Electromagnetism

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This is a graduate course, offered in fall 2013 at Columbia University. Course textbook is Jackson's *Classical electrodynamics*. There would be weekly problem sets 25% and one midterm 30% and a three-hour final 45%. No lowest homework grade would be dropped. Office Hours M.W. 1:30-3:00

Contents

1. Microscopic Maxwell Equations and Lorentz Force

1.1. Microscopic Maxwell Equations

Lecture 1 (9/5/12)

Microscopic here means not to average over material, and should not be interpreted at atomic scale.

$$div\ \vec{E} = 4\pi\rho \tag{1.1}$$

$$div\ \vec{B} = 0 \tag{1.2}$$

$$curl\ \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{1.3}$$

$$curl\ \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{1.4}$$

$\vec{E}$ electric field, $\vec{B}$ magnetic induction. ρ = charge density=charge/volume, and $\vec{j}$ = current density=charge flow/(time)(area$\perp$flow).

Not all four equations are equally useful, for given charge density and current density, and to solve for $\vec{E}$, $\vec{B}$ fields, one may only use the first and the last equations.

1.2. Lorentz force law

$$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c} \times \vec{B} \tag{1.5}$$

Lorentz force law is relativistically correct. See Problem Set 1.2, 1.3.

$$\vec{F} = \frac{d\vec{p}}{dt},\ \vec{p} = m\gamma\vec{v}$$

with m rest mass, $\gamma = 1/\sqrt{1-|v|^2/c^2}$. In quantum mechanical interpretation of electromagnetic field, we take $m_{photon} \equiv 0$. We will see this in problem set 1.1.

1.3. Units

In our course, we use Gaussian CGS (centimeter-gram-second) units.

Electrostatic Unit for charges

1 electrostatic unit (esu) is given by speed of light

$$c = 2.9979 \times 10^{10} \frac{\text{coulomb}}{\text{esu}}$$

For example, charge of a proton

$$c_{proton} = 1.60 \times 10^{-19}\text{coulomb} = 1.60 \times 10^{-19}\frac{1}{c}\text{esu} = 4.8 \times 10^{-10}\text{esu}$$

Gauss Unit for $\vec{E}$ electric field, $\vec{B}$ magnetic flux, $\vec{D}$ electric displacement field, $\vec{H}$ magnetic field, $\vec{P}$ polarization, $\vec{M}$ magnetization.

$$1\, Gauss = 30000\frac{V}{m}$$

E-field in Gauss is a large unit. Electric field of air breakdown is about 100 Gauss; B-field in Gauss is a small unit,

$$1\, Gauss = 10^{-4} tesla$$

B-field of saturation of iron is about 10^4 Gauss, and B-field on the surface of the earth is about 0.3 Gauss.

1.4. Comments

1) M. eq are not valid in rotating frame, only valid in inertial frame.

Suppose we have $\vec{B}$ uniform out of the paper (i.e. Magnet N behind paper, S in front of paper), a charge q is moving clockwise circular around the origin O. In the lab frame, L.F.L says

$$\vec{F} = -qvB/c\hat{r} = -q\omega B/c(x\hat{x} + y\hat{y}),$$

where $\hat{r}$ is the unit position vector from O to the charge q. In the rotating frame (prime frame), q doesn't move, but q experience $\vec{E}'$ field due to moving $\vec{B}$, so

$\vec{F'} = q\vec{E'}$ and $\vec{F'} = \vec{F}$, therefore

$$div\,\vec{E'} = -div\,\omega B/c(x\hat{x} + y\hat{y}) = -2\omega B/c$$

but we know from M. eq

$$div\,\vec{E'} = 4\pi\rho'$$

where $\rho' = 0$. So we showed M. eq are not valid in rotating frame.

2) Maxwell equations must be supplemented by consistent boundary conditions + consistent initial conditions.

Consider a uniform time independent charge density throughout all of space, then by directional symmetric, no direction is special, i.e. $\vec{E} = 0$, then $div\,\vec{E} = 0$, but we assume $\rho \neq 0$. This is a paradox.

3) Conservation of charge and current

M. eq are coupled pde's (given ρ, $\vec{j}$, to find $\vec{E}$, $\vec{B}$), since PDE requires integrability conditions for solutions to exist.

$$0 = div\,curl\,\vec{B} = \frac{4\pi}{c} div\,\vec{j} + \frac{1}{c}\frac{\partial}{\partial t} div\,\vec{E}$$

This implies given ρ, $\vec{j}$ should satisfy charge/current conservation

$$div\,\vec{j} + \frac{\partial \rho}{\partial t} = 0$$

4) M. eq are somewhat redundant.

Take divergent of curl of $\vec{E}$, one gets

$$0 = div\,curl\,\vec{E} = -\frac{1}{c} div \left(\frac{\partial \vec{B}}{\partial t} \right)$$

This implies already $div\,\vec{B}$ = constant in time, so equation 1.2 acts as just an initial condition.

5) Lorentz force law is incorrect.

In Lorentz force law (1.5), $\vec{E}$ and $\vec{B}$ are due by all charges except q on which is calculated. Ignorance of self field makes L.F.L incomplete, but if one wants to consider self field, then one has great trouble, for self field is singular at q.[1]

6) Consider time-independent M. eq (with suitable time-independent ρ, $\vec{j}$)

We are looking at time-indep $\vec{E}$ and $\vec{B}$ fields, then M. eq become

$$div\,\vec{E} = 4\pi\rho \quad curl\,\vec{E} = 0$$

notice ρ must be time-independent, for $\vec{E}$ is time-indep. This implies $div\,\vec{j} = 0$ by the conservation of charge/current.

$$div\,\vec{B} = 0 \quad curl\,\vec{B} = \frac{4\pi}{c}\vec{j}$$

The equations above describe electrostatics and magnetostatics. They can be solved by Gauss and Stokes theorems.

1.5. Problem Set 1 (due 9/12/12)

1)

Let $\vec{\psi} = \vec{E} + i\vec{B}$. Consider Maxwell equations in vacuum. $\rho = 0$, $\vec{j} = 0$.

(a) Show $curl\ \vec{\psi}$ can be written as a Dirac equation

$$\underset{\sim}{H}_{op}\vec{\psi} = i\hbar\frac{\partial\vec{\psi}}{\partial t}.$$

Comparing it to the full Dirac equation $(c\alpha\cdot\vec{p}_{op} + \beta mc^2) = i\hbar\frac{\partial\vec{\psi}}{\partial t}$, it shows mass of photon is zero.

[1]Feynman-Wheeler theory: use "radiation reaction force" to correct L.F.L, in doing this subtraction

$$\frac{1}{2}\text{[retarded field}-\text{advanced field]}$$

singularities cancel, but to use this method, one has to impose boundary conditions at the limit of the universe to get rid of causality radiation.

(b) Show divergent $\vec{\psi}$ is $\vec{p}_{op} \cdot \vec{\psi} = 0$.
(c) Show that for $\vec{\psi}$ satisfying (a) and (b),

$$\underset{\sim op}{H} \cdot \underset{\sim op}{H} \vec{\psi} = c^2 \vec{p}_{op} \cdot \vec{p}_{op} \vec{\psi}.$$

(d) Let $\underset{\sim x}{S} \equiv \hbar \underset{\sim x}{\alpha}$, $\underset{\sim y}{S} \equiv \hbar \underset{\sim y}{\alpha}$, $\underset{\sim z}{S} \equiv \hbar \underset{\sim z}{\alpha}$. Show

$$[S_x, S_y] = i\hbar S_z$$

and

$$S_x S_x + S_y S_y + S_z S_z = 2\hbar^2 I.$$

Since $2\hbar^2 = s(s+1)\hbar^2$ for $s = 1$ spin particle, we conclude that photon is spin 1 particle.

2)

Let $\vec{E} = E_0 \hat{z}$, $\vec{B} = 0$ and $q = e$. Use relativistic Lorentz force law, to find $\vec{v}(t)$ for some given initial $\vec{v}_0$.

3)

Let $\vec{E} = 0$, $\vec{B} = B_0 \hat{z}$ and $q = e$. Use relativistic Lorentz force law, to find $\vec{v}(t)$ for some given initial $\vec{v}_0$. (This slows even in relativity, magnetic fields do no work.)

Lecture 2
(9/10/12)

1.6. Comments (continued)

7) expansion of $\vec{E} + \vec{B}$ fields

Localized ρ, $\vec{j}$ (compact supported, support of ρ, $\vec{j}$ are inside of a ball of radius a about the origin). Let r be the field point.

- Time-independent far field phenomena: 2 length dimensions r, a. ($r \gg a$). One can expand $\vec{E} + \vec{B}$ as power series in $(\frac{a}{r})$. This is the multipole expansion in electrostatic and magnetostatics.

- Time-dependent phenomena in radiation zone: 3 length dimensions r, a, λ. Typical wavelength $\lambda = cT$, T time interval over which ρ, $\vec{j}$ change, $r \gg a$, λ.
  - $a > \lambda$ This is when T is large, meaning small frequency, small energy. One can use multipole expansion, partial wave analysis.
  - $a < \lambda$ This is high frequency phenomena. One will use Born approximation, Eikonal approximation.

8) Limitation of Fourier Method

If one knows ρ, $\vec{j}$ for all spatial and time, plus boundary condition at spatial infinity, one can use Fourier transform to find $\vec{E}$, $\vec{B}$ everywhere. However, if ρ, $\vec{j}$ are not known in some inaccessible regions, then cannot use Fourier transform to solve for $\vec{E}$, $\vec{B}$. This leads to boundary value problem.

Examples of finding $\vec{E}$, $\vec{B}$ in accessible regions from ρ, $\vec{j}$ in accessible regions plus boundary conditions on the surfaces of the inaccessible regions are electrostatics with conductors, resonant cavities, wave guides, and transmission lines.

9) Postulate magnetic charges and magnetic currents

In vacuum (no charge or current) M. eq

$$div\,\vec{E} = 0$$

$$curl\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$div\,\vec{B} = 0$$

$$curl\,\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

are symmetric under exchanging $\vec{E} \to +\vec{B}$ and $\vec{B} \to -\vec{E}$.

The idea of including magnetic charges and currents is to preserve the symmetry, hence we want M. eq to have form

$$div\,\vec{E} = 4\pi\rho_{el} \tag{1.6}$$

$$curl\,\vec{E} = -\frac{4\pi}{c}\vec{j}_{meg} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1.7}$$

$$div\,\vec{B} = 4\pi\rho_{meg} \tag{1.8}$$

$$curl\,\vec{B} = \frac{4\pi}{c}\vec{j}_{el} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{1.9}$$

This requires when $\vec{E} \to +\vec{B}$ and $\vec{B} \to -\vec{E}$, we should have

$$\rho_{el},\ \vec{j}_{el} \to \rho_{meg},\ \vec{j}_{meg}$$

$$\rho_{meg},\ \vec{j}_{meg} \to -\rho_{el},\ -\vec{j}_{el}.$$

We also require L.F.L to be the same under this symmetry,

$$\vec{F}_{q_{el}} = q_{el}\vec{E} + q_{el}\frac{\vec{v}}{c} \times \vec{B} \tag{1.10}$$

$$\vec{F}_{q_{meg}} = q_{meg}\vec{B} - q_{meg}\frac{\vec{v}}{c} \times \vec{E}. \tag{1.11}$$

Exercise

One can show that the generalized M.eq + L.F.L equations (1.6 - 1.11) are symmetric under the duality transformation:

$$\vec{E} \to \cos\alpha\vec{E} - \sin\alpha\vec{B}$$

$$\vec{B} \to \sin\alpha\vec{E} + \cos\alpha\vec{B}$$

with same for

$$\rho_{el} \to \cos\alpha\rho_{el} - \sin\alpha\rho_{meg}$$

$$\rho_{meg} \to \sin\alpha\rho_{el} + \cos\alpha\rho_{meg}$$

and same for $\vec{j}_{el}$, $\vec{j}_{meg}$. Notice before we have $\alpha = -\pi/2$.

2. Mathematical Background & Undergrad EM Review

2.1. Cartesian Tensor Notation

No distinction between contravariant, covariant components for an orthogonal coordinated system.

For non-orthogonal system, Covariant component of vector $\vec{W}$ in space of basis $\{e_1, e_2\}$ are labeled by subscript, W_1, W_2. They are measured from the origin along e_1, e_2 to the foots of $\vec{W}$ orthogonal projections. Contravariant components are $\vec{W}$ labeled by superscript, W^1, W^2, and they are measured from the origin along e_1, e_2 to the parallel projections.

