Surface Integral :

An integral which is to be evaluated over a surface is called a surface integral.

The surface integral of $\vec{F}$ over S is defined by

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \vec{F}\cdot\vec{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

where R_1 is the projection of S on xy plane

Similarly $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_2} \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

where R_2 is the projection of S on yz plane

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_3} \vec{F}\cdot\hat{n}\,\frac{dz\,dx}{|\hat{n}\cdot\vec{j}|}$$

where R_3 is the projection of S on the zx plane.

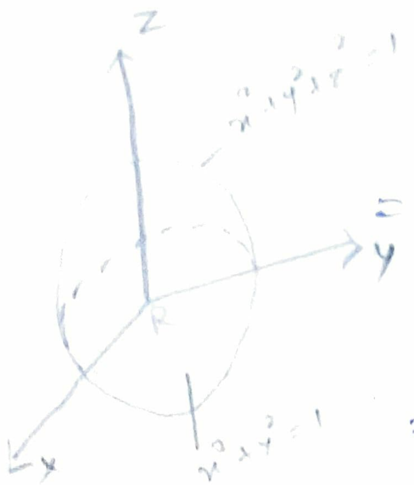
(1) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is that part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Sol: Given $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

Let $\phi = x^2+y^2+z^2-1$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}}$$



$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{1} \quad [\because x^2+y^2+z^2=1]$$

$$= x\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{F}\cdot\hat{n} = xyz + xyz + xyz = 3xyz$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

where R is the projection of S on the xy plane.

Clearly the projection R is bounded by the lines x axis $(y=0)$, y axis $(x=0)$ circle $x^2+y^2=1$, $z=0$.

and the

$$|\vec{n}\cdot\vec{k}| = (x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{k}$$

$$= z$$

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_R 3xyz\,\frac{dx\,dy}{z}$$

$$= 3\iint_R xy\,dx\,dy$$

$$= 3\int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$

$$= 3\int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}}$$

$$= 3\int_0^1 \frac{x(1-x^2)}{2}\,dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= \frac{3}{2}\left(\frac{1}{4}\right)$$

$$= \frac{3}{8} \text{ sq. units.}$$

Evaluate $\iint_S \vec{F} \cdot \vec{n}\, ds$ where $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$ and S is the surface of the cylinder $x^2+y^2=1$ included in the first octant between the planes $z=0$ and $z=2$.

Sol: Given $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$

$$\phi = x^2+y^2-1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2}$$

$$= 2\sqrt{x^2+y^2} = 2\sqrt{1} = 2$$

The unit normal $\vec{n}$ to the surface $= \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2x\vec{i} + 2y\vec{j}}{2}$$

$$= x\vec{i} + y\vec{j}$$

$$\vec{F} \cdot \vec{n} = (z\vec{i} + x\vec{j} - y^2 z\vec{k}) \cdot (x\vec{i} + y\vec{j})$$

$$= xz + xy$$

Now $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

where R is the projection of S on yz plane. (ie $x=0$)

$$= \iint_R (xz+xy)\frac{dy\,dz}{x}$$

$$= \iint_R (z+y)\,dy\,dz$$

$$= \int_0^2 \int_0^1 (z+y)\,dy\,dz$$

$$= \int_0^2 \left[zy + y^2/2\right]_0^1 dz$$

$$= \int_0^2 (z+1/2)\,dz$$

$$= \left(z^2/2 + 1/2\, z\right)_0^2$$

$= 4/2 + 2/2$

$= 3$ //

(3) Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ as S is the part of the plane $2x + 3y + 6z = 12$ which is in the first octant.

<u>Sol:</u> Given $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$

$\phi = 2x + 3y + 6z - 12$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \vec{i}\,\partial\phi/\partial x + \vec{j}\,\partial\phi/\partial y + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$= 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$$|\nabla\phi| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}$$

$$\vec{F}\cdot\vec{n} = (18z\vec{i} - 12\vec{j} + 3y\vec{k})\cdot\left(\frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}\right)$$

$$= \frac{36z - 36 + 18y}{7} \qquad \left[z = \frac{12-2x-3y}{6}\right]$$

$$= \frac{1}{7}\left[36\left(\frac{12-2x-3y}{6}\right) - 36 + 18y\right]$$

$$= \frac{1}{7}\left[6(12-2x-3y) - 36 + 18y\right]$$

$$= \frac{1}{7}\left[36 - 12x\right]$$

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_R \frac{36-12x}{7}\,\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

$$\vec{n}\cdot\vec{k} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}\cdot\vec{k} = 6/7$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,ds = \iint_R \frac{36-12x}{7}\, 7/6\, dx\, dy$$

$$= \iint (6-2x)\, dx\, dy$$

Given plane is $2x+3y+6z=12$ in

Xoy and $z=0$.

$$2x+3y=12$$

$$3y = 12-2x$$

$$y = \frac{12-2x}{3}$$

$\therefore$ y varies from 0 to $\frac{12-2x}{3}$ in

X plane $y=0$, $z=0$,

$$\therefore 2x = 12$$

$$x=6$$

$\therefore$ x varies from 0 to 6.

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\, dy\, dx$$

$$= \int_0^6 \left(6y - 2xy\right)_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \left[6\left(\frac{12-2x}{3}\right) - 2x\left(\frac{12-2x}{3}\right)\right] dx$$

$$= \int_0^6 \left(24 - 4x - 8x + \frac{4x^2}{3}\right) dx$$

$$= \int_0^6 \left(24 - 12x + \frac{4x^2}{3}\right) dx$$

$$= \left(24x - 6x^2 + \frac{4x^3}{9}\right)_0^6$$

$$= 24 \,//$$

Volume Integrals:

An integral which is to be evaluated over a volume bounded by a surface is called a volume integral.

The volume integral of $F(x,y,z)$ over a region enclosing a volume V is given by $\iiint_V F(x,y,z)\,dv$ or

$\iiint_V F(x,y,z)\,dx\,dy\,dz$.