UNIVERSITY OF GALWAY

BACHELOR'S THESIS

Condensed Mathematics

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Declaration of Authorship

We, Monirul Choudhury and Katrina Watson, declare that this project titled, "Condensed Mathematics" and the work presented in it is our own, except where otherwise indicated. We further declare that:

- All sources used in this project have been properly acknowledged and cited.
- No part of this project has previously been submitted for a degree or any other qualification at this University or any other institution.
- We have acknowledged any assistance recieved in the preparation for this project.
- We have read and understood the University's QA220 Academic Integrity Policy

In this report tasks were divided as follows: The contents of Chapters 1, 2, 3 and 5 were joint efforts, researched and authored by both students, while Chapters 4 and 6 were researched and authored by Monirul Choudhury.

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Introduction

Condensed mathematics is a recent theory developed by Dustin Clausen and Peter Scholze to address the following motivating question: "how should one do algebra when algebraic structures carry a topology?" [Sch19b, p. 6] The perspective of condensed mathematics is that when studying these algebro-topological strutures using algebraic techniques, the notion of a topology is "incorrect". For instance, in the setting of abelian groups, it is a fact that a group homomorphism is an isomorphism precisely when it has a trivial kernel and cokernel. If we now endow these abelian groups with a topology compatible with the group structure, the analogues statement in this setting no longer holds true. In particular, we say that the category formed by these topological abelian groups is not an "abelian category". This last point has significant implications for the study of topological abelian groups because it means that we cannot use techniques from homological algebra to study them. Clausen and Scholze aimed to resolve this issue by introducing the idea of a *condensed set* (and related to it: a *condensed abelian group*). This text is an elementary exposition of this theory focusing particularly on the content of [Ásg21, Sections 1.2, 2.1 & 2.2].

In this text we dedicate our time to building up the theory in its own right, neglecting slightly the discussion of its applications in other areas of mathematics. However, we mention here that this theory can be applied to allow tools in algebraic geometry to be used in functional analysis, analytic geometry [Sch19a] and complex geometry [Sch22]. In the latter case, Clausen and Scholze have used techniques in condensed mathematics to reprove a number of classical results such as the Riemann-Roch Theorem and Serre Duality [Mir95, Section VI.3]. Notably, the proofs of these results were analysis-free [Sch22, p. 6].

The structure of the text is organized as follows. In Chapter 1 and 2, we discuss category theory while in Chapter 3 and 4, we discuss the topology required for condensed mathematics. Finally, in Chapter 5, we define a condensed set and provide some theory on the subject. In Chapter 6, we define a condensed abelian group. Notably, in this chapter, we prove a key theorem of this text: the category of condensed abelian groups forms an abelian category. Throughout the course of Chapters 1-4, the material is presented with the overarching goal of developing the theory in the latter two chapters.

The primary references used in this text are as follows. For Chapter 1 and 2, we use [Lei14] and occasionally [Mac71]. For Chapter 3 and 4, we use [Ásg21], [Stacks], [Mun75] and [Bro06]. For Chapter 5 and 6, we primarily use [Ásg21], [Sch19b], [Mac71] and [Stacks]. Condensed mathematics is very much an ongoing recent effort and so our primary references on the subject are lecture notes ([Sch19b], [Sch19a] and [Sch22]) and a Master's thesis ([Ásg21]).

Chapter 1

Basic Category Theory

The theory of condensed mathematics is formulated in the language of category theory. The present chapter is thus an introduction to this theory. In Sections 1.1, 1.2 and 1.3 we define the basic notions of a category, a functor and a natural transformation respectively. Finally, we take a look at the Yoneda embedding in Section 1.4. Our primary reference for this chapter is [Lei14]: Chapter 1 covers Sections 1.1, 1.2 and 1.3 and Sections 4.1 and 4.2 cover Section 1.4.

1.1 Categories

Up till the mid-20th century, broadly speaking, mathematical theory was defined in terms of the internal structure of sets. One could first create a "mathematical object" by endowing a set with structure and then refer to the maps between similar such objects that "preserve this structure". For instance, in the theory of linear algebra, we have vector spaces and linear maps. Instead of the internal structure of these objects, in category theory, we study the mappings between these objects.

Definition 1.1.1 (Category)**.** A *category* $\mathcal{C}$ consists of

1. A class[1] ob($\mathcal{C}$) of *objects*. We often write $A \in \mathcal{C}$ if A is an object in $\mathcal{C}$.

2. For each $A, B \in \mathcal{C}$, a set $\mathrm{Hom}(A, B)$ or $\mathrm{Hom}_{\mathcal{C}}(A, B)$[2] of *morphisms* (or maps / arrows). If f is a morphism from objects A to B in $\mathcal{C}$, we often denote this by $f \in \mathrm{Hom}(A, B)$ or $f : A \to B$ or $A \xrightarrow{f} B$. We additionally insist that $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A', B')$ are disjoint unless $A = A'$ and $B = B'$.

3. For each $A, B, C \in \mathcal{C}$, a *composition* map:

$$\begin{aligned} \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) &\to \mathrm{Hom}(A, C) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

satisfying the following axioms:

1. *Associativity*: for each $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

2. *Identity*: for each $A \in \mathcal{C}$, an identity morphism $\mathrm{id}_A : A \to A$ such that if $f \in \mathrm{Hom}(A, B)$, then $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f$.

[1] As seen in Example 1.1.2, we would like to talk about categories where the objects form a *proper class* rather than a set. For instance, in **Set**, that the set of all sets does not exist is a classical fact in set theory and logic.

[2] We use this notation occasionally to distinguish the Hom-classes of categories that contain common objects, such as in Definitions 1.1.9, 1.1.11.

Example 1.1.2.

(i) The category **Set** where the sets are objects and functions between sets are the morphisms.

(ii) The category **Grp** where the groups are objects and groups homomorphisms are morphisms.

(iii) The category **Top** where topological spaces are objects and continuous maps are morphisms.

In each of the cases **Set**, **Grp** and **Top** above, the composition of morphisms corresponds to the ordinary composition of functions, group homomorphisms and continuous maps (respectively). Associativity holds in each case and the identity morphism is provided by the identity function, identity group homomorphism and identity continuous map (respectively).

A bijective function $f\colon S \to S'$ between sets S, S' has an inverse $f^{-1}\colon S' \to S$ that satisfies $f \circ f' = \mathrm{id}_{S'}$ and $f' \circ f = \mathrm{id}_S$. Similarly, a group isomorphism in **Grp** and a homeomorphism in **Top** satisfies the same property. This observation leads to the definitions of "inverses" and "isomorphisms".

Definition 1.1.3 (Isomorphism)**.** Let $\mathcal{C}$ be a category with objects A, B and a morphism $f\colon A \to B$. We say that f is an *isomorphism* from A to B if there exists a morphism $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. We call g an *inverse* of f.

The following shows that the inverse of an isomorphism in a category, if it exists, is unique.

Proposition 1.1.4. *Let $\mathcal{C}$ be a category and let $f\colon A \to B$ be an isomorphism. If $g\colon B \to A$ and $g'\colon B \to A$ are both inverses of f, then $g = g'$.*

Proof. The following two equalities hold:

$$(g \circ f) \circ g' = (\mathrm{id}_A) \circ g' = g'$$
$$g \circ (f \circ g') = g \circ (\mathrm{id}_B) = g$$

By associativity of composition, they are both equal. □

In the situation of Proposition 1.1.4 above, we say that g is the inverse of f, written $g = f^{-1}$. Additionally we say that A and B are *isomorphic* in $\mathcal{C}$, written $A \cong B$.

Example 1.1.5. In **Set**, isomorphisms are bijective functions. In **Grp**, isomorphisms are group isomorphisms. In **Top**, isomorphisms are homeomorphisms.

In the above categories, we also have a notion of injectivity and surjectivity of maps. The categorical generalization of these two concepts are "monomorphisms" and "epimorphisms" respectively.

Definition 1.1.6. Let $f\colon A \to B$ be a morphism in a category $\mathcal{C}$.

(i) We say f is a *monomorphism* if for all objects $C \in \mathcal{C}$ and morphisms $g_1\colon C \to A$ and $g_2\colon C \to A$,
$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

(ii) We f is an *epimorphism* if for all objects $C \in \mathcal{C}$ and morphisms $g_1 \colon B \to C$ and $g_2 \colon B \to C$,

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

Example 1.1.7.

(i) In **Set**, we can see that every injective function is a monomorphism and every surjective function is an epimorphism. The converse of these two statements can be proven as well. Similarly in **Grp** and **Top**, monomorphisms are the same as injective morphisms and epimorphisms are the same as surjective morphisms [Lei14, pp. 123, 133].

(ii) Any isomorphism $f \colon A \to B$ in a category $\mathcal{C}$ is a monomorphism and an epimorphism. To see that f is a monomorphism, suppose that $C \in \mathcal{C}$ and that $g_1 \colon C \to A$ and $g_2 \colon C \to A$ are morphisms such that

$$f \circ g_1 = f \circ g_2.$$

Then

$$f^{-1} \circ f \circ g_1 = g_1 = f^{-1} \circ f \circ g_2 = g_2.$$

Showing that f is an epimorphism is similar.

Up till now, the categories we have considered consist of objects as sets with some "added structure". This is not generally true for all categories. For instance, consider the following examples.

Example 1.1.8.

(i) Let $\mathbf{0}$ denote the empty category consisting of no objects and no morphisms. We note that $\mathbf{0}$ satisfies the definition of a category vacuously.

(ii) Let $\mathbf{2}$ denote the category consisting of two objects and only the identity morphisms of those objects. We say $\mathbf{2}$ is the *discrete category consisting of two objects*. More generally, we may have a discrete category consisting of an I-indexed family of objects.

Let $\mathcal{C}$ be a category with objects $A, B, C \in \mathcal{C}$. We may have a situation where $f \colon A \to B$, $g \colon B \to C$ and $h \colon A \to C$ are morphisms in $\mathcal{C}$. We say the following diagram *commutes* if $g \circ f = h$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

More generally, we may have a diagram with several directed paths starting from an object A and ending at an object B. Each of these paths gives rise to a morphism from $A \to B$ by composition. We say that the diagram commutes if we have equality of all these morphisms from $A \to B$.

We now discuss two ways of constructing a category $\mathcal{C}'$ from another category $\mathcal{C}$.

Definition 1.1.9 (Dual Category)**.** For a category $\mathcal{C}$, the *opposite* or *dual* category, $\mathcal{C}^{\text{op}}$, is the category where:

1. $\text{ob}(\mathcal{C}^{\text{op}}) = \text{ob}(\mathcal{C})$.
2. For objects $A, B \in \mathcal{C}^{\text{op}}$, $\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$.

That is, the dual category, C^{op}, consists of the same objects as C but where the direction of the morphisms are reversed.

Example 1.1.10. Let X be a topological space. We construct a category Op_X by taking the open sets of X to be the objects and defining a morphism between open sets $U, V \subset X$, $U \to V$ if and only if $U \subset V$. The dual category Op_X^{op} is then the category with open sets of X as objects and a morphism between open sets V and U, $V \to U$, if and only if $U \subset V$.

Definition 1.1.11 (Subcategory)**.** Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ is a category such that:

1. $\text{ob}(\mathcal{D})$ is a subset of $\text{ob}(\mathcal{C})$.
2. For objects $A, B \in \mathcal{D}$, $\text{Hom}_{\mathcal{D}}(A, B)$ is a subclass of $\text{Hom}_{\mathcal{C}}(A, B)$ that is closed under composition and identities in $\mathcal{C}$.

We say that $\mathcal{D}$ is *full* if for each $A, B \in \mathcal{D}$, $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$.

Two examples of importance are the categories **CHaus** and **Ab**.

Example 1.1.12.

(i) Let **CHaus** denote the category of all compact Hausdorff spaces and all continuous maps between them. Then **CHaus** is a full subcategory of **Top**.

(ii) Let **Ab** denote the category of all abelian groups and all group homomorphisms between them. Then **Ab** is a full subcategory of **Grp**.

1.2 Functors

This section introduces the notion of a "functor" which formalizes the idea of "maps between categories".

Definition 1.2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of

1. A map $\text{ob}(\mathcal{C}) \to \text{ob}(\mathcal{D})$, written $A \mapsto F(A)$.
2. For each $A, A' \in \mathcal{C}$, a map $\text{Hom}(A, A') \to \text{Hom}(F(A), F(A'))$, written $f \mapsto F(f)$.

satisfying the following axioms:

1. $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$;
2. $F(\text{id}_A) = \text{id}_{F(A)}$ whenever $A \in \mathcal{C}$.

The definition is best understood through a number of examples, which we outline briefly.

Example 1.2.2. Let $\mathcal{C}$ be a category. We have the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ which maps each object A to itself and each morphism $f \colon A \to B$ in $\mathcal{C}$ to itself.

Example 1.2.3. There exist so-called *forgetful functors*, which "forget structure". For example $F \colon \mathbf{Grp} \to \mathbf{Set}$, which sends a group to its underlying set and a homomorphism to its underlying function.

Example 1.2.4. Let $\mathcal{C}$ be a category and $\mathcal{D}$ a subcategory of $\mathcal{D}$. We define the inclusion functor $i \colon \mathcal{D} \hookrightarrow \mathcal{C}$ as follows. For each $A \in \mathcal{D}$, $i(A) = A$, where here we regard A as an object in $\mathcal{C}$. Similarly, if $f \in \mathrm{Hom}_{\mathcal{D}}(A, A')$, then $i(f) = f$, regarded as a morphism in $\mathrm{Hom}_{\mathcal{C}}(A, A')$.

Further examples can be found in [Lei14, Section 1.2].

1.2.1 Presheaves

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Sometimes we may have a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. We call this a *contravariant functor*. When $\mathcal{D} = \mathbf{Set}$, F is called a *presheaf*. Our motivation for studying these presheaves is as follows: a condensed set is a presheaf on the category **ED** (defined in Section 4.2) satisfying certain extra conditions.

In particular, if $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a contravariant functor, then F reverses the direction of composition. That is, if $A \xrightarrow{f} A' \xrightarrow{f'} A''$ is a morphism in $\mathcal{C}$, then the first functoriality axiom states

$$F(f' \circ f) = F(f) \circ F(f').$$

Example 1.2.5. Let X be a topological space and let Op_X be the category constructed from X in Example 1.1.10 . We construct a presheaf F on Op_X as follows. For each open set U of X, let $F(U) = \{f \colon U \to \mathbb{R} \mid f \text{ continuous}\}$. Recall that for $U, V \in \mathrm{Op}_X$, we have a map $i \colon U \to V$ in $\mathcal{C}$ if $U \subset V$. Define $F(i) \colon F(V) \to F(U)$ by sending each $f \colon V \to \mathbb{R}$ to its restriction on U, $f_{|U} \colon U \to \mathbb{R}$. Then F defines a functor $\mathrm{Op}_X^{\mathrm{op}} \to \mathbf{Set}$ and hence is a presheaf on Op_X.

The importance of the following construction in this text cannot be understated; we refer to it constantly throughout. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. Define a map

$$\mathrm{Hom}(-, A) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

as follows:

1. For objects $B \in \mathcal{C}$, let $\mathrm{Hom}(B, A)$ be the set of morphisms from B to A as before.
2. For morphisms $f \colon B \to B'$ in $\mathcal{C}$, define $\mathrm{Hom}(f, A) \colon \mathrm{Hom}(B', A) \to \mathrm{Hom}(B, A)$ by $p \mapsto p \circ f$ for all $p \colon B' \to A$. We call this *precomposition* by f.

Proposition 1.2.6. *For $A \in \mathcal{C}$, $\mathrm{Hom}(-, A)$ defines a functor $\mathcal{C} \to$ **Set**.*

Proof. We verify the two functoriality axioms.

1. Let $B \xrightarrow{f} B' \xrightarrow{f'} B''$ be morphisms in $\mathcal{C}$. We wish to show that

$$\mathrm{Hom}(f' \circ f, A) = \mathrm{Hom}(f, A) \circ \mathrm{Hom}(f', A).$$

Indeed, if $p \in \mathrm{Hom}(B'', A)$, then

$$\begin{aligned}\mathrm{Hom}(f' \circ f, A)(p) &= p \circ f' \circ f = (p \circ f') \circ f = \big(\mathrm{Hom}(f, A)\big)(p \circ f') \\ &= \mathrm{Hom}(f, A)\Big(\big(\mathrm{Hom}(f', A)\big)(p)\Big) = \big(\mathrm{Hom}(f, A) \circ \mathrm{Hom}(f, A)\big)(p).\end{aligned}$$

2. Let $B \in \mathcal{C}$ and consider id_B. We wish to show that

$$\mathrm{Hom}(\mathrm{id}_B, A) = \mathrm{id}_{\mathrm{Hom}(B,A)}.$$

For $p \in \mathrm{Hom}(B, A)$, we obtain

$$\big(\mathrm{Hom}(\mathrm{id}_B, A)\big)(p) = p \circ \mathrm{id}_B = p.$$

□

Definition 1.2.7. Let $A \in \mathcal{C}$. We call $\mathrm{Hom}(-, A)$ a *contravariant* Hom-*functor* on $\mathcal{C}$.

Example 1.2.8. Let k be a fixed field. Let $\mathbf{Vect}_k$ be the category consisting of vector spaces over k as objects and linear maps between them as morphisms. The contravariant functor $\mathrm{Hom}(-, k)\colon \mathbf{Vect}_k \to \mathbf{Set}$ sends a vector space $V \in \mathbf{Vect}_k$ to the set of linear maps from $V \to k$ and sends a linear map $f\colon V \to W$ to the map

$$\mathrm{Hom}(f, k)\colon \mathrm{Hom}(W, k) \to \mathrm{Hom}(V, k),\, q \mapsto q \circ f$$

The set of linear maps $V \to k$ can be turned into a vector space called the *dual space of* V and denoted V^*. An explicit construction can be found in [Lan89, p. 54]. One can then regard the above example as a contravariant functor from $\mathbf{Vect}_k$ to itself.

Example 1.2.9. Let $(-)^*\colon \mathbf{Vect}_k \to \mathbf{Vect}_k^{\mathrm{op}}$ be defined as follows. $(-)^*$ sends a vector space V to its dual V^* and sends a linear map $f\colon V \to W$ to the linear map $f^*\colon W^* \to V^*, q \mapsto q \circ f$. We verify that f^* is, in fact, a linear map. Let $q, r \in W^*$, $\mu, \lambda \in k$. Then:

$$f^*(\mu q + \lambda r) = (\mu q + \lambda r) \circ f = \mu(q \circ f) + \lambda(r \circ f) = \mu f^*(q) + \lambda f^*(r)$$

We call this the *dual functor* on $\mathbf{Vect}_k$.

1.3 Natural Transformations

In the previous section, we described functors as maps between categories. In the following, we take this abstraction one step further and introduce the notion of a natural transformation as "maps between functors".

Definition 1.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ be functors. A *natural transformation* $\alpha\colon F \to G$ is a family $\Big(F(A) \xrightarrow{\alpha_A} G(A)\Big)_{A \in \mathcal{C}}$ of maps in $\mathcal{D}$ such that for every morphism $f\colon A \to A'$ in $\mathcal{C}$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes i.e. $\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A$. The maps α_A are called the *components* of α.

We now provide an example of a natural transformation. Let k be a fixed field. From a vector space V over k, one can construct the dual of V, V^*. Since V^* is itself a vector space over k, we can take the dual of V^*, which explicitly is the vector space of linear maps from V^* to k [Hal74, Section 1.16]. We call this the double dual of V and denote it V^{**}. Define $\mathbf{FDVect}_k$ to be the full subcategory of $\mathbf{Vect}_k$ consisting of finite dimensional vector spaces. Then we define the *double dual functor* $(-)^{**}\colon \mathbf{FDVect}_k \to \mathbf{FDVect}_k$ as follows:

1. $(-)^{**}$ maps $V \in \mathbf{FDVect}_k$ to its double dual V^{**}
2. If $\phi\colon V \to W$ in $\mathbf{FDVect}_k$, then define $\phi^{**}\colon V^{**} \to W^{**}$ as the map that sends $q \in V^{**}$ to $q \circ \phi^*$ (precomposition by ϕ^*).

