Change of order of integration.

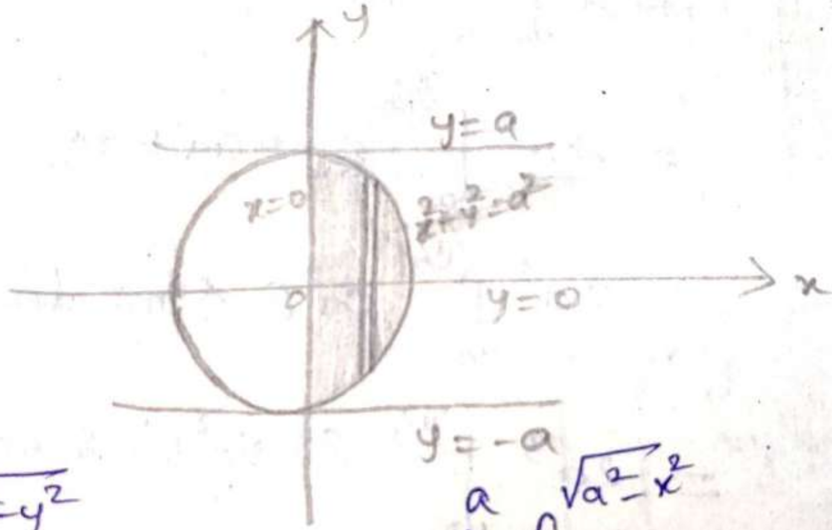
(1) change the order of integration in

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} x\,dx\,dy.$$

Sol:-

$x=0, \quad x=\sqrt{a^2-y^2} \Rightarrow x^2+y^2=a^2$

$y=-a, \quad y=a.$



$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} x\,dx\,dy = \int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x\,dy\,dx$$

$$= 2\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} x\,dy\,dx$$

$$= 2\int_0^a x\,[y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= 2\int_0^a x\sqrt{a^2-x^2}\,dx$$

put $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

when $x=0$, $\theta=0$

when $x=a$, $\theta=\pi/2$

$$= 2\int_0^{\pi/2} a\sin\theta\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$$

$$= 2a^3\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta$$

$$= 2a^3\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta$$

$$= 2a^3\int_0^{\pi/2}(1-\sin^2\theta)\sin\theta\,d\theta$$

$$= 2a^3\left[\int_0^{\pi/2}\sin\theta\,d\theta - \int_0^{\pi/2}\sin^3\theta\,d\theta\right]$$

$$= 2a^3\left[-\cos\theta\right]_0^{\pi/2} - 2/3$$

$$= 2a^3\left[1-2/3\right]$$

$$= \frac{2a^3}{3}$$

② C

hence

Sol:-

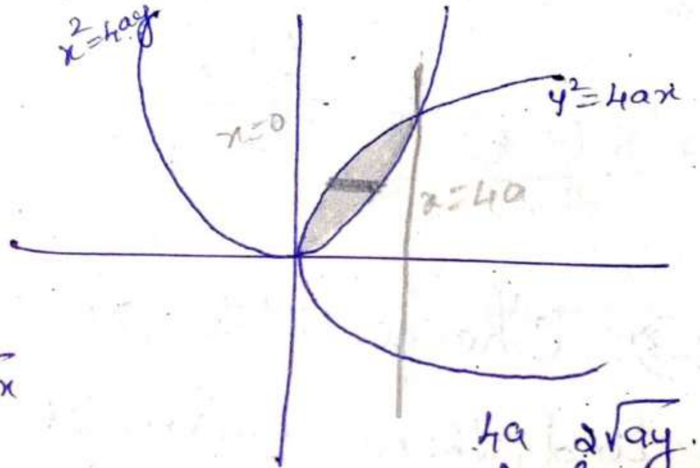
② Change the order of integration and hence evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$.

Sol:-

$$y = x^2/4a \Rightarrow x^2 = 4ay$$

$$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$$

$$x = 0, \quad x = 4a$$



$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a} y\left(\frac{x^2}{2}\right)_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left(y\cdot\frac{4ay}{2} - \frac{y}{2}\frac{y^4}{16a^2}\right)dy$$

$$= \int_0^{4a} \left(2ay^2 - \frac{1}{2}\frac{y^5}{16a^2}\right) dy$$

$$= \left(2a\, y^3/3 - \frac{1}{2}\frac{y^6}{16\times 6a^2}\right)_0^{4a}$$

$$= 2a\,\frac{64a^3}{3} - \frac{\cancel{64}^{4}\times 64a^6}{\cancel{2}\times\cancel{16}\times\cancel{6}_{3}\,a^2}$$

