Integral Calculus:

Introduction:

Calculus deals principally with two geometric problems.

(i) The problem of finding slope of the tangent line to the curve, is studied by the limiting process known as differentiation and

(ii) problem of finding the AREA of a Region under a curve is studied by another limiting process called integration.

Actually integral Calculus was developed into two different directions over a long period independently

(i) Leibnitz and his school of thought approached it as the anti derivative of a differentiable function.

(ii) Archimedes, Eudoxus and others developed it as a numerical value equal to the area under the curve of a function for some interval.

However as far back as the end of the 17th century it become clear that a general method for solution of finding the area under the given curve could be developed in connection with definite problems of integral calculus.

We are already familiar with inverse operations.

(+, -) ; (×, ÷), ($()^n$, $\sqrt{\ }$) are some pair of inverse operations.

Similarly differentiation and integrations are also inverse operations.

The inverse operation of differentiation called anti differentiation.

Simple definite integrals:

First fundamental theorem of Calculus:

Theorem: If $f(x)$ is a Continuous function and $F(x) = \int_a^x f(t)\,dt$ then we have the equation $F'(x) = f(x)$.

Second fundamental theorem of Calculas:

Theorem: If $f(x)$ is a Continuous function with domain $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where F is any anti-derivative of f.

Rules Satisfied by definite int

1. Order of integration $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. Zero width integral $\int_a^a f(x)\,dx = 0$

3. Constant Multiple $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

4. Sum and Difference $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. Additivity $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. Max-Min Inequality: If f has maximum value $\max f$ and minimum value $\min f$ on $[a,b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$

7. Domination $f(x) \ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$

Ex:

Use the Fundamental theorem to find $\frac{dy}{dx}$ if (a) $y = \int_a^x (t^3+1)\,dt$

(b) $y = \int_x^5 3t \sin t \, dt$

(c) $y = \int_1^{x^2} \cos t \, dt$.

Sol:

(a) $$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^x (t^3+1)\,dt\right) = x^3+1$$

(b) $$\frac{dy}{dx} = \frac{d}{dx}\left(\int_x^5 3t\sin t\,dt\right) = \frac{d}{dx}\left(-\int_5^x 3t\sin t\,dt\right)$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$$= -3x\sin x$$

(c)

$$y = \int_1^u \cos t\,dt \quad , \quad u = x^2$$

By chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= \left(\frac{d}{du}\int_1^u \cos t\,dt\right)\cdot\frac{du}{dx}$$

$$= \cos u \cdot \frac{du}{dx}$$

$$= \cos(x^2)\cdot 2x$$

$$= 2x\cos x^2.$$

Properties of Definite integral:

1. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$.

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

7. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

   $= 0$ if $f(2a-x) = -f(x)$

8. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even function

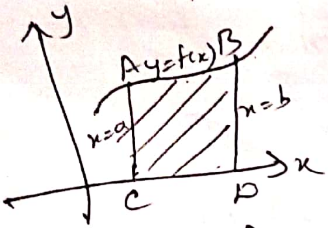
   $= 0$ if $f(x)$ is an odd function

Area of bounded regions

Theorem:

Let $y = f(x)$ be a continuous function defined on $[a, b]$, which is positive ($f(x)$ lies on or above x-axis) on $[a,b]$

Then, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by

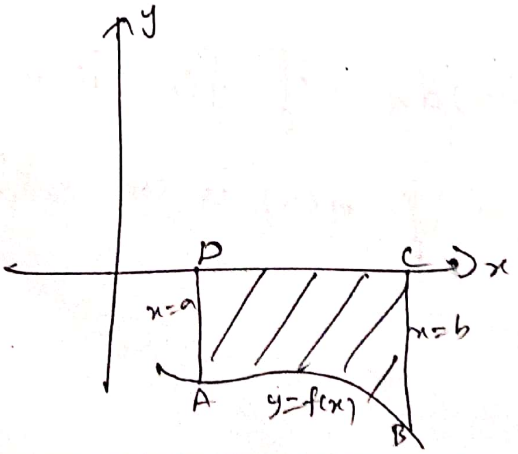


$$\text{Area} = \int_a^b f(x)\,dx \quad (\text{or}) \int_a^b y\,dx.$$

If $f(x) \leq 0$ ($f(x)$ lies on or below x-axis) for all x in $a \leq x \leq b$ then area is given by

$$\text{Area} = \int_a^b (-f(x))\,dx = \int_a^b (-y)\,dx$$

(ie The area below the x-axis is negative)