Similar to Example 1.2.9, one can verify that ϕ^{**} is a linear map.

Example 1.3.2 ([Lei14, Example 1.3.14])**.** We define a natural transformation α from $\mathrm{id}_{\mathbf{FDVect}_k}$ to $(-)^{**}$. The component $\alpha_V\colon V \to V^{**}$ sends $v \in V$ to the evaluation map $\alpha_V(v) \in V^{**}$, which sends $\phi\colon V \to k$ to $\phi(v)$.

In fact, due to a well-known result in linear algebra, the natural transformation defined above is an example of the even stronger notion of "natural isomorphism".

Definition 1.3.3 (Natural Isomorphism)**.** Let $\mathcal{C}, \mathcal{D}$ be categories with functors $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ We say that a natural transformation $\alpha\colon F \to G$ is a *natural isomorphism* if $\alpha_A\colon F(A) \to G(A)$ is an isomorphism for all $A \in \mathcal{C}$. In such a situation, we say that F and G are *naturally isomorphic* and denote this $F \cong G$.

Proposition 1.3.4 ([Hal74, Theorem 1.16.1])**.** *Let V be a finite-dimensional vector space over a field k. The map $\alpha_V\colon V \to V^{**}$, which sends $v \in V$ to the evaluation map $\alpha_V(v) \in V^{**}$, is an isomorphism.*

In particular, it follows that the natural transformation α is a natural isomorphism.

1.3.1 Functor Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. In this section, we construct the *functor category* $[\mathcal{C}, \mathcal{D}]$ to formalize the intuition that natural transformations are morphisms between functors. Here, the functors $F\colon \mathcal{C} \to \mathcal{D}$ are objects and natural transformations between two such functors are morphisms. In order to construct this category, we first define the composition of natural transformations and the identity natural transformation.

Suppose we have functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{C} \to \mathcal{D}$, $H\colon \mathcal{C} \to \mathcal{D}$ with natural transformations

$$\alpha\colon F \to G, \beta\colon G \to H$$

We define the *composite of β and α*

$$\beta \circ \alpha\colon F \to H$$

where for any $A \in \mathcal{C}$, the A-component is defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$. We now check that $\beta \circ \alpha$ defines a natural transformation.

Proposition 1.3.5. *The composite $\beta \circ \alpha$ defines a natural transformation $F \to H$.*

Proof. Let $f \colon A \to A'$ be a morphism in $\mathcal{C}$. From the hypothesis, we obtain the following commutative diagram.

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\beta_A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\ F(A') \arrow[r, "\alpha_{A'}"'] & G(A') \arrow[r, "\beta_{A'}"'] & H(A') \end{tikzcd}$$

In particular, the left square commutes because $\alpha : F \to G$ is a natural transformation and the right square commutes because $\beta : G \to H$ is a natural transformation. The result follows from the commutativity of the diagram. □

Now suppose that $F \colon \mathcal{C} \to \mathcal{D}$ is a functor. We define the *identity of* F $\mathrm{id}_F \colon F \to F$, where for any $A \in \mathcal{C}$, the A-component is defined by $(\mathrm{id}_F)_A = \mathrm{id}_{F(A)}$. This clearly defines a natural transformation, called *the identity natural transformation on* F.

With composition and identity of natural transformations defined, we are now ready to provide the explicit definition of a functor category. To avoid set-theoretic issues while defining the functor category $[\mathcal{C}, \mathcal{D}]$, we require that $\mathcal{C}$ be "small" [Shu08, Section 7].

Definition 1.3.6 (Small Category)**.** Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a *small category* if the class of all morphisms in $\mathcal{C}$ is a set.

If $\mathcal{C}$ is small, then the class of objects is also a set since each object $\mathcal{C}$ is in one-to-one correspondence with the identity morphisms.

Definition 1.3.7 (Functor Category)**.** Let $\mathcal{C}$ be a small category, D any category. The functor category $[\mathcal{C}, \mathcal{D}]$ consists of:

1. Functors $F \colon \mathcal{C} \to \mathcal{D}$ as objects.
2. Natural transformations $\alpha \colon F \to G$ between functors $F, G \in [\mathcal{C}, \mathcal{D}]$ as morphisms between F and G.

Indeed, this satisfies the required axioms for a category [Lei14, Construction 1.3.6]. Isomorphisms in $[\mathcal{C}, \mathcal{D}]$ have a familiar characterization. Namely, they are natural isomorphisms between functors $\mathcal{C} \to \mathcal{D}$ [Lei14, p. 31].

Example 1.3.8. If $\mathcal{C}$ is any small category, the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is called the *category of presheaves of* $\mathcal{C}$. We discuss this further in Section 1.4.

1.3.2 Equivalence of Categories

The following section discusses some theory to compare two categories. In particular, given two categories, we want to know when these categories can be considered the same or *equivalent*. We first describe the composition of functors. Proposition 1.3.9 then shows that this composition defines a functor.

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{D} \to \mathcal{E}$ be functors. We define the *composite of* G *and* F, $G \circ F \colon \mathcal{C} \to \mathcal{E}$, as follows:

1. For each $A \in \mathcal{C}$, let $(G \circ F)(A) = G\big(F(A)\big)$.

2. For each morphism $f\colon A \to A'$ in $\mathcal{C}$, let $(G \circ F)(f) = G\big(F(f)\big)$.

Proposition 1.3.9. *The composition $G \circ F$ is a functor.*

Proof. We verify the two functoriality axioms:

1. Let $g\colon A \to B$ and $f\colon B \to C$ be morphisms in $\mathcal{C}$. Then:

$$\begin{aligned}(G \circ F)(f \circ g) &= G\big(F(f \circ g)\big) = G\big(F(f) \circ F(g)\big) \\ &= G\big(F(f)\big) \circ G\big(F(g)\big) = (G \circ F)(f) \circ (G \circ F)(g)\end{aligned}$$

2. Let $A \in \mathcal{C}$. Then:

$$(G \circ F)(\mathrm{id}_A) = G\big(F(\mathrm{id}_A)\big) = G(\mathrm{id}_{F(A)}) = \mathrm{id}_{G\left(F(A)\right)} = \mathrm{id}_{(G \circ F)(A)}$$

$\square$

We are now ready to define the equivalence of categories.

Definition 1.3.10 (Equivalence of Categories)**.** Let $\mathcal{C}, \mathcal{D}$ be categories. An *equivalence* between $\mathcal{C}$ and $\mathcal{D}$ consists of functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ along with natural isomorphisms:

$$\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F, \epsilon : \mathrm{id}_{\mathcal{D}} \to F \circ G$$

In such a situation, we say that the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, denoted $\mathcal{C} \simeq \mathcal{D}$. We also say that the functors F and G are *equivalences*.

In practice, we often do not need to construct natural isomorphisms η and ϵ to show that G and F are equivalences. There exists an alternative more convenient characterization of equivalences in Lemma 1.3.14. This requires Definitions 1.3.11 and 1.3.13.

Definition 1.3.11 (Faithful and Full Functors)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is *faithful* (respectively, *full*) if for each $A, A' \in \mathcal{C}$, the function

$$\begin{array}{ccc}\mathrm{Hom}(A, A') & \to & \mathrm{Hom}(F(A), F(A')), \\ f & \mapsto & F(f)\end{array}$$

is injective (respectively, surjective). If this function is bijective for all $A, A' \in \mathcal{C}$, we say that F is *fully faithful*.

Example 1.3.12. Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. The inclusion functor $i\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is fully faithful.

Definition 1.3.13 (Essentially Surjective)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is *essentially surjective on objects* if for all $B \in \mathcal{D}$, there exists $A \in \mathcal{C}$ such that $F(A) \cong B$.

Lemma 1.3.14 ([Lei14, Proposition 1.3.18])**.** *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective on objects.*

The following corollary of the above lemma will be of the greatest interest to us and it will feature specifically in Theorem 1.4.5 and Theorem 5.0.7.

Corollary 1.3.15 ([Lei14, Corollary 1.3.19]). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor and let $\mathcal{C}'$ be the full subcategory of $\mathcal{D}$ consisting of objects $B \in \mathcal{D}$ such that $B \cong F(A)$ for some $A \in \mathcal{C}$. Then $\mathcal{C} \simeq \mathcal{C}'$.*

Proof. Define a functor $F'\colon \mathcal{C} \to \mathcal{C}'$ where for $A \in \mathcal{C}, F'(A) = F(A)$ and for a morphism $f\colon A \to A'$ in $\mathcal{C}, F'(f) = F(f)$. Then F' is fully faithful and essentially surjective on objects. □

1.4 Yoneda Embedding

Let $\mathcal{C}$ be a small category. If $A \in \mathcal{C}$, we can consider the contravariant Hom-functor $\mathrm{Hom}(-, A)$, as defined in Definition 1.2.7. The main result of this section, Theorem 1.4.4, states that the mapping $A \mapsto \mathrm{Hom}(-, A)$ can be thought of as a fully faithful functor from $\mathcal{C}$ to the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. We call this the *Yoneda embedding*.

The Yoneda embedding provides a map $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ by mapping each object $A \mapsto \mathrm{Hom}(-, A)$. To turn this map into a functor, we must map each morphism $f\colon A \to A'$ in $\mathcal{C}$ to a natural transformation $\mathrm{Hom}(-, A) \to \mathrm{Hom}(-, A')$.

Let $f\colon A \to A'$ be a morphism in $\mathcal{C}$. Define a family of maps in **Set**

$$\left(\mathrm{Hom}(B, A) \xrightarrow{\mathrm{Hom}(B,f)} \mathrm{Hom}(B, A')\right)_{B\in\mathcal{C}}$$

as follows. For a particular $B \in \mathcal{C}$, define a map

$$\begin{aligned} \mathrm{Hom}(B, f)\colon \mathrm{Hom}(B, A) &\to \mathrm{Hom}(B, A') \\ p &\mapsto f \circ p \end{aligned}$$

Denote this family of maps by $\mathrm{Hom}(-, f)$. We now verify that this defines a natural transformation $\mathrm{Hom}(-, A) \to \mathrm{Hom}(-, A')$.

Proposition 1.4.1. *Let $f\colon A \to A'$ be a morphism in $\mathcal{C}$. The family of maps $\mathrm{Hom}(-, f)$ defined above is a natural transformation $\mathrm{Hom}(-, A) \to \mathrm{Hom}(-, A')$.*

Proof. Let $g\colon B' \to B$ be a morphism in $\mathcal{C}$. We wish to prove that the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}(B, A) \arrow[r, "\mathrm{Hom}(g,A)"] \arrow[d, "\mathrm{Hom}(B,f)"'] & \mathrm{Hom}(B', A) \arrow[d, "\mathrm{Hom}(B',f)"] \\ \mathrm{Hom}(B, A') \arrow[r, "\mathrm{Hom}(g,A')"] & \mathrm{Hom}(B', A') \end{tikzcd}$$

In other words, we wish to show that

$$\mathrm{Hom}(B', f) \circ \mathrm{Hom}(g, A) = \mathrm{Hom}(g, A') \circ \mathrm{Hom}(B, f)$$

Let $h\colon B \to A$. Then

$$\begin{gathered} \big(\mathrm{Hom}(B', f) \circ \mathrm{Hom}(g, A)\big)(h) = \mathrm{Hom}(B', f)\big(\mathrm{Hom}(g, A)(h)\big) = \\ \mathrm{Hom}(B', f)(h \circ g) = f \circ h \circ g \end{gathered}$$

On the other hand

$$\big(\mathrm{Hom}(g, A') \circ \mathrm{Hom}(B, f)\big)(h) = \mathrm{Hom}(g, A')\big(\mathrm{Hom}(B, f)(h)\big) = \\ \mathrm{Hom}(g, A')(f \circ h) = f \circ h \circ g$$

□

Explicitly, the definition of the Yoneda embedding is then as follows.

Definition 1.4.2 (Yoneda Embedding)**.** We define the *Yoneda embedding of* $\mathcal{C}$

$$h^{\bullet} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

as follows:

1. For $A \in \mathcal{C}$, let $h^{\bullet}(A) = \mathrm{Hom}(-, A)$.
2. For a morphism $f \colon A \to A'$, define $h^{\bullet}(f)$ to be

$$\mathrm{Hom}(-, f) \colon \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, A').$$

Proposition 1.4.3. *The Yoneda embedding of $\mathcal{C}$, $h^{\bullet}$, defines a functor $\mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$.*

Proof. We verify the functoriality axioms.

1. Suppose we have morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{C}$. We must verify that

$$h^{\bullet}(f' \circ f) = h^{\bullet}(f') \circ h^{\bullet}(f).$$

In other words, for $B \in \mathcal{C}$:

$$\mathrm{Hom}(B, f' \circ f) = \mathrm{Hom}(B, f') \circ \mathrm{Hom}(B, f)$$

Indeed, if $p \colon B \to A$ is a morphism, then

$$\mathrm{Hom}(B, f' \circ f)(p) = f' \circ f \circ p = \big(\mathrm{Hom}(B, f') \circ \mathrm{Hom}(B, f)\big)(p).$$

2. We wish to show that for $A \in \mathcal{C}$, the natural transformations $h^{\bullet}(\mathrm{id}_A) = \mathrm{Hom}(-, \mathrm{id}_A)$ and $\mathrm{id}_{h^{\bullet}(A)} = \mathrm{id}_{\mathrm{Hom}(-,A)}$ are equal. That is, for $B \in \mathcal{C}$:

$$\mathrm{Hom}(B, \mathrm{id}_A) = \mathrm{id}_{\mathrm{Hom}(B,A)}$$

Let $p \colon B \to A$ be a morphism. Then

$$\big(\mathrm{Hom}(B, \mathrm{id}_A)\big)(p) = \mathrm{id}_A \circ p = p.$$

This proves the claim.

□

The motivation for calling $h^{\bullet}$ an *embedding* is the following. In topology, we say that an injective continuous mapping $f \colon X \to Y$ is an embedding if the function $f' \colon X \to f(X)$ defined by $f'(x) = f(x)$ for all $x \in X$ is a homeomorphism [Mun75, p. 105]. In such a situation, X is topologically equivalent to a subspace of Y. Similarly, in this case, we see that $\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{C}, \mathbf{Set}^{\mathrm{op}}]$. This involves first showing that $h^{\bullet}$ is fully faithful and then applying Corollary 1.3.15.

Theorem 1.4.4 ([Lei14, Corollary 4.3.7]). *Then the Yoneda embedding of $\mathcal{C}$ $h^{\bullet}\colon \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ is fully faithful.*

Proof. We wish to show that for $A, A' \in \mathcal{C}$, the map

$$\begin{aligned} \operatorname{Hom}(A, A') &\to \operatorname{Hom}\big(\operatorname{Hom}(-, A), \operatorname{Hom}(-, A')\big) \\ f &\mapsto \operatorname{Hom}(-, f), \end{aligned}$$

is a bijection. First, we show injectivity. Let $f, g \in \operatorname{Hom}(A, A')$ and suppose $\operatorname{Hom}(-, f) = \operatorname{Hom}(-, g)$. Consider $\operatorname{Hom}(A, f)\colon \operatorname{Hom}(A, A) \to \operatorname{Hom}(A, A')$. Then:

$$\operatorname{Hom}(A, f)(\mathrm{id}_A) = f \circ \mathrm{id}_A = f = \operatorname{Hom}(A, g)(\mathrm{id}_A) = g \circ \mathrm{id}_A = g$$

Now we show surjectivity. Let $\alpha\colon \operatorname{Hom}(-, A) \to \operatorname{Hom}(-, A')$ be a natural transformation. We wish to show that there exists a morphism $f\colon A \to A'$ such that for all $B \in \mathcal{C}$, $\operatorname{Hom}(B, f) = \alpha_B$. In other words, if $p \in \operatorname{Hom}(B, A)$, then:

$$\operatorname{Hom}(B, f)(p) = f \circ p = \alpha_B(p)$$

Consider $\alpha_A\colon \operatorname{Hom}(A, A) \to \operatorname{Hom}(A, A')$. Now define $f \in \operatorname{Hom}(A, A')$ by $f = \alpha_A(\mathrm{id}_A)$. We show that f is the morphism that satisfies the above claim. Let $p\colon B \to A$. This induces the following commutative diagram:

$$\begin{tikzcd} \operatorname{Hom}(A, A) \arrow[r, "\operatorname{Hom}(p, A)"] \arrow[d, "\alpha_A"'] & \operatorname{Hom}(B, A) \arrow[d, "\alpha_B"] \\ \operatorname{Hom}(A, A') \arrow[r, "\operatorname{Hom}(p, A')"'] & \operatorname{Hom}(B, A') \end{tikzcd}$$

Now evaluate at id_A to obtain

$$\begin{aligned} \big(\operatorname{Hom}(p, A') \circ \alpha_A\big)(\mathrm{id}_A) &= \operatorname{Hom}(p, A') \circ f = f \circ p \\ &= \big(\alpha_B \circ \operatorname{Hom}(p, A)\big)(\mathrm{id}_A) = \alpha_B(p). \end{aligned}$$

Thus we obtain the equality $f \circ p = \alpha_B(p)$. □

Let $F \in [\mathcal{C}, \mathbf{Set}^{\mathrm{op}}]$. We say that F is a *representable presheaf* if $F \cong \operatorname{Hom}(-, A)$ for some $A \in \mathcal{C}$. Now due to Corollary 1.3.15, we obtain the following result:

Corollary 1.4.5. *Let $\mathcal{C}$ be a small category. Then $\mathcal{C}$ is equivalent to the full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$ consisting of representable presheaves.*

Chapter 2

Limits & Colimits

This chapter introduces the notions of limits and colimits in a category. From a categorical point of view, the motivation for studying limits and colimits is that they unify many similar constructions in mathematics. For instance, the disjoint union of two sets and the direct product of two abelian groups are two examples of a type of colimit called a "coproduct". From the condensed mathematics point of view, limits and colimits appear very often in theory. In particular, "products / coproducts", "equalizers / coequalizers" and "inverse limits" feature extensively in this text. The definitions of limits and colimits are quite abstract so we follow the presentation found in [Lei14, Chapter 5] by exploring examples of products and coproducts in Section 2.1 and Section 2.2 before presenting the general definition of limits / colimits in Section 2.3 . Finally we examine the frequently mentioned examples of limits and colimits in Section 2.4. Our primary reference for this chapter is [Lei14, Chapter 5].

2.1 Products

The definition of a product is motivated by the following scenario in **Set**. Let X_1 and X_2 be sets. Note that the Cartesian product $X_1 \times X_2$ satisfies the following categorical property. Let $p_1 \colon X_1 \times X_2 \to X_1$, $p_2 \colon X_1 \times X_2 \to X_2$ denote the canonical projections. Suppose that we have a set A with maps $f_1 \colon A \to X_1$ and $f_2 \colon A \to X_2$. Clearly this induces a map $A \to X_1 \times X_2$ defined as follows:

$$f \colon A \to X_1 \times X_2, a \mapsto \big(f_1(a), f_2(a)\big)$$

Moreover, the following diagram commutes:

$$\begin{tikzcd} & A \arrow[d, "F"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\ & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd} \tag{2.1}$$

In fact, it is quite clear that F is the unique map $A \to X \times Y$ such that the above diagram commutes. Suppose otherwise, namely that there exists $G \colon A \to X \times Y$ such that $p_1 \circ G = f_1$ and $p_2 \circ G = f_2$. Let $a \in A$. Then

$$(p_1 \circ G)(a) = f_1(a), (p_2 \circ G)(a) = f_2(a)$$

Thus $G(a) = \big(f_1(a), f_2(a)\big) = F(a)$. To summarize, the Cartesian product of X and Y in **Set** is characterized by the following categorical data:

1. it consists of an object $X \times Y$ in **Set** with morphisms $p_1 \colon X \times Y \to X$, $p_2 \colon X \times Y \to Y$.

2. if there exists an object A in **Set** with morphisms $f \colon A \to X$, $g \colon A \to Y$, then there exists a unique morphism $A \to X \times Y$ such that the diagram (2.1) commutes.

