Volume Integrals:

An integral which is to be evaluated over a volume bounded by a surface is called a volume integral.

The volume integral of $F(x, y, z)$ over a region enclosing a volume V is given by $\iiint_V F(x, y, z)\, dv$ or $\iiint_V F(x, y, z)\, dx\, dy\, dz$.

If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla \times \vec{F}\, dV$ where V is the region bounded by $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

Sol:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0) + \vec{j}(-3+4) + \vec{k}(-2y-0)$$

$$= \vec{j} - 2y\vec{k}$$

$$\therefore \iiint_V \nabla \times \vec{F}\, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k})\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left(z\vec{j} - 2yz\vec{k} \right)_0^{4-2x-2y} dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[(4-2x-2y)\vec{j} - 2y(4-2x-2y)\vec{k}\right] dy\, dx$$

$$= \int_0^2 \left[\left(4y-2xy-y^2\right)\vec{j} - \left(4y^2-2xy^2-\frac{4y^3}{3}\right)\vec{k}\right]_0^{2-x} dx$$

$$= \int_0^2 4(2-x) - 2x(2-x) - (2-x)^2\Big]\vec{j} -$$

$$\left[4(2-x)^2 - 2x(2-x)^2 - \tfrac{4}{3}(2-x)^3\right]\vec{k}\Big]\, dx$$

$$= \int_0^2 \Big[(8-4x-4x+2x^2-4-x^2+4x)\vec{j} -$$

$$(16-16x+4x^2-8x+8x^2-2x^3)$$

$$- \tfrac{4}{3}(8-12x+6x^2-x^3)\vec{k}\Big] dx$$

$$= \int_0^2 (4-4x+x^2)\vec{j} - \vec{k}/3\,(16-24x+12x^2-2x^3]\, dx$$

$$= \left[\left[4x-2x^2+x^3/3\right]\vec{j} - \vec{k}/3\left[16x-12x^2+4x^3-x^4/2\right]\right]_0^2$$

$$= \left(8 - 8 + 8/3\right)\vec{j} - \vec{k}/3\,(32 - 48 + 32 - 8)$$

$$= 8/3\,(\vec{j} - \vec{k})$$ //.

Green's Theorem:

Statement, Verification and its applications.

If $u, v, \partial u/\partial y, \partial v/\partial x$ are continuous and one-valued functions in the Region R enclosed by the Curve C, then

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$$

Corollary (1)

If $\partial v/\partial x = \partial u/\partial y$ the value of the integral $\int_C u\,dx + v\,dy$ is independent of the path of integration.

Corollary (2):

If R is a region bounded by a simply closed curve C then the area of R is given by

$\frac{1}{2} \int_C x\,dy - y\,dx$.