$$= \frac{128a^4}{3} - \frac{64}{3}a^4$$

$$= \frac{64a^4}{3} \;//.$$

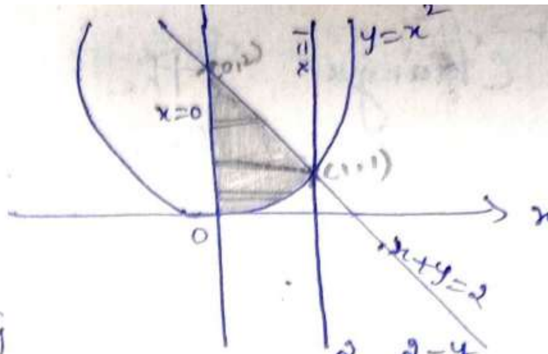
(3) Change the order of integration and then evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$
(AU - 1997 Dec)

Sol:-

$y = x^2$, $y = 2-x \Rightarrow x+y=2$

$x=0$, $x=1$

$\int_0^1 \int_{x^2}^{2-}$



$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2y}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 \left(\frac{x^2y}{2}\right)_0^{2-y} dy$$

$$= \int_0^1 y^2/2\,dy + \int_1^2 \frac{(2-y)^2y}{2}\,dy$$

$$= \left(y^3/6\right)_0^1 + \int_1^2 \frac{(4+y^2-4y)y}{2}\,dy$$

$$= 1/6 + \frac{1}{2}\left(\cancel{4}^{2}y^2/\cancel{2} + y^4/4 - 4y^3/3\right)_1^2$$

$$= 1/6 + 1/2\left(8 + \cancel{16}^{4}/\cancel{4} - 4\times 8/3 - (2 + 1/4 - 4/3)\right)$$

$$= 1/6 + 1/2\left(12 - 32/3 - 2 - 1/4 + 4/3\right)$$

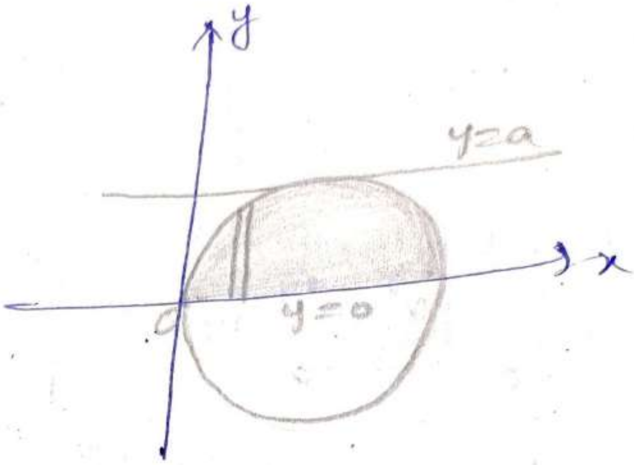
$$= 1/6 + 1/2\left(10 - \frac{28}{3} - 1/4\right)$$

$$= 1/6 + 1/2\left(\frac{120-112-3}{12}\right) = 1/6 + 1/2\,(5/12)$$

$$= 1/6 + 5/24 = \frac{4+5}{24} = 9/24 = 3/8 \;\|$$

A.U.-OCT.-2...

Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$ and hence evaluate it.



$x = a+\sqrt{a^2-y^2}$

$x-a = \sqrt{a^2-y^2}$

$(x-a)^2 = a^2-y^2$

$(x-a)^2+y^2=a^2$

Sol:- $y=0,\ y=a$

$x = a-\sqrt{a^2-y^2}$

$x-a = -\sqrt{a^2-y^2}$

$(x-a)^2 = a^2-y^2$

$(x-a)^2+y^2=a^2$

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy = \int_0^{2a} \int_0^{\sqrt{a^2-(x-a)^2}} dy\,dx$$

$$= \int_0^{2a} \left(y\right)_0^{\sqrt{a^2-(x-a)^2}} dx$$

$$= \int_0^{2a} \sqrt{a^2-(x-a)^2}\, dx$$

$$= \left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^{2a}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

