Note:

1. ∇ is an operator and also it is a vector

2. $\nabla = \sum \vec{i} \, \partial/\partial x = \vec{i} \, \partial/\partial x + \vec{j} \, \partial/\partial y + \vec{k} \, \partial/\partial z$

3. $\nabla \varphi$ is a vector whose three components are $\partial \varphi/\partial x$, $\partial \varphi/\partial y$, $\partial \varphi/\partial z$.

4. If φ is a constant then $\nabla \varphi = 0$

5. $\nabla \varphi$ should not be written as $\varphi \nabla$.

6. $\nabla(c_1 \varphi_1 \pm c_2 \varphi_2) = c_1 \nabla \varphi_1 \pm c_2 \nabla \varphi_2$

where c_1 and c_2 are constants and φ_1, φ_2 are scalar point functions.

7. $\nabla(\varphi_1 \varphi_2) = \varphi_1 \nabla \varphi_2 + \varphi_2 \nabla \varphi_1$

8. $\nabla(\varphi_1/\varphi_2) = \dfrac{\varphi_2 \nabla \varphi_1 - \varphi_1 \nabla \varphi_2}{\varphi_2^2}$, if $\varphi_2 \neq 0$

9. If $v = f(u)$ then $\nabla v = f'(u) \nabla u$.

10. $\nabla(f \pm g) = \nabla f \pm \nabla g$.

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Note:

Gradient of φ or Grad φ is

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\,\partial\varphi/\partial z$$

(1) Find the gradient of $\varphi = xyz$

Sol:

$$\nabla\varphi = \vec{i}\,\partial\varphi/\partial x + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla(xyz) = \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\,\partial/\partial z\,(xyz)$$

$$\nabla(xyz) = \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy$$

(2) Find the grad φ where $\varphi = 3x^2y - y^3z^2$ at $(1,1,1)$.

Sol:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla(3x^2y - y^3z^2) = \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\,\partial/\partial z\,(3x^2y - y^3z^2)$$

$$\nabla(3x^2y - y^3z^2) = 6xy\,\vec{i} + (3x^2 - 3y^2z^2)\vec{j} + (-2y^3z)\vec{k}$$

At $(1,1,1)$

$$= 6\vec{i} + 0\vec{j} - 2\vec{k}$$

$$\nabla\varphi = 6\vec{i} - 2\vec{k}.$$

Note: Unit Normal to the given Surface φ at the point is $\dfrac{\nabla\varphi}{|\nabla\varphi|}$.

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① Find the unit normal to the surface $x^2y + 2xz = 4$ at $(2,-2,3)$.

Sol:

$$\phi = x^2y + 2xz - 4$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla(x^2y + 2xz - y) = \vec{i}\frac{\partial}{\partial x}(x^2y + 2xz - 4) + \vec{j}\frac{\partial}{\partial y}(x^2y + 2xz - 4) + \vec{k}\,\partial/\partial z\,(x^2y + 2xz - 4)$$

$$= (2xy + 2z)\vec{i} + (x^2)\vec{j} + 2x\vec{k}$$

At $(2,-2,3)$

$$\nabla\phi = -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$|\nabla\phi| = \sqrt{4+16+16} = \sqrt{36} = 6.$$

unit normal to the given surface at $(2,-2,3)$

is $\dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6}$$

$$= \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

Note: Direction derivative $= \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$ or $\nabla\phi \cdot \vec{n}$ where $\vec{n} = \dfrac{\vec{a}}{|\vec{a}|}$

⊙ Find the directional derivative of $xy + yz + zx$ at $(1,1,1)$ in the direction $\vec{i} + \vec{j}$.

Sol:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla(xy+yz+zx) = \vec{i}\frac{\partial}{\partial x}(xy+yz+zx) + \vec{j}\frac{\partial}{\partial y}(xy+yz+zx) + \vec{k}\frac{\partial}{\partial z}(xy+yz+zx)$$

$$= \vec{i}(y+z) + \vec{j}(x+z) + \vec{k}(y+x)$$

At (1,1,1)

$$\nabla\phi = 2\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{a} = \vec{i} + \vec{j}$$

$$|\vec{a}| = \sqrt{1+1} = \sqrt{2}$$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= 2\vec{i} + 2\vec{j} + 2\vec{k} \cdot \frac{(\vec{i}+\vec{j})}{\sqrt{2}}$$

$$= \frac{2+2}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

$$D.D = 2\sqrt{2}$$

(2) Find the directional derivative of $\phi = xy^2z^3$ at the point (1,1,1) along the normal to the surface $x^2+xy+z^2=3$ at the point (1,1,1).

solution:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

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$\nabla\varphi$ is the normal to the Surface x^2+xy+z^2-3