Notations:

- Index $i = 1, 2, 3$ are orthogonal Cartesian coordinates $x_1 = x$, $x_2 = y$, and $x_3 = z$.
- $\partial_i = \frac{\partial}{\partial x_i}$
- $\hat{e}_i$ unit vector
- Kronecker delta $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
- Levi-Civita symbol ϵ_{ijk} is defined as (1) Completely antisymmetric interchange any 2 indices; (2) $\epsilon_{123} = +1$

Notice this definition is superior than the one using permutations, because this definition works for any dimensions.

- Einstein summation convention. If a letter index repeated twice, than one sums over the index. (One should never get thrice repeated indices)

e.g. some typical definitions: $\nabla^2\phi = \partial_i\partial_i\phi$, $(\overrightarrow{\nabla\phi})_i = \partial_i\phi$, $div\,\vec{A} = \partial_i A_i$, $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$, and $(curl\,\vec{A})_i = \epsilon_{ijk}\partial_j A_k$.

Useful Theorems:

- $\delta_{ij}A_j = A_i$ summing over j

- $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$
- If $S_{ij} = S_{ji}$ (symmetric with respect to i, j) and $A_{ij} = -A_{ji}$, then $S_{ij}A_{ji} = 0$.

Proof: $S_{ij}A_{ij} \overset{\text{treat } i \text{as } j}{=} S_{ji}A_{ji} \overset{\text{properties of } S,\, A}{=} -S_{ij}A_{ij} \implies S_{ij}A_{ji} = 0.$

Examples

1.
$$div\,(\psi\vec{A}) = \partial_i(\psi A_i) = A_i\partial_i\psi + \psi\partial_i A_i = \vec{A}\cdot\overrightarrow{\nabla\psi} + \psi div\,\vec{A}$$

2.
$$\begin{aligned}\left(curl\,(\vec{A}\times\vec{B})\right)_i &= \epsilon_{ijk}\partial_j\left[\left(\vec{A}\times\vec{B}\right)_k\right]\\ &= \epsilon_{ijk}\epsilon_{klm}\partial_j(A_lB_m) = \partial_j\,(A_iB_j) - \partial_j\,(A_jB_i)\\ &= A_i div\,\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)A_i - B_i div\,\vec{A} + \left(\vec{A}\cdot\vec{\nabla}\right)B_i\end{aligned}$$

$$\begin{aligned}curl\,(\vec{A}\times\vec{B}) &= \hat{e}_i\left(curl\,(\vec{A}\times\vec{B})\right)_i\\ &= \vec{A}div\,\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} - \vec{B}div\,\vec{A} + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}\end{aligned}$$

3.
$$div\,curl\,\vec{A} = \partial_i\left(\epsilon_{ijk}\partial_jA_k\right) = \underbrace{\epsilon_{ijk}}_{\text{antisymm}}\underbrace{\partial_i\partial_j}_{\text{symm}}A_k = 0$$

4.
$$\left[curl\,\overrightarrow{\nabla\psi}\right]_i = \underbrace{\epsilon_{ijk}}_{\text{antisymm}}\underbrace{\partial_j\partial_k}_{\text{symm}}\psi = 0$$

Lecture 3 (9/12/12)

Useful results:

1. $\left(curl\,curl\,\vec{A}\right)_i = \epsilon_{ijk}\partial_j\epsilon_{klm}\partial_lA_m = \partial_j\partial_iA_j - \partial_j\partial_jA_i = \partial_i div\,\vec{A} - \nabla^2A_i$

$$curl\,curl\,\vec{A} = \left(curl\,curl\,\vec{A}\right)_i\hat{e}_i = \hat{e}_i\partial_i div\,\vec{A} - \nabla^2\hat{e}_iA_i = \overrightarrow{\nabla div\,\vec{A}} - \nabla^2\vec{A}$$

2. $curl\,grad\,\psi = 0$

3. $div\, curl\, \vec{A} = 0$

4. Given $\vec{W}(x, y, z)$ s.t. $curl\, \vec{W} = 0$ then there exists a non-unique $\psi(\vec{x})$, such that $\vec{W} = -\overrightarrow{\nabla\psi}$.

5. Given $\vec{W}(x, y, z)$ s.t. $div\, \vec{W} = 0$ then there exists a non-unique $\vec{A}(\vec{x})$, such that $\vec{W} = curl\, \vec{A}$.

2.2. Problem Set 2 (due 9/20/12)

1)

Prove the top 11 vector formulas in Jackson's front cover

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \epsilon_{kij} a_i b_j c_k = c_k \epsilon_{kij} a_i b_j = \vec{c} \cdot (\vec{a} \times \vec{b})$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

$\nabla \times \nabla\psi = 0$

$\nabla \cdot (\nabla \times \vec{a}) = 0$

$\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \Delta\vec{a}$

$\nabla \cdot (\psi\vec{a}) = \vec{a} \cdot \nabla\psi - \psi\nabla \cdot \vec{a}$

$\nabla \times (\psi\vec{a}) = \nabla\psi \times \vec{a} + \psi\nabla \times \vec{a}$

$\nabla(\vec{a} \cdot \vec{b}) = \vec{a} \times (\nabla \times \vec{b}) + \vec{b} \times (\nabla \times \vec{a}) + (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a}$

$\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b})$

$\nabla \times (\vec{a} \times \vec{b}) = \vec{a}(\nabla \cdot \vec{b}) + (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b} - \vec{b}(\nabla \cdot \vec{a})$

2)

$curl\, \vec{W}(\vec{x}) = 0$. Let $\phi(\vec{x}) = -\vec{x} \cdot \int_0^1 d\lambda \vec{W}(\lambda\vec{x})$. Show $\vec{W} = -\nabla\phi$. (This proves useful result 4.)

3)

Given $div\, \vec{W} = 0$. Let $A(\vec{x}) = -\vec{x} \times \int \lambda d\lambda \vec{W}(\lambda\vec{x})$. Show $curl\, \vec{A} = \vec{W}$. (This proves useful result 5.)

2.3. Cartesian Tensor Notation (continued)

D'alembert $\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

Apply vector identities to M.eq. We want to decoupled $\vec{E}$, $\vec{B}$, take

$$curl\, curl\, \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t} curl\, \vec{B}$$

we get

$$grad\, div\, \vec{E} - \nabla^2 \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t} curl\, \vec{B}$$

That is

$$4\pi \overrightarrow{\nabla \rho} - \nabla^2 \vec{E} = -\frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

Hence

$$\Box \vec{E} = -4\pi \overrightarrow{\nabla \rho} - \frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t}$$

Take

$$\begin{aligned} curl\, curl\, \vec{B} &= \frac{4\pi}{c} curl\, \vec{j} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} \\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} - \nabla^2 \vec{B} &= \frac{4\pi}{c} curl\, \vec{j} \end{aligned}$$

Hence

$$\Box \vec{B} = \frac{4\pi}{c} curl\, \vec{j}.$$

Use of scalar potential ψ and vector potential $\vec{A}$

Homogeneous M.eq (i.e. zero charge and current)

$$div\, \vec{B} = 0$$

There exists $\vec{A}$, such that

$$\vec{B} = curl\, \vec{A}$$

Then

$$curl\, \vec{E} = -curl\, \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

That is

$$curl\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=0$$

So there exists ψ, such that

$$\vec{E}=-\overrightarrow{\nabla\psi}-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

Note: (1) If one starts from $div\,\vec{E}=0$, the above derivation will be almost the same, for we have showed for zero charge and current M.eq are symmetric under duality transformation. (2) If the field is time-independent,

$$\vec{B}=curl\,\vec{A}$$

$$\vec{E}=-\overrightarrow{\nabla\psi}$$

Gauge Transformation

Because ψ, $\vec{A}$ are not unique, one can have different ψ, $\vec{A}$ and results same $\vec{E}$, $\vec{B}$.

Suppose

$$\vec{B}=curl\,\vec{A}=curl\,\vec{A}'$$

Therefore

$$\vec{A}'=\vec{A}+\overrightarrow{\nabla\Lambda}$$

for any scalar function $\Lambda(\vec{x},t)$. Then

$$\vec{E}=-\overrightarrow{\nabla\psi}-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}=-\overrightarrow{\nabla\psi'}-\frac{1}{c}\frac{\partial\vec{A}'}{\partial t}$$

implies

$$0=\vec{\nabla}\left(\psi-\psi'+\frac{1}{c}\frac{\partial\Lambda}{\partial t}\right)$$

Hence

$$\psi'=\psi+\frac{1}{c}\frac{\partial\Lambda}{\partial t}+f(t)$$

we can absorb $f(t)$ into $\Lambda(\vec{x},t)$, so we derive Gauge Transformation

$$\vec{A}' = \vec{A} + \overrightarrow{\nabla\Lambda} \tag{2.1}$$

$$\psi' = \psi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \tag{2.2}$$

2.4. Problem set 2 (continued)

4)

Consider a moving charge e in a field generated by a magnetic charge g. So

$$\vec{B} = g\frac{\hat{r}}{r^2}$$

(Recall no coefficients since we use Gauss unit.)

The equation of motion is

$$m\frac{d\gamma\vec{v}}{dt} = \frac{eg}{c}\vec{v}\times\frac{\vec{r}}{r^3}$$

(a) show $|\vec{v}|$ is constant in time.

(b) find $r(t)$ with initial condition $r(0) = b$, where b is the closest approach to g.

(c) Consider the angular momentum

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times(m\gamma\vec{v})$$

Show that $\vec{\mathcal{L}} = \vec{L} - \frac{eg}{c}\frac{\vec{r}}{r}$ is a constant of motion.

(d) Choose $z-$axis along $\vec{\mathcal{L}}$, then show θ is constant in time.

In this problem, we make a magnetic mirror. The electric charge travels with constant speed on the surface of a cone. The charge spirals approaching g for $t < 0$, and receding from g for $t > 0$.

Lecture 4 (9/14/12)

2.5. Cartesian Tensor Notation (continued)

Lorentz Gauge Condition[2]

for give $\vec{E}$,$\vec{B}$, one can find ψ,$\vec{A}$ such that

$$div\,\vec{A} + \frac{1}{c}\frac{\partial\psi}{\partial t} = 0. \tag{2.3}$$

The equation above is called the Lorentz Gauge Condition.

Theorem. *For any $\vec{E}$,$\vec{B}$, one can find ψ,$\vec{A}$ satisfying Lorentz Gauge Condition.*

Proof. Suppose ψ,$\vec{A}$ do not obey L.G.C, so

$$div\,\vec{A} + \frac{1}{c}\frac{\partial\psi}{\partial t} = h(\vec{x}, t) \neq 0$$

Must show using gauge transformation through Λ, we will have that

$$div\,\vec{A}' + \frac{1}{c}\frac{\partial\psi'}{\partial t} = 0$$

that is

$$div\,\vec{A} + \nabla^2\Lambda + \frac{1}{c}\frac{\partial\psi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$$

This is equivalent to

$$\Box\Lambda = h$$

This is well known that the inhomogeneous wave equation has solution for any h. $\Box$

Equations for ψ, $\vec{A}$ in Lorentz Gauge
From M.eq 1.4,

$$curl\,curl\,\vec{A} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Then

$$grad\,div\,\vec{A} - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\left[-\overrightarrow{\nabla\psi} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right]$$

[2]According to Jackson, it should belong to Lorenz.

This can be simplified to

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left[div\,\vec{A} + \frac{1}{c}\frac{\partial \psi}{\partial t}\right]$$

This is same as

$$\Box \vec{A} = \frac{4\pi}{c}\vec{j}.$$

Exercise

Similarly, one can show

$$\Box \psi = 4\pi\rho.$$

Proca Theory

In particle physics, when photon has a non-zero mass m, $\vec{E}$, $\vec{B}$ are still used but they are not fundamental. Introduce a constant

$$\mu = \frac{mc}{\hbar}$$

It has unit of 1/length. (recall Compton wavelength h/mc)

Proca Theory

The following two M.eq are still true

$$\begin{cases} div\,\vec{B} = 0 \\ curl\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \end{cases}$$

Because we used them to derive scalar vector potentials, which means the following are still true

$$\begin{cases} \vec{B} = curl\,\vec{A} \\ \vec{E} = -\overrightarrow{\nabla\psi} - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \end{cases}.$$

In Proca theory, Lorentz force law is still true

$$\frac{d\vec{p}}{dt} = \vec{F} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

But the other two M.eq need modifications

$$\begin{cases} div\,\vec{E} + \mu^2\psi = 4\pi\rho \\ curl\,\vec{B} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \end{cases}$$

and ρ, $\vec{j}$ obey charge/current conservation

$$div\,\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

That is because if one takes

$$div\,curl\,\vec{B} + \mu^2 div\,\vec{A} = \frac{4\pi}{c}div\,\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}div\,\vec{E}$$

That is

$$\mu^2\left(div\,\vec{A} + \frac{1}{c}\frac{\partial\psi}{\partial t}\right) = \frac{4\pi}{c}\left(div\,\vec{j} + \frac{\partial\rho}{\partial t}\right).$$

Find Proca Equation for $\vec{A}$, ψ

From $curl\,\vec{B}$,

$$grad\,div\,\vec{A} - \nabla^2\vec{A} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\psi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)$$

That gives

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left(\frac{1}{c}\frac{\partial\psi}{\partial t} + div\,\vec{A}\right)$$

hence

$$\left(\Box + \mu^2\right)\vec{A} = \frac{4\pi}{c}\vec{j}$$

This is the inhomogeneous Klein-Gordon equation.

