Ordinary Differential Equations

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This is an undergraduate course, offered in Summer 2013 at Columbia University. Course textbook is Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems (Tenth edition).* Grading: assignments 25%; one midterm 30%; final 45%. Office hours: M,T 1:00-2:00.

Contents

Introduction

This course will cover

Lecture 1 (5/28/13)

Chapter 1: sections 1 - 3; Chapter 2: sections 2, 4, 5, 6; Chapter 3: sections 1 - 6; Chapter 4: sections 1 - 4; Chapter 5: sections 2 - 5; Chapter 6: sections 2 - 6; Chapter 7: sections 4 - 9.

Let's start with an example:

Example 1. a falling object of mass m, there are two forces acting on it: gravity and air resistance (drag). Choose down + direction, then Newton's equation gives

$$ma = mg - \gamma v$$

where $a(t) = dv(t)/dt$ acceleration

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v(t) \tag{0.1}$$

which is a 1st order ODE.

The equation (0.1) provides a relation of rate of function with function itself. It contains both quantitative and qualitative information that we will be interested in.

Qualitative: e.g. will the object speed up? or slow down? (common intuition says if the object is initially thrown up, it will speed down; if the object is initial thrown up, it will speed up.) What is the terminal velocity?

They can be answered by the so called "direction field". Easy to see that on the direction field v v.s. t, $v_g = \sqrt{mg/\gamma}$ has 0 slope. If the initial speed $v(0)$ is larger than v_g, the slope is down, so $v(t)$ decreases. If $v(0) < v_g$, or even negative, $v(t)$ has positive slope. All of them will approach to v_g, so it is the terminal velocity.

Qualitative: e.g. If the initial velocity is 5m/s down, what is the velocity after 10s?

To answer this type of question, we need to get a solution of (0.1).

For simplicity, suppose (0.1) is

$$\frac{dv}{dt} = 9.8(1 - v(t)) \tag{0.2}$$

Recall if

$$\frac{df}{dx} = f$$

what is f? If $f(x) \neq 0 \; \forall x$, so we can do

$$\frac{df/dx}{f} = 1 \implies \frac{d}{dx} \ln|f(x)| = 1$$

Integral both side wrt dx

$$\ln|f(x)| = x + c \implies f(x) == \pm e^c e^x = Ae^x$$

for some $A \neq 0$.

If $f(0) = 0$, then clearly from the direction field, it will stay at 0, so Ae^x is a general solution, including $A = 0$.

Apply this to (0.2)

$$-\frac{dv/dt}{1-v} = -9.8 \implies \frac{d}{dt} \ln|1-v| = -9.8$$

Integrate wrt dt

$$v(t) = 1 - Ae^{-9.8t}$$

Definition 2. A differential equation is a relation between a function and its derivatives.

Definition 3. Only ordinary derivatives are involved, it is called ordinary differential equation. If partial derivatives are involved, it is a partial differential equation.

Let $y = y(t)$ be function of t, then an ODE is an equation

$$F(t, y, y', y'', ..., y^{(n)}) = 0 \tag{0.3}$$

where $y^{(k)} = d^{(k)}y/dt^{(k)}$, we will assume we can write above as

$$y^{(n)} = f(t, y', ..., y^{(n-1)}) \tag{0.4}$$

The order of an ODE is the highest order derivative that appears in (0.3).

Example 4. $y''' + yy'' + e^y = 0$ is an order 3 ode. If $y = y(s,t)$,

$$\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 y}{\partial t^2} = 0$$

is an order 2 PDE.

Definition 5. The equation (0.3) is linear if F is linear in $y, y', y'', ..., y^{(n)}$, i.e.

$$F(t, u+v, u'+v', ..., u^{(n)}+v^{(n)}) = F(t, u, u', ..., u^{(n)}) + F(t, v, v', ..., v^{(n)})$$

The general linear ODE of order n has the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + ... + a_n(t)y = g(t) \tag{0.5}$$

Definition 6. A solution of (0.4) on the interval $\alpha < t < \beta$ is a function $\psi(t)$ such that ψ', ..., $\psi^{(n)}$ exist and satisfy

$$\psi^{(n)}(t) = f(t, \psi, \psi', ..., \psi^{(n-1)})$$

for every $t \in (\alpha, \beta)$.

Notice later we will show that not all ODE (0.4) can be solved.

There are important questions we would like to study: Do solution exist? Are they unique? Have we found all solutions? How smooth are the solutions? Maximum domain of existence? If the solution is not ordinary elementary function, can we define the solution as special functions.

1 First Order ODEs

This has a deceptive simple form

$$\frac{dy}{dt} = f(t, y)$$

but it has a lot of interesting examples and techniques.

1.1 1st order linear ODEs

Dealing with 1st order linear ODE, we try to use product rule, method of integrating factor.

Example 1. solve

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

Multiply both sides by an unknown function $\mu(t)$, so that the LHS has product rule type, i.e.

$$LHS = (\mu(t)y)'$$

So

$$\mu' = \frac{1}{2}\mu \implies \mu(t) = e^{t/2}$$

we do not a general solution for the integrating factor, any $\not\equiv 0$ function. Now

$$(e^{t/2}y)' = \frac{1}{2}e^{\frac{5}{6}t} \implies e^{\frac{t}{2}}y = \frac{3}{5}e^{\frac{5}{6}t} + c$$

so

$$y = \frac{3}{5}e^{\frac{1}{3}t} + ce^{-\frac{t}{2}}$$

c is specified by the initial conditions. If $y(0) = 0$, then

$$y = \frac{3}{5}(e^{\frac{1}{3}t} - e^{-\frac{t}{2}})$$

In general, we have

$$\frac{dy}{dt} + p(t)y = g(y)$$

so we find integrating factor so that

$$\mu' = \mu p$$

If $\mu \neq 0$, then

$$\frac{du/dt}{\mu} = p \implies \mu = e^{\int_0^t p(s)ds} \tag{1.1}$$

and

$$y(t) = \frac{1}{\mu}\left(\int_0^t \mu(s)g(s)ds + c\right) \tag{1.2}$$

Lecture 2 (5/29/13)

Example 2. Solve

$$y' - y = 2te^{2t}$$

with $y(0) = 1$.

Apply (1.1), (1.2)

$$\mu(t) = e^{-t}$$

$$y = e^t\left(\int_0^t 2se^s ds + c\right)$$

We can get value of c, now $y(0) = e^0 c = 1 \implies c = 1$

$$y = e^t\left(2te^t - 2e^t + 2 + 1\right) = 2(t-1)e^{2t} + 3e^t$$

1.2 Non-linear Separable 1st Order ODE

The general form

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

We are interested in the case it is repeatable, i.e.

$$M(x, y) = M(x) \quad N(x, y) = N(y)$$

Or

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{1.3}$$

Example 3.

$$\frac{dy}{dx} = \frac{x^2}{y}$$

or

$$0 = x^2 - y\frac{dy}{dx} \implies \frac{d}{dx}\left(\frac{x^3}{3} - \frac{y^2}{2}\right) = 0 \implies \frac{x^3}{3} - \frac{y^2}{2} = c$$

One can also solve the problem by drawing the direction field. One way of doing it is to look for curves where slope is constant. Consider the curve $y = cx^2$, then

$$\frac{dy}{dx} = \frac{1}{c}$$

that is the larger the concavity is, the steeper the slope is. After having the the direction field, we can get draw the solution with some initial condition. The solution is of course given by

$$\frac{x^3}{3} - \frac{y^2}{2} = c$$

In general, to solve (1.3), we need to find anti derivative, i.e.

$$H_1'(x) = M(x) \quad H_2'(y) = N(y)$$

then (1.3) becomes

$$0 = \frac{d}{dx}(H_1(x) + H_2(y)) \implies H_1(x) + H_2(y) = c$$

Example 4. Solve

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \implies \frac{x^4}{4} - 2x^2 + 4y + \frac{y^4}{4} = c$$

1.3 1st order Autonomous Equations

Not t in f

$$\frac{dy}{dt} = f(y)$$

Suppose we have a colony of bacteria living in a Petri dish. It reproduces by division. Suppose each bacteria doubles in every Δt and Δt to be the unit time scale of the problem. Because not all bacterias double at the same times, we can assume that the system is continuously evolved. so we have an analogous situation as continuous compounded interest problem with annal rate 100%, the solution turns out to be a exponential and it satisfies the following ODE

$$\frac{dP}{dt} = P \implies P = P_0 e^t$$

where $P(t)$ is the population at time t.

A more realistic model uses

$$\frac{dP}{dt} = h(P) = \begin{cases} \sim P & \text{under normal sequensentanses} \\ < 0 & \text{if } P\text{i is too large} \end{cases}$$

We choose

$$h(P) = rP(1 - \frac{P}{K}) \tag{1.4}$$

One can draw phase plane $h(P)$ v.s. P. We will only consider $P > 0$. The graph of $h(P)$ shows that it is a concave down parabola, with two roots called critical points: $P = 0$ and $P = K$. We find $P = K$ is a attractive equilibrium, because

$$h(P) : \begin{cases} > 0 & P < K \\ < 0 & P > K \end{cases}$$

From the direction field, one can draw $P(t)$ v.s t for various initial conditions. One sees that these curves are translational invariance.

Question 5. *If $P(0) \neq K$, can P ever reach population K?*

Answer is No. It lies in the heart of following theorem.

Theorem 6. *(existence & uniqueness of 1st order ODE) Consider the 1st order ODE*

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{1.5}$$

Suppose f, $\frac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing (t_0, y_0), then in some interval $(t_0 - h, t_0 + h)$ contained in $\alpha < t < \beta$, there is a unique solution

$$y = \psi(t)$$

to (1.5).

Now solve for (1.4)

$$\frac{dP}{p} + \frac{dP}{K(1-\frac{P}{K})} = \frac{dP}{p(1-\frac{P}{K})K} = rdt$$

Integrate

$$\ln|P| - \ln|1 - \frac{P}{K}| = rt + c$$
$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}}$$

clearly $P(t) \to K \ \forall P_0$.

Example 7.

$$\begin{cases} \frac{dy}{dt} = y(y-1)(y-2) \\ y(0) = y_0 \geq 0 \end{cases}$$

Draw Phase Plane. Critical points: $y = 0$, $y = 1$ asymptotic stable, $y = 2$ repel on both sides.

1.4 Exact Equations

Lecture 3 (5/30/13)

Example 8. Solve

$$2x + y^2 + 2xyy' = 0 \tag{1.6}$$

This is not linear, not separable, not automatons, what can we do.

Observe that $\psi(x, y) = x^2 + xy^2$ satisfies

$$\frac{\partial \psi}{\partial x} = 2x + y^2 \qquad \frac{\partial \psi}{\partial y} = 2xy$$

So (1.6) is nothing but

$$\frac{d}{dx}(\psi(x, y)) = 0$$

so solution

$$x^2 + xy^2 = c$$

In general consider

$$M(x,y) + N(x,y)y' = 0$$

If we can find $\psi(x,y)$ s.t.

$$\frac{\partial \psi}{\partial x} = M(x,y) \qquad \frac{\partial \psi}{\partial y} = N(x,y)$$

Above says $\langle M, N \rangle = \nabla \psi$, iff *curl* $\langle M, N \rangle = 0$, so exact equation is to find ψ so that $\langle M, N \rangle = \nabla \psi$. This is possible iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then the solution is

$$\psi = c$$

Theorem 9. *Let M, N, $\partial M/\partial y$, $\partial N/\partial x$ be continuous in the region*

$$R = (\alpha, \beta) \times (\gamma, \delta)$$

the the equation $M(x,y) + N(x,y)y' = 0$ is exact by which we mean it can be written as

$$\frac{d}{dx}(\psi(x,y)) = 0$$

iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

at each point of R. That is there exists a function $\psi(x,y)$ satisfying

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

Proof. We want t find ψ so that $\frac{\partial \psi}{\partial x} = M$

$$\psi(x,y) = \int_{x_0}^{x} M(s,y)ds + h(y) \text{ for } x \in (\alpha, \beta)$$

But also

$$N(x,y) = \frac{\partial \psi(x,y)}{\partial y} = \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)ds + h'(y) = \int_{x_0}^{x} \frac{\partial N}{\partial s}(s,y)ds + h'(y)$$

so

$$N(x,y) = N(x,y) - N(x_0 - y) + h'(y)$$

thus take solution of

$$h'(y) = N(x_0 - y)$$

yield $\psi(x,y)$ $\square$

Example 10.

$$(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)\frac{dy}{dx} = 0$$

Check it is exact

$$\frac{\partial M}{\partial y} = e^x \cos y - 2\sin x = \frac{\partial N}{\partial x}$$

$$\begin{aligned}
\psi(x,y) &= \int_0^x (e^s \sin y - 2y \sin s)ds + \int_0^y (\cos t + 2)dt \qquad (1.7)\\
&= (e^x - 1)\sin y + 2y(\cos x - 1) + \sin y + 2y\\
&= e^x \sin y + 2y \cos x = c
\end{aligned}$$

In the t integral in (1.7) we did not assume $y > 0$. The form will work for $y < 0$ as well, as long as $\forall\, x, y$ are in the R which is stated in the theorem above.

Separable equation is just a special case of exact equation

$$M(x) + N(y)\frac{dy}{dx} = 0$$

because

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

So we can too use exact equation method to solve

Example 11. Solve

$$(e^x - e^{-x}) + (3 + 4y)\frac{dy}{dx} = 0$$

separable

$$\begin{aligned}
\psi(x,y) &= \int_0^x (e^s - e^{-s})ds + \int_0^y (3 + 4y)dt\\
&= e^x + e^{-x} - 2 + 3y + 2y^2 = c
\end{aligned}$$

Example 12. Separable,

$$\begin{cases} y' = e^y - 1 \\ y(0) = 1 \end{cases}$$

so if $y \neq 0$, we get

$$1 + \frac{1}{1-e^y}\frac{dy}{dx} = 0$$

$$\psi(x,y) = \int_0^x sds + \int_1^y \frac{1}{1-e^t}dt \tag{1.8}$$

In y integral, we realize that that $\frac{1}{1-e^y}$ is not continuous at $y = 0$, so the solution exists only on $y > 0$ or $y < 0$. So for $y > 0$, we use the initial condition and choose 1 to the lower bound for the integral. If we want solution in $y < 0$ region, we need another initial condition e.g. $y(0) = -1$, then the y integral becomes

$$\int_{-1}^y \frac{1}{1-e^t}dt$$

again this works even $y < -1$.