This suggests the following definition in a general category [Lei14, p. 105].

Definition 2.1.1 (Product)**.** Let $\mathcal{C}$ be a category and $(X_i)_{i \in I}$ be an I-indexed family of objects in $\mathcal{C}$. A *product* of $(X_i)_{i \in I}$ consists of an object P and a family of morphisms

$$(P \xrightarrow{p_i} X_i)_{i \in I}$$

such that for all objects A and families of maps

$$(A \xrightarrow{f_i} X_i)_{i \in I}$$

there exists a unique map $f \colon A \to P$ such that $p_i \circ f = f_i$ for all $i \in I$.

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f_i"] & \\ P \arrow[r, "p_i"'] & X_i \end{tikzcd}$$

The morphisms p_i are called the *projections*.

Remark 2.1.2.

(i) A product of $(X_i)_{i \in I}$ consists of an object P along with the projections, but we often refer to the product as P alone.

(ii) Products in a general category may not exist. Consider the discrete category consisting of two objects, **2**. Clearly, the two objects in **2** do not have a product.

However, if products do exist in a category, they are unique up to isomorphism. This is true in general for all limits / colimits and is the statement of Theorem 2.3.5. Hence in such a circumstance, we refer to *the* product of $(X_i)_{i \in I}$ and denote it $\prod_{i \in I} X_i$ without ambiguity. Now we discuss a few examples in some familiar categories.

Example 2.1.3. In **Set**, similar to the binary case, the product of a family of objects $(X_i)_{i \in I}$ is the Cartesian product $\prod_{i \in I} X_i$ of the objects along with the canonical projections $p_j \colon \prod_{i \in I} X_i \to X_j$ for all $j \in I$.

Example 2.1.4. Let $G, H \in$ **Grp**. Define their direct product $G \times H$ as the group where:

1. The underlying set is the Cartesian product $G \times H$

2. For $g_1, g_2 \in G$ and $h_1, h_2 \in H$, the binary operation is defined by:

$$(g_1, h_1) \times (g_2, h_2) = (g_1 \times_G g_2, h_1 \times_H h_2)$$

Then the product of G, H in **Grp** is $G \times H$ along with the canonical projections $p_1 \colon G \times H \to G$, $p_2 \colon G \times H \to H$ [Mac71, p. 69].

2.2 Coproducts

The notion of a coproduct of objects in a category $\mathcal{C}$ is dual to the notion of the product of objects in $\mathcal{C}$. That is, informally speaking, the definition of a coproduct is obtained by reversing the directions of the morphisms in the definition of a product. It is an example of a colimit rather than a limit.

Definition 2.2.1. Let $\mathcal{C}$ be a category and $(X_i)_{i\in I}$ be a I-indexed family of objects in $\mathcal{C}$. A *coproduct* of $(X_i)_{i\in I}$ is an object C and a family of maps

$$(X_i \xrightarrow{p_i} C)_{i\in I},$$

such that for all objects A and families of maps

$$(X_i \xrightarrow{f_i} A)_{i\in I}$$

there exists a unique map $f\colon C \to A$ such that $f_i = f \circ p_i$.

$$\begin{tikzcd} A & \\ C \arrow[u, "f"] & X_i \arrow[l, "p_i"] \arrow[ul, "f_i"'] \end{tikzcd}$$

The morphisms p_i are called the *coprojections.*

As before, the coproduct of objects is not guaranteed to exist in a category. Once again, the discrete category with two objects serves as an example of this. However, if it does exist, it is unique up to isomorphism and is denoted $\coprod_{i\in I} X_i$. We now outline some examples, taken from [Mac71, p. 63].

Example 2.2.2. Let $X_1, X_2 \in \mathbf{Set}$. Define the *disjoint union* $X_1 \sqcup X_2 \in \mathbf{Set}$ of the sets X_1, X_2 by:

$$X_1 \sqcup X_2 = \{(x_1, 1) \mid x_1 \in X_1\} \cup \{(x_2, 2) \mid x_2 \in X_2\}.$$

For $i = \{1, 2\}$, define the canonical injections $\varphi_i\colon X_i \to X_1 \sqcup X_2$ by $\varphi_i(x) = (x, i)$ for all $x \in X_i$. Then $X_1 \sqcup X_2$ along with φ_1 and φ_2 is the coproduct of X_1 and X_2 in $\mathbf{Set}$. More generally, given an I-indexed family $(X_i)_{i\in I}$ family of sets, we define their disjoint union

$$\coprod_{i\in I} X_i = \bigcup_{i\in I} \{(x_i, i) \mid x_i \in X_i\}.$$

We analogously define the canonical injections $\varphi_i\colon X_i \to \coprod_{i\in I} X_i$ for $i \in I$ as in the binary case. Then the coproduct of $(X_i)_{i\in I}$ of objects in $\mathbf{Set}$ is the disjoint union $\coprod_{i\in I} X_i$ of the sets along with the canonical injections.

Example 2.2.3. Let $G, H \in \mathbf{Ab}$. Then the coproduct of G and H is also the direct product.

2.3 Limits & Colimits

Consider the following alternative way to think about the binary product in $\mathbf{Set}$. Let $\mathbf{2}$ be the discrete category consisting of two objects A and B. Then a functor $D\colon \mathbf{2} \to \mathbf{Set}$ simply picks out two objects $X, Y \in \mathbf{Set}$. In the terminology we introduce shortly, the functor D will be called a *diagram in* **Set** *of shape* **2**. If $S \in \mathbf{Set}$ with morphisms $f_1 \colon S \to D(A)$, $f_2 \colon S \to D(B)$, we will call S along with f_1, f_2 a *cone on* D. With this terminology note that $X \times Y$ with the canonical projections p_1, p_2 is a cone on D with the *universal property of the product*. That is, if S is any other cone on D, then there exists a unique morphism $f \colon S \to X \times Y$ such that for $i = \{1, 2\}$.

$$p_i \circ f = f_i.$$

We will then call $X \times Y$ a *limit of* D. The general definitions of a diagram, a cone and a limit mirror this situation. Let $\mathcal{C}$ be a category and $\mathbf{J}$ a small category.

Definition 2.3.1 (Diagram)**.** A functor $D \colon \mathbf{J} \to \mathcal{C}$ is called a *diagram in* $\mathcal{C}$ *of shape* $\mathbf{J}$ (or more simply a diagram).

Definition 2.3.2 (Limit)**.** Let $D \colon \mathbf{J} \to \mathcal{C}$ be a diagram in $\mathcal{C}$ of shape $\mathbf{J}$.

1. A *cone on* D is an object $A \in \mathcal{C}$ along with a family of morphisms in $\mathcal{C}$

   $$\left(A \xrightarrow{f_I} D(I)\right)_{I \in \mathbf{J}},$$

   such that for each morphism $u \colon I \to J$ in $\mathbf{J}$, the following diagram commutes:

$$\begin{tikzcd} & D(I) \arrow[dd, "D(u)"] \\ A \arrow[ur, "f_I"] \arrow[dr, "f_J"'] & \\ & D(J) \end{tikzcd}$$

2. A *limit of* D is a cone $\left(L \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{J}}$ with the property that for each cone $\left(A \xrightarrow{f_I} D(I)\right)_{I \in \mathbf{J}}$ on D, there exists a unique morphism $f \colon A \to L$ such that for each $I \in \mathbf{J}$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f_I"] & \\ L \arrow[r, "p_I"'] & D(I) \end{tikzcd}$$

The morphisms p_I are called the *projections of the limit.*

Remark 2.3.3.

(i) Similar to Remark 2.1.2, a limit of D consists of an object L along with morphisms $\left(p_I \colon L \to D(I)\right)_{I \in \mathbf{J}}$. However, we often refer to L simply as a limit of D.

(ii) The property that a cone must satisfy to be a limit is called the *universal property of the limit*.

Example 2.3.4. If $\mathbf{J}$ is the discrete category consisting of an I-indexed family of objects, then a diagram $D\colon \mathbf{J} \to \mathcal{C}$ consists of an I-indexed family of objects $(X_i)_{i \in I}$ in $\mathcal{C}$. A limit of this diagram, if it exists, is the product $\prod_{i \in I} X_i$.

At an earlier point, we made the claim that the product, if it exists in a category, is unique up to isomorphism. Now we justify this for a general limit.

Theorem 2.3.5 ([Lei14, Corollary 6.1.2])**.** *Let $D\colon \mathbf{J} \to \mathcal{C}$ be a diagram in $\mathcal{C}$ of shape $\mathbf{J}$. Suppose that a limit of D exists. Then it is unique up to isomorphism.*

Proof. Let $\left(L_1 \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{J}}$ and $\left(L_2 \xrightarrow{q_I} D(I)\right)_{I \in \mathbf{J}}$ be limits of D. Firstly, note that since L_1 is a limit of D, there exist a unique $h\colon L_1 \to L_1$ such that for all $I \in \mathbf{J}$

$$p(I) = p(I) \circ h.$$

Clearly $h = \mathrm{id}_{L_1}$. Now since L_1 and L_2 are limits of D, there exists unique morphisms $f\colon L_1 \to L_2$, $g\colon L_2 \to L_1$ such that for all $I \in \mathbf{J}$

$$p(I) = q(I) \circ f,$$

$$q(I) = p(I) \circ g.$$

Then

$$(p(I) \circ g) \circ f = q(I) \circ f = p(I).$$

Thus $g \circ f\colon L_1 \to L_1$ is a morphism such that $p(I) \circ (g \circ f) = p(I)$ for all $I \in \mathbf{J}$. Thus $g \circ f = \mathrm{id}_{L_1}$. Similarly, $f \circ g = \mathrm{id}_{L_2}$. Thus $L_1 \cong L_2$. □

The definition of a colimit is dual to the definition of a limit. Explicitly:

Definition 2.3.6 (Colimit)**.** Let $D\colon \mathbf{J} \to \mathcal{C}$ a diagram in $\mathcal{C}$ of shape $\mathbf{J}$.

1. A *cocone on D* is an object $A \in \mathcal{C}$ along with a family of morphisms in $\mathcal{C}$

$$\left(D(I) \xrightarrow{f_I} A\right)_{I \in \mathbf{J}}$$

   such that for each morphism $u\colon I \to J$ in $\mathbf{J}$, the following diagram commutes.

$$\begin{tikzcd} D(I) \arrow[d, "D(u)"'] \arrow[dr, "f_I"] & \\ D(J) \arrow[r, "f_J"'] & A \end{tikzcd}$$

2. A *colimit of D* is a cocone $\left(D(I) \xrightarrow{p_I} C\right)_{I \in \mathbf{J}}$ with the property that for each cocone $\left(D(I) \xrightarrow{f_I} A\right)_{I \in \mathbf{J}}$ on D, there exists a unique morphism $f\colon C \to A$ such that for each $I \in \mathbf{J}$, the following diagram commutes.

$$\begin{tikzcd} & A \\ D(I) \arrow[ur, "f_I"] \arrow[r, "p_I"'] & C \arrow[u, "f"'] \end{tikzcd}$$

The morphisms p_I are called the *coprojections of the colimit*.

Remark 2.3.7. Similar remarks hold as they did for limits.

(i) The propety that a cocone must satisfy to be a colimit is called the *universal property of the colimit*.

(ii) Once again, as well, a colimit of a diagram, if it exists, is unique up to isomorphism. The proof is dual to the argument presented in Theorem 2.3.5 [Lei14, p. 126].

Finally, an important fact is that in **Set**, **Grp**, **Ab**, all diagrams have limits / colimits [Mac71, Section V.1]. The same is true in **Top** and in fact, the limit / colimit of a diagram is given by the underlying set-theoretic limit / colimit equipped with the "initial" / "final" topologies, respectively. We discuss this further in Chapter 3.

2.4 Examples of Limits & Colimits

This sections outlines the types of limits and colimits that we will encounter in this text. Let $\mathcal{C}$ be a category and recall that $\mathbf{0}$ denotes the empty category. We begin with "terminal" and "initial" objects [[Lei14, p. 48]].

Definition 2.4.1 (Terminal Object & Initial Object)**.** Let $D\colon\ \mathbf{0} \to \mathcal{C}$ be the unique diagram in $\mathcal{C}$ of shape $\mathbf{0}$.

(i) A limit of D in $\mathcal{C}$ is called a *terminal object* in $\mathcal{C}$.

(ii) A colimit of D in $\mathcal{C}$ is called an *initial object* in $\mathcal{C}$.

Suppose that the limit, T, and the colimit, I, of D exists. Note that every object $A \in \mathcal{C}$ is vacuously both a cone and cocone on D. Hence for each $A \in \mathcal{C}$, there exists a unique morphism $A \to T$ and a unique morphism $I \to A$. We note here that sometimes we make reference to an empty product $\prod$ in $\mathcal{C}$. By this, we mean a terminal object in $\mathcal{C}$. Similarly, an empty coproduct $\coprod$ in $\mathcal{C}$ is an initial object.

Example 2.4.2.

(i) The terminal object in Set is the singleton set $\{*\}$ which we sometimes denote simply as $*$. The initial object in **Set** is the empty set $\emptyset$.

(ii) The terminal object in **Top** is the singleton set $*$ endowed with the discrete topology and the initial object is the empty space $\emptyset$.

(iii) The terminal and initial object in **Grp** (and **Ab**) is the trivial group $\{e\}$.

Next, we discuss equalizers and coequalizers. Let $\mathbf{J}$ be the category consisting of two objects, I and J, with the following Hom-sets:

$$\mathrm{Hom}(I,I) = \{\mathrm{id}_I\}; \mathrm{Hom}(J,J) = \{\mathrm{id}_J\}; \mathrm{Hom}(I,J) = \{u,v\}; \mathrm{Hom}(J,I) = \emptyset.$$

Essentially, $\mathbf{J}$ is represented by the following.

$$\begin{tikzcd} I \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & J \end{tikzcd}$$

Definition 2.4.3 (Equalizers & Coequalizers)**.** Let $D\colon \mathbf{J} \to \mathcal{C}$ be a diagram of shape $\mathbf{J}$ in $\mathcal{C}$.

(i) A limit of D in $\mathcal{C}$ is called an *equalizer of* D.

(ii) A colimit of D in $\mathcal{C}$ is called a *coequalizer of* D.

An equalizer and coequalizer of D can be characterized more explicitly as follows [Lei14, Definitions 5.1.11 and 5.2.7]. A cone on D is a morphism $f\colon A \to D(I)$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] & D(I) \arrow[r, shift left, "D(u)"] \arrow[r, shift right, "D(v)"'] & D(J) \end{tikzcd}$$

That is, $D(u) \circ f = D(v) \circ f$. An equalizer of D is a cone $i\colon E \to D(I)$ satisfying the following universal property. If $f\colon A \to D(I)$ is a cone on D, then there exists a unique morphism $\bar{f}\colon A \to E$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[d, "\bar{f}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"'] & D(I) \end{tikzcd}$$

Similarly, a cocone on D is a morphism $f\colon D(J) \to A$ such that the following diagram commutes

$$\begin{tikzcd} D(I) \arrow[r, shift left, "D(u)"] \arrow[r, shift right, "D(v)"'] & D(J) \arrow[r, "f"] & A. \end{tikzcd}$$

That is, $f \circ D(u) = f \circ D(v)$. A coequalizer of D is a cocone $p\colon D(J) \to C$ satisfying the following universal property. If $f\colon D(J) \to A$ is a cocone on D, then there exists a unique morphism $\bar{f}\colon C \to A$ such that the following diagram commutes.

$$\begin{tikzcd} & A \\ D(J) \arrow[ur, "f"] \arrow[r, "p"'] & C \arrow[u, "\bar{f}"'] \end{tikzcd}$$

Example 2.4.4 ([Lei14, Examples 5.1.14 and 5.2.10])**.** Let $G, H \in \mathbf{Ab}$. Let $f\colon G \to H$ be a morphism and let 0_H^G be the trivial group homomorphism. Now consider the following diagram in $\mathbf{Ab}$.

$$\begin{tikzcd} G \arrow[r, shift left, "f"] \arrow[r, shift right, "0_H^G"'] & H \end{tikzcd}$$

An equalizer of this diagram is the inclusion $i\colon \mathrm{Ker}(f) \hookrightarrow G$ from the kernel of f. A coequalizer of this diagram is the canonical surjection $p\colon H \to H/\mathrm{Im}(f)$ into the cokernel of f.

Finally, we define an inverse limit [RZ10, Section 1.1]. This will be useful when describing "profinite sets" in Chapter 4. First, we define a "directed poset".

Definition 2.4.5 (Directed Poset)**.** A *poset* (or *partially ordered set*) $(J, \leq)$ is a set J with a binary relation $\leq$ satisfying:

1. $i \leq i$ for all $i \in J$
2. if $i \leq j$ and $j \leq k$, then $i \leq k$, for all $i, j, k \in J$
3. if $i \leq j$ and $j \leq i$, then $i = j$, for all $i, j \in J$

We additionally say that $(J, \leq)$ is *directed* if for all $i, j \in J$, there exists $k \in J$ such that $i \leq k$ and $j \leq k$.

Example 2.4.6.

(i) $(\mathbb{N}, \leq)$, where $\leq$ is the usual relation.

(ii) Let X be a set and $P(X)$ the power set of X. Then $(P(X), \subseteq)$ is a directed poset where $\subseteq$ is the ordinary set inclusion.

A directed poset $(J, \leq)$ forms a category $\mathbf{J}$ where the objects are elements of J and for $i, j \in J$, there exists a unique morphism $i \to j$ if and only if $i \leq j$. Here, the identity morphism on $i \in \mathbf{J}$ is given by (1) and composition of morphisms is given by (2). We then call $\mathbf{J}$ the *category associated to* $(J, \leq)$.

Definition 2.4.7 (Inverse Systems)**.** An *inverse system over* $\mathbf{J}$ *in* $\mathcal{C}$ is a diagram $D\colon \mathbf{J}^{\mathrm{op}} \to \mathcal{C}$.

More explicitly, an inverse system over $\mathbf{J}$ in $\mathcal{C}$ is a collection of objects $(X_i)_{i \in J}$ such that for each $i \leq j$, there exists a morphism $f_{ij}\colon X_j \to X_i$ satisfying the following:

1. f_{ii} is the identity on X_i for each $i \in I$
2. $f_{ik} = f_{ij} \circ f_{jk}$, if $i \leq j \leq k$

Thus we often denote an inverse system by (X_i, f_{ij}, J).

Definition 2.4.8 (Inverse Limits)**.** Let (X_i, f_{ij}, J) be a inverse system in $\mathcal{C}$. A limit of this inverse system

$$(\varprojlim X_i \xrightarrow{p_j} X_j)_{j \in J}$$

is called an *inverse limit*.

Explicitly, for each $i \leq j$ in J, the following diagram commutes.

$$\begin{tikzcd}
\varprojlim D_i \arrow[r, "p_j"] \arrow[dr, "p_i"'] & X_j \arrow[d, "f_{ij}"] \\
 & X_i
\end{tikzcd}$$

As before, $\varprojlim X_i$ satisfies the universal property of the limit.

Chapter 3

Topology

In this chapter, we begin to outline some of the topology required for condensed mathematics. Often in later sections, we reference limits and colimits in subcategories of **Top**. Thus, in Section 3.2 we provide a useful characterization of limits and colimits in **Top**. To do this, we introduce initial and final topologies in Section 3.1. The theory of final topology will also be useful when we define "compactly generated spaces" in Section 4.4. The aforementioned subcategories of **Top** are defined as full subcategories of **CHaus**. In Section 3.3, we therefore outline some general facts about compact Hausdorff spaces that will be useful later on.

3.1 Initial & Final Topologies

In this section, we define the initial and the final topologies on a set with respect to a class of maps. Our reference for this section is [Bro06, Section 4.2, 5.6]. We first recall some terminology from topology. We begin with a definition.