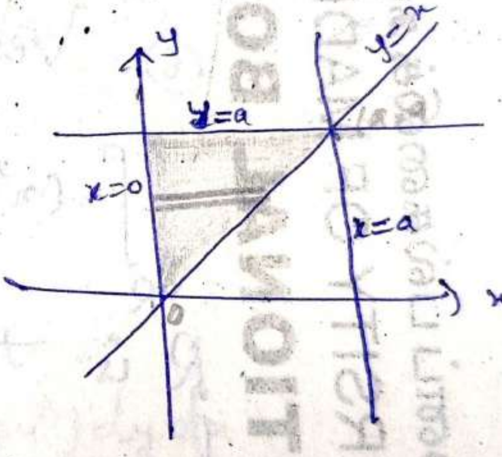
$$= 0 + \frac{a^2}{2}\frac{\pi}{2} + \frac{a^2}{2}\frac{\pi}{2}$$

$$= \frac{\pi a^2}{4} + \frac{\pi a^2}{4}$$

$$= \frac{\pi a^2}{2} \; //.$$

(5) Change the order of integration in

$$\int_0^a \int_x^a (x^2+y^2)\,dy\,dx.$$



Sol:

$y=x,\ y=a$

$x=0;\ x=a$

$$\int_0^a \int_x^a (x^2+y^2)\,dy\,dx = \int_0^a \int_0^y (x^2+y^2)\,dx\,dy.$$

$$= \int_0^a \left(\frac{x^3}{3} + xy^2\right)_0^y dy$$

$$= \int_0^a \left(\frac{y^3}{3} + y^3 \right) dy = \left(\frac{y^4}{12} + \frac{y^4}{4} \right)_0^a$$

$\iint_R$

$$= \frac{a^4}{12} + \frac{a^4}{4}$$

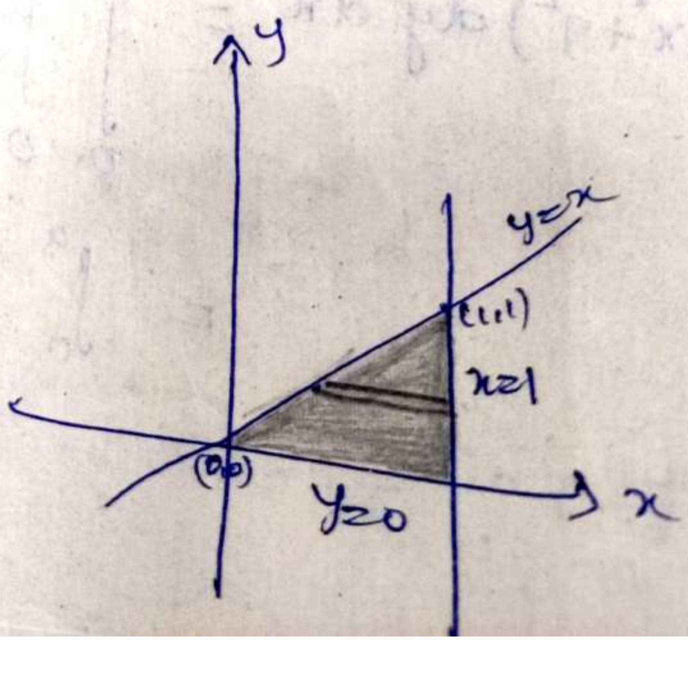
$$= \frac{a^4 + 3a^4}{12}$$

$$= \frac{4a^4}{12}$$

$$= \frac{a^4}{3} \text{ //.}$$

(6) Verify that $\iint_R (x^2+y^2)\,dy\,dx = \iint_R (x^2+y^2)\,dx\,dy$ where the region R is the triangle formed by the lines $y=0$, $x=1$ and $y=x$.

Sol:-



$$\iint_R (x^2+y^2)\,dx\,dy = \int_0^1\int_y^1 (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left(x^3/3 + y^2x\right)_y^1 dy$$

$$= \int_0^1 \left(1/3 + y^2 - y^3/3 - y^3\right) dy$$

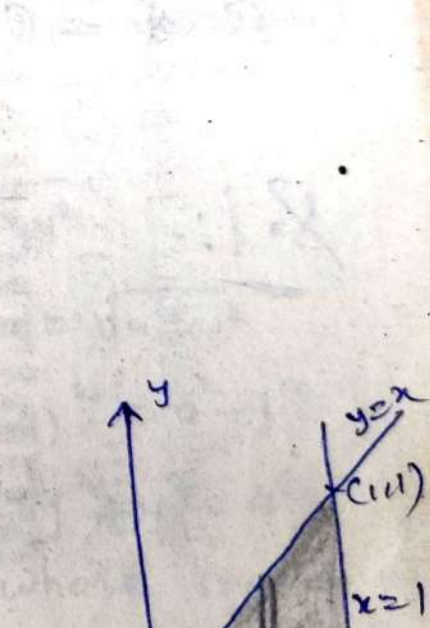
$$= \int_0^1 \left(1/3 + y^2 - 4y^3/3\right) dy$$