Thus (1.8) gives solution

$$\begin{cases} x + y - \ln(e^y - 1) = c & y_0 > 0 \\ 0 & y_0 = 0 \\ x + y - \ln(1 - e^y) = c & y_0 < 0 \end{cases}$$

2 Second Order Linear ODE

2.1 Homogenous with Constant Coefficients

Lecture 4 (5/31/13)

The general 2nd order linear ODE

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

If $P(t) \neq 0$, then we can divide through to obtain

$$y'' + q(t)y' + r(t)y = g(t) \tag{2.1}$$

Initial value problem consist of two piece of data

$$y(t_0) = y_0 \quad y'(\tilde{t}_0) = y_0'$$

the two t_0 need not to be the same for boundary value problem, but for initial value problem they are the same.

If $g(t) \equiv 0$, (2.1) is called homogenous second order ODE, otherwise inhomogeneous.

First we study constant coefficients

$$ay'' + by' + cy = 0 \tag{2.2}$$

$a \neq 0$.

Example 1. Solve

$$y'' - y = 0$$

We guess $y_1 = e^t$, $y_2 = e^{-t}$ are two solutions. For any constant c_1, c_2,

$$y = c_1 y_1 + c_2 y_2$$

is also a solution.

In general to solve (2.2), we try $y = e^{rt}$ plug in

$$ar^2 + br + c = 0$$

called characteristic equation of (2.2). If $r_{1,2}$ are two roots, then $e^{r_{1,2}t}$ are two solutions.

Two Distinct Real Roots

Example 2. $y'' + y' - 2y = 0$ IC $y(0) = 1$, $y'(0) = 1$, then two roots $r_{1,2} = 1, 2$, and

$$c_1 + c_2 = 1 \qquad c_1 - 2c_2 = 1 \implies c_1 = 1 \quad c_2 = 0$$

$$y = e^t$$

Two Complex Roots

Recall Euler's formula

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} (-1) \frac{t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!} = \cos t + i \sin t$$

Example 3. $y'' - 2y' + 2y = 0$, $r = 1 \pm i$

$$y_1 = e^{(1+i)t} = e^t(\cos t + i \sin t) \; y_2 = e^{(1-i)t} = e^t(\cos t - i \sin t)$$

Then

$$y = c_1 y_1 + c_2 y_2$$

for any constant c_1, c_2, real or complex don't matter. Since the solution of physical problem is real, we do

$$\tilde{y}_1 = \frac{y_1 + y_2}{2} \quad \tilde{y}_2 = \frac{y_1 - y_2}{2i}$$

are solutions too. So

$$y = c_1 e^t \cos t + c_2 e^t \sin t$$

This works in general. If $\lambda \pm i\mu$ are roots of the characteristic equation, then

$$y_1 = e^{\lambda t}(\cos \mu t + i \sin \mu t) \quad y_2 = \bar{y}_1$$

so

$$\tilde{y}_1 = e^{\lambda t} \cos \mu t \quad \tilde{y}_2 = e^{\lambda t} \sin \mu t$$

general solution is

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

Example 4. Mass $m = 1$ hangs from a spring with spring constant k, take the equilibrium height to be the origin. Thus

$$\frac{d^2 h}{dt^2} = -kh - \gamma \frac{dh}{dt}$$

$$r_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4k}}{2}$$

Under damp: $\gamma^2 - 4k < 0$. General solution

$$h(t) = e^{-\frac{\gamma}{2}t}(c_1 \cos \frac{\sqrt{4k - \gamma^2}}{2} t + c_2 \sin \frac{\sqrt{4k - \gamma^2}}{2} t) \tag{2.3}$$

Notice if $h(0) = h_0$, $h'(0) = 0$,

$$h(t) \neq e^{-\frac{\gamma}{2}t} h_0 \cos \frac{\sqrt{4k - \gamma^2}}{2} t$$

that is because for graph (2.3), the peaks of cosine are not the points touching the umbrella $e^{-\frac{\gamma}{2}t}$.

Over damp $\gamma^2 - 4k > 0$. General solution

$$h(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$$

where

$$r_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4k}}{2}$$

If $h(0) = h_0 > 0$, $h'(0) = 0$, then the graph of $h(t)$ will never cross the t axis. Because $r_\pm < 0$ and $|r_+| < |r_-|$ $c_1 + c_2 = h_0 > 0$, $r_+c_1 + r_-c_2 = 0$. If the graph did cross 0, then we need at least $c_1 > 0 > c_2$. Since $|c_1| > |c_2|$, $c_1e^{r_+t}$ starting at a higher point than $c_2e^{r_-t}$, and it decreases slower than $c_2e^{r_-t}$, thus the sum will stay positive.

Critical damp $\gamma^2 - 4k = 0$, we will study later.

Lecture 5 (6/3/13)

We want to ask: Do solution exists for all initial conditions? Are they unique?

Theorem 5. *(Existence & uniqueness of 2nd order ODE) Consider the IVP*

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) & 1 \\ y(t_0) = y_0 & 2 \\ y'(t_0) = y'_0 & 3 \end{cases} \tag{2.4}$$

where $p(t)$, $q(t)$, $g(t)$ are continuous on an interval I containing t_0, then there exists an unique $y = \psi(t)$ of the IVP on the interval I.

We want to ask: Are all possible solutions of the form $y = c_1y_1 + c_2y_2$?

Suppose we have two solution y_1, y_2 of the equation (2.4)-1, so $y = c_1y_1 + c_2y_2$ satisfies (2.4)-1, too. We want to be able to find constants $c_{1,2}$ so that $y = c_1y_1 + c_2y_2$ satisfies (2.4)-2 & 3.

Suppose $w(t)$ is some solution of (2.4), by existence theorem $w(t)$ exists.

Define $y_0 = w(t_0)$, $y'_0 = w'(t_0)$, use linear algebra, if

$$W \equiv \det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{pmatrix} \neq 0 \tag{2.5}$$

called Wronskian, then

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

Hence we found $c_{1,2}$. By uniqueness if y and w agrees at one point, then $y(t) = w(t)$. Thus $w(t)$ must be of the form $c_1y_1 + c_2y_2$.

Let's summarize in the following

Theorem 6. *Suppose $y_{1,2}$are solutions* (2.4)*-1 and , then we can find constants $c_{1,2}$ s.t.*

$$y = c_1 y_1 + c_2 y_2$$

satisfies the initial conditions

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

iff $y_{1,2}$ satisfies (2.5)*.*

In this case, we say $y_{1,2}$ form a fundamental set of solutions.

From uniqueness, we can infer that if Wronskian is not 0 at one point, it is not 0 for all $t \in I$. So to check whether $y_{1,2}$ are independent, we can choose any convenience point. cf Abel's Theorem.

Example 7. $y'' + y' - 2y = 0$, $y_1 = e^{-2t}$, $y_2 = e^t$

$$W(0) = \det \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = 3$$

Repeated Roots

Example 8. $y'' = 0$, We find $y_1 = t$, $y_2 = 1$. They are fundamental.

Example 9. $y'' + 4y' + 4y = 0$, $r_{1,2} = -2$. Guess $y_2 = V(t)e^{-2t}$

$$y_2' = V'e^{-2t} - 2Ve^{-2t}$$

$$y_2'' = V''e^{-2t} - 4V'e^{-2t} + 4Ve^{-2t}$$

Thus

$$y_2'' + 4y_2' + 4y_2 = V''e^{-2t} = 0$$

so

$$V'' = 0 \implies V = t$$

we don't need a general solution for $V(t)$.

We found $y_2 = te^{-2t}$. Is this fundamental set?

$$W(0) = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

This clearly works for general repeated root problems.

2.2 Homogenous with Non constant Coefficients

We use the method of undetermined coefficients. This requires to first have y_1, then put $y_2 = V(t)y_1(t)$, and solve for V.

Example 10. Find a fundamental set of solution

$$t^2y'' - 4ty' + 6y = 0$$

$t > 0$, $y_1 = t^2$. Guess $y_2 = V(t)t^2$, then

$$t^2y_2'' - 4ty_2' + 6y_2 = t^4V'' = 0 \implies V = t$$

Alternatively, one can $y = t^a$, then

$$[a(a-1) - 4a + 6]t^a = 0 \implies a = 2 \text{ or } 3$$

Example 11. Find

$$ty'' + 3ty' + y = 0$$

Now guess $y = t^a$ only gives one solution:

$$a(a-1) + 3a + 1 = a^2 + 2a + 1 = 0 \implies a = -1$$

The another solution has to be found through method of undetermined coefficients

$$y_2 = V(t)t^{-1}$$

$$ty_2'' + 3ty_2' + y_2 = tV'' + V' = 0$$

then guess $V' = t^b$, $b + 1 = 0 \implies V' = t^{-1} \implies V = \ln t$,

$$y_2 = \frac{\ln t}{t}$$

2.3 Inhomogeneous 2nd Order Linear ODE

We want to solve

$$y'' + p(t)y' + q(t)y = g(t) \tag{2.6}$$

the corresponding homogenous equation is

$$y'' + p(t)y' + q(t)y = 0 \tag{2.7}$$

Theorem 12. *If $Y_{1,2}$ are solutions of (2.6), then $Y_1 - Y_2$ is a solution of (2.7), thus if $y_{1,2}$ is a fundamental set of solution to (2.6), then*

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

Theorem 13. *Let $y_{1,2}$ be a fundamental set of solution to (2.7) and Y_p is a solution to (2.6), then th general solution to (2.6) is*

$$\psi(t) = c_1 y_1 + c_2 y_2 + Y_p$$

Lecture 6 (6/4/13)

Our goal now is to develop some techniques to find Y_p, a particular solution to (2.6).

Educated Guess: Method of Undetermined Coefficients

Example 14. Solve $y'' - 2y' - 3y = 3e^{2t}$.

Homogenous: $r^2 - 2r - 3 = 0 \implies r = 3, -1$

Inhomogeneous: guess $Y_p = Ae^{2t}$,

$$(4A - 4A - 3A)e^{2t} = 3e^{2t} \implies A = -1$$

$$\psi(t) = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$$

Example 15. Solve $y'' + 2y' + 5y = 3\sin 2t$

Homogenous: $r_\pm = -1 \pm 2i$,

Particular solution: Guess $Y_p = A\sin 2t + B\cos 2t$,

$$-4A\sin 2t - 4B\cos 2t + 4A\cos 2t - 4B\sin 2t + 5A\sin 2t + 5B\cos 2t = 3\sin 2t$$

$$\begin{cases} -4A - 4B + 5A = 3 \\ -4B + 4A + 5B = 0 \end{cases} \implies A = \frac{3}{17} \quad B = -\frac{12}{17}$$

$$\psi(t) = c_1 e^{-t}\sin 2t + c_2 e^{-t}\cos 2t + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$$

Alternatively one can complexity ODE

$$y'' + 2y' + 5y = 3e^{2it}$$

Look for Y_p complex of form $Y_p = \frac{3}{\sqrt{17}} e^{i(2t+\phi)}$, then take the imaginary part of Y_p, should get the same answer.

Example 16. $2y'' + 3y' + y = t^2$

Homogenous: $r_{1,2} = -1, \frac{1}{2}$.

Particular solution: try $At^2 + Bt + C$,

$$4A + 6At + 3B + At^2 + Bt + C = t^2$$

then $A = 1$, $B = -6$, $C = 14$.

$$\psi(t) = c_1 e^{-t} + c_2 e^{\frac{1}{2}t} + t^2 - 6t + 14$$

Above educated guess may not work.

Example 17. Solve $y'' + 2y' + y = 2e^{-t}$

Homogenous: $r^2 + 2r + 1 = 0 \implies r_{1,2} = -1$, $y_1 = e^{-t}$, $y_2 = te^{-t}$.

Thus use $Y_p = e^{-t}$ won't work, try

$$Y_p = V(t)e^{-t}$$

$$y_p'' + 2y_P' + y_p = V''e^{-t} = 2e^{-t} \implies V = t^2$$

So

$$\psi(t) = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t} + t^2 e^{-t}$$

Variation of Parameters

Example 18.

$$y'' + y = \tan t \tag{2.8}$$

$0 < t < \pi/2$

Tangent is not reproducible through derivatives, so the method of undetermined coefficients don't apply.

