Stochastic Discount Factor and Risk Neutral Measure

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1 Overview

This is a short note to prove that the asset pricing under stochastic discount factor (SDF) is consistent with the framework of risk neutral (martingale) measure pricing.

2 Stochastic Discount Factor

- In Lucas tree(1978) model, asset is priced as the present value of future payoff:
$$p = E[m \cdot X] \tag{1}$$
- The payoff $X(\omega)$ and $m(\omega)$ are both random (stochastic) variable. That's why $m(\omega)$ is called SDF or pricing kernel in asset pricing.
- In Arrow-Debreu market, $m(\omega)$ is the state price of Arrow-Debreu security which pays 1 when ω_i happens.

3 Risk Neutral Pricing

- Under the framework of stochastic calculus in financial mathematics, the asset is priced as the present value under risk neutral space discounted by risk-free rate:
$$p = E^Q[\frac{X}{1 + r_f}] \tag{2}$$

- We need to be clear that such pricing method is just mathematical tricks of Feynman-Kac theorem in Black-Scholes PDE. Its price must be exactly the same with equation (1).

- In this note, we prove they are exactly the same under both discrete and continuous probability space.

4 Discrete Probability Space

- Equation (1) can be written as,

$$\begin{aligned} p &= E[m \cdot X] \\ &= \sum_{i=1}^{N} m(\omega_i) \cdot x(\omega_i) \cdot P(\omega_i) \\ &= \sum_{i=1}^{N} x(\omega_i) \cdot \frac{m(\omega_i)P(\omega_i)}{\sum_{i=1}^{N} m(\omega_i)P(\omega_i)} \cdot \sum_{i=1}^{N} m(\omega_i)P(\omega_i) \end{aligned} \tag{3}$$

- It is trivial that the risk-free rate satisfies,

$$\frac{1}{1+r_f} = E[m \cdot 1] = \sum_{i=1}^{N} m(\omega_i) \cdot 1 \cdot P(\omega_i) \tag{4}$$

- Denote $Q(\omega_i) := \frac{m(\omega_i)P(\omega_i)}{\sum_{i=1}^{N} m(\omega_i)P(\omega_i)}$ It is easy to verify that

$$\sum_{i=1}^{N} Q(\omega_i) = 1 \tag{5}$$

  Q is a legal probability measurement.

- Such that equation (3) is rewritten as

$$\sum_{i=1}^{N} \frac{x(\omega_i) \cdot Q(\omega_i)}{1+r_f} := E^Q[\frac{X}{1+r_f}], \tag{6}$$

  The asset is priced under measure Q with risk-free discount factor, economists call such measure risk neutral measure.

5 Continuous Probability Space

- Equation (1) can be rewritten as Lebesgue integral,

$$\begin{aligned} p &= E[m \cdot X] \\ &= \int_\Omega x(\omega)m(\omega)dP(\omega) \\ &= \int_\Omega \frac{x(\omega)m(\omega)}{(1+r_f)\cdot\int_\Omega m(\omega)dP(\omega)}dP(\omega) \end{aligned} \tag{7}$$

- We denote the $Z(\omega)$ as

$$Z(\omega) = \frac{m(\omega)}{\int_\Omega m(\omega)dP(\omega)} \tag{8}$$

- It is trivial that

$$E[Z] = \int_\Omega Z(\omega)dP(\omega) = \int_\Omega \frac{m(\omega)}{\int_\Omega m(\omega)dP(\omega)}dP(\omega) = 1. \tag{9}$$

  It is the Random Nickodym derivative of measure transformation, the new risk measure is following,

$$\frac{dQ(\omega)}{dP(\omega)} = Z(\omega) \tag{10}$$

- Finally, (7) can be rewritten as the expectation under risk neutral measure with risk-free discount factor:

$$p = \int_\Omega \frac{x(\omega)}{(1+r_f)}dQ(\omega) := E^Q[\frac{X}{1+r_f}] \tag{11}$$

6 Reference

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