$\nabla(x^2+xy+z^2-3) = \vec{i}\frac{\partial}{\partial x}(x^2+xy+z^2-3)+\vec{j}\frac{\partial}{\partial y}(x^2+xy+z^2-3)+$

$\vec{k}\frac{\partial}{\partial z}(x^2+xy+z^2-3)$

$= \vec{i}\,(2x+y)+\vec{j}\,(x)+\vec{k}\,(2z)$

At $(1,1,1)$

$= 3\vec{i}+\vec{j}+2\vec{k}$

To find the directional derivative of $\varphi = xy^2z^3$ at $(1,1,1)$ in the direction of $\vec{a} = 3\vec{i}+\vec{j}+2\vec{k}$

$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}$

$\nabla(xy^2z^3) = \vec{i}\,\partial/\partial x\,(xy^2z^3)+\vec{j}\frac{\partial}{\partial y}(xy^2z^3)+\vec{k}$

$\partial/\partial z\,(xy^2z^3)$

$= (y^2z^3)\vec{i}+\vec{j}\,(2xyz^3)+\vec{k}(3xy^2z^2)$

At $(1,1,1)$

$= \vec{i}+2\vec{j}+3\vec{k}$

$D.D = \nabla\varphi\cdot\frac{\vec{a}}{|\vec{a}|}$

$= (\vec{i}+2\vec{j}+3\vec{k})\cdot\frac{(3\vec{i}+\vec{j}+2\vec{k})}{\sqrt{3^2+1^2+2^2}}$

$= \frac{3+2+6}{\sqrt{14}}$

$= 11/\sqrt{14}$ //

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Magnitude of Maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad}\,\varphi|$.

(1) In what direction from the point $(1,-1,2)$ is the directional derivative of $\varphi = x^2y^2z^3$ a maximum? what is the magnitude of this maximum?

Let $\varphi = x^2y^2z^3$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$\frac{\partial\varphi}{\partial x} = 2xy^2z^3, \quad \frac{\partial\varphi}{\partial y} = 2x^2yz^3,$$

$$\frac{\partial\varphi}{\partial z} = 3z^2x^2y^2$$

$$\nabla\varphi = 2xy^2z^3\vec{i} + 2x^2yz^3\vec{j} + 3x^2y^2z^2\vec{k}$$

At $(1,-1,2)$

$$\nabla\varphi = 16\vec{i} - 16\vec{j} + 12\vec{k}$$

The directional derivative is maximum is the direction $16\vec{i} - 16\vec{j} + 12\vec{k}$ and the magnitude of this maximum is $|\nabla\varphi|$

$$|\nabla\varphi| = \sqrt{256+256+144} = \sqrt{656}.$$

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Angle between the Surfaces:

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$$

Note: Two surfaces are said to cut orthogonally at a point of intersection if the respective normals at that point are perpendicular.

Since two surfaces cut orthogonally,

$$\nabla\varphi_1 \cdot \nabla\varphi_2 = 0.$$

① Find the angle between the surfaces $x^2+y^2+z^2=9$, $z=x^2+y^2-3$ at $(2,-1,2)$.

Solution:

Let $\varphi_1 = x^2+y^2+z^2-9$, $\varphi_2 = x^2+y^2-z-3$

$\frac{\partial\varphi_1}{\partial x} = 2x$, $\frac{\partial\varphi_1}{\partial y} = 2y$, $\frac{\partial\varphi}{\partial z} = 2z$, $\frac{\partial\varphi_2}{\partial x} = 2x$,

$\frac{\partial\varphi_2}{\partial y} = 2y$

$\frac{\partial\varphi_2}{\partial z} = -1$

$$\nabla\varphi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\varphi_1(2,-1,2) = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\nabla\varphi_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\varphi_2(2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

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$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16+4+16}\ \sqrt{16+4+1}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

(2) Find a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.

Solution: Let $\phi_1 = ax^2 - byz - (a+2)x$

$\phi_2 = 4x^2y + z^3 - 4$

$\nabla\phi_1 = (2xa - a - 2)\vec{i} - bz\vec{j} - by\vec{k}$

$(\nabla\phi_1)_{(1,-1,2)} = (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$

$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$

$(\nabla\phi_2)_{(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$

Since the surfaces cut orthogonally, $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$$\left((a-2)\vec{i} - 2b\vec{j} + b\vec{k}\right) \cdot \left(-8\vec{i} + 4\vec{j} + 12\vec{k}\right) = 0$$

$$-(a-2)8 - 8b + 12b = 0$$

$$-8a + 16 - 8b + 12b = 0$$

$$-8a + 4b = -16$$

$$2a - b = 4 \longrightarrow (1)$$

Since the point $(1, -1, 2)$ lies on ϕ_1

$$\cancel{a} + 2b - \cancel{a} - 2 = 0$$

$$2b = 2$$

$$\boxed{b = 1} \longrightarrow (2)$$

Using (2) in (1), we get

$$2a - 1 = 4$$

$$2a = 4 + 1$$

$$\boxed{a = 5/2}$$

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Note:

Unit tangent vector $= \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$

(1) Find a unit tangent vector to the following surfaces at the specified points $x = t^2+1$, $y = 4t-3$, $z = 2t^2-6t$ at $t=2$.