Definition 3.1.1 (Initial Topology)**.** Let X be a set and let $\{Y_i\}_{i\in I}$ be a class of I-indexed topological spaces with associated functions

$$f_i \colon X \to Y_i,$$

for each $i \in I$. We say that a topology τ on X is *initial with respect to the class of functions* $F = \{f_i \colon X \to Y_i\}_{i\in I}$ if for each topological space Z and function $g\colon Z \to X$, g is continuous if and only if $f_i \circ g \colon Z \to Y_i$ is continuous for all $i \in I$.

$$\begin{tikzcd} X \arrow[r, "f_i"] & Y_i \\ Z \arrow[u, "g"] \arrow[ur, "f_i \circ g"'] & \end{tikzcd}$$

Of course, at this point, we do not know if an initial topology on X with respect to F exists or if one does, if it is unique. The following proposition proves the latter statement.

Proposition 3.1.2 ([Bro06, Proposition 5.6.6])**.** *If τ is an initial topology on X with respect to F, then it is the coarsest topology on X such that $f_i \colon X \to Y_i$ is continuous for all $i \in I$.*

Proof. Let $\mathrm{id}_{(X,\tau)} \colon (X,\tau) \to (X,\tau)$ be the identity morphism on (X,τ). Then since $\mathrm{id}_{(X,\tau)}$ is continuous, it follows that $f_i = f_i \circ \mathrm{id}_{(X,\tau)}$ is continuous for all $i \in I$. Now let τ' be a topology on X such that $f_i \colon (X,\tau') \to Y_i$ is continuous for all $i \in I$. Let $i\colon (X,\tau') \to (X,\tau)$ be the set-theoretic identity map on X. Since $f_i \circ i = f_i$ is a continuous map $(X, \tau') \to Y_i$ for all $i \in I$, and since τ is initial with respect to F, it follows that i is continuous. $\square$

To provide a characterization for the initial topology on X with respect to F, we first recall some terminology from topology.

Definition 3.1.3 (Subbasis)**.** Let X be a topological space. A *subbasis* $\mathcal{B}$ is a collection of subsets of X such that the union of these subsets is equal to X.

Define a collection τ of subsets of X as follows. We declare $U \subset X$ to be an element of τ if U can be written as the union of a finite intersection of elements of $\mathcal{B}$. One can then obtain that τ is a topology, called the *topology generated by* $\mathcal{B}$ [Mun75, pp. 82–83].

Proposition 3.1.4 ([Bro06, Proposition 5.6.7])**.** *The initial topology on X with respect to F is the topology τ on X whose subbase is generated by the sets*

$$\{f_i^{-1}(U) \mid i \in I \text{ and } U \subset Y_i \text{ is open in } Y_i\}.$$

Proof. We show that τ is initial with respect to F. Let $g\colon Z \to X$ be a function from a topological space Z. Note firstly that f_i is continuous for all $i \in I$: if $U \subset Y_i$ is open for some $i \in I$, then $f_i^{-1}(U)$ is an element of the subbase of τ and is hence open in X. Thus if g is continuous, then the composition $f_i \circ g$ is continuous for all $i \in I$. Now suppose that $f_i \circ g$ is continuous for all $i \in I$. We show that g is continuous. Let $V \subset X$ be open. Then V can be written as

$$V = \bigcup_{j \in I} \bigcap_{k \in I_j} f_k^{-1}(U_k),$$

where each I_j is finite and each U_k is open in Y_k. Then the preimage of V under g is

$$g^{-1}(V) = \bigcup_{j \in I} \bigcap_{k \in I_j} g^{-1}(f_k^{-1}(U_k)).$$

We note that each $g^{-1}(f_k^{-1}(U_k)$ is open in Z by continuity of $f_i \circ g$ and since $g^{-1}(V)$ is the union of a finite intersection of these sets, it follows that $g^{-1}(V)$ is open in Z. Therefore g is continuous. $\square$

Example 3.1.5. Let $(X_i)_{i \in I}$ be an I-indexed family of topological spaces. The initial topology on the set $\prod_{i \in I} X_i$ with respect to the projections $\{p_j\colon \prod_{i \in I} X_i \to X_j\}_{j\ I}$ has as subbase the sets

$$\{p_i^{-1}(U) \mid i \in I \text{ and } U \subset Y_i \text{ is open in } Y_i\}.$$

We call this the *product topology on* $\prod_{i \in I} X_i$.

Our treatment of the final topology is essentially dual to the above discussion.

Definition 3.1.6. Let X be a set and let $\{Y_i\}_{i \in I}$ be a class of I-indexed topological spaces with associated functions

$$f_i\colon Y_i \to X,$$

for each $i \in I$. We say that a topology τ on X is *final with respect to the class of functions* $F = \{f_i\colon Y_i \to X\}_{i \in I}$ if for each topological space Z and function $g\colon X \to Z$, g is continuous if and only if $g \circ f_i$ is continuous for all $i \in I$.

$$\begin{tikzcd} Y_i \arrow[r, "f_i"] \arrow[dr, "g\circ f_i"'] & X \arrow[d, "g"] \\ & Z \end{tikzcd}$$

The proof of the following proposition is analogues to the proof of Proposition 3.1.2.

Proposition 3.1.7 ([Bro06, Proposition 4.2.1])**.** *If τ is a final topology on X with respect to F, then it is the finest topology on X such that $f_i\colon Y_i \to X$ is continuous for all $i \in I$.*

Given the notation as above, we define a family τ of subsets of X as follows. For $U \subset X$, we say $U \in \tau$ if and only $f_i^{-1}(U)$ is open in Y_i for all $i \in I$. Then τ defines a topology on X. We can similarly define a topology τ' on X by declaring $V \subset X$ to be closed if and only if $f_i(V)$ is closed in Y_i for all $i \in I$ [Bro06, p. 102].

Proposition 3.1.8. *The topologies τ and τ' on X are the same.*

Proof. Let $U \subset X$. Then $U \in \tau'$ if and only if $X \setminus U$ is closed in τ' if and only if $f_i^{-1}(X \setminus U) = Y_i \setminus \big(f_i^{-1}(U)\big)$ is closed in Y_i for all $i \in I$. This is true if and only if $f_i^{-1}(U)$ is open in Y_i for all $i \in I$ if and only if $U \in \tau$. □

We then obtain that the final topology with respect to F on X exists and the following characterization for it.

Proposition 3.1.9 ([Bro06, Proposition 4.2.2])**.** *The final topology on X with respect to F is the topology τ on X such that a subset U of X is declared to be open (or closed respectively) if and only if $f^{-1}(U)$ is open (or closed respectively) in Y_i for all $i \in I$.*

Proof. We show that τ is final with respect to F. Let $g\colon X \to Z$ be a function for a topological space Z. Note firstly that f_i is continuous for all $i \in I$; indeed, if $U \subset X$ is open, then $f_i^{-1}(U)$ is open in Y_i for all $i \in I$. Thus if g is continuous, then the composition $g \circ f_i$ is certainly continuous. Suppose now that $g \circ f_i$ is continuous for all $i \in I$. Let $U \subset Z$ be open. Then $f_i^{-1}\big(g^{-1}(U)\big)$ is open in Y_i for all $i \in I$. Hence $g^{-1}(U)$ is open in X. Therefore g is continuous. □

Example 3.1.10. Let $(X_i)_{i\in I}$ be an I-indexed family of topological spaces and let $\coprod_{i\in I} X_i$ denote their disjoint union. Let τ denote final topology on $\coprod_{i\in I} X_i$ with respect to the canonical injections

$$\varphi_j\colon X_j \to \textstyle\coprod_{i\in I} X_i.$$

Then $U \subset \coprod_{i\in I} X_i$ is in τ if and only if $\varphi_j(U)$ is open in X_j for each $j \in I$. We call this the *the disjoint union topology on* $\coprod_{i\in I} X_i$.

3.2 Limits & Colimits in Top

The goal of this section is to provide a useful characterizations for limits and colimits in **Top**. Let **J** be a small category and $D\colon \mathbf{J} \to \mathbf{Top}$ a diagram. Suppose for each $I \in \mathbf{J}$, $D(I) = (S(I), \tau_I)$ where $S(I) \in \mathbf{Set}$ and τ_I is a topology on $S(I)$. Recall that the forgetful functor on **Top**, $F\colon \mathbf{Top} \to \mathbf{Set}$, maps each topological space to its underlying set and each continuous map to its underlying function. Then we obtain a diagram $F \circ D\colon \mathbf{J} \to \mathbf{Set}$ of the underlying sets for which the limit and colimit is guaranteed to exist. Let $\big(L \xrightarrow{p_I} S(I)\big)_{I\in\mathbf{J}}$ denote the limit of $F \circ D$ and $\big(S(I) \xrightarrow{q_I} C\big)_{I\in\mathbf{J}}$ the colimit of D.

Theorem 3.2.1. *The limit of D exists. Moreover, it is the underlying set-theoretic limit endowed with the initial topology with respect to the projections $\{p(I)\colon L \to D(I)\}_{I\in\mathbf{J}}$.*

Proof. Since L is the limit of $F \circ D$, for $u\colon I \to J$ in $\mathbf{J}$, the following diagram commutes in **Set**.

$$\begin{tikzcd}
& S(I) \arrow[dd, "(F\circ D)(u)"] \\
L \arrow[ur, "p_I"] \arrow[dr, "p_J"'] & \\
& S(J)
\end{tikzcd}$$

Endow L with the initial topology with respect to the class of maps

$$\{p_I\colon L \to D(I)\}_{I\in\mathbf{J}}.$$

Firstly, note that due to Proposition 3.1.2, each p_I is then continuous. Hence the following diagram commutes in **Top** for each morphism $u\colon I \to J$ in $\mathbf{J}$.

$$\begin{tikzcd}
& D(I) \arrow[dd, "D(u)"] \\
L \arrow[ur, "p_I"] \arrow[dr, "p_J"'] & \\
& D(J)
\end{tikzcd}$$

Thus, L is a cone on D. Now we must show that it satisfies the universal property of the limit. Let $(A \xrightarrow{f_I} D(I))_{I\in\mathbf{J}}$ be a cone on D. Then since L is a limit of $F \circ D$, there exists a unique function $f\colon A \to L$ such that for each $I \in \mathbf{J}$, the following diagram commutes in **Set**.

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "f_I"] & \\
L \arrow[r, "p_I"'] & S(I)
\end{tikzcd}$$

Each $f_I\colon A \to D(I)$ is continuous and hence by the initial topology on L, it follows that f is continuous. $\square$

We omit the proof of the next theorem since it is dual to the proof of the previous theorem.

Theorem 3.2.2 ([Rie17, Example 3.5.3])**.** *The colimit of D exists. Moreover, it is the underlying set-theoretic colimit endowed with the final topology with respect to the coprojections $\{q(I)\colon D(I) \to L\}_{I\in\mathbf{J}}$.*

Example 3.2.3. Let $(X_i)_{i\in I}$ be an I-indexed family of topological spaces. Then due to Examples 3.1.5 and 3.1.10:

(i) The product of $(X_i)_{i\in I}$ in **Top** is $\prod_{i\in I} X_i$ endowed with the product topology;

(ii) The coproduct of $(X_i)_{i\in I}$ in **Top** is $\coprod_{i\in I} X_i$ endowed with the disjoint union topology.

This characterizes limits and colimits in **Top** but not necessarily in subcategories of **Top**. In the next section, we show that the product and coproduct of objects of **CHaus** are exactly the product and coproduct of the objects regarded as objects of **Top**.

3.3 Compact Hausdorff Spaces

Compact Hausdorff spaces are "nice" in that they satisfy certain useful properties. This section will outline some of these properties that will be useful in later chapters. Our primary reference here is [Mun75, Section 3.5]. We begin with the following characterization of closed sets in compact Hausdorff spaces.

Proposition 3.3.1. *Let X be a topological space.*

(i) If X is a compact space and if $Y \subset X$ is closed, then Y is compact [Mun75, Section 3.5, Theorem 5.2].

(ii) If X is a Hausdorff space and if $Y \subset X$ is compact, then Y is closed [Mun75, Section 3.5, Theorem 5.3].

Hence, $Y \subset X$ is closed in a compact Hausdorff space X if and only if it is compact.

Proposition 3.3.2 ([Mun75, Section 3.5, Theorem 5.5])**.** *Suppose X, Y are topological spaces and that X is compact. Let $f\colon X \to Y$ be a continuous function. Then the image $f(X)$ is compact.*

We are now ready for our first major result about compact Hausdorff spaces. In topology, it is well known that continuous bijections between topological spaces are not guaranteed to be homeomorphisms. This statement does not hold when the topological spaces are compact Hausdorff. More precisely:

Theorem 3.3.3 ([Mun75, Section 3.5, Theorem 5.6])**.** *Suppose X is a compact space and Y is a Hausdorff space. Every continuous bijection $f\colon X \to Y$ is a homeomorphism.*

Proof. We show the continuity of $f^{-1}\colon Y \to X$. Let $U \subset X$ be closed. By Proposition 3.3.1 (i), U is compact. The inverse image of U by f^{-1} is the image $f(U)$. By Proposition 3.3.2, $f(U)$ is compact. By Proposition 3.3.1 (ii), this set is closed. □

We reference the following theorem in Proposition 4.3.7.

Theorem 3.3.4. *Suppose X is a compact space and Y is a Hausdorff space. Every continuous surjection $f\colon X \to Y$ is a quotient map.*

Proof. By Proposition A.2.3, it is sufficient to show that f is a closed map. Let $U \subset X$ be closed. By Proposition 3.3.2, $f(U)$ is compact. By Proposition 3.3.1 (ii), $f(U)$ is closed. □

We concluded the previous section by claiming that the product and coproduct of objects of **CHaus** is the product and coproduct of the spaces when regarded as objects of **Top**. We now provide justification for this claim. We first show that the product / coproduct of compact Hausdorff spaces in **Top** is compact Hausdorff.

Proposition 3.3.5 ([Stacks, Tag 08ZU])**.** *Let $(X_i)_{i \in I}$ be a family of compact spaces. Then the product $\prod_{i \in I} X_i$ of these spaces in* **Top** *is compact.*

Proposition 3.3.6. *Let $(X_i)_{i\in I}$ be a family of Hausdorff spaces. Then the product $X = \prod_{i\in I} X_i$ of these spaces in* **Top** *is Hausdorff.*

Proof. Let $p_i \colon X \to X_i$ denote the projections for $i \in I$. Let $x = (x_i)_{i\in I}, y = (y_i)_{i\in I}$ be distinct. Then for some $j \in I$, $x_j \neq y_j$. By the Hausdorff condition on X_j, there exists disjoint open sets U_j and V_j in X_j containing x_j and y_j respectively. Then the open set $p_j^{-1}(U_j)$ in X containing x and the open set $p_j^{-1}(V_j)$ in X containing y are disjoint. $\square$

Proposition 3.3.7. *Let $(X_i)_{i\in I}$ be compact spaces. Then $X = \coprod_{i\in I} X_i$ is compact.*

Proof. Let $\varphi_i \colon X_i \to X$, $x \mapsto (x, i)$ denote the canonical injections for $i \in I$. Let $\mathcal{C}$ be an open cover of X. For $i \in I$, by Proposition 3.3.2, $\varphi_i(X_i)$ is compact and thus admits a finite subcover of $\mathcal{C}$, $\mathcal{C}_i$. Since $X = \bigcup_{i\in I} \varphi_i(X_i)$, it follows that the union of $\mathcal{C}_i$ is a finite open cover of X that is also a subcover of $\mathcal{C}$. $\square$

Proposition 3.3.8. *Let $(X_i)_{i\in I}$ be Hausdorff spaces. Then $X = \coprod_{i\in I} X_i$ is Hausdorff.*

Proof. Consider distinct $(x, i) \in \varphi(X_i), (y, j) \in \varphi(X_j)$. If $i = j$, then the Hausdorff condition on X_i implies that we can find disjoint open sets U_x containing x and V_y containing y. Clearly then $\varphi(U_x)$, $\varphi(V_y)$ are disjoint open sets containing $(x, i), (y, i)$ respectively. If $i \neq j$, then the open sets $\varphi(X_i), \varphi(X_j)$ containing $(x, i), (y, j)$ respectively are disjoint. $\square$

Hence, if $(X_i)_{i\in I}$ are compact Hausdorff spaces, then their product $\prod_{i\in I} X_i$ and coproduct $\coprod_{i\in I} X_i$ in **Top** is compact Hausdorff as well. Since **CHaus** is a full subcategory of **Top**, the canonical projections and coprojections of the product and coproduct (respectively) are morphisms in **CHaus**. Moreover, if we have a cone (or cocone respectively) on the diagram in **CHaus**, then the unique morphism obtained from the universal property of the product (or coproduct respectively) in **Top** is a morphism in **CHaus** as well. Hence, the product / coproduct of objects of **CHaus** is just the product / coproduct of the objects regarded as objects of **Top**.

Chapter 4

Profinite Sets & Extremally Disconnected Spaces

In this chapter, we describe the two main classes of topological spaces required for the theory of condensed mathematics: profinite sets and extremally disconnected spaces. The full subcategories of **CHaus** consisting of profinite sets and extremally disconnected spaces form categories called **Prof** and **ED**, respectively. In Chapter 5, we use the category **ED** to define a condensed set. An important fact about these extremally disconnected compact Hausdorff spaces is that they can be defined without relying on classical topology. To do this, we first describe a non-topological construction of **Prof** and then show that objects of **ED** are "projective objects" in **Prof** in Section 4.2. To provide an example of an object of **ED**, we explain the Stone-Čech compactification of a topological space in Section 4.3. Finally, we define and discuss compactly generated spaces in Section 4.4, which we once again will use in Chapter 5.

4.1 Profinite Sets

In this section, we define a profinite set. We first provide a topological description of these spaces, and then a characterization that does not reply on topology.

Definition 4.1.1 (Totally Disconnected)**.** Let X be a topological space. X is said to be *totally disconnected* if the connected components of X are singletons.

Theorem 4.1.2 ([Stacks, Tag 08ZY])**.** *Let X be a topological space. The following are equivalent.*

(i) X is homeomorphic to an inverse limit (in **Top***) of finite discrete spaces.*

(ii) X is a compact Hausdorff, totally disconnected space.

Definition 4.1.3 (Profinite Set)**.** Let X be a topological space. If X satisfies either of the conditions of Theorem 4.1.2, it is called a *profinite set*.

Example 4.1.4. Every finite discrete space is a profinite set.

A second example of a profinite set is the Cantor set. We first recall the definition. Let

$$C_0 = [0,1],$$
$$C_1 = [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1].$$

In general, for $n \geq 1$ define

$$C_n = \tfrac{C_{n-1}}{3} \cup \left(\tfrac{2}{3} + \tfrac{C_{n-1}}{3}\right)$$

In other words, C_n is obtained by removing the open middle third interval of each interval in C_{n-1}. The Cantor set is then defined as $C = \bigcap_{n=0}^{\infty} C_n$. To show that the Cantor set is profinite, we first state the following result.

Lemma 4.1.5 ([Cla20, Lemma 2.103]). *If $\{0,1\}$ is endowed with the discrete topology, then the Cantor set is homeomorphic to the product $\prod\limits_{n\in\mathbb{N}} \{0,1\}$.*

Proposition 4.1.6. *The Cantor set C is a profinite set.*

Proof. It suffices to show that $X = \prod_{n\in\mathbb{N}}\{0,1\}$ is profinite. For every $i \in \mathbb{N}$, let $p_i \colon X \to \{0,1\}$ be the projection onto the ith component. Due to Propositions 3.3.5 and 3.3.6, it follows that X is compact Hausdorff. Now we show that X is totally disconnected. Let $x = (x_n)_{n\in}, y = (y_n)_{n\in\mathbb{N}} \in X$ and suppose that $x \neq y$. Then $x_i \neq y_i$ for some $i \in \mathbb{N}$. Then $p_i^{-1}(x_i)$ and $p_i^{-1}(y_i)$ are disjoint open sets whose union is X. Then due to Proposition A.1.5, it follows that x and y are in distinct connected components. Hence the connected components of X are the singletons. ☐

One immediately obtains that **Prof** is a full subcategory of **CHaus**. From the definition of profinite sets, we further see that profinite sets inherit the nice properties of compact Hausdorff spaces described in Section 3.3.

