Concept of differentiation:

There are two ways of introducing this concept, the geometrical way (as the slope of curve) and the physical way (as a rate of change)

Applications:

* Temperature change at a particular time.

* Velocity of a falling object at a particular time.

* current through a circuit at a particular time.

* population growth at a particular time.

Calculus Introduction:

Calculus was invented for the purpose of solving problems that deal with continuously changing quantities.

Calculus is used in calculating the rate of change of velocity of a vehicle with respect to time, the rate of change of growth of population with respect to time, etc.

Calculus also helps us to maximise profits or minimise losses.

Isacc Newton of England and Gottfried wilhelm Leibnitz of Germany invented calculus in the 17th century independently.

Leibnitz, a great Mathematician of all times, approached the problem of settling tangents geometrically; but Newton approached it Calculus physical concepts.

Newton, one of the greatest Mathematicians and physicists of all time, applied the Calculus to formulate his laws of motion and gravitation.

Note:

Every differentiable function is continuous;

Differentiation Rules (or) Differentiation Techniques:

1. Derivative of a Constant function:

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof:

$$f'(c) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

2. The derivative of x^n is $n x^{n-1}$, where n is a rational number. (Power Rule for positive integers:)

Note: $z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + z x^{n-2} + x^{n-1})$

Differentiate the following powers of x.

1. x^3 Ans $3x^2$
2. $x^{2/3}$ — $\frac{2}{3} x^{-1/3}$
3. $x^{\sqrt{2}}$ — $\sqrt{2}\, x^{\sqrt{2}-1}$
4. $1/x^4$ — $-4/x^5$
5. $x^{-4/3}$ — $-\frac{4}{3} x^{-7/3}$
6. $\sqrt{x^{2+\pi}}$ — $\frac{1}{2}(2+\pi)\sqrt{x^{\pi}}$

3. The derivative of $\sin x$ is $\cos x$.

4. " " " $\cos x$ is $-\sin x$.

Theorem:

(5) If f and g are differentiable functions of x and c is any constant, then the following are true.

(i) $\frac{d(cf(x))}{dx} = c\frac{d(f(x))}{dx}$

(ii) $\frac{d(f(x) \pm g(x))}{dx} = \frac{d(f(x))}{dx} \pm \frac{d(g(x))}{dx}$.

In particular.

$$\frac{d}{dx}(cx^n) = cnx^{n-1}$$

Note:

If $y = \log_a x$ then $\frac{dy}{dx} = \frac{1}{x}\log_a e$

If $y = \log_e x$ then $\frac{dy}{dx} = \frac{1}{x}$.

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(1) Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Ans. $y' = 3x^2 + \frac{4}{3} 2x - 5 + 0.$

Product Rule:

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Ex:

Find the derivative of

$$y = (x^2+1)(x^3+3)$$

$$y' = (x^2+1)(3x^2) + (x^3+3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

Quotient Rule:

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

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Ex:

(1) Find the derivative of $y = \dfrac{t^2-1}{t^3+1}$

$$\frac{dy}{dt} = \frac{(t^3+1)(2t) - (t^2-1)(3t^2)}{(t^3+1)^2}$$

$$= \frac{2t^4+2t-3t^4+3t^2}{(t^3+1)^2}$$

$$= \frac{-t^4+3t^2+2t}{(t^3+1)^2}.$$

(2) Find the derivative of $y = \dfrac{(x-1)(x^2-2x)}{x^4}$

Ans: $-\dfrac{1}{x^2} + \dfrac{6}{x^3} - \dfrac{6}{x^4}$. without using Quotient Rule.

Second and Higher order Derivative:

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''.

So $f'' = (f')'$.

The function f'' is called the second derivative of f because it is the derivative of the first derivative. It is written in several ways.

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y') = y'' = D^2(f)(x) = D_x^2 f(x)$$

Ex:

The first four derivatives of $y = x^3 - 3x^2 + 2$

$y' = 3x^2 - 6x$

$y'' = 6x - 6$

$y''' = 6$

$y^{iv} = 0$.

H.W

Find the first and second derivatives

1. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$

2. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

3. $w = 3z^{-2} - \frac{1}{z}$.

4. $p = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right)$

5. $p = \frac{q^2+3}{(q-1)^3 + (q+1)^3}$

6. $w = \left(\frac{1+3z}{3z}\right)(3-z)$

1. $u = \frac{(x^2+x)(x^2-x+1)}{x^4}$

$r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3}$

Find the derivative of the following

1. $y = (x^2+1)(x+5+1/x)$

2. $y = (1+x^2)(x^{3/4} - x^{-3})$

3. $v = (1-t)(1+t^2)^{-1}$

4. $w = (2x-7)^{-1}(x+5)$

5. $u = \frac{5x+1}{2\sqrt{x}}$

① (4) $\frac{dp}{dq} = q/6 + \frac{1}{6q^3} + \frac{1}{q^5}$ ✓ Ans.

