Maxima & Minima for a function of two Variables:

① Find the local extreme values of

$f(x,y) = x^2 + y^2 - 4y + 9.$

Solution:

$\partial f/\partial x = 2x$, $\partial f/\partial y = 2y - 4$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x = 0$, $\partial f/\partial y = 0 \Rightarrow 2y - 4 = 0$

$\Rightarrow x = 0$ $\Rightarrow y = 2.$

The only possible point is $(0,2)$.

$A = \frac{\partial^2 f}{\partial x^2} = 2$, $B = \frac{\partial^2 f}{\partial x \partial y} = 0$, $C = \frac{\partial^2 f}{\partial y^2} = 2$

$AC - B^2 = (2)(2) - 0 = 4 > 0$

and $A > 0$.

$\therefore$ f has a minimum Value at $(0,2)$.

$\therefore$ Minimum Value $= 0 + 2^2 - 8 + 9$

Local minimum value $= 5$.

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(2) Find the local extreme values of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Solution:

$$\frac{\partial f}{\partial x} = y - 2x - 2 \quad , \quad \frac{\partial f}{\partial y} = x - 2y - 2$$

$\partial f / \partial x = 0 \Rightarrow -2x + y - 2 = 0$

$\Rightarrow -2x + y = 2 \longrightarrow (1)$

$\frac{\partial f}{\partial y} = 0 \Rightarrow x - 2y - 2 = 0$

$\Rightarrow x - 2y = 2 \longrightarrow (2)$

$(1) \times 2 \Rightarrow -4x + \cancel{2y} = 4$

$(2) \Rightarrow x - \cancel{2y} = 2$

$-3x = 6$

$\boxed{x = -2}$

$x = -2$ sub in (2).

$\therefore \boxed{y = -2}$

$\therefore$ The only possible point is $(-2,-2)$

$$A = \partial^2 f / \partial x^2 = -2 \quad , \quad B = \frac{\partial^2 f}{\partial x \partial y} = 1 \quad , \quad C = \frac{\partial^2 f}{\partial y^2} = -2$$

$$AC - B^2 = (-2)(-2) - 1 = 4 - 1 = 3 > 0$$

$\&$ $A < 0$

$\therefore$ f has a maximum at $(-2,-2)$.

$\therefore$ Maximum Value $= (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$

$= 4 - \cancel{4} - \cancel{4} + \cancel{4} + \cancel{4} + 4$

$= 8$.

(3) Find the local extreme values of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

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Solution:

$$\frac{\partial f}{\partial x} = -6x + 6y, \quad \frac{\partial f}{\partial y} = 6y - 6y^2 + 6x$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow -6x + 6y = 0, \quad \frac{\partial f}{\partial y} = 0 \Rightarrow 6y - 6y^2 + 6x = 0 \rightarrow (2)$$

$$\Rightarrow x = y \rightarrow (1)$$

sub (1) in (2), $6x - 6x^2 + 6x = 0$.

$12x - 6x^2 = 0$

$6x(2 - x) = 0$.

$\Rightarrow x = 0, x = 2$.

when $x = 0, y = 0$

when $x = 2, y = 2$.

$\therefore$ The points are $(0,0), (2,2)$.

$$A = \frac{\partial^2 f}{\partial x^2} = -6, \quad B = \frac{\partial^2 f}{\partial x \partial y} = 6, \quad C = \frac{\partial^2 f}{\partial y^2} = 6 - 12y$$

At $(0,0)$ $A = -6, B = 6, C = 6$

$$AC - B^2 = (-6)(6) - 6^2 = -36 - 36 < 0.$$

$\therefore$ f has neither a maximum nor a minimum at $(0,0)$.

$\therefore$ $(0,0)$ is called saddle point.

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At $(2,2)$

$A = -6, \quad B = 6, \quad C = -18$

$AC - B^2 = (-6)(-18) - 6^2 = 108 - 36 > 0.$

and $A < 0$.

$\therefore$ f has a maximum at $(2,2)$.

$\therefore$ Maximum value $= 3(2)^2 - 2(2)^3 - 3(2)^2 + 6(2)(2)$

$= \cancel{12} - 16 - \cancel{12} + 24$

$= 8.$

Find the local extreme values (if any) of $f(x,y) = y^2 - x^2$.

Solution: $f_x = -2x, \quad f_y = 2y$

$f_x = 0 \Rightarrow -2x = 0 \qquad f_y = 0 \Rightarrow 2y = 0$

$\Rightarrow x = 0 \qquad \Rightarrow y = 0.$

$\therefore$ The only possible point is $(0,0)$.

$A = \frac{\partial^2 f}{\partial x^2} = -2, \quad B = \frac{\partial^2 f}{\partial x \partial y} = 0, \quad C = \frac{\partial^2 f}{\partial y^2} = 2$

At $(0,0)$

$A = -2, \quad B = 0, \quad C = 2$

$AC - B^2 = (-2)(2) - 0 = -4 < 0,$

$\therefore$ f has neither a maximum nor a minimum at $(0,0)$.

$\therefore$ $(0,0)$ is called saddle point.

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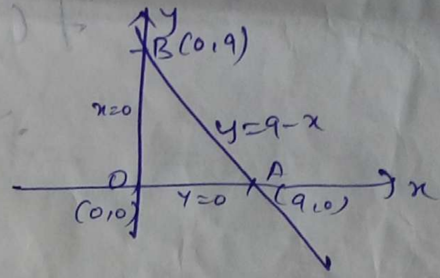
① Find the absolute maximum and minimum value of $f(x,y) = 2+2x+2y-x^2-y^2$ on the triangular region in the first quadrant bounded by the lines $x=0, y=0, y=9-x$.