Homogenous $y_1 = \sin t$, $y_2 = \cos t$

Find Y_p, try

$$Y_p = U_1 y_1 + U_2 y_2$$

this gives more freedom than necessary. Plug in (2.8), we need to first find

$$Y_p' = \underbrace{U_1' y_1 + U_2' y_2} + U_1 y_1' + U_2 y_2'$$

Setting

$$U_1' y_1 + U_2' y_2 \equiv 0$$

so that no $U''_{1,2}$ terms appear in Y''_p, then

$$Y''_p + Y_p = \underbrace{U_1 y''_1 + U_1 y_1} + \underbrace{U_2 y''_2 + U_2 y_2} + U'_1 y'_1 + U'_2 y'_2$$

The first two terms on RHS are 0 because $y_{1,2}$ are fundamental solutions. By inversion of Wronskian

$$\begin{cases} U'_1 y_1 + U'_2 y_2 = 0 \\ U'_1 y'_1 + U'_2 y'_2 = \tan t \end{cases} \implies U'_1 = \sin t \; U'_2 = -\frac{\sin^2 t}{\cos t}$$

Easy to get $U_1 = -\cos t$, U_2 requires integration

$$\begin{aligned} U_2 &= -\int_0^t \frac{\sin^2 s}{\cos s} ds = \sin t - \int_0^t \frac{1}{\cos s} ds \\ &= \sin t - \int_0^t \frac{\cos s}{\cos^2 s} ds = \sin t - \frac{1}{2} \ln \left| \frac{1+\sin t}{1-\sin t} \right| = \sin t - \ln(\sec t + \tan t) \end{aligned}$$

$$\begin{aligned} Y_p &= -\cos t \sin t + \sin t \cos t - \cos t \ln(\sec t + \tan t) \\ &= -\cos t \ln(\sec t + \tan t) \end{aligned}$$

In general situation is summarized below

Theorem 19. *If $p(t)$,$q(t)$, $g(t)$ are continuous on the interval I, and if functions $y_{1,2}(t)$ are a fundamental set of solutions to (2.6), then a particular solution of (2.6) is given by*

$$Y_p(t) = -y_1(t) \int_{t_0}^t \underbrace{\frac{y_2(s) g(s)}{W(y_1, y_2)(s)}}_{U'_1} ds + y_2(t) \int_{t_0}^t \underbrace{\frac{y_2(s) g(s)}{W(y_1, y_2)(s)}}_{U'_1} ds$$

for any $t_0 \in I$.

Lecture 7 (6/5/13)

As we see from the general, this method works in general, but the integration can be hard to compute.

Example 20. $y'' - 2y' + y = \frac{e^t}{1+t^2}$

Homogenous: $y_1 = e^t$, $y_2 = te^t$

$$\begin{cases} U'_1 e^t + U'_2 te^t = 0 \\ U'_1 e^t + U'_2 (1+t) e^t = \frac{e^t}{1+t^2} \end{cases} \implies U'_1 = -\frac{t}{1+t^2} \; U'_2 = \frac{1}{1+t^2}$$

$$\psi(t) = c_1 e^t + c_2 te^t - \frac{\ln(1+t^2)}{2} e^t + (\tan^{-1} t) te^t$$

so the educated guess method won't work, since no genius knows Y_p of such form.

3 Higher Order Linear ODEs

We won't do much on this topic, since not much interesting model arise from higher order ODEs, moreover the techniques of higher order ODEs are conceptually similar to 2nd order but only the computations become more tactical.

cf (0.5) an nth order linear ODE, if the leading coefficient is never 0 (later we will deal with the case of 0 leading coeff)

$$y^{(n)} + p_1 y^{(n-1)} + ... + p_{n-1} y' + p_n y = g(t) \tag{3.1}$$

an initial value problem consists of n initial conditions

$$y(t_0) = y_0 \quad y'(t_0) = y_0' \quad ... \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{3.2}$$

Theorem 1. *If $p_1, ..., p_n$, g are continuous in the open interval I, then there exists a unique solution $y = \psi(t)$ of* (3.1)*, satisfying* (3.2)*, and $\psi(t)$ exists on the interval I.*

3.1 Homogenous Nth Order ODEs

It is useful to study homogenous

$$y^{(n)} + p_1 y^{(n-1)} + ... + p_{n-1} y' + p_n y = 0 \tag{3.3}$$

If $y_{1,..,n}$ are solutions to (3.3), then so is

$$c_1 y_1 + ... + c_n y_n$$

Question 2. *Is every solution of* (3.3) *of this form?*

Yes, provided that for any $t_0 \in I$ and for any choice of initial conditions, there are constants $c_1, ..., c_n$ s.t. $y(t) = c_1 y_1 + ... + c_n y_n$ satisfies the IVP with specified initial conditions.

As in the case $n = 2$, this is equivalent to the Wronskian being non-zero

$$W(y_1, ..., y_n)(t) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & & & \\ \vdots & & & \vdots \\ y_1^{(n)} & & ... & y_n^{(n)} \end{pmatrix}$$

From the uniqueness of nth order ODE, one can show if Wronskian is not zero at one point, it is non zero for all $t \in I$. cf Abel's identity.

In this case, we call $\{y_1, ..., y_n\}$ a fundamental set of solution.

If (3.3) has only constant coefficients, try $y = e^{rt}$. Solve characteristic equations.

Real and complex distinct roots, treated the same way.

Repeated roots: say r_i is a repeated root with multiplicity s, then

$$y_1 = e^{r_i t},\ y_2 = te^{r_i t}, ..., y_s = t^{s-1}e^{r_i t}$$

Example 3. $y''' - 3y'' + 3y' - y = 0$

characteristic equation $r^3 - 3r^2 + 3r - 1 = 0 \implies r_{1,2,3} = 1$

$$y_1 = e^t, y_2 = te^t, y_3 = t^2 e^t$$

Are they linearly independent? i.e. can one find constants $c_{1,2,3}$ s.t.

$$c_1 y_1 + c_2 y_2 + c_3 y_3 \equiv 0 \implies c_{1,2,3} = 0$$

That is

$$c_1 e^t + c_2 te^t + c_3 t^2 e^t \equiv 0$$

hence

$$c_1 + c_2 t + c_3 t^2 \equiv 0 \implies c_{1,2,3} = 0$$

3.2 Non homogenous nth Order ODEs

If $W_{1,2}$ are solutions of (3.1) then as before $W_1 - W_2$ solves (3.3) so if $\{y_1, ..., y_n\}$ is a fundamental set of solution to (3.3), then

$$W_1 - W_2 = c_1 y_1 + ... + c_n y_n$$

To find a particular solution, we can do educated guess, or in general use variation of parameters. Guess

$$Y_p(t) = \sum_{k=1}^{n} U_k y_k$$

Impose

$$Y_p'(t) = \underbrace{\sum_{k=1}^{n} U_k' y_k}_{0} + \sum_{k=1}^{n} U_k y_k'$$

$$Y_p''(t) = \underbrace{\sum_{k=1}^{n} U_k' y_k'}_{0} + \sum_{k=1}^{n} U_k y_k''$$

and so on

$$Y_p^{(n)}(t) = \sum_{k=1}^{n} U_k' y^{(n-1)}{}_k + \sum_{k=1}^{n} U_k y^{(n)}{}_k$$

After plug them into (3.3), the second terms on the right canceled, one term remains

$$\sum_{k=1}^{n} U_k' y^{(n-1)}{}_k = g(t)$$

Therefore we obtain

$$\begin{cases} \sum_{k=1}^{n} U_k' y_k = 0 \\ \sum_{k=1}^{n} U_k' y_k' = 0 \\ \vdots \\ \sum_{k=1}^{n} U_k' y_k^{(n-2)} = 0 \\ \sum_{k=1}^{n} U_k' y^{(n-1)}{}_k = g(t) \end{cases}$$

solve for U_k' and integrate.

4 Linear 2nd Order Homogenous ODE Power Series Way

Lecture 8 (6/6/13)

We want to develop power series technique for solving

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{4.1}$$

$P(x)$ may have singularity. So the existence & unique theorem we studied before may not apply. The method we are going to study will work for P, Q, R analytic functions, i.e. have power series representation. But for computational simplicity, we will work with P, Q, R are polynomials.

4.1 Power Series Review

Definition 1. A power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to coverage at a point x if

$$\lim_{m\to\infty}\sum_{n=0}^{m} a_n(x-x_0)^n$$

exists.

Definition 2. The power series converges absolutely at x if

$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

converges.

Absolutely converges $\implies$ converges.

Theorem 3. *(ratio test) If $a_n \neq 0$ and for fixed x, we have*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L$$

then i) If $|x-x_0|L < 1$, then the series converge absolutely. ii) If $|x-x_0|L > 1$, then the series diverges. iii) If $|x-x_0|L = 1$, the test is inconclusive.

Example 4. Where does $\sum_{n=0}^{\infty}\frac{n}{2^n}x^n$ converge? By ratio

$$|x|\lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| < 1 \implies |x| < 2$$

endpoints $x = \pm 2$ diverges, thus the series converges $|x| < 2$.

Theorem 5. *If $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at x_1, then it converges absolutely for $|x-x_0| < |x_1-x_0|$; if it diverges at x_1, then it diverges for $|x-x_0| > |x_1-x_0|$.*

Definition 6. There is a non negative number ρ called the radius of convergence such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0| < \rho$ and diverges if $|x-x_0| > \rho$.

If the series converges only at $x = x_0$ then $\rho = 0$. If $\rho > 0$ then the interval $|x-x_0| < \rho$ is called the interval of convergence. The series can either converge or diverge at $|x-x_0| = \rho$.

Series can be added or subtracted term wise

Theorem 7. *If $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge to $f(x)$ and $g(x)$ for $|x-x_0| < \rho$, then*

$$f \pm g = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-x_0)^n$$

and this converge for $|x-x_0| < \rho$.

We can also multiply series

$$fg = \sum_{n=0}^{\infty} c_n(x-x_0)^n$$

where

$$c_n = a_0b_n + a_1b_{n-1} + ... + a_{n-1}b_1 + a_nb_0$$

This converges for $|x-x_0| < \rho$.

We can differentiate power series

Theorem 8. *$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has derivative of all orders on $|x-x_0| < \rho$ and these can be computed by differentiating the sum term wise.*

Example 9. $f(x) = \sum_{n=0}^{\infty} \frac{n}{2^n}x^n$, $x \in (-2, 2)$.

$$f'(x) = 0 + \sum_{n=1}^{\infty} \frac{n^2}{2^n}x^{n-1}$$

$$f''(x) = 0 + 0 + \sum_{n=2}^{\infty} \frac{n^2(n-1)}{2^n}x^{n-2} \tag{4.2}$$

We can that the power series of $f(x)$ is really the Taylor series of $f(x)$ at x_0, i.e.

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

After taking derivatives, we may want to shift index back to 0.

Example 10. We can rewrite (4.2), $n \to n+2$

$$f''(x) = \sum_{n=0}^{\infty} \frac{(n+2)^2(n+1)}{2^{n+2}}x^n$$

Example 11. Suppose we can

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

what does this imply about a_n's?

Shift index of LHS

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n \quad \forall\, x$$

so

$$(n+1) a_{n+1} = a_n$$

or

$$a_{n+1} = \frac{a_n}{n+1} = \frac{a_{n-1}}{(n+1)(n)} = ... = \frac{a_0}{(n+1)!}$$

that is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$$

4.2 Solve Homogenous Near Ordinary Points

Back to (4.1).

Definition 12. If x_0 is an ordinary point if $P(x_0) \neq 0$, otherwise we say x_0 is a singular point.

By the existence & uniqueness theorem we are guaranteed to have solutions near an ordinary points, because we can divide out $P(x)$ to get the form (2.4).

Example 13. Solve $y'' - xy = 0$, try $y = \sum_{n=0}^{\infty} a_n x^n$,

$$\begin{aligned} y'' - xy &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \end{aligned}$$

The key is to reindex so that x^n is the same, so the coefficients can be compared

So

$$\begin{aligned} n &= 0 \qquad 2a_2 = 0 \\ n &\geq 1 \qquad (n+2)(n+1)a_{n+2} = a_{n-1} \end{aligned}$$

Thus

$$\begin{cases} a_3 = \frac{a_0}{2\cdot 3} & a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}\cdots \\ a_4 = \frac{a_0}{3\cdot 4} & a_7 = \frac{a_3}{6\cdot 7} = \frac{a_0}{3\cdot 4\cdot 6\cdot 7}\cdots \\ a_2 = 0 & a_5 = 0 \ldots \end{cases}$$

General formula $n \geq 1$

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot \ldots \cdot (3n-1)(3n)} \qquad a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot \ldots \cdot (3n)(3n+1)} \qquad a_{3n+2} = 0$$

Or

$$\begin{aligned} \psi(t) &= a_0\left(1 + \frac{x^3}{2\cdot 3} + \ldots + \frac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot \ldots \cdot (3n-1)(3n)} + \ldots\right) \\ &\quad + a_1\left(x + \frac{x^4}{3\cdot 4} + \ldots + \frac{x^{3n+1}}{3\cdot 4\cdot 6\cdot 7\cdot \ldots \cdot (3n)(3n+1)} + \ldots\right) \end{aligned}$$

Lecture 9 (6/10/13)

Example 14. Find a power series representation for the solution to

$$(1-x)y'' + y = 0$$

about $x_0 = 0$. Set $y = \sum_{n=0}^{\infty} a_n x^n$,

$$\begin{aligned} (1-x)y'' + y &= (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n \qquad (4.3) \end{aligned}$$

So

$$\begin{aligned} n &= 0 \qquad 2a_2 = -a_0 \\ n &\geq 1 \qquad (n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n = 0 \end{aligned}$$

(Note: we could reindex the second term in (4.3) as

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

so we would not have to separate the two cases: $n = 0$ and $n \geq 1$.)

So

$$a_3 = \frac{1}{6}(2a_2 - a_1) = -\frac{1}{6}(a_0 + a_1)$$
$$a_4 = \frac{1}{2}a_3 - \frac{1}{12}a_2 = -\frac{1}{12}(a_1 + \frac{a_0}{2})$$

Thus

$$y(x) = a_0\underbrace{(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + ...)}_{y_1} + a_1\underbrace{(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + ...)}_{y_2} \tag{4.4}$$

Is $\{y_{1,2}\}$ a fundamental set of solution?

$$W(y_{1,2}) = \det\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{4.5}$$

(4.4), (4.5) are true in general. (4.4) reflects the fact that $y(0) = a_0$, $y'(0) = a_1$.

