Total derivatives:

Composite functions:

If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then u is said to be a composite function of the variable t.

u can be expressed as a function of t alone by substituting the values of x and y in $f(x, y)$.

Thus we can find the ordinary derivative du/dt which is called the total derivative of u.

$\frac{du}{dt}$ can be evaluated using

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

(or)

$$df/dt = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Total Differentiation

If $u = f(x, y)$ is a function of x and y where $x = f(t)$ and $y = g(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \rightarrow (1)$$

In the differential form, (1) can be written as

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}$$

du is called the total differential of u.

Note:- If $u = f(x, y, z)$, where x, y, z are all functions of t. Then the total differential

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz .$$

Note :-

Taking $t = x$ in (1), we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\boxed{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}}$$

Composite Functions :-

(i) If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$.

(ii) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$

then z is called a composite function of (two variables) u and v so that we can find $\cancel{\frac{\partial z}{\partial u}}$ and $\frac{\partial z}{\partial v}$.

Differentiation of Composite functions

If u is composite function of t, defined by $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$ then

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

Implicit functions :-

Let f be a function of two variables. By solving $f(x, y) = 0$, we express y as a function of x.

$\therefore$ $f(x, y) = 0$ defines y as an implicit function of x.

Differentiation of Implicit Functions:-

If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x

Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$ becomes

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\therefore \boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

Note :-

Second derivative of implicit function given by

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$$

where

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial f}{\partial x} = f_x = f_x(x,y)$$

$$f_{yy} = \partial^2 f / \partial y^2 \qquad \frac{\partial f}{\partial y} = f_y = f_y(x,y)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

① If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$

Prove that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Sol:-

Here u is a composite function of t.

$\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\,3 + \frac{1}{\sqrt{1-(x-y)^2}}(-1)\,12t^2$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

Chain rule of Function of Function rule :-

This rule is very useful in partial differentiation.

Let u be a function of z and z be a function of two independent variable x and y. Then

$$\frac{\partial u}{\partial x} = \frac{du}{dz} \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{du}{dz} \cdot \frac{\partial z}{\partial y}$$

(Note that the straight d is used in $\frac{du}{dz}$ as u is a function of only one variable z while the curved ∂ is used in $\partial z/\partial x$, $\partial z/\partial y$ as z is a function of two independent variables).

② If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$.

Sol:-

Here $f(x,y) = x^3 + y^3 + 3xy - 1 = 0$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2+3y}{3y^2+3x}$$

Given $u = x \log xy$

By total derivative,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} \longrightarrow (1)$$

Now $\frac{\partial u}{\partial x} = \log(xy) + x \frac{1}{xy} y$

$= \log xy + 1$

and $\frac{\partial u}{\partial y} = x \frac{1}{xy} x = x/y$

$$\therefore (1) \Rightarrow \frac{du}{dx} = \log xy + 1 + \frac{x}{y}\left(-\frac{3x^2+3y}{3y^2+3x}\right)$$

Introduction

Jacobians

(*) If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x and y and is denoted by

$$J\left(\frac{u, v}{x, y}\right) \text{ or } \frac{\partial(u, v)}{\partial(x, y)}.$$

If u, v and w are functions of x, y, z then

$$J\left(\frac{u, v, w}{x, y, z}\right) \text{ or } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Two Important properties of Jacobians :-

① If u and v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$ (Jacobian of composite functions)

② If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$

then $J \cdot J' = 1$.

③ If the functions u, v, w of three independent variables x, y, z are not independent, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

① If $u = x^2 - 2y$, $v = x + y$ prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = 2x + 2$.

Proof:- Given $u = x^2 - 2y$, $v = x + y$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \partial v/\partial x = 1$$

$$\partial u/\partial y = -2 \quad , \quad \partial v/\partial y = 1$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \partial v/\partial y \end{vmatrix}$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 2x + 2.$$

② If $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$, evaluate $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

Sol:-

Given $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}(2x)$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}(2y)$$

$$= \frac{y}{\sqrt{x^2+y^2}}$$

$$\partial\theta/\partial x = \frac{1}{1+y^2/x^2}\left(-y/x^2\right) = \frac{-y}{x^2+y^2}$$

Similarly $\partial\theta/\partial y = \frac{x}{x^2+y^2}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\sqrt{x^2+y^2}}$$

(3) If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

evaluate $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Sol:- Given $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial z}{\partial r} = 0$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta, \quad \partial z/\partial\theta = 0$$

$$\partial z/\partial z = 1, \quad \frac{\partial x}{\partial z} = 0, \quad \frac{\partial y}{\partial z} = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta\,(r\cos\theta - 0) + r\sin\theta(\sin\theta - 0) + 0(0-0)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

(4) If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, Compute $\dfrac{\partial(F,G,H)}{\partial(u,v,w)}$.

Sol:-

$F = ux + v - y$, $G = u^2 + vy + w$,

$H = zu - v + vw$

$$\frac{\partial F}{\partial u} = x \qquad \frac{\partial g}{\partial u} = 2u \qquad \frac{\partial H}{\partial u} = z$$

$$\partial F/\partial v = 1 \qquad \frac{\partial G}{\partial v} = y \qquad \frac{\partial H}{\partial v} = -1+w$$

$$\partial F/\partial w = 0 \qquad \frac{\partial G}{\partial w} = 1 \qquad \frac{\partial H}{\partial w} = v.$$

$$\frac{\partial(F, G, H)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \partial f/\partial v & \partial f/\partial w \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} x & 1 & 0 \\ 2u & y & 1 \\ z & -1+w & v \end{vmatrix}$$

$$= x\left[yv - (-1+w)\right] - 1\left[2uv - z\right)$$

$$= xyv + x - xw - 2uv + z$$

(5) If $x = u(1+v)$, $y = v(1+u)$ find

$\frac{\partial(x, y)}{\partial(u, v)}$. A.U 2002 - Nov.

Sol:

Given that

$x = u(1+v) \quad y = v(1+u)$

$\frac{\partial x}{\partial u} = 1+v \quad \frac{\partial y}{\partial v} = 1+u$

$\frac{\partial x}{\partial v} = u \quad \frac{\partial y}{\partial u} = v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= u + v + \cancel{uv} + 1 - \cancel{uv}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = u+v+1$$

(6) Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3. if

$$y_1 = \frac{x_2 x_3}{x_1}, \quad y_2 = \frac{x_3 x_1}{x_2}, \quad y_3 = \frac{x_1 x_2}{x_3}.$$

(A.U. May 2003)

Sol:-

Given that $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$,

$$y_3 = \frac{x_1 x_2}{x_3}$$

$$\rightarrow \frac{\partial y_1}{\partial x_1} = \frac{-x_2 x_3}{x_1^2}, \quad \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}, \quad \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}, \quad \frac{\partial y_2}{\partial x_2} = -\frac{x_3 x_1}{x_2^2}, \quad \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}, \quad \frac{\partial y_3}{\partial x_2} = x_1/x_3, \quad \frac{\partial y_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2}$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_2 x_1 \\ x_3 x_2 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(\underbrace{1-1}_{0}) - 1(-1-1) + 1(1+1)$$

$$= 2+2 = 4.$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4.$$