Exercise

Similarly, one can show

$$(\Box + \mu^2)\psi = 4\pi\rho.$$

2.6. Fourier Transform

Fourier integral representation of $f(x)$,

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} f(x)$$

Theorem. *"Any" complex valued function $f(x)$, can be written*

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikx} F(k)$$

Example. Fourier transform of Gaussian is Gaussian

$$F(k) = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} C e^{-\alpha x^2} = \frac{C}{\sqrt{2\pi}} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{k^2}{4\alpha}}$$

Useful Fact:

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}$$

for Re$\{\alpha\} > 0$, and any $\beta \in \mathbb{C}$.

Properties

(1) $\int \frac{dk}{\sqrt{2\pi}} e^{ikx} F(k) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} G(k)$ a.e. $\implies$ $F(k) = G(k)$ a.e. This is because $\{e^{ikx}\}$ are linearly independent.

Proof. $f(x) = g(x) \implies F(k) = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} f(x) = \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} g(x) = G(k)$ □

(2) Reality Condition $f(x)$ is real, then $F^*(k) = F(-k)$

Proof. $\int \frac{dk}{\sqrt{2\pi}} e^{ikx} F^*(-k) = \int \frac{dk}{\sqrt{2\pi}} e^{-ikx} F^*(k) = f^*(x) = f(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} F(k)$ □

(3) Parseval's Theorem $\int dx f^*(x) g(x) = \int dk F^*(k) G(k)$. In Dirac notation $\langle f|g\rangle_{position\,space} = \langle F|G\rangle_{momentum}$

Proof.

$$\begin{aligned}\int dx f^*(x)g(x) &= \int dx \int \frac{dk}{\sqrt{2\pi}} e^{-ikx} F^*(k) g(x) \\ &= \int dk F^*(k) \int \frac{dx}{\sqrt{2\pi}} e^{-ikx} g(x) = \int dk F^*(k) G(k)\end{aligned}$$

□

(4) Convolution

Given complex-valued $f(x)$, $g(x)$, the convolution of f and g is another function of x,

$$(g \circ f)(x) = \int_{-\infty}^{\infty} dx' g(x - x') f(x')$$

g is called Green function or propagator.

Theorem. $\widetilde{(g \circ f)}(k) = \sqrt{2\pi} G(k) F(k)$

Proof. consider

$$\begin{aligned}(g \circ f)(x) &= \int dx' g(x - x') f(x') = \int dx' \int \frac{dk}{\sqrt{2\pi}} e^{ik(x-x')} G(k) f(x') \\ &= \int \frac{dk}{\sqrt{2\pi}} e^{ikx} G(k) \int dx' e^{-ikx'} f(x') = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} \sqrt{2\pi} G(k) F(k)\end{aligned}$$

and $(g \circ f)(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} \widetilde{(g \circ f)}(k)$, so property (1), we have proved theorem. □

(5) Riemann-Lebesgue Lemma

If $f(x)$ is absolutely integrable, then $|F(k)| \to 0$ as $k \to \pm\infty$. If $F(k)$ is absolutely integrable, then $|f(x)| \to 0$ as $x \to \pm\infty$.

Intuitive Statement: if $F(k)$ is sufficiently smooth and $F(k)$ goes to zero sufficiently fast at infinity, then $|f(x)| \to 0$ as $x \to \pm\infty$. Similar statement can be made for $F(k)$.

Generalized Riemann-Lebesgue lemma

Singularities of $F(k)$ determine the non-zero asymptotic behavior of $f(x)$

e.g. $F(k) = A\delta(k)$ has singularity at $k = 0$.

$$f(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} A\delta(k) = \frac{A}{\sqrt{2\pi}} \nrightarrow 0$$

Lecture 5
(9/19/12)

Fourier Transform of Derivatives and Vector Functions

$$\left.\begin{array}{c} f(x) \\ \frac{df(x)}{dx} \\ \frac{d^2 f(x)}{dx^2} \end{array}\right\} = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikx} \left\{\begin{array}{c} F(k) \\ ikF(k) \\ -k^2 F(k) \end{array}\right.$$

$$\left.\begin{array}{c} F(k) \\ ikF(k) \\ -k^2 F(k) \end{array}\right\} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} \left\{\begin{array}{c} f(x) \\ \frac{df(x)}{dx} \\ \frac{d^2 f(x)}{dx^2} \end{array}\right.$$

In two Fourier Transform above, d/dx, d^2/dx^2 pass through the integral, and the inverse Fourier is using integration by parts. This is one of the reasons why we need boundary conditions at infinity that $f(x) \to 0$ sufficiently fast. If one deals with a wave coming from the infinity, there may be some extra delta function added to the F.T.

$$\left.\begin{array}{c} f(\vec{x}) \\ \overrightarrow{\nabla f(\vec{x})} \\ \nabla^2 f(\vec{x}) \end{array}\right\} = \int_{all\ space} \frac{d^3 k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \left\{\begin{array}{c} F(\vec{x}) \\ i\vec{k}F(\vec{x}) \\ (i\vec{k})^2 F(\vec{x}) \end{array}\right.$$

$$\left.\begin{array}{c} F(\vec{x}) \\ i\vec{k}F(\vec{x}) \\ (i\vec{k})^2 F(\vec{x}) \end{array}\right\} = \int_{all\ space} \frac{d^3 x}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} \left\{\begin{array}{c} f(\vec{x}) \\ \overrightarrow{\nabla f(\vec{x})} \\ \nabla^2 f(\vec{x}) \end{array}\right.$$

Notice we treat each component independently. This is the very reason why we insist on using Cartesian, so is in quantum mechanics. Also notice we didn't assume axis of $\vec{k}$ to be the same of $\vec{x}$.

In the derivation of showing $\overrightarrow{\nabla f(\vec{x})}$,

$$\begin{aligned}
\overrightarrow{\nabla f(\vec{x})} &= \hat{x}_i \partial_{x_i} \int\limits_{all\,space} \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} F(k) \\
&= \left(i\vec{k}\cdot\hat{x}_i\right)\hat{x}_i \int\limits_{all\,space} \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} F(k) \\
&= i\vec{k} \int\limits_{all\,space} \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} F(k)
\end{aligned}$$

We used $\sum\left(\vec{k}\cdot\hat{x}_i\right)\hat{x}_i = \vec{k}$.

$$\left.\begin{array}{c} \vec{f}(\vec{x}) \\ div\,\vec{f}(\vec{x}) \\ curl\,\vec{f}(\vec{x}) \end{array}\right\} = \int\limits_{all\,space} \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \left\{\begin{array}{c} \vec{F}(\vec{k}) \\ i\vec{k}\cdot\vec{F}(\vec{k}) \\ i\vec{k}\times\vec{F}(\vec{k}) \end{array}\right.$$

$$\left.\begin{array}{c} \vec{F}(\vec{k}) \\ i\vec{k}\cdot\vec{F}(\vec{k}) \\ i\vec{k}\times\vec{F}(\vec{k}) \end{array}\right\} = \int\limits_{all\,space} \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} \left\{\begin{array}{c} \vec{f}(\vec{x}) \\ div\,\vec{f}(\vec{x}) \\ curl\,\vec{f}(\vec{x}) \end{array}\right.$$

Let us prove the $div\,\vec{f}(\vec{x})$ formula

$$\begin{aligned}
div\,\vec{f}(\vec{x}) &= \frac{\partial}{\partial x_i}\vec{f}(\vec{x})\cdot\vec{x}_i = \frac{\partial}{\partial x_i}\int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \vec{F}(\vec{k})\cdot\vec{x}_i \\
&= \left(i\vec{k}\cdot\hat{x}_i\right)\int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \vec{F}(\vec{k})\cdot\vec{x}_i \\
&= \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} i\vec{k}\cdot\vec{F}(\vec{k})
\end{aligned}$$

Because of the freedom in choosing axes for $\vec{k}$ and $\vec{F}(\vec{k})$, we choose three mutually orthogonal unit vectors, $\hat{\epsilon}_1(\vec{k})$, $\hat{\epsilon}_2(\vec{k})$, $\hat{\epsilon}_3(\vec{k})$, so that $\hat{\epsilon}_3(\vec{k}) = \hat{k}$. We write $\vec{F}(\vec{k}) = a_i(\vec{k})\hat{\epsilon}_i(\vec{k})$. The benefit of doing so is that the Fourier transform of

$$div\,\vec{f} \leftrightarrow i|\vec{k}|a_3(\vec{k})$$

$$curl\,\vec{f} \leftrightarrow i|\vec{k}|\left[a_1(\vec{k})\hat{\epsilon_2}(\vec{k}) - a_2(\vec{k})\hat{\epsilon_1}(\vec{k})\right]$$

Note: (1) $div\,\vec{f}$ determines longitudinal $F.T.$ components (along $\vec{k}$) and $curl\,\vec{f}$ determines transverse $F.T.$ components ($\perp \vec{k}$).

(2) $div\,\vec{f} = 0 \implies F.T.$ is transverse, $curl\,\vec{f} = 0 \implies F.T.$ is longitudinal.

(3) Helmholtz theorem: $div\,\vec{f}$ and $curl\,\vec{f}$ completely determine $\vec{f}$.

Proof. $\vec{F}(\vec{k})$ is completely determined. So $\vec{f}$ is determined.[3] □

Time-dependent Fourier Transform

Vector function of x, y, z, t.

$$\vec{f}(\vec{x}, t) = \int_{all\,\vec{k}} \frac{d^3k}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{dw}{(2\pi)^{1/2}} e^{i\vec{k}\cdot\vec{x}} e^{-iwt} \vec{F}(\vec{k}, w)$$

$$\vec{F}(\vec{k}, w) = \int_{all\,\vec{x}} \frac{d^3x}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{dt}{(2\pi)^{1/2}} e^{-i\vec{k}\cdot\vec{x}} e^{iwt} \vec{f}(\vec{x}, t)$$

The convention for using $+$ and $-$ is for doing Lorentz invariance easily $k^\mu x_\mu = wt - \vec{k}\cdot\vec{x}$.

Use F.T. to Solve Linear Inhomogeneous PDE

Consider an example electrostatics

$$curl\,\vec{E} = 0$$

$$div\,\vec{E} = 4\pi\rho$$

here ρ is time independent. Assumes $\vec{E} \to 0$ sufficiently fast as $|\vec{x}| \to \infty$. Additionally ρ is compact supported. We want to find $\vec{E}$

$$\vec{E}(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \tilde{E}(\vec{k})$$

[3] This is correct except for boundary conditions at $|\vec{x}| \to \infty$. This is related to singularities at $\vec{k} = 0$ where $\hat{\epsilon_i}$ are not well defined.

$curl\,\vec{E} = 0$ implies $\tilde{E}(\vec{k}) = a_3\hat{k}$. F.T. of $div\,\vec{E} = 4\pi\rho$ is[4]

$$i\vec{k}a_3 \cdot \hat{k} = 4\pi\tilde{\rho}$$

So

$$\begin{aligned}
\vec{E}(\vec{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}(-i\vec{k})\frac{4\pi}{k^2}\tilde{\rho} \\
&= -\vec{\nabla}\int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}\frac{4\pi}{k^2}\tilde{\rho} \\
&= -\vec{\nabla}\psi(\vec{x})
\end{aligned}$$

We define potential

$$\begin{aligned}
\psi(\vec{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}\frac{4\pi}{k^2}\tilde{\rho}(\vec{k}) \\
&= \frac{4\pi}{(2\pi)^3}\int d^3k e^{i\vec{k}\cdot\vec{x}}\frac{1}{k^2}\int d^3x' e^{-i\vec{k}\cdot\vec{x}'}\rho(\vec{x}') \\
&= \frac{4\pi}{(2\pi)^3}\int d^3x'\rho(\vec{x}')\int d^3k\frac{1}{k^2}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&= \frac{4\pi}{(2\pi)^3}\int d^3x'\rho(\vec{x}')\iiint k^2\sin\theta dk d\theta d\phi dk\frac{e^{ikR\cos\theta}}{k^2} \\
&= \frac{4\pi}{(2\pi)^3}2\pi\int d^3x'\rho(\vec{x}')\int_0^\infty dk\frac{2i\sin kR}{ikR} \\
&= \int d^3x'\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}
\end{aligned}$$

In the third equation above, we defined $\vec{R} = \vec{x} - \vec{x}'$, which is fixed during d^3k integration, then we chose $\hat{z}$ direction of $\vec{k}$ to be along $\vec{R}$. We also used $\int_{-\infty}^{\infty}\frac{\sin kR}{k}dk = \pi$.