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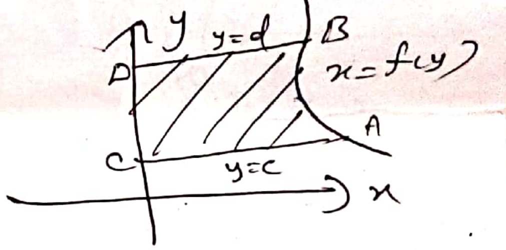
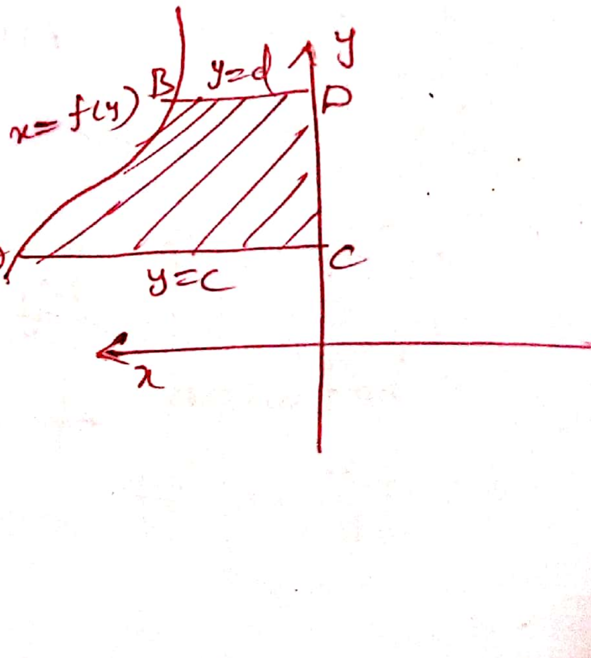
Area between a Continuous Curve and y-axis:

Let $x = f(y)$ be a continuous function of y on $[c, d]$. The area bounded by the curve $x = f(y)$ and the ordinates $y = c$, $y = d$ to the right of y-axis is given by $\int_c^d x\,dy$.



If the curve lies to the left of y-axis between the lines $y = c$ and $y = d$, the area is given by

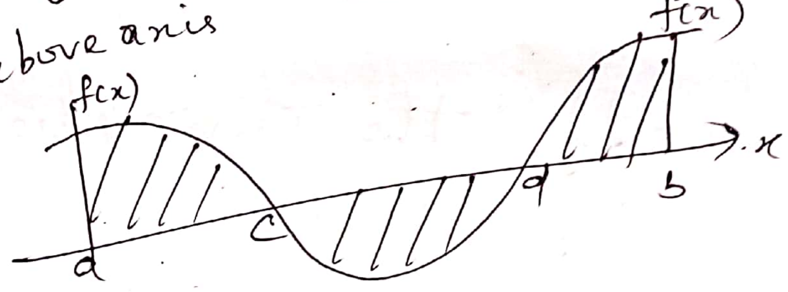
$$\int_c^d (-x)\,dy.$$



Remark:

If the Continuous curve f crosses the x-axis, then the integral $\int_a^b f(x)dx$ gives the algebraic sum of the areas between the curve and the axis, counting area above as positive and below as negative.

$$\int_a^b f(x)dx = \underbrace{\int_a^c f(x)dx}_{\text{above axis}} + \underbrace{\int_c^d (-f(x))dx}_{\text{below axis}} + \underbrace{\int_a^b f(x)dx}_{\text{above axis}}$$



General Area principle:

Let f and g be two continuous curves with f lying above g. Then the area R between f and g, from $x=a$ to $x=b$ is given by

$$R = \int_a^b (f-g)dx$$

No restriction on f and g where they lie.

Both may be lie above or below the x-axis or g lies below and f lies above the x-axis.

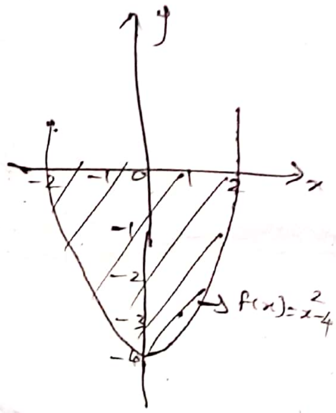
1) (i) Evaluate the integral $\int_{-2}^{2} x^2-4\, dx$

(ii) Find the area between the graph $f(x)=x^2-4$ and the x-axis over $[-2,2]$.

