Quantum Field Theory II

Lam Hui

Transcribed by Ron Wu

This is a graduate course. Offered in Fall 2014 at Columbia University. Required Course textbooks: Srednicki, *Quantum Field Theory*; Zee, *Quantum Field Theory in a Nutshell*; Schwartz, *Quantum field Theory and the Standard Model.* Office hours: Mon 11:30-12:20.

Contents

Part I.
Spin 1/2 Field Theory (Cont.)

1. Fermion Interaction

1.1. Fermion Propagator

Lecture 1 (9/3/14)

Continue from last semester, for a free Dirac spinor $\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$ and Dirac equation

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$

As usual, propagator, S, is the green's function of EOM

$$(-i\gamma^\mu\partial_\mu^{(x)} + m)_{\alpha\beta}S_{\beta\eta}(x-y) = \delta^{(4)}(x-y)\delta_{\alpha\eta} \tag{1.1}$$

where subscript $\alpha\beta\eta$ are spin indices, and superscript x indicates the derivative is wrt x variable.

Recall in the scalar case $(-\Box^{(x)} + m^2)\Delta(x-y) = \delta^{(4)}(x-y)$, then Fourier

$$(k^2+m^2)\tilde{\Delta} = e^{ik(x-y)} \implies \Delta = \int \frac{d^4x}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon} \tag{1.2}$$

Now Fourier (1.1)

$$(\underbrace{\gamma^\mu k_\mu}_{\not{k}} + m)\tilde{S}_{\beta\eta} = e^{ik(x-y)}$$

What is the inverse of matrix $\gamma^\mu k_\mu + m$? Try $-\gamma^\mu k_\mu + m$

$$(\gamma^\mu k_\mu + m)_{\alpha\beta}(-\gamma^\nu k_\nu + m)_{\beta\eta} = -(\gamma^\mu)_{\alpha\beta}(\gamma^\nu)_{\beta\eta}k_\mu k_\nu + m^2$$

this is kind of analogous to taking square root of KG equation.

Since $k_\mu k_\nu$ is symmetric,

$$[-(\gamma^\mu)_{\alpha\beta}(\gamma^\nu)_{\beta\eta}k_\mu k_\nu]_{\alpha\eta} = -\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{-2\eta^{\mu\nu}}k_\mu k_\nu = k^2\delta_{\alpha\eta}$$

so the inverse is $\frac{-\gamma^\mu k_\mu+m}{k^2+m^2}$ and the propagator is

$$S_{\beta\eta}=\int\frac{d^4k}{(2\pi)^4}\underbrace{\frac{(-\gamma^\mu k_\mu+m)_{\beta\eta}}{k^2+m^2-i\epsilon}}_{(\frac{1}{\not k+m-i\epsilon})_{\beta\eta}}e^{ik(x-y)}$$

As usual, what propagator gives us is to compute 2 point function. For the scalar case, we derived the following

$$\left\langle 0\middle|T\hat{\phi}(x)\hat{\phi}(y)\middle|0\right\rangle=\frac{1}{i}\Delta(x-y) \tag{1.3}$$

by 2 seemingly independent methods:

1) Canonical quantization

$$\hat{\phi}(x)=\int\frac{d^3k}{(2\pi)^3 2w_k}(a_k e^{ikx}+a_k^\dagger e^{-ikx}) \tag{1.4}$$

where $w_k=\sqrt{\vec{k}^2+M^2}$. Then plugging in (1.3) to verify (1.3).

2) Path integral

$$\left\langle 0\middle|T\hat{\phi}(x)\hat{\phi}(y)\middle|0\right\rangle=\int D\phi\phi(x)\phi(y)e^{i\int d^4x-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2\phi^2}$$

We want to do the same here. We want to show

$$\left\langle 0\middle|T\Psi_\alpha(x)\bar{\Psi}_\beta(y)\middle|0\right\rangle=\frac{1}{i}S_{\alpha\beta}(x-y)$$

We've already done canonical quantization (see last semester HW13 problem 4) with

$$\Psi_\alpha(x)=\int\frac{d^3k}{(2\pi)^3 2w_k}\sum_s(b_s(k)u_{s,\alpha}(k)e^{ikx}+d_s^\dagger(k)v_{s,\alpha}(k)e^{-ikx}) \tag{1.5}$$

where $w_k=\sqrt{\vec{k}^2+m^2}$

Our present goal is to derive

$$\int D\Psi D\bar{\Psi}\Psi_\alpha(x)\bar{\Psi}_\beta(y)e^{iS_{Dirac}} = \frac{1}{i}S_{\alpha\beta}(x-y) \tag{1.6}$$

via path integral. But we have to be careful, because as mentioned last semester, $\Psi, \bar{\Psi}$ are Grassmann variables.

1.2. Grassmann Variables Integration

Suppose a is a Grassmann variables, we want to integral $\int daf(a)$. Think of $f(a)$ as a Taylor series

$$f(a) = \# + \#a + \underbrace{\dots}_{0}$$

higher terms are 0 because $a^2 = 0$. Hence all we need to know how to integral $\int da1$ and $\int daa$, because we assume Grassmann integral has the usual properties:

1)

$$\int da(\alpha f(a) + \beta g(a)) = \alpha \int daf(a) + \beta \int dag(a) \tag{1.7}$$

2)

$$\int daf(a+b) = \int daf(a) \text{ for some (Grassman) constant } b \tag{1.8}$$

Therefore by (1.7), $\int da(a+1) = \int daa + \int da1$ by (1.8), $\int da(a+1) = \int daa$ so

$$\int da1 = 0 \tag{1.9}$$

and we normalize such that

$$\int daa = 1 \tag{1.10}$$

here we have implicitly use the fact that multiplying two Grassmann numbers (i.e. da, a) gives an ordinary number. This is justified physically that tensor product of 2 fermions is a boson.

Somehow (1.9), (1.10) look like derivative in ordinary sense: $1' = 0$ and $a' = 1$.

What about integration of 2 variables? By Taylor series, we only need to know

$\int da_2 da_1 1 = 0$, $\int da_2 da_1 a_1 = 0$, $\int da_2 da_1 a_2 = \int da_2 \int \underbrace{da_1 1}_{0} a_2 = 0$ or $\int da_2 da_1 a_2 =$ $-\underbrace{\int da_2 a_2}_{0} \int da_1 = 0and\int da_2 da_1 a_1 a_2 = 1$, $\int da_2 da_1 a_1 a_2 = -1$.

Next we study Gaussian Grassmann integral

$$\int da_2 da_1 e^{\frac{1}{2}\vec{a}^T M \vec{a}} \tag{1.11}$$

where M is 2×2 real matrix. Since $a_1^2 = a_2^2 = 0$,

$$\frac{1}{2}\vec{a}^T M \vec{a} = \frac{1}{2}(a_1 M_{12} a_2 + a_2 M_{21} a_1) = \frac{1}{2} a_1 (M_{12} - M_{21}) a_2$$

suggesting only antisymmetric part of M contributes, so set $M_{12} = -M_{21}$, then

$$\frac{1}{2}\vec{a}^T M \vec{a} = a_1 M_{12} a_2$$

so

$$(1.11) = \int da_2 da_1 (1 + a_1 M_{12} a_2 + \underbrace{(a_1 M_{12} a_2)^2 + ...}_{0}) = M_{12} = \sqrt{\det M} \tag{1.12}$$

This turns out to be true for n variables

$$\int da_n ... da_1 e^{\frac{1}{2}a^T M a} = \sqrt{\det M}$$

comparing this to normal Gaussian integral

$$\int da_n ... da_1 e^{\frac{1}{2}a^T M a} = \frac{(2\pi)^{n/2}}{\sqrt{\det M}}$$

the minus sign of the square root is the origin of the supersymmetry argument of 0 cosmological constant, because recall $\int da_n ... da_1 e^{\frac{1}{2}a^T M a}$ counts all bobble diagrams, giving vacuum energy.

Lecture 2 (9/8/14)

Next we compute 2 point function, so we need

$$\int da_n ... da_1 a_i a_j e^{\frac{1}{2}a^T M a} = -\sqrt{\det M}(M^{-1})_{ij} \tag{1.13}$$

which is almost the same as for the real case

$$\int da_n ... da_1 a_i a_j e^{\frac{1}{2}a^T M a} = (M^{-1})_{ij}$$

Check (1.13) for $n = 2$. From (1.12)

$$\begin{aligned} (LHS\quad)_{12} &= \int da_2 da_1 a_1 a_2 = 1 \\ (RHS)_{12} &= -M_{12} \begin{pmatrix} 0 & -\frac{1}{M_{12}} \\ \frac{1}{M_{12}} & 0 \end{pmatrix}_{12} = 1 \end{aligned}$$

In general Gaussian Grassmann integral is

$$\int da_n ... da_1 e^{\frac{1}{2}a^T M a + \eta^T a} = \sqrt{\det M} e^{\frac{1}{2}\eta^T M^{-1}\eta} \tag{1.14}$$

where generating η is too real. Compare to real integral

$$\int da_n ... da_1 e^{\frac{1}{2}a^T M a + \eta^T a} = (2\pi)^{n/2} \frac{e^{\frac{1}{2}\eta^T M^{-1}\eta}}{\sqrt{\det M}}$$

Proof of (1.14), complete the square

$$\frac{1}{2}a^T M a + \eta^T a = \frac{1}{2}(a + \Delta)^T M (a + \Delta) - \frac{1}{2}\Delta^T M \Delta$$

the second part is independent of a, so we take it out of integral. To satisfy above, we need

$$\frac{1}{2}\Delta^T M a + \frac{1}{2}a^T M \Delta = \eta^T a = \frac{1}{2}\eta^T a - \frac{1}{2}a^T \eta$$

so

$$\Delta^T = \eta^T M^{-1} \text{ and } \Delta = -M^{-1}\eta$$

fortunately the two conditions agree, because M is antisymmetric, so is M^{-1}, i.e. $(M^{-1})^T = -M^{-1}$. QED

To use (1.14) as generating function, we need to choose a consistent derivative rule, although η is real, we still need to distinguish e.g.

$$\frac{\partial}{\partial\eta_i}(\eta a) = a_i \text{ and } (\eta a)\frac{\partial}{\partial\eta_i} = -a_i$$

1.3. Fermion 2 Point Function

Now we consider each a is a pair, $\int da_n d\bar{a}_n ... da_1 d\bar{a}_1 e^{\bar{a}^T M a}$ using the same logic before, it is easy to see that if $n = 2$

$$\begin{aligned} \bar{a}^T M a &= (\bar{a}_1\ \bar{a}_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \bar{a}_1 M_{11} a_1 + \bar{a}_1 M_{12} a_2 + \bar{a}_2 M_{21} a_1 + \bar{a}_2 M_{22} a_2 \end{aligned} \tag{1.15}$$

and

$$\int da_2 d\bar{a}_2 da_1 d\bar{a}_1 e^{\bar{a}^T M a} = \int da_2 d\bar{a}_2 da_1 d\bar{a}_1 (1 + \bar{a}^T M a + \frac{1}{2}(\bar{a}^T M a)^2 + ...)$$

From (1.15), only $\frac{1}{2}(\bar{a}^T M a)^2$ gives non-zero value because only it contains $\bar{a}_2 \bar{a}_1 a_2 a_1$ terms.

Hence

$$\int da_2 d\bar{a}_2 da_1 d\bar{a}_1 e^{\bar{a}^T M a} = \det M$$

Note that unlike before here in (1.15) clearly we don't need M to be antisymmetric. In fact in application to Dirac equation $\bar{a}^T M a = i \int \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$, we use M that is strictly symmetric.

In general,

$$\begin{aligned} \int da_n d\bar{a}_n ... da_1 d\bar{a}_1 e^{\bar{a}^T M a} &= \det M \\ \int da_n d\bar{a}_n ... da_1 d\bar{a}_1 a_i \bar{a}_j e^{\bar{a}^T M a} &= -\det M (M^{-1})_{ij} \\ \int da_n d\bar{a}_n ... da_1 d\bar{a}_1 e^{\bar{a}^T M a + \bar{\eta}^T a + \bar{a}^T \eta} &= \det M e^{-\bar{\eta}^T M^{-1} \eta} \end{aligned} \tag{1.16}$$

last line assumes M is symmetric.

One can check (1.16) for $n = 2$ just like what we did before.

We are ready to compute 2 point function

$$\left\langle \Psi_\alpha(x)\bar{\Psi}_\beta(y) \right\rangle = \frac{\int D\Psi D\bar{\Psi}\Psi_\alpha(x)\bar{\Psi}_\beta(y)e^{iS_{Dirac}}}{\int D\Psi D\bar{\Psi}e^{iS_{Dirac}}} = -(M^{-1})_{\alpha x,\beta y}$$

where the measure

$$D\Psi D\bar{\Psi} = d\psi_1(x_1)d\bar{\psi}_1(x_1)d\psi_2(x_1)d\bar{\psi}_2(x_1)...d\psi_1(x_2)d\bar{\psi}_1(x_2)... ... ...$$

On the RHS x, y are spatial indices and α, β are spinor indices.

Let's find out what M is for Dirac operator,

$$iS_{Dirac} = i\int d^4x\bar{\Psi}(i\partial\!\!\!/ - m)\Psi = \sum_{\alpha x,\beta y} \bar{\Psi}_\alpha(x)M_{\alpha x,\beta y}\Psi_\beta(y)$$

so

$$M_{\alpha x,\beta y} = d^4x\delta_{xy}i(i\partial\!\!\!/^{(y)} - m)_{\alpha\beta}$$

The inverse of M is by (1.1), so

$$\left\langle \Psi_\alpha(x)\bar{\Psi}_\beta(y) \right\rangle = \frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{(-k\!\!\!/ + m)_{\alpha\beta}}{k^2+m^2-i\epsilon}e^{ik(x-y)}$$

One can see from (1.15), e.g. unlike (1.16)

$$\int da_2 d\bar{a}_2 da_1 d\bar{a}_1 a_1 a_2 e^{\bar{a}^T M a} = 0$$

or $\langle \Psi_\alpha(x)\Psi_\beta(y) \rangle = 0$ also $\left\langle \bar{\Psi}_\alpha(x)\bar{\Psi}_\beta(y) \right\rangle = 0$. However they are not true for Majorana particles for Ψ, $\bar{\Psi}$ are related.

Similarly one can show for any odd numbers of fields

$$\left\langle \Psi_\alpha(x)\bar{\Psi}_\beta(y)\Psi_\gamma(z) \right\rangle = 0 \tag{1.17}$$

and

$$
\begin{aligned}
\left\langle \Psi_{\alpha_1}(x_1)\bar{\Psi}_{\alpha_2}(x_2)\Psi_{\alpha_3}(x_3)\bar{\Psi}_{\alpha_4}(x_4)\right\rangle &= \langle 1\bar{2}\rangle\langle 3\bar{4}\rangle + \underbrace{\langle 13\rangle}_{0}\langle \bar{2}\bar{4}\rangle + \langle 1\bar{4}\rangle\langle \bar{2}3\rangle \\
&= \langle 1\bar{2}\rangle\langle 3\bar{4}\rangle - \langle 1\bar{4}\rangle\langle 3\bar{2}\rangle
\end{aligned}
$$

so always pair Ψ with $\bar{\Psi}$.

1.4. A Baby Yukawa Theory

We are not QED yet; we study interaction between spin 1/2 and spin 0

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + g\phi\bar{\Psi}\Psi \tag{1.18}$$

Recall free $\Psi_\alpha(x)$ is (1.5) and $\phi(x)$ is (1.4).

Consider

$$\underbrace{e^-}_{p,s}\underbrace{\phi}_{k} \to \underbrace{e^-}_{p',s'}\underbrace{\phi}_{k'} \tag{1.19}$$

There are two associated Feynman diagrams, see Srednicki figure 45.4 on page 286. The internal line has to be a fermion because we discussed in (1.17), any odd numbers of fermion field is 0.

We claim

$$\text{scatter amp} = (ig)^2\frac{1}{i}\bar{u}_{s'}(p')\left(\frac{-(p\!\!\!/ + k\!\!\!/) + m}{(p+k)^2 + m^2 - i\epsilon} + \frac{-(p\!\!\!/ - k\!\!\!/') + m}{(p-k')^2 + m^2 - i\epsilon}\right)u_s(p) \tag{1.20}$$

everything picks nicely, if without u, $\bar{u}$, it would be a matrix.

To justify above, let's recall LSZ.

Lecture 3 (9/10/14)

An outgoing ϕ with momentum k can be thought of $\langle 0|\, a_k$, created by

$$a_k \to i\int d^4x e^{-ikx}(-\Box_x + m^2)\phi_x \tag{1.21}$$

$a_k^\dagger |0\rangle$ creates incoming particle with momentum k, by

$$a_k^\dagger \quad\rightarrow\quad i\int d^4x e^{ikx}(-\Box_x + m^2)\phi_x = i\int d^4x \phi_x(-\overleftarrow{\Box}_x + m^2)e^{ikx}$$

For Ψ, create outgoing e^- with momentum k, $\langle 0| b_s$ by

$$b_s(k) \rightarrow i\int d^4x e^{-ikx}\bar{u}_s(k)(-i\partial\!\!\!/_x + m)\Psi_x \tag{1.22}$$

comparing to (1.21), where does the addition $\bar{u}_s(k)$ come from? Because of (1.5) and the orthonormal of u,v, $\bar{u}$, and $\bar{v}$.

Create incoming e^+ with momentum k by

$$d_s^\dagger(k) \rightarrow -i\int d^4x e^{ikx}\bar{v}_s(k)(-i\partial\!\!\!/_x + m)\Psi(x) \tag{1.23}$$

Create incoming e^- with momentum k by

$$b_s^\dagger(k) \rightarrow i\int d^4x \bar{\Psi}(x)(-i\overleftarrow{\partial\!\!\!/}_x + m)u_s(k)e^{ikx} \tag{1.24}$$

Create outgoing e^+ with momentum k by

$$d_s^\dagger(k) \rightarrow -i\int d^4x \bar{\Psi}(x)(-i\overleftarrow{\partial\!\!\!/}_x + m)v_s(k)e^{-ikx} \tag{1.25}$$

Why do we move $\bar{\Psi}$ to the left of operator? because pedantically speaking $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$ is a row vector.

Now back to (1.18),

$$
\begin{aligned}
{}_{out}\langle k';p',s'|p,s;k\rangle_{in} - 1 \;&\sim\; \left\langle 0\middle| a_{k'}b_{s'}(p')b_s^\dagger(p)a_k^\dagger \middle|0\right\rangle \qquad (1.26)\\
&= \int ...T\left\langle 0\middle|\hat{\phi}\hat{\Psi}\hat{\bar{\Psi}}\hat{\phi}\middle|0\right\rangle\\
&= \int ...\int D\phi D\Psi D\bar{\Psi}\phi\Psi\bar{\Psi}\phi e^{i(S_{free}+S_{int})}\\
&= \int ...\int D\phi D\Psi D\bar{\Psi}\phi\Psi\bar{\Psi}\phi e^{iS_{free}}(\underbrace{1+i\underbrace{S_{int}}_{\int g\phi\Psi\bar{\Psi}}}_{0}+\frac{1}{2}(iS_{free})^2+...)
\end{aligned}
$$

where $\int ... = \int$ KG & Dirac operators.

The first term in the perturbation is 0, because we subtract $'-1'$ no scattering. The second term in the perturbation is 0, because it doesn't have internal line. When KG & Dirac act on it, removing all external lines; it gives 0, see proof of LSZ from last semester. Another reason the paring of three ϕ is bad; Gaussian of $\langle\phi\phi\phi\rangle = 0$.

So

$$
{}_{out}\langle k';p',s'|p,s;k\rangle_{in} - 1 \sim \left\langle \overbrace{\underbrace{\phi}_{a}}^{k'}\overbrace{\underbrace{\Psi}_{b}}^{p',s'}\overbrace{\underbrace{\bar{\Psi}}_{c}}^{p,s}\overbrace{\underbrace{\phi}_{d}}^{k}\left(\int g\underbrace{\phi}_{e}\underbrace{\Psi}_{f}\underbrace{\bar{\Psi}}_{g}\right)\left(\int g\underbrace{\phi}_{h}\underbrace{\Psi}_{i}\underbrace{\bar{\Psi}}_{j}\right)\right\rangle \qquad (1.27)
$$

Why does $\bar{\Psi}$ associate incoming p,s, and Ψ associate with outcoming? Because of (1.22), (1.24).

We see that there are 2 nonequivalent ways to connect the diagram.

1)

$$a \leftrightarrow e, b \leftrightarrow g, j \rightarrow f, c \leftrightarrow i, d \leftrightarrow h$$

in this way $c \leftrightarrow d$, $a \leftrightarrow b$, so we get the first figure 45.4 on Srednicki page 286. And the internal line is $j \rightarrow f$

2)

$$a \leftrightarrow e, b \leftrightarrow j, g \rightarrow i, c \leftrightarrow f, d \leftrightarrow h$$

in this way $a \leftrightarrow c$, $b \leftrightarrow d$, so we get the second figure 45.4 on Srednicki page 286. And the internal line is $g \rightarrow i$.

Why not add a third $f \leftrightarrow g, i \leftrightarrow j$? This gives disconnected diagram. Bad

The reason for putting $j \to f$ instead of $f \to j$ and putting $g \to i$ instead of $i \to g$ is that we follow the direction of fermion propagator. Because the flipping of fermions will result possible sign change, so we first fix the direction of the incoming fermions (Feynman rule 11 on Srednicki page 288). Hence in this case, always start from c. Then the direction $j \to f$ and $g \to i$ is fixed. Therefore we see that both diagrams have the same $-$ sign for internal line $\langle \bar{\Psi}\Psi \rangle = -\langle \Psi\bar{\Psi} \rangle$. This justifies the $+$ sign in (1.20). In general if two processes differed by switching bosons have the same sign; if by switching fermions have opposite sign.

What about $\bar{u}_{s'}$ and u_s? Why one is on the most right and the other is on the most left in (1.20)? Because $\bar{u}$ is a row vector.

In the end

$$_{out}\langle k'; p', s' | p, s; k \rangle_{in} - 1 = (2\pi)^4 \delta^4(p' + k' - p - k) i\mathcal{M}$$

where $i\mathcal{M} = (1.20)$.

1.5. Three More Examples

Consider

$$\underbrace{e^+}_{p,s} \underbrace{\phi}_{k} \to \underbrace{e^+}_{p',s'} \underbrace{\phi}_{k'} \tag{1.28}$$

Clearly from (1.23), (1.25), change $\bar{u} \to v$ and $u \to \bar{v}$, so

$$i\mathcal{M} = (ig)^2 \frac{1}{i} \bar{v}_s(p) \left(\frac{-(\not{p} + \not{k}) + m}{(p+k)^2 + m^2 - i\epsilon} + \frac{-(\not{p} - \not{k}') + m}{(p-k')^2 + m^2 - i\epsilon} \right) v_{s'}(p')$$

Next example: $e^{\pm}$

$$\underbrace{e^-}_{p_1,s_1} \underbrace{e^+}_{p_2,s_2} \to \underbrace{\phi}_{k_1} \underbrace{\phi}_{k_2} \tag{1.29}$$

There are two processes, see figure 45.6 on Srednicki page 289. They are differed by switching bosons. LSZ looks like

$$\langle 0 | a(k_1) a(k_2) d^\dagger_{s_2}(p_2) b^\dagger_{s_1}(p_1) | 0 \rangle$$

$$i\mathcal{M} = (ig)^2 \frac{1}{i} \bar{v}_{s_2}(p_2) \left(\frac{-(\not{p}_1 - \not{k}_1) + m}{(p_1 - k_1)^2 + m^2 - i\epsilon} + \frac{-(\not{p}_1 - \not{k}_2) + m}{(p_1 - k_2)^2 + m^2 - i\epsilon} \right) u_{s_1}(p_1)$$

Lastly consider coulomb

$$\underbrace{e^-}_{p_1, s_1} \underbrace{e^-}_{p_2, s_2} \to \underbrace{e^-}_{p_1', s_1'} \underbrace{e^-}_{p_2', s_2'}$$

By early remark the propagator has to be boson. There are two processes, see figure 45.7 on Srednicki page 290. They are differed by switching fermions.

$$i\mathcal{M} = (ig)^2 \frac{1}{i} \left(\frac{(\bar{u}_{s_1'} u_{s_1})(\bar{u}_{s_2'} u_{s_2})}{(p_1 - k_1)^2 + M^2 - i\epsilon} - \frac{(\bar{u}_{s_2'} u_{s_1})(\bar{u}_{s_1'} u_{s_2})}{(p_1 - k_2)^2 + M^2 - i\epsilon} \right)$$

Problem set 1 problems 1&2 ask to consider

$$e^- \phi \to e^+ \phi$$

and

$$e^+ e^- \to e^+ e^-$$

one can show explicitly the formal has 0 scattering amplitude at tree level.

1.6. Cross-Section

Lecture 4 (9/15/14)

We want to square $|i\mathcal{M}|^2$ to get cross section. We see a typically $i\mathcal{M}$ is made of

$$\bar{u}'(k_\mu)\gamma^\mu u, \text{ or } \bar{u}'(k_\mu)\gamma^\mu v, \text{ etc}$$

for each μ, k_μ is a number, so

$$\begin{aligned}
|\mathcal{M}|^2 &\sim (\bar{u}'\gamma^\mu u)^\dagger(\bar{u}'\gamma^\nu u)k_\mu k_\nu \\
&= \underbrace{u^\dagger\gamma^{\mu\dagger}(u'^\dagger\gamma^0)^\dagger}_{u^\dagger \underbrace{\gamma^0\gamma^0}_{\text{insert } I} \gamma^{\mu\dagger} \underbrace{\gamma^{0\dagger}}_{\gamma^0} u'} \;\; \bar{u}'\gamma^\nu u k_\mu k_\nu \\
&= \underbrace{\bar{u}\underbrace{\gamma^0\gamma^{\mu\dagger}\gamma^0}_{\gamma^\mu}u'\bar{u}'\gamma^\nu u}_{\text{some }\#} k_\mu k_\nu \\
&= \mathrm{Tr}(\bar{u}\gamma^\mu u'\bar{u}'\gamma^\nu u)k_\mu k_\nu \\
&= \mathrm{Tr}(\gamma^\mu \underbrace{u'\bar{u}'}\gamma^\nu \underbrace{u\bar{u}})k_\mu k_\nu
\end{aligned}$$

Use the orthonormality and completeness

$$\sum_s u_s(p)\bar{u}_s(p) = -\gamma^\mu p_\mu + m \qquad \sum_s v_s(p)\bar{v}_s(p) = -\gamma^\mu p_\mu - m$$

So we end up computing

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\underbrace{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}_{\{\gamma^\mu,\gamma^\nu\}=-2\eta^{\mu\nu}}) = -4\eta^{\mu\nu}$$

and $\mathrm{Tr}(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta)$, which is outlined in Srednicki chapter 47.

