_3) = 0 \Rightarrow x = \frac{x_1+x_2+x_3}{3}$$

$$\frac{\partial v}{\partial y} = 0 \Rightarrow 3y-(y_1+y_2+y_3) = 0 \Rightarrow y = \frac{y_1+y_2+y_3}{3}$$

Thus $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$ is the stationary point

Now $\quad r = \dfrac{\partial^2 v}{\partial x^2} = 6;\ t = \dfrac{\partial^2 v}{\partial y^2} = 6$ and $s = \dfrac{\partial^2 v}{\partial x \partial y} = 0$

$\therefore rt - s^2 = 6 \times 6 = 36 > 0$ and $r > 0$

$\therefore \left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$ f has minimum.

This stationary point is called the Centroid of the triangle.

Example 11: Find the maximum and minimum values of x^3+y^3-3axy

Solution: Let $f(x,y)=x^3+y^3-3axy$. Then

$$\frac{\partial f}{\partial x}=3x^2-3ay,\ \frac{\partial f}{\partial y}=3y^2-3ax$$

For stationary points put $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0$

$\Rightarrow 3x^2-3ay=0 \Rightarrow x^2=ay$

$\Rightarrow 3y^2-3ax=0 \Rightarrow y^2=ax$

Solving these we get $x=0, y=0$ and $x=a, y=a$. Therefore (0,0) and (a,0) are the stationary points.

Now $r==6x, \quad t=s==-3a.$

At (0,0), $rt-s^2=(0)\,(0)-9a^2<0$

Hence at (0,0), f has neither maximum nor minimum.

At (a,a) $rt-s^2=36a^2-9a^2>0$ and $r=6a$

If $a<0\Rightarrow r<0$. Thus f has maximum, If $a>0\Rightarrow r>0$. Thus f has minimum.

Example 12: Find the maximum and minimum values of the functions $x^3y^2(1-x-y)$

Solution: The given function $f(x,y)=x^3y^2-x^4y^2-x^3y^3$.

Now $\frac{\partial f}{\partial x}=3x^2y^2-4x^3y^2-3x^2y^3$

$$\frac{\partial f}{\partial y}=2x^3y-2x^4y-3x^3y^2.$$

The stationary points are obtained from the equation $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. So

$$\frac{\partial f}{\partial x}=3x^2y^2-4x^3y^2-3x^2y^3=0$$

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 = 0$$ solving these two equations we get

$$x^2y^2(3-4x-3y) = 0 \text{ and } x^3y(2-2x-3y) = 0 \Rightarrow 3 = 4x+3y,\ 2 = 2x+3y$$

$$2x = 1 \Rightarrow x = \frac{1}{2} \text{ then } y = \frac{1}{3}$$

$\therefore \left(\frac{1}{2}, \frac{1}{3}\right)$ is stationay point.

$$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

$$r\left(\frac{1}{2}, \frac{1}{3}\right) = 6\left(\frac{1}{2}\right)\left(\frac{1}{9}\right) - 12\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) - 6\left(\frac{1}{2}\right)\left(\frac{1}{27}\right)$$

$$= \frac{2(18) - 36 - 12}{54(2)} = -\frac{12}{108} = -\frac{1}{9}$$

$$t\left(\frac{1}{2}, \frac{1}{3}\right) = 2\left(\frac{1}{8}\right) - 2\left(\frac{1}{16}\right) - 6\left(\frac{1}{8}\right)\left(\frac{1}{3}\right) = -\frac{1}{8}$$

$$s\left(\frac{1}{2}, \frac{1}{3}\right) = 6\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) - 8\left(\frac{1}{8}\right)\left(\frac{1}{3}\right) - 9\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) = -\frac{1}{12}$$

$$\therefore \text{At } \left(\frac{1}{2}, \frac{1}{3}\right),\ rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$$

and $r = -\frac{1}{9} < 0$

$\therefore$ at $\left(\frac{1}{2}, \frac{1}{3}\right)$ thus f has maximum.

Example 13: Locate the stationary points of $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature.

Sloution: The given function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. Then

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

The stationary points are obtained from equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$4x^3 - 4x + 4y = 0 \qquad (1)$$

$$4y^3 + 4x - 4y = 0 \qquad (2)$$

Adding (1) and (2) we get $4(x^3+y^3) = 0 \Rightarrow (x+y)(x^2-xy+y^2) = 0$

$y = -x$ ($\because$ other gives two imaginary results)

Putting $y = -x$ in (1) $\quad 4x^3 - 4x - 4x = 0$

$$\Rightarrow \quad 4x^3 - 8x = 0$$

$$\Rightarrow \quad 4(x^3 - 2x) = 0 \text{ (or) } x(x^2-2) = 0$$

$$x = 0, \sqrt{2}, -\sqrt{2}$$

since $y = -x$, we have $y = 0, \sqrt{2}, -\sqrt{2}$

$\therefore$ the stationary points are $(0,0)\ (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4; \quad t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4 \quad \text{and} \quad s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

At $(0,0)$, $r = -4, t = -4, s = 4$

$(rt - s^2) = 0$ and hence the further investigations are needed.

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At $\left(\sqrt{2}, -\sqrt{2}\right)$, $r = 12(2) - 4 = 20, t = 20, s = 4$

$$\therefore (rt - s^2) = (20)(20) - 16 = 384 > 0$$

and r at $\left(\sqrt{2}, -\sqrt{2}\right) = 20 > 0$

Hence f has minimum at $\left(\sqrt{2}, -\sqrt{2}\right)$

At $\left(-\sqrt{2}, \sqrt{2}\right)$

$$(rt - s^2) = (20)(20) - 16 > 0 \text{ and } r = 20 > 0.$$

Hence at $\left(-\sqrt{2}, \sqrt{2}\right)$, also f has minimum.

Example 14: Find the maximum and minimum values of $\sin x + \sin y + \sin(x+y)$

Solution: Let $f(x,y) = \sin x + \sin y + \sin(x+y)$. Then

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$$

The stationary points are obtained on solving the two equations $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0.$

i.e $\quad \cos x + \cos(x+y) = 0 \quad$ .... (1)

$\cos y + \cos(x+y) = 0 \quad$ .... (2)

$\cos x - \cos y = 0$

$\cos x = \cos y$ or $x = y$

Substituting in (1) we get $\cos x + \cos(2x) = 0$

$$\cos 2x = -\cos x = \cos(\pi - x)$$

$$2x = \pi - x \Rightarrow 3x = \pi \Rightarrow x = \frac{\pi}{3}$$

Since as $x = y$, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a stationary point. Now

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $r = \sqrt{3}$, $t = -\sqrt{3}$, $s = \frac{\sqrt{3}}{2}$.

$$r^2 - s^2 = \frac{9}{4} > 0; \; r < 0.$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$; f has maximum and maximum value of f is given by

$$f = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin 2\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 3\frac{\sqrt{3}}{2}.$$