[4]The general solution of this should be $ika_3 = 4\pi\tilde{\rho} + Ak\delta(\vec{k})$. That is because if $\rho = 0$, then one has a homogeneous differential equation $div\,\vec{E} = 0$, so $\vec{E}$ =constant. Thus $\tilde{E}(\vec{k}) = A\delta(\vec{k})$. Since we set the boundary condition so that $\vec{E} \to 0$, we don't have to include the homogeneous solution.

Therefore

$$\begin{aligned}\vec{E}(\vec{x}) &= -\int d^3x'\vec{\nabla}\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\\ &= -\int d^3x'\rho(\vec{x}')\frac{-(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\end{aligned}$$

This result is Coulomb's law

$$\vec{E}(\vec{x}) = \sum_{source}(dq')\frac{\vec{R}}{R^3}$$

and potential

$$\psi(\vec{x}) = \int d^3x'\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|}\circ\rho(\vec{x})$$

$G(\vec{x}-\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$ is the propagator, which propagates source points of ρ to the field point of $\vec{E}$.

Lecture 6 (9/21/12)

2.7. Generalized Functions

Consider

$$\delta_\epsilon(x) = \begin{cases}\frac{1}{\epsilon} & -\frac{\epsilon}{2} < x < \frac{\epsilon}{2}\\ 0 & \text{elsewhere}\end{cases}$$

(ϵ is a small real number.)

$\lim_{\epsilon\to 0^+}\delta_\epsilon(x)$ doesn't exist as ordinary function, but for some test function $T(x)$ continuous at 0, one has

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}dx\delta_\epsilon(x)T(x) &= \lim_{\epsilon\to 0^+}\int_{-1/\epsilon}^{1/\epsilon}dx\frac{1}{\epsilon}\left(T(0)+o(\epsilon)\right)\\ &= \lim_{\epsilon\to 0^+}\left(T(0)+o(\epsilon)\right) = T(0)\end{aligned}$$

When $\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}dx\delta_\epsilon(x)T(x)$ exists, we write

$$\int dx\delta(x)T(x)$$

There are other functions e.g. $\delta_\epsilon(x) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{x^2}{2\epsilon^2}}$ give $\delta(x)$.

Properties

(1) $\int_{-\infty}^{\infty} dx\delta(x)T(x) = T(0)$ ($T(x)$ continuous at 0)

(2) $\delta(x)$ is real.

(3) $\delta(x) = \delta(-x)$

(4) $g(x)\delta(x) = g(0)\delta(x)$

(5) $\int_{-\infty}^{\infty} dx\delta(x-a)T(x) = T(a)$

(6) $\int_{x_1}^{x_2} dx\delta(x-a)T(x) = \begin{cases} T(a) & x_1 < a < x_2 \\ 0 & x_1 > a \text{ or } x_2 < a \\ \text{undefined} & x_1 = a \text{ or } x_2 = a \end{cases}$

(7) Given real ordinary function $h(x)$ that has zero at $x_1, x_2, ..., x_n$, and $h'(x)$ doesn't vanish at these points, then

$$\delta(h(x)) = \frac{1}{\left|\frac{dh(x)}{dx}\right|} \left[\delta(x-x_1) + \delta(x-x_2) + ... + \delta(x-x_n)\right].$$

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} dx\delta(h(x))T(x) &= \sum_i \int_{\epsilon\text{-nbhd of } x_i} dx\delta(h(x))T(x) \\ &= \sum T(x_i) \int_{\epsilon\text{-nbhd of } h(x_i)} dh \frac{\delta(h)}{\left|\frac{dh(x)}{dx}\right|} \\ &= \sum \frac{T(x_i)}{\left|\frac{dh(x)}{dx}\right|_{x=x_i}} = \sum \int \frac{dxT(x)}{\left|\frac{dh(x)}{dx}\right|} \delta(x-x_i) \end{aligned}$$

□

(8) Derivative of δ function

$$\int dx \frac{d^N\delta(x-a)}{dx^N} T(x) = (-1)^N \int dx\delta(x-a) \frac{d^N T(x)}{dx^N} = (-1)^N \left.\frac{d^N T(x)}{dx^N}\right|_{x=a}$$

(9) Fourier Transform of δ function and derivative of δ function

$\delta(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} \tilde{\delta}(k)$, $\tilde{\delta}(k) = \int \frac{dk}{\sqrt{2\pi}} e^{-ikx} \delta(x) = 1/\sqrt{2\pi}$, Fourier representation

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx}$$

or

$$\delta(x) = \int \frac{dk}{2\pi} e^{-ikx}$$

for $\delta(x)$ is real.

Do conditionally convergent integration by putting a convergent factor $e^{-\frac{\epsilon^2}{2}k^2}$,

$$\begin{aligned}
\delta(x) &= \lim_{\epsilon \to 0^+} \delta_\epsilon(x) = \lim_{\epsilon \to 0^+} \int \frac{dk}{2\pi} e^{\pm ikx} e^{-\frac{\epsilon^2}{2}k^2} \\
&= \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \sqrt{\frac{2\pi}{\epsilon^2}} e^{-\frac{k^2}{2\epsilon^2}} = \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{k^2}{2\epsilon^2}}
\end{aligned}$$

(10) 3-dimension δ function

$$\int_{all\,space} d^3x \delta^3(\vec{x}) T(\vec{x}) = T(\vec{0})$$

$$\int_{finite\,region\,D\,of\,space} d^3x \delta^3(\vec{x} - \vec{a}) T(\vec{x}) = \begin{cases} T(\vec{0}) & \vec{a} \in D \\ 0 & \vec{a}\,\text{outside}\,D \\ \text{undefined} & \vec{a}\,\text{on}\,\partial D \end{cases}$$

If one uses Cartesian coordinate, $d^3x = dxdydz$, and $\delta^3(\vec{x}) = \delta(x)\delta(y)\delta(z)$.

Fourier representation

$$\delta^3(\vec{x}) = \int \frac{d^3k}{(2\pi)} e^{\pm i\vec{k}\cdot\vec{x}}$$

2.8. Problem Set 3 (due 9/27/12)

1)

Use F.T. solve electrostatics with non-zero mass. (see Proca theory) We first want to solve scalar potential. ρ is localized in a finite region of space, and $\vec{E} \to 0$

sufficiently fast.

$$-\Delta\psi + \mu^2\psi = 4\pi\rho,$$

ANS

$$\Phi(x) = \int d\vec{x'} \frac{e^{-\mu\left|\vec{x}-\vec{x'}\right|}}{\left|\vec{x}-\vec{x'}\right|}\rho(\vec{x'})$$

2)

Use F.T. solve for magnetostatics, $\vec{j}$ is localized in a finite region of space, and $\vec{B} \to 0$ sufficiently fast. Should get

$$\begin{aligned}\vec{B} &= curl\,\vec{A} \\ &= curl \int \frac{1}{c} d^3x' \frac{\vec{j}(\vec{x'})}{\left|\vec{x}-\vec{x'}\right|} \\ &= \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x'}) \times (\vec{x}-\vec{x'})}{\left|\vec{x}-\vec{x'}\right|^3}\end{aligned}$$

This is the Biot-Savart law.

3)

Diffusion Equation

$$\frac{\partial\rho}{\partial t} - D\nabla^2\rho = 0$$

is from $\frac{\partial\rho}{\partial t} + div\,\vec{j} = 0$ and Fick's law $\vec{j} = -D\overrightarrow{\nabla\rho}$. D is a positive constant. Initial time $\rho(\vec{x},0) = A\delta^3(\vec{x})$.

(a) solve for $\tilde{\rho}(\vec{k},t)$ with the initial condition. If you F.T. in $\vec{x}$ variables not in t, then solve ode for $\tilde{\rho}(\vec{k},t)$. If you F.T. in $\vec{x}$, and t. then you have simple equation for $\tilde{\rho}(\vec{k},t)$, and condition between k and w.

(b) solve $\rho(\vec{x},t)$.

4)

Consider M.eq in vacuum,

$$div\,\vec{E} = 0$$

$$div\,\vec{B} = 0$$

$$curl\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$curl\,\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

with the following initial conditions

$$\vec{E}(\vec{x},0) = \hat{x}E_0\delta(z) \quad \vec{B}(\vec{x},0) = 0$$

(a) find M.eq for $\tilde{E}(\vec{k},t)$, $\tilde{B}(\vec{k},t)$.

(b) solve these differential equations

(c) find $\vec{E}$, $\vec{B}$. The answer shows $\vec{E}$, $\vec{B}$ are non-zero on two sheets, one is moving along $+z$, the other along $-z$. Both moves at speed of c. On both sheets $\vec{E}$ are along $+x$, and $\vec{B}$ is along $+y$ on one sheet and $-y$ on the other sheet, and $|\vec{E}| = |\vec{B}| =$ constant. Additionally $\vec{E}\times\vec{B}$ gives the direct of the sheets moving.

Lecture 7
(9/24/12)

2.9. Generalized Functions (continued)

Heaviside Step function

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\frac{d\Theta(x)}{dx} = \delta(x)$$

Proof.

$$\begin{aligned} \int dx \frac{d\Theta(x)}{dx} T(x) &= \Theta(x)\,T(x)|_{-\infty}^{\infty} - \int_0^\infty dx \frac{dT(x)}{de} \\ &= T(\infty) - 0 - T(\infty) + T(0) \\ &= T(0) \end{aligned}$$

□

2.10. Review of Analyticity and Contour Integration

$f(z)$ is analytic iff $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists and is independent of the direction how $\Delta z \to 0$, then $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{df}{dz}$.

Theorem: If f is analytic everywhere, then f is infinitely differentiable.

Cauchy-Goursat: If $f(z)$ is analytic within D or on ∂D, then $\oint_C f(z)dz = 0$. This gives deformation of path.

Cauchy Residue: $\oint dz f(z) = 2\pi i \sum Res(f, z_i)$.

Cauchy theorem: $f(z)$ is analytic, $\oint dz \frac{f(z)}{(z-z_0)^{N+1}} = \frac{2\pi i}{N!}\left[\frac{d^N f(z)}{dz^N}\right]_{z_0}$.

Jordan's Lemma: $Re(\alpha) > 0$ $\int_{\text{upper half cicle}} g(z)e^{i\alpha z}dz = 0$,$\int_{\text{lower half}} g(z)e^{-i\alpha z}dz = 0$

Inhomogeneous Wave Equations

Recall $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

$$\Box\vec{E} = -4\pi\overrightarrow{\nabla\rho} - \frac{4\pi}{c^2}\frac{\partial\vec{j}}{\partial t}$$

$$\Box\vec{B} = \frac{4\pi}{c} curl\,\vec{j}$$

In Lorentz gauge

$$\Box\vec{A} = \frac{4\pi}{c}\vec{j}$$

$$\Box\psi = 4\pi\rho$$

We obtained total 10 inhomogeneous wave equations (1 for scalar potential, and 3 for spatial Cartesian components of $\vec{E}$,$\vec{B}$, and $\vec{A}$) of the form

$$\Box\psi = 4\pi f(\vec{x}, t)$$

Note:

(1) Given $f(\vec{x}, t)$ where f is non-zero in a finite region of $\vec{x}$ space during a finite time interval. ("source" is localized in space + time)

(2) Given spatial boundary $|\psi(\vec{x}, t)| \to 0$ sufficiently fast as $|\vec{x}| \to \infty$.

(3) For $\frac{\partial^2}{\partial t^2}$ need initial conditions for $\psi(\vec{x}, t_0)$ and $\frac{\partial\psi}{\partial t}(\vec{x}, t_0)$ for all $\vec{x}$.

(3a) Let $\psi(\vec{x}, t_0 = -\infty) = 0$ and $\frac{\partial\psi}{\partial t}(\vec{x}, t_0 = -\infty) = 0$, find $\psi(\vec{x}, t)$ $\forall t > t_0$. This gives retarded solution.

(3b) Let $\psi(\vec{x}, t_0 = \infty) = 0$ and $\frac{\partial\psi}{\partial t}(\vec{x}, t_0 = \infty) = 0$, find $\psi(\vec{x}, t)$ $\forall t < t_0$. This gives advanced solution.