1.7. Renormalization & Loop Corrections

Back to baby Yukawa (1.18). As before for renormalization (1 loop correction), we need to add Z_ϕ, Z_Ψ, Z_m, Z_M, Z_g, Z_λ, etc. Moreover, we want to add all possible allowed interactions to (1.18): ϕ, ϕ^3, and ϕ^4.

Why no terms like $\phi^2\Psi\bar{\Psi}$, $(\Psi\bar{\Psi})^2$, ϕ^5? Because $[\phi] = 1$, $[\Psi] = 3/2$. These terms are finite for $d = 4$. They will be automatically included in Z_g or Z_λ There is trick to eliminate ϕ and ϕ^3 terms by invoking parity symmetry. We will replace the $g\phi\bar{\Psi}\Psi$ term in (1.18) by $ig\phi\bar{\Psi}\gamma_5\Psi$. In this way we can impose (1.18) to be parity invariant, so that there should not be ϕ and ϕ^3, provided ϕ is a pseudoscalar, i.e.

$$\phi(x) \to -\phi(\mathbb{P}x)$$

with $\mathbb{P} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

Let's first check $ig\phi\bar{\Psi}\gamma_5\Psi$ is real (that is the reason of the additional i)

$$\begin{aligned} (ig\phi\bar{\Psi}\gamma_5\Psi)^\dagger &= -ig\phi\bar{\Psi}^\dagger\gamma^0\gamma^0\gamma_5\gamma^0\Psi \\ &= -ig\phi\bar{\Psi}\underbrace{\gamma^0\gamma_5\gamma^0}_{-\gamma_5}\Psi \end{aligned}$$

Next check this term is invariant under parity.

$$\bar{\Psi}\gamma_5\Psi \xrightarrow{\mathbb{P}} \bar{\Psi}\gamma^0\gamma_5\gamma^0\Psi = \Psi^\dagger\gamma^0\gamma^0\gamma_5\gamma^0\Psi = \bar{\Psi}\underbrace{\gamma^0\gamma_5\gamma^0}_{-\gamma_5}\Psi$$

Now putting thing together using dimensional regulation

$$\begin{aligned} \mathcal{L} &= iZ_\Psi\bar{\Psi}\partial\!\!\!/\Psi - Z_m m\bar{\Psi}\Psi - \frac{1}{2}Z_\phi(\partial\phi)^2 - \frac{1}{2}Z_M M^2\phi^2 \\ &\qquad + iZ_g g\phi\bar{\Psi}\gamma_5\Psi\mu^{\epsilon/2} - \frac{1}{4!}Z_\lambda\lambda\phi^4\mu^\epsilon \end{aligned} \tag{1.30}$$

introducing μ, ϵ to keep g,λ dimensionless. $d = 4 - \epsilon$.

Actually without imposing parity, one can show directly (problem set 2 problem 2) ϕ^3 term won't be generated at the one-loop level from (1.30).

We now proceed. First compute loop correction to $\langle\phi\phi\rangle$ propagator, see Srednicki figure 51.1 on page 316. First and second diagrams are $\phi\Psi\bar{\Psi}$ and ϕ^4 interaction and the last diagrams are counter correction to absorb UV divergence. Clearly for the second diagram

$$i\Pi_{second\,diagram} = \mu^\epsilon(-i\lambda)\frac{1}{2}\frac{1}{i}\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2 + M^2 - i\epsilon} \tag{1.31}$$

here M is the renormalized mass.

For the first diagram, we claim

$$i\Pi_{first\,diagram} = \mu^{\epsilon}(i(ig))^2\frac{1}{i^2}(-1)\int\frac{d^dl}{(2\pi)^d}\mathrm{Tr}\left(\gamma_5\frac{-(\not k+\not l)+m}{(k+l)^2+m^2-i\epsilon}\gamma_5\frac{-\not l+m}{l^2+m^2-i\epsilon}\right) \tag{1.32}$$

Idea of proof, just like before cf (1.27)

$$\begin{aligned}\langle\phi_1\phi_2\rangle_{full}-1 &= \int D\phi D\Psi D\bar\Psi\phi_1\phi_2 e^{iS}-1\\ &= \left\langle\underbrace{\phi_1}_{a}\underbrace{\phi_2}_{b}\frac{1}{2!}(i(ig))^2(\int g\underbrace{\phi}_{c}\underbrace{\Psi}_{d}\gamma_5\underbrace{\bar\Psi}_{e})(\int g\underbrace{\phi}_{f}\underbrace{\Psi}_{g}\gamma_5\underbrace{\bar\Psi}_{h})\right\rangle_{free}\end{aligned} \tag{1.33}$$

There are two possibilities. One is

$$a\leftrightarrow c, b\leftrightarrow f, d\to h\,(\langle\Psi\bar\Psi\rangle), e\to g\,(\langle\bar\Psi\Psi\rangle)$$

which makes a fermion loop. The second possibility is switching a and b

$$a\leftrightarrow f, b\leftrightarrow c, h\to d\,(\langle\bar\Psi\Psi\rangle), g\to e\,(\langle\Psi\bar\Psi\rangle)$$

both processes have negative sign. This justifies the overall - in (1.32). The 2! in (1.33) is killed by 2 symmetry factor, so no 2 in (1.32). The overall $-$ is important because it indicates the sign difference between (1.31), (1.32). Recall the full ϕ propagator is

$$\begin{aligned}\frac{1}{i}\tilde\Delta_{full\,\phi\,prop} &= \frac{1}{i}\tilde\Delta_{free}(k^2)+\frac{1}{i}\tilde\Delta_{free}(k^2)i\Pi(k^2)\tilde\Delta_{free}(k^2)+...\\ &= \frac{1}{i}\frac{1}{k^2+M^2-\Pi(k^2)-i\epsilon}\end{aligned} \tag{1.34}$$

where $\Pi(k^2)$ is 1PI, contains the sum of (1.31), (1.32), etc.

Similarly one can compute loop correction to fermion propagator $\langle\Psi\bar\Psi\rangle$, see

Srednicki figure 51.2 on page 319.

$$i\Sigma_{\alpha\beta}(\not{p}) = (iig)^2\mu^\epsilon(\frac{1}{i})^2\int\frac{d^dl}{(2\pi)^d}\left(\gamma_5\frac{-(\not{p}+\not{l})+m}{(p+l)^2+m^2-i\epsilon}\gamma_5\right)_{\alpha\beta}\frac{1}{l^2+M^2-i\epsilon} \tag{1.35}$$

note there is no (-1) because the fermion loop is not closed entirely by fermion.

In problem set 2 problem 1, we show that despite Σ is matrix now, we still have the similar structure like (1.34)

$$\begin{aligned}\frac{1}{i}\tilde{S}^{full\,\Psi\,prop}_{\alpha\beta}(p) &= \frac{1}{i}\tilde{S}^{free}_{\alpha\beta}(p)+\frac{1}{i}\tilde{S}^{free}_{\alpha\kappa}(p)i\Sigma_{\kappa\rho}(\not{p})\tilde{S}^{free}_{\rho\beta}(p)+...\\ \left(\tilde{S}^{full\,\Psi\,prop}(p)\right)^{-1}_{\alpha\beta} &= \left(\tilde{S}^{free}(p)\right)^{-1}_{\alpha\beta}-\Sigma_{\alpha\beta}(\not{p})\end{aligned}$$

One can also compute vertex correction, see Srednicki figure 51.3 on page 321.

From here we can get Z_ϕ, Z_Ψ, Z_m, Z_M, Z_g, Z_λ.

Part II. Spin 1 Field Theory

2. QED Formulate

2.1. Massive Spin 1 Particle

Lecture 5 (9/17/14)

We start from building a theory of a massive spin 1 particle. The natural choice for the fields variables is A^μ, $\mu = 0,1,2,3$. As in quantum mechanics, spin 1 particle have spins $|+1\rangle, |0\rangle, |-1\rangle$, with respect to the vertical axis. Two requirements: theory is Lorentz invariant with only 3 dof, and there is a massive particle interpretation, like $k^2+m^2=0$.

First guess

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{b}{2}\partial_\mu A^\nu\partial_\nu A^\mu - \frac{1}{2}m^2A_\nu A^\nu \tag{2.1}$$

This consists of all possible contractions, e.g. $\partial_\mu A^\mu \partial_\nu A^\nu$ ($\partial_\mu A_\nu \partial^\nu A^\mu$) can be obtained by integration by parts from $\partial_\mu A^\nu \partial_\nu A^\mu$ ($\partial_\mu A_\nu \partial^\mu A^\nu$). Here we see that mass dimension of A_μ is 1 for $d = 4$.

Now we fix the parameter b in the kinetic term. Varying the action $S = \int d^4x \mathcal{L}$ for A_ν.

$$\delta S = \int d^4x \left((\Box A^\nu) - b\partial_\mu \partial^\nu A^\mu - m^2 A^\nu \right) \delta A_\nu$$

We get EOM

$$-\Box A^\nu + b\partial^\nu \partial_\mu A^\mu + m^2 A^\nu = 0 \text{ for all } \nu \tag{2.2}$$

We want to free fields to look like

$$A^\mu(x) = \sum_{\lambda=\pm 1,0} \int \frac{d^3k}{(2\pi)^3 2w_k} (\epsilon^\mu_\lambda(k) a_\lambda(k) e^{ikx} + h.c.) \tag{2.3}$$

Plugging into EOM

$$k^2 \epsilon^\nu - b k^\nu k_\mu \epsilon^\mu + m^2 \epsilon^\nu = 0$$

Want to get

$$k^2 + m^2 = 0 \tag{2.4}$$

This limits to 2 choices:

1) $b = 0$, then (2.1) describes nothing new. It just 4 copies of ϕ of KG Lagrangian for each μ.

2)

$$k_\mu \epsilon^\mu = 0 \tag{2.5}$$

This indeed limits to 3 dofs. So we choose this one. Or

$$\partial_\mu A^\mu = 0 \tag{2.6}$$

Taking ∂_ν of (2.2),

$$-\Box(\partial_\nu A^\nu) + b\Box(\partial_\mu A^\mu) + m^2(\partial_\nu A^\nu) = 0 \tag{2.7}$$

so $b = 1$ implies (2.6). This fixes value of b.

The canonical choice for the polarization vectors ϵ^μ is from (2.5), going to rest frame $k^\mu = (m, 0, 0, 0)$

$$\epsilon^\mu = (0, \vec{\epsilon})$$

with

$$\vec{\epsilon} = \left((0,0,1), \frac{1}{\sqrt{2}}(0,1,\pm i, 0) \right)$$

$\epsilon^\mu = (0,0,0,1)$ is related to gauge mode when $m = 0$. That will be clear later. $\epsilon^\mu = \frac{1}{\sqrt{2}}(0,0,1,\pm i, 0)$ are left and right circular polarization. In this language, we see something special about spin 1 particle that the spin index μ is the same as spatial index μ in the polarization vector,

$$\sum_i R^j_i \epsilon^i_s = \sum_{s'} \underbrace{D_{ss'}}_{e^{i\vec{J}\cdot\vec{\theta}}} \epsilon^j_{s'}$$

RHS is usual spatial rotation; LHS is spin rotation. Both share the same indices. This only happens to spin 1, because $SU(2) \simeq O(3)$.

Now we can write (2.1) more compactly

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\nu A^\nu$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this way, EOM (2.2) by varying A_ν, becomes

$$-\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

Clearly $F_{\mu\nu}$ is antisymmetric, so Lagrangian has no $\dot{A}^0$ term, because $\partial_0 A^0 \leftrightarrow F_{00} = 0$. That is

$$\frac{\partial \mathcal{L}}{\partial \dot{A}^0} = 0 \tag{2.8}$$

i.e. A^0 has no conjugate momentum. This will become significant later, because it indicates that although A^μ has seemingly 4 dofs, A^0 is special, which turns out to a redundant dof, so the number of physical dofs is 3.

2.2. Classical E&M

Consider adding a source term

$$\mathcal{L} = -\frac{1}{4}\underbrace{F_{\mu\nu}F^{\mu\nu}}_{F^2} - \frac{1}{2}m^2 A_\nu A^\nu + A_\nu J^\nu \tag{2.9}$$

and EOM becomes

$$-\partial_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu \tag{2.10}$$

Let's now take $m \to 0$, we get

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\nu J^\nu \tag{2.11}$$

and EOM

$$\partial_\mu F^{\mu\nu} = -J^\nu \tag{2.12}$$

Recall from classical E&M,

$$A^\mu = (\phi, \vec{A}) \qquad J^\mu = (-\rho, \vec{J})$$

ϕ =electric potential, $\vec{A}$ =vector potential, ρ =charge density, $\vec{J}$ =current density.

And compute $F^{\mu\nu}$

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -\partial_t A^i - \partial^i \phi = E^i$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = \epsilon^{ijk} B_k$$

From (2.12), putting $\nu = 0$

$$\vec{\nabla} \cdot \vec{E} = \rho$$

in our convention, we don't have 4π in front of ρ. If we want 4π, add it in front of $A_\nu J^\nu$ in (2.9).

Putting $\nu = i$

$$-\partial_t \vec{E} + \vec{\nabla} \times \vec{B} = \vec{J}$$

The other two Maxwell equations follows from the definitions of $\vec{E}$ and $\vec{B}$ in terms of A^μ, e.g. divergence of curl is 0. Or follow from

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{2.13}$$

cf problem set 3 problem 3. $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. To see (2.13)

$$\partial_\mu \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \partial_\mu (\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 0$$

because $\partial_\mu \partial_\alpha$ is symmetric wrt $\mu\alpha$, while as in $\epsilon^{\mu\nu\alpha\beta}$, $\mu\alpha$ is antisymmetric.

Using the same trick, we show

$$\partial_\mu J^\mu = \partial_\mu \partial_\nu F^{\nu\mu} = 0$$

hence charge conservation.

Now contemplate (2.11), it has U(1) symmetry, i.e. $\mathcal{L}$ is invariant under

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{2.14}$$

for any scalar function λ. This is true provided (1) $m = 0$, (2) J is conserved, because of the use of IBP for $A_\nu J^\nu$ term.

What about (2.6)? Before we say putting $b = 1$ forces (2.6) because $m \neq 0$. Now by (2.7), putting $b = 1$ allow $\partial_\mu A^\mu$ to be anything, so we have to impose

$$\partial_\mu A^\mu = 0 \tag{2.15}$$

as an additional condition by choice, which is called Landau gauge. So if we are given an A^μ s.t. $\partial_\mu A^\mu \neq 0 = \chi(x)$, we put

$$A^\mu \to A^\mu + \partial^\mu \lambda \text{ with } \Box\lambda = -\chi \tag{2.16}$$

Furthermore we have (2.5), and (2.4) gives $k^2 = 0$. The natural choice now is

$$k^\mu = (k, 0, 0, k)$$

and polarization

$$\epsilon^\mu = \left((1,0,0,1), \frac{1}{\sqrt{2}}(0,1,\pm i,0)\right)$$

Now it is clear $\epsilon^\mu = (1,0,0,1) \parallel k^\mu$ is a pure gauge mode, because (2.5) implies

$$\epsilon^\mu \to \epsilon^\mu + \# k^\mu \tag{2.17}$$

changes nothing, so $\epsilon^\mu = (1,0,0,1)$ can be made to $(0,0,0,0)$. So there are only 2 physical dofs. Although we show there are 2 physical dofs using a specific gauge, problem set 3 problem 5 gives an gauge independent way to show it.

In formal language, choose a gauge like (2.15) limits unphysical dofs, and gauge symmetry like (2.14) creates redundancy. This makes gauge symmetry to be very different than ordinary symmetry.

Remark 1) (2.15) itself doesn't completely specify λ in (2.14), because we can add

$$\lambda \to \lambda + \lambda'$$

with $\Box\lambda' = 0$. The residual freedom is fixed by the specific choice of # in (2.17).

Remark 2) one can run the same argument using

$$\vec{\nabla} \cdot \vec{A} = 0$$

coulomb gauge.

2.3. Cauchy Problem

Gauge symmetry has special implication of Cauchy problem. Naively one may think that since (2.12) is a system of 2nd order odes, if one has the initial conditions $A^\mu(0,\vec{x})$, $\dot{A}^\mu(0,\vec{x})$, and know $J^\mu(t,\vec{x})$, one can obtain a unique solution $A^\mu(t,\vec{x})$. But by gauge transformation (2.14), we know

$$A_\mu(t,\vec{x}) + \partial_\mu\lambda(t,\vec{x})$$

is just as good as $A^\mu(t,\vec{x})$, provided $\lambda(0,\vec{x}) = 0$ and $\dot{\lambda}(0,\vec{x}) = 0$. This gives serious distinction between massive and massless theories.

Lecture 6 (9/22/14)

To explore the origin of the problem, for the massive theory let's go back to (2.8) which is true for $m =$ or $\neq 0$. A^0 has no conjugate momentum; A^i has conjugate momentum

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = -F^0{}_i = -F^{0i}$$

Back to EOM (2.10), which is too a system of 2nd order odes. We claim it has unique solutions for the initial value problem.

Putting $\nu = 0$

$$\partial_i F^{0i} + m^2 A^0 = J^0 \tag{2.18}$$

this has no time evolution wrt dynamical variables A^0,A^i.

Putting $\nu = i$,

$$\partial_t F^{i0} + \partial_j F^{ij} + m^2 A^i = J^i \tag{2.19}$$

Since $F^{0i} = \partial^0 A^i - \partial^i A^0 = -\partial_t A^i - \partial^i A^0$,

$$\partial_t A^i = -F^{0i} - \partial^i A^0 \tag{2.20}$$

Cauchy problem: specify $J^0(t,\vec{x})$, $J^i(t,\vec{x})$, and at $t = 0$, specify $A^\mu(0,\vec{x})$ and $\dot{A}^\mu(0,\vec{x})$. For minimal initial conditions, we can just specify $A^i(0,\vec{x})$, $F^{0i}(0,\vec{x})$. Because $A^i(0,\vec{x}) \implies F^{ij}(0,\vec{x})$, and by (2.18) $A^0(0,\vec{x})$ is too fixed. We claim we will get unique solution $A^\mu(t,\vec{x})$.

We solve the initial value problem as follows:

1) solves (2.19) for $F^{i0}(\Delta t,\vec{x})$

2) solves (2.20) for $A^i(\Delta t,\vec{x})$

3) solves (2.18) for $A^0(\Delta t,\vec{x})$

Note that if $m = 0$, there will be no way to do time evolution to $A^0(\Delta t,\vec{x})$, even we are given $A^0(0,\vec{x})$. Without knowing $A^0(\Delta t,\vec{x})$, we cannot solve (2.20) for later $A^i(\Delta t',\vec{x})$.

Let us see what kind of difficulty arises from gauge symmetry on path integral for massless theory.

Put $m = 0$, by (2.1), the action is

$$\begin{aligned} S &= \int d^4x \left(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A^\nu \partial_\nu A^\mu + A_\mu J^\mu \right) \\ &= \int d^4x [-\frac{1}{2} A_\mu \underbrace{(\eta^{\mu\nu}(-\Box) + \partial^\mu \partial^\nu)}_{(-\Box)\mathbb{P}^{\mu\nu}} A_\nu + A_\mu J^\mu] \end{aligned} \tag{2.21}$$

where

$$\mathbb{P}^{\mu\nu} = \eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box}$$

is a projection $\mathbb{P}^2 = \mathbb{P}$, and EOM is

$$(-\Box)\mathbb{P}^{\mu\nu} A_\nu = -J^\mu$$

and path integral naively is

$$Z = \int DA e^{iS[A]} = \int DA e^{i \int d^4x [-\frac{1}{2} A_\mu (-\Box)\mathbb{P}^{\mu\nu} A_\nu + A_\mu J^\mu]} \tag{2.22}$$

Then the usual scheme for finding propagator (cf (1.2) for scalar and (1.1) for spin 1/2) is to find the inverse of $(-\Box)\mathbb{P}^{\mu\nu}$, i.e.

$$((-\Box)\mathbb{P}^{\mu\nu})_x \Delta_{\nu\alpha}(x-y) = \delta^\mu_\alpha \delta^{(4)}(x-y)$$

however $(-\Box)\mathbb{P}^{\mu\nu}$ is not invertible, because

$$(-\Box)\mathbb{P}^{\mu\nu}(\partial_\mu \lambda) = 0 \text{ for any } \lambda$$

hence $(-\Box)\mathbb{P}^{\mu\nu}$ has infinitely many eigenvectors $\partial_\mu \lambda$ with 0 eigenvalue.

Moreover (2.22) is not well defined, because A_μ & $A_\mu + \partial_\mu \lambda$ should not be treated as if they are independent integration variables.

There are three ramifications:

1) assume $m \neq 0$, then

$$(-\Box)\mathbb{P}^{\mu\nu} + m^2 \eta^{\mu\nu}$$

is invertible, so find Green function, then send $m \to 0$. This is quantization of a massive vector field, see Zee I.5

2) impose a gauge to eliminate gauge freedom, e.g.

$$\partial_\mu A^\mu = 0 \text{ Landau/Lorenz gauge}$$

$$\vec{\nabla} \cdot \vec{A} = 0 \text{ Coulomb gauge}$$

this is what we will do, following Srednicki. This is ugly, because it breaks Lorentz symmetry, but good to see polarization.

There are more

$$A_0 = 0 \text{ temporal gauge}$$

$$A_3 = 0 \text{ axial gauge}$$

3) Faddeev-Popov quantization. It keeps expression covariant at the cost of introducing extra particle, see Coleman 5.2.

2.4. Path Integral (Coul. Gauge)

Applying Coulomb gauge, in (2.21) $A_\mu \partial^\mu \partial^\nu A_\nu$ term becomes $A_0 \partial^0 \partial_0 A^0$ after IBP, we get

$$S = \int d^4x \underbrace{\underbrace{-\frac{1}{2} A_\mu(-\Box)A^\mu - \frac{1}{2} A_0 \partial^0 \partial_0 A^0}_{-\frac{1}{2}A_i(-\Box)A^i + \frac{1}{2}A_0 \vec{\nabla}^2 A^0}}_{-\frac{1}{2}A^i(-\Box)A^i - \frac{1}{2}A^0\vec{\nabla}^2 A^0} + A_\mu J^\mu$$

since $\Box = -\partial_t^2 + \vec{\nabla}^2$.

Therefore we break path integral in two parts,

$$Z = \int DA^i \underbrace{e^{i\int d^4x\left(-\frac{1}{2}A^i(-\Box)A^i + A^i J^i\right)}}_{I} \int DA^0 \underbrace{e^{i\int d^4x\left(-\frac{1}{2}A^0\vec{\nabla}^2 A^0 - A^0 J^0\right)}}_{II} \tag{2.23}$$

In this way, both $(-\Box)$ & $\vec{\nabla}^2$ are invertible. Notice that the integral $\int DA^i$ should consist of divergence free A^i. So we should replace A^i by

$$A_\perp^i = P^{ij} A^j$$

where

$$P^{ij} = \delta^{ij} - \frac{\partial^i \partial^j}{\vec{\nabla}^2} \tag{2.24}$$

is the projection onto divergence free A^i, i.e. check

$$\partial_i A_\perp^i = \partial_i A^i - \partial^j A^j = 0$$

So $A^i J^i$ in (2.23) should be $A_\perp^i J^i = A_\perp^i J_\perp^i$. Then the rest is just usual Gaussian integral

$$\begin{aligned} I &= \exp\left(\frac{i}{2} \int d^4x d^4y \underbrace{J_\perp^i(x)}_{(\delta^{im} - \frac{\partial_x^i \partial_y^j}{\vec{\nabla}_x^2}) J^m(x)} \left(\delta^{ij} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - i\epsilon} \right) \underbrace{J_\perp^j(y)}_{(\delta^{jn} - \frac{\partial_x^j \partial_y^n}{\vec{\nabla}_y^2}) J^n(y)} \right) & (2.25) \\ &= e^{\frac{i}{2} \int d^4x d^4y J^m(x) \left(\int \frac{d^4k}{(2\pi)^4} \frac{\delta^{mn} - \hat{k}^m \hat{k}^n}{k^2 - i\epsilon} e^{ik(x-y)} \right) J^n(y)} & (2.26) \end{aligned}$$

where $\hat{k}^m = k^m / \left| \vec{k} \right|$.

$$II = \exp\left(\frac{i}{2} \int dt d^3y J^0(\vec{x}, t) \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{-e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2}}_{-\frac{1}{4\pi|\vec{x}-\vec{y}|}} J^0(\vec{y}, t) \right) \tag{2.27}$$

Combining I, II and using $\partial_\mu J^\mu = 0$, we can simplify (2.23)

$$Z = \exp \int \frac{i}{2} \int d^4x d^4y J_\mu(x) \underbrace{\left(\int \frac{d^4k}{(2\pi)^4} \frac{\eta^{\mu\nu}}{k^2 - i\epsilon} e^{ik(x-y)} \right)}_{\Delta^{\mu\nu}(x-y)} J_\nu(y)$$

Note that

1) We mentioned before Landau gauge doesn't completely fix dof, so is Coulomb

gauge. In (2.24) the inversion of $\vec{\nabla}^2$ depends on boundary (if we use our favorite ∞ decay boundary, then $\vec{\nabla}^2 f = 0 \implies f = 0$) but usually this does introduce residual dof.