The content of Theorem 4.1.2 suggests a non-topological viewpoint for the objects of **Prof**. We want a similar viewpoint for the morphisms of **Prof**. First, we state the following lemma. Recall that if (X_i, f_{ij}, J) is an inverse system in a category $\mathcal{C}$, we denote the inverse limit of this inverse system by $\varprojlim X_i$. Additionally, we let $p_j \colon \varprojlim X_i \to X_j$ denote the canonical projections for each $j \in J$.

Lemma 4.1.7 ([RZ10, Lemma 1.1.16 (a)]). *Let (X_i, f_{ij}, J) be an inverse system of profinite sets and let $\varprojlim X$ denote the inverse limit in* **Top**. *If $f \colon X_i \to Y$ is a continuous map, then there exists a $j \in J$ and a continuous map $f_j \colon X_j \to Y$ such that the following diagram commutes.*

$$\begin{tikzcd}
\varprojlim X_i \arrow[r, "f"] \arrow[d, "p_j"'] & Y \\
X_j \arrow[ur, "f_j"'] &
\end{tikzcd}$$

Theorem 4.1.8. *Let $(X_i, f_{ij,I})$ and (Y_j, g_{ij}, J) be inverse systems of finite discrete spaces and let $\varprojlim X_i$ and Y_j be their inverse limits in* **Top**, *respectively. Every map $f \colon \varprojlim X_i \to \varprojlim Y_j$ is continuous if and only if for every $j \in J$, there exists an $i \in I$ and a map $f_{ij} \colon X_i \to Y_j$ such that the following diagram commutes:*

$$\begin{tikzcd}
\varprojlim X_i \arrow[r, "f"] \arrow[d, "p_i"'] & \varprojlim Y_j \arrow[d, "p_j"] \\
X_i \arrow[r, "f_{ij}"'] & Y_j
\end{tikzcd}$$

Proof. Let $f\colon \varprojlim X_i \to \varprojlim Y_j$ be a map. Suppose that f is continuous. Then for each $j \in J$, $p_j \circ f\colon \varprojlim X_i \to Y_j$ is continuous. Since $\varprojlim$ is an inverse limit of finite discrete spaces (and hence profinite sets), it follows from the previous lemma that there exists some $i \in I$ and a continuous map $f_{ij}\colon X_i \to Y_j$ such that the above diagram commutes. Now suppose that for each $j \in J$, there exists some $i \in I$ and a map $f_{ij}\colon X_i \to Y_j$ such that the above diagram commutes. Since each X_i is discrete, it follows that each f_{ij} is continuous. Hence $p_j \circ f = f_{ij} \circ p_i$ is continuous for each $j \in J$. Since $\varprojlim Y_j$ is endowed with the initial topology with respect to the canonical projections, it follows that f is continuous. □

The significance of the this theorem is that morphisms in **Prof** are essentially characterized by the existence of certain morphisms between finite discrete spaces; these are simply functions between sets.

4.2 Extremally Disconnected Spaces

This section describes extremally disconnected spaces and the category **ED**. Similar to section 4.1, we first describe **ED** using topology and then categorically.

Definition 4.2.1 (Extremally Disconnected)**.** Let X be a topological space. We say X is *extremally disconnected* if the closure of every open set is open.

Example 4.2.2. Every discrete space is extremally disconnected.

Example 4.2.3. Let X be an infinite set with the cofinite topology

$$\tau = \{A \subset X \mid A = \emptyset \text{ or } (X \setminus A) \text{ is finite}\}$$

Let $A \subset X$ be open. We show that $\overline{A} = X$ and hence X is extremally disconnected. Let $x \in X$ and let $U \subset X$ be an open neighbourhood of x. Since U is open, $X \setminus U$ is finite. Since A is infinite, $A \not\subset X \setminus U$. Hence, there exists $a \in A$ such that $a \in U$. This proves the claim. Observe also that X cannot be written as the union of disjoint open sets (the complement of an open set is finite) and hence X is connected. In particular, X is not totally disconnected.

The above example demonstrates that an extremally disconnected space X is not necessarily totally disconncected. The following provides a sufficient condition for X to be totally disconnected.

Proposition 4.2.4 ([Ásg21, Proposition 1.2.5])**.** *Let X be an extremally disconnected Hausdorff space. Then X is totally disconnected.*

Proof. Let $x \in X$ and let C_x be the connected component of x. Let $y \in X$ with $y \neq x$. By the Hausdorff condition, there exists an open set U containing x and an open set V containing y such that $U \cap V = \emptyset$. It follows then that $y \notin \overline{U}$. Note by Proposition A.1.4, $C_x = \overline{C_x}$. Since X is extremally disconnected, $\overline{U}$ is open and closed and $C_x = \overline{C_x}$ is open. In particular $C_x \cap \overline{U}, C_x \cap \overline{U}^c$ are open. Now write $C_x = (C_x \cap \overline{U}) \cup (C_x \cap \overline{U}^c)$. If $C_x \cap \overline{U}^c \neq \emptyset$, then C_x can be written as the union of two disjoint open sets. Thus $C_x \cap \overline{U}^c = \emptyset$. It follows then that $C_x \subset \overline{U}$ and therefore $y \notin C_x$. □

In particular, if X is an extremally disconnected compact Hausdorff space, then it is a profinite set. We thus define **ED** to be the full subcategory of **Prof** consisting of extremally disconnected spaces as objects. We now provide a non-topological description of these objects. First, some terminology.

Definition 4.2.5 (Projective Objects)**.** Let $\mathcal{C}$ be a category. We say that an object $A \in \mathcal{C}$ is a *projective object* in $\mathcal{C}$ if for any morphism $f\colon A \to B$ and any epimorphism $g\colon C \to B$, there exists a morphism $\overline{f}\colon A \to C$ such that the following diagram commutes

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[ur, "\overline{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

With this terminology, we state the following theorem.

Theorem 4.2.6 ([Stacks, Tag 08YN])**.** *Let X be a compact Hausdorff space. Then the following are equivalent.*

(i) X is extremally disconnected.

(ii) X is a projective object in **Prof**.

(iii) Every surjective continuous map $f\colon Y \to X$ from a compact Hausdorff space Y has a continuous right inverse.

We postpone providing an example of an object of **ED** until we have discussed the Stone-Čech compactification. Instead, we conclude this section with the following analogue to Propositions 3.3.7 and 3.3.8.

Proposition 4.2.7. *Let $I = \{1, 2, \ldots, n\}$ and let $(X_i)_{i\in I}$ be a collection of extremally disconnected spaces. Then the coproduct $X = \coprod_{i\in I} X_i$ in* ***Top*** *is extremally disconnected.*

Proof. Let U be an open set in X. We can write $U = \bigcup_{i=1}^{n} \varphi_i(U_i)$ where each U_i is open in X_i, and $\varphi_i\colon X_i \to X$ is the canonical injection for $i \in I$. Note that $\overline{U} = \bigcup_{i=1}^{n} \overline{\varphi_i(U_i)}$. Each $\overline{\varphi_i(U_i)}$ is open in X as each preimage $\overline{U_i}$ is open in X_i and thus $\overline{U}$ is open in X. □

Hence, due to a similar reasoning to Section 3.3, we obtain that the coproduct of a finite collection of objects of **ED** is simply the disjoint union of the objects.

4.3 Stone-Čech Compactification

The goal of this section is to exhibit an example of an object of **ED**. We use the Stone-Čech compactification to do this. The topological motivation for studying the Stone-Čech compactification is as follows: compact Hausdorff spaces satisfy some nice properties (some that we have outlined previously). Hence, we would like a canonical way of mapping a general topological space to a compact Hausdorff space.

Definition 4.3.1 (Stone-Čech Compactification)**.** A Stone-Čech compactification of a topological space X is a compact Hausdorff space βX along with a continuous map $i_X\colon X \to \beta X$ such that the following property holds. For any continuous map $f\colon X \to Y$ where Y is a compact Hausdorff space, there exists a continuous map $\bar{f}\colon \beta X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & \beta X \arrow[dl, "\bar{f}"] \\ Y & \end{tikzcd}$$

If a Stone-Čech compactification of a topological space X exists, then due to a similar reasoning to the uniqueness of limits / colimits, it is unique up to homeomorphism.

Theorem 4.3.2 ([Stacks, Tag 0908])**.** *The Stone-Čech compactification of every topological space exists.*

Example 4.3.3. The Stone-Čech compactification of a compact Hausdorff space X is itself along with the identity map $\mathrm{id}_X \colon X \to X$. This can be seen with the universal property of the Stone-Čech compactification. Indeed, for every map $f \colon X \to Y$ into a compact Hausdorff space Y, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[dl, "f"] \\ Y & \end{tikzcd}$$

In general, it is very hard to provide an explicit example of a Stone-Čech compactification of a topological space X. The situation we are particularly interested in is when X is discrete. We will explain shortly why in this case, βX cannot be explicitly constructed.

Let $f \colon X \to Y$ be a continuous map of topological spaces. We say that f is a *dense embedding* if X is homeomorphic to the image $f(X)$ and if the closure $\overline{f(X)} = Y$ [Mun75, p. 239]. For "Tychonoff spaces", the Stone-Čech compactification is a dense embedding.

Definition 4.3.4 (Tychonoff Space)**.** We say a topological space X is *completely regular* if for each $x \in X$ and closed subset A of X not containing x, there exists a continuous map $f \colon X \to [0,1]$ such that $f(x) = 1$ and $f(A) = \{0\}$. If X is also Hausdorff, we say X is a *Tychonoff space*.

Proposition 4.3.5 ([Čec37, p. 9])**.** *If X is a Tychonoff space, then the Stone-Čech compactification $i_X \colon X \to \beta X$ is a dense embedding.*

If X is discrete and $A \subset X$ is closed, then the function $f \colon X \to [0,1]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{otherwise} \end{cases}$$

is a continuous map. Since X is also Hausdorff, it is certainly Tychonoff and the above propositon applies.

The reason that βX cannot be constructed explicitly is then as follows. It can be shown that the set βX corresponds to the set of *ultrafilters* on X. Here, the points

$i_X(X)$ correspond to the *principal ultrafilters* whereas the points $\beta X \setminus i_X(X)$ correspond to the *non-principal ultrafilters*. These non-principal ultrafilters cannot be defined without the axiom of choice, hence the difficulty in providing an explicit construction of βX [Hin98, Section 3.1-3.3]. In any case, we are ready to prove the main result of this section.

Theorem 4.3.6 ([Ásg21, Example 1.2.7])**.** *The Stone-Čech compactification of a discrete space X is extremally disconnected.*

Proof. Since X is homeomorphic to $\mathrm{Im}(i_X) \subset \beta X$, we may suppose that $X \subset \beta X$. Let $g\colon Y \to \beta X$ be a continuous surjection from a compact Hausdorff space Y. We wish to show that there exists a continuous map $f'\colon \beta X \to Y$ such that $g \circ f'$ is the identity morphism $\mathrm{id}_{\beta X}$ of βX in **Top**. Define a function $f\colon X \to Y$ as follows. For each $x \in X$, let $f(x)$ be any $y \in Y$ such that $g(y) = x$. Then since X is discrete, f is continuous. This has two consequences: firstly, $g \circ f$ is the identity morphism i_X of X in **Top**. Secondly, since Y is compact Hausdorff, there exists a unique morphism $\bar{f}\colon \beta X \to Y$ such that the following diagram commutes in **Top**.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_X"'] & Y \\ \beta X \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Hence, the following diagram commutes as well.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_X"'] & Y \arrow[r, "g"] & X \\ \beta X \arrow[ur, "\bar{f}"'] & & \end{tikzcd}$$

Thus the restriction $(g \circ \bar{f})_{|X}\colon X \to \beta X$ is the identity on X. Since $\overline{X} = \beta X$ and βX is Hausdorff, the map $g \circ \bar{f}$ extends uniquely (due to [Mun75, p. 112]) to the identity on βX. □

Using the above theorem, we obtain the following proposition.

Proposition 4.3.7 ([Ásg21, Example 1.2.8])**.** *Let $X \in$ **CHaus**. There exists an object $X' \in$ **ED** such that when X' is viewed as an object in **CHaus**, there exists a quotient map $f\colon X' \to X$ in **CHaus**.*

Proof. Let X_{Discrete} denote the underlying set of X equipped with the discrete topology and let $\beta X_{\mathrm{Discrete}}$ denote the compact Hausdorff space obtained from the Stone-Čech compactification of X_{Discrete}. By the above, $\beta X_{\mathrm{Discrete}} \in$ **ED**. Let $i\colon X_{\mathrm{Discrete}} \to X$ denote the identity map on the underlying sets of X_{Discrete} and X. Clearly i is a continuous map into a compact Hausdorff space. By the universal property of Stone-Čech compactification, there exists a continuous map $f\colon \beta X_{\mathrm{Discrete}} \to X$ such that in **CHaus**, the following diagram commutes

$$\begin{tikzcd} X_{\text{Discrete}} \arrow[r, "i_X"] \arrow[d, "i"'] & \beta X_{\text{Discrete}} \arrow[dl, "f"] \\ X & \end{tikzcd}$$

Since i is a surjection, it follows that f is a surjection too. By Theorem 3.3.4, it follows that f is the quotient map that we desire. $\square$

4.4 Compactly Generated Spaces

In Chapter 5, we describe a functor $\square$: **Top** $\to$ **Cond(Set)**. A reasonable question to ask is whether $\square$ is fully faithful or not. More informally, this is equivalent to asking if we lose any information about **Top** by moving into **Cond(Set)** through $\square$. The answer, derived in Lemma 5.0.5 and Theorem 5.0.6, is that $\square$ is in general faithful and only fully faithful when restricted to the full subcategory of **Top** consisting of compactly generated spaces as objects. Roughly speaking, a topological space X is compactly generated if the topology on X is determined entirely by maps from compact Hausdorff spaces. This section firstly describes their construction and secondly enumerates a few examples of these spaces.

Remark 4.4.1. There exist at least three non-equivalent definitions of a compactly generated space in the literature. We use the definition outlined in Scholze's original lectures on condensed mathematics [Sch19b, p. 8]. Our primary reference for this section is [Bro06, Section 5.9].

Definition 4.4.2 (Compactly Generated)**.** Let X be a topological space and

$$F = \{f_i \colon Y_i \to X\}_{i \in I}$$

be the class of all continuous maps from all compact Hausdorff spaces into X. We say X is *compactly generated* if the topology τ on X is equal to the final topology τ_F with respect to F.

Explicitly, X is compactly generated if it satisfies the following equivalent conditions:

1. $U \subset X$ is open (resp. closed) if and only if $f^{-1}(U)$ is open (resp. closed) for each map $f \colon Y \to X$ in F.

2. For any topological space Z, a function $g \colon X \to Z$ is continuous if and only if $g \circ f \colon Y \to Z$ is continuous for each map $f \colon Y \to X$ in F.

In fact, we can reduce the class F in the preceding discussion to the class F' of all continuous maps from all objects of **ED** into X. This fact is used later in Chapter 5.

Proposition 4.4.3. *Let X be a topological space. Denote the final topology on X induced by F and F' by τ_F and $\tau_{F'}$ respectively. Then $\tau_F = \tau_{F'}$.*

Proof. Let $U \subset X$ such that $U \in \tau_F$. Since $F' \subset F$, for all $f' \in F'$, the preimage $f'^{-1}(U)$ is open in each $Y' \in$ **ED**. Thus $U \in \tau_{F'}$. Now suppose $U \in \tau_{F'}$. Consider a map $f \colon Y \to X$ in F. Since $Y \in$ **CHaus**, by Proposition 4.3.7, there exists $Y' \in$ **ED** and a quotient map $g \colon Y' \to Y$. Then $f \circ g \colon Y' \to X$. Since $U \in \tau_{F'}$, the preimage $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$ is open in Y'. As g is a quotient map, $f^{-1}(U)$ is open in Y. Since this is true for all $f \in F$, it follows that $U \in \tau_F$. $\square$

We then define the category **CompGen** as the full subcategory of **Top** consisting of compactly generated spaces. The next part of this section is devoted to enumerating examples of compactly generated spaces. The first example is quite obvious.

Example 4.4.4. Let X be a compact Hausdorff space. We use the second criterion in Definition 4.4.2 to show that X is compactly generated. Suppose we have a function $f\colon X \to Z$ where Z is a topological space. Suppose for every $Y \in$ **CHaus** and every continuous map $g\colon Y \to X$, the composition $f \circ g\colon Y \to Z$ is continuous. Let $Y = X$ and $g = \mathrm{id}_X$. Then the composition $f \circ \mathrm{id}_X = f$ is continuous mapping between X and Z.

The main example we discuss are "first countable" spaces, which includes all metric spaces [Mun75, p. 190].

Definition 4.4.5 (First Countable Space)**.** Let X be a topological space. We say a point $x \in X$ has a *countable basis* if there exists a countable collection $\{B_n(x)\}_{n\in\mathbb{N}}$ of open neighbourhoods of x such that if U is an open neighbourhood of x, then there exists $i \in \mathbb{N}$ with $B_i(x) \subset U$. We say X is *first countable* if every point in X has a countable basis.

Example 4.4.6. Let (X, d) be a metric space. For each $x \in X$, consider the collection $\{B_d(x, \frac{1}{n})\}$. Now let U be an open neighbourhood of x. Then there exists $r > 0$ such that $B_d(x,r) \subset U$. By the Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < r$ and hence

$$B_d(x, \tfrac{1}{N}) \subset B_d(x,r) \subset U.$$

Note in the example above, $B_d(x, \frac{1}{n+1}) \subset B_d(x, \frac{1}{n})$ for each $n \in \mathbb{N}$. If X is a general countable space, we can construct a countable basis for each $x \in X$ that satisfies the analogous property in X. Let $\{B_n(x)\}_{n\in\mathbb{N}}$ be a countable basis at $x \in X$. Define a new basis $\{B'_n(x)\}_{n\in\mathbb{N}}$ at x by

$$\begin{aligned} B'_1(x) &= B_1(x), \\ B'_n(x) &= B'_{n-1}(x) \cap B_n(x), \end{aligned}$$

for all $n \in \mathbb{N}$ and $n > 1$. We then obtain $B'_{n+1}(x) \subset B'_n(x)$ for all $n \in \mathbb{N}$. We call $\{B'_n(x)\}$ *a collection of cofinal open neighbourhoods of* x. In particular, if we have a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in B'_n(x)$ for all $n \in \mathbb{N}$, then (x_n) converges to x.

To show that any first countable space is compactly generated, we first define the "Alexandroff extension" of $\mathbb{N}$, which satisfies the useful property described in Proposition 4.4.8.

Definition 4.4.7. Let $\mathbb{N}$ denote the set of natural numbers with the discrete topology. Define $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, where $\{\infty\}$ is the singleton set. We define a topology on $\mathbb{N}^*$ by declaring the open sets to be:

1. Open sets in $\mathbb{N}$
2. Sets of the form $\mathbb{N}^* \setminus C = (\mathbb{N} \setminus C) \cup \{\infty\}$, where C contains a finite number of elements of $\mathbb{N}$.

We call this the *Alexandroff extension* of $\mathbb{N}$.

The above defines a topology on $\mathbb{N} \cup \{\infty\}$. Moreover, $\mathbb{N} \cup \{\infty\}$ is compact Hausdorff [Kel75, Theorerm 21].

Proposition 4.4.8 ([nLa24, Example 4.2])**.** *Let X be a topological space with a sequence $(x_n)_{n\in\mathbb{N}}$. For $x \in X$, define the function $f\colon \mathbb{N}^* \to X$ by*

$$f(n) = x_n \text{ if } n \in \mathbb{N} \text{ and } f(\infty) = x$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if f is continuous.

