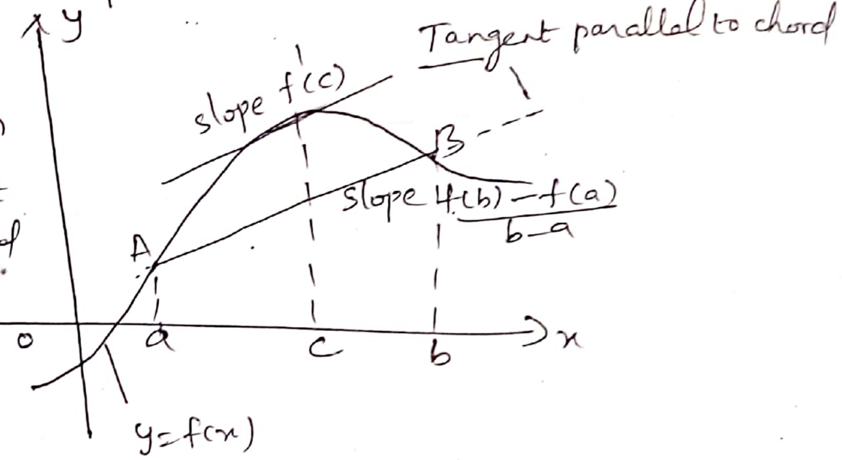
Mean value theorem: (Law of the mean due to Lagrange)

The mean value theorem, which was first stated by Joseph-Louis (1736–1813) Lagrange, is a slanted version of Rolle's theorem (Fig). The mean value theorem guarantees that there is a point where the tangent line is parallel to the chord AB.

Geometrically, the mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to chord AB.



Mean value theorem:

Let $f(x)$ be a real valued function that satisfies the following conditions.

(i) $f(x)$ is continuous on the closed interval $[a, b]$

(ii) $f(x)$ is differentiable on the open interval (a, b)

Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(1) verify Lagrange's law of the mean for $f(x) = x^3$ on $[-2, 2]$.

Ans: f is a polynomial, hence continuous on $[-2,2]$ and differentiable on $(-2, 2)$.

$$f(2) = 2^3 = 8 \;; \quad f(-2) = (-2)^3 = -8 .$$

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$$

By law of the mean there exist an element $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 = \frac{8 - (-8)}{4} = 4$$

$$c^2 = 4/3$$

$$c = \pm 2/\sqrt{3}$$

The required 'c' in the law of mean are $+2/\sqrt{3}$ and $-2/\sqrt{3}$ as both lie in $[-2, 2]$.

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Verify Lagrange's law of mean for the following functions.

(i) $f(x) = 2x^3 + x^2 - x - 1$, $[0, 2]$

Ans True, $c = \frac{-1 \pm \sqrt{61}}{6}$

(ii) $f(x) = x^3 - 5x^2 - 3x$, $[1, 3]$

True, $c = 7/3$

(iii) $f(x) = x^{2/3}$, $[-2, 2]$ Fails, Function is not differentiable at $x=0$

(iv) $f(x) = 1 - x^2$, $[0, 3]$, True $c = 3/2$

(v) $f(x) = \frac{1}{x}$, $[1, 2]$, True $c = \sqrt{2}$.

Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x, how large can $f(2)$ possibly be?

Since by hypothesis f is differentiable and f is continuous everywhere.

We can Apply Lagrange's Law of the mean on the interval $[0, 2]$. There exist atleast one 'c' $\in (0, 2)$ such that

$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

$$f(2) = f(0) + 2f'(c)$$

$$f(2) = -3 + 2f'(c)$$

Given that $f'(x) \leq 5$ for all x.

In particular we know that $f'(c) \leq 5$.

multiplying both sides of the inequality by 2, we have

$$2f'(c) \leq 10.$$

$$f(2) = -3 + 2f'(c)$$

$$f(2) \leq -3 + 10 = 7$$

ie the largest possible value of $f(2)$ is 7.

Increasing and Decreasing Functions:

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval.

As another Corollary to the mean value theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions.

A function that is increasing or decreasing on an interval is said to be monotonic on the interval.

<u>Corollary:</u> Suppose that f is continuous on [a, b] and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is strictly increasing on [a, b].