Lecture 7 (9/24/14)

2) (2.25) looks just like 3 scalar fields $i, j = 1, 2, 3$

$$\phi^i \Delta \phi^j$$

see Srednicki equation (56.13).

3) $-i\epsilon$ in (2.25) is necessary, and $-i\epsilon$ is unnecessary in (2.27). It is only important $\int dt$ is involved, because the position of pole decides whether it's an advanced or retarded solution.

2.5. Quantization (Coul. Gauge)

Naively (but wrong) put

$$[A^\mu(t, \vec{x}), \Pi^\nu(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})\eta^{\mu\nu}$$

Two problems: 1) A^0 has no conjugate momentum. 2)

$$[A^i(t, \vec{x}), \underbrace{\Pi^j(t, \vec{y})}_{\dot{A}^j}] = i\delta^{(3)}(\vec{x} - \vec{y})\eta^{ij}$$

is inconsistent with Coulomb gauge. Apply $\partial_i^{(x)}$ to both sides, $LHS = 0$ and $RHS \neq 0$.

What is $[A^i(t, \vec{x}), \dot{A}^j(t, \vec{y})]$? For a rigorous approach, see Weinberg chapter 7. He used Dirac brackets–generalized Poisson brackets for systems with constraints.

We will instead guess based on path integral. Claim

$$[A^i(t, \vec{x}), \dot{A}^j(t, \vec{y})] = i\left(\delta^{ij} - \frac{\partial^i \partial^j}{\vec{\nabla}^2}\right)\delta^{(3)}(\vec{x} - \vec{y}) \tag{2.28}$$

Pf. By (2.26)

$$\underbrace{\left\langle A^i(t_x,\vec{x}), A^j(t_y,\vec{y})\right\rangle}_{T\left\langle 0\middle|\hat{A}^i(x)\hat{A}^j(y)\middle|0\right\rangle} = \frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{\delta^{ij}-\hat{k}^i\hat{k}^j}{k^2-i\epsilon}e^{-ik^0(t_x-t_y)+i\vec{k}\cdot(\vec{x}-\vec{y})} \tag{2.29}$$

Consider small $\eta > 0$,

$$\begin{aligned}\left\langle 0\middle|\hat{A}^i(t_y+\eta,\vec{x})\dot{\hat{A}}^j(t_y,\vec{y})\middle|0\right\rangle &= \frac{\partial}{\partial t_y}(2.29)\\ &= \frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{\delta^{ij}-\hat{k}^i\hat{k}^j}{k^2-i\epsilon}(ik^0)e^{-ik^0\eta+i\vec{k}\cdot(\vec{x}-\vec{y})}\end{aligned} \tag{2.30}$$

Next consider going back in time η. Because of time order, we have to switch

$$\begin{aligned}\left\langle 0\middle|\dot{\hat{A}}^j(t_y,\vec{y})\hat{A}^i(t_y-\eta,\vec{x})\middle|0\right\rangle &= \frac{\partial}{\partial t_y}(2.29)\\ &= \frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{\delta^{ij}-\hat{k}^i\hat{k}^j}{k^2-i\epsilon}(ik^0)e^{-ik^0(-\eta)+i\vec{k}\cdot(\vec{x}-\vec{y})}\end{aligned} \tag{2.31}$$

Integrating over k^0,

$$\begin{aligned}k^2-i\epsilon &= -(k^0+\sqrt{\vec{k}^2-i\epsilon})(k^0-\sqrt{\vec{k}^2-i\epsilon})\\ &= -(k^0+\sqrt{\vec{k}^2}-i\epsilon)(k^0-\sqrt{\vec{k}^2}+i\epsilon)\end{aligned}$$

For (2.30), we have to close the lower half plane, pole is $\left|\vec{k}\right|$. For (2.31), we have to close the upper half plane, enclosed pole is $-\left|\vec{k}\right|$, therefore

$$
\begin{aligned}
\lim_{\eta\to 0}\left\langle 0\left|\left[\hat{A}^i(t_y+\eta,\vec{x}),\dot{\hat{A}}^j(t_y,\vec{y})\right]\right|0\right\rangle &= \frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{\delta^{ij}-\hat{k}^i\hat{k}^j}{-2\left|\vec{k}\right|}(-2\pi i)(i\left|\vec{k}\right|)e^{-i\left|\vec{k}\right|\eta+i\vec{k}\cdot(\vec{x}-\vec{y})} \\
&\quad -\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{\delta^{ij}-\hat{k}^i\hat{k}^j}{2\left|\vec{k}\right|}(2\pi i)(-i\left|\vec{k}\right|)e^{-i\left|\vec{k}\right|\eta+i\vec{k}\cdot(\vec{x}-\vec{y})} \\
&= i\int\frac{d^3\vec{k}}{(2\pi)^3}(\delta^{ij}-\hat{k}^i\hat{k}^j)e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\
&= i\left(\delta^{ij}-\frac{\partial^i\partial^j}{\vec{\nabla}^2}\right)\delta^{(3)}(\vec{x}-\vec{y})
\end{aligned}
$$

QED.

Use similar thing like (2.3)

$$
A^\mu(x)=\sum_{\lambda=\pm 1}\int\frac{d^3\vec{k}}{(2\pi)^3 2w_k}(\epsilon_\lambda^{i*}(k)a_\lambda(k)e^{ikx}+\epsilon_\lambda^i(k)a_\lambda^\dagger(k)e^{-ikx}) \tag{2.32}
$$

where $w_k=\left|\vec{k}\right|$, and here that we put ϵ_λ^{i*} instead of ϵ_λ^i follows Srednicki convention, so (2.33) $\pm$ is switched.

Let's work out what ϵ, a should satisfy.

Massless

$$
k^2=0
$$

Coulomb gauge

$$
k_i\epsilon_\lambda^i=0
$$

Typical solution

$$
k^\mu=(k,0,0,k)\quad \vec{k}=(0,0,k)
$$

and put

$$
\vec{\epsilon}^{\,i}_{\lambda=\pm}=\frac{1}{\sqrt{2}}(1,\mp i,0) \tag{2.33}
$$

for circular polarization. Particle physics prefer them, because they are eigenstates of spatial rotation helicity.

For linear polarization

$$\vec{\epsilon}^{\,i}_{\lambda=\pm} = (1,0,0) \quad (0,1,0)$$

one can always write linear polarization as linear combination of circular polarization.

$$\alpha a_R^\dagger |0\rangle + \beta a_L^\dagger |0\rangle$$

We have orthonormality

$$\sum_i \epsilon^i_{\lambda'}(k)\epsilon^{i*}_\lambda(k) = \delta_{\lambda\lambda'}$$

completeness

$$\sum_\lambda \epsilon^{i*}_\lambda(k)\epsilon^j_\lambda(k) = \delta^{ij} - \hat{k}^i\hat{k}^j$$

For our previous choices of k,

$$\delta^{ij} - \hat{k}^i\hat{k}^j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

Plugging (2.32) into (2.3), we also get usual particle creation/annihilation

$$\begin{aligned} [a_\lambda(k), a^\dagger_{\lambda'}(k')] &= (2\pi)^3 2w_k \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}') \\ [a_\lambda(k), a_{\lambda'}(k')] &= 0 \\ [a^\dagger_\lambda(k), a^\dagger_{\lambda'}(k')] &= 0 \end{aligned}$$

2.6. Faddeev-Popov

This is an alternative approach that makes Lorentz invariant manifest. This approach is not so important for QED, but it is extremely important later for QCD.

Recall the problem of path integral with gauge symmetry is that we are naively integrating too much. First consider a simple equation.

$$\int dadbe^{(a\,b)\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}} \tag{2.34}$$

here matrix $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ is not invertible. One may isolate integral of b

$$(2.34) = \int dae^{-a^2}\int db$$

so we see that integration too much gives unwanted ∞.

To get rid of $\int db$, we change (2.34) to

$$\int dadbe^{(a\,b)\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}}\delta(b-\#)$$

for any number #. Or even

$$\int dadbe^{(a\,b)\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}}\delta(\underbrace{b-f(a)}_{F(a,b)})$$

since for any a, $\int db\delta(b-f(a))=1$, or actually we want

$$\int dadbe^{(a\,b)\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}}\underbrace{\delta(F(a,b))\det\frac{\partial F}{\partial b}}_{\delta(b-b_0)} \tag{2.35}$$

requiring for any given a, there is one and only one b_0 s.t. $F(a,b_0)=0$, then

$$\int db\delta(F(a,b))\det\frac{\partial F}{\partial b}=1$$

we put det instead of just $\partial F/\partial b$ for alluding to that b can be a vector.

Now back to (2.22), we change it to

$$Z = \int DAe^{iS[A,J]}\delta(\underbrace{\partial A - \chi}_{F(A)}) \det \frac{\delta F}{\delta \lambda} \tag{2.36}$$

We should think $DA = D\bar{A}D\lambda$, where $\bar{A}$ is any equivalent class, modules by gauge. λ is any function with ∞ decay at boundary. And

$$F(A) = \partial(\underbrace{\bar{A} - \partial\lambda}_{A}) - \chi(x) \tag{2.37}$$

for some arbitrary fixed function χ. Cf (2.16), one can see that F satisfies the condition in remark after (2.35).

We put in $\delta(F(A)) \det \frac{\delta F}{\delta\lambda}$ to get rid of $\int D\lambda$. First compute $\det \frac{\delta F}{\delta \lambda}$,

$$\frac{\delta F}{\delta \lambda} = \frac{F(\bar{A} + \partial\lambda) - F(\bar{A})}{\delta\lambda} = \Box \implies \det \frac{\delta F}{\delta\lambda} = \det \Box \tag{2.38}$$

which is independent of A, so we could absorb it into the measure, but because later when we will do Faddeev-Popov for QCD, in contrast $\det \frac{\delta F}{\delta \lambda}$ will depend on A. For that reason, we will leave the $\det \Box$ factor in the integrand, not to absorb into the measure.

Because clearly Z is independent of the choice of χ, we can add in a Gaussian integral of χ

$$Z = \int D\chi e^{-\frac{i}{2\xi}\int d^4x\chi^2} \int DAe^{iS[A,J]}\delta(\partial A - \chi) \det \Box \tag{2.39}$$

with ξ arbitrary and $\int D\chi e^{-\frac{i}{2\xi}\int d^4x\chi} = \#$ for which we absorb any $\#$ into the measure, then we combine above, and integrate out the delta function, we get

$$Z = \int DAe^{iS[A,J]-\frac{i}{2\xi}\int d^4x(\partial A)^2} \det \Box \tag{2.40}$$

Lecture 8 (9/29/14)

$S[A, J]$ is (2.21),

$$\begin{aligned}
Z &= \int DA e^{\int d^4x[-\frac{1}{2}A_\mu(\eta^{\mu\nu}(-\Box)+\partial^\mu\partial^\nu)A_\nu - \frac{1}{2\xi}\underbrace{(-A_\mu\partial^\mu\partial^\nu A_\nu)}_{(\partial A)^2}+A_\mu J^\mu}\det\Box \\
&= \int DA e^{\int d^4x[-\frac{1}{2}A_\mu\left(\eta^{\mu\nu}(-\Box)+(1-\frac{1}{\xi})\partial^\mu\partial^\nu\right)A_\nu+A_\mu J^\mu}\det\Box
\end{aligned}$$

Recall previously we said the propagator was not invertible, but what about now?

$$\left(\eta^{\mu\nu}(-\Box)+(1-\frac{1}{\xi})\partial_x^\mu\partial_x^\nu\right)\Delta_{\nu\alpha}(x-y)=\delta^\mu_\alpha\delta^{(4)}(x-y)$$

Fourier above

$$\left(k^2\eta^{\mu\nu}-(1-\frac{1}{\xi})k^\mu k^\nu\right)\tilde{\Delta}_{\nu\alpha}=\delta^\mu_\alpha \tag{2.41}$$

We guess

$$\tilde{\Delta}_{\nu\alpha}(k)=\frac{1}{k^2}\eta_{\nu\alpha}+ck_\nu k_\alpha$$

actually by Lorentz symmetry, they are the only possible terms on the RHS, so plug in (2.41)

$$\left(k^2\eta^{\mu\nu}-(1-\frac{1}{\xi})k^\mu k^\nu\right)\left(\frac{1}{k^2}\eta_{\nu\alpha}+ck_\nu k_\alpha\right)=\delta^\mu_\alpha-\left(-(1-\frac{1}{\xi})(\frac{1}{k^2}+ck^2)+ck^2\right)k^\mu k_\alpha$$

so pick $c=\frac{\xi-1}{(k^2)^2}$. Therefore

$$\tilde{\Delta}_{\nu\alpha}(k)=\frac{1}{k^2-i\epsilon}\left(\eta_{\nu\alpha}-(1-\xi)\frac{k_\nu k_\alpha}{k^2}\right) \tag{2.42}$$

with added $i\epsilon$ to signal it's Feynman green function.

From (2.39), if we pick $\xi=0$, we force $\chi\to 0$, so $\partial A=0$, hence we get Lorentz gauge. If we pick $\xi=1$, we get so called Feynman gauge.

One can also make this to Coulomb gauge and it agrees what we did before, see Brown, *Quantum Field Theory.*

2.7. Spinor QED

So far we have not had much physics. In the AJ term, where the physical content of J remains unspecified. In fact all we assumed was that unconditionally $\partial J=0$.

Now we want to consider current made of real electrons. We use qft 1 note equation (6.38) conserved current from global $U(1)$ symmetry, i.e. $\Psi \to e^{i\theta}\Psi$, and assume EOM

$$J^\mu = e\bar{\Psi}\gamma^\mu\Psi$$

with $e = -0.302822$ or $e^2/4\pi = 1/137$. (see (3.10) for why it is defined)

Recall the proof.

$\mathcal{L}_{Dirac}(\Psi, \partial_\mu\Psi) = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$ is invariant under $\Psi \to \Psi + \underbrace{\Delta\Psi}_{i\theta\Psi}$. Then

$$\Delta\Psi\frac{\partial\mathcal{L}}{\partial\Psi} + \underbrace{\Delta\partial_\mu}_{\partial_\mu\Delta}\Psi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = 0$$

Use EL (or EOM)

$$\frac{\partial\mathcal{L}}{\partial\Psi} = \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)})$$

we get

$$\Delta\Psi\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}) + \partial_\mu\Delta\Psi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = 0$$

or

$$\partial_\mu[\underbrace{\Delta\Psi(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)})}_{-i\theta\Psi(i\bar{\Psi}\gamma^\mu)\equiv J}] = 0 \tag{2.43}$$

the minus sign in J is from switching order. QED.

The spinor Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^2 + i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi + \underbrace{e\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\mathcal{L}_{int}} \qquad (2.44)\\ &= -\frac{1}{4}F^2 + i\bar{\Psi}\gamma^\mu(\underbrace{\partial_\mu - ieA_\mu}_{D_\mu})\Psi - m\bar{\Psi}\end{aligned}$$

D_μ is called gauge covariant derivative.

Now we have a problem. It turns out that

$$\begin{cases} A_\mu \to A_\mu + \partial\lambda \\ \Psi \to \Psi \end{cases}$$

is not a symmetry of (2.44), under above

$$\delta\mathcal{L}_{int} = \underbrace{e\bar{\Psi}\gamma^\mu\Psi}_{J}\partial_\mu\lambda = -\partial J\lambda \tag{2.45}$$

because $e\bar{\Psi}\gamma^\mu\Psi$ is only conserved when EOM is satisfied cf proof above, but symmetry should be independent of EOM (in the literature, it says symmetry is an off-shell statement.)

However there is an useful theorem:

Theorem. *Suppose a theory under a certain transformation has the action vary by something proportional to eom, then a suitably deformed transformation can be formed such that it is a symmetry.*

We can check that (2.44) is invariant under this symmetry, called gauged $U(1)$ symmetry.

$$\begin{cases} A_\mu \to A_\mu + \partial\lambda \\ \Psi \to e^{ie\lambda}\Psi \end{cases} \tag{2.46}$$

Note: under gauged $U(1)$, $D_\mu\Psi \to e^{ie\lambda}D_\mu\Psi = e^{ie\lambda}D_\mu e^{-ie\lambda}e^{ie\lambda}\Psi$ or more abstractly

$$D_\mu \to e^{ie\lambda}D_\mu e^{-ie\lambda}$$

and

$$[D_\mu, D_\nu]\Psi = -ieF_{\mu\nu}\Psi \text{ or } \frac{i}{e}[D_\mu, D_\nu] = F_{\mu\nu} \tag{2.47}$$

this is similar to a statement in general relativity regarding Riemann tensor

$$[\nabla_\mu, \nabla_\nu]V_\alpha = R^\lambda{}_{\mu\nu\alpha}V_\lambda$$

for any vector field V_α. Since $R^\lambda{}_{\mu\nu\alpha}$ has to do geometry and $R^\lambda{}_{\mu\nu\alpha}$ is invariant under diffeomorphism (coordinate transformation), one may infer that $F_{\mu\nu}$ has to be geometry too. This is the starting point of gauge theory and geometry.

Lecture 9 (10/1/14) Let's elaborate more on the theorem above (2.46) and show the rationale of (2.46) that how to come up with such symmetry.

For example $S[\phi, \psi]$ consider a transformation

$$\phi \to \phi + \Delta\phi \text{ and } \psi \to \psi + \Delta\psi$$

such that

$$\delta S = \int d^4x \frac{\partial \mathcal{L}}{\partial \psi} \Delta\phi F(\phi, \psi) \propto \frac{\partial \mathcal{L}}{\partial \psi} \propto \frac{\partial S}{\partial \psi} \text{ is the EOM for } \psi$$

deformed transformation

$$\phi \to \phi + \Delta\phi \text{ and } \psi \to \psi + \Delta\psi - \Delta\phi F(\phi, \psi)$$

we will get $\delta S = 0$. Here we should assume $F(\phi, \psi)$ is a local function, i.e. when $\phi(x_0) \to \phi'(x_0)$, F depends only on x_0, so the deformed transformation is local.

PF.

$$\begin{aligned}
\delta S_{new} &= \Delta\phi \int d^4x \frac{\partial \mathcal{L}}{\partial \phi} + [\Delta\psi - \Delta\phi F(\phi, \psi)] \int d^4x \frac{\partial \mathcal{L}}{\partial \psi} \\
&= \underbrace{\int d^4x \frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \frac{\partial \mathcal{L}}{\partial \psi} \Delta\psi}_{\delta S_{old}} - \underbrace{\int d^4x \Delta\phi F(\phi, \psi) \frac{\partial \mathcal{L}}{\partial \psi}}_{\delta S_{old}}
\end{aligned}$$

QED.

Apply to (2.44), under $A_\mu \to A_\mu + \partial_\mu \lambda$, $\Psi \to \Psi$ and $\bar{\Psi} \to \bar{\Psi}$. Cf (2.45), one can check

$$\begin{aligned}
\delta S &= -\int d^4x \partial_\mu (\bar{\Psi} \gamma^\mu \Psi) e\lambda \\
&= \int d^4x \left(-ie\lambda \Psi \frac{\partial \mathcal{L}}{\partial \Psi} + ie\lambda \bar{\Psi} \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} \right)
\end{aligned}$$

the minus sign comes from switching Φ, $\bar{\Psi}$.Therefore the deformed transformation should be

$$\Psi \to \Psi + ie\lambda \text{ and } \bar{\Psi} \to \bar{\Psi} - ie\lambda$$

so we arrive (2.46).

One may ask if the new gauged $U(1)$ symmetry implies new conserved current? The answer is no, nothing non-trivial. This is an interesting aspect of gauge symmetry, redundancy of dof. One can show that

$$J^{\mu}_{gaugeU(1)} = J^{\mu}_{globalU(1)} + something\, trivial$$

Similar thing happens in general relativity. In GR language, gauge transformation is coordinate transformation; conversed current is Killing vector. Choosing exotic frame means doing some gauge transformation, but no non-trivial Killing vectors will be found.

We can also write (2.46) in more abstract terms,

$$\Psi \to \underbrace{e^{ie\lambda(x)}}_{U}\Psi \implies \Psi \to U\Psi$$

$$D_\mu \to e^{ie\lambda} D_\mu e^{-ie\lambda} \implies D_\mu \to U D_\mu U^{-1}$$

and

$$D_\mu \Psi \to U D_\mu \Psi$$

gauge transformation is simply pulled out of covariant derivative.

And

$$A_\mu \to A_\mu + \partial_\mu \lambda \implies A_\mu \to U A_\mu U^{-1} + \frac{i}{e} U \partial_\mu U^{-1} \tag{2.48}$$

Later when we study non-abelian theory, we will make U to be matrix (no longer a scalar function), A_μ D_μ will be matrices, then we will have these same formulas

3. QED Calculation

3.1. Scattering Calculation

Example $e^{\pm}$ annihilation, compare to (1.29)

$$\underbrace{e^-}_{\substack{p_1, s_1 \\ (\alpha)}} \quad \underbrace{e^+}_{\substack{p_2, s_2 \\ (\beta)}} \quad \to \quad \underbrace{\gamma}_{\substack{k_1, \lambda_1 \\ (\mu)}} \quad \underbrace{\gamma}_{\substack{k_2, \lambda_2 \\ (\nu)}}$$

Refer to (2.32), (1.5), and use LSZ

$$\begin{aligned} i\mathcal{M} &\sim \big\langle 0\big| a_{\lambda_1}(k_1) a_{\lambda_2}(k_2) d^\dagger_{s_2}(p_2) b^\dagger_{s_1}(p_1) \big|0\big\rangle \\ &\sim \epsilon^\nu_{\lambda_2}(k_2)\epsilon^\mu_{\lambda_1}(k_1) u^\alpha_{s_1}(p_1)\bar{v}^\beta_{s_2}(p_2) \qquad \underbrace{\big\langle A_\nu A_\mu \bar{\Psi}^\alpha \Psi^b \big\rangle}_{\frac{1}{2}\left\langle AA\bar{\Psi}\Psi(\int e\bar{\Psi}\gamma\Psi A)(\int e\bar{\Psi}\gamma\Psi A)\right\rangle} \end{aligned}$$

Do the same pairing like (1.27). There are two processes, see Srednicki page 358 figure 59.1. First diagram gives

$$i\mathcal{M} = \frac{(ie)^2}{i}\epsilon^\nu_{\lambda_2}(k_2)\epsilon^\mu_{\lambda_1}(k_1)\bar{v}_{s_2}(p_2)\gamma_\nu \frac{-(\not{p}_1 - \not{k}_1) + m}{(p_1 - k_1)^2 + m^2 - i\epsilon}\gamma_\mu u_{s_1}(p_1)$$

The another, differed by flipping γ, so the two have same sign.

$$i\mathcal{M} = \frac{(ie)^2}{i}\epsilon^\nu_{\lambda_2}(k_2)\epsilon^\mu_{\lambda_1}(k_1)\bar{v}_{s_2}(p_2)\gamma_\nu \frac{-(\not{p}_1 - \not{k}_2) + m}{(p_1 - k_2)^2 + m^2 - i\epsilon}\gamma_\mu u_{s_1}(p_1)$$

Then square gets the total.

One can also do

$$e^-\gamma \to e^-\gamma$$

like (1.28). See Srednicki problem 59.1.

3.2. Renormalization & Propagator Correction

Lecture 10 (10/8/14)

We do renormalization and dimension regulation, $d = 4 - \epsilon$,

$$\mathcal{L} = -\frac{1}{4}Z_3F^2 + iZ_2\bar{\Psi}\partial\!\!\!/\Psi - Z_m m\bar{\Psi}\Psi + \tilde{\mu}^{\epsilon/2}Z_1 e\bar{\Psi}\gamma^\mu\Psi A_\mu \tag{3.1}$$

Similar to the comment after (2.1), where we argued that we had to consider all possible contraction. Now we ask if there are other operators with mass dimension ≤ 4 we can include? $A_\mu A^\mu$, $(A_\mu A^\mu)^2$,... forbidden by gauge symmetry (i.e. gauge symmetry forces photons to have no mass, there is no mass renormalization for A_μ). Likewise $\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$ violates parity, and the previous $\bar{\Psi}\gamma_5\Psi$ also violates parity. When $\epsilon \to 0$, we want the second and fourth terms in (3.1), becomes the usual covariant derivative (i.e. gauge invariance is preserved by renormalization)

$$iZ_2\bar{\Psi}(\underbrace{\partial_\mu - ieA_\mu}_{D_\mu})\gamma^\mu\Psi$$

so we want

$$Z_1 = Z_2 \tag{3.2}$$

Loop Correction to A^μ Propagator

$$\frac{1}{i}\Delta^{full}_{\rho\sigma}(k) = \frac{1}{i}\Delta^{free}_{\rho\sigma}(k) + \frac{1}{i}\Delta^{free}_{\rho\sigma}(k)i\Pi^{\mu\nu}(k)\frac{1}{i}\Delta^{free}_{\rho\sigma}(k) + ... \tag{3.3}$$

The second term on the RHS $i\Pi^{\mu\nu}(k)$ includes the lowest order–first figure 62.1 in Srednicki page 378 and all other higher order 1PI. The counter term (second figure 62.1 in Srednicki page 378) is from $(Z_3 - 1)$ term.