(i) $\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} (x^2-4)\,dx = \left(\frac{x^3}{3} - 4x\right)_{-2}^{2}$

$= \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3}+8\right) = \frac{-32}{3}$

(ii) Area $= \int_{-2}^{2} -f(x)\,dx$

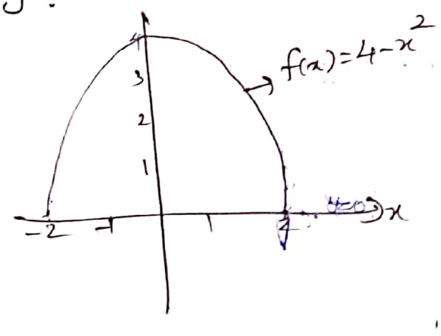


$= -\int_{-2}^{2} f(x)\,dx$

$= -\left(\frac{-32}{3}\right) = \frac{32}{3}.$

2) (i) Evaluate the integral $\int_{-2}^{2} 4-x^2\,dx$.

(ii) Find the area between the graph $f(x)=4-x^2$ and the x-axis over $[-2,2]$.



(i) $\int_{-2}^{2} (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$= (8 - 8/3) - (-8 + 8/3)$

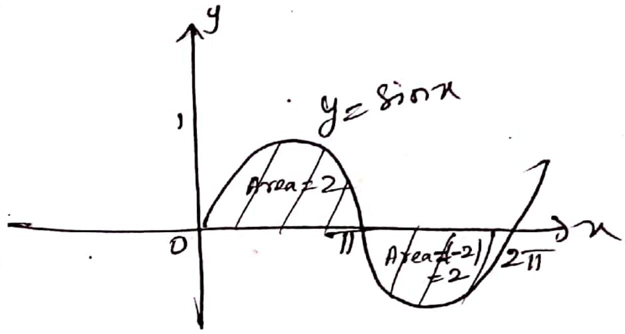
$= \frac{32}{3}.$

(ii) Area $= \int_{-2}^{2} f(x)\,dx = \frac{32}{3}$.

(3) The function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$.

Compute (i) the definite integral of $f(x)$ over $[0, 2\pi]$

(ii) The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.



(i). $\int_0^{2\pi} \sin x \, dx = \left(-\cos x\right)_0^{2\pi}$

$= -\left(\cos 2\pi - \cos 0\right)$

$= -(1-1)$

$= 0.$

$$\text{Area} = \int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} -f(x)\,dx$$

$$= \int_0^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} -\sin x \, dx$$

$$= \left(-\cos x\right)_0^{\pi} + \left(\cos x\right)_{\pi}^{2\pi}$$

$\cos\pi = -1$

$\cos 0 = 1$

$\cos 2\pi = 1$

$$= 1 + 1 + 1 + 1$$

$$= 4.$$

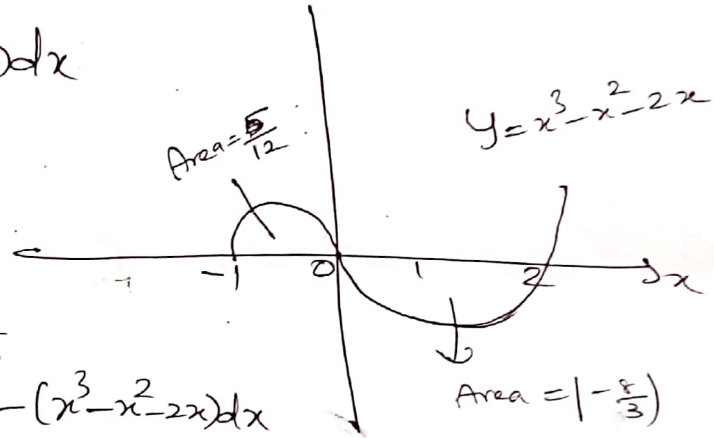
4. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

$$f(x) = x^3 - x^2 - 2x$$
$$= x(x^2 - x - 2) = x(x+1)(x-2).$$

x-axis is $y = 0$
ie $f(x) = 0$.
So $x = 0, -1, 2$

$x = 0, -1, 2$

$$\text{Area} = \int_{-1}^{0} f(x)\,dx + \int_{0}^{2} -f(x)\,dx$$



$$= \underbrace{\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx}_{I_1} + \underbrace{\int_{0}^{2} -(x^3 - x^2 - 2x)\,dx}_{I_2}$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right]_{0}^{2}$$

$$= -\frac{1}{4} - \frac{1}{3} + 1 - \frac{16}{4} + \frac{8}{3} + 4 \rightarrow$$

$$= -\frac{17}{4} + \frac{7}{3} + 5$$

$$= \frac{-51 + 28 + 60}{12} = \frac{37}{12}$$

$I_1 = \frac{5}{12}$

$I_2 = -\frac{8}{3}$

The total enclosed area is obtained by adding the absolute values of the calculated integrals. $\frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$