$$= \left(1/3\,y + y^3/3 - \cancel{4}/3\; y^4/\cancel{4}\right)_0^1$$

$$= 1/3 + 1/3 - 1/3$$

$$\iint_R (x^2+y^2)\,dx\,dy = 1/3$$

$$\iint_R (x^2+y^2)\,dy\,dx = \int_0^1\int_0^x (x^2+y^2)\,dy\,dx$$



$$= \int_0^1 \left(x^2y + y^3/3\right)_0^x dx$$

$$= \int_0^1 \left(x^3 + x^3/3\right) dx$$

$$= \int_0^1 4x^3/_3 \, dx$$

$$= \left(\frac{\cancel{4}x^4}{\cancel{4}\times 3}\right)_0^1$$

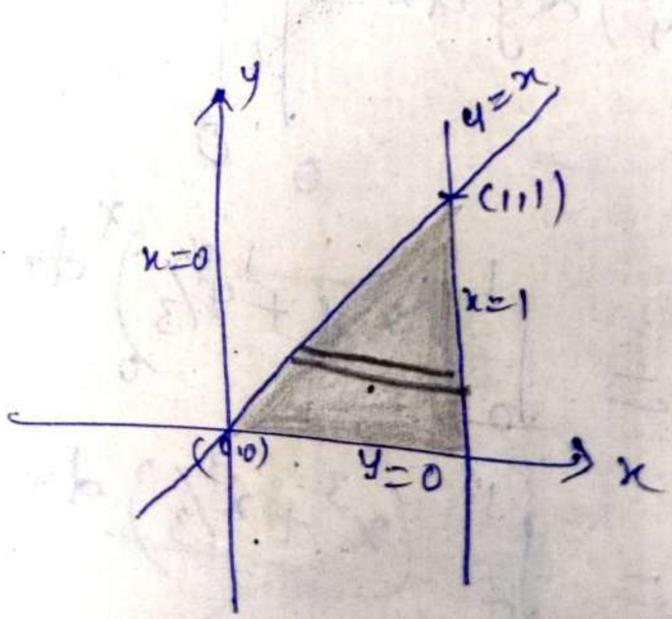
$$\iint_R (x^2+y^2)\,dy\,dx = 1/3$$

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \iint_R (x^2+y^2)\,dy\,dx.$$

Change the order of integration and evaluate $\int_0^1 \int_0^x dy\,dx$.

Sol:-

$y=0,\ y=x,\ x=0,\ x=1$



$$\int_0^1\int_0^x dy\,dx = \int_0^1\int_y^1 dx\,dy$$

$$= \int_0^1 [x]_y^1\, dy$$

$$= \int_0^1 (1-y)\,dy = \left(y - y^2/2\right)_0^1$$

$$= 1 - 1/2$$

$$= 1/2$$

Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy$ and then evaluate it. (AU oct 2001).

Sol :-

$x = a - \sqrt{a^2-y^2}$, $x = a + \sqrt{a^2-y^2}$

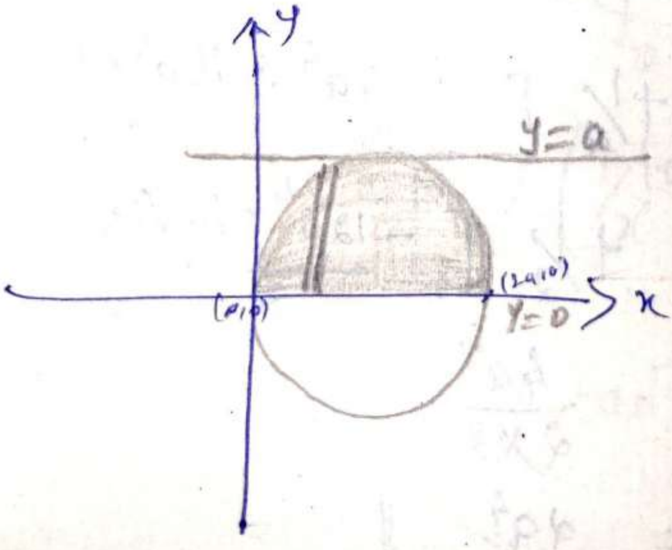
$y = 0$, $y = a$.

$x - a = \sqrt{a^2-y^2}$

$(x-a)^2 = a^2 - y^2$

$(x-a)^2 + y^2 = a^2$

whose centre is $(a, 0)$ & radius is a.



