Vector Integration :

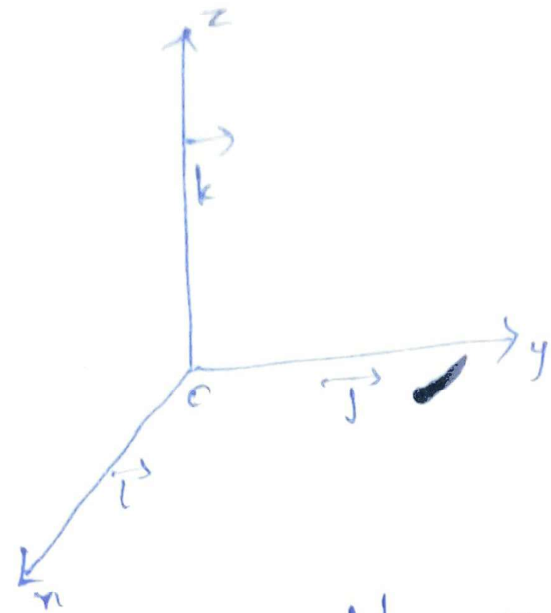
Line integral : (path integral)

An integral which is to be evaluated along a curve C is called a line integral.

Let $\vec{F}(x,y,z)$ be a continuous vector function defined at each point of the curve C where C is $\vec{r} = \vec{r}(t)$.

1. The line integral of $\vec{F}$ along C is defined as $\int_C \vec{F} \cdot d\vec{r}$

Note:



Along ox

$d\vec{r} = \vec{i}\,dx$

Along xoy

$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$

$\hat{n}\,ds = \vec{k}\,dx\,dy$

Along oy

$d\vec{r} = \vec{j}\,dy$

Along yoz

$d\vec{r} = \vec{j}\,dy + \vec{k}\,dz$

$\hat{n}\,ds = \vec{i}\,dy\,dz$

Along oz

$d\vec{r} = \vec{k}\,dz$

Along xoz

$d\vec{r} = \vec{i}\,dx + \vec{k}\,dz$

$\hat{n}\,ds = \vec{j}\,dx\,dz$.

2. Work done by the force =

$$\int_C \vec{F} \cdot d\vec{r}$$

3. $\int_C \vec{F} \cdot d\vec{r} = 0$ means $\vec{F}$ is a Conservative vector field also $\vec{F}$ is independent path of the integration.

(1) If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x=t$, $y=t^2$, $z=t^3$.

Sol:

The end points are $(0,0,0)$ and $(1,1,1)$.

These points Correspond to $t=0$ and $t=1$.

$\therefore dx = dt \quad , \quad dy = 2t\,dt ,$

$dz = 3t^2\,dt$

$$\int_c \vec{F} \cdot d\vec{r} = \int_c (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

$$= \int_0^1 (3t^2 + 6t^2)\,dt - 14t^5(2t\,dt) + 20t^7(3t^2)\,dt$$

$$= \int_0^1 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= (3 - 4 + 6 - 0)$$

$$= 5$$

(2) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ from $t = 0$ to $t = 1$ along the curve $x = 2t^2$, $y = t$, $z = 4t^3$.

Sol:

Work done $= \int_C \vec{F} \cdot d\vec{r}$

$\vec{F} = 3x^2 \vec{i} + (2xz - y)\vec{j} - z\vec{k}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$x = 2t^2$	$y = t$	$z = 4t^3$
$dx = 4t\,dt$	$dy = dt$	$dz = 12t^2\,dt$

$\vec{F} \cdot d\vec{r} = 48t^5 dt + (16t^5 - t)\,dt - 48t^5 dt$

$\vec{F} \cdot d\vec{r} = (16t^5 - t)\,dt$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (16t^5 - t)\,dt$$

$$= \left(16\,t^6/6 - t^2/2\right)_0^1$$

$$= 8/3 - 1/2$$

$$= \frac{16-3}{6}$$

$$= 13/6$$

(3) If $\vec{F} = x^2\vec{i} + y^3\vec{j}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve 'c' is the xy plane $y = x^2$ from the point $(0,0)$ to $(1,1)$.