Later we will use Ward identity to show (also Srednicki problem 68.1.)

$$k_\mu\Pi^{\mu\nu}(k) = 0 \tag{3.4}$$

which implies

$$\Pi^{\mu\nu}(k) = \Pi(k)k^2P^{\mu\nu} \tag{3.5}$$

where the projection $P^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$, and $\Pi(k)$ is some scalar. By (2.42),

$$\Delta^{free}_{\rho\sigma}(k) = \frac{P_{\rho\sigma} + \xi\frac{k_\rho k_\sigma}{k^2}}{k^2 - i\epsilon}$$

thus (3.3) gives

$$\begin{aligned}\Delta^{full}_{\rho\sigma}(k) &= \frac{\xi\frac{k_\rho k_\sigma}{k^2}}{k^2-i\epsilon}+\frac{P_{\rho\sigma}}{k^2-i\epsilon}\left(1+\frac{\Pi(k^2)k^2}{k^2-i\epsilon}+(\frac{\Pi(k^2)k^2}{k^2-i\epsilon})^2+...\right)\\ &= \frac{P_{\rho\sigma}+\xi\frac{k_\rho k_\sigma}{k^2}\left(1-\Pi(k^2)\right)}{k^2\left(1-\Pi(k^2)\right)-i\epsilon}\end{aligned} \tag{3.6}$$

and

$$i\Pi^{\mu\nu}=i\Pi^{\mu\nu}_{first\,diag}+i\Pi^{\mu\nu}_{first\,diag}+O(e^4)$$

Let's compute $\Pi(k^2)$ at the lowest order, On-shell renormalization,

$$\Pi(k^2=0)=0 \tag{3.7}$$

Now compute the first figure 62.1 in Srednicki page 378.

$$\langle A_\rho A_\sigma\rangle_{full}\sim\langle A_\rho A_\sigma\rangle_{free}+\frac{1}{2!}\left\langle A_\rho A_\sigma ie\int\bar{\Psi}\gamma^\mu\Psi A_\mu ie\int\bar{\Psi}\gamma^\nu\Psi A_\nu\right\rangle_{free}$$

Similar to (1.32)

$$i\Pi^{\mu\nu}_{first\,diag}=(ieZ_1\tilde{\mu}^\epsilon)^2\frac{1}{i^2}(-1)\int\frac{d^dl}{(2\pi)^d}\mathrm{Tr}\left(\gamma^\mu\frac{-(\not{k}+\not{l})+m}{(k+l)^2+m^2-i\epsilon}\gamma^\nu\frac{-\not{l}+m}{l^2+m^2-i\epsilon}\right)$$

the (-1) is because it is a closed fermion loop. Using (3.5), one can show

$$i\Pi^{\mu\nu}_{first\,diag}\propto P^{\mu\nu}$$

Likewise we compute the second figure 62.1 in Srednicki page 378, which is due to

$$\mathcal{L}_{counterterm}=-\frac{1}{4}(Z_3-1)F^2=-\frac{1}{2}(Z_3-1)\left(A^\mu(-\Box\eta_{\mu\nu})A^\nu+A^\mu\partial_\mu\partial_\nu A^\nu\right)$$

The gives

$$i\Pi^{\mu\nu}_{second\,diag}=-i(Z_3-1)(k^2\eta^{\mu\nu}-k^\mu k^\nu)$$

which turns out $i\Pi^{\mu\nu}_{second\,diag}\propto P^{\mu\nu}$, so it suffices to cancel divergence in $\Pi^{\mu\nu}_{first\,diag}$.

(PS4 problem 4) Applying Feynman's trick, using Tr(γ's) and dimension regulation, one finds

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx x(1-x)\left(\frac{1}{\epsilon} - \frac{1}{2}\ln\frac{D}{\mu^2}\right) - (Z_3 - 1)$$

where $D = x(1-x)k^2 + m^2 - i\epsilon$, $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$, γ =Euler-Mascheroni constant. And

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} + finite\right) + O(e^4)$$

Applying on-shell (3.7)

$$\begin{aligned} Z_3^{OS} &= 1 - \frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{m}{\mu}\right) \\ \Pi^{OS}(k^2) &= \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\ln\frac{D}{m^2} < \infty \\ &= \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\ln\left(1 + \frac{k^2}{m^2}x(1-x)\right) \end{aligned} \tag{3.8}$$

Implication for Coulomb Law

Back to (3.6), choose Landau gauge $\xi = 0$, we will see Coulomb propagator $1/k^2$ gets modified

$$\frac{1}{k^2\left(1 - \Pi(k^2)\right) - i\epsilon}$$

What does it imply change in potential? Answer is

Later we will use this technique to compute electron magnetic momentum.

$$\left\langle 0\middle|e^{-i\int dtH}\middle|0\right\rangle_J = \int DAD\Psi D\bar{\Psi}e^{iS_{QED}+i\int d^4xeJ^\mu A_\mu} = e^{iW[J]} \tag{3.9}$$

where e is coupling strength and

$$iW[J] = \text{connected diagrams} = \frac{ie^2}{2}\int d^4xd^4yJ^\mu(x)\Delta_{\mu\nu}^{full}(x-y)J^\nu(y) + HigerOrderJ$$

assume external J is small (ignoring higher order in J) and assume J is static, i.e.

no time dependence

$$J^{\mu} = \begin{cases} 0 & \text{for } \mu \neq 0 \\ \rho(\vec{x}) & \text{for } \mu = 0 \end{cases}$$

Lecture 11 (10/13/14)

Thus by (3.6), choose Landau gauge $\xi = 0$,

$$iW[J] = \frac{ie^2}{2}\int d^4x \underbrace{d^4y}_{dt_y d^3\vec{y}} \rho(\vec{x})\rho(\vec{y}) \int \frac{d^4k}{(2\pi)^4} \frac{\eta_{00} - \frac{k_0 k_0}{k^2}}{k^2\left(1 - \Pi(k^2)\right) - i\epsilon} e^{ik(x-y)}$$

Integrating dt_y with $e^{ik(x-y)}$ gives $\delta(k_0 = 0)$, time dependence completely drop out, so the denominator $k^2 = \vec{k}^2$, as mentioned before, in this way no need to do contour integral, so no retarded or advanced solutions. This is why static source J is special. Therefore

$$iW[J] = -\frac{ie^2}{2}\int dt_x \int d^3x d^3y \rho(\vec{x})\rho(\vec{y}) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2\left(1 - \Pi(\vec{k}^2)\right)}$$

Comparing to (3.9), E associated with turning on J

$$H = E = \frac{1}{2}\int d^3x d^3y \rho(\vec{x})\rho(\vec{y}) \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{e^2}{\vec{k}^2\left(1 - \Pi(\vec{k}^2)\right)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}_{V(|\vec{x}-\vec{y}|)}$$

We can integrate$V(|\vec{x} - \vec{y}|)$ as the potential energy between two electrons, i.e. when $\rho(\vec{x}) = \delta(\vec{x}), \rho(\vec{y}) = \delta(\vec{y})$.

Now let's approx it in non-relativistic situation $k^2 \ll m^2$, so from (3.8)

$$\Pi^{OS}(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\frac{k^2}{m^2}x(1-x) = \frac{e^2}{2\pi^2}\frac{k^2}{m^2}\frac{1}{30}$$

so

$$\begin{aligned} V(|\vec{x} - \vec{y}|) &= e^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{\vec{k}^2}\left(1 + \Pi(\vec{k}^2)\right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\ &= \frac{e^2}{4\pi\left|\vec{x} - \vec{y}\right|} + e^4\frac{\delta^{(3)}(\vec{x} - \vec{y})}{60\pi^2 m^2} \end{aligned}$$

Define fine structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137} \tag{3.10}$$

then

$$V(|\vec{x} - \vec{y}|) = \frac{\alpha}{|\vec{x} - \vec{y}|} + \frac{4\alpha^2}{m^2}\delta^{(3)}(\vec{x} - \vec{y}) \tag{3.11}$$

Comments:

1) Clearly $V > 0$, i.e. repulsion interaction between two electrons. Contrast with potential between two spin 0 or spin 2 $V < 0$, see PS 4 problem 2.

2) The loop correction increase as energy increases. At high energy electrons get closer, $V \uparrow$. The effective strength $e \uparrow$. This is opposite to QCD. Actually to be completely self-consistent as energy $\uparrow$, the analysis needs to go beyond $k^2 \ll m^2$. However that the loop correction increase as energy increases is still true. Read β function of QED Srednicki chapter 66.

Loop Correction to Ψ Propagator

(PS4 problem 4) Using similar arguments, we can find fermion propagator. For diagrams, see Srednicki figure 62.2 on page 381.

$$\begin{aligned} Z_2 &= 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + finite\right) + O(e^4) \\ Z_m &= 1 - \frac{e^2}{2\pi^2}\left(\frac{1}{\epsilon} + finite\right) + O(e^4) \end{aligned}$$

3.3. Vertex Correction

From

$$\mathcal{L}_{int} = \bar{\Psi} A_\mu (Z_1 e \gamma^\mu) \Psi$$

in (3.1), the vertex function is

$$iV^\mu(p, p') = iZ_1 e\gamma^\mu + iV^\mu_{loop}(p, p') \tag{3.12}$$

where p, p' are in Srednicki figure 62.3 on page 383. Now we want to compute $Z_1 = 1 + O(e^2)$. The $O(e^2)$ is to cancel divergence in V^μ_{loop}.

$$\begin{aligned} iV^\mu_{loop}(p,p') &\sim \left\langle bb^\dagger a^\dagger \right\rangle \\ &\sim \bar{u}u\epsilon_\mu \left\langle \Psi\bar{\Psi}A^\mu \right\rangle \end{aligned}$$

so $V^\mu(p,p')$ has both spin indices and space time indices. There 3 internal propagators

$$\left\langle \Psi\bar{\Psi}A^\mu \right\rangle \sim \left\langle \Psi\bar{\Psi}A^\mu \left(ie \int A_\rho \bar{\Psi}\gamma^\rho\Psi \right) \left(ie \int A_\alpha \bar{\Psi}\gamma^\alpha\Psi \right) \left(ie \int A_\nu \bar{\Psi}\gamma^\nu\Psi \right) \right\rangle$$

gives 3 different parings, and we find

$$iV^\mu_{loop}(p,p') = \frac{(ie)^3}{i^3} \int \frac{d^d l}{(2\pi)^d} \gamma^\rho \frac{-(\not{p}' + \not{l}) + m}{(p'+l)^2 + m^2 - i\epsilon} \gamma^\mu \frac{-(\not{p} + \not{l}) + m}{(p+l)^2 + m^2 - i\epsilon} \gamma^\nu \tilde{\Delta}^{free}_{\rho\nu}(l)$$

where $\tilde{\Delta}^{free}_{\rho\nu}(l)$ is (3.6), using the low order photons, because we are doing perturbation. Note $V^\mu_{loop}(p,p')$ is here a 4×4 matrix, not taken trace, so order matters. And because it's a matrix, we'll need to sandwich it by u, v to get a number.

In the end, we find

$$Z_1 = 1 - \frac{e^2}{8\pi^2} \left(\frac{1}{\epsilon} + finite \right) + O(e^4)$$

This sort of proves (3.2) via perturbation. Later we will give a better proof without using perturbation.

Apply on shell renormalization for V^μ, one can show

$$\bar{u}_{s'} V^\mu(p,p') u_s(p) \Big|_{\substack{p^2 = -m^2 \\ p'^2 = -m^2 \\ q^2 = (p'-p)^2 = 0}} = e_{phys} \bar{u}_{s'} \gamma^\mu u_s(p) \tag{3.13}$$

as if $Z_1 = 1$ and $V^\mu_{loop} = 0$ in (3.12). This actually defines physically what value e is.

Let's see what physical process corresponding to (3.13). Let's go to a frame where $\vec{p}' = \vec{p}$, hence $p'^0 = p^0$ hence equal energy, so $q^\mu = p'^\mu - p^\mu = 0$. Since if a 4-vector is 0 in one frame, it's $\equiv 0$ in all frames, so we have a situation that photons are very soft, not just $(k, 0, 0, k)$, but $\sim (0, 0, 0, 0)$.

From (3.13), one can determine (3.12), the vertex function

$$V^\mu(p, p') = e(F_1(q^2)\gamma^\mu - \frac{i}{m} F_2(q^2)\Sigma^{\mu\nu} q_\nu)) \tag{3.14}$$

where $q = p' - p$, $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, and form factors (problem set 5 problem 1)

$$F_1(q^2) = F_2 1 + O(e^2) \qquad (q^2) = O(e^2) \tag{3.15}$$

and in particular to accord to (3.13),

$$F_2(0) = \frac{\alpha}{2\pi}$$

Using Gordon identities (Srednicki equation 38.18), we get a number

$$\bar{u}_{s'} V^\mu(p, p') u_s(p) = \bar{u}(p') \left(\frac{e}{2m}(p' + p)^\mu F_1(q^2) - \frac{ie}{m}\Sigma^{\mu\nu} q_\nu (1 + F_2(q^2)) \right) u_s(p) \tag{3.16}$$

Later we will see that the first term on the right relates to electric potential; second term relates to magnetic moment. See discussion after (3.24).

3.4. Magnetic Moment of Electrons

Lecture 12 (10/15/14)

This was the first confirmation of QED and one of the biggest triumphant in QFT, achieved by Schwinger in 1948. Since whether it is electric or magnetic field depends on the frame, we should use lab frame to talk about the conventional meaning of magnetic moment. That is in non-relativistic limit, hence $KE \ll m$.

Last semester we did non-relativistic limit of Dirac equation. Now we redo that but with covariant derivative

$$(i\gamma^\mu \underbrace{(\partial_\mu - ieA_\mu)}_{D_\mu} - m)\psi = 0 \tag{3.17}$$

square above, we get some sort of Klein Gordon equation.

$$\underbrace{(i\gamma^\mu D_\mu + m)(i\gamma^\nu D_\nu - m)}_{-\gamma^\mu\gamma^\nu D_\mu D_\nu - m^2}\psi = 0 \tag{3.18}$$

use $\gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = -2i\Sigma^{\mu\nu} - \eta^{\mu\nu}$, then

$$-\gamma^\mu\gamma^\nu D_\mu D_\nu = 2i\Sigma^{\mu\nu} \underbrace{D_\mu D_\nu}_{\underbrace{[D_\mu, D_\nu]}_{-ieF_{\mu\nu}}} + D^\mu D_\mu$$

since $\Sigma^{\mu\nu}$ is antisymmetric, replace $D_\mu D_\nu$ by $[D_\mu, D_\nu]$. Consider

$$A^\mu = (0, -\frac{1}{2}By, \frac{1}{2}Bx, 0)$$

magnetic field along $\hat{z}$ with strength B. Assume B is weak,

$$\begin{aligned} D^\mu D_\mu &= -\partial_t^2 + (\partial^j - ieA^j)(\partial_j - ieA_j) \\ &= -\partial_t^2 + \vec{\nabla}^2 - eB(\underbrace{xi\partial_y - yi\partial_x}_{-L_z}) + O(A^2) \end{aligned}$$

L_z =angular momentum in $\hat{z}$. Or

$$D^\mu D_\mu = -\partial_t^2 + \vec{\nabla}^2 + e\vec{B}\cdot\vec{L}$$

Next

$$\Sigma^{0i} = \frac{1}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix},\ \Sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix} = \text{ and } F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & B_3 & -B_2 \\ & & 0 & B_1 \\ & & & 0 \end{pmatrix}$$

assume weak magnetic field, no electric field,

$$\Sigma^{\mu\nu}F_{\mu\nu} = 2\vec{S}\cdot\vec{B}$$

$\vec{S}$ =usual spin operator for 1/2 spin particles, then (3.18) is completely diagonalized and proportional to identity

$$[-\partial_t^2 + \vec{\nabla}^2 + e\vec{B}\cdot(\vec{L}+2\vec{S}) - m^2]\psi = 0$$

Now taking non-relativistic limit, so we want to change KG equation into Schrodinger equation. Is ψ a 4 component object? Yes, but as we will see in the leading order 2 of the 4 components are the same and the other 2 are the same, so if we just consider e^-, not e^+. Then 1 of the 2 component is 0. Also see qft 1 note page 113. Try $\psi = e^{-imt}\chi$ and assume $\dot{\chi} \ll m\chi$. Dominated term gives

$$(m^2 - m^2)\psi = 0$$

Next leading order

$$-(-im)2e^{-imt}\dot{\chi} + \nabla^2 e^{-imt}\chi + e\vec{B}\cdot(\vec{L}+2\vec{S})e^{-imt}\chi = 0$$

we thus get Schrodinger equation:

$$i\dot{\chi} = \left(-\frac{\nabla^2}{2m} - \frac{e\vec{B}\cdot(\vec{L}+2\vec{S})}{2m}\right)\chi$$

and we say

$$H_B = -\vec{B}\cdot\vec{\mu}$$

with magnetic moment

$$\vec{\mu} = \frac{e}{2m}(\vec{L} + \underbrace{2}_{g}\vec{S}) \tag{3.19}$$

why is that magnetic moment? Classically consider a circular current loop in magnetic field

$$\mu = \text{current}\cdot\text{area} = \frac{e}{2\pi r/v}\pi r^2 = \frac{e}{2}rv = \frac{eL}{2m}$$

The "2" in (3.19) is called gyromagnetic ratio, first obtained by Dirac. We are going to show that the actual value computed from tree level QED is

$$g = 2(1 + F_2(q^2 = 0)) = 2(1 + \frac{\alpha}{2\pi})$$

The idea is the same as above, to get an expression involve energy. To do that we use the same trick in deriving modification of Coulomb law (3.11), but we will use a different external source (not static charge) and we need to compute loop to cubic vertex (not to propagator).

For diagram, see Srednicki figure 62.3 on page 383. Forget about the upper l, we get a tree diagram The scattering amplitude in the presence of J is

$$\langle p'|S - I|p\rangle_J^{tree} = i\tilde{J}_\mu(q)\frac{ie}{i}\frac{\eta^{\mu\nu}}{q^2 - i\epsilon}\bar{u}(p')e\gamma_\nu u(p)$$

where $\tilde{J}_\mu(q) = \int d^4y e^{-iqy} J_\mu(y)$. Then include loop correction, we get

$$\left\langle p' \middle| e^{-i\int dtH} \middle| p \right\rangle = \langle p'|S - I|p\rangle_J = i\tilde{J}_\mu(q)\frac{1}{q^2 - i\epsilon}\bar{u}(p')V^\mu(p,p')u(p) \tag{3.20}$$

consistent with (3.14), (3.15).

Lecture 13 (10/20/14)

First we show how $\tilde{J}_\mu$ is related to $\tilde{A}_\mu$. Starting from (2.12), applying Landau gauge,

$$-\Box A^\mu = J^\mu \implies q^2\tilde{A}^\mu(q) = \tilde{J}^\mu(q) \tag{3.21}$$

so (3.20) becomes

$$\langle p'|S - I|p\rangle_J = i\tilde{A}_\mu(q)\bar{u}(p')V^\mu(p,p')u(p) \tag{3.22}$$

What if we use a different gauge

$$A_\mu \to A_\mu + q_\mu \times (something)?$$

(3.22) will be the same, because

$$q_\mu V^\mu(p,p') = 0 \tag{3.23}$$

We will derive using that in the next section using Ward-Takahashi identity. (3.23) is a generalization of $q_\mu \tilde{J}^\mu(q) = 0$, which is conservation of J^μ.

Therefore by (3.16)

$$\begin{aligned}
\langle p'|S-I|p\rangle_J &= \left\langle p'\middle|T\left(e^{-i\int dt(H_E+H_B)}\right)-I\middle|p\right\rangle_J = -i\int dt\left\langle p'\middle|\hat{H}_E\middle|p\right\rangle_J - i\int dt\left\langle p'\middle|\hat{H}_B\middle|p\right\rangle_J \\
&= i\int d^4y A_\mu(y)e^{-iqy}\bar{u}_{s'}(p')\left(\frac{e}{2m}(p'+p)^\mu F_1(q^2) - \frac{ie}{m}\Sigma^{\mu\nu}q_\nu(1+F_2(q^2))\right)u_s(p) \qquad (3.24)
\end{aligned}$$

We break the integral into two.

Taking non-relativistic limit, $p, p' \sim (m,0,0,0)$, $q = p'-p$, $F_1 \approx 1$, first part

$$\int dt\left\langle p'\middle|\hat{H}_E\middle|p\right\rangle_J = -\int dt d^3y \sum_\alpha \left(\bar{u}_{s'\alpha}(p')e^{-ip'y}\right)\underbrace{eA_0(t,\vec{y})}_{e\Phi}\left(u_{s\alpha}(p)e^{ipy}\right) \qquad (3.25)$$

Φ =electrostatic potential.

To get an expression for H_E, we modify LHS

$$\int dt\left\langle p'\middle|\hat{H}_E\middle|p\right\rangle_J = \int dt d^3y d^3y' \sum_{\alpha\alpha'}\langle p's'|y'\alpha'\rangle\left\langle y'\alpha'\middle|\hat{H}_E\middle|y\alpha\right\rangle\langle y\alpha|ps\rangle$$

so we should have

$$\langle p's'|y'\alpha'\rangle = \bar{u}_{s'\alpha}(p')e^{-ip'y} \quad \langle y\alpha|ps\rangle = u_{s\alpha}(p)e^{ipy}$$

They are wave functions for e^- and

$$\left\langle y'\alpha'\middle|\hat{H}_E\middle|y\alpha\right\rangle = -\delta_{\alpha\alpha'}\delta^{(3)}(y-y')e\Phi$$

which is indeed correct electrostatic energy. Sanity check completes.

Now we compute the second part of (3.24),

$$\int dt\left\langle p'\middle|\hat{H}_B\middle|p\right\rangle_J = \int d^4y A_\mu(y)e^{-iqy}\bar{u}_{s'}(p')\left(\frac{ie}{m}\Sigma^{\mu\nu}q_\nu(1+F_2(q^2))\right)u_s(p) \qquad (3.26)$$

We will do similar thing like (3.25),

$$\begin{aligned}
RHS \text{ of } (3.26) &= \int d^4y A_\mu(y) \frac{\partial}{\partial(-iy^\nu)} e^{-iq_\nu y^\nu} \bar{u}_{s'}(p') \left(\frac{ie}{m} \Sigma^{\mu\nu}(1+F_2(q^2)) \right) u_s(p) \\
&= -i \int d^4y \underbrace{\partial_\nu A_\mu(y)}_{\frac{1}{2}F_{\nu\mu}} e^{-iq_\nu y^\nu} \bar{u}_{s'}(p') \left(\frac{ie}{m} \Sigma^{\mu\nu}(1+F_2(q^2)) \right) u_s(p)
\end{aligned}$$

because $\Sigma^{\mu\nu}$ is antisymmetric, we can replace $\partial_\nu A_\mu$ by $\partial_{[\nu} A_{\mu]}$, and recall $F_{\mu\nu}\Sigma^{\mu\nu} = 2\vec{B}\cdot\vec{S}$

$$RHS \text{ of } (3.26) = -i \int dt d^3y \left(e^{-ip'y} \bar{u}_{s'}(p') \right) \left[\vec{B}(t,\vec{y}) \cdot \vec{S} \frac{e}{2m} 2(1+F_2(q^2)) \right] \left(e^{ipy} u_s(p) \right)$$

so compare to (3.19),

$$H_B = \vec{B}(t,\vec{y}) \cdot \vec{S} \frac{e}{2m} \underbrace{2(1+F_2(q^2))}_{=g}$$

putting $q^2 = 0$,

$$g = 2(1+\frac{\alpha}{2\pi})$$

3.5. Ward-Takahashi Identity

Lecture 14 (10/22/14)

In this section, we want to prove a few things via non-perturbation:

1) Scattering amplitude is independent of gauge

$$k_\mu \mathcal{M}^\mu = 0 \tag{3.27}$$

2) Gauge invariant is preserved by renormalization

$$Z_1 = Z_2$$

problem set 5 problem 4. In en route we will show (3.23).

3) Loop correction is transverse, so we can use projection to eliminate gauge redundancy, cf (3.4)

$$k_\mu \Pi^{\mu\nu} = 0$$

see Srednicki problem 68.1.

All of them are derived from Dyson-Schwinger identity (see qft 1 note page 90) and Ward-Takahashi identity (see qft 1 note page 87). In simple terms, the formal one says although classically we know

$$eom \implies \frac{\delta S}{\delta \phi} = 0$$

in qm

$$\begin{aligned} i\left\langle \left.\frac{\delta S}{\delta \phi}\right|_{x_0} \phi(x_1)\phi(x_2)...\phi(x_n)\right\rangle &= -\delta(x_0 - x_1)\left\langle \phi(x_2)...\phi(x_n)\right\rangle \\ &\quad -\delta(x_0 - x_2)\left\langle \phi(x_1)\phi(x_3)...\phi(x_n)\right\rangle \\ &\quad ... \\ &\quad -\delta(x_0 - x_n)\left\langle \phi(x_2)...\phi(x_{n-1})\right\rangle \end{aligned}$$

which will be 0 if $x_0 \neq x_i$ for all $i = 1, ..., n$. The RHS side is called "contact term". We can apply the identity to different fields with

$$\frac{\delta \phi_i(x_i)}{\delta \phi_0(x_0)} = \begin{cases} 0 & \text{different fields} \\ \delta(x_i - x_0) & \text{same fields} \end{cases}$$

thus the contact terms will be trivially 0 if $\phi_1, ..., \phi_n$ are different fields from ϕ_0, regardless $x_0 \neq x_i$ or not, i.e.

$$\begin{aligned} i\left\langle \left.\frac{\delta S}{\delta \phi}\right|_{x_0} \phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\right\rangle &= -\delta(x_0 - x_1)\delta_{\phi_0\phi_1}\left\langle \phi_2(x_2)...\phi_n(x_n)\right\rangle \\ &\quad -\delta(x_0 - x_2)\delta_{\phi_0\phi_2}\left\langle \phi_1(x_1)\phi_3(x_3)...\phi_n(x_n)\right\rangle \\ &\quad ... \\ &\quad -\delta(x_0 - x_n)\delta_{\phi_0\phi_n}\left\langle \phi_2(x_2)...\phi_{n-1}(x_{n-1})\right\rangle \end{aligned}$$

The latter identity says although classically we know if $\phi \to \phi + \Delta\phi$ is a symmetry, then there is a conserved J (cf (2.43)), $\partial_\mu J^\mu = 0$ on the EOM, in QM

$$\begin{aligned}
i\partial_\mu^{(x_0)} \langle J^\mu(x_0)\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\rangle &= \delta(x_0 - x_1)\delta_{\phi_0\phi_1} \langle \Delta\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\rangle \\
&\quad \delta(x_0 - x_2)\delta_{\phi_0\phi_2} \langle \phi_1(x_1)\Delta\phi_2(x_2)...\phi_n(x_n)\rangle \\
&\quad ... \\
&\quad \delta(x_0 - x_n)\delta_{\phi_0\phi_n} \langle \phi_1(x_1)\phi_2(x_2)...\Delta\phi_n(x_n)\rangle
\end{aligned}$$

Let's prove 1). We used (3.27) in previous scattering amplitude calculation of vertex, where we had one fermion p in, one fermion p' out and one photon q in or one external source J^μ. But (3.27) is true in general, where there is still only one photon $A^\mu(k)$ leg and many other (≥ 2) fermion legs at the vertex.