Definition:

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

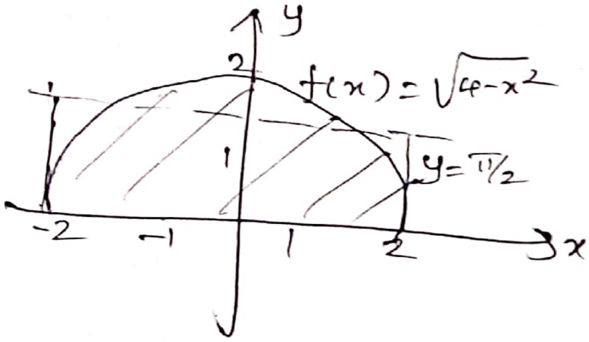
Definition:

If f is integrable on $[a, b]$, then its average value on $[a, b]$, also called its mean, is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$

We recognize $f(x) = \sqrt{4-x^2}$ as a function whose graph is the upper semi-circle of radius 2 centered at the origin.



$x = -2$, $y = 0$
$x = -1$, $y = \sqrt{3} = 1.73$
$x = 0$, $y = 2$
$x = 1$, $y = \sqrt{3}$
$x = 2$, $y = 0$

The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2}\cdot \pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi,$$

Because f is non negative, the area is also the value of the integral of f from -2 to 2,

$\int \sqrt{a^2-x^2}\,dx$
$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c$

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\pi.$$

Therefore, the average value of f is

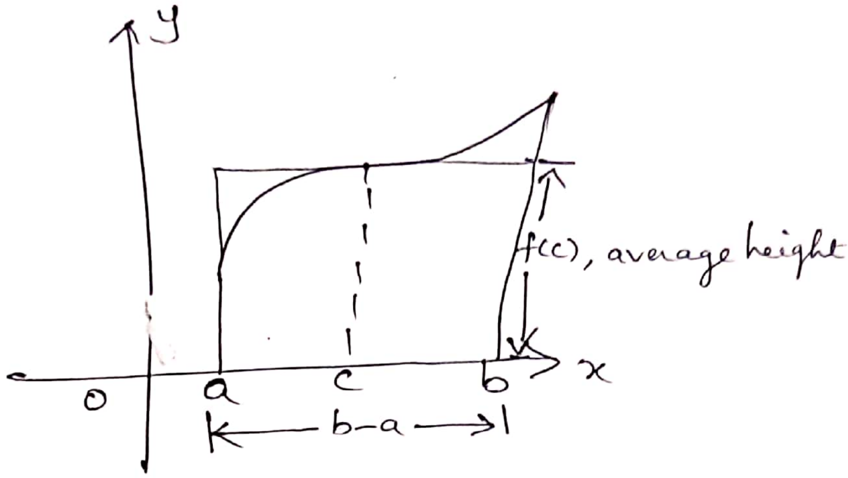
$$av(f) = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= \frac{1}{4}(2\pi) = \pi/2.$$

[Mean value theorem asserts that the area of the upper semicircle over $[-2,2]$ is the same as the area of the rectangle whose height is the average value of f over $[-2,2]$]

Mean value theorem for definite integrals: If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$



The value $f(c)$ in the mean value theorem is, in a sense, the average (or mean) height of f on $[a,b]$.

When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)\,(b-a) = \int_a^b f(x)\,dx.$$

Definition:

A function $F(x)$ is called an anti derivative or integral of a function $f(x)$ on an interval I if

$F'(x) = f(x)$ for every value of x in I

ie If the derivative of a function $F(x)$ w.r.t x is $f(x)$, then we say that the integral of $f(x)$ w.r.t x is $F(x)$.

ie $$\int f(x)\,dx = F(x).$$

Formulae:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\int \frac{1}{x^n}\,dx = -\frac{1}{(n-1)x^{n-1}} + c \quad (n \neq 1)$$

$$\int \frac{1}{x}\,dx = \log x + c$$

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\log a} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$

Properties of integrals:

(i) If k is any constant then

$$\int k f(x) \, dx = k \int f(x) dx$$

(ii) If $f(x)$ and $g(x)$ are any two functions in x then

$$\int [f(x) + g(x)] \, dx = \int f(x) dx + \int g(x) dx$$

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Substitution Rule:

① Find the integral $\int (x^3+x)^5 (3x^2+1)\,dx$.

put $u = x^3+x$

$du = (3x^2+1)\,dx \quad \left(\because du = \frac{du}{dx}dx\right)$

$$\int (x^3+x)^5 (3x^2+1)\,dx = \int u^5\,du$$

$$= \frac{u^6}{6} + c$$

$$= \frac{(x^3+x)^6}{6} + c$$

② Find $\int \sqrt{2x+1}\,dx$

put $u = 2x+1$

$du = \frac{du}{dx}dx = 2\,dx$

$\frac{du}{2} = dx$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{3/2} + c$$

$$= \frac{1}{3}(2x+1)^{3/2} + C.$$

Substitution Rule:

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

Find $\int 5\sec^2(5t+1)\,dt$

$$\int \sec^2(5t+1).5dt = \int \sec^2 u\, du$$

$u = 5t+1$

$du = 5dt$

$= \tan u + c$

$= \tan(5t+1) + c.$

HW

1. Find $\int \cos(7\theta+3)\,dx$. Ans: $\frac{1}{7}\sin(7\theta+3)+c$

2. Find $\int x^2 \sin(x^3)\,dx$ Ans: $\left[\begin{array}{l} u = x^3 \\ du = 3x^2 dx \\ -\frac{1}{3}\cos(x^3)+c \end{array}\right.$

3. " $\int x\sqrt{2x+1}\,dx$

$u = 2x+1$

$du = 2dx$

$= \int \frac{u-1}{2}\sqrt{u}\frac{du}{2}$

$= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du$

$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + c$

4. " $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

Ans: $\frac{3}{2}(z^2+1)^{2/3}+c$

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5. Find $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + c$$

6. Find $\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$

Substitution in Definite integrals:

If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Evaluate $\int_{-1}^{1} 3x^2 \sqrt{x^3+1}\,dx$

Put $u = x^3+1$ $\quad$ when $x=-1$, $u=0$

$du = 3x^2\,dx$ $\quad$ when $x=1$, $u=2$

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}\left(u^{3/2}\right)_0^2$$

$$= \frac{2}{3}\left(2^{3/2} - 0\right) = \frac{2}{3}\left(2\sqrt{2}\right) = \frac{4\sqrt{2}}{3}.$$

2. Find $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \int_1^0 u(-du)$

$u = \cot\theta$

$du = -\csc^2\theta \, d\theta$

$$= -\int_1^0 u\,du$$

$$= -\left(\frac{u^2}{2}\right)_1^0$$

$$= -\left(0 - \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Definite integrals of symmetric functions:

Theorem: Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$

$$= 2\int_0^2 (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

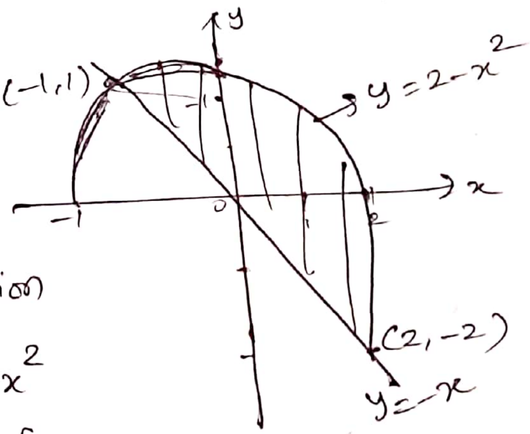
Areas Between Curves:

Definition: If f and g are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then the area of the region between the curves $y=f(x)$ and $y=g(x)$ from a to b is the integral of $(f-g)$ from a to b:

$$A = \int_a^b \left(f(x) - g(x)\right) dx.$$

Find the area of the region enclosed by the parabola $y = 2-x^2$ and the line $y=-x$.

First we sketch the two curves.



The limits of integration are found by solving $y=2-x^2$ and $y=-x$ simultaneously for x.

$$2-x^2 = -x$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x=-1, \quad x=2.$$

The region runs from $x=-1$ to $x=2$.
The limits of integration are $a=-1$, $b=2$.