$y = x^2$

$dy = 2x\,dx$

Sol:

$$\vec{F} = x^2\vec{i} + y^3\vec{j}$$

$$\vec{F} = x^2\vec{i} + x^6\vec{j}$$

$$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$$

$$= \vec{i}\,dx + \vec{j}\,2x\,dx$$

$$\vec{F}\cdot d\vec{r} = x^2\,dx + x^6(2x\,dx)$$

$$= x^2\,dx + 2x^7\,dx$$

$$\vec{F}\cdot d\vec{r} = (x^2 + 2x^7)\,dx$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (x^2 + 2x^7)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{\cancel{2}x^8}{\cancel{8}_4}\right)_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

(4) If $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight line from $A(0,0,0)$ to $B(2,1,3)$.

Sol:

$$\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

Straight line formula

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\frac{x - 0}{0 - 2} = \frac{y - 0}{0 - 1} = \frac{z - 0}{0 - 3}$$

$$\frac{x}{-2} = \frac{y}{-1} = \frac{z}{-3}$$

$$\frac{x}{2} = \frac{y}{1} = z/3 = t \text{ (say)}$$

$x = 2t$, $y = t$, $z = 3t$

$dx = 2dt$, $dy = dt$, $dz = 3dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 3(2t)^2\,2dt + (12t^2-t)dt + 9t\,dt$$

$$= \left(\cancel{24}^{8}\frac{t^3}{\cancel{3}} + \cancel{12}^{4}\frac{t^3}{\cancel{3}} - \frac{t^2}{2} + \frac{9t^2}{2}\right)_0^1$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2}$$

$$= 12 + \frac{\cancel{8}^{4}}{\cancel{2}}$$

$$= 16$$

(5) If $\vec{F} = x\vec{j} - y\vec{i}$ find $\int_C \vec{F}\cdot d\vec{r}$ along the arc of the circle $x^2+y^2=1$ from $(1,0)$ to $(0,1)$.

Sol: Given $\vec{F} = x\vec{j} - y\vec{i}$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = -y\,dx + x\,dy$$

$$x^2+y^2=1$$

$$2x\,dx + 2y\,dy = 0$$

$$2x\,dx = -2y\,dy$$

$$x\,dx = -y\,dy$$

$$\vec{F}\cdot d\vec{r} = -y\,dx + x\left(-\frac{x}{y}\,dx\right)$$

$$= -y\,dx - \frac{x^2}{y}\,dx$$

$$= -\left[y + x^2/y\right]dx$$

$$= -\left[\frac{y^2+x^2}{y}\right]dx$$

$$= -1/y\,dx$$

$$= -\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^0 \frac{-1}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \left(\sin^{-1}x\right)_0^1$$

$$= \sin^{-1}1 - \sin^{-1}0$$

$$= \pi/2 - 0 = \pi/2$$

(b) Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ the curve C is the rectangle in the xy plane bounded by $x=0$, $x=a$, $y=b$, $y=0$.

Solution:

Given $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

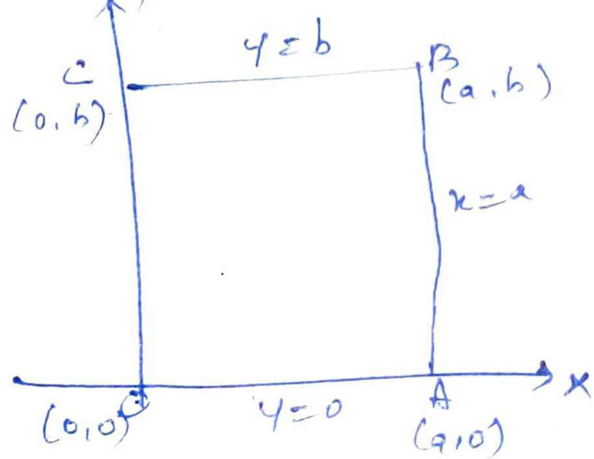
$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy$