$y'' - xy' - y = 0$ about $x_0 = 1$, put $y = \sum_{n=0}^{\infty} a_n(x-1)^n$

$$\begin{aligned}
y'' - xy' - y &= \sum_{n=2}^{\infty}(n-1)(n)a_n(x-1)^{n-2} - x\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \\
&\qquad - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n \\
&\qquad - \sum_{n=0}^{\infty} na_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n
\end{aligned}$$

(Here we rewrite $x = 1 + (x-1)$ i.e. Taylor expend x about $x = 1$. That is what we will study later.)

$$\forall n \quad (n+1)(n+2)a_{n+2} - (n+1)a_{n+1} - na_n - a_n = 0$$

$$(n+2)a_{n+2} = a_n + a_{n+1}$$

So

$$
\begin{aligned}
a_2 &= \frac{1}{2}(a_0 + a_1) \\
a_3 &= \frac{1}{3}(a_1 + a_2) = \frac{1}{6}a_0 + \frac{1}{2}a_1 \\
a_4 &= \frac{1}{4}(a_2 + a_3) = \frac{1}{6}a_0 + \frac{1}{4}a_1
\end{aligned}
$$

$$
y(x) = a_0 \left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + ...\right) + a_1 \left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + ...\right)
$$

What about the radius of convergence?

Suppose we want to solve (4.1) with $\psi(x_0) = y_0$, $\psi(x_0) = y_0'$, write $\psi(x) = \sum_0^\infty a_n(x-x_0)^n$ then automatically $y_0 = a_0$, $y_0' = a_1$. Since $\psi''(x_0) = 2a_2$, from (4.1)

$$
P(x_0)\psi''(x_0) + Q(x_0)a_1 + R(x_0)a_0 = 0
$$

Hence if $P(x_0) \neq 0$, we can solve for a_2.

Differentiate (4.1), yield

$$
\begin{aligned}
&P(x_0)\psi'''(x_0) + P'(x_0)\psi''(x_0) + Q(x_0)\psi''(x_0) + Q'(x_0)\psi'(x_0) \\
&\qquad + R'(x_0)\psi(x_0) + R(x_0)\psi(x_0) = 0
\end{aligned}
$$

This allows to solve for a_3, and so on.

Does the resulting poser series converge?

Answer: NO! not in general. It does if

$$
q(x) = \frac{Q(x)}{P(x)} \quad r(x) = \frac{R(x)}{P(x)}
$$

are analytic.

Definition 15. A point x_0 is an ordinary point if $q(x)$, $r(x)$ are analytic at x_0, otherwise x_0 is said to be a singular point.

The definition is slightly more general than before. This allows $P(x_0) = 0$, but after canceling common factors $q(x_0) \neq \infty$.

Theorem 16. *If x_0 is an ordinary point of (4.1), then the general solution of (4.1) is*

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2$$

where a_0, a_1 are arbitrary constants, and y_1,y_2 are power series solutions. They are analytic near x_0. The set $\{y_1, y_2\}$ forms fundamental set of solution and the radius of convergence is at least as large as the minimum of the radius of convergence of $q(x)$, $r(x)$ about x_0.

Example 17. $y'' + xy' + 2y = 0$, about $x_0 = 0$

$$\begin{aligned} y'' + xy' + 2y &= \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0 \end{aligned}$$

$$(n+1)a_{n+2} = -a_n$$

n even

$$a_{2k+2} = -\frac{a_{2k}}{2k+1} = ... = (-1)^k \frac{a_0}{(2k+1)!!}$$

n odd

$$a_{2k+3} = -\frac{a_{2k+1}}{2k+2} = ... = (-1)^k \frac{a_1}{(2k+2)!!}$$

4.3 Euler's Equation

Lecture 10 (6/11/13)

Before we do ODE with singularity, we study Euler's equation

$$x^2 y'' + \alpha x y' + \beta y = 0 \tag{4.6}$$

$\alpha, \beta \in \mathbb{R}$. $x = 0$ is the only singular point.

Assume we want solution $x > 0$, guess

$$y = x^r$$

then

$$[r(r-1) + \alpha r + \beta]x^r = 0 \tag{4.7}$$

$$r_{1,2} = \frac{-(\alpha - 1) \pm \sqrt{(\alpha-1)^2 + 4\beta}}{2}$$

Case 1: $r_1 \neq r_2$, real

$$W(y_1, y_2) = (r_2 - r_1)x^{r_1+r_2-1} \neq 0$$

so general solution

$$y = c_1 x^{r_1} + c_2 x^{r_2} \qquad x > 0$$

Example 18. $2x^2y'' + 3xy' - y = 0$ $r_{1,2} = -1, \frac{1}{2}$,

$$y_1 = x^{-1},\ y_2 = x^{\frac{1}{2}}$$

Case 2: repeated roots

Hence (4.7) gives

$$x^2y'' + \alpha xy' + \beta y = (r - r_1)^2 x^r = 0$$

Above equation is identically to 0 for all $x > 0$ when . But if we view above as a function of r for some fixed x, above is not identically to 0, best we can say

$$x^2y'' + \alpha xy' + \beta y = F(r)x^r$$

where $F(r) = (r - r_1)^2$, now take d/dr of above

$$x^2(\frac{d}{dr}y)'' + \alpha x(\frac{d}{dr}y)' + \beta(\frac{d}{dr}y) = F'(r)x^r + F(r)\frac{d}{dr}x^r$$

Because $F(r_1) = F'(r_1) = 0$,

$$\left(\frac{d}{dr}x^r\right)_{r=r_1} = x^{r_1}\ln x$$

is another solution. Check Wronskian

$$W(y_1, y_2) = x^{2r_1-1} \neq 0$$

so they form a fundamental set of solution.

Case 3: complex roots

$r_{1,2} = \lambda \pm i\mu$, we get

$$y_1 = x^{r_1} = x^{\lambda}x^{i\mu} = x^{\lambda}e^{i\mu\ln x} = x^{\lambda}(\cos\mu\ln x + i\sin\mu\ln x)$$

Similar for y_2. As before we get real valued solutions

$$\tilde{y}_1 = \frac{y_1 + y_2}{2} = x^\lambda \cos \mu \ln x \qquad \tilde{y}_2 = x^\lambda \sin \mu \ln x$$

They form a fundamental set of solution.

Now we look for $x < 0$ solutions of (4.6).

Substitute $\xi = -x$ then the solution $u(\xi)$ for $\xi > 0$ of

$$\xi^2 \frac{d^2 u}{d\xi^2} + \alpha \xi \frac{du}{d\xi} + \beta u = 0$$

is

$$u(\xi) = \begin{cases} c_1 \xi^{r_1} + c_2 \xi^{r_2} & r_1 \neq r_2 \in \mathbb{R} \\ c_1 \xi^{r_1} + c_2 \xi^{r_1} \ln \xi & r_1 = r_2 \\ c_1 \xi^\lambda \cos(\mu \ln \xi) + c_2 \xi^\lambda \sin(\mu \ln \xi) & r_{1,2} \in \mathbb{C} \end{cases}$$

Thus the solution for $x \neq 0$ of (4.6) is

$$y(x) = \begin{cases} c_1 |x|^{r_1} + c_2 |x|^{r_2} & r_1 \neq r_2 \in \mathbb{R} \\ c_1 |x|^{r_1} + c_2 |x|^{r_1} \ln |x| & r_1 = r_2 \\ c_1 |x|^\lambda \cos(\mu \ln |x|) + c_2 |x|^\lambda \sin(\mu \ln |x|) & r_{1,2} \in \mathbb{C} \end{cases}$$

4.4 Solve Homogenous Near Regular Singularities

Can we generalize this to other equations with singular points?

Yes, if the equation looks infinitesimally like the Euler equation, meaning that the singularity is a regular singularity, i.e. simple pole.

Definition 19. x_0 is a regular singular point of (4.1) with P, Q, R are analytic functions if both

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \qquad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

are finite. More generally we say that x_0 is a regular singularity if

$$(x - x_0) \frac{Q(x)}{P(x)} \qquad (x - x_0)^2 \frac{R(x)}{P(x)}$$

have convergent power series at x_0.

Definition 20. Any singular points that is not regular is called irregular.

Example 21. Lagrange Equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

$x = \pm 1$ are regular singular points.

Example 22. $x^2(1-x)^2y'' + 2xy' + 4y = 0$, $x = 0, 1$ singular, and $x = 0$ regular.

Consider (4.1), suppose $x = 0$ is a regular singularity, for more general case just shift $\tilde{x} = x - x_0$, then for some radius ρ, $|x| < \rho$, we have

$$xq(x) = \sum q_n x^n \quad x^2 r(x) = \sum b_n x^n$$

Dividing (4.1) by $P(x)$ and multiply by x^2,

$$\begin{aligned} 0 &= x^2y'' + x(xq(x))y' + (x^2r(x))y \\ &= x^2y'' + xq_0y' + b_0y + ... \end{aligned}$$

The first three terms resemble to Euler equation, the remaining terms is 1st order, so we guess the general solution for $x > 0$ is

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \tag{4.8}$$

where x^r solves the indicial equation: $x^2y'' + xq_0y' + b_0y$, giving $r(r-1) + q_0r + b_0 = 0$, called "indicial equation". In addition to to determine r, we want to find a_n's, and the radius of convergence.

Example 23.

$$3x^2y'' + 2xy' + x^2y = 0 \tag{4.9}$$

near $x = 0$

$$q_0 = \lim_{x\to 0} x\frac{2x}{3x^2} = \frac{2}{3}, \quad b_0 = \lim_{x\to 0} x^2\frac{x^2}{3x^2} = 0$$

So the indicial equation

$$x^2y'' + x\frac{2}{3}y' = 0$$

$$r_{1,2} = 0, \frac{1}{3}$$

Now use (4.8),

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Cautionary Note: n in the sum starts at 0, not 1, 2, because now whether $n = 0$ or 1 term vanishes depends on r.

Plug in

$$\begin{aligned}
(4.9) \quad &= \sum_{n=0}^{\infty} 3a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 2a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} \\
&= (\underbrace{3a_0 r(r-1) + 2a_0 r}_{0 \because \text{indicial eq}})x^r + (3a_1(1+r)r + 2a_1(1+r))x^{1+r} \\
&\quad + \sum_{n=2}^{\infty} 3a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=2}^{\infty} 2a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} \\
&= (3a_1(1+r)r + 2a_1(1+r))x^{1+r} + \sum_{n=0}^{\infty} 3a_{n+2}(n+r+2)(n+r+1)x^{n+r+2} \\
&\quad + \sum_{n=0}^{\infty} 2a_{n+2}(n+r+2)x^{n+r+2} + \sum_{n=0}^{\infty} a_n x^{n+r+2}
\end{aligned}$$

therefore

$$3a_1(1+r)r + 2a_1(1+r) = 0 \implies a_1 = 0$$

$$3a_{n+2}(n+r+2)(n+r+1) + 2a_{n+2}(n+r+2) + a_n = 0 \; \forall \, n \geq 0$$

a_0 is free

$$a_{n+2}(r = 0, \frac{1}{3}) = -\frac{a_n}{(n+r+2)(3n+3r+5)}$$

General solution

$$y(x) = c_1 \sum_{n=0}^{\infty} a_n(r=0)x^n + c_2 x^{\frac{1}{3}} \sum_{n=0}^{\infty} a_n(r = \frac{1}{3})x^n$$

Lecture 11 (6/12/13)

In general, we want to solve

$$x^2 y'' + x[xq(x)]y' + [x^2 r(x)]y = 0 \tag{4.10}$$

where $xq(x) = \sum q_n x^n$ and $x^2 r(x) = \sum b_n x^n$. $x = 0$ is a regular singularity.

We obtain the associated Euler equation

$$x^2y'' + xq_0y' + by = 0$$

and the indicial equation

$$F(r) = r(r-1) + q_0r + b_0 = 0$$

Ansatz $y = x^r \sum a_n x^n$, substituting into (4.10),

$$\begin{aligned}
0 &\equiv \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \left(\sum_{n=0}^{\infty} a_n(n+r)x^{n+r}\right)\left(\sum_{n=0}^{\infty} q_nx^n\right) \\
&\quad + \left(\sum_{n=0}^{\infty} a_nx^{n+r}\right)\left(\sum_{n=0}^{\infty} b_nx^n\right) \\
&= a_0(\underbrace{r(r-1)+q_0r+b_0}_{0 \because \text{indicial eq}})x^r + \sum_{n=1}^{\infty} a_n(n+r)(n+r-1)x^{n+r} \\
&\quad + \left(\sum_{n=1}^{\infty} a_n(n+r)x^{n+r}\right)q_0 + \left(\sum_{n=0}^{\infty} a_n(n+r)x^{n+r}\right)\left(\sum_{n=1}^{\infty} q_nx^n\right) \\
&\quad + \left(\sum_{n=1}^{\infty} a_nx^{n+r}\right)b_0 + \left(\sum_{n=0}^{\infty} a_nx^{n+r}\right)\left(\sum_{n=1}^{\infty} b_nx^n\right) \\
&= aF(r)x^r + \sum_{n=1}^{\infty}\left\{F(r+n)a_n + \sum_{k=0}^{n-1} a_k\left[(r+k)q_{n-k} + b_{n-k}\right]\right\}x^{n+r} \qquad (4.11)
\end{aligned}$$

last step we used uniform convergence so we can swap summations.

Hence

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k\left[(r+k)q_{n-k} + b_{n-k}\right] = 0 \qquad (4.12)$$

This gives a way to find a_n's. Note $a_0 \neq 0$ is free, it is just a common factor in $y_{1,2} = x^{r_{1,2}} \sum a_n(r_{1,2})x^n$. The method doesn't work all the time. Suppose $r_{1,2}$ are two roots of $F(r)$, and if $r_1 = r_2 + m$ for some $m \in \mathbb{N}$, then $F(r_2 + m) = 0$ so a_m and all a_n beyond are undetermined, so we cannot find y_2 right a way.