Theorem 4.4.9 ([Sch19b, Remark 1.6])**.** *Let X be a first countable space. Then X is compactly generated.*

Proof. Let $V \subset X$ such that for all maps $f\colon Y \to X$ in F, $f^{-1}(V)$ is closed in Y. We wish to show that V is closed. Let $x \in \overline{V}$ and let $\{B_n(x)\}$ be a collection of cofinal open neighbourhoods of x. Since $x \in \overline{V}$

$$V \cap B'_n(x) \neq \emptyset,$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $x_n \in V \cap B'_n(x)$. We then obtain a map $g\colon \mathbb{N}^* \to X$ defined by

$$g(n) = x_n \text{ if } n \in \mathbb{N}, \text{ and } g(\infty) = x.$$

Since $(x_n)_{n\in\mathbb{N}}$ converges to x, by Proposition 4.4.8, it follows that g is continuous and therefore is in F. Hence the preimage $g^{-1}(V)$ is closed in $\mathbb{N}^*$ and that $\mathbb{N} \subset g^{-1}(V)$. Since $\{\infty\}$ is not an open set in $\mathbb{N}^*$, it follows that $g^{-1}(V) = \mathbb{N}^*$. Thus $x \in V$ and V is closed. $\square$

Chapter 5

Condensed Sets

In this short chapter, we define a condensed set and the category **Cond(Set)**. We also describe a functor **Top** $\to$ **Cond(Set)** which will be the main way to translate from topology into the condensed setting. Using this functor, we prove one of the highlights of this project: **CompGen** is equivalent to a full subcategory of **Cond(Set)**. The main references for this section are [Ásg21, Section 1.2], [Sch19b, pp. 6–9] and [Sch22, pp. 11–15].

Definition 5.0.1 (Condensed Set)**.** A *condensed set* is a presheaf on **ED**

$$T\colon \mathbf{ED}^{\mathrm{op}} \to \mathbf{Set}$$

such that for every finite collection $(S_i)_{i\in I}$ of objects of **ED**, the canonical morphism

$$T(\coprod_{i\in I} S_i) \to \prod_{i\in I} T(S_i)$$

is a bijection.

Note that a finite collection of objects of **ED** certainly includes an empty collection. In Section 2.4, we mentioned that the empty coproduct in a category $\mathcal{C}$ is the initial object and the empty product is the terminal object in $\mathcal{C}$. In $\mathbf{ED}^{\mathrm{op}}$, the initial object is $\emptyset$ and in **Set**, the terminal object is $*$. Hence a condensed set T satisfies $T(\emptyset) = *$.

We now explain what we mean by the canonical morphism in the definition above. For $j \in I$, we have (in **ED**) the canonical injections

$$S_j \to \coprod_{i\in I} S_i.$$

If T is a presheaf, then by applying T, we obtain in **Set**

$$T(S_j) \leftarrow T(\coprod_{i\in I} S_i).$$

Since such a morphism exists for all $j \in I$, by property of the product, there exists a canonical morphism

$$T(\coprod_{i\in I} S_i) \to \prod_{i\in I} T(S_i),$$

obtained by the universal property of the product. For T to be a condensed set, we want this canonical morphism to be a bijection for every finite collection $(S_i)_{i\in I}$ of objects of **ED**.

Definition 5.0.2. The category **Cond(Set)** is the full subcategory of $[\mathbf{ED}^{\mathrm{op}}, \mathbf{Set}]$ consisting of condensed sets.

Remark 5.0.3.

1. The definitions above present set-theoretic issues. Namely, if $\mathcal{C}$ and $\mathcal{D}$ are categories, recall that we can only consider the functor category $[\mathcal{C}, \mathcal{D}]$ if $\mathcal{C}$ is small. However, $\mathbf{ED}^{\mathrm{op}}$ is not small; this fact was explained to us by Dr. Tobias Rossmann. Every set can be endowed with the discrete topology and hence gives rise to an object of **ED** via the Stone-Čech compactification. The class of objects in $\mathbf{ED}^{\mathrm{op}}$ thus contains sets of arbitrary size and hence is not a set. This issue can be resolved by introducing a "restriction" on the cardinality of objects of **ED** and only considering objects with cardinality less than this restriction. We can then use these objects to construct a slightly different variant of the definitions above. In our text, we ignore this technicality, instead simply noting to the reader that these set-theoretic issues can be resolved, as in [Ásg21, Remark 1.2.1] and [Sch19b, Remark 1.3].

2. One can use the categories **CHaus** and **Prof** to define equivalent notions of a condensed set. This is the approach presented in [Ásg21, Section 1.2]. The definition of a condensed set in terms of **ED** is the simplest and so we opt for this choice.

For T a condensed set, we call $T(*)$ the *underlying set of* T. We now discuss a way to construct a condensed set $\underline{X}$ from any topological space X.

Proposition 5.0.4. *Let X be a topological space. Then the contravariant Hom-functor on X restricted to* $\mathbf{ED}^{\mathrm{op}}$

$$\mathrm{Hom}(-, X)_{|\mathbf{ED}^{\mathrm{op}}} \colon \mathbf{ED}^{\mathrm{op}} \to \mathbf{Set}$$

is a condensed set.

Proof. First note that $\mathrm{Hom}(\emptyset, X)_{|\mathbf{ED}^{\mathrm{op}}} = *$. Now let $I = \{1, 2, \dots, n\}$ and let $(S_i)_{i \in I}$ be objects of ED. For simplicity, let

$$\coprod S_i = \coprod_{i \in I} S_i \text{ and } \prod \mathrm{Hom}(S_i, X) = \prod_{i \in I} \mathrm{Hom}(S_i, X).$$

Let $\varphi_i \colon S_i \to \coprod S_i$ denote the canonical injections. Let $g \colon \mathrm{Hom}(\coprod S_i, X) \to \prod \mathrm{Hom}(S_i, X)$ be the morphism such that for all $j \in I$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(\coprod S_i, X) \arrow[d, "g"'] \arrow[dr, "\mathrm{Hom}(\varphi_j, X)"] & \\
\prod \mathrm{Hom}(S_i, X) \arrow[r, "p_j"'] & \mathrm{Hom}(S_j, X)
\end{tikzcd}$$

Then g is given by

$$g(f) = (f \circ \varphi_1, f \circ \varphi_2, \dots, f \circ \varphi_n),$$

for $f \in \mathrm{Hom}(\coprod S_i, X)$. We wish to show that g is a bijection. First we prove injectivity of g. Let $f_1, f_2 \in \mathrm{Hom}(\coprod S_i, X)$ and suppose that

$$(f_1 \circ \varphi_1, f_1 \circ \varphi_2, \dots, f_1 \circ \varphi_n) = (f_2 \circ \varphi_1, f_2 \circ \varphi_2, \dots, f_2 \circ \varphi_n).$$

Let $s \in S$. Then s is of the form $s = (s_i, i)$ for $i \in I$ and so

$$(f_1 \circ \varphi_i)(s_i) = f_1(s_i, i) = f_1(s) = (f_2 \circ \varphi_i)(s_i) = f_2(s_i, i) = f_2(s).$$

Hence, $f_1 = f_2$. Next we prove surjectivity of g. Let $(f_1, f_2, \ldots, f_n) \in \prod \mathrm{Hom}(S_i, X)$. By the universal property of the coproduct, this induces a unique morphism $f\colon \coprod S_i \to X$ such that for $j \in I$, the following diagram commutes

$$\begin{tikzcd} S_j \arrow[d, "\varphi_j"'] \arrow[dr, "f_j"] & \\ \coprod S_i \arrow[r, "f"'] & X \end{tikzcd}$$

Then $g(f) = (f_1, f_2, \ldots, f_n)$. □

We call the condensed set $\mathrm{Hom}(-, X)_{|\mathbf{ED}^{\mathrm{op}}}$ the *associated condensed set to* X and denote it $\underline{X}$. Similar to the Yoneda embedding, the mapping $X \to \underline{X}$ defines a functor $\underline{\square}$: **Top** $\to$ **Cond(Set)**. Once again, a morphism $f\colon X \to Y$ in **Top** is mapped to the natural transformation $\mathrm{Hom}(-, f)_{|\mathbf{ED}^{\mathrm{op}}}\colon \underline{X} \to \underline{Y}$, which we more simply denote as $\underline{f}$.

In [Sch22, pp. 12–13], Clausen and Scholze provide the central motivation behind the definitions presented thus far. The following is a summary of their discussion. The perspective of condensed mathematics is to replace each topological space X with the associated condensed set $\underline{X}$. From this viewpoint, the concepts presented above can be interpreted intuitively:

1. Contravariant functoriality of a condensed set: continuous maps $S_1 \to X$ and $S_2 \to S_1$ should compose to give a continuous map $S_2 \to X$, where $S_1, S_2 \in$ **ED**.

2. Axiom for a condensed set: defining a continuous map $\coprod_{i \in I} S_i \to X$ should be the same as defining continuous maps $S_i \to X$ separately for each $i \in I$.

3. Underlying set: the set of continuous maps $* \to X$ is in one-to-one correspondence with the set of points in X via the evaluation map

$$\begin{aligned} \mathrm{ev}_X\colon \underline{X}(*) &\to X \\ p &\mapsto p(*), \end{aligned}$$

   so it makes sense to call $\underline{X}(*)$ the underlying set of $\underline{X}$.

In Theorem 1.4.4, we obtained that the Yoneda embedding on a small category $\mathcal{C}$ is a fully faithful functor. The next two results can be thought of as analogues to 1.4.4 for the functor S: **Top** $\to$ **Cond(Set)** [Sch19b, Proposition 1.7].

Lemma 5.0.5. *The functor* $\underline{\square}$: **Top** $\to$ **Cond(Set)** *is faithful.*

Proof. We wish to show that if $X, Y \in$ **Top**, the map

$$\begin{aligned} \mathrm{Hom}(X, Y) &\to \mathrm{Hom}(\underline{X}, \underline{Y}) \\ f &\mapsto \mathrm{Hom}(-, f), \end{aligned}$$

is injective. Let $f, g \in \mathrm{Hom}(X, Y)$ and suppose that $\mathrm{Hom}(-, f) = \mathrm{Hom}(-, g)$. Consider the singleton set $*$ and the maps

$$\begin{aligned}\mathrm{Hom}(*, f)\colon \underline{X}(*) &\to \underline{Y}(*)\\ p &\mapsto f \circ p\\ \mathrm{Hom}(*, g)\colon \underline{X}(*) &\to \underline{Y}(*)\\ p &\mapsto g \circ p\end{aligned}$$

Since $\underline{X}(*)$ and X are in one-to-one correspondence via ev_X, we obtain that $f \circ (p(*)) = g \circ (p(*))$ implies that $f(x) = g(x)$ for all $x \in X$, and hence that $f = g$. $\square$

Theorem 5.0.6. *The restriction of* $\underline{\square}$: **Top** $\to$ **Cond(Set)** *to* **CompGen** *is a fully faithful functor* **CompGen** $\to$ **Cond(Set)**. .

Proof. Let X, Y be compactly generated spaces. It suffices to show that the map

$$\begin{aligned}\mathrm{Hom}(X, Y) &\to \mathrm{Hom}(\underline{X}, \underline{Y})\\ f &\mapsto \mathrm{Hom}(-, f),\end{aligned}$$

is surjective. In other words, if $\alpha\colon \underline{X} \to \underline{Y}$ is a natural transformation, then there exists a morphism $f\colon X \to Y$ such that for all $S \in \mathbf{ED}$ and $p\colon S \to X$:

$$f \circ p = \alpha_S(p). \tag{5.1}$$

Let $S \in \mathrm{ED}$ and $t\colon * \to S$. Since α is a natural transformation, we obtain the following commutative diagram.

$$\begin{tikzcd}
\underline{X}(S) \arrow[r, "\mathrm{Hom}(t,X)"] \arrow[d, "\alpha_S"'] & \underline{X}(*) \arrow[d, "\alpha_*"] \\
\underline{Y}(S) \arrow[r, "\mathrm{Hom}(t,Y)"'] & \underline{Y}(*)
\end{tikzcd}$$

Let $p\colon S \to X$. Then

$$\begin{aligned}&(\alpha_* \circ \mathrm{Hom}(t, X))(p) = \alpha_*(\mathrm{Hom}(t, X)(p)) = \alpha_*(p \circ t)\\ = &(\mathrm{Hom}(t, Y) \circ \alpha_S)(p) = \mathrm{Hom}(t, X)(\alpha_S(p)) = \alpha_S(p) \circ t\end{aligned}$$

and hence we obtain the equality

$$\alpha_*(p \circ t) = \alpha_S(p) \circ t. \tag{5.2}$$

Now define a map $f\colon X \to Y$ by taking the composite

$$X \xrightarrow{(\mathrm{ev}_X)^{-1}} \underline{X}(*) \xrightarrow{\alpha_*} \underline{Y}(*) \xrightarrow{\mathrm{ev}_Y} Y.$$

We show that f is the morphism that satisfies (5.1). Suppose $t(*) = s \in S$. Then due to (5.2)

$$(f \circ p)(s) = (\mathrm{ev}_Y \circ \alpha_* \circ (\mathrm{ev}_X)^{-1} \circ p)(s) = (\mathrm{ev}_Y \circ \alpha_*)(p \circ t) = \alpha_*(p \circ t)(*) = \alpha_S(p)(s)$$

and therefore $f \circ p = \alpha_S(p)$. Note that $\alpha_S(p)$ is a continuous map $S \to Y$ and also that this is true for all S and for all $p\colon S \to X$. Since X is compactly generated, it follows from Proposition 4.4.3 that $f\colon X \to Y$ is continuous. $\square$

The final paragraph of the previous proof was inspired by [Lin23, Proposition 4.2.8]. Now due to Corollary 1.3.15, we immediately obtain the following theorem.

Theorem 5.0.7. **CompGen** *is equivalent to a full subcategory of* **Cond(Set)**.

Chapter 6

Condensed Abelian Groups

In this chapter, we discuss the theory of condensed abelian groups. The definition of a condensed abelian group is analogues to the definition of a condensed set presented in Chapter 5 and is provided in Section 6.3 following some prior theory. Firstly, in Section 6.1, we discuss the central notion of an "abelian category". Roughly speaking, abelian categories are categories that generalize certain "nice" properties of **Ab**. In Section 6.2, we present the idea of a "topological abelian group". The category **AbTop** formed from these objects is not an abelian category. In contrast, a key result of this text is that the category formed from condensed abelian groups, **Cond(Ab)**, is an abelian category. The significance of these last two points is discussed further in Section 6.3. Our primary references for this chapter are [Ásg21, Section 2.1-2.2] and [Mac71, pp. 187–198].

6.1 Abelian Categories

The prototypical example of an abelian category is **Ab**. For a general category to be abelian, it must satisfy four properties. We explain each of these properties in this section, all while showing that they hold true in **Ab**. We begin with a definition.

Definition 6.1.1 (Zero Object)**.** Let $\mathcal{C}$ be a category. An object that is both an initial and a terminal object in $\mathcal{C}$, if it exists, is called a *zero object*.

Note that if such an object exists in $\mathcal{C}$, it is unique up to isomorphism due to the uniqueness of limits and colimits. Throughout this section, unless otherwise stated, we assume that $\mathcal{C}$ is a category with a zero object 0. If $A \in \mathcal{C}$, we denote the unique morphisms $0 \to A$ and $A \to 0$ by 0_A^0 and 0_0^A respectively. The existence of a zero object in $\mathcal{C}$ implies certain properties of $\mathcal{C}$. We discuss these now.

If $A, B \in \mathcal{C}$, then there exists a morphism $A \to B$ obtained by the composition

$$A \xrightarrow{0_0^A} 0 \xrightarrow{0_B^0} B$$

We call this a *zero morphism* from A to B, denoted 0_B^A. In $\mathcal{C}$, we may have more than one choice for a zero object. A zero morphism from A to B is independent of this choice, as proved in the following proposition.

Proposition 6.1.2 ([Mac71, p. 20])**.** *Let* 0 *and* $0'$ *be zero objects in* $\mathcal{C}$. *If* $A, B \in \mathcal{C}$, *then* $0_B^A = 0'^A_B$.

Proof. The uniqueness of the morphisms $A \to 0'$ and $0 \to B$ results in the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathrm{id}_A"'] \arrow[rr, bend left, "0_B^A"] & 0 \arrow[r] \arrow[d] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r] \arrow[rr, bend right, "0'^A_B"'] & 0' \arrow[r] & B
\end{tikzcd}$$

The result follows from the commutativity of the diagram. □

The above commutative diagram was explained to us by Dr. Tobias Rossmann.

We hence call 0_B^A the zero morphism from A to B. We use the following property of zero morphisms numerous times in this section.

Proposition 6.1.3 ([Mac71, p. 20])**.** *Let $A, B, C \in \mathcal{C}$ and suppose $f\colon A \to B$ and $g\colon B \to C$ are morphisms in $\mathcal{C}$. Then:*

(i) $0_C^B \circ f = 0_C^A$,

(ii) $g \circ 0_B^A = 0_C^A$.

Proof. We prove (i) noting that (ii) is very similar. Using the definition of $0_C^B\colon B \to C$ and noting that 0_0^A is the unique morphism $A \to 0$, we obtain the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0_0^A"'] & B \arrow[r, "0_C^B"] \arrow[d, "0_0^B"] & C \\
& 0 \arrow[ur, "0_C^0"'] &
\end{tikzcd}$$

This proves the claim. □

Example 6.1.4. In **Grp**, the zero object is given by the trivial group $\{e\}$. If $G, H \in$ **Grp**, the zero morphism $G \to H$ is the homomorphism that sends each $g \in G$ to the identity in H.

The existence of zero morphisms implies the existence of certain other morphisms. Suppose that $A, B \in \mathcal{C}$ and that the coproduct $A \sqcup B$ and the product $A \times B$ exist. Since we have morphisms $\mathrm{id}_A\colon A \to A$ and $0_B^A\colon A \to B$ in $\mathcal{C}$, it follows, by property of the product, that there exists a unique morphism $A \to A \times B$. Similarly, there exists a unique morphism $B \to A \times B$. Hence, by property of the coproduct, there exists a unique morphism $A \sqcup B \to A \times B$ such that the following diagram commutes.

$$\begin{tikzcd}
& A \times B & \\
& A \sqcup B \arrow[u] & \\
A \arrow[ur] \arrow[uur, bend left] & & B \arrow[ul] \arrow[uul, bend right]
\end{tikzcd}$$

We call this the canonical morphism $A \sqcup B \to A \times B$.

Example 6.1.5. We noted previously in Example 2.2.3 that in **Ab**, if $G, H \in$ **Ab**, then the product $G \times H$ and the coproduct $G \sqcup H$ are both the direct product of G and H. In this case, the morphisms $\varphi_1 \colon G \to G \times H$ and $\varphi_2 \colon H \to G \times H$ are precisely the canonical injections defined by

$$\varphi_1(g) = (g, 0) \text{ and } \varphi_2(h) = (0, h).$$

We note that in this case is the unique morphism $f \colon G \times H \to G \times H$ such that the following diagram commutes

$$\begin{tikzcd} & G \times H & \\ & G \times H \arrow[u, "f"] & \\ G \arrow[ur, "\varphi_1"'] \arrow[uur, bend left, "\varphi_1"] & & H \arrow[ul, "\varphi_2"] \arrow[uul, bend right, "\varphi_2"'] \end{tikzcd}$$

is an isomorphism. More precisely, f is $\mathrm{id}_{G \times H}$.

Once $\mathcal{C}$ has a notion of zero morphisms, we can define "kernels" and "cokernels" of morphisms.

Definition 6.1.6 (Kernels & Cokernels)**.** Let $f \colon A \to B$ be a morphism in $\mathcal{C}$.

1. A *kernel* of f is a morphism $\operatorname{Ker}(f) \xrightarrow{i} A$ that satisfies (i) $f \circ i = 0_B^{\operatorname{Ker}(f)}$ and (ii) if $i' \colon C \to A$ is a morphism such that $f \circ i' = 0_A^C$, then there exists a unique morphism $g \colon C \to \operatorname{Ker}(f)$ such that the following diagram commutes.

$$\begin{tikzcd} C \arrow[d, "g"'] \arrow[dr, "i'"] & \\ \operatorname{Ker}(f) \arrow[r, "i"'] & A \end{tikzcd}$$

2. A *cokernel* of f is a morphism $B \xrightarrow{p} \operatorname{Coker}(f)$ that satisfies (i) $p \circ f = 0_{\operatorname{Coker}(f)}^A$ and (ii) if $p' \colon B \to C$ is a morphism such that $p' \circ f = 0_C^A$, then there exists a unique morphism $g \colon \operatorname{Coker}(f) \to C$ such that the following diagram commutes.

$$\begin{tikzcd} & C \\ B \arrow[ur, "p'"] \arrow[r, "p"'] & \operatorname{Coker}(f) \arrow[u, "g"'] \end{tikzcd}$$

Alternatively, consider the following diagram D

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0_B^A"'] & B. \end{tikzcd}$$

A kernel of f is an equalizer of D, whereas a cokernel of f is a coequalizer of D. This justifies our notation: if either the kernel or the cokernel exist, then they are unique up to isomorphism.