$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy = \int_0^{2a}\int_0^{\sqrt{a^2-(x-a)^2}} xy\,dy\,dx$$

$$= \int_0^{2a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-(x-a)^2}} dx$$

$$= \frac{1}{2}\int_0^{2a} x\left[a^2-(x-a)^2\right] dx$$

$$= \frac{1}{2}\int_0^{2a} x\left[\cancel{a^2}-x^2-\cancel{a^2}+2ax\right] dx$$

$$= \frac{1}{2}\int_0^{2a}\left[-x^3+2ax^2\right] dx$$

$$= \frac{1}{2}\left[-\frac{x^4}{4}+\frac{2ax^3}{3}\right]_0^{2a}$$

$$= \frac{1}{2}\left[-\frac{16a^4}{4}+2a\frac{8a^3}{3}\right]$$

$$= +\frac{1}{2}\left[-4a^4+\frac{16a^4}{3}\right]$$

$$= +\frac{1}{2}\left[\frac{-12a^4+16a^4}{3}\right]$$

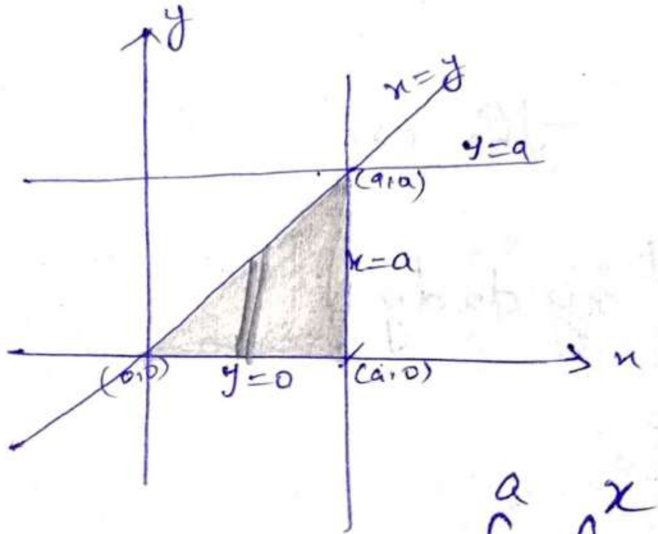
$$= \frac{4a^4}{2\times 3}$$

$$= \frac{2a^4}{3}$$

(7) Change the order of integration

$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ and then evaluate.

Sol:- $x = y$, $x = a$, $y = 0$, $y = a$.



$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_0^a x\left(\int_0^x \frac{1}{x^2+y^2}\, dy\right) dx$$

$$= \int_0^a \cancel{x}\left(\frac{1}{\cancel{x}}\tan^{-1} y/x\right)_0^x dx$$

$$= \int_0^a \left(\tan^{-1} 1 - \tan^{-1} 0\right) dx$$

$$= \int_0^a (\pi/4 - 0)\, dx$$

$$= \pi/4 \int_0^a dx$$

$$= \pi/4 \, (x)_0^a$$

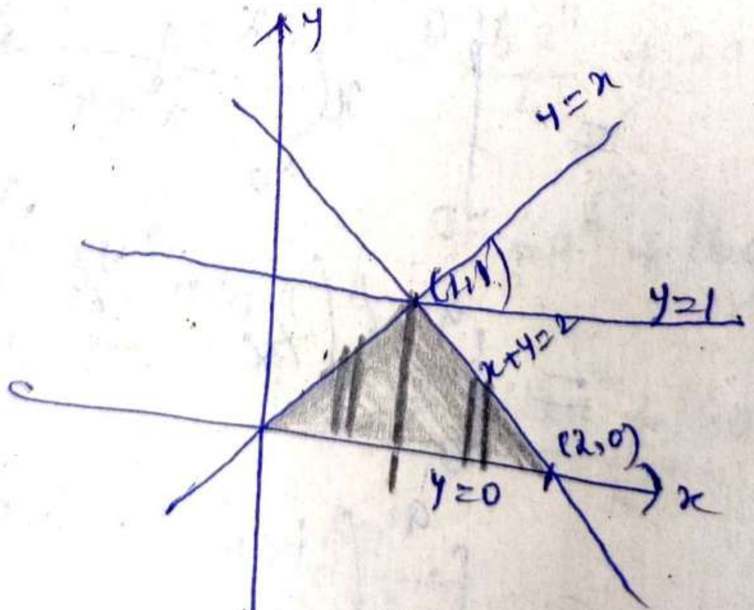
$$= \pi a/4 \, //.$$

Change the order of integration in $\int_0^1 \int_y^{2-y} xy \, dx\, dy$ and hence evaluate it. (AU Nov-2001)