Question 24. *What can we say about convergence?*

Suppose we can find $y_{1,2} = e^{r_{1,2}} \sum a_n(r_{1,2})x^n$, the radius of convergence is at least the minimum of the radii of convergence of $\sum q_nx^n$ and $\sum b_nx^n$.

We now deal with the case $r_1 = r_2$. The case of $r_1 = r_2 + m$ for some $m \in \mathbb{N}$ is more advanced, we won't do it.

So using (4.12), we get y_1, one of the two fundamental solutions. Claim

$$y_2 = \frac{d}{dr}\bigg|_{r=r_1} \sum a_n(r) x^{n+r} \tag{4.13}$$

where $a_n(r)$ are coefficients in y_1. Indeed vary r, by (4.11) $y(x,r)$ satisfies

$$x^2 y'' + x[xq(x)]y' + [x^2 r(x)]y = a_0 F(r) x^r \tag{4.14}$$

We don't include $\sum_{n=1}^{\infty} \left\{ F(r+n)a_n + \sum_{k=0}^{n-1} a_k \left[(r+k)q_{n-k} + b_{n-k}\right] \right\} x^{n+r}$, because $a_n(r)$'s vary with r so that each term in the sum is identically 0. This remains true even we take derivatives. e.g. $x - x \equiv 0$, take derivative wrt x, $1 - 1 \equiv 0$.

Take r derivative of (4.14)

$$x^2 (\frac{dy}{dr})'' + x[xq(x)](\frac{dy}{dr})' + [x^2 r(x)](\frac{dy}{dr}) = a_0 F' x^r + a_0 F x^r \ln x$$

Setting $r = r_1$, $F(r_1) = F'(r_1) = 0$, so $\frac{dy_1}{dr}\Big|_{r=r_1}$ is another solution, proving (4.13).

$$\frac{dy_1}{dr}\bigg|_{r=r_1} = y_1 \ln x + \sum \frac{da_n}{dr}\bigg|_{r=r_1} x^{n+r_1} \tag{4.15}$$

Example 25. Find series solution to $xy'' + y' - y = 0$ near 0.

Write in standard form

$$x^2 y'' + x[1]y' - [x]y = 0 \tag{4.16}$$

so $x = 0$ is a regular singularity. Indicial equation

$$r(r-1) + r + 0 = 0 \implies r_{1,2} = 0$$

repeated roots. Use ansatz $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\begin{aligned} x^2 y'' + x[1]y' - [x]y &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} \\ &= a_0 r^2 x^r + \sum_{n=1}^{\infty} \left[a_n (n+r)(n+r-1) + a_n (n+r) - a_{n-1}\right] x^{n+r} \end{aligned}$$

$r = 0$

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0$$

or

$$a_n = \frac{a_{n-1}}{(n+r)^2} = \frac{a_{n-1}}{n^2} = \frac{a_0}{(n!)^2}$$

Hence

$$y_1 = \sum \frac{1}{(n!)^2} x^{n+r}$$

To find y_2, we can either use the formula (4.15)

$$\begin{aligned} d_n &= \left.\frac{da_n}{dr}\right|_{r=r_1} = \frac{d}{dr}\frac{a_{n-1}}{(n+r)^2} \\ &= \frac{\frac{d}{dr}a_{n-1}(r)}{n^2} - 2\frac{a_{n-1}(0)}{n^3} = \frac{d_{n-1}}{n^2} - \frac{2a_0}{n^3(n!)^2} \end{aligned}$$

and so on.

Or we can plug $y_2 = y_1 \ln x + \sum d_n x^{n+r_1}$ in (4.16), and get the recursively d_n.

Lecture 12 (6/13/13)

Example 26. $xy'' + (1-x)y' - y = 0$

In standard form

$$x^2y'' + x[1-x]y' - xy = 0$$

$q_0 = 1$ and $b_0 = 0$. Indicial

$$r(r-1) + r = 0 \implies r_{1,2} = 0$$

Let $y = \sum a_n x^n$

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^n + (1-x)\sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=1}^{\infty}(n-1)a_{n-1}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n \end{aligned}$$

For $n \geq 1$

$$n(n-1)a_n + na_n - (n-1)a_{n-1} - a_{n-1} = 0$$

$$a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n!}$$

$$y_1 = e^x$$

Second solution instead of using formula (4.15), we do $y_2 = \sum d_n x^n + y_1 \ln x$ plug in

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty} n(n-1)d_n x^n + 2xy_1' + x^2 y_1'' \ln x - y_1 \\
&\quad +(1-x)\sum_{n=0}^{\infty} n d_n x^n + x(1-x)y_1' \ln x + (1-x)y_1 \\
&\quad -x\sum_{n=0}^{\infty} d_n x^n - xy_1 \ln x \\
&= \ln x[\underbrace{x^2 y_1'' + x(1-x)y_1' - xy_1}_{0}] + \sum_{n=0}^{\infty} n^2 d_n x^n - \sum_{n=0}^{\infty} n d_n x^{n+1} \\
&\quad +2xy_1' - xy_1 - \sum_{n=0}^{\infty} d_n x^{n+1} \\
&= \sum_{n=1}^{\infty} n^2 d_n x^n - \sum_{n=1}^{\infty}(n-1)d_{n-1}x^n + \sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n - \sum_{n=1}^{\infty} d_{n-1}x^n
\end{aligned}
$$

For $n \geq 1$

$$n^2 d_n - (n-1)d_{n-1} + 2a_{n-1} - a_{n-1} - d_{n-1} = 0$$

$$d_n = \frac{d_{n-1}}{n} - \frac{a_{n-1}}{n^2}$$

There is a better way to get y_2. Since we know the analytic form of $y_1 = e^x$, try $y_2 = V(x)y_1$

$$x(V'' + 2V' + V) + (1-x)(V' + V) - V = 0$$

$$xV'' + xV' + V' = 0$$

Integrating factor

$$\frac{dV'}{V'} = -\frac{1+x}{x}dx$$

$$\ln V' = -\ln x + x + c$$

choose $c = 0$

$$V' = \frac{1}{x}e^x$$

Then find V, then get y_2.

Example 27. $x^2y'' + (x^2 + \frac{1}{4})y = 0$ near 0

$q_0 = 0$, $b_0 = \frac{1}{4}$, indicial

$$r(r-1) + \frac{1}{4} = 0 \implies r_{1,2} = \frac{1}{2}$$

$$0 = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} + \sum_{n=0}^{\infty} \frac{a_n}{4}x^{n+r}$$

$$\begin{array}{lcl} n & = & 0 \qquad r(r-1)a_0 + \dfrac{a_0}{4} = 0 \implies a_0 \text{ anything} \\ n & = & 1 \qquad (r(r+1) + \dfrac{1}{4})a_1 = 0 \implies a_1 = 0 \\ n & > & 1 \qquad \left[(n+r)(n+r-1) + \dfrac{1}{4}\right] a_n = -a_{n-2} \end{array}$$

$$a_n = -\frac{a_{n-2}}{n^2} \qquad a_1 = 0$$

i.e. y_1 is an even function.

Second solution

$$d_n = \frac{d}{dr}\left(-\frac{a_{n-2}(r)}{(n+r)(n+r-1)+\frac{1}{4}}\right)$$

$$y_2 = x^{\frac{1}{2}}\sum_{n=0}^{\infty} d_n x^n + y_1 \ln x$$

5 Laplace Transform Solve Linear ODEs

5.1 Laplace Transform

Lecture 13 (6/19/13)

Given a function $f(t)$, the Laplace transform of $f(t)$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt$$

This is useful in study ODEs, it translates a differential equation to an algebraic equations.

Given a function $f(t)$, the (improper) integral $\int_0^{\infty} f(t)dt$ is defined to be

$$\lim_{A\to\infty}\int_0^{A} f(t)dt$$

If the limit exists, the integral is said to converges, otherwise it is said to diverge. E.g.

$$\int_0^\infty e^{ct} dt$$

converges when $c < 0$.

$$\int_1^\infty t^{-p} dt$$

converges iff $p > 1$.

$$\int_0^\infty \sin t dt$$

diverges.

Definition 1. $f(t)$ is piecewise continuous on $[\alpha, \beta]$, (α, β are real numbers, because proper Riemann integral works only on some bounded domain) iff $\exists\, t_i \in [\alpha, \beta]$ $\alpha = t_0 < t_1 < ... < t_n = \beta$ s.t. $f(t)$ is continuous on (t_i, t_{i+1}) and

$$\lim_{t \to t_i^+} f(t), \qquad \lim_{t \to t_i^-} f(t)$$

exist and finite, i.e. $f(t)$ has finitely many jump discontinuities on any bounded subintervals.

Example 2. Piecewise continuous $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ k & t = 1 \\ 0 & t > 1 \end{cases}$

Theorem 3. *Suppose that 1) f is piecewise continuous on $[0, A]$ for any A 2) $\exists\, k, a, M \in \mathbb{R}$, $|f(t)| \leq Ke^{at}$ for $t \geq M$ then the Laplace transform exists for $s > a$.*

Example 4. i)

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

for $s > 0$.

ii)

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

for $s > a$.

iii) Compute Laplace $f(t)$ in example 2

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st} dt + \int_1^\infty 0 e^{-st} dt = \frac{1-e^{-s}}{s}$$

for $s \neq 0$. This turns piecewise function into a continuous function. Later we will use this function to solve inhomogeneous ODE with discontinuous driving force.

iv)

$$\mathcal{L}\{\sin at\} = \int_0^\infty \sin at e^{-st} dt = \frac{1}{a} - \frac{s^2}{a^2}\mathcal{L}\{\sin at\}$$

so

$$\mathcal{L}\{\sin at\} = \frac{a}{a^2+s^2}$$

for $s > 0$. More

$$\mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}$$

change $s \to s-b$,

$$\mathcal{L}\{e^{bt}\sin at\} = \frac{a}{(s-b)^2+a^2} \qquad \mathcal{L}\{e^{bt}\cos at\} = \frac{s-b}{(s-b)^2+a^2} \tag{5.1}$$

The key point is that $\mathcal{L}\{f'\}$ and $\mathcal{L}\{f\}$ are simply related,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

More generally

Theorem 5. *Suppose $f, f', ..., f^{(n-1)}$ continuous, and $f^{(n)}$ is piecewise continuous on interval $0 \leq t \leq A$ for any A, and suppose $\exists K, a, M$ s.t. $|f^{(k)}(t)| < Ke^{at}$ for all $0 \leq k \leq n-1$, for $t \geq M$, then $\mathcal{L}\{f^{(n)}\}$ exists for $s > a$,*

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - ... - s^2 f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Theorem 6.

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$$

5.2 Solve Linear Homogeneous ODEs

Example 7. Solve $y'' - y' - 2y = 0$ $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

so

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - [s\mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = 0$$

so

$$\mathcal{L}\{y\} = \frac{s-1}{s^2-s-2} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1} = \mathcal{L}\{\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}\}$$

Hence

$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Here the solution is continuous so the inverse of Laplace is unique, if inverse of Laplace turns out to be a piecewise continuous function, but at the points of discontinuities the values are not definite.

Strategy:

1) Use the Laplace transform to turn a linear ODE into an algebraic equation for $\mathcal{L}\{y\} = F(s)$

2) Try to invert the Laplace transform by looking up the table.

Example 8. $y'' - 2y' + 2y = \sin t$, $y(0) = 1$ $y'(0) = 0$

take Laplace

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\} = \frac{1}{1+s^2}$$

$$(s^2 - 2s + 2)\mathcal{L}\{y\} = \frac{1}{1+s^2} + s - 2$$

$$\mathcal{L}\{y\} \quad = \quad \frac{1}{(s^2-2s+2)(1+s^2)} + \frac{s}{s^2-2s+2} - \frac{2}{s^2-2s+2}$$

$$\frac{1}{(s^2-2s+2)(1+s^2)} = \frac{\frac{2}{5}s + \frac{1}{5}}{1+s^2} + \frac{-\frac{2}{5}s + \frac{3}{5}}{s^2-2s+2}$$

$s^2 - 2s + 2$ has no real root, it would not be correct to be partial fraction for complex roots, since no corresponding inverse Laplace for them. Instead we do $s^2 - 2s + 2 = (s-1)^2 + 1$ so they look like (5.1)

$$\mathcal{L}\{y\} = \frac{2}{5}\frac{s}{1+s^2} + \frac{1}{5}\frac{1}{1+s^2} + \frac{3}{5}\frac{s-1}{(s-1)^2+1} - \frac{4}{5}\frac{1}{(s-1)^2+1}$$

$$y = \frac{2}{5}\cos t + \frac{1}{5}\sin t + \frac{3}{5}e^t\cos t - \frac{4}{5}e^t\sin t$$

5.3 Piecewise Continuous Driving Forces

Step Functions

Definition 9. The unit Step function or Heaviside function, $c > 0$

$$u_c = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Its Laplace

$$\mathcal{L}\{u_c(t)\} = \int_c^{\infty} e^{-st} = \frac{e^{-sc}}{s}$$

for $s > 0$.

Example 10. $h(t) = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t < \infty \end{cases}$, solve

$$y'' - y = h(t)$$

$y(0) = 0$, $y'(0) = 0$, take the Laplace transform

$$s^2\mathcal{L}\{y\} - \mathcal{L}\{y\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$\begin{aligned} \mathcal{L}\{y\} &= \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2-1)} = (e^{-\pi s} - e^{-2\pi s})\left(\frac{1}{s} - \frac{1}{s^2-1}\right) \\ &= (e^{-\pi s} - e^{-2\pi s})\left(\mathcal{L}\{1\} - \mathcal{L}\{\cos s\}\right) \end{aligned} \tag{5.2}$$

Theorem 11. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ (a is defined in theorem 3) then for c a positive constant*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

conversely if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

Back to (5.2)

$$y = u_\pi(t) - u_\pi(t)\cos(t-\pi) - u_{2\pi}(t) + u_{2\pi}(t)\cos(t-2\pi)$$

$$y = \begin{cases} 0 & t < \pi \\ 1 + \cos t & \pi < t < 2\pi \\ 2\cos t & t > 2\pi \end{cases}$$

the solution is continuous.