We mentioned in Example 2.4.4 that in **Ab**, a kernel of a morphism $f\colon G \to H$ is provided by the inclusion $i\colon \mathrm{Ker}(f) \hookrightarrow G$. A cokernel of f is the canonical surjection $p\colon H \to H/\mathrm{Im}(f)$. Note that f is injective if and only if $\mathrm{Ker}(f) = \{0\}$ and f is surjective if and only if $\mathrm{Coker}(f) = \{0\}$. If $\mathcal{C}$ is also a "preadditive" category, then an analogous statement is true.

Definition 6.1.7 (Preadditive)**.** We say that a category $\mathcal{A}$ is *preadditive* if for each $A, B \in \mathcal{A}$, the following two conditions hold:

1. $\mathrm{Hom}(A, B)$ is an abelian group.
2. If $C \in \mathcal{A}$, then the composition map
$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(B, C)$$
is bilinear.

If $f_1, f_2 \in \mathrm{Hom}(A, B)$ and $g_1, g_2 \in \mathrm{Hom}(B, C)$, then the last point means that
$$(g_1 + g_2) \circ (f_1 + f_2) = g_1 \circ f_1 + g_1 \circ f_2 + g_2 \circ f_1 + g_2 \circ f_2.$$

Example 6.1.8. Let $G, H \in \mathbf{Ab}$. Then $\mathrm{Hom}(G, H)$ is an abelian group with the group operation $+$ defined as follows. For each $f, g \in \mathrm{Hom}(G, H)$, define $(f + g)$ as the group homomorphism where
$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in G.$$

With this abelian group structure on the Hom-sets, we further obtain that the composition map for $G, H, I \in \mathbf{Ab}$
$$\mathrm{Hom}(H, I) \times \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, I)$$
is bilinear. Hence, **Ab** is preadditive [Mac71, p. 28].

If $\mathcal{A}$ contains a zero object 0, then one finds that the group-theoretic identity in $\mathrm{Hom}(A, B)$ is precisely the zero morphism 0_B^A. To see this, first note that $\mathrm{Hom}(A, 0)$ and $\mathrm{Hom}(0, B)$ are both the trivial groups containing the group-theoretic identities 0_0^A and 0_B^0 respectively. Now by bilinearity, we obtain
$$0_B^A = 0_B^0 \circ 0_0^A = (0_B^0) \circ (0_0^A + 0_0^A) = 0_B^0 \circ 0_0^A + 0_B^0 \circ 0_0^A = 0_B^A + 0_B^A.$$

Hence the composition $0_B^A = 0_B^0 \circ 0_0^A$ is the identity in $\mathrm{Hom}(A, B)$.

Proposition 6.1.9. *Let $\mathcal{A}$ be a preadditive category with a zero object 0 and let $f\colon A \to B$ be a morphism in $\mathcal{C}$. If the kernel and the cokernel of f exist, then the following is true:*

(i) f is a monomorphism if and only if the kernel of f is $0_A^0\colon 0 \to A$.

(ii) f is an epimorphism if and only if the cokernel of f is $0_0^B\colon B \to 0$.

Proof. We prove (i) noting that the argument for (ii) is dual to this. Firstly, suppose that f is a monomorphism. Due to Proposition 6.1.3, $f \circ 0_A^0 = 0_B^0$. Now suppose that for $C \in \mathcal{A}$, a morphism $i' \colon C \to A$ in $\mathcal{A}$ satisfies $f \circ i' = 0^C_B$. But note also that $f \circ 0^C_A = 0^C_B$. Since f is a monomorphism, we obtain that $i' = 0^C_A$ and the following commutative diagram.

$$\begin{tikzcd} C \arrow[d, "0^C_0"'] \arrow[dr, "0^C_A"] & \\ 0 \arrow[r, "0^0_A"'] & A, \end{tikzcd}$$

Since 0^C_0 is unique, it is certainly the unique morphism $C \to 0$ such that the above diagram commutes. Hence, the kernel of f is 0^0_A. Now suppose that the kernel of f is 0^0_A. Suppose that morphisms $g_1, g_2 \in \mathrm{Hom}(C, A)$ satisfy $f \circ g_1 = f \circ g_2$. By bilinearity, we obtain

$$f \circ (g_1 - g_2) = 0^B_C.$$

Let $g = g_1 - g_2$. By universal property of the kernel, we obtain the following commutative diagram.

$$\begin{tikzcd} C \arrow[d, "0^C_0"'] \arrow[dr, "g"] & \\ 0 \arrow[r, "0^0_A"'] & A \end{tikzcd}$$

Hence $g = g_1 - g_2 = 0^C_A$ which proves the claim. □

The category $\mathcal{C}$ need only satisfy an additional property to ensure that it is preadditive.

Lemma 6.1.10 ([Ásg21, pp. 22-23])**.** *If for each $A, B \in \mathcal{C}$, the product $A \times B$ and the coproduct $A \sqcup B$ exist and if the canonical map $A \sqcup B \to A \times B$ is an isomorphism, then $\mathcal{C}$ admits the structure of a preadditive category.*

Now we introduce some more terminology.

Definition 6.1.11 (Image & Coimage)**.** Let $f \colon A \to B$ be a morphism in $\mathcal{C}$ and suppose that the kernel and the cokernel of f exist.

1. An *image* of f is a kernel $\mathrm{Im}(f) \xrightarrow{i} B$ of the cokernel of f.

2. A *coimage* of f is a cokernel $A \xrightarrow{p} \mathrm{Coim(f)}$ of the kernel of f.

Example 6.1.12. Let $f \colon G \to H$ be a morphism in **Ab**. Since a cokernel of f is the canonical surjection $H \to H/\mathrm{im}(f)$, a kernel of this morphism (and hence the image of f) is given by the inclusion $\mathrm{im}(f) \hookrightarrow H$. A kernel of f is the inclusion $\mathrm{Ker}(f) \hookrightarrow G$. A cokernel of this morphism (and hence the coimage of f) is the surjection $G \to G/\mathrm{Ker}(f)$. Note further that, due to the First Isomorphism Theorem of Groups, we obtain a canonical decomposition of f into

$$G \to G/\mathrm{Ker}(f) \to \mathrm{im}(f) \to H,$$

where $G/\mathrm{Ker}(f) \to \mathrm{im}(f)$ is an isomorphism. Provided that the necessary objects and morphisms exist in a general preadditive category $\mathcal{C}$, a similar decomposition holds for a morphism $g \colon A \to B$ in $\mathcal{A}$, though we are not generally guaranteed that the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism. An example of such a category is provided in Section 6.2.

Proposition 6.1.13 ([Stacks, Tag 0107]). *Let $f\colon A \to B$ be a morphism in a preadditive category $\mathcal{A}$. If the kernel, the cokernel, the image, and the coimage of f exist, then there exists a canonical morphism $f'\colon \operatorname{Coim}(f) \to \operatorname{Im}(f)$ such that f can be written as the composition*

$$A \xrightarrow{p} \operatorname{Coim}(f) \xrightarrow{f'} \operatorname{im}(f) \xrightarrow{i} B.$$

In the above proposition, the morphism p is the cokernel of the kernel and the morphism i is the kernel of the cokernel. We are now ready to present the definition of an abelian category [Ásg21, Definition 2.1.11].

Definition 6.1.14 (Abelian Category)**.** We say a category $\mathcal{A}$ is *abelian* if it satisfies the following properties:

1. $\mathcal{A}$ has a zero object.
2. For all $A, B \in \mathcal{C}$, the product $A \times B$ and the coproduct $A \sqcup B$ exist and the canonical morphism $A \sqcup B \to A \times B$ is an isomorphism.
3. Every morphism in $\mathcal{A}$ has a kernel and a cokernel.
4. If $f\colon A \to B$ is a morphism in $\mathcal{A}$, then the canonical morphism $\operatorname{Coim}(f) \xrightarrow{f'} \operatorname{im}(f)$ is an isomorphism.

Over the course of this section, we have shown that conditions 1-4 hold true in **Ab**. Hence, **Ab** is an abelian category.

In **Ab**, an injective and surjective morphism is an isomorphism. The analogous statement is true for abelian categories.

Theorem 6.1.15 ([Mac71, p. 195]). *Let $\mathcal{A}$ be an abelian category and let $f\colon A \to B$ be a morphism in $\mathcal{A}$. If f is a monomorphism and an epimorphism, then f is an isomorphism.*

Proof. First note that we can canonically decompose f into

$$A \xrightarrow{p} \operatorname{Coim}(f) \xrightarrow{f'} \operatorname{Im}(f) \xrightarrow{i} B.$$

Since f' is an isomorphism, it suffices to show that p and i are isomorphisms. We show that p is an isomorphism noting to the reader once again that showing i is an isomorphism is very similar. Our strategy is to show that $\mathrm{id}_A\colon A \to A$ is a cokernel of $\operatorname{Ker}(f) \to A$ and then use the uniqueness of limits / colimits to establish that p is an isomorphism. Since f is a monomorphism, the kernel of f is $0^0_A\colon 0 \to A$. Firstly, note that $\mathrm{id}_A \circ 0^0_A = 0^0_A$. Now let $p'\colon A \to C$ be a morphism that satifies $p' \circ 0^0_A = 0^0_C$. Then p' is the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
 & C \\
A \arrow[ur, "p'"] \arrow[r, "\mathrm{id}_A"'] & A \arrow[u, "p'"']
\end{tikzcd}$$

This proves that $\mathrm{id}_A\colon A \to A$ is a cokernel of the kernel of f. Since $p \circ 0^0_A = 0^0_{\operatorname{Coim}(f)}$, there exists a unique isomorphism $g\colon A \to \operatorname{Coim}(f)$ such that the following diagram commutes.

$$
\begin{tikzcd}
 & \mathrm{Coim}(f) \\
A \arrow[ur, "p"] \arrow[r, "\mathrm{id}_A"'] & A \arrow[u, "g"']
\end{tikzcd}
$$

Clearly $g = p$. □

In Example 1.1.7, we showed that an isomorphism is an epimorphism and a monomorphism. Thus in abelian categories, a morphism is an isomorphism if and only if it is an epimorphism and a monomorphism. In particular, due to Proposition 6.1.9, we obtain the following corollary.

Corollary 6.1.16. *Let $\mathcal{A}$ be an abelian category. A morphism $f\colon A \to B$ is an isomorphism if and only if it has trivial kernel and cokernel.*

6.2 Topological Abelian Groups

The main goal of this section is to provide an example of a preadditive category that is not an abelian category: the category **AbTop** formed from topological abelian groups. Our main reference is [Pro99, Section 2]. We begin with a definition.

Definition 6.2.1 (Topological Abelian Groups)**.** A *topological group* G is a group G endowed with a topology such that:

1. The group operation $\cdot\colon G \times G \to G$ is continuous, where $G \times G$ is endowed with the product topology.
2. The group-theoretic inversion map $(-)^{-1}\colon G \to G, x \mapsto x^{-1}$ is continuous.

If, in addition, the group structure on G is abelian, we say G is a *topological abelian group*.

Remark 6.2.2. If G is a topological abelian group, then we often use additive notation. The group operation is denoted by $+$, the group theoretic inversion map is denoted by $-\square$ and the identity is denoted by 0.

Example 6.2.3.

(i) If G is any group, then G endowed with the discrete topology is a topological group. To see this, note that $G \times G$ is also discrete (since singletons in $G \times G$ are open by the product topology). Hence the group operation $\colon G \times G \to G$ is continuous. Also, the inversion $(-)^{-1}\colon G \to G$ is also continuous since G is discrete.

(ii) Let $\mathbb{R}_{\text{Euclidean}}$ denote $\mathbb{R}$ with the Euclidean topology. Then $\mathbb{R}_{\text{Euclidean}}$ under $+$ is a topological abelian group.

We then define the category **AbTop** as the category consisting of topological abelian groups as objects and continuous group homomorphisms as morphisms. We show that **AbTop** does not satisfy condition 4 of Definition 6.1.14. First, observe that if 0 is endowed with the discrete topology, then 0 is the zero object in **AbTop**. Indeed, if $X \in$ **AbTop**, then the unique group homomorphism $X \to 0$ is continuous. Furthermore, the unique group homomorphism $0 \to X$ is continuous as well since every constant function is continuous. It is clear also, due to a similar reasoning to Example 6.1.8, that **AbTop** admits the structure of a preadditive category.

Proposition 6.2.4 ([Pro99, Proposition 2.4]). *Let $f\colon A \to B$ be a morphism in* **AbTop**. *Then the following is true.*

(i) *The subgroup* $\mathrm{Ker}(f)$ *of A endowed with the subspace topology along with the inclusion* $\mathrm{Ker}(f) \hookrightarrow A$ *is a kernel of f.*

(ii) *The quotient group* $B/\mathrm{Im}(f)$ *endowed with the quotient topology along with the canonical surjection* $B \to B/\mathrm{Im}(f)$ *is a cokernel of f.*

Hence, due to Proposition 6.1.13, it follows that every morphism $f\colon A \to B$ in **AbTop** has a canonical decomposition into

$$A \xrightarrow{p} \mathrm{Coim}(f) \xrightarrow{f'} \mathrm{Im}(f) \xrightarrow{i} B.$$

For a general morphism f, the canonical morphism $f'\colon \mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not necessarily an isomorphism. Consider, for instance, the following example.

Example 6.2.5. Let $\mathbb{R}_{\text{Discrete}}$ denote $\mathbb{R}$ with the discrete topology. Consider the set-theoretic identity map $f\colon \mathbb{R}_{\text{Discrete}} \to \mathbb{R}_{\text{Euclidean}}$. Firstly, note that this mapping is a continuous homomorphism and hence is a morphism in **AbTop**. We also note that this map is injective and surjective (and hence a monomorphism and an epimorphism in **AbTop**). Due to Proposition 6.1.9, we obtain that the categorical kernel of f and the categorical cokernel of f are the zero morphisms $0 \to \mathbb{R}_{\text{Discrete}}$ and $\mathbb{R}_{\text{Euclidean}} \to 0$ respectively. Hence, the coimage of f is the identity morphism $\mathrm{id}_{\mathbb{R}_{\text{Discrete}}}$ and the image of f is the identity morphism $\mathbb{R}_{\text{Euclidean}}$. The canonical decomposition of f in this case is given by

$$\mathbb{R}_{\text{Discrete}} \xrightarrow{\mathrm{id}_{\mathbb{R}_{\text{Discrete}}}} \mathbb{R}_{\text{Discrete}} \xrightarrow{f} \mathbb{R}_{\text{Euclidean}} \xrightarrow{\mathrm{id}_{\mathbb{R}_{\text{Euclidean}}}} \mathbb{R}_{\text{Euclidean}}.$$

However, f is not an isomorphism in **AbTop**: the set-theoretic inverse of f is not continuous.

6.3 Condensed Abelian Groups

In this section, we define condensed abelian groups and the category they form, **Cond(Ab)**. The presentation of this section will be analogous to our presentation of condensed sets in Chapter 5. In particular, we will observe that the definition of a condensed abelian group is very similar to the definition of a condensed set. Furthermore, we will obtain a functor **AbTop** $\to$ **Cond(Ab)** and an analogue to Theorem 5.0.6. In the previous section, we proved that **AbTop** is not an abelian category. In contrast, we prove that **Cond(Ab)** is an abelian category in Theorem 6.3.7.

We begin with the definition of a condensed abelian group.

Definition 6.3.1 (Condensed Abelian Group)**.** A *condensed abelian group* is a functor

$$T\colon \mathbf{ED}^{\mathrm{op}} \to \mathbf{Ab}$$

such that for every finite collection $(S_i)_{i\in I}$ of objects of **ED**, the canonical morphism

$$T(\coprod_{i\in I} S_i) \to \prod_{i\in I} T(S_i)$$

is an isomorphism.

We define the category **Cond(Ab)** to be the full subcategory of $[\mathbf{ED}^{\mathrm{op}}, \mathbf{Ab}]$ consisting of condensed abelian groups as objects.

Let $X \in \mathbf{AbTop}$. We wish to obtain an "associated condensed abelian group"

$$\underline{X} \colon \mathbf{ED}^{\mathrm{op}} \to \mathbf{Ab}$$

from X in much the same way that we did in the case of topological spaces and condensed sets. In particular, for each $S \in \mathbf{ED}$, $\underline{X}(S)$ should be an abelian group whose underlying set $\mathrm{Hom}_{\mathbf{Top}}(S, X)$ is the set of continuous maps $S \to X$, which we will more simply denote as $\mathrm{Hom}(S, X)$. The abelian group structure on $\mathrm{Hom}(S, X)$ is essentially inherited from the group operation $+\colon X \to X \to X$ as follows. If $f, g \in \mathrm{Hom}(S, X)$, define the function $f + g \colon S \to X$ by

$$(f+g)(s) = f(s) + g(s) \text{ for all } s \in S.$$

Note that $f + g$ is the composition

$$S \xrightarrow{(f,g)} X \times X \xrightarrow{+} X,$$

where $(f, g)\colon S \to X \times X$ is continuous by the product topology on $X \times X$. Therefore $f + g$ is a continuous map from S to X. In this way, we obtain a binary operation $+\colon \mathrm{Hom}(S, X) \times \mathrm{Hom}(S, X) \to \mathrm{Hom}(S, X)$ defined by $(f, g) \mapsto f + g$. Under this binary operation, we check that $\mathrm{Hom}(S, X)$ is an abelian group.

Proposition 6.3.2. *Let $S \in \mathbf{ED}$. The set of continuous maps from S to X, $\mathrm{Hom}(S, X)$, is an abelian group under the binary operation $+$ defined above.*

Proof. We first check the three group axioms.

(i) Identity. Define the function $e\colon S \to X$ by $e(s) = 0$ for all $s \in S$, where 0 is the group theoretic identity in X. This map is continuous since every constant function is continuous. Now let $f \in \mathrm{Hom}(S, X)$. We find that

$$\begin{aligned}(f+e)(s) &= f(s) + e(s) = f(s) + e = f(s) \\ &= e + f(s) = e(s) + f(s) = (e+f)(s).\end{aligned}$$

Therefore $f + e = e + f = f$ and e is an identity element in $\mathrm{Hom}(S, X)$.

(ii) Inverses. Let $f \in \mathrm{Hom}(S, X)$. Define $-f\colon S \to X$ by $-f(s) = -\big(f(s)\big)$. Note $-f$ is the composition

$$S \xrightarrow{f} X \xrightarrow{-\Box} X,$$

and is therefore continuous. Now we check that $-f$ is the inverse of f:

$$(f + -f)(s) = f(s) + -f(s) = 0 = -f(s) + f(s) = (-f + f)(s).$$

Therefore $f + -f = -f + f = e$.

(iii) Associativity. Let $f, g, h \in \mathrm{Hom}(S, X)$. We wish to show that

$$(f+g) + h = f + (g+h).$$

For $s \in S$, we obtain

$$\big((f+g)+h\big)(s) = (f+g)(s)+h(s) = \big(f(s)+g(s)\big)+h(s) =$$
$$f(s)+\big(g(s)+h(s)\big) = f(s)+(g+h)(s) = \big(f+(g+h)\big)(s).$$

Finally let $f, g \in \mathrm{Hom}(S, X)$. For $s \in S$, note that

$$(f+g)(s) = f(s)+g(s) = g(s)+f(s) = (g+f)(s),$$

and hence $\mathrm{Hom}(S, X)$ is abelian. □

The definition of an associated condensed abelian group then proceeds as we would expect. Let X be a topological abelian group. The associated condensed abelian group of X is the functor $\underline{X}\colon \mathbf{ED}^{\mathrm{op}} \to \mathbf{Ab}$ defined as follows.