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is strictly decreasing on [a, b]

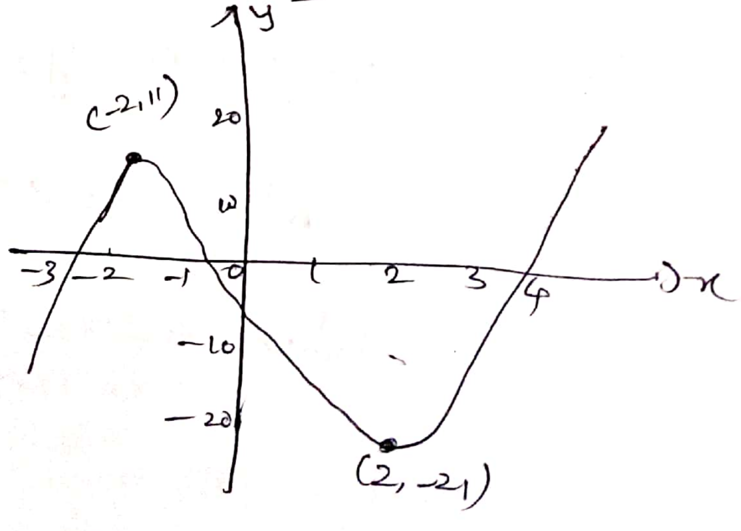
Find the critical points of $f(x) = x^3 - 12x - 15$ and identify the intervals on which f is increasing and on which f is decreasing.

$f'(x) = 3x^2 - 12$

$f'(x) = 0 \Rightarrow 3x^2 - 12 = 0$

$\Rightarrow 3(x^2 - 4) = 0$

$\Rightarrow x = \pm 2$



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The values $x = -2$ ad $x = 2$ divide the real line into intervals $(-\infty, -2), (-2, 2), (2, \infty)$.

$-\infty \quad -2 \quad 0 \quad 2 \quad \infty$

Interval	f' evaluated	Sign of f'	Behavior of f
$-\infty < x < -2$	$f'(-3) = 15$	$+$	increasing
$-2 < x < 2$	$f'(0) = -12$	$-$	decreasing
$2 < x < \infty$	$f'(3) = 15$	$+$	increasing.

Note-

For a function $y = f(x)$

when $x_1 < x_2$ then $f(x_1) \leq f(x_2)$ Increasing

When $x_1 < x_2$ then $f(x_1) < f(x_2)$ Strictly Increasing

For a function $y = f(x)$

when $x_1 < x_2$ then $f(x_1) \geq f(x_2)$ Decreasing

When $x_1 < x_2$ then $f(x_1) > f(x)$ strictly decreasing

Ex: $f(x) = x^3 - 4x$ in $[-1, 2]$.

$f'(x) = 3x^2 - 4$

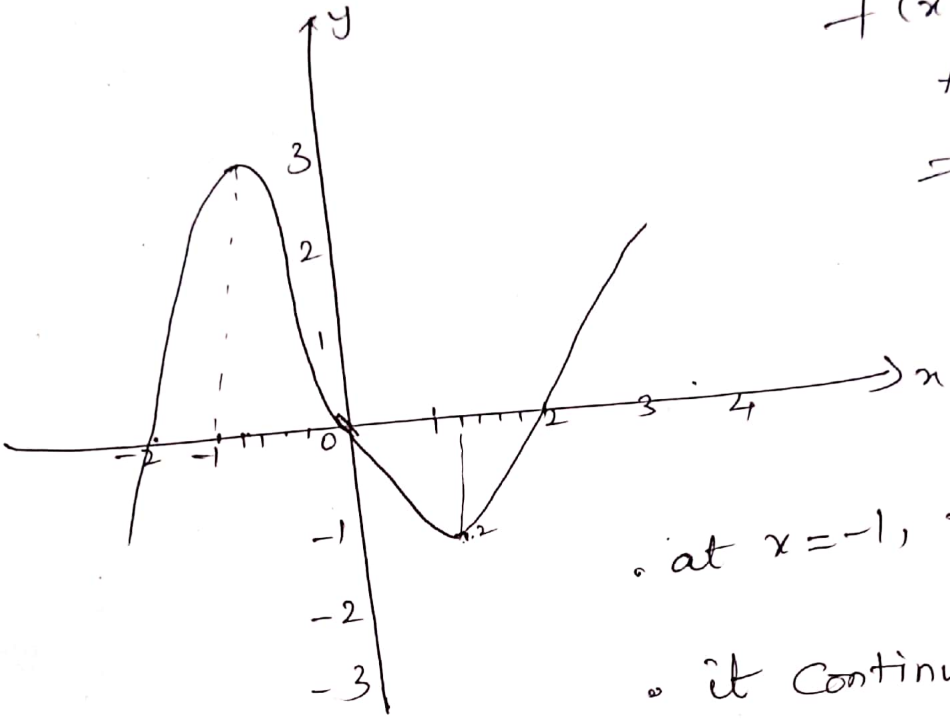
$f'(x) = 0$.