Sol: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2-6t)\vec{k}$

$\dfrac{d\vec{r}}{dt} = 2t\,\vec{i} + 4\vec{j} + (4t-6)\vec{k}$

$\left[\dfrac{d\vec{r}}{dt}\right]_{t=2} = 4\vec{i} + 4\vec{j} + 2\vec{k}$

$\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{16+16+4} = \sqrt{36} = 6$

Unit tangent vector $= \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$

$= \dfrac{4\vec{i} + 4\vec{j} + 2\vec{k}}{6}$

$= \dfrac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$ //.

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Note:

Normal derivative $= |\nabla \varphi|$.

(1) Find the normal derivative of $\varphi = xy + yz + zx$ at $(-1,1,1)$.

Sol.

Given $\varphi = xy + yz + zx$

$$\nabla \varphi = \sum \vec{i} \frac{\partial}{\partial x}(xy+yz+zx)$$

$$= \sum \vec{i}(y+z)$$

$$= (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$$

$$\nabla \varphi_{(-1,1,1)} = 2\vec{i} + 0\vec{j} + 0\vec{k}$$
$$= 2\vec{i}$$

Normal derivate is $|\nabla \varphi| = \sqrt{4} = 2$.

Note: The vector equation of the tangent plane and normal line to the surface

(i) Equation of the tangent plane is $(\vec{r}-\vec{a})\cdot \nabla\varphi = 0$

(ii) Equation of the normal line is $(\vec{r}-\vec{a}) \times \nabla\varphi = 0$

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(1) Find the equation of the tangent plane and normal line to the surface $xyz = 4$ at the point $\vec{i}+2\vec{j}+2\vec{k}$.

<u>Sol:</u> Given $\varphi = xyz-4$

$$\nabla\varphi = yz\vec{i}+xz\vec{j}+xy\vec{k}$$

$$\nabla\varphi_{(1,2,2)} = 4\vec{i}+2\vec{j}+2\vec{k}$$

Equation of the tangent plane at the point $\vec{a} = \vec{i}+2\vec{j}+2\vec{k}$ is $(\vec{r}-\vec{a})\cdot\nabla\varphi = 0$

$$\left[(x\vec{i}+y\vec{j}+z\vec{k})-(\vec{i}+2\vec{j}+2\vec{k})\right]\cdot(4\vec{i}+2\vec{j}+2\vec{k})=0$$

$$\left((x-1)\vec{i}+(y-2)\vec{j}+(z-2)\vec{k}\right)\cdot(4\vec{i}+2\vec{j}+2\vec{k})=0$$

$$(x-1)4+(y-2)2+(z-2)2=0$$

$$4x-4+2y-4+2z-4=0$$

$$4x+2y+2z-12=0$$

$$2x+y+z-6=0$$

$$2x+y+z=6$$

The equation of the normal line is

$$(\vec{r}-\vec{a})\times\nabla\varphi=0$$

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$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = 0$$

$$\vec{i}\left[2(y-2) - 2(z-2)\right] - \vec{j}\left[2(x-1) - 4(z-2)\right] + \vec{k}\left[2(x-1) - 4(y-2)\right] = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Equating $\vec{i}, \vec{j}, \vec{k}$ components on both sides,

$2(y-2) - 2(z-2) = 0$, $2(x-1) - 4(z-2) = 0$,

$2(x-1) - 4(y-2) = 0$

$\cancel{2}(y-2) = \cancel{2}(z-2)$, $\cancel{2}(x-1) = \cancel{4}^{2}(z-2)$, $\cancel{2}(x-1) = \cancel{4}^{2}(y-2)$

ie $(y-2) = (z-2)$, $(x-1) = 2(z-2)$, $(x-1) = 2(y-2)$

ie $(y-2) = (z-2)$, $\dfrac{(x-1)}{2} = (z-2)$, $\dfrac{(x-1)}{2} = (y-2)$

ie $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-2}{1}$ which is the

required equation of the normal line.

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Note:

Property of Dot product:

1) $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$

2) $\vec{i}.\vec{j} = \vec{j}.\vec{i} = \vec{j}.\vec{k} = \vec{k}.\vec{j} = \vec{k}.\vec{i} = \vec{i}.\vec{k} = 0$

property of cross product:

1) $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

2) $\vec{j} \times \vec{i} = -\vec{k}$, $\vec{k} \times \vec{j} = -\vec{i}$, $\vec{i} \times \vec{k} = -\vec{j}$

3) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$.

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