C is the rectangle OABC and

C consists of four different paths

viz.,



OA $(y=0)$

AB $(x=a)$

BC $(y=b)$

CO $(x=0)$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA, $y=0$, $dy=0$

AB, $x=a$, $dx=0$

BC, $y=b$, $dy=0$

CO, $x=0$, $dx=0$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} x^2\,dx + \int_{AB} -2ay\,dy + \int_{BC} (x^2+b^2)\,dx + \int_{CO} 0$$

$$= \int_0^a x^2\,dx - 2a\int_0^b y\,dy + \int_a^0 (x^2+b^2)\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^a - 2a\left(\frac{y^2}{2}\right)_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^0$$

$$= \frac{a^3}{3} - \frac{2ab^2}{2} + \left(0 - \frac{a^3}{3} - ab^2\right)$$

$$= \frac{a^3}{3} - \frac{\cancel{2}ab^2}{\cancel{2}} - \frac{a^3}{3} - ab^2$$

$$= -2ab^2$$

(7) Evaluate $\int x\,dy - y\,dx$ around the circle $x^2+y^2=1$.

Sol:

The parametric equation of the given circle is

$x = \cos t \qquad y = \sin t$

$dx = -\sin t\,dt \qquad dy = \cos t\,dt$

Around the circle, t varies from 0 to 2π.

$$\therefore \int (x\,dy - y\,dx) = \int_0^{2\pi} \cos t \cdot \cos t\,dt - \sin t(-\sin t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt$$

$$= [t]_0^{2\pi}$$

$$= 2\pi$$

Condition for $\vec{F}$ to be Conservative:

Note: If $\vec{F}$ is an irrotational vector, then it is conservative

1. If $\int_C \vec{F} \cdot d\vec{r}$ be independent of the path is that curl $\vec{F} = 0$.

2. If $\vec{F}$ is conservative then $\nabla \times \vec{F} = 0$.

(1) Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

Sol:

If $\vec{F}$ is conservative then

$\nabla \times \vec{F} = 0$

Now,
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$= 0$

$\therefore$ $\vec{F}$ is a conservative vector field.

(e) Show that the value of the integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path C where $\vec{F} = (e^x z - 2xy)\vec{i} + (1 - x^2)\vec{j} + (e^x + z)\vec{k}$.

Sol:

$\int_C \vec{F} \cdot d\vec{r}$ is independent of the path C, if curl $\vec{F} = 0$.

Now curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x z - 2xy & 1 - x^2 & e^x + z \end{vmatrix}$

$= \vec{i}(0-0) - \vec{j}(e^x - e^x) + \vec{k}(-2x + 2x) = 0$

Hence $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path C.

(14) (3) Find the circulation of $\vec{F}$ round the curve C. where $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ and C is the circle $x^2+y^2=1$, $z=0$.

Sol

Sol:

The circulation of $\vec{F}$ is $\int_C \vec{F}\cdot d\vec{r}$ where C is the closed curve.

Now $\vec{F}\cdot d\vec{r} = (y\vec{i}+z\vec{j}+x\vec{k})\cdot(dx\,\vec{i}+dy\,\vec{j}+dz\,\vec{k})$

$= y\,dx + z\,dy + x\,dz$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (y\,dx + z\,dy + x\,dz)$$

$$= \int_C y\,dx \quad \left[\because z=0,\ dz=0\right]$$

For the circle $x^2+y^2=1$,

$x=\cos\theta$, $y=\sin\theta$

$dx=-\sin\theta\, d\theta$

and θ varies from 0 to 2π.

$$= \int_0^{2\pi} \sin\theta(-\sin\theta)\, d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= -\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\int_0^{2\pi} \frac{1}{2}\, d\theta + \int_0^{2\pi} \frac{\cos 2\theta}{2}\, d\theta$$

$$= -\frac{1}{\cancel{2}}(\cancel{2}\pi) + \left(\frac{\sin 2\theta}{4}\right)_0^{2\pi}$$

$$= -\pi + 0$$

$$= -\pi$$

Hence the circulation of $\vec{F}$ is $-\pi$.