Sol:-

$$x = y, \quad x = 2 - y \Rightarrow x + y = 2$$

$$y = 0, \quad y = 1$$



$$\int_0^1\int_y^{2-y} xy\,dx\,dy = \int_0^1\int_0^x xy\,dy\,dx + \int_1^2\int_0^{2-x} xy\,dy\,dx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^x dx + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \int_0^1 \frac{x^3}{2}\,dx + \int_1^2 \frac{x(2-x)^2}{2}\,dx$$

$$= \left(\frac{x^4}{8}\right)_0^1 + \int_1^2 \frac{x(4+x^2-4x)}{2}\,dx$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{2\cancel{4}x^2}{\cancel{2}} + \frac{x^4}{4} - \frac{4x^3}{3}\right]_1^2$$

$$= \frac{1}{8} + \frac{1}{2}\left[8 + \frac{\cancel{16}^{\,4}}{\cancel{4}} - 4\times\frac{8}{3} - \left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[12 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{120-112-3}{12}\right]$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{5}{12}\right]$$

$$= \frac{1}{8} + \frac{5}{24} = \frac{3+5}{24} = \frac{8}{24} = \frac{1}{3} \;//$$

7 Change the order of integration and hence evaluate $\int_0^b \int_0^{a/b\sqrt{b^2-y^2}} xy\, dx\, dy$.

Sol:-

$x=0\,,\quad x = a/b\sqrt{b^2-y^2}$

$xb = a\sqrt{b^2-y^2}$

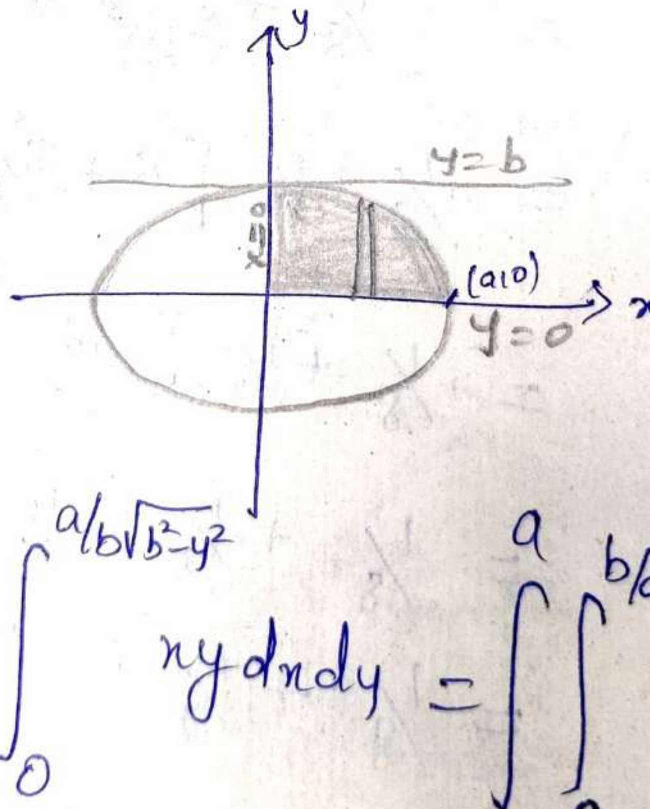
$y=0,\ y=b \qquad x^2b^2 = a^2(b^2-y^2)$

$x^2b^2 = a^2b^2 - a^2y^2$

$x^2b^2 + a^2y^2 = a^2b^2$

$\div\, a^2b^2$

$x^2/a^2 + y^2/b^2 = 1$



$$\int_0^b \int_0^{a/b\sqrt{b^2-y^2}} xy\, dx\, dy = \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= \int_0^a x \left(\frac{y^2}{2}\right)_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x}{2}\, b^2/a^2 \,(a^2-x^2)\, dx.$$

$$= \frac{b^2}{2a^2} \int_0^a (xa^2 - x^3)\, dx$$

$$= b^2/2a^2 \left[\frac{x^2a^2}{2} - x^4/4\right]_0^a$$

$$= b^2/2a^2 \left[a^4/2 - a^4/4\right]$$

$$= b^2/2a^2 \left[\frac{2a^4-a^4}{4}\right]$$

$$= b^2/2\cancel{a^2}\ \cancel{a^4}^{a^2}/4$$

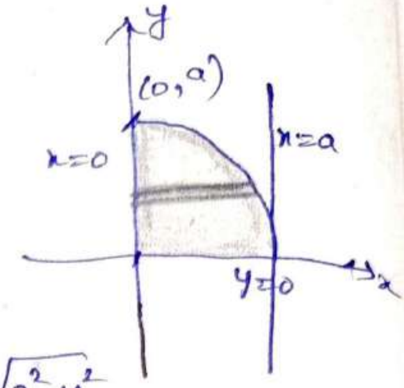
$$= \frac{a^2b^2}{8} \;//.$$

Change the order of integration and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$.