From LSZ,

$$\begin{aligned}
(2\pi)^d \delta(\text{conserved mom}) i\mathcal{M} &= {}_{out}\langle ...|...\rangle_{in} - 1 \\
&= i\int d^4x e^{-ikx}\epsilon_\mu(-\Box)_x(\underbrace{... ... ...}_{\substack{\text{like } \Box_x \text{ to kill} \\ \text{external legs}}})\langle A^\mu\,(x)\underbrace{... ... ...}_{\text{other legs}}\rangle
\end{aligned}$$

The strategy of the proof is very similar to (3.20) & (3.21). First find a relation between A^μ and J^μ, via global $U(1)$ transformation on photons and local $U(1)$ transformation on fermions. Lucky that will give $A^\mu \to \frac{1}{p^2}J^\mu$, i.e. the leg has a $1/p^2$ which will be killed by the operators in front, $e^{-ikx}(-\Box)_x \to e^{-ikx}p^2$. In other words all poles in the legs will be killed, so we will get finite answer as $p \to 0$.

However if we multiply $k_\mu \mathcal{M}^\mu$,

$$\begin{aligned} k\mathcal{M} &\to \int d^4x e^{-ikx} k(\dots\dots\dots\dots) \langle J^\mu(x)\dots\dots\dots\rangle \\ &\to \int d^4x \partial_x e^{-ikx}(\dots\dots\dots\dots) \langle J^\mu(x)\dots\dots\dots\rangle \\ &\to \int d^4x e^{-ikx}(\dots\dots\dots\dots) \underbrace{\partial_x \langle J^\mu(x)\dots\dots\dots\rangle}_{\substack{\downarrow \\ \text{ward} \\ \downarrow \\ 0}} \end{aligned}$$

now applying Ward, we are supposed to get some contact terms. Since ∂_x field is photon field, functional derivative of ∂_x on fermion fields, $\phi_1\phi_2\dots\phi_n$, gives 0, so no such contact term. In the proof we don't need $p \to 0$, neither do we assume that in proving 2).

However 1) is even correct when there are (≥ 2) photon $A^\mu(k)$ legs and many other (≥ 2) fermion legs at the vertex. Then we need $p \to 0$, because then

$$\begin{aligned} k\mathcal{M} &\to \int d^4x e^{-ikx} k(\dots\dots\dots\dots) \langle \underbrace{J^\mu(x)A^\mu(y)\dots}_{\text{photons}} \underbrace{\dots\dots\dots\dots}_{\text{fermions}} \rangle \\ &\to \int d^4x e^{-ikx}(\dots\dots\dots\dots) \quad \underbrace{\partial_x \langle J^\mu(x)A^\mu(y)\dots\dots\dots\rangle}_{\substack{\downarrow \\ \text{ward} \\ \downarrow \\ \delta(x-y)\langle \Delta A^\mu(y)\dots\rangle + \underbrace{\dots\dots\dots}_{\text{other } \gamma}}} \end{aligned}$$

In global $U(1)$ symmetry, $\Delta A^\mu(y) \sim A^\mu(y) \sim J^\mu/p^2$. This $1/p^2$ pole will be killed by $-\Box_y$, but we still have $-\Box_x$ hanging around, so putting $p \to 0$ will give $k\mathcal{M} \to 0$. We will use this

$$\partial_\mu \langle 0|J^\mu(x)J^\nu(y)|0\rangle \to 0$$

to prove 3).

Some old books called (3.27) the Ward identity.

Let's prove 2). Here we will restrict to 1 fermion in, 1 fermion out, and 1 photon.

Recall our full renormalized Lagrangian in Faddeev-Popov gauge

$$\mathcal{L} = iZ_2\bar{\Psi}\gamma^\mu\partial_\mu\Psi - Z_m m\bar{\Psi}\Psi - \frac{1}{4}Z_3F^2 + Z_1e\bar{\Psi}\gamma^\mu A_\mu\Psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

applying the same $U(1)$ symmetry, eom for A^μ, and be careful about factors, we get

$$D^\mu{}_\nu A^\nu = \frac{Z_1}{Z_2}J^\mu \tag{3.28}$$

where $D^\mu{}_\nu = -Z_3\delta^\mu{}_\nu\Box + (Z_3 - \frac{1}{\xi})\partial^\mu\partial_\nu$, $J^\mu = eZ_2\bar{\Psi}\gamma^\mu\Psi$.

The full photon-fermion-fermion three point function

$$\left\langle A^\nu(x_0)\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\right\rangle = \int d^4y_0d^4y_1d^4y_2\frac{1}{i}\Delta^\nu{}_{\nu'}(x_0-y_0)\frac{1}{i}\Sigma_{\alpha\alpha'}(x_1-y_1)iV^{\nu'}{}_{\alpha'\beta'}(y_0,y_1,y_2)\frac{1}{i}\Sigma_{\beta\beta'}(x_2-y_2)$$

Also Ward identity implies

$$\frac{\partial}{\partial x_0^\mu}\left\langle J^\mu(x_0)\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\right\rangle = -e\left\langle\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\right\rangle(\delta(x_0-x_1)-\delta(x_0-x_2))$$

combining the two above with (3.28), with a bit of manipulation, one can get

$$\frac{Z_2}{Z_1}\frac{\partial}{\partial x_0^\nu}V^\nu{}_{\alpha\beta}(x_0,y_1,y_2) = -ei\Sigma^{-1}_{\alpha\beta}(y_1-y_2)(\delta(x_0-y_1)-\delta(x_0-y_2))$$

or

$$q_\nu V^\nu{}_{\alpha\beta}(q,p,p') = \frac{Z_1}{Z_2}e\left(\Sigma^{-1}_{\alpha\beta}(p') - \Sigma^{-1}_{\alpha\beta}(p)\right) \tag{3.29}$$

so clearly on-shell, $p' = p$, we have proven (3.23).

In (3.29) V, Σ^{-1} are finite, so $\frac{Z_1}{Z_2}$ is finite. We have shown previously in the $\overline{\text{MS}}$ scheme, in the next leading order, both $Z_{1,2}$ take the form of $1+O(1/d-4)+...$ Therefore for $\frac{Z_1}{Z_2}$ to be finite, $Z_1 = Z_2$.

What about on-shell renormalization? We suppose $q \ll 1$,

$$\Sigma^{-1}_{\alpha\beta}(p) = \not{p} + m - i\epsilon - \Sigma^{-1}_{loop\,\alpha\beta}(\not{p}) = \not{p} + m_{phys}$$

where $\Sigma^{-1}_{loop\,\alpha\beta}$ is loop correction. Thus (3.29) gives

$$q_\nu V^\nu{}_{\alpha\beta}(q,p,p') = \frac{Z_1}{Z_2} e\gamma^\nu q_\nu \implies V^\nu{}_{\alpha\beta}(q,p,p') = \frac{Z_1}{Z_2} e\gamma^\nu$$

combining (3.13), we get

$$e_{phys} = \frac{Z_1}{Z_2} e$$

so on-shell renormalization, too, $\implies Z_1 = Z_2$. Or assume $Z_1 = Z_2 \implies e_{phys} = e$.

For an alternative proof see problem set 5 problem 4b.

Part III. Non-abelian Gauge Theory

4. Group Theory & QCD

4.1. Group & Transformation

SO(N)

Lecture 15 (10/27/14)

Suppose we have N real scalar fields ϕ_i, we want to consider the following transformation.

$$\phi_i \to R_{ij}\phi_j$$

with

$$R^T R = 1 \tag{4.1}$$

This will ensure, e.g. $\sum_i \partial_\mu\phi_i\partial^\mu\phi_i$, $\sum_i \phi_i\phi_i$ are invariant. We assume R is real, so $\phi_i \to \phi'_j$ remains real.

In addition we assume $\det R = 1$, which says R is connected to I, so

$$R_{ij} = \delta_{ij} \underbrace{- i\theta^a T^a_{ij}}_{\Theta_{ij}} + O(\theta^2)$$

T^a_{ij} is generator, and θ^a is angle of rotation wrt rotation axes a, and θ is real. From (4.1)

$$(1+\Theta)^T(1+\Theta) = 1 \implies \Theta^T = -\Theta \tag{4.2}$$

and Θ is real, so T is purely imaginary and it is antisymmetric. A basis for T^a_{ij},

$$\underbrace{\begin{pmatrix} 0 & i & & & \\ -i & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}}_{\substack{\downarrow \\ \text{rot. in } xy\text{plane}}}, \underbrace{\begin{pmatrix} 0 & & & & \\ & 0 & i & & \\ & -i & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}}_{\substack{\downarrow \\ \text{rot. in } yz\text{plane}}}, \ldots \tag{4.3}$$

so total number of independent generators is

$$\frac{N(N-1)}{2}$$

For $N = 3$, $SO(3)$ has 3 generators.

SU(N)

Suppose we have N complex scalar fields ϕ_i, we want to consider the following transformation.

$$\phi_i \to U_{ij}\phi_j$$

with

$$U^\dagger U = 1 \tag{4.4}$$

This will ensure, e.g. $\sum_i \partial_\mu \phi_i^* \partial^\mu \phi_i$, $\sum_i \phi_i^* \phi_i$ are invariant.

In addition we assume $\det U = 1$, so

$$U_{ij} = \delta_{ij} - i\theta^a T^a_{ij} + O(\theta^2) \tag{4.5}$$

T^a_{ij} is generator, and θ^a is "angle" of rotation. From (4.4)

$$(1 + i\theta^a T^{aT}_{ij})(1 - i\theta^a T^a_{ij}) = 1 \implies T^{aT}_{ij} = T^a_{ij}$$

so T is hermitian and traceless, because

$$\det U = \det(e^{i\theta T}) = e^{i\theta \mathrm{Tr} T} = 1 \implies \mathrm{Tr} T = 0$$

Clearly

$$\text{generators of } SO(N) \subset \text{generators of } SU(N)$$

A basis for T^a_{ij},

$$\left\{ \underbrace{\begin{pmatrix} 0 & i & & & \\ -i & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}}_{\frac{N(N-1)}{2}}, \ldots + \underbrace{\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}}_{\frac{N(N-1)}{2}}, \ldots + \underbrace{\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -n & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}}_{N-1}, \ldots \right\} \tag{4.6}$$

with some factors in front so that (4.8) is satisfied.

so total number of independent generators is

$$\frac{N(N-1)}{2} 2 + N - 1 = N^2 - 1 \tag{4.7}$$

For $N = 2$, $SU(2)$ has 3 generators.

Structure of T

We choose normalization condition

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{4.8}$$

this follows Srednicki equation 69.8 for generator of $SU(N)$, however in Srednicki equation 24.5 $\mathrm{Tr}(T^a T^b) = 2\delta^{ab}$ for generators of $SO(N)$. The difference is only conventional.

To exploit some group properties, let's consider two rotations for either $SO(N)$ or $SU(N)$, assuming θ is small, so we keep the linear order

$$U(\theta) = 1 - i\theta^a T^a - \frac{1}{2}\theta^a\theta^b T^{ab} + ... \qquad U(\theta') = 1 - i\theta'^a T^a - \frac{1}{2}\theta'^a\theta'^b T^{ab} + ... \tag{4.9}$$

note even in this form, $T^{ab} = T^{ba}$, doesn't imply that for small θ or θ' above rotation in xy plane then yz plane is the same as rotation in yz plane then xy plane. Because e.g.

$$e^{\theta^x T^x + \theta^y T^y} \neq e^{\theta^x T^x} e^{\theta^y T^y}$$

cf Baker–Campbell–Hausdorff formula. However

$$e^{i(\theta^x T^x + \theta^y T^y)} \to ... - \frac{1}{2}\theta^x\theta^y (\underbrace{T^x T^y + T^y T^x}_{T^{xy}}) + ...$$

so T^{ab} is symmetric.

We impose that

$$U(\theta')U(\theta) = U(g(\theta', \theta)) \tag{4.10}$$

where $g(\theta', \theta)$ is some other angle.

Claim (4.10) $\implies$

$$[T^a, T^b] = i f^{abc} T^c$$

This is Lie algebra. f^{abc} is called structure constant. It is real, completely anti-symmetric, and obey Jacobi identity,

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \tag{4.11}$$

If the group has only 3 generators, then $f^{abc} \propto \epsilon^{abc}$, the usual rotation.

When $f \neq 0$, the group is non-abelian.

Proof.

From (4.9), we propose

$$g(\theta', \theta) = \theta'^a + \theta^a + g^{abc}\theta'^b\theta^c$$

this makes sense when one of (θ', θ) is 0. Then (4.10) $\Longrightarrow$

$$(1 - i\theta'^a T^a - \frac{1}{2}\theta'^a\theta'^b T^{ab})(1 - i\theta^a T^a - \frac{1}{2}\theta^a\theta^b T^{ab}) =$$

$$1 - i(\theta'^a + \theta^a + g^{abc}\theta'^b\theta^c)T^a - \frac{1}{2}(\theta'^a + \theta^a)(\theta'^b + \theta^b)T^{ab}$$

Equate the coefficients of $\theta'\theta$ terms,

$$-\theta'^a\theta^b T^a T^b = -ig^{abc}\theta'^b\theta^c T^a - \underbrace{\frac{1}{2}(\theta'^a\theta^b + \theta'^b\theta^a)T^{ab}}_{\theta'^a\theta^b T^{ab}}$$

so

$$T^a T^b = ig^{cab}T^c + T^{ab}$$

or

$$T^b T^a = ig^{cba}T^c + T^{ba}$$

hence

$$[T^a, T^b] = i\underbrace{(g^{cab} - g^{cba})}_{f^{abc}}T^c \tag{4.12}$$

this shows f^{abc} is antisymmetric wrt bc. Now multiply T^d from the right to (4.12), and take trace, by (4.8)

$$\mathrm{Tr}(\underbrace{[T^a, T^b]T^d}_{T^aT^bT^d - T^bT^aT^d}) = if^{abc}\frac{1}{2}\delta^{cd} = \frac{i}{2}f^{abd}$$

so f is antisymmetric wrt ab. QED

Similar technique to prove (4.11).

$$LHS \text{ of } (4.11) = \mathrm{Tr}\left[T^e \underbrace{\left([[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b]\right)}_{0\ \because \text{original Jacobi identity}}\right] = 0$$

4.2. Yang-Mills Theory

Lecture 16 (10/29/14)

We want to find a gauge for non-abelian theory. Recall the logic of deriving covariant derivative and $U(1)$ transformation before. We have fermion fields Ψ and we want to couple them to photons field A^μ. Then we get a form of covariant derivative then we have to figure out the correct $U(1)$ transformation, because $U(1)$ global is no longer correct.

Now we study quarks interact with gluons. The logic is not the same. People know how photons gauge works (2.14), but when Yang & Mills proposed non-abelian Yang-Mills theory, people knew nothing about gluon fields A_μ. Yang-Mills started from some

$$\phi_a \quad a = 1, ..., N$$

N scalar or spinor; real or complex fields. (For quarks $N = 3$) They said

$$\begin{aligned} \phi &\rightarrow U\phi \\ A_\mu &\rightarrow A'_\mu \end{aligned} \tag{4.13}$$

and Yang-Mills imposed

$$D_\mu \phi \equiv (\partial_\mu - igA_\mu)\phi \tag{4.14}$$

where A_μ is matrix (turns out there are 8 gluon, so A^a_μ, $a = 1, ..., 8$, total 8 metrics). g is some coupling number. They also imposed

$$D_\mu \rightarrow D'_\mu = UD_\mu U^{-1}$$

so that $D_\mu\phi \rightarrow UD_\mu\phi$.

Let's see what A'_μ (4.13) should be?

$$\underbrace{\underbrace{U(\partial_\mu - igA_\mu)U^{-1}}_{D'}f}_{(U\partial_\mu U^{-1}f+\partial_\mu - igUA_\mu U^{-1})f} = (\underbrace{\partial_\mu - igA'_\mu}_{D'})f$$

hence

$$A'_\mu = UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1} \tag{4.15}$$

cf (2.48).

They also demanded

$$\mathrm{Tr}A_\mu = 0$$

Because subtracting $\frac{\mathrm{Tr}A}{N}I$ will not change physics, just like in E&M one can freely adding constants to A_μ. Also see that in (4.15) adding constant will not the transformation.

What are the gauge invariant terms under (4.15)?

If $\phi's$ are scalars, kinetic term

$$(D_\mu\phi)^\dagger D_\mu\phi \xrightarrow{U} (UD_\mu\phi)^\dagger UD_\mu\phi = (D_\mu\phi)^\dagger U^\dagger UD_\mu\phi$$

mass term

$$m\phi^\dagger\phi \xrightarrow{U} m\phi^\dagger U^\dagger U\phi$$

If $\phi's$ are spinor, kinetic term

$$i\bar{\Psi}\gamma^\mu D_\mu\Psi \xrightarrow{U} i\overline{U\Psi}\gamma^\mu UD_\mu\Psi$$

$$\overline{U\Psi} = (U\Psi)^\dagger\gamma^0 = \Psi^\dagger U^\dagger\gamma^0 = \Psi^\dagger\gamma^0 U^\dagger = \bar{\Psi}U^\dagger$$

we can switch $\gamma^0 U^\dagger$ because one is in spinor space, the other is in N (particle) space. Thus

$$i\bar{\Psi}\gamma^\mu D_\mu\Psi \xrightarrow{U} i\bar{\Psi}\gamma^\mu\underbrace{U^\dagger U}_{I}D_\mu\Psi$$

Likewise the mass term

$$m\bar{\Psi}\Psi \xrightarrow{U} m\bar{\Psi}U^\dagger U\Psi$$

As we have seen that the index structure in above, e.g. $i\bar{\Psi}\gamma^\mu D_\mu\Psi$, is quite confusing.

$$\begin{aligned} i\bar{\Psi}\gamma^\mu D_\mu\Psi &= i\bar{\Psi}_a\gamma^\mu \left(D_\mu\right)_{ab}\Psi_b,\ a,b=1,...,N \text{ for \# of fields} \\ &= i\left(\bar{\Psi}_a\right)_\alpha \left[\gamma^\mu\right]_{\alpha\beta}\left[\left(D_\mu\right)_{ab}\right]\left(\Psi_b\right)_\beta \quad \alpha,\beta \text{ for spinor indices} \end{aligned}$$

There is no spinor indices for D_μ. Although we discussed before that A_μ of photon has both spin index and spatial index, this is not the case here.

What is the equivalent $F_{\mu\nu}$ for Yang-Mills?

Recall (2.47), motivated by EM,

$$F_{\mu\nu}f = \frac{i}{g}[D_\mu, D_\nu]f = \partial_\mu A_\nu f - \partial_\nu A_\mu f - A_\nu\partial_\mu f + A_\mu\partial_\nu f - ig[A_\mu, A_\nu]f$$

so

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{4.16}$$

The additional term $[A_\mu, A_\nu]$ indicates the non-abelianness of A_μ.

Check if (4.16) is gauge invariant,

$$F_{\mu\nu} \xrightarrow{U} \frac{i}{g}[UD_\mu U^{-1}, UD_\nu U^{-1}] = UF_{\mu\nu}U^{-1}$$

hence not gauge invariant, and $F_{\mu\nu}F^{\mu\nu}$ is not even a number.

How to write kinetic term? Take trace

$$\text{Tr}(F_{\mu\nu}F^{\mu\nu}) = \text{Tr}(UF_{\mu\nu}U^{-1}UF^{\mu\nu}U^{-1})$$

is gauge invariant.

For $SU(N)$

$$\mathcal{L}_{kinetic\ for\ A_\mu} = -\frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu})$$

For $SO(N)$

$$\mathcal{L}_{kinetic\ for\ A_\mu} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

From (4.16), one sees that the kinetic terms contains not only quadratic but cubic, quartic terms. They will give self-interaction. This makes the theory more complicated. In particular graviton can produce gravitons. Classical example is the Schwarzschild solution of a black hole which is the vacuum solution that contains nothing but gravitons.

Now we know what D_μ & $F_{\mu\nu}$ are. But what is A_μ? A_μ is a traceless metric. Since we want A_μ to represent actual particles, we want A_μ to be hermitian. Recall generators for $SU(N)$ or $SO(N)$ are traceless hermitian or antisymmetric matrix. Therefore it is natural to put

$$A_\mu = \sum_{a \in \text{all generators}} A_\mu^a T^a$$

similar to (4.5). A_μ^a are purely real numbers. T^a are generators of $SU(N)$ or $SO(N)$. Since they are 8 gluons, natural choice $SU(3)$ for strong interaction.

Using (4.8), one can get A_μ^a from A_μ,

$$A_\mu^a = 2\mathrm{Tr}(A_\mu T^a)$$

There are changes to D_μ & $F_{\mu\nu}$. (4.14) is now

$$D_\mu = \partial_\mu - igA_\mu^a T^a \tag{4.17}$$

Similarly we define

$$F_{\mu\nu} = F_{\mu\nu}^a T^a$$

and so

$$F_{\mu\nu}^a = 2\mathrm{Tr}(F_{\mu\nu}T^a)$$

how to relate $F^a_{\mu\nu}$ to A^a_μ, (4.16)

$$F^a_{\mu\nu}T^a = \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a - ig\underbrace{[A^b_\mu T^b, A^c_\nu T^c]}_{A^b_\mu A^c_\nu i \underbrace{f^{bca}}_{f^{abc}} T^a}$$

so

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{4.18}$$

so the kinetic term

$$\mathcal{L}_{kinetic\ for\ A_\mu} = -\frac{1}{2}\mathrm{Tr}(F^a_{\mu\nu}T^a F^{b\mu\nu}T^b) = -\frac{1}{2}F^a_{\mu\nu}F^{b\mu\nu}\mathrm{Tr}(T^aT^b) = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{4.19}$$

In the case of strong interaction, the Lagrangian looks like

$$\mathcal{L} = \underbrace{\sum_{I=1,\ldots,6} i\bar{\Psi}^I\gamma^\mu D_\mu\Psi - m^I\bar{\Psi}^I\Psi^I}_{\text{fermions-quarks}} - \underbrace{\sum_{a=1,\ldots,8}\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}_{\text{Yang-Mills gauge}} \tag{4.20}$$

and $\Psi = (\psi_i) = (\psi_1, \psi_2, \psi_3)$, for the 3 colors (red, green, blue) of quarks, which have same mass. I indicates 6 flavors (up,down,strange,...); they don't have same masses, so we have m^I. However for all flavors the 8 gluons are the same $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$. Question: what is the dimension of T^a? By (4.17), it looks like it should be 3×3. This is actually a coincidence, because mathematically one can represent $SU(3)$ by 3×3 or 4×4 matrix... any higher dimensional. However in any dimension the number of generators is the same.

4.3. Group & Transformation (cont.)

Lecture 17 (11/3/14)

Let's recap what we have done and we want to make them more precise. We showed that under

$$\phi \to U\phi \tag{4.21}$$

where ϕ is mulch-component, and U can be global or local. If local, we introduce A_μ and D_μ (cf (4.17)) such that

$$D_\mu \phi \to U D_\mu \phi$$

We mentioned that if our symmetry group is $SU(N)$,$SO(N)$, ϕ need not be $N-$component, or A_μ or U need not be $N \times N$. We recall that a representation of Lie algebra is a set of matrices, one for each generator that satisfies Lie algebra (4.12).

Now we want to consider mixing. Consider a different linear combination of $\phi_i's$

$$\chi_i = V_{ij}\phi_j$$

where V is any invertible matrix. Let's think about how χ transforms under symmetry U,

$$\chi_i \to \chi_i' = V_{ij}U_{jm}\phi_m = V_{ij}U_{jm}V^{-1}_{mn}\underbrace{V_{np}\phi_p}_{\chi_n}$$

In other words

$$\chi \to VUV^{-1}\chi$$

contrast to (4.21).

Therefore if we replace U by SUS^{-1}, then $\phi \to SUS^{-1}\phi$, and $\chi \to (VS)U(SV)^{-1}\chi$, so physically U and SUS^{-1} have the equivalent effects by changing to a different χ. We say U, SUS^{-1} are equivalent representation.

Another important concept is reducible representation. Suppose we have a representation U, by applying SUS^{-1} for some S, we make SUS^{-1} block diagonal, so that there are some components of ϕ transform separately. Then we say U is reducible.

We are interested in enumerating nonequivalent & irreducible representations. One confusion: in math people say representations, they mean the matrices U. In physics community, people say representations, they refer to the objects being transformed, ϕ_i. Because that is what we care about. But in both communities people the dimension of the representation, they mean the dimension of ϕ.

Fundamental Representations (Defining Representation)

The usual $N \times N$ metrics representation of $SU(N)$ & $SO(N)$.

Adjoint Representations of $SU(N)$

$$T_A^a = [T_A^a]^{bc} = -f^{abc} \tag{4.22}$$

A is only a label for adjoin. Since b, c runs from 1 to $N^2 - 1$, the dimension (i.e. number of generator) of representation is $(N^2 - 1)(N^2 - 1)$. First prove adjoint representation as defined in (4.22) is indeed a representation.