Lecture 8 (10/1/12)

Solve Retarded Solution of

$$\Box\psi = 4\pi f(\vec{x}, t)$$

use F.T. in x, y, z, t,

$$\left(-\frac{w^2}{c^2} + k^2\right)\tilde{\psi}(\vec{k}, w) = 4\pi\tilde{f}(\vec{k}, w)$$

If $f = 0$, then we have a homogenous equation. Then we must have $w = \pm kc$. ($k \equiv |\vec{k}|$) Solution

$$\psi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}\left[A(\vec{k})e^{-ikct} + B(\vec{k})e^{ikct}\right]$$

A, B are determined from initial time $\psi(\vec{x}, t_0)$ and $\frac{\partial\psi}{\partial t}(\vec{x}, t_0)$.

Now we solve inhomogeneous equation,

$$\tilde{\psi}(\vec{k}, w) = \frac{-4\pi c^2\tilde{f}(\vec{k}, w)}{(w - kc)(w + kc)}$$

$$\begin{aligned}
\psi(\vec{x}, t) &= \frac{-4\pi c^2}{(2\pi)^4}\int d^3k \int dw \frac{e^{i\vec{k}\cdot\vec{x} - iwt}}{(w - kc)(w + kc)} \int d^3x' \int dt' e^{-i\vec{k}\cdot\vec{x}' + iwt'} f(\vec{x}', t') \\
&= \frac{-4\pi c^2}{(2\pi)^4}\int d^3x' \int dt' f(\vec{x}', t') \int d^3k e^{i\vec{k}\cdot(\vec{x} - \vec{x}')} \int dw \frac{e^{-iw(t - t')}}{(w - kc)(w + kc)}
\end{aligned}$$

One has trouble to integrate

$$\int dw \frac{e^{-iw(t - t')}}{(w - kc)(w + kc)}$$

we use analytically continue that equation into complex contour integral and move poles off the real axis,[5]

$$\int dw \frac{e^{-iw(t-t')}}{[w-(kc-i\epsilon)]\,[w+(kc-i\epsilon)]}$$

where we take $\epsilon \to 0^+$.

$$\lim_{\epsilon\to 0^+}\int dw \frac{e^{-iw(t-t')}}{[w-(kc-i\epsilon)]\,[w+(kc-i\epsilon)]} = \begin{cases} 0 & \begin{array}{c} t<t' \\ \text{upper half circle} \end{array} \\ -2\pi i\left(\frac{e^{ikc(t-t')}}{-2kc}+\frac{e^{-ikc(t-t')}}{2kc}\right) & \begin{array}{c} t>t' \\ \text{lower half circle} \end{array} \end{cases}$$

Then

$$\begin{aligned}
&\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x'})}\int dw \frac{e^{-iw(t-t')}}{(w-kc)(w+kc)} \\
&\quad = \int d^3k e^{i\vec{k}\cdot\vec{R}}\left(\frac{e^{ikc(t-t')}}{-2kc}+\frac{e^{-ikc(t-t')}}{2kc}\right) \\
&\quad = \int k^2 dk d\phi d\cos\theta e^{ikR\cos\theta}\left(\frac{e^{ikc(t-t')}}{-2kc}+\frac{e^{-ikc(t-t')}}{2kc}\right) \\
&\quad = \frac{2\pi}{2iRc}\int_0^\infty dk\left(e^{ikR}-e^{-ikR}\right)\left(-e^{ikc(t-t')}+e^{-ikc(t-t')}\right) \\
&\quad = \frac{\pi}{2iRc}\int_{-\infty}^\infty dk\left(-2e^{\pm ik(R+c(t-t'))}+2e^{\pm ik(R-c(t-t'))}\right) \\
&\quad = \frac{2\pi}{2iRc}(2\pi)\left(-\delta(R+c(t-t'))+\delta(R-c(t-t'))\right)
\end{aligned}$$

[5]There are four possible ways to move the poles off the axis: (1) $[w-(kc-i\epsilon)]\,[w+(kc-i\epsilon)]$ gives retarded solution; (2) $[w-(kc+i\epsilon)]\,[w+(kc+i\epsilon)]$ gives advanced solution; (3) $[w-(kc+i\epsilon)]\,[w+(kc-i\epsilon)]$ is used in Feynman integral; (4) $[w-(kc-i\epsilon)]\,[w+(kc+i\epsilon)]$ anti-Feynman integral.

$$\begin{aligned}
\psi(\vec{x},t) &= \frac{-4\pi c^2}{(2\pi)^4}(-2\pi i)\int d^3x' \int_{-\infty}^{t} dt' f(\vec{x}',t')\frac{2\pi}{2iRc}(2\pi)\left(-\delta(R+c(t-t'))+\delta(R-c(t-t'))\right) \\
&= \frac{-4\pi c^2}{(2\pi)^4}(-2\pi i)\frac{2\pi}{2ic}(2\pi)\int d^3x' \int_{-\infty}^{t} dt' \frac{f(\vec{x}',t')}{R}\delta(R-c(t-t')) \\
&= \frac{-4\pi c^2}{(2\pi)^4}(-2\pi i)\frac{2\pi}{2ic}(2\pi)\int d^3x' \int_{-\infty}^{t} dt' \frac{f(\vec{x}',t')}{R}\frac{\delta(t'-t+R/c)}{\left|\frac{d}{dt'}\left[R-c(t-t')\right]\right|} \\
&= \frac{-4\pi c^2}{(2\pi)^4}(-2\pi i)\frac{2\pi}{2ic}(2\pi)\frac{1}{c}\int d^3x' \frac{f(\vec{x}',t'=t-R/c))}{R} \\
&= \int d^3x' \frac{f(\vec{x}',t'=t-\left|\vec{x}-\vec{x'}\right|/c))}{\left|\vec{x}-\vec{x'}\right|}
\end{aligned}$$

Note:

(1) If f source is independent of time,

$$\psi(\vec{x},t) = \int d^3x' \frac{f(\vec{x}')}{\left|\vec{x}-\vec{x'}\right|}$$

This is the solution to Poisson equation

$$-\nabla^2\psi = 4\pi f(\vec{x})$$

(2) We can write

$$\psi(\vec{x},t) = \int d^3x' dt' G(\vec{x}-\vec{x'},t-t')f(\vec{x'},t') = G\circ f(\vec{x},t)$$

(3) $(\vec{x},t)$ refers to the field point where $\psi(\vec{x},t)$ is detected; $(\vec{x'},t')$ refers to source point which produces field at $(\vec{x},t)$. hence, any $\vec{x'}$ contributes to $(\vec{x},t)$ is when $f(\vec{x'},t')$ is at $t' = t - \left|\vec{x}-\vec{x'}\right|/c$.

Example $f(\vec{x},t) = A\delta^3(\vec{x})\delta(t)$

$$\psi = \int d^3x' \frac{A\delta(\vec{x'})}{\left|\vec{x}-\vec{x'}\right|}\delta(t-\left|\vec{x}-\vec{x'}\right|/c) = \frac{A}{|\vec{x}|}\delta(t-|\vec{x}|/c)$$

(This is a flash source. Someone asked what this differed from the case $f = 0$ and initial condition $\psi(\vec{x}, 0) = A\delta^3(\vec{x})$ and what the deeper reasons were. ANS: they appear on different sides of equation.)

2.11. Problem Set 4 (due 10/3/12)

1)

Jackson 6.1 part (a), $f = \delta(x)\delta(y)\delta(t)$, line source. Find ψ.

2)

Jackson 6.1 part (b), $f = \delta(x)\delta(t)$, sheet source. Find ψ.

3)

$f = \delta(\vec{x} - vt\hat{z})$, moving point source. Find ψ. In this problem we assume $v < c$ speed of wave in the medium, which gives one retarded solution. If $v > c$, one could have either no retarded solution or two retarded solutions (this effect is called "Cherenkov Radiation", similar to supersonic boom.)

Lecture 9 (10/3/12)

3. Retarded Solutions in Lorentz gauge

Recall what we have done

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \overrightarrow{\nabla\psi} \tag{3.1}$$

$$\vec{B} = curl\, \vec{A} \tag{3.2}$$

Satisfying Lorentz gauge condition $div\, \vec{A} + \frac{1}{c}\frac{\partial\psi}{\partial t} = 0$ implies,

$$\Box\vec{A} = \frac{4\pi}{c}\vec{j} \tag{3.3}$$

$$\Box\psi = 4\pi\rho \tag{3.4}$$

3.1. Retarded Solutions

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x'}, t' = t - \left|\vec{x} - \vec{x'}\right|/c)}{\left|\vec{x} - \vec{x'}\right|} \tag{3.5}$$

$$\psi(\vec{x},t) = \int d^3x' \frac{\rho(\vec{x'}, t' = t - \left|\vec{x} - \vec{x'}\right|/c)}{\left|\vec{x} - \vec{x'}\right|} \tag{3.6}$$

The converse is true. Equations 3.3, 3.4 imply Lorentz gauge condition. Let $h = div\,\vec{A} + \frac{1}{c}\frac{\partial\psi}{\partial t}$

$$\Box h = \frac{4\pi}{c}\left(div\,\vec{j} + \frac{\partial\rho}{\partial t}\right) = 0$$

Boundary condition on $\vec{A}$, ψ at $t = -\infty$ give $h = 0$ for all time and space.

Exercise

Alternative Method. Verify equations 3.5, 3.6 satisfy $div\,\vec{A} + \frac{1}{c}\frac{\partial\psi}{\partial t} = 0$. use chain rule to get $\frac{\partial}{\partial x'}\left|\vec{x} - \vec{x'}\right|$, then integration by parts.

Example

Two infinitely large and infinitely close sheets parallel to each other at $z = 0$. Top one has charge density σ, moving with velocity $\vec{v} = v_0\hat{x}\cos wt$. Bottom sheet with $-\sigma$ doesn't move. Find $\vec{E}$ and $\vec{B}$.

The general method is to complexify the equations and take real parts at the end of calculation. It is because in equations 3.3, 3.4, $\vec{A}$, $\vec{j}$, ψ,ρ are linear , thus the linear operations (e.g. derivative, integration) act on real and imaginary parts independently.

Let $\vec{v}_{physical} = Re\,(\vec{v})$, $\vec{v} = v_0 e^{-iwt}\hat{x}$. We have charge density$\rho = \sigma\delta(z) - \sigma\delta(z) = 0$. (notice we have the right dimension dimension of σ is charge/area, dimension of $\delta(z)$ is 1/length). We deduce that

$$\psi = 0.$$

Current density is produced from the top sheet

$$\vec{j}(\vec{x},t) = \sigma v_0\delta(z)e^{-iwt}\hat{x}$$

$$\vec{A}(\vec{x},t) = \frac{\sigma v_0 \hat{x}}{c} e^{-iwt} \int d^3x' \frac{\delta(z) e^{ik|\vec{x}-\vec{x'}|}}{|\vec{x}-\vec{x'}|}$$

Without lost of generality, we choose $\vec{x}$ to be on z-axis, and use cylindrical coordinate. Let $r = \sqrt{x'^2 + y'^2}$

$$\begin{aligned}
\vec{A}(\vec{x},t) &= \frac{\sigma v_0 \hat{x}}{c} e^{-iwt} \int d\phi \int_0^\infty r dr' \frac{e^{ik\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} \\
&= \frac{2\pi\sigma v_0 \hat{x}}{c} e^{-iwt} \int_{|z|}^\infty du e^{iku}
\end{aligned}$$

set $u = \sqrt{r^2+z^2}$, $du = rdr/\sqrt{r^2+z^2}$

To integrate this, we put in a convergent factor

$$\begin{aligned}
\int_{|z|}^\infty du e^{iku} &= \lim_{\epsilon\to 0^+} \int_{|z|}^\infty du e^{iku} e^{-\epsilon u} = \lim_{\epsilon\to 0^+} \left(\frac{e^{(ik-\epsilon)u}}{ik-\epsilon} \right)_{|z|}^\infty \\
&= \lim_{\epsilon\to 0^+} \frac{-e^{(ik-\epsilon)|z|}}{ik-\epsilon} = \frac{-e^{ik|z|}}{ik}
\end{aligned}$$

So

$$\vec{A} = \frac{2\pi\sigma v_0 \hat{x}}{c} e^{-iwt} \frac{-e^{ik|z|}}{ik} \overset{Re}{=} -\frac{2\pi\sigma v_0 \hat{x}}{w} \sin(k|z| - wt)$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{2\pi\sigma v_0 \hat{x}}{c} \cos(k|z| - wt)$$

$$\vec{B} = curl\, \vec{A} = \hat{y}\frac{\partial A_x}{\partial z} = -\frac{2\pi\sigma v_0 \hat{y}}{c} \cos(k|z| - wt) \begin{cases} 1 & z > 0 \\ -1 & z < 0 \end{cases}$$

We obtain a *plane-wave*. It has

(1) Linear polarized in z-direction

(2) $\vec{E}$ field perpendicular to $\vec{B}$ field

(3) $|\vec{E}| = |\vec{B}|$

(4) $\vec{E} \times \vec{B}$

(5) $\vec{E}$, $\vec{B}$ are in phase.