Solution:

$$f_x = 2-2x, \quad f_y = 2-2y$$

$$f_x = 0 \Rightarrow 2-2x=0 \Rightarrow \boxed{x=1}, \quad f_y = 0 \Rightarrow 2-2y=0 \Rightarrow \boxed{y=1}$$

Interior point $f(1,1) = 4$

Boundary points:

(i) on OA, $y=0, \quad 0 \le x \le 9$.

The function is $f(x,0) = 2+2x-x^2$

When $x=0$, $f(0,0) = 2$

when $x=9$, $f(9,0) = 2+18-81 = -61$

$$f'(x,0) = 0 \Rightarrow 2-2x = 0$$
$$\Rightarrow x = 1.$$

$\therefore$ The point is $(1,0)$.

$$f(1,0) = 3$$

(ii) <u>On OB</u>

$x=0, \quad 0 \le y \le 9.$

The function is $f(0,y) = 2+2y-y^2$

when $y=0$, $f(0,0) = 2$

When $y=9$, $f(0,9) = -61$

$f'(0,y) = 0 \implies 2-2y=0 \implies y=1.$

$\therefore$ The point $(0,1)$.

$f(0,1) = 3$

(iii) <u>On AB</u>

$y = 9-x$

$f(x, 9-x) = 2+2x+2(9-x)-x^2-(9-x)^2$

$= -61+18x-2x^2$

$f'(x, 9-x) = 18-4x=0 \implies x = \frac{18}{4} = \frac{9}{2}$

$\therefore y = 9 - 9/2 = 9/2.$

$\therefore$ The point is $(9/2, 9/2)$

$\therefore f(9/2, 9/2) = -\frac{41}{2}$

absolute Max is 4 at $(1,1)$

absolute Mini is -61 at $(0,9)$ & $(9,0)$.

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(2) Find the point $P(x, y, z)$ on the plane $2x+y-z-5=0$ that is closest to the origin.

$f = (\text{Distance})^2 = (x-0)^2+(y-0)^2+(z-0)^2 = x^2+y^2+z^2$

$$g = x^2+y^2+z^2+\lambda(2x+y-z-5)=0$$

$$\frac{\partial g}{\partial x}=0 \Rightarrow 2x+2\lambda=0 \Rightarrow x=-\lambda$$

$$\frac{\partial g}{\partial y}=0 \Rightarrow 2y+\lambda=0 \Rightarrow y=-\lambda/2$$

$$\frac{\partial g}{\partial z}=0 \Rightarrow 2z-\lambda=0 \Rightarrow z=\lambda/2$$

$$\frac{\partial g}{\partial \lambda}=0 \Rightarrow 2x+y-z-5=0$$

$$\Rightarrow 2x+y-z=5$$

$$2(-\lambda)-\lambda/2-\lambda/2=5$$

$$-3\lambda=5$$

$$\boxed{\lambda=-5/3}$$

$\therefore\ x=5/3,\ y=5/6,\ z=-5/6$

$$(\text{Distance})^2 = x^2+y^2+z^2 = \frac{25}{9}+\frac{25}{36}+\frac{25}{36}$$

$$= \frac{25}{9}+\frac{25}{18}$$

$$(\text{Distance})^2 = \frac{50+25}{18} = \frac{75}{18} = \frac{25}{6}$$

$$\text{Distance} = \frac{5}{\sqrt{6}}\ //.$$

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(3) Find the maximum & minimum values of the function $f(x,y) = 3x+4y$ on the circle $x^2+y^2=1$.

$$g = 3x+4y+\lambda(x^2+y^2-1)$$

$$\frac{\partial g}{\partial x} = 0 \Rightarrow 3+2x\lambda = 0 \Rightarrow x = \frac{-3}{2\lambda}$$

$$\frac{\partial g}{\partial y} = 0 \Rightarrow 4+2y\lambda = 0 \Rightarrow y = \frac{-2}{\lambda}$$

$$\frac{\partial g}{\partial \lambda} = 0 \Rightarrow x^2+y^2-1 = 0$$

$$\Rightarrow x^2+y^2 = 1$$

$$\Rightarrow \frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1$$

$$\Rightarrow \frac{9+16}{4\lambda^2} = 1$$

$$\Rightarrow 25 = 4\lambda^2$$

$$\Rightarrow \lambda^2 = \frac{25}{4}$$

$$\lambda = \pm\frac{5}{2}$$

When $\lambda = 5/2$, $x = -\frac{3}{5}$, $y = -\frac{4}{5}$

When $\lambda = -5/2$, $x = 3/5$, $y = 4/5$

At $(3/5, 4/5)$

$$f(3/5, 4/5) = 5$$

$\therefore$ Max. value $= 5$ at $(3/5, 4/5)$

At $(-3/5, -4/5)$

$$f(-3/5, -4/5) = -5$$

Min. value $= -5$ at $(-3/5, -4/5)$

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Taylor's Series:

Find a quadratic approximation to $f(x,y) = \sin x \sin y$ near the origin.

$f(0,0) = 0$ $\quad f_{xx}(0,0) = 0$

$f_x(0,0) = 0$ $\quad f_{xy}(0,0) = 1$

$f_y(0,0) = 0$ $\quad f_{yy}(0,0) = 0$

$\sin x \sin y = \frac{1}{\cancel{2}} \cancel{2} xy = xy$ //.

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