Proof. By Jacobi identity (4.11)

$$(-if^{abd})(-if^{cbe}) - (-if^{cbd})(-if^{ade}) = (if^{acd})(-if^{dbe})$$

so

$$[T_A^a]^{bd}[T_A^c]^{be} - [T_A^c]^{bd}[T_A^a]^{be} = if^{acd}[T_A^d]^{be}$$

that is

$$[T_A^a, T_A^c] = if^{acd}T_A^d$$

QED.

It is also clear from (4.22) T_A^a is hermitian.

Trivial Representation

This is the singlet representation, because it is 1 dimensional. So $T^a = 0$, so

$$U^{iT} = I$$

In SM, Leptons has no strong interaction, so people say leptons are singlets under color $SU(3)$ or sometimes say leptons have no color charges.

One way to build further representation is to multiply existing ones. E.g. suppose ϕ_i & ψ_j transform according to defining representation of $SO(N)$. Consider

$$\chi_{ij} = \phi_i \psi_j \tag{4.23}$$

under $N \otimes N$. N indicates fundamental representation $SO(N)$

χ_{ij} seems to belong to a larger representation but it's reducible.

Claim

$$\chi_{ij} = \underbrace{\frac{\chi_{ij} + \chi_{ji}}{2}}_{S_{ij}} + \underbrace{\frac{\chi_{ij} - \chi_{ji}}{2}}_{A_{ij}} \tag{4.24}$$

S_{ij} & A_{ij} don't mix under U. Or

$$N \otimes N = \frac{N(N+1)}{2} \oplus \frac{N(N-1)}{2} \tag{4.25}$$

Pf

$$\chi_{ij} \to U_{im} U_{jn} \underbrace{\phi_m \psi_n}_{\chi_{mn}}$$

or in metric notation

$$\chi \to \chi' = U \chi U^T = USU^T + UAU^T$$

in writing this way we should think χ_{ij} as a N^2 component column vector, S_{ij} is $N(N+1)/2$ component column vector, and A_{ij} is $N(N-1)/2$ component column vector.

Now we just have to show USU^T is symmetric and UAU^T is antisymmetric. Indeed

$$(USU^T)^T = USU^T$$

Up to this point the derivation would be exactly the same if we started with $SU(N)$ instead of $SO(N)$. But the following steps only work for $SO(N)$.

We show S_{ij} in (4.24) is reducible. Consider

$$\mathrm{Tr}\chi \to \mathrm{Tr} U \chi U^T = \mathrm{Tr}\chi$$

hence $\text{Tr}\chi = \chi_{ii}$ (i.e. $\chi_{ij} \subset S_{ij}$) is invariant under $N \otimes N$. So we have

$$\chi = \underbrace{\frac{1}{N}\text{Tr}(S)I}_{\text{trace}} + \underbrace{S - \frac{1}{N}\text{Tr}(S)I}_{\text{traceless sym}} + A \tag{4.26}$$

or

$$N \otimes N = 1 \oplus \frac{N(N+1)}{2} - 1 \oplus \frac{N(N-1)}{2}$$

E.g. $N = 3$, $SO(3)$, we know from QM, total spins of 2 spin 1 particles can be spin 0, spin 1 & spin 2

$$3 \otimes 3 = \underbrace{1}_{\text{spin } 0} \oplus \underbrace{5}_{\text{spin } 2} \oplus \underbrace{3}_{\text{spin } 1}$$

Lecture 18 (11/5/14)

Now we want to get a similar decomposition like (4.26) for $SU(N)$. Start from (4.9)

$$U(\theta) = 1 - i\theta^a T_R^a \tag{4.27}$$

where R labels what kind of representations. E.g $R = N$ for fundamental representation. Recall to get (4.26) we used $UU^T = 1$, but for $SU(N)$ we need $UU^\dagger$, so we first take complex conjugate of (4.12)

$$[-T_R^{a*}, -T_R^{b*}] = if^{abc}(-T_R^{c*})$$

This looks like an algebra for another representation

$$T_{\bar{R}}^a := -T_R^{a*} = -(T_R^a)^T$$

because T is hermitian. This turns out to be in general a new representation for $SU(N)$, i.e.

$$T_{\bar{R}}^a \neq S T_R^a S^{-1}$$

with exception of $SU(2)$ because

$$T_2^a = \sigma^a, \sigma = \text{pauli}$$

$$T^a_{\bar{2}} = (-\sigma^a)^* = (\sigma^2)^{-1}\sigma^a(\sigma^2)$$

but this is not a new representation for $SO(N)$. Because for $SO(N)$, T is purely imaginary and antisymmetric cf (4.2),

$$T^a_{\bar{R}} := -T^{a*}_R = T^a_R$$

The reason we put bar for adjoint representation, because it has to do with antiparticles. This will be clear later.

Back to (4.27), take hermitian conjugate

$$U^\dagger = 1 + ig\theta^a T^{a\dagger}_R = 1 - ig\theta^a (\underbrace{-T^{a*}_R}_{T^a_{\bar{R}}})^T$$

which looks just like (4.27) and take hermitian conjugate of $\phi \to U\phi$, we get

$$\phi^\dagger \to \phi'^\dagger U^\dagger$$

thus we get ϕ transforms under T^a_R while $\phi^\dagger$ transforms under $T^a_{\bar{R}}$. Borrowing covariant/controvariant index for column and row vectors in GR, we write

$$\begin{aligned} \phi_i &\to U_i^{\ j}\phi_j = (\delta_i^{\ j} - ig\theta^a (T_R)_i^{\ j})\phi_j \\ \phi^{\dagger i} &\to \phi^{\dagger j} U^{\dagger\ j}_{\ i} = \phi^{\dagger j}(\delta_j^{\ i} - ig\theta^a (T_{\bar{R}})_j^{\ i}) = (\delta^i_{\ j} - ig\theta^a (T_{\bar{R}})^i_{\ j})\phi^{\dagger j} \end{aligned}$$

So in particular when $R = N$, there are in general 2 fundamentals N & $\bar{N}$, with exception for $SU(2)$, that 2 & $\bar{2}$ turn out to be the same.

We have done $N \otimes N$. Let's try $N \otimes \bar{N}$. So adopt (4.23) notation

$$\chi_i^{\ j} = \phi_i \psi^j$$

so

$$\chi \to U\chi U^\dagger$$

Can we still break it into symmetric and antisymmetric? No, because

$$(USU^\dagger)^T = U^* S (U^\dagger)^* \neq (USU^\dagger)^T$$

is not symmetric. We have to look for other decomposition. consider

$$\mathrm{Tr}\chi \to \mathrm{Tr}U\chi U^{\dagger} = \mathrm{Tr}\chi$$

is invariant. Also ${\delta_j}^i$ is invariant, i.e.

$$I \to UIU^{\dagger} = I$$

therefore we put

$$\chi = \underbrace{\frac{1}{N}\mathrm{Tr}(\chi)I}_{\text{trace}} + \underbrace{\chi - \frac{1}{N}\mathrm{Tr}(\chi)I}_{\text{traceless}} \tag{4.28}$$

or

$$N \otimes \bar{N} = \underbrace{1}_{\text{singlet}} \oplus \underbrace{N^2 - 1}_{\text{adjoint}}$$

For more general case

$$R \otimes \bar{R} = \underbrace{1}_{\text{singlet}} \oplus \text{adjoint} \oplus ...$$

Besides ${\delta_j}^i$, there is another object that is invariant under $SO(N)$ and $SU(N)$ that is $\epsilon_{i_1 i_2,\ldots,i_N}$ because

$$\epsilon_{i_1 i_2,\ldots,i_N} \to {U_{i_1}}^{j_1}{U_{i_2}}^{j_2}...{U_{i_N}}^{j_N}\epsilon_{j_1 j_2,\ldots,j_N} = \underbrace{\det U}_{1}\epsilon_{i_1 i_2,\ldots,i_N} = \epsilon_{i_1 i_2,\ldots,i_N}$$

We will use the valuable information to decompose $SU(2)$ and $SU(3)$.

4.4. SU(2) & SU(3)

SU(2)

As explained before 2 & $\bar{2}$ are the same representations, e.g. instead of studying ${\chi_{ij}}^k$ under $2 \otimes 2 \otimes \bar{2}$, we can focus on objects with only lower indices. To do so, we multiply

$${\chi_{ij}}^k \epsilon_{km} \tag{4.29}$$

then work in $2 \otimes 2 \otimes 2$ and at the end we multiply ϵ^{ml} to get it back, because ϵ_{km} is invariant under $SU(2)$.

Furthermore we only have to focus on

$$\chi_{i_1 i_2,\ldots,i_m} = \chi_{\{i_1,i_2\},\ldots,i_m} + \chi_{[i_1,i_2],\ldots,i_m} \tag{4.30}$$

whose indices are completely symmetric under exchange of any pair of indices. Because of (4.25), they are invariant under $SU(2)$. One can also see that by multiplying $\epsilon^{i_1 i_2}$

$$\chi_{\{i_1,i_2\},\ldots,i_m}\epsilon^{i_1 i_2} = 0$$

The antisymmetric part in (4.30) is invariant too.

$$\chi_{[i_1,i_2],\ldots,i_m} = \underbrace{\frac{1}{2}\chi_{[p,q],\ldots,i_m}\epsilon^{pq}}_{\tilde{\chi}_{i_3\ldots i_m}}\epsilon_{i_1 i_2} \tag{4.31}$$

$\tilde{\chi}_{i_3\ldots i_m}$ has no $i_1 i_2$ dependence, so it is singlet under the first 2 of $2 \otimes 2 \otimes 2 \otimes$ .....

Let's see how this agrees what we know about particle spin. Consider χ_i has 1 index, so it can be

$$\chi_1 \text{ or } \chi_2$$

so it has dimension 2, which is the spin-1/2 particle. Next consider χ_{ij} has 2 indices. It can be

$$\chi_{11}, \chi_{22}, \text{ or } \chi_{12} + \chi_{21}$$

it has dimension 3, which is the spin-1 particle. Anther way to think about this is to use (4.25)

$$2 \otimes 2 = 1 \oplus \underbrace{3}_{\text{sym}}$$

Next χ_{ijk},

$$\chi_{111}, \chi_{222}, \chi_{112} + \chi_{121} + \chi_{211} \text{ or } \chi_{221} + \chi_{212} + \chi_{122}$$

it has dimension 4, which is the spin-3/2 particle. One can also add in $\chi_{\text{no indice}}$, the trivial representation, it has dimension 1, which is the spin-0 particle. The general rule is easy to deduce

$$\chi_{i_1 i_2,\ldots,i_m}$$

has $m+1$ dimension.

SU(3)

Lecture 19 (11/10/14)

There is the trivial representation for singlet

$$\phi \to \phi$$

and the fundamental 3

$$\phi_i \to U_i{}^j \phi_j$$

and $\bar{3}$

$$\phi^{\dagger i} \to \phi^{\dagger j} U^{\dagger}{}_j{}^i$$

Next $3 \otimes 3$, using the same trick (4.30), (4.31)

$$\chi_{ij} = \chi_{\{i,j\}} + \underbrace{\frac{1}{2}\chi_{[p,q]}\epsilon^{pqk}}_{\tilde{\chi}^k}\epsilon_{ijk}$$

$\tilde{\chi}^k$ has one index, so it is $\bar{3}$. Thus

$$3 \otimes 3 = \bar{3} \oplus 6$$

Similarly

$$\bar{3} \otimes \bar{3} = 3 \oplus \bar{6}$$

They agree with (4.25).

Next

$$3 \otimes \bar{3} = 1 \oplus 8 \tag{4.32}$$

or

$$\chi_i{}^j = \frac{1}{3}\chi_k{}^k \delta_i{}^j + (\chi_i{}^j - \frac{1}{3}\chi_k{}^k \delta_i{}^j)$$

What about we have more indices? The general case is treated by Young diagrams, but for $\chi_{i_1 i_2,\ldots,i_m}$ under $SU(3)$, we can still use the trick (4.30),

$$\chi_{i_1 i_2,\ldots,i_m} = \chi_{\{i_1,i_2\},\ldots,i_m} + \underbrace{\chi_{[i_1,i_2],\ldots,i_m}}_{\underbrace{\frac{1}{2}\chi_{[p,q],\ldots,i_m}\epsilon^{pqk}}_{\tilde{\chi}_{i_3\ldots i_m}{}^{k}}\epsilon_{i_1 i_2 k}}$$

We can see why it does not work for higher $SU(N)$. For example $SU(4)$

$$\chi_{i_1 i_2,\ldots,i_m} = \chi_{\{i_1,i_2\},\ldots,i_m} + \underbrace{\chi_{[i_1,i_2],\ldots,i_m}}_{\underbrace{\frac{1}{2}\chi_{[p,q],\ldots,i_m}\epsilon^{pqkh}}_{\tilde{\chi}_{i_3\ldots i_m}{}^{kh}}\epsilon_{i_1 i_2 kh}}$$

so $\tilde{\chi}_{i_3\ldots i_m}{}^{kh}$ still have 2 indices $i_1 i_2$, not reduce indices. But what if we group 3 indices

$$\chi_{i_1 i_2,\ldots,i_m} = \chi_{\{i_1,i_2,i_3\},\ldots,i_m} + \underbrace{\chi_{[i_1,i_2,i_3],\ldots,i_m}}_{\underbrace{\frac{1}{2}\chi_{[p,q,k],\ldots,i_m}\epsilon^{pqkh}}_{\tilde{\chi}_{i_4\ldots i_m}{}^{h}}\epsilon_{i_1 i_2 i_3 h}} + \chi_{(\underbrace{\ldots\ldots\ldots}_{i_1,i_2,i_3}),\ldots,i_m}$$

then looks like $\tilde{\chi}_{i_4\ldots i_m}{}^{h}$ is down to 1 index, but the problem for 3 indices is that it is not just sum of complete symmetric and complete antisymmetric. There will be extra terms which may not be invariant under $SU(4)$, so we may not have completely block diagonal form.

Now let's do the same discussion following (4.31) for completely symmetric $\chi_{i_1 i_2,\ldots,i_m}$. We claim: the dimension is $\frac{1}{2}(m+1)(m+2)$. Let's check for $m = 2$. We can list them

$$\chi_{11}, \chi_{22}, \chi_{33}, \chi_{12} + \chi_{21}, \chi_{23} + \chi_{32}, \text{ and } \chi_{13} + \chi_{31}$$

To prove this in general, we use a combinatorial trick, we associate each with

$$\begin{aligned}
\chi_{11} &\leftrightarrow i_1 i_2 \,\|\, | \\
\chi_{22} &\leftrightarrow \,\|\, i_1 i_2 | \\
\chi_{33} &\leftrightarrow \,\|\, | i_1 i_2 \\
\chi_{12} + \chi_{21} \leftrightarrow \chi_{12} &\leftrightarrow i_1 \,\|\, i_2 | \\
\chi_{23} + \chi_{32} \leftrightarrow \chi_{23} &\leftrightarrow \,\|\, i_1 | i_2 \\
\chi_{13} + \chi_{31} \leftrightarrow \chi_{13} &\leftrightarrow i_1 \,\|\, | i_2
\end{aligned}$$

the rule is left to $\|$ is 1, in between $\|$ & $|$ is 2, right to $|$ is 3. So there are $m+1$ spots to insert $\|$ & $|$ for

$$i_1 i_2, ..., i_m$$

i.e. pick 2 out of $m+1$ order doesn't matter. This is the problem of combinations with repetition, so

$$\begin{pmatrix} m+1+2-1 \\ 2 \end{pmatrix} = \frac{(m+2)(m+1)}{2}$$

Similarly we can have completely symmetry up indices, or have both up and down indices completely symmetric separately. We're also interested in traceless case, therefore

$$(m,n) := \chi_{i_1 \dots i_m}{}^{j_1 \dots j_n} - \frac{1}{3} \chi_{k i_2 \dots i_m}{}^{k j_2 \dots j_n} \delta_{i_1}{}^{j_1} \tag{4.33}$$

has dimension

$$\frac{(m+2)(m+1)}{2} \frac{(n+2)(n+1)}{2} - \frac{(m+1)m}{2} \frac{(n+1)n}{2} = \frac{(m+1)(n+1)(m+n+2)}{2}$$

Some lowest members of (4.33) are

$$
\begin{array}{ll}
(0,0) & 1 \text{ dim singlet} \\
(1,0) & 3 \\
(0,1) & \bar{3} \\
(2,0) & 6 \\
(0,2) & \bar{6} \\
(3,0) & 10 \\
(0,\bar{3}) & \overline{10} \\
(1,1) & 8 \text{ this is the symm part of (4.32)} \\
(2,2) & 27 \\
\quad \cdots &
\end{array}
$$

What are the generators of $SU(3)$ in the fundamental representation?

$$T^a_{N=3} = \frac{1}{2}\lambda^a$$

λ^a are gell-mann matrices. Knowing they have to be, traceless, hermitian, we can write them down easily.

$$
\lambda^1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}
$$

$$
\lambda^4 = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}
$$

$$
\lambda^6 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}
$$

The coefficients are chosen so that T^a satisfies (4.8).

Notices $\lambda^{1,2,3}$ (they are Pauli) form sub algebra $SU(2)$, meaning commutators of $\lambda^{1,2,3}$ remains within $\lambda^{1,2,3}$. Also notice that λ^8 commutes with $\lambda^{1,2,3}$. $\lambda^{1,2,3,8}$ form a

$$SU(2) \times U(1) \text{ subalgebra} \tag{4.34}$$

$U(1)$ means λ^8 commutes with $\lambda^{1,2,3}$. Sometimes people denote (4.34) as

$$3 \text{ under } SU(3)|_{SU(2)} \to 2^{\frac{1}{3}} \oplus 1^{-\frac{2}{3}}$$

2 for upper 2×2 (u, d) block and 1 for singlet (s) under $SU(2)$. The 1/3, $-2/3$ label hypercharges for u, d, s, see later (4.35).

We will use this idea later when we study $SU(5)$ symmetry of SM.

4.5. Flavor SU(3)

Unlike color $SU(3)$, flavor $SU(3)$ is not an exact symmetry (called approximate or global symmetry), because

$$\phi_i = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

the masses are not the same. First observe that $T^{3,8}$ form a maximal commuting set

$$[T^3, T^8] = 0$$

that is no more matrix that commute with them. So we can find simultaneous eigenstates of $T^{3,8}$, we call them the fundamental representation

$$u, d, s$$

Hypercharge

Define hypercharge

$$Y \equiv \frac{2}{\sqrt{3}} \text{eigenvale}(T^8)$$

and actual charge

$$Q = \text{eigenvalue}(T^3) + \frac{Y}{2}$$

then

	eigenvalue(T^3)	eigenvalue(T^8)	Y	Q	Strangeness $= \frac{1}{3} - Y$
u	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{3}$	$\frac{2}{3}$	0
d	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{3}$	$-\frac{1}{3}$	0
s	0	$-\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$	$-\frac{1}{3}$	1

(4.35)

Assuming flavor $SU(3)$ is symmetry of H, we say these are conserved quantum numbers.

Why are they conserved?

Lecture 20 (11/12/14)

Let's use T^3 for example. We associate T^3 with an operator $\hat{Q}^3$ i.e. we are in Heisenberg picture,

$$e^{i\theta\hat{Q}^3}\hat{\phi}_i e^{-i\theta\hat{Q}^3} = (e^{i\theta T^3})_i^{\ j}\hat{\phi}_j \tag{4.36}$$

consider θ small, and if we put $\hat{\phi}_i = \hat{u}$, we have

$$[\hat{Q}^3, \hat{u}] = \frac{1}{2}\hat{u} \tag{4.37}$$

the right hand side is because u is eigenstate of T^3. Similarly

$$[\hat{Q}^3, \hat{d}] = -\frac{1}{2}\hat{d}$$

$$[\hat{Q}^3, \hat{s}] = 0$$

In the Fourier way,

$$\hat{u} \sim \int \hat{a}_u^+ e^{(\dots)} + h.c.$$

so (4.37) says

$$[\hat{Q}^3, \hat{a}_u^+] = \frac{1}{2}\hat{a}_u^+ \tag{4.38}$$

This will help us to build quark composites–mesons. Does (4.38) make sense? If we put $|u\rangle = \hat{a}_u^+|0\rangle$,

$$\hat{Q}^3|u\rangle = \hat{Q}^3\hat{a}_u^+|0\rangle = \hat{a}_u^+\underbrace{\hat{Q}^3|0\rangle}_{0} + \frac{1}{2}\underbrace{\hat{a}_u^+|0\rangle}_{|u\rangle} = \frac{1}{2}|u\rangle$$

we need $\hat{Q}^3 |0\rangle = 0$ hence no spontaneous symmetry breaking.

Let's see why (4.35) are conserved. We know e.g.

$$[\hat{Q}, H_{strange}] \sim 0 \tag{4.39}$$

because $m_{strange} > m_{u,d}$. The symmetry is not exact, but close to 0 if we study fast decay systems. Suppose we have such scattering process

$$T^3 = \left\langle u', d', ... \middle| e^{i\theta \hat{Q}^3} (S - I) e^{-i\theta \hat{Q}^3} \middle| u, d, ... \right\rangle$$

S being the scattering matrix and it is function of H. If we have (4.39), then clearly total T^3 charge is conserved.

Next we take hermitian conjugate of (4.36),

$$e^{i\theta \hat{Q}^3} \hat{\phi}^{\dagger i} e^{-i\theta \hat{Q}^3} = \hat{\phi}^{\dagger i} (e^{i\theta T^3})_i^{\ j}$$

so the only change is that $T^3 \to -T^3$, so the eigenvalues are flipped. So they are the antiparticles. Thus

$$\bar{u}, \bar{d}, \bar{s}$$

have same table (4.35) with opposite signs.

Let's build mesons by (4.32). The '1' is the singlet, trace

$$\phi_i \phi^{\dagger i} = u\bar{u} + d\bar{d} + s\bar{s}$$

called η' particle.

mass	957.6MeV
T^3	0
Y	0
Q	0

so looks like T^3 of η' is the sum of its constitutes. Proof is one line. By (4.36)

$$e^{i\theta \hat{Q}^3} u\bar{u} e^{-i\theta \hat{Q}^3} = e^{i\theta \hat{Q}^3} u e^{-i\theta \hat{Q}^3} e^{i\theta \hat{Q}^3} \bar{u} e^{-i\theta \hat{Q}^3} = e^{-i\theta(\frac{1}{2} - \frac{1}{2})} u\bar{u}$$

The '8' is the famous octant 8.

Octant 8

			mass(MeV)	T^3	Y	Q_e
$\phi_1\phi^2 = u\bar{d}$	$\to$	Π^+	139.6	1	0	1
$\phi_2\phi^1 = d\bar{u}$	$\to$	Π^-	139.6	-1	0	-1
$\left(\phi_1\phi^1 - \frac{1}{3}\mathrm{Tr}(\phi_1\phi^1)\right) - \left(\phi_1\phi^1 - \frac{1}{3}\mathrm{Tr}(\phi_1\phi^1)\right) = u\bar{u} - d\bar{d}$	$\to$	Π^0	134.96	0	0	0
$\phi_3\phi^3 - \frac{1}{3}\mathrm{Tr}\phi_3\phi^3 = \frac{2}{3}s\bar{s} - \frac{1}{3}d\bar{d} - \frac{1}{3}u\bar{u}$	$\to$	η	548.8	0	0	0
$\phi_1\phi^3 = u\bar{s}$	$\to$	K^+	493.67	$\frac{1}{2}$	1	1
$\phi_3\phi^1 = s\bar{u}$	$\to$	K^-	493.67	$-\frac{1}{2}$	-1	-1
$\phi_2\phi^3 = d\bar{s}$	$\to$	K^0	497.72	$-\frac{1}{2}$	1	0
$\phi_3\phi^2 = s\bar{d}$	$\to$	$\bar{K}^0$	497.72	$\frac{1}{2}$	-1	0

That Π's have no s explains the masses difference between Π and K's. Because they have no s, they form the 3 of $2 \otimes \bar{2} = 1 \oplus 3$. The neutral particles are slighter than their charged counterpart, due to EM interaction. One can draw the octant 8 on a plane with horizontal axis: T^3 and vertical axis: Y, then the shape looks like an octant.

To get protons neutrons, we need three quarks. So instead of (4.32), we have

$$\underbrace{3 \otimes 3}_{\bar{3}\oplus 6} \otimes 3 = \underbrace{\bar{3} \otimes 3}_{1\oplus 8} \oplus \underbrace{6 \otimes 3}_{8\oplus 10}$$

We have to add colors. Why do we need colors? Before Greenberg hypothesized color in 1964, there was a big problem related to proton isospins.

$$p = 2u(\uparrow)u(\uparrow)d(\downarrow) - u(\uparrow)u(\downarrow)d(\uparrow) - u(\downarrow)u(\uparrow)d(\uparrow)$$

The first term on the right violates Pauli exclusion. The reason why there are three colors so that we get anomaly free is still a mystery.

We can add color to (4.32) too,

$$\phi_{iI}\phi^{\dagger jJ}$$

and we impose color singlet (no individual color is seen), thus we are interested in complete symmetric traceless color,

$$\phi_{iI}\phi^{\dagger jJ} - \frac{1}{3}\delta_I^{\ J}\phi_{iK}\phi^{\dagger jK}$$

4.6. Yang-Mills Theory (cout.)

Lecture 21 (11/17/14)

Let's continue with Yang-Mills for $SU(N)$ symmetry. Recall $\phi \to U\phi$ and $D_\mu\phi = (\partial_\mu - igA_\mu)\phi \to UD_\mu\phi$,

$$U = e^{-ig\theta^a T^a}$$

where g = coupling, θ^a = real angle, $a = 1, ..., N^2 - 1$ number of generators T^a is fixed for $SU(N)$, but the dimension of the matrix T^a depends on the kind of representations, e.g. for fundamental, dimension is N and for adjoint dimension is $N^2 - 1$.