Lecture 14 (6/20/13)

Proof. (of theorem 11) Change of variables

$$\begin{aligned}
\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty u_c(t)f(t-c)e^{-st}dt \\
&= \int_c^\infty f(t-c)e^{-st}dt \\
&= \int_0^\infty f(t)e^{-s(t+c)}dt \\
&= e^{-sc}\mathcal{L}\{f(t)\}
\end{aligned}$$

□

Theorem 12. *If $F(s) = \mathcal{L}\{f(t)\}$ exists $s > a \geq 0$, (a is defined in theorem 3) then for c constant,*

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$$

for $s > a + c$, conversely

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Proof.

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{ct}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-c)t}dt = F(s-c)$$

□

Example 13.

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s-2)^2 + 1}$$

find $\mathcal{L}^{-1}\{G(s)\}$. Recall

$$\mathcal{L}\{\sin t\} = \frac{1}{1+s^2} = F(s)$$

By theorem 12

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{F(s-2)\} = e^{2t}\sin t$$

Example 14. solve $2y'' + y' + 2y = g(t)$ $y(0) = 0$ $y'(0) = 0$

$$g(t) = u_5(t) - u_{20}(t)$$

$$2s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s}$$
$$\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}$$
$$\frac{1}{s(2s^2 + s + 2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{(s + \frac{1}{4}) + \frac{1}{4}}{(s + \frac{1}{4})^2 + \frac{15}{16}}$$
$$h(t) = \mathcal{L}^{-1}\{\frac{1}{s(2s^2 + s + 2)}\} = \frac{1}{2} - \frac{1}{2}[e^{-\frac{t}{4}} \cos \frac{\sqrt{15}}{4}t + \frac{\sqrt{15}}{15}e^{-\frac{t}{4}} \sin \frac{\sqrt{15}}{4}t]$$

By theorem 11
$$y(t) = u_5(t)h(t - 5) - u_{20}(t)h(t - 20)$$

Impulse Functions

Possible model, $\tau > 0$
$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{else} \end{cases}$$

What we really want is an ideal version of this called Direct-delta function or call distribution function,
$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\delta(t)dt &= \lim_{\tau\to 0} \int_{-\infty}^{\infty} f(t)d_\tau(t)dt \\ &= \lim_{\tau\to 0} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(t)dt \\ &= f(0) \end{aligned}$$

last step is given by mean value theorem.

The Laplace of δ is
$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$$

Example 15. $2y'' + y' + 2y = \delta(t - 5)$, $y(0) = y'(0) = 0$
$$\mathcal{L}\{y\} = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2}\frac{1}{(s + \frac{1}{4})^2 + \frac{15}{16}}$$
$$f(t) = \mathcal{L}^{-1}\{\frac{1}{(s + \frac{1}{4})^2 + \frac{15}{16}}\} = \frac{4}{\sqrt{15}}e^{-\frac{t}{4}} \sin \frac{\sqrt{15}}{4}t$$
$$y(t) = u_5(t)\frac{f(t - 5)}{2} = \frac{2}{\sqrt{15}}u_5(t)e^{-\frac{t-5}{4}} \sin \frac{\sqrt{15}}{4}(t - 5)$$

Example 16. $y'' + 4y = \sin t - u_\pi(t)\sin t,\ y(0) = y'(0) = 0$

$$u_\pi(t)\sin t = -u_\pi(t)\sin(t-\pi)$$

so apply theorem 11

$$\mathcal{L}\{y\} = \frac{1+e^{-\pi s}}{(s^2+1)(s^2+4)} = \frac{1+e^{-\pi s}}{3}\left(\frac{1}{1+s^2} - \frac{1}{4+s^2}\right)$$

$$\begin{aligned} y(t) &= \frac{1}{3}\left(\sin t - \frac{1}{2}\sin 2t + u_\pi(t)\sin(t-\pi) - \frac{1}{2}u_\pi(t)\sin 2(t-\pi)\right) \\ &= \begin{cases} \frac{1}{3}(\sin t - \frac{1}{2}\sin 2t) & 0 < t < \pi \\ -\frac{2}{3}\sin 2t & t > \pi \end{cases} \end{aligned}$$

6 First Order Linear System of ODEs

6.1 Linear Algebra Review

Definition 1. A vector space V over $\mathbb{R}$ (or $\mathbb{C}$) is a set V together with an operation

$$+ : V \times V \to V$$

and an operation

$$\cdot : \mathbb{R} \times V \to V$$

s.t. $u, v, w \in V$ and $a, b, c \in \mathbb{R}$

1) $u + (v + w) = (u + v) + w$

2) $u + v = v + u$

3) $\exists \vec{0} \in V$ s.t. $0 + V = V$

4) For every $v \in V$, $\exists -v \in V$ s.t. $v + (-v) = 0$

5) $a \cdot (u + v) = a \cdot u + a \cdot v$

6) $(a + b) \cdot u = a \cdot u + b \cdot u$

7) $a \cdot (b \cdot u) = (ab) \cdot u$

8) $1 \cdot u = u$

Examples of vector spaces: vectors in $\mathbb{R}^n$ with dimension n;

$$\{f : [0,1] \to \mathbb{R},\ f \text{ smooth}\}$$

∞ dimensional.

Definition 2. A basis of V is an ordered set $B = \{v_i\}_{i \in I}$ s.t. for any $w \in V$ we write

$$w = \sum_{i \in I} a_i v_i$$

the dimension of V is the Cardinality of the smallest basis.

Definition 3. A linear map between vector space V, W is a map $T : V \to W$ s.t.

$$T(av_1 + bv_2) = aTv_1 + bTv_2$$

Lecture 15 (6/24/13)

Example 4. Examples of linear maps

$$\begin{aligned} T : \mathbb{R}^n &\to \mathbb{R}^n \\ v &\mapsto v \end{aligned}$$

identity map.

$$\begin{aligned} T : C^\infty(0,1) &\to C^\infty(0,1) \\ f &\mapsto \frac{df}{dx} \end{aligned}$$

derivative map.

$$\begin{aligned} T : C^\infty(0,1) &\to \mathbb{R} \\ f &\mapsto f(\frac{1}{2}) \end{aligned}$$

evaluation map.

If f is finite dimensional, $\dim V = n$, $\dim W = m$ and $\{v_1, ..., v_n\}$,$\{w_1, ..., w_m\}$ are basis of V, W. Then a linear map gives rise to a matrix $T : V \to W$

$$T(v) = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T(v_i)$$

set

$$T(v_i) = \sum_{j=1}^m b_{ji} w_j$$

then

$$T(v) = \sum_{j=1}^m b_{ji} a_i w_j$$

Write

$$B = (b_{ji}) = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ \vdots & & & \\ b_{m1} & & ... & b_{mn} \end{pmatrix}$$

and

$$T(v) = B \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Properties of Matrices

1) Equality a $m \times n$ matrix $A = (a_{ij})$ is equal to $B = (b_{ij})$ if

$$a_{ij} = b_{ij} \; \forall \, i, j$$

2) The zero metric has all its entries zero.

3) Addition $A + B = (a_{ij} + b_{ij})$

4) $\alpha A = \alpha(a_{ij}) \implies \alpha \in \mathbb{R}$ (or $\mathbb{C}$)

5) $A - B = A + (-B)$

6) Multiplication

$$AB = (c_{ij}) \quad c_{ij} = \sum_{k=1}^{n} {a_i}^k b_k^j$$

$$(AB)C = A(BC) \quad A(B + C) = AB + AC \quad AB \neq BA$$

7) Identity $I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

8) A square matrix $(n \times n)$ matrix B is invertible if there is a matrix A s.t.

$$AB = BA = I$$

We set $A = B^{-1}$. Some matrices don't have inverse, we call them singular.

Theorem 5. *A is invertible iff* $\det A \neq 0$.

In the 2×2 case

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

9) Matrix function

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1} & & a_{nn}(t) \end{pmatrix}$$

$$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt}\right) \quad \frac{d}{dt}(A+B) = \frac{dA}{dt}+\frac{dB}{dt} \quad \frac{d}{dt}(CA) = \left(\frac{dC}{dt}\right)\cdot A+C\cdot\left(\frac{dA}{dt}\right)$$

10) Matrices are useful for encoding the information of linear algebraic equations

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases}$$

Put $A = (a_{ij})$, then the linear system becomes

$$A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad A\vec{x} = \vec{b}$$

If A is invertible,

$$\vec{x} = A^{-1}\vec{b}$$

In particular if $\det A \neq 0$, then $A\vec{x} = \vec{0}$ has only solution $\vec{x} = 0$.

Definition 6. A set of k-vector $\vec{x}_1, \dots, \vec{x}_k$ is said to be linearly dependent. If there are complex numbers $c_1, \dots, c_k$ not all zero such that

$$c_1\vec{x}_1 + \dots + c_k\vec{x}_k = 0 \tag{6.1}$$

Suppose $\dim V = k$ write $\vec{x}_l = \begin{pmatrix} x_{1l} \\ \vdots \\ x_{kl} \end{pmatrix}$ then (6.1) becomes

$$\underbrace{\begin{pmatrix} x_{11} & \dots & x_{1k} \\ \vdots & \ddots & \vdots \\ x_{k1} & \dots & x_{kk} \end{pmatrix}}_{\equiv X} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = 0$$

so $\det X \neq 0$, iff the column vectors are linear independent.

Definition 7. Given a $n \times n$ matrix A corresponds to a linear map $T : V \to V$, an eigenvector $\vec{w} \neq \vec{0}$ with eigenvalue $\lambda \in \mathbb{R}$ is a pair $(\vec{w}, \lambda)$ satisfying

$$A\vec{w} = \lambda\vec{w}$$

so $(A - \lambda)\vec{w} = 0$. Since $\vec{w} \neq \vec{0}$, $\det(A - \lambda I) = 0$.

Definition 8. Given a matrix A, the characteristic equation is

$$\det(A - \lambda I) = 0$$

This is a polynomial equation in λ.

Definition 9. If λ is a solution of the characteristic equation, then the algebraic multiplicity is its multiplicity as a root.

Definition 10. If λ is a root of the characteristic equation, then the geometric multiplicity is the number of linearly independent eigenvectors with eigenvalue λ.

Example 11. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\lambda = 1$ algebraic multiplicity is 2 and geometric multiplicity is 1.

6.2 Existence & Uniqueness of Linear ODE System

Example 12. Consider

$$ay'' + by + c = 0$$

We turn this into 1st order system as follows. Set $u = y'$ then

$$\begin{cases} u' = y'' = -\frac{b}{a}y' - \frac{c}{a}y = -\frac{b}{a}u - \frac{c}{a}y \\ y' = u \end{cases}$$

or

$$\begin{pmatrix} y \\ u \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} \tag{6.2}$$

If we have an nth order linear equation

$$y^{(n)} = F(t, y, y', ..., y^{(n-1)}) \tag{6.3}$$

then set $x_1 = y$, $x_2 = x_1'$, $x_3 = x_2'$, ..., $x_n = x_{n-1}'$ so

$$x_n' = y^{(n)}$$

So the matrix form of (6.3) will be as simple as (6.2) with a lot of 0's.

We actually have existence & uniqueness theorem for much more general 1st order system. Suppose

$$\begin{cases} x_1' = F_1(t, x_1, ..., x_n) \\ x_2' = F_2(t, x_1, ..., x_n) \\ \vdots \\ x_n' = F_n(t, x_1, ..., x_n) \end{cases} \tag{6.4}$$

A solution is a set of n function $\psi_1(t), ..., \psi_n(t)$ on $\alpha < t < \beta$ s.t. $x_1 = \psi_1, ..., x_n = \psi_n$ satisfying (6.4)

Theorem 13. *Let each of the function $F_1, ..., F_n$, $\frac{\partial F_1}{\partial x_1}, ..., \frac{\partial F_1}{\partial x_n}, ..., \frac{\partial F_n}{\partial x_1}, ..., \frac{\partial F_n}{\partial x_n}$ are continuous in a region R of $(t, x_1, ..., x_n)$ space $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, ...,$\alpha_n < x_n < \beta_n$ with $(t_0, x_1^0, ..., x_n^0) \in R$, then there exists an unique interval $|t - t_0| < h$ on which there is a unique solution of (6.4) initial value*

$$\psi_1(t_0) = x_1^0, \psi_2(t_0) = x_2^0, ..., \psi_n(t_0) = x_n^0$$

Lecture 16 (6/25/13)

We are actually interested in linear case.

$$\begin{cases} x_1' = P_{11}(t)x_1 + ... + P_{1n}(t)x_n + g_1(t) \\ x_2' = P_{21}(t)x_1 + ... + P_{2n}(t)x_n + g_2(t) \\ \vdots \\ x_n' = P_{n1}(t)x_1 + ... + P_{nn}(t)x_n + g_n(t) \end{cases}$$

Or

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

The system is homogenous if $\vec{g} = 0$,

$$\vec{x}' = P(t)\vec{x} \tag{6.5}$$

If $\vec{x}^{(1)}$, $\vec{x}^{(2)}$ are two solutions of (6.5), then so is $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$.

Question 14. *Are these linear combinations all possible solutions?*

Before we saw that $y_{1,2}$ form a fundamental set of solution to 2nd linear homogenous ODE if

$$W(y_1, y_2) \neq 0$$

Definition 15. Let $\vec{x}^{(1)}$,..., $\vec{x}^{(n)}$ be solutions of (6.5) we define the Wronskian of $\vec{x}^{(1)}$,..., $\vec{x}^{(n)}$ to be $W = \det X(t)$ where $X(t)$ is the matrix whose columns are $\vec{x}^{(1)}, ..., \vec{x}^{(n)}$, then $\{\vec{x}^{(1)}, ..., \vec{x}^{(n)}\}$ is a fundamental set of solution if $W(t_0) \neq 0$.