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The area between the curve is

$$A = \int_a^b (f(x) - g(x))\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx$$

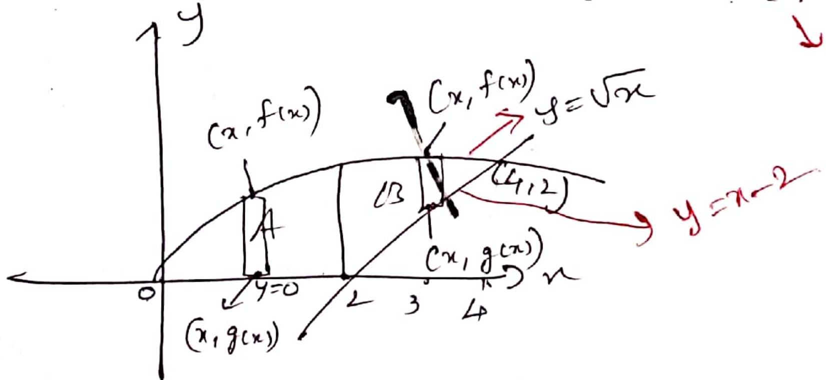
$$= \int_{-1}^{2} (2 + x - x^2)\,dx$$

$$= \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right)_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{9}{2}.$$

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



The sketch shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$).

We subdivide the region at $x=2$ into subregions A and B.

The limits of integration for region A are $a=0$ and $b=2$.

The left-hand limit for region B is $a=2$. To find the right-hand limit, we solve the equations $y=\sqrt{x}$ and $y=x-2$ simultaneously for x.

$$\sqrt{x}=x-2$$
$$x=(x-2)^2$$
$$x=x^2-4x+4$$
$$x^2-5x+4=0$$
$$(x-1)(x-4)=0$$
$$x=1,4.$$

Only the value $x=4$ satisfies the equation $\sqrt{x}=x-2$. The value $x=1$ is an extraneous root introduced by squaring.

The right hand limit is $b=4$.

For $0 \leq x \leq 2$: $f(x)-g(x)=\sqrt{x}-0=\sqrt{x}$

For $2 \leq x \leq 4$: $f(x)-g(x)=\sqrt{x}-(x-2)=\sqrt{x}-x+2$

we add the areas of subregions A and B to find the total area:

$$\text{Total area} = \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx$$

$$= \frac{2}{3}\left(x^{3/2}\right)_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)_2^4$$

$$= \frac{2}{3}(2^{3/2}) - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = 10/3 .$$

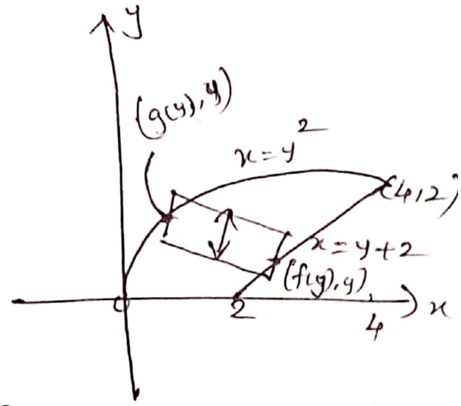
Integration with respect to y:

$$A = \int_c^d (f(y) - g(y))\,dy .$$

($f-g$ is nonnegative)

In the above problem, Find the area of the region w.r.t y.

$$x = y^2, \quad x = y+2$$



$$y + 2 = y^2$$

$$y^2 - y - 2 = 0$$

$$y = -1, 2 .$$

(-1 is a point of intersection below the x-axis)

$$A = \int_c^d [f(y) - g(y)]\,dy = \int_0^2 (y + 2 - y^2)\,dy = \frac{10}{3} .$$

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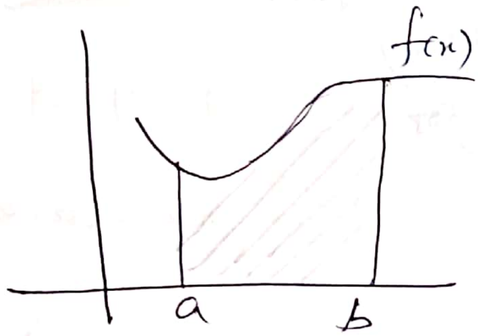
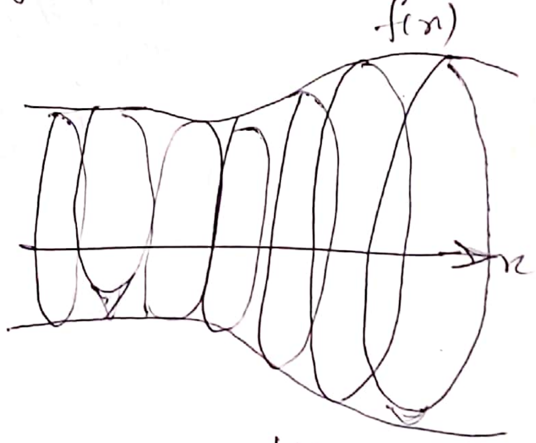
Volume of Solids of revolution:

Let f be a non-negative and continuous curve on $[a, b]$ and let R be the region bounded above by the graph of f, below by the x-axis and on the sides by the lines $x=a$ and $x=b$ [Fig a]





(b)

When this region is revolved about the x-axis, it generates a solid having circular cross sections (Fig b). Since the cross section at x has radius $f(x)$, the cross-sectional area is $A(x) = \pi[f(x)]^2 = \pi y^2$.

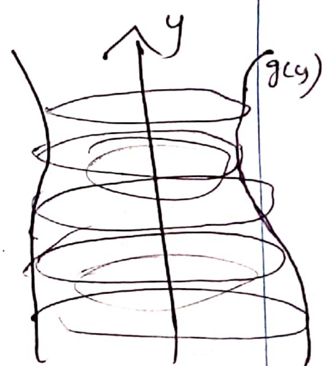
The volume of the solid is generated by moving the plane circular disc (Fig b) along x-axis perpendicular to the disc.

Therefore volume of the solid is $V = \int_a^b \pi [f(x)]^2 dx$

$= \int_a^b \pi y^2 dx$.

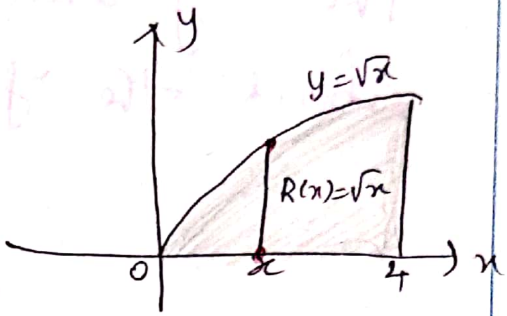
(ii) If the region bounded by the graph of $x = g(y)$, the y-axis and on the sides by the lines $y = c$ and $y = d$ (Fig c) then the volume of the solid generated is given by

$$V = \int_c^d \pi [g(y)]^2 dx = \int_c^d \pi x^2 dy$$



[The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a solid of revolution.

Ex. The region between the curve $y=\sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

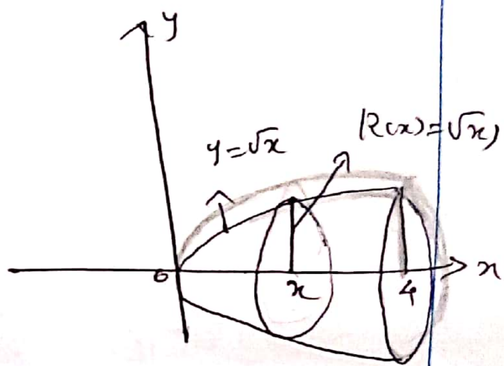


$$V = \int_a^b \pi (f(x))^2 dx$$

$$= \int_0^4 \pi (\sqrt{x})^2 dx$$

$$= \pi \int_0^4 x\, dx$$

$$= \pi \left(\frac{x^2}{2}\right)_0^4 = 8\pi$$



2) The circle $x^2+y^2=a^2$ is rotated about the x-axis to generate a sphere. Find its volume.

$$V = \int_a^b \pi (f(x))^2 dx = \int_a^b \pi y^2 dx$$

$$= \int_{-a}^{+a} \pi (a^2 - x^2) dx = \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

③ Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

$$\text{Volume } V = \int_1^4 \pi (f(y))^2 \, dy$$

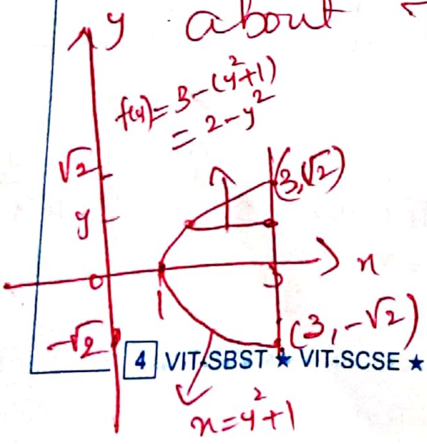
$$= \int_1^4 \pi x^2 \, dy$$

$$= \int_1^4 \pi \left(2/y\right)^2 dy = 4\pi \int_1^4 \frac{1}{y^2} \, dy$$

$$= 4\pi \left(-\frac{1}{y}\right)_1^4$$

$$= 4\pi \left[\frac{3}{4}\right] = 3\pi.$$

④ Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.