We know A_μ should transfer according to (4.15). So

$$\begin{aligned} A_\mu \to A'_\mu &= UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1} \\ &= A_\mu - ig\theta^c[T^c, A_\mu] - \partial_\mu\theta^a T^a \end{aligned}$$

And using $A_\mu = A_\mu^a T^a$,

$$A_\mu^a \to A_\mu^a \quad \underbrace{\underbrace{\underbrace{-ig\theta^c f^{cba} A_\mu^b}_{igA_\mu^b(-if^{bac})\theta^c} - \underbrace{\partial_\mu\theta^a}_{\delta^{ac}\partial_\mu\theta^c}}_{-\left(\delta^{ac}\partial_\mu - igA_\mu^b(-if^{bac})\right)\theta^c}}_{\equiv D_\mu^{ac}} \tag{4.40}$$

which looks kind of EM gauge $A_\mu \to A_\mu + \partial_\mu\lambda$, (2.14). One can show that there is another analogy thing here–charge conservation. Recall field strength (4.18) and kinetic term for A_μ(4.19), we write

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + A_\mu^a J_a^\mu$$

where $J^\mu = J_a^\mu T^a$. Find EOM, assume charge is conserved,

$$\begin{aligned}
0 = \delta S &= \delta \int \mathcal{L} = \int -\frac{1}{2} F_{\mu\nu}^a \delta F_a^{\mu\nu} + \delta A_\mu^a J_a^\mu \\
&= \int -F_{\mu\nu}^a \left(-\partial^\nu \delta A_a^\mu + g f^{abc} \delta A_b^\mu A_c^\nu\right) + \delta A_\mu^a J_a^\mu \\
&= \int \delta A_b^\mu \left(-\partial^\nu F_{\mu\nu}^b - g F_{\mu\nu}^a f^{abc} A_c^\nu + J_\mu^b\right)
\end{aligned}$$

we get

$$-\underbrace{\left(\delta^{ba}\partial^\nu - ig(-if^{cba})A_c^\nu\right)}_{\equiv D_\nu^{ba}} F_{\mu\nu}^a + J_\mu^b = 0$$

reminiscing of charge conservation.

Recall in EM we had to deal with redundancy of gauge invariant in path integral. One way was to use Faddeev-Popov. We need to do that here too, because of (4.40)

$$A_\mu^a \to A_\mu^a - \underbrace{D_\mu^{ab}\theta^b}_{\delta A_\mu^a}$$

(2.36), (2.37) gain index a

$$Z = \int DA e^{iS[A,J]} \prod_a \delta F^a(A) \det \frac{\delta F^a}{\delta \theta^b}$$

and

$$F^a(A) = \partial_\mu A^{a\mu} - \chi^a(x)$$

for some arbitrary fixed function χ^a. So we get (2.40)

$$Z = \int DA e^{\underbrace{iS[A]}_{\text{YM}} - \underbrace{\frac{i}{2\xi}\int d^4x (\partial^\mu A_\mu^a)(\partial_\nu A_a^\nu)}_{\text{gauge fixing}}} \det \frac{\delta F^a}{\delta \theta^b}$$

What is $\det \frac{\delta F^a}{\delta \theta^b}$? Recall for fermions

$$\int D\bar{\Psi} D\Psi e^{-i\int \bar{\Psi} M \Psi} \sim \det M$$

and

$$\delta F^a = -\partial^\mu (D_\mu^{ab}\theta^b) \implies \frac{\delta F^a}{\delta\theta^b} = -\partial^\mu (D_\mu^{ab}\delta^a(x-y))$$

notice that we see that explicitly $\frac{\delta F^a}{\delta\theta^b}$ depends on D_μ^{ab} so depends on A^μ. That was the reason in remark after (2.38) where we claimed det depends on A^μ.

Therefore

$$\begin{aligned}
\det \frac{\delta F^a}{\delta\theta^b} &\sim \int D\bar{c}Dc e^{i\int d^4x d^4y \bar{c}^a(x)\partial^\mu (D_\mu^{ab}\delta^a(x-y))c^b(y)} \\
&= \int D\bar{c}Dc e^{i\int d^4x \bar{c}^a(x)\partial^\mu \underbrace{D_\mu^{ab}c^b(x)}_{\delta^{ab}\partial_\mu c^b + gf^{acb}A_\mu^c c^b}} \\
&= \int D\bar{c}Dc e^{i\int d^4x \left(-\partial^\mu \bar{c}^a \partial_\mu c_a + gf^{abc}\partial_\mu \bar{c}^a c^b A_c^\mu\right)}
\end{aligned}$$

thus

$$Z = \int DAD\bar{c}Dc e^{\underbrace{iS[A]}_{\text{YM}} \underbrace{-\frac{i}{2\xi}\int d^4x (\partial^\mu A_\mu^a)(\partial_\nu A_a^\nu)}_{\text{gauge fixing}} \underbrace{+i\int d^4x - \partial^\mu \bar{c}^a \partial_\mu c_a + gf^{abc}\partial_\mu \bar{c}^a c^b A_c^\mu}_{\text{ghost}}}$$

for abelian theory $f^{abc} = 0$, $\bar{c}$, c are completely decoupled to the gauge, so we don't need ghosts.

Therefore the full strong interaction Lagrangian is not just (4.20), but

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{gaugefixing} + \mathcal{L}_{ghost} + \mathcal{L}_{quarks} \tag{4.41}$$

4.7. Feynman Rules, Propagator & Vertex

Lecture 22 (11/19/14)

Starting from

$$\begin{aligned}\mathcal{L}_{YM} &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a \\ &= -\frac{1}{2}(\partial^\mu A^{a\nu}\partial_\mu A^a_\nu - \partial^\mu A^{a\nu}\partial_\nu A^a_\mu) \\ &\quad -gf^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu \qquad (4.42) \\ &\quad -\frac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu \qquad (4.43)\end{aligned}$$

The second line is free gluons; third line is a cubic gluon self interaction; fourth line is a quartic gluon self interaction.

Free Gluon Propagator

$$-\frac{1}{2}(\partial^\mu A^{a\nu}\partial_\mu A^a_\nu - \partial^\mu A^{a\nu}\partial_\nu A^a_\mu)\underbrace{-\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2}_{\mathcal{L}_{gauge\ fix}}$$

and

$$\left\langle A^a_\mu(x)A^b_\nu(y)\right\rangle := \int \frac{1}{i}\tilde{\Delta}^{ab}_{\mu\nu}(k)e^{ik(x-y)}\frac{d^4k}{(2\pi)^4}$$

we get similar to abelian theory (2.42)

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

with one addition δ^{ab}. That is because $a = 1, ..., N^2 - 1$ color index, $\mu = 1, ..., 4$ spacetime index, for free theory color should not change. Putting $\xi = 1$ for Feynman gauge.

Gluon Cubic Self-Interaction Vertex

From (4.42)

$$i\mathcal{L}_{cubic\ int} = -igf^{mnl}\eta_{\beta\gamma}A^{m\alpha}A^{n\beta}\partial_\alpha A^{l\gamma}$$

see Feynman diagram, Srednicki figure 72.1 on page 436. One can rigorously compute $iV^{abc}_{\mu\nu\rho}(p,q,r)$ from $\left\langle A^{a\mu}A^{b\nu}A^{c\rho}\int i\mathcal{L}_{cubic\ int}\right\rangle_{free}$ and removing external propagators and overall delta functions.

We will do a quick way. Choose Feynman gauge, $\tilde{\Delta}^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}\eta_{\mu\nu}$, then it will look like very similar to scale cubic vertex $\left\langle \phi_1\phi_2\phi_3 \int \frac{\lambda}{3!}\phi\phi\phi \right\rangle_{free}$, $iV = i\lambda$, or to Fermion-Fermion-photon cubic, just the coupling, here is not that different.

Considering one possible pairing, (there are total 6 possible pairings)

$$\underbrace{A^{a\mu}}_{1}\underbrace{A^{b\nu}}_{2}\underbrace{A^{c\rho}}_{3}\int(-ig)f^{mnl}\eta_{\beta\gamma}\underbrace{A^{m\alpha}}_{1}\underbrace{A^{n\beta}}_{2}\underbrace{\partial_\alpha A^{l\gamma}}_{3} \tag{4.44}$$

which gives vertex

$$(-ig)f^{abc}\eta_{\nu\rho}(-ir_\mu)$$

To figure out the indices, we just look at the pairing. Indices $abc, \nu\rho$ are simple to find, thus μ (comes from α) has to come from $1 \leftrightarrow 1$ pairing, not directly from α in $3 \leftrightarrow 3$. Actually r_μ is the momentum of $A^{c\rho}$ (see Srednicki first figure 72.1 the upper right leg) which is connected to $\partial_\alpha A^{l\gamma}$, then through α it goes to $1 \leftrightarrow 1$ pairing. Another thing to remember is that why is $-ir_\mu$ not $+ir_\mu$? The sign is determined as follows if the derivative interaction ∂A is out of vertex gives a minus, if going in, gives a plus.

After computing all 6 possible pairing, we get

$$iV^{abc}_{\mu\nu\rho}(p,q,r) = gf^{abc}[(q-r)_\mu\eta_{\nu\rho} + (r-p)_\nu\eta_{\rho\nu} + (p-q)_\rho\eta_{\mu\nu}]$$

Gluon Quartic Self-Interaction Vertex

From (4.43), we get

$$\begin{aligned} iV^{abcd}_{\mu\nu\rho\sigma} &= ig^2[f^{abe}f^{cde}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) + f^{ace}f^{dbe}(\eta_{\mu\sigma}\eta_{\rho\sigma} - \eta_{\mu\nu}\eta_{\sigma\rho}) \\ &\qquad + f^{ade}f^{bce}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})] \end{aligned}$$

indices label see Feynman diagram, Srednicki figure 72.1 on page 436. We see already at tree level quartic is suppressed by g^2 comparing to cubic.

Free Ghost Propagator

$$\mathcal{L}_{ghost} = -\partial^\mu\bar{c}^a\partial_\mu c_a + gf^{abc}\partial_\mu\bar{c}^a c^b A^\mu_c$$

free ghost is

$$\langle c^a(x)\bar{c}^b(y)\rangle := \int \frac{1}{i}\tilde{\Delta}^{ab}(k^2)e^{ik(x-y)}\frac{d^4k}{(2\pi)^4}$$

$$\tilde{\Delta}^{ab}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}$$

Ghost-Ghost-Gluon Vertex

Similar to before

$$iV_\mu^{abc}(q,r) = igf^{abc}(-iq_\mu)$$

see Feynman diagram, Srednicki figure 72.2 on page 437.

Lecture 23 (11/24/14)

We have done $\mathcal{L}_{YM}, \mathcal{L}_{gauge\ fixing}, \mathcal{L}_{ghost}$ in (4.41). We are ready to see gluon-quark interaction. Let's first rewrite (4.41) with renormalization coefficients, which will be useful later.

$$\begin{aligned}
\mathcal{L}_{QCD} &= \mathcal{L}_{YM} + \mathcal{L}_{gauge\ fixing} + \mathcal{L}_{ghost} + \mathcal{L}_{quarks} \\
\mathcal{L}_{freeYM} &= -\frac{1}{2}\underbrace{Z_3(\partial^\mu A^{a\nu}\partial_\mu A_\nu^a} - \partial^\mu A^{a\nu}\partial_\nu A_\mu^a) - \frac{1}{2\xi}(\partial_\mu A^{a\nu})^2 \qquad (4.45) \\
\mathcal{L}_{selfinteractYM} &= -Z_{3g}gf^{abc}A^{a\mu}A^{b\nu}\partial_\mu A_\nu^c\tilde{\mu}^{\epsilon/2} - \frac{1}{4}Z_{4g}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A_\mu^c A_\nu^d\tilde{\mu}^\epsilon \qquad (4.46) \\
\mathcal{L}_{ghost} &= -Z_2'\partial^\mu\bar{c}^a\partial_\mu c^a + Z_1' gf^{abc}A_c^\mu\partial_\mu\bar{c}^a c^b\tilde{\mu}^{\epsilon/2} \qquad (4.47) \\
\mathcal{L}_{quarks} &= \sum_{I=1,\ldots,6} iZ_2\bar{\Psi}_i^I\gamma^\mu D_\mu\Psi_i - Z_m m^I\bar{\Psi}_i^I\Psi_i^I \\
&= \sum_{I=1,\ldots,6} i\underbrace{Z_2\bar{\Psi}_i^I\not{\partial}\Psi_i^I} - Z_m m^I\bar{\Psi}_i^I\Psi_i^I + \underbrace{gZ_1A_\mu^a\bar{\Psi}_i^I\gamma^\mu T_{ij}^a\bar{\Psi}_j^J\tilde{\mu}^{\epsilon/2}} \qquad (4.48)
\end{aligned}$$

as usual $d = \epsilon - 4$.

Free Quarks Propagator

That is just the fermions propagator

$$\langle\Psi_i(x)\bar{\Psi}_j(y)\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{1}{i}S_{ij}^{free}(\not{p})e^{ip(x-y)}$$

where

$$S_{ij}^{free} = \frac{-\not{p} + m}{p^2 + m^2 - i\epsilon}\delta_{ij}$$

We have done all pieces except the last term in (4.48):

Quark-Quak-Gluon Vertex

For Feynman diagram see Srednicki figure 72.3 on page 437. Similar to before, we use Feynman gauge

$$iV_{ij\mu}^{aIJ} = \left\langle \Psi_i^I(x)\bar{\Psi}_j^J(y)A_\mu^a \int (ig)A_\nu^b \bar{\Psi}_m^M \gamma^\nu T_{mn}^b \bar{\Psi}_n^N \right\rangle_{free}$$

we have to paring $A \leftrightarrow A$, $\Psi \leftrightarrow \bar{\Psi}$, and $\bar{\Psi} \leftrightarrow \Psi$, so

$$iV_{ij\mu}^{aIJ} = (ig)T_{ij}^a \gamma_\mu \delta^{IJ}$$

the reason up stair ν become down stair μ is in part due to $A_\mu^a \leftrightarrow A_\nu^b$, in Feynman gauge, it becomes $\eta_{\mu\nu}$. Why is δ^{IJ}? Because we're in color $SU(N)$. it has nothing to do with flavor $SU(N)$, that is flavor structure is completely independent of color.

4.8. Renormalization & QCD Beta Functions

The trick is to relate bare fields & couplings to the renormalized ones.

Using the first term in (4.45)

$$A_{bare}^{a\mu} = Z_3^{1/2} A^{a\mu}$$

using the first term in (4.48),

$$\Psi_{bare} = Z_2^{1/2}\Psi$$

therefore combining the two above and the last term in (4.48), we get

$$g_{bare}^2 = g^2 \frac{Z_1^2}{Z_2^2 Z_3}\tilde{\mu}^\epsilon \tag{4.49}$$

We can also work with (4.47),

$$g_{bare}^2 = g^2 \frac{Z_1'^2}{Z_2'^2 Z_3} \tilde{\mu}^\epsilon$$

or use (4.46)

$$g_{bare}^2 = g^2 \frac{Z_{3g}^2}{Z_3^3} \tilde{\mu}^\epsilon = g^2 \frac{Z_{4g}}{Z_3^2} \tilde{\mu}^\epsilon$$

Let's work on (4.49). Since g_{bare} independent of $\tilde{\mu}$, we can work out how g depends on $\tilde{\mu}$ if we know how Z_1,Z_2,Z_3 depend on g.

Let's do Z_2, first term in (4.48). We will use quadratic Casimir $C(R)$, defined in problem set 6 problem 1.

$$(T_R^a T_R^a)_{ij} = \delta_{ij} C(R) \tag{4.50}$$

See Srednicki figure 73.1 on page 440 for one-loop gluon correction and counterterm to quark free propagator.

$$i\Sigma_{ij} = i\Sigma_{counter} + i\Sigma_{loop}$$

The counterterm is usual

$$i(Z_2 - 1)\bar{\Psi}_i \partial\!\!\!/ \Psi_i - m(Z_m - 1)\bar{\Psi}_i \Psi_i$$

thus

$$i\Sigma_{counter} = ii(Z_2 - 1)\gamma^\mu (ip_\mu)\delta_{ij} - im(Z_m - 1)\delta_{ij}$$

Loop

$$\left\langle \Psi_i \bar{\Psi}_j \frac{(ig)^2}{\cancel{2}!} \int A_\mu^a \bar{\Psi}_m \gamma^\mu T_{mn}^a \Psi_n \int A_\nu^b \bar{\Psi}_k \gamma^\nu T_{kq}^b \Psi_q \right\rangle_{free} \tilde{\mu}^\epsilon$$

there are two possible pairing, so the 1/2 in front cancels. $\Psi_i \to \bar{\Psi}_m$ or $\Psi_i \to \bar{\Psi}_k$. Thus using Feynman gauge

$$i\Sigma_{loop} = \tilde{\mu}^\epsilon \frac{(ig)^2}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{\delta^{ab}\eta_{\mu\nu}}{l^2 - i\epsilon} \gamma^\mu \frac{[-(p\!\!\!/ + l\!\!\!/) + m]}{(p+l)^2 + m^2 - i\epsilon} \gamma^\nu T_{in}^a T_{nj}^b$$

The last TT can be simplified by (4.50). And use the old Feynman trick

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}$$

to isolate l and p to get logarithmic divergence, so

$$\frac{1}{(l^2 - i\epsilon)((p+l)^2 + m^2 - i\epsilon)} = \int_0^1 dx \frac{1}{[\underbrace{(l-xp)^2}_{\equiv q^2} + \underbrace{x(1-x)^2 p + xm^2 - i\epsilon}_{\equiv D}]^2}$$

thus

$$i\Sigma_{loop} = \tilde{\mu}^\epsilon g^2 C(R)\delta_{ij} \int_0^1 dx \underbrace{\int \frac{d^d q}{(2\pi)^d} \frac{N}{(q^2+D)^2}}_{\downarrow \, \frac{1}{\epsilon}}$$

where

$$\begin{aligned} N &= \gamma^\mu[-(\not{p} + \not{l}) + m]\gamma_\mu \\ &= -(d-2)(\underbrace{\not{p} + \not{l}}_{\not{q} + (1-x)\not{p}}) - dm \end{aligned}$$

The dm part gives logarithmic divergence, and the $\not{p} + \not{l}$ part is not divergence so adding some finite term. In $\overline{MS}$, we don't care them, therefore the counter term must cancel divergence

$$Z_2 = 1 - \frac{g^2}{8\pi^2} C(R) \frac{1}{\epsilon}$$

Likewise we can find Z_1.

$$Z_1 = 1 - \frac{g^2}{8\pi^2}(C(R) + T(A))\frac{1}{\epsilon}$$

Starting from last term in (4.48), we add gluon vertex correction, see Feynman diagram, Srednicki figure 73.2 on page 440.The second figure of 73.2 is a cubic vertex self gluon interaction, and this cubic vertex is due to $f^{abc} \neq 0$. And f^{abc} is related to adjoint representation. This explains why $T(A)$ appears in Z_1, $T(R)$ is defined as

$$\mathrm{Tr}(T^a(R)T^b(R)) = T(R)\delta^{ab}$$

Lastly we find Z_3. There are 4 Feynman diagrams (one quark loop, one ghost loop, and two gluon loops). All are at g^2 level. See Srednicki figure 73.3 on page 443.

$$Z_3 = 1 + \frac{g^2}{8\pi^2}\left(\frac{5}{3}T(A) - \frac{4}{3}n_F T(R)\right)\frac{1}{\epsilon}$$

n_F =number of quark flavors. Why is there n_F? Because in the last term of (4.48), there are 8 gluons for each flavors.

Plug $Z_{1,2,3}$ back in (4.49), and demand g_{bare} independent of $\tilde{\mu}$, let

$$\alpha = \frac{g^2}{4\pi}$$

same definition of coupling constant for QED. We find β function, (since $d\ln\mu = d\ln\tilde{\mu}$)

$$\begin{aligned}\beta(\alpha): &= \frac{d\alpha}{d\ln\mu} \\ &= \frac{\alpha^2}{2\pi}\left(\frac{11}{3}T(A) - \frac{4}{3}n_F T(R)\right)\end{aligned}$$

For $SU(3)$, quarks in fundamental representation: $T(R) = \frac{1}{2}$, $T(A) = 3$, see problem set 6 problem 2,

$$\beta(\alpha) < 0 \text{ as long as } n_F < 33/2$$

this shows QCD is asymptotically free. However at low energy the coupling is hard to solve perturbatively, so people use lattice theory.

This calculation was famously done by 2 graduate students and 1 professor. 30 years later, 2004, they got nobel prize.

5. Standard Model & Beyond

5.1. Spontaneous Symmetry Breaking

Abelian

Lecture 24 (11/26/14)

Recall we did last semester, with this semester's covariant derivative and F^2 field

$$\mathcal{L} = -D^\mu \phi^+ D_\mu \phi - V(\phi) - \frac{1}{4}F^2$$

ϕ =complex scalar.

$$D_\mu = \partial_\mu - igA_\mu \tag{5.1}$$

$$V(\phi) = m^2\phi^+\phi + \frac{1}{4}\lambda(\phi^+\phi)^2$$

with $m^2 < 0$, $\lambda > 0$. Minimum at $|\phi| = v/\sqrt{2}$ with $v = \sqrt{4\,|m^2|\,/\lambda}$.

This theory has global $U(1)$. Also it has gauged $U(1)$

$$\phi(x) \to e^{-ig\Lambda(x)}\phi(x) \text{ with } A_\mu(x) \to A_\mu(x) - \partial_\mu\Lambda(x) \tag{5.2}$$

If we pick a vacuum state:

$$\langle\phi\rangle = v/\sqrt{2}$$

i.e. on the real axis. Recall last semester we said global $U(1)$ acting on $\langle\phi\rangle$ change to different vacuum, i.e. $\langle\phi\rangle$ is not invariant under global $U(1)$. This was called spontaneous symmetry breaking

We also did

$$\phi(x) = \langle\phi\rangle + \text{fluctuations} = \frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v}$$

ρ =radial fluctuation, χ =phase/angle fluctuation.

If $U(1)$ is not gauged, $\chi(x)$ is the Goldstone ghost, i.e. $V(\phi)$ gives no mass term for χ. If $U(1)$ is gauged, pick Λ of (5.2) so that $\chi(x) = 0$, called unitary gauge. In the literature, people say massless A_μ eats the Goldstone and becomes massive. Let's how it gain mass.

$$
\begin{aligned}
-D^\mu\phi^+ D_\mu\phi &= -\frac{1}{2}(\partial^\mu\rho + ig(v+\rho)A^\mu)(\partial_\mu\rho - ig(v+\rho)A_\mu) \\
&= -\frac{1}{2}\partial^\mu\rho\partial_\mu\rho - \underbrace{\frac{1}{2}g^2(v+\rho)^2A^\mu A_\mu}
\end{aligned}
\tag{5.3}
$$

This shows A^μ has mass gv, and also is coupled to ρA^2, $\rho^2 A^2$. The $V(\phi)$ is function of ρ only, it gives mass ρ and self interaction. F^2 terms are usually kinetic of A^μ.

The similar SSB theory was developed to explain superconductor. Photon becomes massive, and it repels magnet.

Let's count dof. Before SSB

$$
\begin{aligned}
1 \text{ complex } \phi &\rightarrow 2 \text{ dof} \\
1 \text{ massless } A_\mu &\rightarrow 2 \text{ dof}
\end{aligned}
$$

so total 4 dofs. After SSB

$$
\begin{aligned}
1 \text{ real } \rho &\rightarrow 1 \text{ dof} \\
1 \text{ massive } A_\mu &\rightarrow 3 \text{ dof}
\end{aligned}
$$

so still total 4 dofs.

Non-abelian

ϕ has multiple components & A_μ is a matrix, D_μ is (5.1).

$$
\mathcal{L} = -\left(\quad\right) D^\mu\phi^+ D_\mu\phi - V(\phi) - \frac{1}{4}F^2
$$

For $SO(N)$, $\phi's$ are real, and $\frac{1}{2}$ in front of $D^\mu\phi^+ D_\mu\phi$. For $SN(N)$, $\phi's$ are complex, no need for $\frac{1}{2}$ in front.

SO(N)

First we do $SO(N)$, where the idea of broken and unbroken generators is more intuitive.

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

$\phi_i' s$ are real scalar fields, lie in the fundamental of $SO(N)$. Suppose $V(\phi)$ is not like $V(\left|\vec{\phi}\right|)$ that respects $SO(N)$ symmetry, but it takes a form that its minimum value $\langle\phi\rangle$ of ϕ breaks $SO(N)$. Assume

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \tag{5.4}$$

$v \neq 0$. Recall generators of $SO(N)$ are (4.3). What are the broken and unbroken generators, i.e. $\langle\phi\rangle \to e^{-i\theta^a T^a}\langle\phi\rangle = \langle\phi\rangle$ and $\neq \langle\phi\rangle$ respectively? Since

$$e^{-i\theta^a T^a}\langle\phi\rangle = (1 - i\theta^a T^a)\langle\phi\rangle$$

unbroken T^a means $\langle\phi\rangle$ is invariant under $SO(N)$ generated by T^a, thus

$$\begin{aligned} \text{broken generators} \quad &\Longleftrightarrow \quad T^a\langle\phi\rangle \neq 0 \\ \text{unbroken generators} \quad &\Longleftrightarrow \quad T^a\langle\phi\rangle = 0 \end{aligned}$$

So clearly from (5.4)

$$\text{unbroken generators} = \begin{pmatrix} \boxed{\phantom{\begin{matrix} x & x \\ x & x \end{matrix}}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}$$

have no elements on the last column and row, so there are

$$\frac{(N-1)(N-2)}{2}$$

of them, they are the generators of $SO(N-1)$ fundamental representation.

$$\text{broken generators} = \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} i \\ \\ \end{matrix} \\ \begin{matrix} & -i \end{matrix} & 0 \end{pmatrix}, etc$$

so there are $N-1$ broken generators.

