

Stochastic Discount Factor and Risk Neutral Measure

Ruming Liu

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1 Overview

This is a short note to prove that the asset pricing under stochastic discount factor (SDF) is consistent with the framework of risk neutral (martingale) measure pricing.

2 Stochastic Discount Factor

- In Lucas tree(1978) model, asset is priced as the present value of future payoff:

$$p = E[m \cdot X] \tag{1}$$

- The payoff $X(\omega)$ and $m(\omega)$ are both random (stochastic) variable. That's why $m(\omega)$ is called SDF or pricing kernel in asset pricing.
- In Arrow-Debreu market, $m(\omega)$ is the state price of Arrow-Debreu security which pays 1 when ω_i happens.

3 Risk Neutral Pricing

- Under the framework of stochastic calculus in financial mathematics, the asset is priced as the present value under risk neutral space discounted by risk-free rate:

$$p = E^Q[\frac{X}{1 + r_f}] \tag{2}$$

- We need to be clear that such pricing method is just mathematical tricks of Feynman-Kac theorem in Black-Scholes PDE. Its price must be exactly the same with equation (1).
- In this note, we prove they are exactly the same under both discrete and continuous probability space.

4 Discrete Probability Space

- Equation (1) can be written as,

$$\begin{aligned}
p &= E[m \cdot X] \\
&= \sum_{i=1}^N m(\omega_i) \cdot x(\omega_i) \cdot P(\omega_i) \\
&= \sum_{i=1}^N x(\omega_i) \cdot \frac{m(\omega_i)P(\omega_i)}{\sum_{i=1}^N m(\omega_i)P(\omega_i)} \cdot \sum_{i=1}^N m(\omega_i)P(\omega_i)
\end{aligned} \tag{3}$$

- It is trivial that the risk-free rate satisfies,

$$\frac{1}{1 + r_f} = E[m \cdot 1] = \sum_{i=1}^N m(\omega_i) \cdot 1 \cdot P(\omega_i) \tag{4}$$

- Denote $Q(\omega_i) := \frac{m(\omega_i)P(\omega_i)}{\sum_{i=1}^N m(\omega_i)P(\omega_i)}$ It is easy to verify that

$$\sum_{i=1}^N Q(\omega_i) = 1 \tag{5}$$

Q is a legal probability measurement.

- Such that equation (3) is rewritten as

$$\sum_{i=1}^N \frac{x(\omega_i) \cdot Q(\omega_i)}{1 + r_f} := E^Q\left[\frac{X}{1 + r_f}\right], \tag{6}$$

The asset is priced under measure Q with risk-free discount factor, economists call such measure risk neutral measure.

5 Continuous Probability Space

- Equation (1) can be rewritten as Lebesgue integral,

$$\begin{aligned}
 p &= E[m \cdot X] \\
 &= \int_{\Omega} x(\omega)m(\omega)dP(\omega) \\
 &= \int_{\Omega} \frac{x(\omega)m(\omega)}{(1+r_f) \cdot \int_{\Omega} m(\omega)dP(\omega)}dP(\omega)
 \end{aligned} \tag{7}$$

- We denote the $Z(\omega)$ as

$$Z(\omega) = \frac{m(\omega)}{\int_{\Omega} m(\omega)dP(\omega)} \tag{8}$$

- It is trivial that

$$E[Z] = \int_{\Omega} Z(\omega)dP(\omega) = \int_{\Omega} \frac{m(\omega)}{\int_{\Omega} m(\omega)dP(\omega)}dP(\omega) = 1. \tag{9}$$

It is the Random Nickodym derivative of measure transformation, the new risk measure is following,

$$\frac{dQ(\omega)}{dP(\omega)} = Z(\omega) \tag{10}$$

- Finally, (7) can be rewritten as the expectation under risk neutral measure with risk-free discount factor:

$$p = \int_{\Omega} \frac{x(\omega)}{(1+r_f)}dQ(\omega) := E^Q[\frac{X}{1+r_f}] \tag{11}$$

6 Change of Numeraire

- In the framework of interest rate structure, if the risk free rate is not available on the market, we may use other asset as the discount factor, e.g. the forward rate.
- Assume an asset is priced as

$$p_f = E[m \cdot F] = \int_{\Omega} f(\omega)m(\omega)dP(\omega) \tag{12}$$

- Then equation (7) can be rewritten as,

$$\begin{aligned}
\frac{p}{p_f} &= \int_{\Omega} \frac{x(\omega)m(\omega)}{\int_{\Omega} f(\omega)m(\omega)dP(\omega)} dP(\omega) \\
&= \int_{\Omega} \frac{x(\omega)f(\omega)m(\omega)}{\int_{\Omega} f(\omega)m(\omega)dP(\omega)} dP(\omega) \\
&= \int_{\Omega} \frac{x(\omega)}{f(\omega)} T(\omega) dP(\omega) \\
&= E^T\left[\frac{x(\omega)}{f(\omega)}\right]
\end{aligned} \tag{13}$$

where $T(\omega) = \frac{f(\omega)m(\omega)}{\int_{\Omega} f(\omega)m(\omega)dP(\omega)}$ is the forward measure.

7 Reference

- Chapter “Stochastic Discount Factor”: Björk, T., 2009. Arbitrage theory in continuous time. Oxford university press.
- <https://economics.stackexchange.com/questions/27325/does-the-lucas-1978-ass>
- <https://www.zhihu.com/question/51149348/answer/529541218>
- <https://www.zhihu.com/question/21941452/answer/1072053714>