$\Rightarrow 3x^2 - 4 = 0$

$x^2 = 4/3$

$x = \pm 2/\sqrt{3}$

$x = 1.2 \in (-1, 2)$



- at $x = -1$, the function is decreasing
- it continuous to decrease until about 1.2.
- it then increases from there, past $x = 2$.

without exact analysis we cannot pinpoint where the curve turns from decreasing to increasing, so let us just say within the interval $[-1, 2]$:

- the curve decreases in $[-1, \text{approx } 1.2]$
- the " increases in $[\text{approx } 1.2, 2]$

Note: If the critical number are not included in the intervals, then the intervals of increasing (decreasing) becomes strictly increasing (strictly decreasing)

Note: If a function changes its signs at different points of a region (interval) then the function is not monotonic in that region.

So to prove the non-monotonicity of a function, it is enough to prove that f' has different signs at different points.

Theorem: 1)

A function $f(x)$ increases on an interval I if $f(b) \geq f(a)$ for all $b > a$ where $a, b \in I$.

If $f(b) > f(a)$ for all $b > a$ then the function is said to be strictly increasing.

Conversely, a function $f(x)$ decreases on an interval I if $f(b) \leq f(a)$ for all $b > a$ with $a, b \in I$.

If $f(b) < f(a)$ for all $b > a$, then the function is said to be strictly decreasing.

Find the interval in which $f(x) = 2x^3 + x^2 - 20x$ is increasing and decreasing.

Ans:
$$f'(x) = 6x^2 + 2x - 20$$

$$f'(x) = 0 \Rightarrow 6x^2 - 2x - 20$$

$$\Rightarrow x = -2, 5/3.$$

The values -2 and $5/3$ divide the real line into intervals $(-\infty, -2)$, $(-2, 5/3)$ and $(5/3, \infty)$

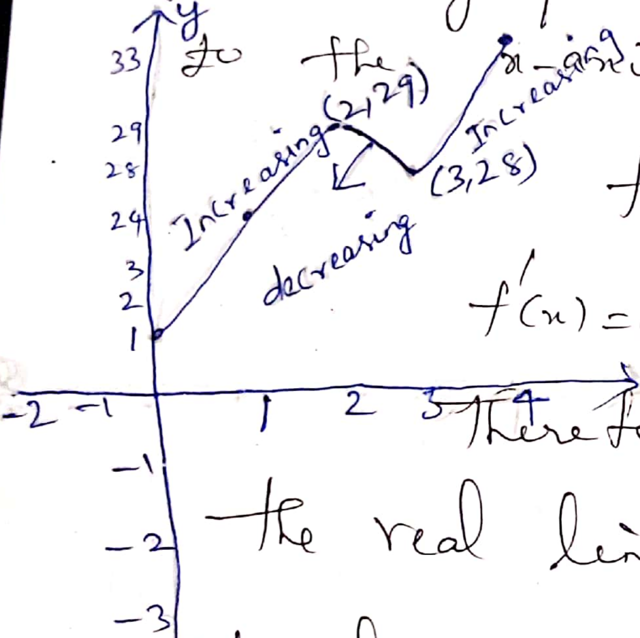
$-\infty \quad -2 \quad 0 \quad 5/3 \quad \infty$.

$-\infty < x < -2$, $f'(x) > 0$ increasing on $(-\infty, -2]$

$-2 < x < 5/3$, $f'(x) < 0$ decreasing on $[-2, 5/3]$

$5/3 < x < \infty$, $f'(x) > 0$ increasing on $[5/3, \infty)$

Determine for which value of x, the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing and for which it is decreasing. Also determine the points where the tangents to the graph of the function are parallel to the x-axis.



$f'(x) = 6x^2 - 30x + 36$ $f(0) = 1$

$f'(x) = 0 \implies x = 2, 3.$

$f(1) = 24$

$f(2) = 29$

$f(3) = 28$

$f(4) = 33$

Therefore the points 2 & 3 divide the real line into $(-\infty, 2)$, $(2, 3)$, $(3, \infty)$.

Interval	$f'(x)$	intervals of inc/dec
$-\infty < x < 2$	+	increasing on $(-\infty, 2]$
$2 < x < 3$	−	decreasing on $[2, 3]$
$3 < x < \infty$	+	increasing on $[3, \infty)$

The points where the tangent to the graph of the function are parallel to the x-axis are given by $f'(x) = 0$ ie when $x = 2, 3$

Therefore the required points are $(2, 29)$, $(3, 28)$. $f(2) = 29$

$f(3) = 28$