Theorem 16. *If $\vec{x}^{(1)}$,..., $\vec{x}^{(n)}$ are a fundamental set of solution to* (6.5) *on $\alpha < t < \beta$, (α, β are defined in theorem 13) then any solution $\psi(t)$ to* (6.5) *can be written uniquely as*

$$\vec{\psi}(t) = c_1\vec{x}^{(1)} + ... + c_n\vec{x}^{(n)}$$

for $c_1, ..., c_n \in \mathbb{R}$.

What about our old Wronskian:

take $y'' + p(t)y' + q(t)y = 0$, set $u = y'$

$$\begin{pmatrix} y \\ u \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}$$

Let $\vec{x}^{(1)}$, $\vec{x}^{(2)}$ be two solutions of above, then $x_1^{(1)} = y_1$, $x_2^{(2)} = y_2$

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2)$$

agreeing old Wronskian.

Theorem 17. *If $\vec{x}^{(1)}$,..., $\vec{x}^{(n)}$ are solutions of* (6.5) *on $\alpha < t < \beta$, (α, β defined in theorem 13) then $W(\vec{x}^{(1)}, ..., \vec{x}^{(n)})$ is either identical zero or never zero.*

Now we want to solve (6.5).

6.3 Constant Coefficients Linear Homogenous System

A $n \times n$ matrix with constant entries, solve

$$\vec{x}' = A\vec{x} \tag{6.6}$$

Try $\vec{x} = \vec{V}e^{rt}$, then

$$r\vec{V} = A\vec{V}$$

so $\vec{V}$ is an eigenvector with eigenvalue r.

We will focus on 2×2 matrix.

Example 18. Solve

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$$

Find eigenvalues and eigenvectors

$$\vec{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

There are 3 cases:

1) eigenvalues are real and distinct. Then two eigenvectors are automatically linearly independent

2) eigenvalues are complex.

3) repeated eigenvalues. We'll do this later.

If $r_{1,2}$ are complex, since $\det(A - rI) = 0$ is a real equation, we have $r_{1,2} = \lambda \pm \mu i$ are complex conjugate.

Let $\vec{V}_1$ be eigenvector associated to r_1, then

$$A\vec{V}_1 = r_1 \vec{V}_1 \implies A\overline{\vec{V}}_1 = \bar{r}_1 \overline{\vec{V}}_1$$

so $\overline{\vec{V}}_1$ is an eigenvector associated to r_2, hence $\vec{V}_{1,2} = \vec{a} \pm i\vec{b}$

$$\begin{aligned} \vec{V}_{1,2} e^{r_{1,2}t} &= (\vec{a} \pm i\vec{b}) e^{\lambda t \pm i\mu t} \\ &= e^{\lambda t}(\vec{a} \pm i\vec{b})(\cos \mu t \pm i \sin \mu t) \\ &= e^{\lambda t}(\vec{a} \cos \mu t - \vec{b} \sin \mu t) \pm i e^{\lambda t}(\vec{b} \cos \mu t + \vec{a} \sin \mu t) \end{aligned}$$

As before put

$$\vec{x}^{(1)} = e^{\lambda t}(\vec{a} \cos \mu t - \vec{b} \sin \mu t), \ \vec{x}^{(2)} = e^{\lambda t}(\vec{b} \cos \mu t + \vec{a} \sin \mu t)$$

Example 19. Solve

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}$$

this is matrix form of $y'' + y = 0$.

Eigenvalue $r = \pm i$, $\vec{V}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\vec{V} e^{rt} = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] (\cos t + i \sin t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Hence

$$\vec{x}^{(1)} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \vec{x}^{(2)} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

6.4 Fundamental Matrix of Constant Coefficients System

We develop a symmetric way to solve (6.6). Suppose that A is an $n \times n$ matrix and $\vec{x}^{(1)}$,..., $\vec{x}^{(n)}$ are fundamental set of solution to (6.6).

Definition 20. The fundamental matrix of (6.6) is the matrix

$$\Psi(t) = \begin{pmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{pmatrix}$$

A solution $\vec{x}(t) = c_1\vec{x}^{(1)}+...+c_n\vec{x}^{(n)}$ is given by $\Psi(t)\vec{c}$ where $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

In particular if $\vec{x}(t_0) = \vec{x}^{(0)}$, initial conditions, then

$$\Psi(t_0)\vec{c} = \vec{x}^{(0)}$$

Since Wronskian $\det \Psi \neq 0$,

$$\vec{c} = \Psi^{-1}(t_0)\vec{x}^{(0)}$$

So for later t

$$\vec{x}(t) = \Psi(t)\Psi^{-1}(t_0)\vec{x}^{(0)}$$

The fundamental matrix $\Psi(t)$ satisfies

$$\Psi'(t) = A\Psi(t) \tag{6.7}$$

because each column of Ψ satisfies (6.6).

Later we will show it is possible to do a basis transformation so that

$$\vec{x}^{(j)}(t_0) = \vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \qquad \forall\, 1 \leq j \leq n \tag{6.8}$$

$\vec{e}_j$ standard basis: all entries are 0 but jth row is 1. For now let's assume that we have (6.8), then

$$\Psi(t_0) = I \tag{6.9}$$

Therefore

$$\vec{x}(t) = \Psi(t)\vec{x}^{(0)}$$

The normalization process (6.9) is important because we can now solve (6.7) by exponentiation. Notice this can not be done directly on (6.6), so defining fundamental matrix is important too.

Solution to (6.7)

$$\Psi(t) = e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

Computing e^{At} can be difficult, special case

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

is diagonal, then

$$A^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} \implies e^{At} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_k} \end{pmatrix}$$

More generally assume $A = TDT^{-1}$ diagonalize

$$A^k = TD^kT^{-1} \implies \Psi(t) = e^{At} = Te^{Dt}T^{-1}$$

Later we'll show Te^{Dt} is the old fundamental matrix without normalization.

Lecture 17 (6/26/13)

Example 21. Solve

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

eigenvalues $\lambda_{1,2} = 3, -1$, eigenvectors $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ notice no needs to normalize the eigenvectors, if we do normalize the eigenvectors then $\det T = 1$, and finding the inversion T^{-1} will be a little bit simpler. Put

$$T = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} \text{-} & \vec{u}_1 & \text{-} \\ \text{-} & \vec{u}_2 & \text{-} \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

then

$$T^{-1}AT = \begin{pmatrix} \text{-} & \vec{u}_1 & \text{-} \\ \text{-} & \vec{u}_2 & \text{-} \end{pmatrix} \begin{pmatrix} | & | \\ 3\vec{v}_1 & -\vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 3\langle \vec{u}_1, \vec{v}_1 \rangle & -\langle \vec{u}_1, \vec{v}_2 \rangle \\ 3\langle \vec{u}_2, \vec{v}_1 \rangle & -\langle \vec{u}_2, \vec{v}_2 \rangle \end{pmatrix} = \begin{pmatrix} 3 & \\ & -1 \end{pmatrix} = D$$

Or

$$A = TDT^{-1}$$

Hence

$$\Psi(t) = e^{At} = T \begin{pmatrix} e^{3t} & \\ & e^{-t} \end{pmatrix} T^{-1}$$

Question 22. *How does this normalized fundamental matrix relate to our old fundamental matrix?*

Suppose $\vec{V}_1, ..., \vec{V}_n$ are linearly independent eigenvectors of the $n \times n$ matrix A with eigenvalues $\lambda_1, ..., \lambda_n$ (not necessary distinct) Define a new variable $\vec{y}$ by

$$\vec{x} = T\vec{y} \tag{6.10}$$

where $T = \begin{pmatrix} | & & | \\ \vec{V}_1 & ... & \vec{V}_n \\ | & & | \end{pmatrix}$. Since $\vec{x}' = A\vec{x}$,

$$\vec{y}' = T^{-1}\vec{x}' = T^{-1}A\vec{x} = T^{-1}AT\vec{y} = D\vec{y}$$

(the first equation assumes T doesn't on t, so eigenvectors are constant, hence A doesn't dependent on t, so the method works only for constant coefficients) So in the $\vec{y}$ coordinate (called normal node solutions), the ODE system is decoupled, that is

$$\vec{y}^{(1)} = e^{\lambda_1 t}\vec{e}_1, ..., \vec{y}^{(n)} = e^{\lambda_n t}\vec{e}_n$$

where $\vec{e}_1, ..., \vec{e}_n$ are the standard basis.

Therefore the old fundamental matrix

$$\Psi(t) = \begin{pmatrix} | & & | \\ \vec{x}^{(1)} & ... & \vec{x}^{(n)} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ T\vec{y}^{(1)} & ... & T\vec{y}^{(n)} \\ | & & | \end{pmatrix} = Te^{Dt} = \begin{pmatrix} | & & | \\ \vec{V}_1 e^{\lambda_1 t} & ... & \vec{V}_n e^{\lambda_n t} \\ | & & | \end{pmatrix}$$

(hence we see the orders of columns in T, Ψ matter, they also correspond to the positions of the diagonal elements in D)

This is indeed the old fundamental matrix

$$\Psi(0) = T = \begin{pmatrix} | & & | \\ \vec{V}_1 & ... & \vec{V}_n \\ | & & | \end{pmatrix}$$

and

$$\Psi'(t) = TDe^{Dt} = TDT^{-1}Te^{Dt} = A\Psi(t)$$

Example 23. Find a fundamental matrix

$$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}$$

Method 1: Find 2 linearly independent solution, $\lambda = 1, -1$, $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$, $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

$$\Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

Method 2: Matrix exponential

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\Psi(t) = Te^{Dt} = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

6.5 Jordan Normal Form

What if we cannot diagonalize A? If A has repeated eigenvalues then it might be the case that its geometric multiplicity < algebraic multiplicity. Then there is not enough linearly independent eigenvectors.

Suppose $\lambda = r$ is a root of multiplicity 2 of $\det(A - \lambda I) = 0$ but eigenspace of r admits only one eigenvector $\vec{V}$.

Guess the second solution may be

$$\vec{x}^{(2)} = \vec{V}te^{rt} + \vec{\eta}e^{rt}$$

then

$$A\vec{x}^{(2)} = r\vec{V}te^{rt} + A\vec{\eta}e^{rt}$$

and

$$\vec{x}'^{(2)} = \vec{V}e^{rt} + r\vec{V}te^{rt} + r\vec{\eta}e^{rt}$$

so $\vec{x}^{(2)}$ is another solution iff

$$A\vec{\eta}e^{rt} = \vec{V}e^{rt} + r\vec{\eta}e^{rt}$$

or

$$(A - rI)\vec{\eta} = \vec{V} \tag{6.11}$$

Definition 24. We say $\vec{\eta}$ is a generalized eigenvector with eigenvalue r. We can write (6.11) as

$$(A - rI)^2\vec{\eta} = (A - rI)\vec{V} = 0$$

where $\vec{V}$ is a genuine eigenvector with eigenvalue r.

Example 25.

$$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$$

eigenvalue $\lambda_{1,2} = 2$, only 1 eigenvector . find $\vec{\eta}$

$$\begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies \vec{\eta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} \qquad \vec{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t}$$

$$\Psi(t) = \begin{pmatrix} e^{2t} & te^{2t} - e^{2t} \\ -e^{2t} & -te^{2t} \end{pmatrix} \tag{6.12}$$

When a matrix does not admit a basis of eigenvectors it cannot be diagonalized, however, it does have a Jordan normal form.

Definition 26. A matrix A is in Jordan normal form

$$A = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n \end{pmatrix}$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

To put a matrix in Jordan normal form, we need to find all the eigenvectors and all the generalized eigenvectors until we have a basis

Example 27. back to example 25

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \vec{\eta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Put

$$T = \begin{pmatrix} | & | \\ \vec{v} & \vec{\eta} \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$$

then

$$\begin{aligned} T^{-1}AT &= \begin{pmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \end{pmatrix} \begin{pmatrix} | & | \\ 2\vec{v} & \vec{v} + 2\vec{\eta} \\ | & | \end{pmatrix} \\ &= \begin{pmatrix} 2\langle \vec{u}_1, \vec{v} \rangle & \langle \vec{u}_1, \vec{v} \rangle + 2\langle \vec{u}_1, \vec{\eta} \rangle \\ 2\langle \vec{u}_2, \vec{v} \rangle & \langle \vec{u}_2, \vec{v} \rangle + 2\langle \vec{u}_2, \vec{\eta} \rangle \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = J \end{aligned}$$

Similar to (6.10), we let

$$\vec{x} = T\vec{y}$$

Suppose $\vec{V}_1, ..., \vec{V}_n$ are linearly independent generalized eigenvectors of the $n \times n$ matrix A with eigenvalues $\lambda_1, ..., \lambda_n$ (not necessary distinct) Define a new variable $\vec{y}$ such that

$$\vec{y}' = T^{-1}\vec{x}' = T^{-1}A\vec{x} = T^{-1}AT\vec{y} = J\vec{y}$$

So in the $\vec{y}$ coordinate the ODE system is partially decoupled, e.g. in example 27

$$\vec{y}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{y}$$

$$\vec{y}^{(1)} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \qquad \vec{y}^{(2)} = \begin{pmatrix} te^{2t} \\ e^{2t} \end{pmatrix}$$

Therefore the old fundamental matrix

$$\Psi(t) = \begin{pmatrix} | & & | \\ \vec{x}^{(1)} & ... & \vec{x}^{(n)} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ T\vec{y}^{(1)} & ... & T\vec{y}^{(n)} \\ | & & | \end{pmatrix} = Te^{Jt}$$

This is indeed the old fundamental matrix

$$\Psi(0) = T = \begin{pmatrix} | & & | \\ \vec{V}_1 & \dots & \vec{V}_n \\ | & & | \end{pmatrix}$$

and

$$\Psi'(t) = TJe^{Jt} = TJT^{-1}Te^{Jt} = A\Psi(t)$$

Jordan normal form is relatively simple to compute. Prove by induction

$$J = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} \implies J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ & \lambda^n \end{pmatrix}$$

Then

$$e^{Jt} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} & \sum_{n=0}^{\infty} \frac{n\lambda^{n-1}t^n}{n!} \\ & \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \end{pmatrix}$$

$$\sum_{n=0}^{\infty} \frac{n\lambda^{n-1}t^n}{n!} = t\sum_{n=1}^{\infty} \frac{\lambda^{n-1}t^{n-1}}{(n-1)!} = t\sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} = te^{\lambda t}$$

So

$$e^{Jt} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ & e^{\lambda t} \end{pmatrix}$$

Example 28. Back to example 27.

$$\Psi(t) = Te^{Jt} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ & e^{\lambda t} \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} - e^{2t} \\ -e^{2t} & -te^{2t} \end{pmatrix}$$

agreeing (6.12).