$d^2p/dq^2 = \frac{1}{6} - \frac{1}{2q^4} - 5/q^6$

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find 1^{st} & 2^{n} derivative of $p = \dfrac{q^2+3}{(q-1)^3(q+1)^3}$

$$p = \frac{q^2+3}{(q^2-1)^3}$$

$$\frac{dp}{dq} = \frac{-4q(q^2+5)}{(q^2-1)^4} \quad \text{(or)} \quad \frac{-4q^3-20q}{(q^2-1)^4}$$

$$\frac{d^2p}{dq^2} = \frac{4(5q^4+38q^2+5)}{(q^2-1)^5}$$ //

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Chain Rule:

If $u = f(x)$ and $y = F(u)$
then $y = F(f(x))$ is the composition of f and F.

In the expression $y = F(u)$, u is called intermediate argument.

This chain rule can further be extended to ie if $y = F(u)$, $u = f(t)$, $t = g(x)$

then $\dfrac{dy}{dx} = F'(u)\, u'(t)\, t'(x)$

ie $\dfrac{dy}{dx} = \dfrac{dF}{du} \dfrac{du}{dt} \dfrac{dt}{dx}$.

Ex:

(1) Differentiate $e^{\sin x^2}$.

Let $y = e^{\sin x^2}$, $u = \sin x^2$, $t = x^2$

Then $y = e^{u}$, $u = \sin t$, $t = x^2$

$\therefore$ By Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dt} \frac{dt}{dx}$$

$$= e^{u} \cos t \; 2x$$

$$\frac{dy}{dx} = 2x\, e^{\sin x^2} \cos(x^2).$$

(1) Differentiate $\sin(ax+b)$ w.r.t x.

Ans $\frac{dy}{dx} = a\cos(ax+b)$.

(2) Differentiate $\sin(\log x)$

Ans $\frac{dy}{dx} = \frac{\cos(\log x)}{x}$

(3) Differentiate $\log\sqrt{x}$ w.r.t x.

Ans: $\frac{dy}{dx} = \frac{1}{2x}$.

Derivatives of trigonometrical functions:

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
6. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Derivatives of inverse trigonometrical functions.

1. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$
5. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
6. $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$

Differentiation of implicit functions:

If the relation between x and y is given by an equation of the form $f(x,y)=0$ and this equation is not easily solvable for y, then y is said to be an implicit function of x.

In case y is given interms of x, then y is said to be an explicit function of x. In case of implicit function also, it is possible to get $\frac{dy}{dx}$.

Ex: (1) obtain $\frac{dy}{dx}$ when $x^3+8xy+y^3=64$.

$$3x^2+8\left[x\frac{dy}{dx}+y.1\right]+3y^2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{3x^2+8y}{8x+3y^2}$$

(2) Find $\frac{dy}{dx}$ when $\tan(x+y)+\tan(x-y)=1$.

$$\sec^2(x+y)\left(1+\frac{dy}{dx}\right)+\sec^2(x-y)\left(1-\frac{dy}{dx}\right)=0$$

$$\frac{dy}{dx}=-\frac{\sec^2(x+y)+\sec^2(x-y)}{\sec^2(x+y)-\sec^2(x-y)}$$

(3) Find $\frac{dy}{dx}$ if $xy+xe^{-y}+ye^{x}=x^2$

$$x\frac{dy}{dx}+y\cdot 1+xe^{-y}\left(-\frac{dy}{dx}\right)+e^{-y}\cdot 1+ye^{x}+e^{x}\frac{dy}{dx}=2x$$

$$\frac{dy}{dx}=-\frac{ye^{x}+y+e^{-y}-2x}{e^{x}-xe^{-y}+x}.$$

Power chain Rule:

$$\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$$

H.W

Chain Rule:

(1) Diff $x(t)=\cos(t^2+1)$ w.r.t t.

Ans: $\frac{dx}{dt}=-2t\sin(t^2+1)$

(2) Diff $\sin(x^2+x)$ w.r.t x.

$$\frac{dy}{dx}=(2x+1)\cos(x^2+x)$$

(3) Find the derivative $g(t)=\tan(5-\sin 2t)$

$$\frac{dg}{dt}=-2(\cos 2t)\sec^2(5-\sin 2t).$$

Ex.

① Find the derivative of $(5x^3-x^4)^7$.

Using Power Chain Rule,

$$\frac{d}{dx}(5x^3-x^4)^7 = 7(5x^3-x^4)^6 \frac{d}{dx}(5x^3-x^4)$$

$$= 7(5x^3-x^4)^6(15x^2-4x^3).$$

② Find the ~~derivative~~ of $\frac{1}{3x-2}$.

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

$$= -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$$

$$= -(3x-2)^{-2}(3)$$

$$= -\frac{3}{(3x-2)^2}.$$

③ Find the derivative of $\sin^5 x$.

$$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \frac{d}{dx}(\sin x)$$

$$= 5\sin^4 x \cos x.$$

Implicit functions:

① Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Ans: $\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$

② Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.

Ans: $y' = \frac{x^2}{y}$.

$y'' = \frac{2x}{y} - \frac{x^4}{y^3}$