E.g. $SO(3) \to SO(2)$

$$\text{unbroken generators} = J^z \ \text{broken generators} = J^x, J^y \tag{5.5}$$

Next we show unbroken generators give massless gauge bosons, broken give massive gauge bosons. Because

$$-\frac{1}{2}D^\mu \phi^+ D_\mu \phi \ = \ -\frac{1}{2}[(\partial^\mu - igA^{a\mu}T^a)\phi]^+[(\partial_\mu - igA^b_\mu T^b)\phi]$$

similar to (5.3), put in fluctuation around $\langle\phi\rangle$. The mass of gauge bosons come from the last term,

$$-\frac{1}{2}A^{a\mu}A^b_\mu \underbrace{g^2 \langle\phi\rangle^+ T^{a+}T^b \langle\phi\rangle}_{:=(M^2)^{ab}}$$

It turns out

$$(M^2)^{ab} = \begin{cases} g^2v^2\delta^{ab} & a \in \text{broken} \\ 0 & a, b \in \text{unbroken} \end{cases}$$

check

$$g^2 \begin{pmatrix} 0 & \cdots & v \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} -i \\ \\ \end{matrix} \\ \begin{matrix} & i \end{matrix} & 0 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} i \\ \\ \end{matrix} \\ \begin{matrix} & -i \end{matrix} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ v \end{pmatrix} = g^2v^2$$

so all massive gauge bosons share the same mass.

Let's count dofs for $SO(3) \to SO(2)$, (5.3). Before SSB

$$\begin{aligned}
3 \text{ real } \phi_i &\to 3 \text{ dof} \\
3 \text{ massless } A_\mu &\to 3 \times 2 = 6\text{dof}
\end{aligned}$$

so total 9 dofs. After SSB

$$\begin{aligned}
1 \text{ real scalar} &\to 1 \text{ dof} \\
1 \text{ massless } A_\mu &\to 2\text{dof} \\
2 \text{ massive } A_\mu &\to 6 \text{ dof}
\end{aligned}$$

so still total 9 dofs.

SU(N)

Let's do the same for $SU(N)$

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

$\phi_i's$ are complex scalar fields, lie in the fundamental of $SU(N)$. This is not always the case. For Higgs it longs to adjoint representation. cf discussion surround (5.7). But for now let's consider $SU(N) \to SU(N-1)$, and pick

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ \vdots \\ v \end{pmatrix}$$

v is real. One can follow the same derivation above and use (4.6) to find that

$$\text{broken generators} = \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \\ & -n \end{pmatrix}$$

so there are $2(N-1)+1$. Another way to see it, by (4.7), there are N^2-1 generator of $SU(N)$ and there are $(N-1)^2-1$ number of generators of $SU(N-1)$, so broken generators

$$N^2 - 1 - [(N-1)^2 - 1] = 2(N-1) + 1$$

One can then find the mass matrix $(M^2)^{ab}$, the $2(N-1)$ forms representation of a smaller $SU(\tilde{N})$ and they have same mass. The 1 forms a singlet, it has different mass than $2(N-1)$. E.g.

$$SU(5) - \underbrace{SU(4)}_{\text{unbroken } SU(5)} \to \underbrace{SU(3) \times U(1)}_{\text{broken } SU(5)}$$

SU(5) with a Special Vacuum

Lecture 25 (12/1/14)

There is an alternative formulation to the usual

$$D_\mu \phi^a = (\partial_\mu - igA_\mu^b T^b)^{ac}\phi^c \qquad \phi \to U\phi$$

we define $\Phi = \phi^a T^a$, then

$$D_\mu \Phi = \partial_\mu \Phi - ig[A_\mu^b T^b, \Phi] \qquad \Phi \to U\Phi U^{-1} \approx \Phi - ig[\theta^b T^b, \Phi] \tag{5.6}$$

If we use this formulation, Φ is a matrix. Cf problem set 7 problem 1.

Consider a theory with $SU(5)$ gauge symmetry, this also means that it has $SU(5)$ global symmetry (let $\lambda(x) = \text{const}$). Suppose the vacuum is

$$\langle \Phi \rangle = N \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix} \tag{5.7}$$

N some real number. $\langle \Phi \rangle$ is traceless as it should be. What are the unbroken generators? Previously (un)broken is found by $T^a \langle \Phi \rangle = 0$ or not. Now by (5.6).

$$\text{unbroken generators } T^a \iff [T^a, \langle \Phi \rangle] = 0 \tag{5.8}$$

so clearly

$$\text{unbroken generators}=\begin{pmatrix} \boxed{\quad SU(3) \quad} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \boxed{\quad SU(2) \quad} \end{pmatrix} \otimes \underbrace{\langle \Phi \rangle}_{U(1)} \tag{5.9}$$

what about others e.g. $\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3 & \\ & & & & 0 \end{pmatrix}$? It looks like satisfies (5.8). But we don't include it, for answer see problem set 7 problem 2.

Please read Srednicki ch85,86 for quantization SSB YM.

5.2. Higgs

We study electroweak, so not to include gluons $SU(3)$. We want to show

$$SU(2) \times U(1)_Y \to U(1)_{EM} \tag{5.10}$$

the $U(1)_Y$ on the left is the $U(1)$ in (5.9), which is part of SSB of Higgs ϕ. The $U(1)_{EM}$ on the right is the $U(1)$ EM gauge we studied early. From (5.10) we get

$$D_\mu \phi = \left(\partial_\mu - i[g_2 A_\mu^a T^a + g_1 B_\mu Y]\right) \phi \tag{5.11}$$

where $T^a = \sigma^a/2$ and $Y = -\frac{1}{2}I$, here Higgs ϕ is 2 component complex scalar fields (4 dofs). Choose such Y is to refer to hypercharge. A_μ^a & B_μ are the gauge bosons which permeate forces. People often denote the representation of $SU(2) \times U(1)$ that ϕ belongs to as

$$\phi \in (2, -\frac{1}{2})$$

2 is for 2 dimensional fundamental representation, $\sigma^a/2$, of $SU(2)$, and $-\frac{1}{2}$ is the hypercharge of $U(1)$.

Higgs Lagrangian

$$\mathcal{L} = \underbrace{-D^\mu \phi^+ D_\mu \phi - V(\phi)}_{\mathcal{L}_{Higgs}} \underbrace{- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}}_{\mathcal{L}_{gauge}} \tag{5.12}$$

where

$$V(\phi) = \frac{1}{4}\lambda(\phi^+\phi - \frac{1}{2}v^2)^2 \tag{5.13}$$
$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

Later we will add in leptons and quarks.

Use the global $SU(2) \times U(1)$ to rotate vacuum into this form

$$\langle\phi\rangle = \frac{1}{\sqrt{2}} v \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.14}$$

v =real.

What are the unbroken generators?

$$\begin{pmatrix} 0 & 0 \\ 0 & anything \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

The only one is

$$(\underbrace{T^3 + Y}_{U(1)_{EM}}) \langle\phi\rangle = 0$$

Plugging (5.11) into (5.12), we get gauge bosons mass

$$\mathcal{L}_{mass} = -\frac{1}{8} v^2 g_2^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} A^3_\mu - \frac{g_1}{g_2} B_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -(A^3_\mu + \frac{g_1}{g_2} B_\mu) \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

the square denotes matrix multiplication with one μ up and one μ down. To diagonalize the matrix, we define

$$\begin{aligned} W^\pm_\mu &= A^1_\mu \mp iA^2_\mu \\ Z_\mu &= \cos\theta_w A^3_\mu - \sin\theta_w B_\mu \\ A_\mu &= \sin\theta_w A^3_\mu + \cos\theta_w B_\mu \end{aligned}$$

In this way we will also see that $W^{\pm}$ have charges and Z, A are neutral. The A_μ turns out to have 0 mass, it is photon.

$$\mathcal{L}_{mass} = -\frac{1}{2}M_Z^2 Z^\mu Z_\mu - M_W^2 W^{+\mu}W_{-\mu}$$

where θ_w is electroweak or Weinberg angle.

$$\tan\theta_w = \frac{g_1}{g_2} \quad M_W = \frac{g_2 v}{2} \quad M_Z = \frac{M_w}{\cos\theta_w}$$

experiment says $\cos\theta_w \sim 0.8 - 0.9$

To get the mass of Higgs, we add fluctuation to the Higgs field, from (5.14), we write

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

H =real, Higgs fluctuation. Then (5.13) gives

$$V(\phi) = \frac{1}{4}\lambda v^2 H^2 + \frac{1}{4}\lambda v H^3 + \frac{1}{16}\lambda H^4$$

hence

$$m_H = \sqrt{\frac{\lambda}{2}}v$$

(5.12) gives coupling of H to the gauge fields. One can also rewrite (5.12) using $W^\pm$, Z, & A. One interesting result is that $W_\mu^\pm$ has kinetic term, comes with $D_\nu W_\mu^\pm$, with

$$D_\nu = \partial_\nu - ig_2(\cos\theta_w Z_\mu + \sin\theta_w A_\mu)$$

implies

$$e = g_2 \sin\theta_w \tag{5.15}$$

5.3. Leptons

Lecture 26 (12/3/14)

Last time we worked out mass of electroweak gauge bosons, use no Higgs mechanism, now we want to get leptons mass. The kinetic term is similar to last semester qft1 note equation 6.25 on page 106.

$$\mathcal{L}_{leptons} = \underbrace{il^+\bar{\sigma}^\mu D_\mu l + i\bar{e}^+\bar{\sigma}^\mu D_\mu \bar{e}}_{\mathcal{L}_{kinetic}} \underbrace{-y\epsilon^{ij}\phi_i l_j \bar{e} + h.c.}_{\mathcal{L}_{Yukawa}(cf\ (1.18))} \tag{5.16}$$

where y =real number, $\epsilon^{ij} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ is the rising operator in (4.29). l =lepton is singlet under $SU(3)$ and $SU(2)$ doublet

$$l = \begin{pmatrix} \nu \\ e \end{pmatrix} \in (1, 2, -\frac{1}{2})$$

where ν, e are left-handed weyl spinors. Just like Higgs ϕ, here 2 means in the fundamental representation of $SU(2)$, $-\frac{1}{2}$ is hypercharge. $\bar{e}$ is related to anti electron, it doesn't mean

$$\bar{e} = e^+\gamma^0$$

because we are not in Dirac spinor. And

$$\bar{e} \in (1, 1, 1)$$

it is $SU(2)$ singlet, and has hypercharge 1.

Question, why not put in an explicitly mass term in (5.16)? What is wrong with ll or $\bar{e}\bar{e}$? Total hypercharge ll is -1; total hypercharge $\bar{e}\bar{e}$ is 2, not 0. Hence such terms are not invariant under $U(1)_Y$. Check $\phi l \bar{e}$ is good.

So hypercharge forbids putting in explicit mass terms, so any mass has to come from SSB via Higgs. That's what people say Higgs give leptons masses.

Let's use unitary gauge

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix} \tag{5.17}$$

then

$$\mathcal{L}_{Yukawa} = -y\frac{1}{\sqrt{2}}(v + H)[e\bar{e} + (e\bar{e})^+]$$

hence neutrino gets no mass. To see e is electron, let's use 4 component Dirac spinor, 2 for electron 2 for anti-electron, cf page 110 qft1 notes

$$\mathcal{E} = \begin{pmatrix} e \\ \epsilon\bar{e}^{+} \end{pmatrix}$$

so

$$\mathcal{L}_{Yukawa} = -y\frac{1}{\sqrt{2}}(v+H)\underbrace{\mathcal{E}^{+}\gamma^{0}}_{\bar{\mathcal{E}}}\mathcal{E} \tag{5.18}$$

so mass of electron $yv/\sqrt{2}$. Let's their EM charges.

$$Q_{EM}l = (\frac{T^3}{2}+Y)l = \left[\frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right] l = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} l$$

so ν has EM charge 0 and e has -1. Likewise

$$Q_{EM}\bar{e} = (0+1)\bar{e}$$

has charge 1.

We now show in (5.16)

$$\begin{aligned} \mathcal{L}_{kinetic} &= ie^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}e + i\bar{e}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\bar{e} + i\nu^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\nu \\ &\quad + g_2 W_{\mu}^{+}\underbrace{\nu^{\dagger}\bar{\sigma}^{\mu}e}_{J_{-}^{\mu}} + g_2 W_{\mu}^{-}\underbrace{\bar{e}^{\dagger}\bar{\sigma}^{\mu}\nu}_{J_{+}^{\mu}} \\ &\quad + eA_{\mu}[\underbrace{-e^{\dagger}\bar{\sigma}^{\mu}e + \bar{e}^{\dagger}\bar{\sigma}^{\mu}\bar{e}}_{J_{EM}^{\mu}}] \\ &\quad + Z_{\mu}J_{Z}^{\mu} \end{aligned} \tag{5.19}$$

simply expand covariant derivative (5.11)

$$\begin{aligned} g_2 A_{\mu}^{a}T^{a} &= g_2 W_{\mu}^{+}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + g_2 W_{\mu}^{-}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + eA_{\mu}(\frac{T^3}{2}+Y) \\ &\quad + \frac{e}{\sin\theta_w\cos\theta_w}(T^3 - \sin^2\theta_w(T^3+Y))Z_{\mu} \\ g_1 B_{\mu}Y &= e\left(A_{\mu} - \frac{\sin\theta_w}{\cos\theta_w}Z_{\mu}\right)Y \end{aligned}$$

So in particular

$$l^\dagger \bar{\sigma}^\mu D_\mu l \to g_2 l^\dagger W_\mu^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} l \to g_2 W_\mu^+ \underbrace{\nu^\dagger \bar{\sigma}^\mu e}_{J_-^\mu}$$

$J_\pm^\mu$ is called charged current in comparison to J_Z, neutral current. We can check all terms in $\mathcal{L}_{kinetic}$ have total EM charge 0. E.g.

$$\underbrace{W_\mu^+}_{1} \underbrace{\nu^\dagger}_{0} \bar{\sigma}^\mu \underbrace{e}_{-1}$$

Just like the mass Yukawa (5.18), can we put (5.19) in Dirac spinor? Define

$$N_L = \begin{pmatrix} \nu \\ 0 \end{pmatrix}, \quad E_L = \begin{pmatrix} e \\ 0 \end{pmatrix}$$

then

$$\begin{aligned} \mathcal{L}_{kinetic} + \mathcal{L}_{Yukawa} &= i\mathcal{E}^\dagger \gamma^0 \gamma^\mu \partial_\mu \mathcal{E} + (\text{neutrino Majorana kinetic}) + m_e \mathcal{E}^\dagger \gamma^0 \mathcal{E} \\ &\quad + g_2 W_\mu^+ \underbrace{N_L^\dagger \gamma^0 \gamma^\mu E_L}_{J_-^\mu} + g_2 W_\mu^- \underbrace{E_L^\dagger \gamma^0 \gamma^\mu N_L}_{J_+^\mu} \\ &\quad + eA_\mu [\underbrace{-\mathcal{E}^\dagger \gamma^0 \mathcal{E}}_{J_{EM}^\mu}] \\ &\quad + Z_\mu J_Z^\mu \end{aligned}$$

If we don't have neutrino, we should get back QED.

Recall (5.15), g_2 and e are not to different, so why photon scattering is so much stronger. That is because don't forget $W^\pm$, A in front. So the propagator

$$\tilde{A} = \frac{1}{k^2 - i\epsilon} \tilde{W} = \frac{1}{k^2 + M_W^2 - i\epsilon}$$

so at low energies, they are very different. At high energy however $k \gg M_W$, two scatterings approach the same.

Lastly we want to put in 3 generations of fermions

$$\begin{aligned} l &= \begin{pmatrix} \nu_e \\ e \end{pmatrix} & \bar{e} \\ l &= \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} & \bar{\mu} \\ l &= \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} & \bar{\tau} \end{aligned}$$

Change (5.16)

$$\mathcal{L}_{leptons} = il_I^+ \bar{\sigma}^\mu D_\mu l_I + i\bar{e}_I^+ \bar{\sigma}^\mu D_\mu \bar{e}_I \underbrace{-\epsilon^{ij}\phi_i l_{jI} y_{IJ} \bar{e}_J + h.c.}_{\text{Yukawa}} \tag{5.20}$$

we put in flavor index. We can safely assume flavor is diagonal in kinetic term, since it is true for the free theory. But no reason to assume that in the Yukawa, interacting with the Higgs, so y number becomes y_{IJ} matrix.

But y_{IJ} is diagonalizable. As we will see, this is not true for the quarks.

There is a math theorem helping us. We can transform: unitary rotation

$$\begin{aligned} l_I &\to \mathbb{L}_{IJ} l_J \\ \bar{e}_I &\to \bar{\mathbb{E}}_{IJ} \bar{e}_J \end{aligned}$$

this will keeps kinetic term diagonal because of l^+l, $\bar{e}^+\bar{e}$. By choosing good $\mathbb{L}$, $\bar{\mathbb{E}}$, one can always diagonalize y_{IJ}

$$y_{IJ} \to \mathbb{L}^T y_{IJ} \bar{\mathbb{E}}$$

thus no mixing among lepton of different flavors. This is supported by the assumption as we did in the derivation, assuming neutrino has no mass.

5.4. Quarks

Similar to before, still 2 components, doublet q

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \quad \bar{u} \quad \bar{d}$$

before ν has no anti-neutrino.

$$q \in (3, 2, \frac{1}{6}) \quad \bar{u} \in (\bar{3}, 1, -\frac{2}{3}) \quad \bar{d} \in (\bar{3}, 1, \frac{1}{3})$$

Lecture 27 -Last Lec- (12/8/14)

Very thing is similar to (5.20),

$$\mathcal{L}_{\text{Yukawa}} = -\epsilon^{ij}\phi_i q_{j\alpha} y' \bar{d}^\alpha - \epsilon^{ij}\phi_i^+ q_{j\alpha} y'' \bar{u}^\alpha + h.c. \tag{5.21}$$

note: ij are $SU(2)$ indices, α is $SU(3)$ index which couples to gluons. Now both terms will contribute to masses, and u, d will both get masses. Also note the second term contains ϕ^+ because of hypercharge conservation.

We can then introduce flavors, eigenstates of kinetic terms, like in (5.20),

$$\mathcal{L}_{\text{Yukawa}} = -\epsilon^{ij}\phi_i q_{j\alpha I} y'_{IJ} \bar{d}^\alpha_J - \phi^{+j} q_{j\alpha I} y''_{IJ} \bar{u}^\alpha_J + h.c.$$

now not possible to diagonalize both terms simultaneously. This gives he famous CKM matrix. What does it mean physically?

$$\begin{array}{ccc} & & u \\ & \nearrow & \\ \dashrightarrow & & \\ W & \searrow & \\ & & ? \end{array}$$

there will be mixing among generations/flavors, so e.g. in the above vertex, not just d, but other generation may enter.

5.5. Massive Neutrinos

It is not hard to extent SM to include massive neutrinos. But this is now all speculation, we will follow one option. It may or may not be correct. There are other options, like saying Higgs doesn't lie in fundamental $SU(5)$.

Use the same idea for quark, we add $\bar{\nu}$,

$$l = \begin{pmatrix} \nu \\ e \end{pmatrix} \quad \bar{e} \quad \bar{\nu} \in (1, 1, Y = ?)$$

At the moment we don't know what $\bar{\nu}$ is? It should be singlet under $SU(3)$, $SU(2)$, just like $\bar{e}$, what about its hypercharge? We assume we have same form of the second term in (5.21), because we pretend $\bar{u} \leftrightarrow \bar{\nu}$

$$\mathcal{L}_{\text{Yukawa}} = ... - \phi^{+i} l_i \tilde{y} \bar{\nu} + h.c. \tag{5.22}$$

then we get the same quark model. What is the hypercharge? $Y = 0$. Hence

$$\bar{\nu} = (1, 1, 0)$$

is sterile. It doesn't participate in any of the $SU(3), SU(2), U(1)_Y$ interaction except it enters the Yukawa term (5.22) to gain mass.

So if we choose the same unitary gauge (5.17), we get

$$\mathcal{L}_{Yukawa} = ... + -\tilde{y}\frac{1}{\sqrt{2}}(v + H)[\nu\bar{\nu} + (\nu\bar{\nu})^{+}]$$

hence ν gains mass $\tilde{m} = \frac{1}{\sqrt{2}} v \tilde{y}$. We know ν mass is very small. Instead of just blaming $\tilde{y}$ is so small, there is a better explanation.

There is another term can be added to the $\mathcal{L}$,

$$-\frac{1}{2} M(\bar{\nu}\bar{\nu} + \bar{\nu}^{+}\bar{\nu}^{+})$$

previously hypercharge forbid us to do so for leptons and quarks, but $\bar{\nu}$ has 0 hypercharge. Therefore we should combine the two mass terms

$$-\frac{1}{2}\begin{pmatrix} \nu & \bar{\nu} \end{pmatrix}\begin{pmatrix} 0 & \tilde{m} \\ \tilde{m} & M \end{pmatrix}\begin{pmatrix} \nu \\ \bar{\nu} \end{pmatrix} + h.c.$$

assuming $M \gg \tilde{m}$ then two eigenvalues are

$$M \qquad \frac{\tilde{m}^2}{M}$$

E.g. $\tilde{m}$ =MeV (similar to e^-), M =TeV (E&W scale),

$$\frac{\tilde{m}^2}{M} = 1\text{eV}$$

sounds right. This is known as seesaw mechanism.

Also note that in this way we will get lepton generation mixing too.

5.6. Solitons

1+1 Dimension

i.e. 1 spatial dimension. Consider a real scalar field ϕ in a double well potential

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \underbrace{\frac{1}{8}\lambda(\phi^2 - v^2)^2}_{V(\phi)} \tag{5.23}$$

it has global discrete symmetry $\phi \to -\phi$. Choose one of two minimum energy states $\pm v$ to be the vacuum and add fluctuation

$$\phi = v + \delta\phi$$

plugging in V,

$$V(\phi) = \frac{\lambda}{2}v^2\delta\phi^2 + \frac{\lambda}{2}v\delta\phi^3 + \frac{\lambda}{8}\delta\phi^4 \tag{5.24}$$

so the fluctuation has mass $\sqrt{\lambda}v$. Let's look at another solution, called kink solution ϕ_{kink}

$$\phi_{kink}(x \to \infty) = v \qquad \phi_{kink}(x \to -\infty) = -v$$

around $x = x_0$, ϕ_{kink} jumps. This is not a vacuum state, but this is a stable state that will not decay to vacuum.

Let's find the analytic form. Assume it is static

$$\frac{\partial \phi}{\partial t} = 0$$

EOM of (5.23) implies

$$\Box \phi = \frac{\partial V}{\partial \phi} \text{ or } \partial_x \phi \partial_x (\partial_x \phi) = \partial_x \phi \frac{\partial V}{\partial \phi}$$

that is

$$\partial_x \left(\frac{1}{2}(\partial_x \phi)^2 - V\right) = 0$$

or

$$\frac{1}{2}(\partial_x \phi)^2 - V = const := 0 \tag{5.25}$$

since $\partial_x \phi$, $V \to 0$ at $\pm\infty$, so $const = 0$. Therefore

$$\phi = v \tanh \frac{1}{2} m (x - x_0) \tag{5.26}$$

so the width of the kink is $1/m$, so we say the soliton is localized around x_0 with uncertainty $1/m$. This agrees the Compton wavelength$= h/mc$.

Now show why it is stable. We find the energy is set at local minimum

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx \frac{1}{2} \underbrace{(\partial_t \phi)^2}_{0} + \frac{1}{2}(\partial_x \phi)^2 + V(\phi) \\ &= \int dx \frac{1}{2}(\partial_x \phi - \sqrt{2V})^2 + \sqrt{2V}\partial_x \phi \\ &= \int dx \frac{1}{2}(\underbrace{\partial_x \phi - \sqrt{2V}}_{0\ (5.25)})^2 + \int_{-v}^{v} d\phi \sqrt{2V(\phi)} \qquad (5.27) \\ &= \frac{2}{3}\frac{m^3}{\lambda} = \text{finite} \qquad (5.28) \end{aligned}$$

so we see from line (5.27), the second term is independent of form of $\phi(x)$, so by choosing (5.25), we get a local minimum energy. Check that the dimension of the coupling λ work out fine in (5.28).

In sum, we see that soliton is a stable configuration, i.e. fluctuation (perturbation) around the soliton profile costs energy. The mass of soliton $1/\lambda$ is obtained through (5.28). This is not from usual SSB or any perturbatively methods. In fact no perturbatively method will give soliton mass. So soliton is a non-pertubtive object. That makes it so interesting.

2+1 Dimensions

i.e. 2 spatial dimensions. ϕ is complex scalar field with Mexican hat potential

$$\mathcal{L} = -(\partial^{\mu}\phi)^{+}(\partial_{\nu}\phi) - \underbrace{\frac{1}{4}\lambda(|\phi|^2 - v^2)^2}_{V(\phi)}$$

it has $U(1)$ global symmetry.

In $1+1$D we say that soliton solution is some 1D disturbance, localized at e.g. $x_0 = 0$ and the two ends connect to $\pm v$. In $2+1$D we say that soliton is some 2D disk disturbance, localized at the origin, and radially $r \to \infty$, it connects to the S^1_{vacuum}, or symbolically

$$S^1_{r\to\infty} \longrightarrow S^1_{vacuum}$$

we call it a string (Dirac string later for monopoles), the soliton form is

$$\phi(r,\theta) = vf(r)e^{i\alpha(r,\theta)}$$

as $r \to \infty$,

$$f(r) \to 1, \alpha \to n\theta$$

the $n \in \mathbb{Z}$ is called wedding number. That $n \in \mathbb{Z}$ is because we need ϕ be single valued. One will find

$$E = \int r dr d\theta [(\partial^i \phi)^+ (\partial_i \phi)] = \text{infinite}$$

3+1 Dimensions

$\mathcal{L}$ has $SU(2)$ symmetry. This is the monopole.

Unfortunately we have to wait till next semester, qft3, to talk about them.