Lecture 18 (6/27/13)

We can also find the normalized fundamental matrix

$$\Psi(t) = Te^{Jt}T^{-1} = \begin{pmatrix} e^{2t} - te^{2t} & -te^{2t} \\ te^{2t} & e^{2t} + te^{2t} \end{pmatrix}$$

The normalized fundamental matrix satisfies

$$\begin{cases} \Psi'(t) = A\Psi(t) \\ \Psi(0) = I \end{cases}$$

Its columns are solutions to

$$\vec{x}' = A\vec{x}$$

and

$$\vec{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Psi(t)\vec{c}$ is a solution to

$$\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{c} \end{cases}$$

This completes our study of constant coefficients linear 1st order ODE system.

6.6 Inhomogeneous Linear System

Theorem 29. *Let $\vec{x}^{(1)}, ..., \vec{x}^{(n)}$ be a fundamental set of solution to*

$$\vec{x}' = P(t)\vec{x}$$

and $\vec{x}_p(t)$ is a particular solution to

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

then the general solution is

$$\vec{x} = c_1\vec{x}^{(1)} + ... + c_n\vec{x}^{(n)} + \vec{x}_p(t)$$

We study 4 methods of finding a particular solution.

Diagonalization

This works only for constant matrices

Consider

$$\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \vec{g}(t)$$

then

$$\begin{cases} x_1' = 2x_1 + g_1(t) \\ x_2' = 2x_2 + g_2(t) \end{cases}$$

use integrating factor to solve.

In general, A is diagonalizable with eigenvectors $\vec{V}_1, ..., \vec{V}_n$, set $T = [\vec{V}_1, ..., \vec{V}_n,]$, $\vec{x} = T\vec{y}$, then

$$\vec{x}' = T\vec{y}' = A\vec{x} + \vec{g}(t) = AT\vec{y} + \vec{g}(t)$$

$$\vec{y}' = T^{-1}AT\vec{y} + T^{-1}\vec{g}(t) = D\vec{y} + T^{-1}\vec{g}(t)$$

use integrating factor to solve for $\vec{y}$, then get $\vec{x}$.

This works even A is not diagonalizable. So A has Jordan normal form. E.g.

$$\vec{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \vec{x} + \vec{g}(t)$$

so

$$\begin{cases} x_1' = 2x_1 + x_2 + g_1(t) \\ x_2' = 2x_2 + g_2(t) \end{cases}$$

Solve for x_2 and then solve for x_1.

Example 30.

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

eigenvalue $\lambda_{1,2} = -3, -1$ $\vec{v}_{1,2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\vec{y}' = D\vec{y} + T^{-1}\vec{g} = \begin{pmatrix} -3 & \\ & -1 \end{pmatrix} \vec{y} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{g}$$

use integrating factor to solve $\vec{y}$.

$$\begin{cases} y_1' = -3y_1 + \frac{1}{2}(2e^{-t} - 3t) \\ y_2' = -y_2 + \frac{1}{2}(2e^{-t} + 3t) \end{cases}$$

The Method of Undetermined Coefficients

The method doesn't assume A is constant, but requires some luckiness. This is identical to the case of linear equations of order ≥ 2.

Example 31. Same as example 30

$$\begin{aligned} \vec{x}' &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + 2\begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ t \end{pmatrix} \end{aligned}$$

Since $\lambda = -1$ is one of the eigenvalue, and e^{-t} also appears in the driving term, so guess particular solution

$$\vec{x}_p(t) = \vec{a}te^{-t} + \vec{b}e^{-t} + \vec{c}t + \vec{d}$$

plug in and solve for $\vec{a}, \vec{b}, \vec{c}, \vec{d}$.

Variation of Parameters

This does not assume A is constant, but we need to know the fundamental matrix $\Psi(t)$ for the homogenous system, then try

$$\vec{x}_p = \Psi(t)\vec{u}(t)$$

To find $\vec{u}(t)$, we do

$$\begin{aligned}
\vec{x}_p' &= \Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t) \\
&= P(t)\Psi(t)\vec{u}(t) + \Psi(t)\vec{u}'(t) \\
&= P(t)\vec{x}_p + \vec{g}(t)
\end{aligned}$$

hence

$$\vec{g}(t) = \Psi(t)\vec{u}'(t)$$

or

$$\vec{u}'(t) = \Psi(t)^{-1}\vec{g}(t)$$

Integrate to find $\vec{u}(t)$.

Example 32. Same as example 30,

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

$$\vec{u}'(t) = \frac{1}{2}\begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

Then integrate.

Laplace Transform

The method would not be much doable if A is not independent of time.

We define the Laplace transform of $\vec{x}(t)$

$$\vec{y}(s) = \mathcal{L}\{\vec{x}(t)\}$$

whose entries are the Laplace transform of the entries of $\vec{x}(t)$.

E.g.

$$\mathcal{L}\{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\} = \begin{pmatrix} \int_0^\infty e^{-st}x_1(t)dt \\ \int_0^\infty e^{-st}x_2(t)dt \end{pmatrix}$$

Then obviously

$$\mathcal{L}\{\vec{x}'\} = s\vec{y}(s) - \vec{x}(0)$$

Then $\vec{x}' = A\vec{x} + \vec{g}(t)$ becomes

$$\mathcal{L}\{\vec{x}'\} = A\mathcal{L}\{\vec{x}\} + \mathcal{L}\{\vec{g}(t)\}$$

we take out A because it is independent of t. Thus

$$s\vec{y}(s) - \vec{x}(0) = A\vec{y}(s) + \vec{G}(s)$$

that is

$$\vec{y}(s) = (s - A)^{-1}(\vec{x}(0) + \vec{G}(s))$$

then invert the Laplace to find $\vec{x}(t)$.

Example 33. Same as example 30,

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

for simplicity, assume initial condition $\vec{x}(0) = 0$, then

$$\vec{y}(s) = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

then invert the Laplace.

6.7 Qualitative Technique for Linear Systems

Lecture 19 -Last Lec- (7/1/13)

We will force on

$$\vec{x}' = A\vec{x} \tag{6.13}$$

A is constant 2×2 matrices, and $\vec{x} \in \mathbb{R}^2$, so that we can draw the vector $\vec{x}$ on a plane.

Pick a point (x_1, x_2), then any solution $\vec{\psi}(t)$ of (6.13) which passes through (x_1, x_2) at time t_0 must satisfying

$$\vec{\psi}'(t) = A\vec{\psi}(t) = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

so in the (x_1, x_2) plane, we can put a small vector $\vec{V}$ at $\vec{x}$ pointing in the direction $A\vec{x}$. This yields the direction field. A solution is an integral curve of the direction field, we call the (x_1, x_2) plane the phase plane as before.

A sketch of the characteristic trajectory is called a phase portrait.

Example 34. Draw the phase portrait for

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$$

eigenvalue

$$\lambda_{1,2} = 2, -1 \quad \vec{v}_{1,2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution

$$\psi(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

On the phase plane, we see that there are two straight trajectory lines: $t\vec{v}_1$ and $t\vec{v}_2$, i.e. if the initial starting point is on the two lines the solution will stay on the lines. $t\vec{v}_1$ has arrows away from the origin and $t\vec{v}_2$ has arrows pointing into the origin. Say for any other initial $\vec{x}_0$, we can write $\vec{x}_0$ as linear combination of $\vec{v}_1$ and $\vec{v}_2$,

$$\vec{x}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

then the direction of how $\vec{x}_0$ moves is given by

$$\vec{x}_0' = A\vec{x}_0 = 2c_1\vec{v}_1 - c_2\vec{v}_2$$

In other words, if $\vec{x}_0$ is closer to $\vec{v}_1$ then $c_1 \gg c_2$, then $\vec{x}_0' \approx 2\vec{v}_1$, hence $\vec{x}_0$ moves in the direction parallel to $2\vec{v}_1$ which is what the arrows on line $t\vec{v}_1$ represents. So for region separated by $t\vec{v}_1$ and $t\vec{v}_2$, we roughly draw the direction field in the same tendency given by the arrows on $t\vec{v}_1$ and $t\vec{v}_2$.

Therefore we have seen that in general the qualitative behavior depends on the eigenvalues and eigenvectors.

case 1: Real Distinct Eigenvalues

1. $r_1 < r_2 < 0$

Clearly near the origin all trajectories go into the origin, which is called nodal sink. What about far region?

Solution

$$\vec{x}(t) = c_1\vec{v}_1 e^{r_1 t} + c_2\vec{v}_2 e^{r_2 t} = e^{r_2 t}(c_1\vec{v}_1 e^{(r_1 - r_2)t} + c_2\vec{v}_2) \to c_2\vec{v}_2 e^{r_2 t}$$

as $t \to \infty$. $\vec{x}$ moves asymptotically in the direction $\vec{v}_2$.

2. $r_1 < 0 < r_2$

Everything is the same but the arrows are reversed, or think of it as time reversal. This is called a nodal source.

3. $0 < r_1 < r_2$

Solution
$$\vec{x}(t) = c_1\vec{v}_1 e^{r_1 t} + c_2\vec{v}_2 e^{r_2 t}$$
$t \to \infty$, all solutions except $t\vec{v}_1$ approach asymptotically close to $t\vec{v}_2$. $t \to -\infty$ all solution except $t\vec{v}_2$ approach asymptotically close to $t\vec{v}_1$.

Case 2: Real repeated eigenvalues

1. Two independent eigenvectors.

Solution
$$\vec{x}(t) = (c_1\vec{v}_1 + c_2\vec{v}_2)e^{rt}$$
The trajectories are rays. If $r > 0$, ray outwards, called source. If $r < 0$, ray inwards, called sink.

2. One eigenvector, one generalized eigenvector.

Solution
$$\vec{x}(t) = e^{rt}(c_1\vec{v}_1 + c_2\vec{v}_1 t + c_2\vec{\eta})$$
Suppose $r > 0$, then $t \to -\infty$, all trajectories $\vec{x} \to 0$. As $t \to \infty$,
$$\vec{x}(t) \approx e^{rt}c_2(\vec{v}_1 t + \vec{\eta})$$
$\vec{\eta}$ is a constant vector, so all solution except $t\vec{v}_2$ approach asymptotically parallel to $t\vec{v}_1$.

If $r < 0$, reverse the time.

This configuration is called "improper node" or "degenerate node".

Case 3: Complex eigenvalues

1. Pure imaginary eigenvalues

Example 35.
$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x} \tag{6.14}$$
$\lambda = \pm i$, so for any point (x_1, x_2) the direction vector is $(-x_2, x_1)$ hence they are orthogonal. Trajectories are concentric circles, called "center".

In general if eigenvalues are pure imaginary, with some coordinate transformation we can get A looks like (6.14), hence the phase portrait consists of ellipses (circle is a special case).

2. General complex eigenvalues

A has eigenvalue $\lambda \pm \mu i$ and eigenvector $\vec{a} \pm i\vec{b}$, then $\vec{a}$, $\vec{b}$ are two independent vectors

$$A(\vec{a} + i\vec{b}) = (\lambda + i\mu)(\vec{a} + i\vec{b})$$

compare real and imaginary parts

$$A\vec{a} = \lambda\vec{a} - \mu\vec{b} \qquad A\vec{b} = \lambda\vec{b} + \mu\vec{a}$$

In the basis $\{\vec{a}, \vec{b}\}$, after change of basis, A looks like

$$A = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}$$

Go to polar coordinate, set

$$r^2 = x_1^2 + x_2^2$$

if $\vec{x}(t)$ is a solution, then $(r(t), \theta(t))$ satisfies

$$\begin{aligned} 2rr' &= 2x_1x_1' + 2x_2x_2' \\ &= 2x_1(\lambda x_1 + \mu x_2) + 2x_2(-\mu x_1 + \lambda x_2) = 2\lambda r^2 \end{aligned}$$

or

$$r' = \lambda r \implies r(t) = ce^{\lambda t}$$

Set

$$\tan\theta = \frac{x_2}{x_1}$$

take derivative

$$\begin{aligned} \sec^2\theta\theta' &= \frac{x_2'x_1 - x_1'x_2}{x_1^2} = \frac{(-\mu x_1 + \lambda x_2)x_1 - (\lambda x_1 + \mu x_2)x_2}{x_1^2} \\ \frac{r^2}{x_1^2}\theta' &= -\frac{\mu r^2}{x_1^2} \end{aligned}$$

thus

$$\theta = -\mu t + c$$

hence the system differential equation becomes separable. That is in the basis of $\vec{a}$, $\vec{b}$

$$\begin{cases} r(t) = c_1e^{\lambda t} \\ \theta(t) = -\mu t + c_2 \end{cases}$$

such configuration is called spirals. Whether it goes counterclockwise or clockwise dependence on the sign of μ, and the sign of λ tells it is spiral sink or spiral source.