1. For $S \in \mathbf{ED}$, $\underline{X}(S)$ is defined to be the abelian group $(\mathrm{Hom}(S, X), +)$ or more simply denoted $\mathrm{Hom}(S, X)$.
2. For a morphism $S_1 \to S_2$ in $\mathbf{ED}$, we define $\underline{X}(f)\colon \mathrm{Hom}(S_2, X) \to \mathrm{Hom}(S_1, X)$ by $p \mapsto p \circ f$ for all $p \in \mathrm{Hom}(S_2, X)$.

In the second point, we are claiming that $\underline{X}(f)$ is a group homomorphism. Indeed, if $p_1, p_2 \in \mathrm{Hom}(S_2, X)$, then for all $s \in S$

$$\big((p_1+p_2)\circ f\big)(s) = (p_1+p_2)\big(f(s)\big) = (p_1 \circ f)(s) + (p_2 \circ f)(s) = \big((p_1 \circ f)+(p_2 \circ f)\big)(s).$$

Therefore $\big(\underline{X}(f)\big)(p_1+p_2) = \big(\underline{X}(f)\big)(p_1) + \big(\underline{X}(f)\big)(p_2)$.

For a condensed abelian group T, we call $T(*)$ the *underlying abelian group* of T. If $X \in \mathbf{AbTop}$, then the bijection between the underlying sets $\mathrm{ev}_X\colon \underline{X}(*) \to X, p \mapsto p(*)$ is a morphism in $\mathbf{Ab}$. To see this, let $p_1, p_2 \in \underline{X}(*)$. Then

$$\mathrm{ev}_X(p_1+p_2) = (p_1+p_2)(*) = p_1(*)+p_2(*) = \mathrm{ev}_X(p_1)+\mathrm{ev}_X(p_2).$$

Hence, ev_X is a morphism in $\mathbf{Ab}$ and since it is also a bijection, it is an isomorphism in $\mathbf{Ab}$.

Once again, the mapping $X \to \underline{X}$ defines a functor $\underline{\square}\colon \mathbf{AbTop} \to \mathbf{Cond(Set)}$. With this functor, we wish to prove an analogue to Theorem 5.0.6. First, we define **AbCompGen** to be the full subcategory of **AbTop** consisting of objects whose underlying topological space is compactly generated. Then we obtain the following theorem.

Theorem 6.3.3. *The functor* $\underline{\square}$: **AbTop** $\to$ **Cond(Ab)** *is faithful. Moreover, the restriction of* $\underline{\square}$ *to* **AbCompGen** *is a fully faithful functor* **AbCompGen** $\to$ Cond(Ab).

Proof. The proof that $\underline{\square}$ is faithful is analogous to the proof of Lemma 5.0.5. We therefore only prove the second statement. Let $X, Y \in \mathbf{AbCompGen}$. We wish to show that if $\alpha\colon \underline{X} \to \underline{Y}$ is a natural transformation, then there exists a morphism $f\colon X \to Y$ in **AbCompGen** such that for all $S \in \mathbf{ED}$ and $p\colon S \to X$:

$$f \circ p = \alpha_S(p).$$

In Theorem 5.0.6, we proved that if f is the composite

$$X \xrightarrow{(\mathrm{ev}_X)^{-1}} \underline{X}(*) \xrightarrow{\alpha_*} \underline{Y}(*) \xrightarrow{\mathrm{ev}_Y} Y,$$

then f is a continuous map that satisfies the above equation. Hence, it suffices to show that f is a group homomorphism. We know that α_* is a group homomorphism and from our above comments, ev_Y and ev_X^{-1} are both group homomorphism as well. Since f is their composite, it follows that f is a group homomorphism too. □

We now show that in contrast to **AbTop**, **Cond(Ab)** forms an abelian category. We will use the fact that the conditions provided in Definition 6.1.14 can be rephrased in terms of limits and colimits. Thus, we need some theory on limits and colimits in **Cond(Ab)**. Firstly, we introduce some terminology. If $S \in \mathbf{ED}$, then we claim that "evaluating at S" defines a functor $\mathbf{Cond(Ab)} \to \mathbf{Ab}$. More precisely, define ev_S as follows:

1. For every $T \in \mathbf{Cond(Ab)}$, let $\mathrm{ev}_S(T)$ be $T(S)$.
2. For every morphism $\alpha\colon T \to T'$ in **Cond(Ab)**, let $\mathrm{ev}_S(\alpha)$ be the S-component $\alpha_S\colon T(S) \to T'(S)$.

Proposition 6.3.4. *Let $S \in \mathbf{ED}$. Then ev_S is a functor $\mathbf{Cond(Ab)} \to \mathbf{Ab}$.*

Proof. We verify the two functoriality axioms.

(i) Let $\alpha\colon T \to T'$ and $\alpha'\colon T' \to T''$ be morphisms in **Cond(Ab)**. Then

$$\mathrm{ev}_S(\alpha' \circ \alpha) = (\alpha' \circ \alpha)_S = \alpha'_S \circ \alpha_S = \mathrm{ev}_S(\alpha') \circ \mathrm{ev}_S(\alpha).$$

(ii) Let $T \in \mathbf{Cond(Ab)}$ and consider id_T. Then

$$\mathrm{ev}_S(\mathrm{id}_T) = (\mathrm{id}_T)_S = \mathrm{id}_{T(S)}.$$

This proves the claim. □

Theorem 6.3.5 ([Ásg21, Theorem 2.2.3])**.** *Let $\mathbf{J}$ be a small category and let $D\colon \mathbf{J} \to \mathbf{Cond(Ab)}$ be a diagram. Then a limit $\big(p_I\colon L \to D(I)\big)_{I\in\mathbf{J}}$ and colimit $\big(c_I\colon D(I) \to C\big)_{I\in\mathbf{J}}$ of D exist. Moreover if $S \in \mathbf{ED}$, then:*

(i) The evaluation functor ev_S applied to $\big(p_I\colon L \to D(I)\big)_{I\in\mathbf{J}}$ is the limit of the diagram $ev_S \circ D\colon \mathbf{J} \to \mathbf{Ab}$.

(ii) The evaluation functor ev_S applied to $\big(c_I\colon D(I) \to C\big)_{I\in\mathbf{J}}$ is the colimit of the diagram $ev_S \circ D\colon \mathbf{J} \to \mathbf{Ab}$.

In other words, limits and colimits in **Cond(Ab)** are formed objectwise on objects of **ED**. The following is an example.

Example 6.3.6. Let $\mathbf{2}$ denote the discrete category consisting of two objects and let $D\colon \mathbf{2} \to \mathbf{Cond(Ab)}$ be the diagram which simply picks out objects $A, B \in \mathbf{Cond(Ab)}$. The first claim is that the product $A \times B$ exists. If $S \in \mathbf{ED}$, the second claim is that $(A \times B)(S)$ is the limit of the diagram $\mathrm{ev}_S \circ D\colon \mathbf{2} \to \mathbf{Ab}$. Hence, $(A \times B)(S)$ is just the product $A(S) \times B(S)$. Furthermore, if we denote the canonical projections $A \times B \to A$ and $A \times B \to B$ by p_1 and p_2 respectively, then the S-components p_{1S} and p_{2S} are the canonical projections $A(S) \times B(S) \to A(S)$ and $A(S) \times B(S) \to B(S)$ respectively. Now let $T \in \mathbf{Cond(Ab)}$ and suppose $f_1\colon T \to A$ and $f_2\colon T \to A$ are morphisms in **Cond(Ab)**. Then, by the universal property of the product, there exists a canonical morphism $f\colon A \times B \to T$. If we apply ev_S, we obtain the following commutative diagram in **Ab**.

$$\begin{tikzcd}
& T(S) \arrow[d, "f_s"] \arrow[ddl, bend right, "f_{1s}"'] \arrow[ddr, bend left, "f_{2s}"] & \\
& A(S) \times B(S) \arrow[dl, "p_{1s}"] \arrow[dr, "p_{2s}"'] & \\
A(S) & & B(S)
\end{tikzcd}$$

We see then that then the S-component f_S is the canonical morphism $T(S) \to A(S) \times B(S)$ obtained from the universal property of the product. This last point is significant; it tells us that morphisms obtained from the universal property of limits / colimits in **Cond(Ab)** evaluate (at a particular $S \in \mathbf{ED}$) to morphisms obtained from the universal property of limits / colimits in **Ab**. We use this in (ii) of Theorem 6.3.7.

We use the above result in the proof of the following theorem.

Theorem 6.3.7 ([Sch19b, Theorem 1.10])**. Cond(Ab)** *is an abelian category.*

Proof. We verify the four conditions presented in Definition 6.1.14.

(i) Let $\mathbf{0}$ be the empty category and $D\colon \mathbf{0} \to \mathbf{Cond(Ab)}$ the unique diagram $\mathbf{0} \to \mathbf{Cond(Ab)}$. The limit T of D and the colimit I of D exist. Note that T is the terminal object in **Cond(Ab)** and I is the initial object in **Cond(Ab)**. For $S \in \mathbf{ED}$, $T(S)$ is the limit of $\mathrm{ev}_S \circ D\colon \mathbf{0} \to \mathbf{Ab}$, which is just the trivial group 0. Similarly, $I(S) = 0$. Therefore, $T \cong I$ is a zero object in **Cond(Ab)**. Let 0 denote a zero object for latter parts of this proof.

(ii) Let $A, B \in \mathbf{Cond(Ab)}$. Then $A \sqcup B$ and $A \times B$ exist and moreover, due to (i), there exists a canonical morphism $f\colon A \sqcup B \to A \times B$. If we apply ev_S to f, we obtain in **Ab**
$$f_S\colon A(S) \times B(S) \to A(S) \times B(S).$$
Recall that f is obtained by the universal property of the coproduct $A \sqcup B$. Hence, f_S is the canonical morphism $A(S) \times B(S) \to A(S) \times B(S)$ in **Ab** and is therefore an isomorphism. Since this is true for all $S \in \mathbf{ED}$, it follows that f is a natural isomorphism. As mentioned in Section 1.3.1, natural isomorphisms are isomorphisms in functor categories. Thus, we conclude that f is an isomorphism in **Cond(Ab)**.

(iii) Let $f\colon A \to B$ be a morphism in **Cond(Ab)** and consider the following diagram D
$$A \underset{0^A_B}{\overset{f}{\rightrightarrows}} B.$$
The equalizer and coequalizer of D exist and therefore the kernel $\mathrm{Ker}(f)$ of f and cokernel $\mathrm{Coker}(f)$ of f exist.

(iv) Let $f\colon A \to B$ be a morphism in **Cond(Ab)**. Due to Lemma 6.1.10 and Proposition 6.1.13, we find that f has a canonical decomposition
$$A \xrightarrow{p} \mathrm{Coim}(f) \xrightarrow{f'} \mathrm{Im}(f) \xrightarrow{i} B.$$

If we apply ev_S to f, we obtain that in **Ab**, the morphism $f_S \colon A(S) \to B(S)$ decomposes into

$$A(S) \xrightarrow{p_S} \big(\mathrm{Coim}(f)\big)(S) \xrightarrow{f'_S} \big(\mathrm{Im}(f)\big)(S) \xrightarrow{i_S} B(S).$$

Since limits and colimits in **Cond(Ab)** are computed objectwise, $p_S \colon A(S) \to \big(\mathrm{Coim}(f)\big)(S)$ is equal to the coimage of f_S. Similarly, $i_S \colon \big(\mathrm{Im}(f)\big)(S) \to B(S)$ is the image of f_S. Thus f'_S is the canonical morphism $\mathrm{Coim}(f_S) \to \mathrm{Im}(f_S)$ and hence is an isomorphism in **Ab**. Since this is true for all $S \in \mathbf{ED}$, due to a similar reasoning to (ii), we conclude that f' is an isomorphism.

This concludes the proof. □

Let us now revisit Example 6.2.5.

Example 6.3.8 ([Sch19b, Example 1.9])**.** We noted in Corollary 6.1.16 that in abelian categories, the failure of morphisms to be isomorphisms is detected by non-trivial kernels or cokernels. The set-theoretic identity map $f \colon \mathbb{R}_{\text{Discrete}} \to \mathbb{R}_{\text{Euclidean}}$ had both a trivial kernel and cokernel. However, it is not an isomorphism. If we apply the functor $\underline{\square} \colon$ **AbTop** $\to$ **Cond(Ab)**, we obtain the morphism

$$\underline{f} \colon \underline{\mathbb{R}_{\text{Discrete}}} \to \underline{\mathbb{R}_{\text{Euclidean}}},$$

which for each $S \in \mathbf{ED}$, evaluates to the following morphism in **Ab**.

$$\underline{f}(S) \colon \underline{\mathbb{R}_{\text{Discrete}}}(S) \to \underline{\mathbb{R}_{\text{Euclidean}}}(S)$$
$$p \mapsto f \circ p$$

Let $Q \in$ **Cond(Ab)** be the cokernel of $\underline{f}$. Then at a particular $S \in \mathbf{ED}$, due to Theorem 6.3.5, $Q(S)$ is given by the cokernel of $\underline{f}(S)$, which is the quotient

$$\Big(\underline{\mathbb{R}_{\text{Euclidean}}}(S)\Big) / \Big(\mathrm{Im}\big(\underline{f}(S)\big)\Big).$$

Every map in $\mathrm{Im}\big(\underline{f}(S)\big)$ is of the form $f \circ p$, where $p \colon S \to \mathbb{R}_{\text{Discrete}}$ is a continuous map. Since p is continuous and S is compact, it follows, due to Proposition 3.3.2, that the image of p is compact in $\mathbb{R}_{\text{Discrete}}$ and hence is just a finite subset of $\mathbb{R}$. Since f is the set-theoretic identity, it follows that for every $S \in \mathbf{ED}$, all maps in $\mathrm{Im}\big(\underline{f}(S)\big)$ have finite image. This is not true for every morphism $S \to \mathbb{R}_{\text{Euclidean}}$. Namely, the following diagram commutes.

$$\begin{tikzcd}
{[0,1]_{\text{Discrete}}} \arrow[r, "f_{[0,1]}"] \arrow[d, "g"'] & {[0,1]_{\text{Euclidean}}} \arrow[r, hook, "i"] & \mathbb{R}_{\text{Euclidean}} \\
\beta([0,1]_{\text{Discrete}}) \arrow[ur, "\bar{g}"'] & &
\end{tikzcd}$$

In the above diagram, $g \colon [0,1]_{\text{Discrete}} \to \beta([0,1]_{\text{Discrete}})$ is the Stone-Čech compactification of $[0,1]_{\text{Discrete}}$. Due to Proposition 4.3.6, $\beta([0,1]_{\text{Discrete}}) \in \mathbf{ED}$. The morphism $\bar{g} \colon \beta([0,1]_{\text{Discrete}}) \to [0,1]_{\text{Euclidean}}$ is obtained from the universal property of the Stone-Čech compactification. Finally, since the diagram commutes, $[0,1]$ is a subset of the image of $i \circ \bar{g} \colon \beta([0,1]_{\text{Discrete}}) \to \mathbb{R}_{\text{Euclidean}}$. Hence, for $S = \beta([0,1]_{\text{Discrete}})$, $Q(S)$ is non-trivial and therefore Q is non-trivial as a condensed abelian group.

The particular details of the above example was explained to us by Dr. Tobias Rossmann, including the commutative diagram.

The significance of the fact that **Cond(Ab)** forms an abelian category (and **AbTop** does not) extends beyond the above example. Suppose, firstly, that G is a group. A *G-module* is an abelian group M along with a group action $G \times M \to M$ a *G-module* such that

$$g \cdot (m + m') = g \cdot m + g \cdot m'$$

for all $m, m' \in M$ and all $g \in G$. Given a G-module, one can explicitly form a sequence of morphisms in **Ab**

$$0 \xrightarrow{d_0} C^0(G,M) \xrightarrow{d_1} C^1(G,M) \xrightarrow{d_2} \dots \xrightarrow{d_n} C^n(G,M) \xrightarrow{d_{n+1}} C^{n+1}(G,M) \xrightarrow{d_{n+1}} \dots,$$

where $d_n \circ d_{n-1}$ is the trivial group homomorphism for each $n \in \mathbb{N}$ [Mil20, Proposition 1.17]. We call this a *cochain complex*. We then define the *nth cohomology group* $H^n(G,M)$ to be the quotient $\mathrm{Ker}(d^{n+1})/\mathrm{Im}(d^n)$. By studying these cohomology groups using tools from homological algebra, one can extract a lot of information about G. However, these tools rely on the fact that **Ab** is an abelian category. Now, if X is a topological group acting on a topological abelian group N, we can similarly construct cohomology groups $H^n(X,N)$ of topological abelian groups [Sch19b, Question 1.1 (ii)]. But since **AbTop** is not an abelian category, the tools in homological algebra are no longer available to us, so we cannot study X using them! If we now take the associated condensed abelian groups of each $H^n(X,N)$, the theory of homological algebra is once again freely available to us because **Cond(Ab)** forms an abelian category. For further details on cohomology of condensed objects, see [Ásg21].

Appendix A

Basic Topology

This appendix is a compilation of some basic results in topology that the reader should be familiar with before reading Chapters 3 and 4.

A.1 Connected Spaces

Definition A.1.1 (Connected)**.** Let X be a topological space. We say X is *connected* if it cannot be written as $X = A \cup B$, where A, B are non-empty open sets and $A \cap B = \emptyset$.

Proposition A.1.2 ([Mun75, Theorem 3.1.4])**.** *Let X be a topological space and let $A \subset X$ be connected. If a subset B of X satisfies $A \subset B \subset \overline{A}$, then B is also connected.*

Definition A.1.3 (Connected Components)**.** Let X be a topological space and let $x \in X$. The connected component of x, C_x, is defined as the union of all connected subsets of X that contain x.

Proposition A.1.4. *Let X be a topological space and let $x \in X$. Then C_x is closed.*

Proof. Note $C_x \subset \overline{C_x} \subset \overline{C_x}$. By Proposition A.1.2, it follows that $\overline{C_x}$ is a connected set containing x. Since C_x is the union of all such connected sets, it follows that $\overline{C_x} \subset C_x$ and hence $C_x = \overline{C_x}$. □

Proposition A.1.5. *Let X be a topological space with $X = A \cup B$, where A, B are non-empty and disjoint. If $x \in A$ and $y \in B$, then $y \notin C_x$.*

Proof. Suppose $y \in C_x$. Then y is contained in a connected set U containing x. But then we can write $U = (A \cap U) \cup (B \cap U)$, where $(A \cap U), (B \cap U)$ are disjoint. In particular, $x \in A \cap U$ and $y \in B \cap U$ so these sets are non-empty. Since these sets are open in U, it follows that U is disconnected. Contradiction. □

For further reading, see [Mun75, pp. 147–159].

A.2 Quotient Topology

Definition A.2.1 (Quotient Topology)**.** Let X be a topological space and let A be a set. If there exists a surjection $f : X \to A$, then we can define a topology on A, where we declare a set $U \subset A$ to be open if and only if $f^{-1}(U)$ is open in X.

Section 2.11 of [Mun75] verifies this is, in fact, a topology on A. Note then that f is made continuous relative to this topology on A. We refer to f as a *quotient map* and the resulting topology on A as the *quotient topology induced by f*.

Given topological spaces X and Y and a continuous surjection $f : X \to Y$, a reasonable question to ask is when the topology on Y is equal to the quotient topology induced by f on Y viewed as a set. The following is a condition on f that guarantees this.

Definition A.2.2 (Closed Maps)**.** Let X, Y be topological spaces. If $f : X \to Y$ sends each closed set U to a closed set $f(U)$ in Y, then it is called a *closed map*.

Proposition A.2.3 ([Mun75, Section 2.11])**.** *Let X, Y be topological spaces and suppose $f : X \to Y$ is a continuous surjection. If f is a closed map, then it is also a quotient map.*

For further reading, see [Mun75, Section 2.11].

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