$f(y) = 3 - (y^2+1)$
$= 2 - y^2$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi (f(y))^2 \, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi (2 - y^2)^2 \, dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[4 - 4y^2 + y^4\right] dy = \frac{64\pi\sqrt{2}}{15}.$$

Volume by washer for Rotation about the x-axis

$$V = \int_a^b \pi \left[(R(x))^2 - (r(x))^2 \right] dx$$

$R(x)$ - outer Radius
$r(x)$ - inner "

The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

$$V = \int_a^b \pi \left[(R(x))^2 - (r(x))^2 \right] dx$$

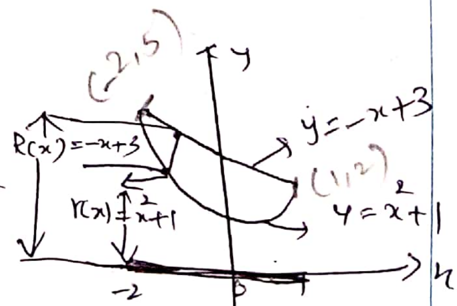
$x^2+1 = -x+3$

$x^2+x-2=0$

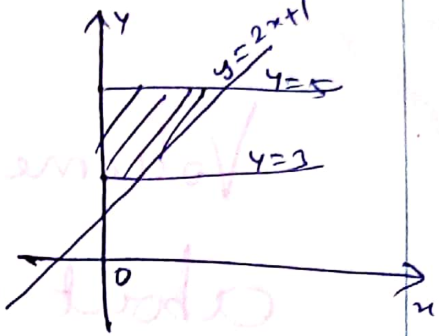
$x = -2, 1.$

$$= \int_{-2}^{1} \pi \left((-x+3)^2 - (x^2+1)^2 \right) dx$$

$$= \pi \int_{-2}^{1} \left(8 - 6x - x^2 - x^4 \right) dx = \frac{117\pi}{5}.$$



① Find the area of the region bounded by $y = 2x+1$, $y=3$, $y=5$ and y-axis.

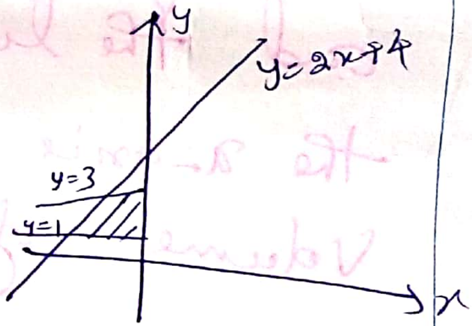


$$A = \int_c^d x\,dy$$

$$= \int_3^5 \frac{y-1}{2}\,dy = \frac{1}{2}(8-2) = 3 \text{ sq. units}$$

② Find the area of the region bounded by $y = 2x+4$, $y=1$ and $y=3$ and y-axis

$$A = \int_1^3 (-x)\,dy = \int_1^3 -\left(\frac{y-4}{2}\right) dy = 2 \text{ sq. units}$$



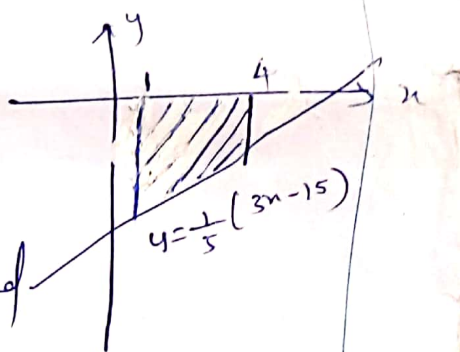
① Find the area of the region bounded by the lines $3x-2y+6=0$, $x=1$, $x=3$ and x-axis.

$$A = \int_1^3 y\,dx = \int_1^3 \frac{3x+6}{2}\,dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2}+2x\right)_1^3 = 12 \text{ sq.units.}$$

② Find the are of the region bounded by the line $3x-5y-15=0$, $x=1$, $x=4$ and x-axis

$$A = \int_1^4 (-y)\,dx = \frac{9}{2} \text{ sq.units}$$



③ Find the area of the region bounded $y=x^2-5x+4$, $x=2$, $x=3$ and x-axis.

$$A = \int_2^3 (-y)\,dx$$

$$= \int_2^3 -(x^2-5x+4)\,dx = \frac{13}{6} \text{ sq.units}$$