(6) Phases of $\vec{E}$, $\vec{B}$ are constant.

Wavefront (constant phase plane) is in which the phase is constant, i.e. for $z > 0$

$$(kz - wt) = (k(z + \Delta z) - w(t + \Delta t)$$

so

$$v_{phase} = \Delta z / \Delta t = w/k = c$$

3.2. Radiation Zone

In optics, this is called "Fraunhofer zone". Field points $|\vec{x}| = r$

$$r \gg a, \lambda$$

Sources are confined within ball of radius a, $\lambda = cT$.

We approximate

$$\begin{aligned}
\left|\vec{x} - \vec{x'}\right| &= r\left[\hat{r} - \frac{\vec{x'}}{r}\right] = r\sqrt{\left(\hat{r} - \frac{\vec{x'}}{r}\right)\left(\hat{r} - \frac{\vec{x'}}{r}\right)} \\
&= r\sqrt{1 - \frac{2\hat{r}\cdot\vec{x'}}{r} + \frac{\vec{x'}\cdot\vec{x'}}{r^2}} \approx r\sqrt{1 - \frac{2\hat{r}\cdot\vec{x'}}{r}} \\
&\approx r\left(1 - \frac{\hat{r}\cdot\vec{x'}}{r}\right) + o(\frac{1}{r}) \approx r - \hat{r}\cdot\vec{x'}
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{1}{\left|\vec{x} - \vec{x'}\right|} &\approx \frac{1}{r}\left(1 - \frac{2\hat{r}\cdot\vec{x'}}{r}\right)^{-1/2} \\
&\approx \frac{1}{r} + o(\frac{1}{r^2}) \approx \frac{1}{r}
\end{aligned}$$

Now we can modify equations3.5, 3.6.

$$\psi(\vec{x}, t) = \frac{1}{r}\int d^3x' \rho(\vec{x'}, t' = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{x'}}{r})$$

$$\vec{A}(\vec{x},t) = \frac{1}{c}\frac{1}{r}\int d^3x'\vec{j}(\vec{x}',t' = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{x}'}{r})$$

3.3. Problem Set 5 (due 10/10/12)

1)

Inhomogeneous Diffusion Equation

$$\frac{\partial\rho(\vec{x},t)}{\partial t} - D\nabla^2\rho(\vec{x},t) = 4\pi f(\vec{x},t)$$

f is non-zero within finite region, ρ goes to 0 sufficiently fast, initial value $\rho(\vec{x},-\infty) = 0$.

2)

Two infinitely large and infinitely close sheets parallel to each other at $z = 0$. Top one has charge density σ, moving with velocity $\vec{v} = v_0\hat{x}\Theta(t)$. Bottom sheet with $-\sigma$ doesn't move. Find $\vec{E}$ and $\vec{B}$. Step function $\Theta(t)$. Assume $v_0 \ll c$, so that the moving sheet has the same surface charge density in its rest frame as the laboratory frame.

3)

Bessel Function will be used in radiation problems involving circular loops and circular plates.

Definition:

$$J_N(x) = \frac{1}{2\pi i^N}\int_0^{2\pi} d\phi e^{ix\cos\phi}e^{-iN\phi}$$

$N = 0, \pm1, \pm2, ..., x \in \mathbb{R}$.

(a) Show $J_{-N}(x) = (-1)^N J_N(x)$, $J_N(-x) = (-1)^N J_N(x)$, $J_N(x) \in \mathbb{R}$

(b) Show $J_N(x)$ satisfies

$$J_N'' + \frac{1}{x}J_N' + \left(1 - \frac{N^2}{x^2}\right)J_N = 0$$

(c) Show for $|x|$ small

$$J_N(x) \approx A_N x^N + ...$$

find A_N.

Lecture 10 (10/8/12)

3.4. Radiation Theory - Scalar Wave

Scalar potential retarded solution

$$\begin{aligned} \psi(\vec{x},t) &= \int dx'^3 \frac{\rho(\vec{x'}, t' = t - |\vec{x} - \vec{x'}|/c)}{|\vec{x} - \vec{x'}|} \\ &= \frac{1}{r} \int d^3x' \rho(\vec{x'}, t' = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{x'}}{r}) \end{aligned}$$

Example 1

Given source

$$\rho(\vec{x},t) = A\cos(wt)\frac{d\delta(x)}{dx}\delta(y)\delta(z)$$

we solve this using exact retarded solution and the radiation approximation

$$\begin{aligned} \psi(\vec{x},t) &= \int d^3x' \frac{Ae^{-iw\left(t-|\vec{x}-\vec{x'}|/c\right)}}{|\vec{x}-\vec{x'}|} \frac{d\delta(x')}{dx'}\delta(y')\delta(z') \\ &= -Ae^{-iwt}\int_{-\infty}^{\infty} dx'\delta(x')\frac{d}{dx'}\frac{e^{ik\sqrt{(x-x')^2+y^2+z^2}}}{\sqrt{(x-x')^2+y^2+z^2}} \\ &= -Ae^{-iwt}\left(ik\frac{-(x-x')e^{ik\sqrt{(x-x')^2+y^2+z^2}}}{(x-x')^2+y^2+z^2} - \frac{-(x-x')e^{ik\sqrt{(x-x')^2+y^2+z^2}}}{\left((x-x')^2+y^2+z^2\right)^{3/2}} \right)_{x'=0} \\ &= Ae^{-iwt}\left(\frac{ikx}{r^2} - \frac{x}{r^3}\right)e^{ikr} \\ &= \frac{A}{r}(-k)\sin\theta\cos\phi\sin\left(kr-wt\right) - \frac{A}{r^2}x\sin\theta\cos\phi\cos\left(kr-wt\right) \end{aligned}$$

set $r = \sqrt{x^2+y^2+z^2}$, $(\vec{x})_x = x = r\sin\theta\cos\phi$.

In radiation approximation

$$\begin{aligned}
\psi(\vec{x},t) &= \frac{A}{r}\int d^3x' e^{-iw\left(t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{x}'}{c}\right)}\frac{d\delta(x')}{dx'}\delta(y')\delta(z') \\
&= \frac{A}{r}e^{i(kr-wt)}\int_{-\infty}^{\infty} dx' e^{-ik\hat{r}\cdot\hat{x}x'}\frac{d\delta(x')}{dx'} \\
&= \frac{A}{r}e^{i(kr-wt)}\left(ik\hat{r}\cdot\hat{x}e^{-ik\hat{r}\cdot\hat{x}x'}\right)_{x'=0} \\
&= \frac{A}{r}e^{i(kr-wt)}(ik\sin\theta\cos\phi) \\
&= \frac{A}{r}(-k)\sin\theta\cos\phi\sin(kr-wt)
\end{aligned}$$

Example 2

Given two point sources oscillating in phase

$$\rho(\vec{x},t) = Ae^{-iwt}\left(\delta^3(\vec{x})+\delta^3(\vec{x}-L\hat{z})\right)$$

In radiation approximation

$$\begin{aligned}
\psi(\vec{x},t) &= \frac{A}{r}e^{i(kr-wt)}\int d^3x' e^{-ik\hat{r}\cdot\vec{x}'}\left(\delta^3(\vec{x})+\delta^3(\vec{x}-L\hat{z})\right) \\
&= \frac{A}{r}e^{i(kr-wt)}\left(1+e^{-ikL\cos\theta}\right) \\
&= \frac{2A}{r}\cos(kr-wt-\frac{kL\cos\theta}{2})\cos\frac{kL\cos\theta}{2}
\end{aligned}$$

Let's look at the effect on the radiation field due to a change of origin.

If in the above example the origin is chosen at the middle of the two point sources, we have

$$\psi(\vec{x},t) = \frac{2A}{r}\cos(kr-wt)\cos\frac{kL\cos\theta}{2}$$

Just charge phase, not affect on the physical nature of radiation field, such as power radiated average.

This is true in general, say we shift origin by $\vec{a}$ ($|\vec{a}| \ll |\vec{r}|$). Then the new potential

$$\psi = \frac{1}{|\vec{r}-\vec{a}|}\int d^3x' f\left(\vec{x'}-\vec{a}, t-|\vec{r}-\vec{a}|/c+\left(\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|}\cdot\left(\vec{x'}-\vec{a}\right)\right)/c\right)$$

Use the same approximation technique

$$\frac{1}{|\vec{r}-\vec{a}|} \approx \frac{1}{r}$$

$$|\vec{r}-\vec{a}| \approx r - \vec{r}\cdot\vec{a}$$

$$\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|} \approx \frac{1}{r}\left(\vec{r}-\vec{a}\right) = \hat{r} - \frac{\vec{a}}{r} \approx \hat{r}$$

Hence

$$\begin{aligned}\psi &= \frac{1}{r}\int d^3x' f\left(\vec{x'}-\vec{a}, t-r/c+\vec{r}\cdot\vec{a}/c+\left(\hat{r}\cdot\left(\vec{x'}-\vec{a}\right)\right)/c\right) \\ &= \frac{1}{r}\int d^3x' f\left(\vec{x'}, t-r/c+\vec{r}\cdot\vec{a}/c+\hat{r}\cdot\vec{x'}/c\right)\end{aligned}$$

Compare this new potential with the old one, we see that change of origin adds a constant term $\vec{r}\cdot\vec{a}/c$ to t'. This change affects $t=0$ time. For harmonic sources, it just changes phase. Notice in the example above $\vec{r}\cdot\vec{a}/c = L\cos\theta/2c = Lk\cos\theta/2w$.

Scalar Potential Wave Intensity

Waves like $\vec{E}$, $\vec{B}$, sound wave, water wave, plasma wave, wave on string, wave on membrane carry energy. We define

$$\text{wave intensity} = \frac{\text{energy flow}}{(\text{time})(\text{area}\perp\text{flow})} = \text{energy flux}$$

Linear waves ($\vec{E}$, $\vec{B}$) are easy to handle, because they agree the mathematical equations of waves.

But many waves are too complicated to be analyzed. For small amplitude wave, one can treat complex wave as if it was linear. Examples: water wave, plasma wave, wave on string, wave on membrane. And sound wave is already almost linear.

Recall linear wave: $intensity \propto \psi^2_{physical}$.

For radiated wave $intensity = C\psi^2_{physical}\hat{r}$, C depends on the types of waves

One is most interested in dP =$power = \frac{energy}{time}$ incident on dA of the detector. $dA = r^2 d\Omega$ the perpendicular area r distance away from the radiated source with solid angle $d\Omega$.

$$dP = \left(C\psi^2_{phys}\right) dA$$

(Notice since $dP/d\Omega$ is not linear ψ, one cannot use the complex ψ then take the real part of the final value.)

$$\frac{dp}{d\Omega} = \left(C\psi^2_{phys}\right) r^2$$

Since

$$\psi = o(\frac{1}{r}) + o(\frac{1}{r^2}),$$

only the $o(\frac{1}{r})$ radiation field contributes to $dP/d\Omega$. Moreover $dP/d\Omega$ is independent of r.

Use ψ from example 2, which is analogy to double slits interference.

$$\psi(\vec{x}, t) = \frac{2A}{r}\cos(kr - wt - \frac{kL\cos\theta}{2})\cos\frac{kL\cos\theta}{2}$$

$$\frac{dp}{d\Omega} = 4CA^2\cos^2(kr - wt - \frac{kL\cos\theta}{2})\cos^2\frac{kL\cos\theta}{2}$$

Time average over one or more cycles

$$\begin{aligned}
\left\langle\frac{dp}{d\Omega}\right\rangle &= \frac{1}{T}\int_0^T \frac{dp}{d\Omega}dt \\
&= 2CA^2\cos^2\frac{kL\cos\theta}{2} \\
&= 2CA^2\cos^2\frac{kL\sin\alpha}{2}
\end{aligned}$$

define $\alpha = \pi/2 - \theta$. This is the angle measured from $x - y$ plane to the field point, and it is normally used to indicate the interference pattern on a screen.

Maximum occur at $\frac{kL\sin\alpha}{2} = 0, \pm\pi, ..., \pm N\pi$.

$$L\sin\alpha = 2N\pi/k = N\lambda,$$

which agrees double slits interference.

3.5. Problem Set 6 (due 10/17/12)

1)

There are N point sources on z-axis, oscillating in phase

$$f(\vec{x},t) = A\sum_{n=0}^{N-1}\delta^3(\vec{x}-nL\hat{z})\cos wt$$

Find ψ, $\left\langle\frac{\partial P}{\partial\Omega}\right\rangle$, at what angle α intensity at a maxima?

$$L\sin\alpha = m\lambda$$

This problem is analogy to N-slit interference.