Sol:- $y=0,\ y=\sqrt{a^2-x^2}$

$y^2=a^2-x^2$

$x^2+y^2=a^2$



$x=0,\ x=a.$

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\,dx=\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy$$

$$=\int_0^a\left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2}+\frac{a^2-y^2}{2}\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}}dy$$

$$\left[\because\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=\int_0^a\left[\left(0+\frac{a^2-y^2}{2}\frac{\pi}{2}\right)-(0+0)\right]dy$$

$$=\int_0^a\frac{a^2-y^2}{2}\frac{\pi}{2}\,dy$$

$$=\frac{\pi}{4}\int_0^a(a^2-y^2)\,dy$$

$$= \frac{\pi}{4}\left(a^2y - \frac{y^3}{3}\right)_0^a$$

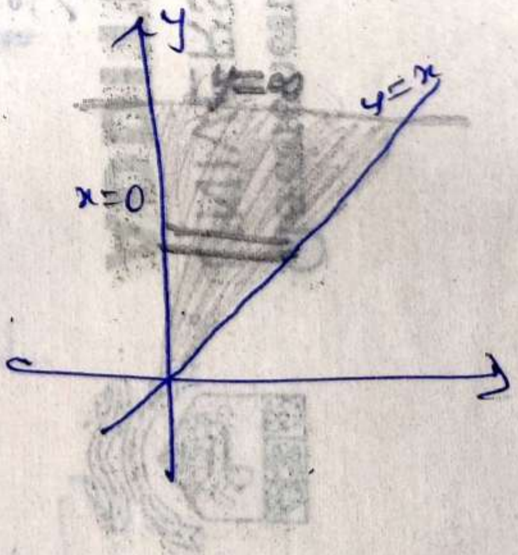
$$= \frac{\pi}{4}\left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{\pi}{4}\left(\frac{2a^3}{3}\right)$$

$$= \frac{\pi a^3}{6} \,//.$$

Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ and hence evaluate it.

Sol:- $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = \int\int \frac{e^{-y}}{y}\,dx\,dy$



$y = x, \; y = \infty$

$x = 0, \; x = \infty$

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (x)_0^y \, dy$$

$$= \int_0^\infty \frac{e^{-y}}{\cancel{y}} \cancel{y} \, dy$$

$$= \left(-e^{-y}\right)_0^\infty$$

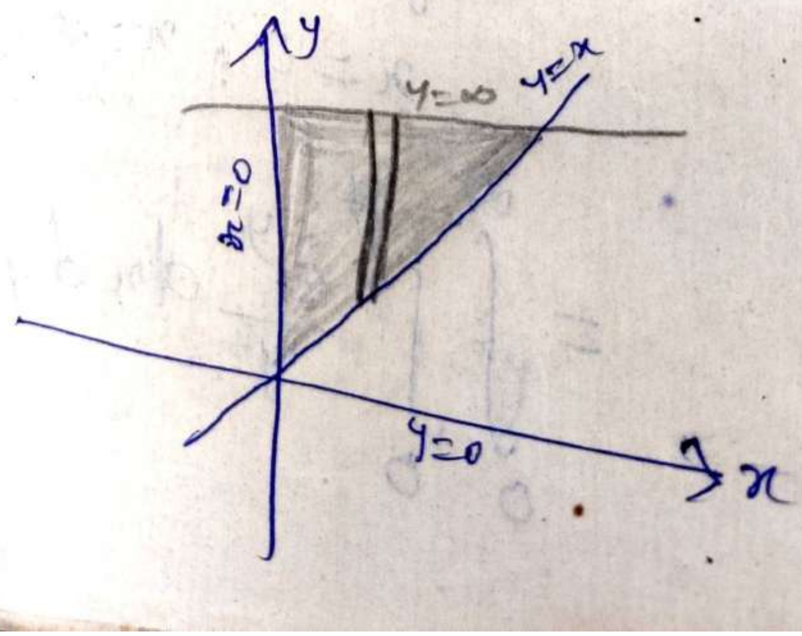
$$= 1 \; //.$$

Change the order of integration in $\int_0^\infty \int_0^y y e^{-y^2/x} \, dx\, dy$ and hence evaluate it. (Au)

