Solenoidal and irrotational vector Functions:

Definition: A vector function $\vec{F}$ is said to be solenoidal if $\nabla \cdot \vec{F} = 0$ and irrotational if $\nabla \times \vec{F} = 0$

If $\nabla \times \vec{F} = 0$ then the field F is called a Conservative field.

If $\vec{F}$ is irrotational then a scalar function ϕ can be found so that $\vec{F} = \nabla\phi$ and ϕ is called the scalar potential of F

1) prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

To prove $\nabla \cdot \vec{F} = \vec{0}$

$$\nabla \cdot \vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) \cdot (z\vec{i} + x\vec{j} + y\vec{k})$$

$$= \partial/\partial x\,(z) + \partial/\partial y\,(x) + \partial/\partial z\,(y)$$

$$= 0$$

Hence $\vec{F}$ is solenoidal.

(2) If $\vec{v} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+\lambda z)\vec{k}$ is solenoidal find the value of λ. (Au 1996 may)

Sol:

Given $\nabla \cdot \vec{v} = 0$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+\lambda z) = 0$$

$$1 + 1 + \lambda = 0$$

$$\lambda = -2.$$

(3) Show that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right]$$

$$= \Sigma \vec{i}(x-x)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

Hence $\vec{F}$ is irrotational.

(4) Find the constants a, b, c so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Sol:

Given $\nabla \times \vec{f} = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0\vec{i} + 0\vec{j} + 0\vec{k}$

ie $c+1=0$, $4-a=0$, $b-2=0$

$c=-1$, $a=4$, $b=2$.

(5) Prove that $\vec{f} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$ is solenoidal as well as irrotational. Also find the scalar potential of $\vec{f}$.

Sol: Given $\vec{f} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$

$$\nabla\cdot\vec{f} = \frac{\partial}{\partial x}(2x+yz) + \frac{\partial}{\partial y}(4y+zx) - \frac{\partial}{\partial z}(6z-xy)$$

$$= 2+4-6=0$$

$\therefore \vec{f}$ is solenoidal.

$$\nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+yz & 4y+zx & -6z+xy \end{vmatrix}$$

$$= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z)$$

$$=0$$

$\therefore \vec{f}$ is irrotational.

Hence $\vec{f}$ is solenoidal as well as irrotational

Now to find φ such that $\vec{f} = \nabla\varphi$

$$(2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k} = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\frac{\partial\varphi}{\partial x} = 2x+yz \quad , \frac{\partial\varphi}{\partial y} = 4y+zx \quad , \frac{\partial\varphi}{\partial z} = -(6z-xy)$$

Integrating (1), (2) & (3) partially w.r.t x, y, z respectively we get

$$\varphi = x^2 + xyz + c_1 \longrightarrow (4)$$

$$\varphi = 2y^2 + xyz + c_2 \longrightarrow (5)$$

$$\varphi = -3z^2 + xyz + c_3 \longrightarrow (6)$$

From (4), (5), (6) we get

$$\varphi = x^2 + 2y^2 - 3z^2 + xyz + c \quad \text{where } c = c_1 + c_2 + c_3 \text{ is a constant,}$$

φ is the scalar potential of $\vec{f}$.

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