2)

Line source from 0 to L,

$$f(\vec{x},t) = \begin{cases} A\delta(x)\delta(y)\cos wt & 0 < z < L \\ 0 & otherwise \end{cases}$$

Find ψ, $\left\langle\frac{\partial P}{\partial\Omega}\right\rangle$, at what angle α intensity at a minima (intensity$=0$)?

$$L\sin\alpha = m\pi$$

This problem is analogy to single slit diffraction.

3)

(a)

Ring source

$$f(\vec{x},t) = A\delta(r_\perp - R)\delta(z)\cos wt$$

Find ψ, $\left\langle \frac{\partial P}{\partial \Omega} \right\rangle$. Answer in form of Bessel Function of order 0 (J_0).

(b)

Plate source

$$f(\vec{x},t) = \begin{cases} A\delta(z)\cos wt & 0 < r_\perp < R \\ 0 & otherwise \end{cases}$$

Find ψ, $\left\langle \frac{\partial P}{\partial \Omega} \right\rangle$. Answer in form of Bessel Function of order 1 (J_1).

4)

consider two point sources oscillating out of phase,

$$f(\vec{x},t) = A\left(\delta^3(\vec{x} + \frac{L}{2}\hat{y})\cos wt + \delta^3(\vec{x} - \frac{L}{2}\hat{y})\cos{(wt - \alpha)}\right)$$

Find ψ, $\left\langle \frac{\partial P}{\partial \Omega} \right\rangle$. Unlike any examples we have seen before, in this problem $\left\langle \frac{\partial P}{\partial \Omega} \right\rangle$ has ϕ dependence.

Broadside array $\alpha = 0$, $L = \pi c/w = \lambda/2$, maximum intensity is along x, perpendicular to dipole.

End-fire array $\alpha = \pi/2$, $L = \pi c/2w = \lambda/4$, maximum intensity is in $+y$ direction.

Lecture 11 (10/10/12)

3.6. Vector Potential in Radiation zone

Generalized Orthogonal Coordinates Review

Denote (q_1, q_2, q_3) to be a orthogonal coordinate with unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$. Let $dl_i, i = 1, 2, 3$ be arc length when q_i changes infinitesimally dq_i and other q's are kept constant. Denote $h_i(q_1, q_2, q_3)$, $i = 1, 2, 3$ to be the proportionality, so

$$dl_i(q_1, q_2, q_3) = h_i(q_1, q_2, q_3)dq_i$$

e.g. spherical polar coordinates $(q_1, q_2, q_3) = (r, \theta, \psi)$, $\hat{e}_1, \hat{e}_2, \hat{e}_3 = \hat{r}, \hat{\theta}, \hat{\psi}$, and $h_1, h_2, h_3 = 1, r, r\sin\theta$

- Gradient

  It is defined as directional derivative.

$$\frac{d\psi}{dl_{\hat{n}}} = \frac{\text{change in } \psi}{\text{arc length in } \hat{n}\text{direction}}$$

  Therefore

$$\overrightarrow{\nabla\psi} = \sum \hat{e}_i \frac{1}{h_i} \frac{\partial \psi}{\partial q_i}$$

  e.g. in spherical coordinates $grad = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

- Divergence

  It is defined as the flux per volume, and take limit of volume to zero

$$\begin{aligned}
div\,\vec{W} &= \frac{1}{\text{volume}} \oint \vec{W} \cdot \hat{n} da \\
&= \frac{1}{dl_1 dl_2 dl_3} (W_1(q_1 + dq_1) dl_2(q_1 + dq_1) dl_3(q_1 + dq_1) - W_1(q_1) dl_2(q_1) dl_3(q_1) \\
&\quad + W_2(q_2 + dq_2) dl_1(q_2 + dq_2) dl_3(q_1 + dq_1) - W_2(q_2) dl_1(q_2) dl_3(q_2) \\
&\quad + W_3(q_3 + dq_3) dl_1(q_3 + dq_3) dl_2(q_3 + dq_3) - W_3(q_3) dl_1(q_3) dl_2(3_2)) \\
&= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial q_1}(W_1 h_2 h_3) + \frac{\partial}{\partial q_2}(W_2 h_1 h_3) + \frac{\partial}{\partial q_3}(W_3 h_1 h_2) \right)
\end{aligned}$$

  e.g. in spherical coordinates

$$\begin{aligned}
div\,\vec{W} &= \frac{1}{r^2 \sin\theta} \left(\frac{\partial}{\partial r}\left(W_r r^2 \sin\theta\right) + \frac{\partial}{\partial \theta}\left(W_\theta r \sin\theta\right) + \frac{\partial}{\partial \phi}\left(W_\phi r\right) \right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(W_r r^2\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(W_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(W_\phi\right)
\end{aligned}$$

- Laplacian

  $div\,grad\,\psi = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}(h_2 h_3 \frac{1}{h_1}\frac{\partial \psi}{\partial q_1}) + \frac{\partial}{\partial q_2}(h_1 h_3 \frac{1}{h_2}\frac{\partial \psi}{\partial q_2}) + \frac{\partial}{\partial q_3}(h_1 h_2 \frac{1}{h_3}\frac{\partial \psi}{\partial q_3})\right)$

e.g. in spherical coordinates

$$\begin{aligned}
div\, grad\, \psi &= \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi}\right)\right) \\
&= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right]
\end{aligned}$$

- Curl

is defined as circulation per area, $\hat{n}$ direction is given by the right hand rule from the direction of the closed curve in the integral. Limit area goes to 0.

$$\left(curl\, \vec{W}\right)_{\text{in } \hat{n} \text{ direction}} = \frac{1}{\text{area}} \oint \vec{W} \cdot \vec{dl}$$

Let's calculate

$$\begin{aligned}
\left(curl\, \vec{W}\right)_3 &= \frac{1}{dl_1 dl_2}\left(W_1(q_2) dl_1(q_2) + W_2(q_1 + dq_1) dl_2(q_1 + dq_1)\right) \\
&= -W_1(q_2 + dq_2) dl_1(q_2 + dq_2) - W_2(q_1) dl_2(q_1)) \\
&= \frac{1}{h_1 h_2}\left(\frac{\partial W_2 h_2}{\partial q_1} - \frac{\partial W_1 h_1}{\partial q_2}\right)
\end{aligned}$$

Therefore

$$curl\, \vec{W} = det \begin{pmatrix} \frac{\hat{e}_1}{h_2 h_3} & \frac{\hat{e}_2}{h_1 h_3} & \frac{\hat{e}_3}{h_1 h_2} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 W_1 & h_2 W_2 & h_3 W_3 \end{pmatrix}$$

e.g. in spherical coordinates

$$
\begin{aligned}
curl\,\vec{W} &= \frac{\hat{r}}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\left(r\sin\theta W_\phi\right) - \frac{\partial}{\partial\phi}\left(rW_\theta\right)\right) \\
&\quad + \frac{\hat{\theta}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\left(W_r\right) - \frac{\partial}{\partial r}\left(r\sin\theta W_\phi\right)\right) \\
&\quad + \frac{\hat{\phi}}{r}\left(\frac{\partial}{\partial r}\left(rW_\theta\right) - \frac{\partial}{\partial\theta}\left(W_r\right)\right) \\
&= \frac{\hat{r}}{r\sin\theta}\left(\frac{\partial r\sin\theta W_\phi}{\partial\theta} - \frac{\partial W_\theta}{\partial\phi}\right) + \frac{\hat{\theta}}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(W_r\right) - \frac{\partial}{\partial r}\left(rW_\phi\right)\right) \\
&\quad + \frac{\hat{\phi}}{r}\left(\frac{\partial rW_\theta}{\partial r} - \frac{\partial W_r}{\partial\theta}\right)
\end{aligned}
$$

Quadratic Expression in fields

We mentioned last lecture, expressions for electromagnetic fields store energy (or linear momentum, angular momentum) and transport energy (or linear momentum, angular momentum) from one point to another involve quadratic.

For example, Poynting vector in cgs

$$
\begin{aligned}
\vec{S} &= \frac{c}{4\pi}\vec{E}\times\vec{B} \\
&= \frac{\text{field energy flow}}{(\text{time})(\text{area}\perp\text{flow})} = \text{energy flux}
\end{aligned}
$$

Since quadratic expressions are not as easy to be handle as linear expression, the following theorem will be helpful.

Fields that vary harmonically in time, as before, use complex fields, and then take the real part to get physical field.

Theorem. *If* $f = |f|e^{-i(wt+\alpha)}$, $g = |g|e^{-i(wt+\beta)}$, *i.e.* $f_{phys} = |f|\cos(wt+\alpha)$, $g_{phys} = |g|\cos(wt+\beta)$, *then*

$$\langle f_{phys}\cdot g_{phys}\rangle_{time\,average} = \frac{1}{2}Re\{f^* g\}.$$

Proof.

$$\begin{aligned}
\langle f_{phys} \cdot g_{phys} \rangle &= \frac{1}{T} \int_0^T |f||g| \left(\cos\left(wt + \alpha\right) \cos\left(wt + \beta\right) \right) dt \\
&= \frac{1}{T} \int_0^T |f||g| \left(\frac{\cos\left(2wt + \alpha + \beta\right)}{2} + \frac{\cos\left(\alpha - \beta\right)}{2} \right) dt \\
&= |f||g| \frac{\cos\left(\alpha - \beta\right)}{2} \\
&= \frac{1}{2} Re\{f^* g\}
\end{aligned}$$

□

4. Electromagnetic Radiation Theory

We're now ready to derive formal radiation theory. References are Jackson's chapters 9, 10–antennas with currents in wire, and chapter 14, 15–point particle in arbitrary motion.

In radiation zone, we keep only $1/r$ term. This means $\vec{A}(\vec{x}, t)$ has the form

$$\vec{A}(\vec{x}, t) = \frac{\vec{a}(\hat{r},\ t - r/c)}{r}$$

so the spatial variable of $\vec{a}$ has only θ, ϕ dependence.

Theorem. *In radiation zone,*

$$curl\, \vec{A} = -\frac{1}{c} \frac{\partial}{\partial t} \left(\hat{r} \times \vec{A} \right).$$

Proof. Ignoring any $1/r^2$ term or higher

$$
\begin{aligned}
curl\,\vec{A} &= det\begin{pmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{a_r}{r} & \frac{a_\theta}{r}r & \frac{a_\phi}{r}r\sin\theta \end{pmatrix} \\
&= \frac{\hat{\theta}}{r\sin\theta}\left[\frac{1}{r}\frac{\partial a_r}{\partial\phi} - \sin\theta\frac{\partial a_\phi}{\partial r}\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial a_\theta}{\partial r} - \frac{1}{r}\frac{\partial a_r}{\partial\theta}\right] \\
&= -\frac{\hat{\theta}}{r}\frac{\partial a_\phi}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial a_\theta}{\partial r}
\end{aligned}
$$

where we use

$$\frac{\partial a_\phi}{\partial r} = \frac{\partial a_\phi}{\partial t}\frac{dt}{dr} = -\frac{1}{c}\frac{\partial a_\phi}{\partial t}$$

We get,

$$
\begin{aligned}
\vec{B} = curl\,\vec{A} &= \frac{\hat{\theta}}{r}\frac{1}{c}\frac{\partial a_\phi}{\partial t} - \frac{\hat{\phi}}{r}\frac{1}{c}\frac{\partial a_\theta}{\partial t} \\
&= \frac{1}{c}\frac{\partial}{\partial t}\left[A_\phi\hat{\theta} - A_\theta\hat{\phi}\right] \\
&= -\frac{1}{c}\frac{\partial}{\partial t}\left(\hat{r}\times\vec{A}\right) \\
&= -\frac{1}{c}\hat{r}\times\dot{\vec{A}}
\end{aligned}
$$

$\square$

Find $\vec{E}(\vec{x},t)$.

We use

$$curl\,\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

for no current at field point.

$$
\begin{aligned}
\vec{E} &= c\int dt curl\,\vec{B} \\
&= -\int dt\hat{r}\times curl\,\dot{\vec{A}} \\
&= \int dt\hat{r}\times\frac{1}{c}\left(\hat{r}\times\ddot{\vec{A}}\right) \\
&= \frac{1}{c}\hat{r}\times\left(\hat{r}\times\dot{\vec{A}}\right)+\vec{W}(\vec{x})
\end{aligned}
$$

where $\vec{W}(\vec{x})$ constant of integration, independent of t. This means $\vec{W}(\vec{x})$ is electrostatic, and it behaves like coulomb's law, i.e. $\vec{W}(\vec{x})\sim 1/r^2$, so far as radiation is concerned, we drop this term.