Sol:-

$x = 0, \; x = y$

$y = 0, \; y = \infty$



$$\int_0^{\infty}\int_0^{y} y e^{-y^2/x}\,dx\,dy = \int_0^{\infty}\left(\int_x^{\infty} y e^{-y^2/x}\,dy\right)dx$$

put $y^2/x = t$

$$\frac{2y\,dy}{x} = dt$$

$$y\,dy = \frac{x}{2}\,dt$$

when $y = x$, $t = x$

$y = \infty$, $t = \infty$

$$= \int_0^{\infty}\left(\int_x^{\infty} \frac{x}{2}\, e^{-t}\,dt\right)dx$$

$$= \int_0^{\infty} \frac{x}{2}\left(\int_x^{\infty} e^{-t}\,dt\right)dx$$

$$= \int_0^{\infty} \frac{x}{2}\left(-e^{-t}\right)_x^{\infty} dx$$

$$= \int_0^{\infty} \frac{x e^{-x}}{2}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} x e^{-x}\,dx$$

$$= \frac{1}{2}\left[\left(-x e^{-x}\right)_0^{\infty} + \int_0^{\infty} e^{-x}\,dx\right]$$

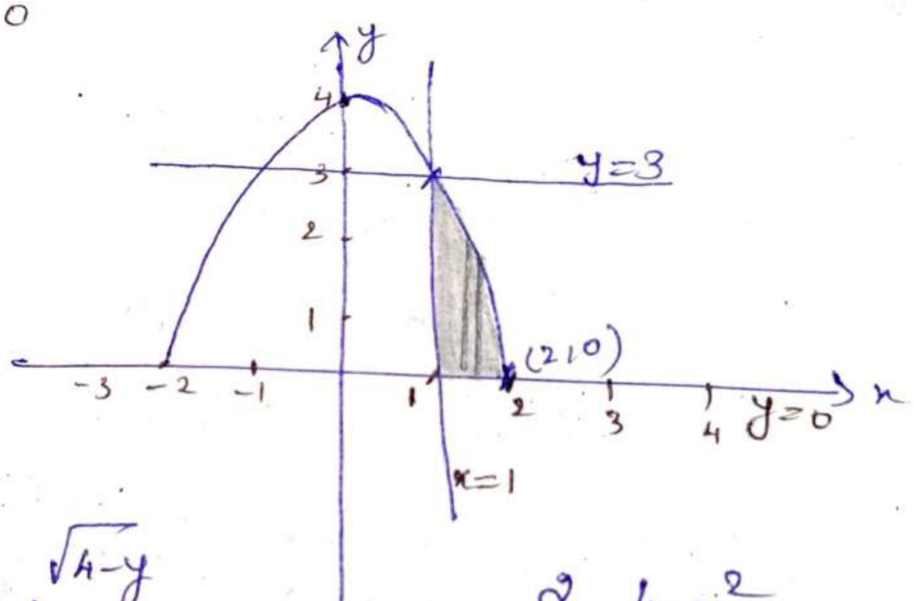
$$= \frac{1}{2}\left[-e^{-x}\right]_0^{\infty} = \frac{1}{2}(1) = \frac{1}{2}$$

Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$ by changing the order of integration.

Sol:- $x=1$, $x=\sqrt{4-y}$

$y=0$, $y=3$ $\quad x^2 = 4-y$

x	0	1	-1	2	-2
y	4	3	3	0	0



$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy = \int_1^2 \int_0^{4-x^2} (x+y)\, dy \cdot dx$$

$$= \int_1^2 \left(xy + \frac{y^2}{2}\right)_0^{4-x^2} dx$$

$$= \int_1^2 x(4-x^2) + \frac{(4-x^2)^2}{2}\, dx$$

$$= \int_1^2 \left(4x - x^3 + \frac{16+x^4-8x^2}{2}\right) dx$$

$$= \int_1^2 \left(4x - x^3 + 8 + x^4/2 - 4x^2\right) dx$$

$$= \left(2\cancel{4}x^2/\cancel{2} - x^4/4 + 8x + x^5/10 - 4x^3/3\right)_1^2$$

$$= 2\times 4 - 16/4 + 16 + 32/10 - 4\times 8/3 - 2 + 1/4 - 8 - 1/10 + 4/3$$

$$= \cancel{8} - 4 + 16 + 16/5 - 32/3 - 2 + 1/4 - \cancel{8} - 1/10 + 4/3$$

$$= 10 + 16/5 - 28/3 + 1/4 - 1/10$$

$$= \frac{50+16}{5} + \frac{-112+3}{12} - 1/10$$

$$= 66/5 - \frac{109}{12} - 1/10$$

$$= \frac{792-545-6}{60}$$

$$= 241/60$$