We have shown in radiation zone

$$\vec{E}=\frac{1}{c}\hat{r}\times\left(\hat{r}\times\dot{\vec{A}}\right)$$

$$\vec{B}=-\frac{1}{c}\hat{r}\times\dot{\vec{A}}$$

(1) $\vec{E}$, $\vec{B}$ are $\propto 1/r$, because $\vec{A}$ is
(2) $\vec{E}\perp\hat{r}$, $\vec{B}\perp\hat{r}$
(3) $|\vec{E}|=|\vec{B}|$
(4) $\vec{E}\times\vec{B}$ is in $\hat{r}$ direction
(5) $\vec{E}\perp\vec{B}$
(6) $\vec{S}=\frac{c}{4\pi}|\vec{E}|^2\hat{r}$

Lecture 12 (10/15/12)

power detected $= dP = (\vec{S}\cdot\hat{r})da$

$$
\begin{aligned}
\frac{dP}{d\Omega} &= \frac{c}{4\pi}|\vec{E}|^2r^2 \\
\left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{c}{8\pi}\left(\vec{E}^*\vec{E}\right)r^2
\end{aligned}
$$

Both are independent of r.

$$\langle P\rangle = \int d\Omega \left\langle \frac{dP}{d\Omega}\right\rangle$$

4.1. Antennas

Let's pay some special attention to harmonic field.

In this lecture, we'll look at harmonic fields that have same frequency and same phase at each source point.

Example 1

Small antennas at origin

$$\vec{j}(\vec{x},t) = \hat{z}J_0 e^{-iwt}\delta^3(\vec{x})$$

Find

$$\begin{aligned}\vec{A} &= \frac{1}{r}\frac{1}{c}\int d^3x'\hat{z}J_0 e^{-iw(t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{x}}{c})}\delta^3(\vec{x}) \\ &= \frac{e^{i(kr-wt)}}{cr}\hat{z}J_0\end{aligned}$$

$$\vec{E} = \frac{1}{c}\hat{r}\times(\hat{r}\times\dot{\vec{A}}) = \frac{-iwe^{i(kr-wt)}}{c^2 r}J_0\hat{\theta}\sin\theta$$

$$\left\langle \frac{dP}{d\Omega}\right\rangle = \frac{c}{8\pi}\frac{w^2J_0^2\sin^2\theta}{c^4}$$

note: Radiation field vanishes at $\theta = 0, \pi$. In $\vec{E}$ equation, $\hat{\theta}\sin\theta$ means from observer's point view, $\vec{E}$ field is polarized perpendicular to the line from origin to the observers. At $\theta = 0, \pi$, the observer sees no oscillation. Intensity is maximum at $\theta = \pi/2$.

Example 2

Finite size antenna along z-axis

$$\vec{j} = \begin{cases} \hat{z} J_1 e^{-iwt} \delta(x)\delta(y) & 0 < z < L \\ 0 & otherwise \end{cases}$$

Find

$$\begin{aligned} \vec{A} &= \frac{e^{i(kr-wt)}\hat{z}J_1}{rc} \int_0^L dz' e^{-ikz'\cos\theta} \\ &= \frac{e^{i(kr-wt)}\hat{z}J_1}{rc} \frac{e^{-ikL\cos\theta}-1}{-ik\cos\theta} \end{aligned}$$

$$\vec{E} = \frac{1}{c}\frac{-iwe^{i(kr-wt)}J_1}{rc}\frac{e^{-ikL\cos\theta}-1}{-ik\cos\theta}\hat{\theta}\sin\theta$$

$$\begin{aligned} \left\langle \frac{dP}{d\Omega} \right\rangle &= \frac{c}{8\pi}\frac{w^2 J_1^2 \sin^2\theta}{c^4 k^2 \cos^2\theta}\left[(e^{-ikL\cos\theta}-1)(e^{ikL\cos\theta}-1)\right] \\ &= \frac{w^2 J_1^2 \sin^2\theta}{4\pi c^3 k^2 \cos^2\theta}\left[1-\cos\left(kL\cos\theta\right)\right] \\ &= \frac{w^2 J_1^2 L^2 \sin^2\theta}{8\pi c^3}\left(\frac{\sin\left(\frac{kL}{2}\cos\theta\right)}{\frac{kL}{2}\cos\theta}\right)^2 \end{aligned}$$

note:

(1) In example 1, J_0 has dimension (current density)(length)^3,because $\hat{z}$, exponential function have no dimension, and dimension of δ^3 is (1/length)^3. In example 2, J_1 has dimension, (current density)(length)^2.

(2) If $kL \ll 1$, that is L is very small, example 2 becomes example 1, with $J_0 = J_1 L$.

(3) If $kL \gg 1$, then $\frac{\sin\left(\frac{kL}{2}\cos\theta\right)}{\frac{kL}{2}\cos\theta}$ factor becomes significant, which gives extra lobes besides the main largest top.

Lecture 13 (10/17/12)

In this lecture, we'll look at harmonic fields that have oscillation in different directions or phases. This gives rise to polarization fields.

Suppose at a particular spatial point, we have a vector function

$$\vec{F}(t) = \hat{x}C_1 \cos(wt + \alpha) + \hat{y}C_2 \cos(wt + \beta) + \hat{z}C_3 \cos(wt + \gamma)$$

assuming $C_{1,2,3} \geq 0$.

There is a better way to characterize $\vec{F}$.

Let $\vec{F}_0 = \hat{x}C_1 e^{-i\alpha} + \hat{y}C_2 e^{-i\beta} + \hat{z}C_3 e^{-i\gamma}$, then $\vec{F}(t) = Re\{\vec{F}_0 e^{-iwt}\}$.

Consider

$$\begin{aligned} \vec{F}_0 \cdot \vec{F}_0 &= C_1^2 e^{-2i\alpha} + C_2^2 e^{-2i\beta} + C_3^2 e^{-2i\gamma} \\ &= |\vec{F}_0 \cdot \vec{F}_0| e^{-i\delta} \end{aligned}$$

We set

$$\vec{F}_0 = e^{-i\delta/2} \left(\vec{a} + \vec{b}i \right)$$

$\vec{a}$, $\vec{b}$ are real 3-dimensional vectors. This new definition is legitimate, because original $\vec{F}_0$ has 6 parameters. Now we have 7. If we calculate

$$\vec{F}_0 \cdot \vec{F}_0 = e^{-i\delta} \left(|\vec{a}|^2 - |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}i \right),$$

we find out the constraints, $\vec{a} \perp \vec{b}$, and $|\vec{a}| \geq |\vec{b}|$. Substitute this new definition into $\vec{F}$ yields

$$\vec{F}(t) = \vec{a} \cos \left(wt + \frac{\delta}{2} \right) + \vec{b} \sin \left(wt + \frac{\delta}{2} \right)$$

This characterizes $\vec{F}(t)$ as rotation on an ellipse with semi major axis $\vec{a}$, semi minor axis $\vec{b}$ at *average* angular frequency w.

If $|\vec{a}| = |\vec{b}|$, circular rotation at speed w, circular polarization; if $|\vec{b}| = 0$, one has a linear polarization.

Suppose current density $\vec{j}$ is harmonic in time, $\vec{j} = \vec{j}_w(\vec{x})e^{-iwt'}$

$$\begin{aligned}\vec{A} &= \frac{1}{rc}\int d^3x'\vec{j}(\vec{x}', t - r/c + \hat{r}\cdot\vec{x}/c) \\ &= \frac{e^{i(kr-wt)}}{rc}\int d^3x'\vec{j}_w(\vec{x}')e^{-ik\hat{r}\cdot\vec{x}'}\end{aligned}$$

This is replica of $\vec{F}(t)$ that rotates on an ellipse.

$$\vec{E} = \frac{1}{c}\hat{r}\times\left(\hat{r}\times\dot{\vec{A}}\right)$$

$$\vec{B} = -\frac{1}{c}\hat{r}\times\dot{\vec{A}}$$

Observations: $\vec{A}$, $\vec{E}$, and $\vec{B}$ are harmonic in time. $\vec{E}$, and $\vec{B}$ are in phase. $\vec{E}$, and $\vec{B}$ rotate on their own ellipsis, but they keep perpendicular to each other. The factor $e^{i(kr-wt)}$ in $\vec{A}$ will be in $\vec{E}$, and $\vec{B}$ also. This factor gives period and wavelength of $\vec{E}$, $\vec{B}$ waves.

Example

2 tiny linear antennas at origin, oscillating 90 degree out of phase.

$$\vec{j} = \hat{x}J_0\cos wt\delta^3(\vec{x}) + \hat{y}J_0\sin\delta^3(\vec{x})$$

Complexify

$$\vec{j} = J_0\delta^3(\vec{x})(\hat{x}+i\hat{y})e^{-iwt}$$

$$\begin{aligned}\vec{A} &= \frac{1}{rc}\int d^3x' J_0\delta^3(\vec{x'})(\hat{x}+i\hat{y})e^{-iw(t-r/c+\hat{r}\cdot\vec{x}/c)} \\ &= \frac{e^{i(kr-wt)}J_0}{rc}(\hat{x}+i\hat{y})\end{aligned}$$

$$\vec{E} = \frac{1}{c}\frac{-iwe^{i(kr-wt)}J_0}{rc}\hat{r}\times(\hat{r}\times(\hat{x}+i\hat{y}))$$

$\hat{r}\times(\hat{r}\times(\hat{x}+i\hat{y}))$ indicates polarization.

If $\hat{r}=\hat{x}$, polarization direction is $\hat{y}$. This says one on x axis can only see oscillation in y direction.

If $\hat{r}=\hat{z}$, $\hat{r}\times(\hat{r}\times(\hat{x}+i\hat{y}))=-x-iy$. This is circular polarization.

We also calculate power radiation

$$\begin{aligned}\left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{c}{8\pi}\frac{w^2J_0^2}{c^4}\left[(\hat{r}\times(\hat{r}\times\hat{x}))^2+(\hat{r}\times(\hat{r}\times\hat{y}))^2\right]\\ &= \frac{w^2J_0^2}{8\pi c^3}\left[|\hat{r}\times\hat{x}|^2+|\hat{r}\times\hat{y}|^2\right]\end{aligned}$$

$|\hat{r}\times\hat{x}|^2=\sin^2\angle(\hat{r},\hat{x})=1-\cos^2\angle(\hat{r},\hat{x})=1-(\hat{r}\cdot\hat{x})^2=1-\sin^2\theta\cos^2\phi$, $|\hat{r}\times\hat{y}|^2=1-\sin^2\theta\sin^2\phi$, thus

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{w^2J_0^2}{8\pi c^3}\left(2-\sin^2\theta\right)=\frac{w^2J_0^2}{8\pi c^3}\left(1+\cos^2\theta\right)$$

Maximum intensity is at $\theta=0,\pi$. Minimum is at $\theta=\pi/2$.

4.2. Problem Set 7 (due 10/26/12)

Problem 1. Standing wave antenna of length L on z axis,

$$\vec{j}=J_1\hat{z}\sin\left(\frac{N\pi}{L}z\right)\cos wt\delta(x)\delta(y)\Theta(z)\Theta(L-z)$$

$N=1,2,3...$

Find $\vec{E}$, $\langle\frac{dP}{d\Omega}\rangle$, show that total power radiated into the upper hemisphere is equal to total power into the lower hemisphere.

Problem 2.

Traveling wave current in antenna

$$\vec{j}=J_1\hat{z}\cos\left(\frac{N\pi}{L}z-xwt\right)\delta(x)\delta(y)\Theta(z)\Theta(L-z)$$

Find $\vec{E}$, $\langle\frac{dP}{d\Omega}\rangle$, show that total power radiated into the upper hemisphere is greater to total power into the lower hemisphere.

Problem 3. Circular antenna

$$\vec{j}=J_1\hat{\phi}\delta(z)\delta(r_\perp-R)\cos wt$$

$r_\perp = \sqrt{x^2+y^2}$.

Find $\vec{E}$, $\left\langle \frac{dP}{d\Omega} \right\rangle$.

ANS $\left\langle \frac{dP}{d\Omega} \right\rangle \sim \left(J_1(kR\sin\theta)\right)^2$, involving Bessel function of order 1. If $kR \ll 1$, $J_1(x)$ behaves like a linear function with positive slope, so the intensity increases gradually as θ goes from 0 to $\pi/2$, then intensity decreases. If kR is not small, $J_1(x)$ can reach many zeros as $\sin\theta$ goes from 0 to 1. This gives many lobes to the intensity, but the 4 largest intensity lobes are near $\theta = 0, \pi$.

Lecture 14
(10/22/12)

4.3. ρ, $\vec{j}$ and Dipole Moments Review

Relating ρ and $\vec{j}$

suppose that ρ moves with velocity $\vec{v}$ at a given point and given time. Consider a small cube with edge vdt, area of faces $da = vdt$, and two faces (left and right) of the cube are perpendicular to flow. All charges in this volume pass through right side da in time dt. Then by definition of current density

$$\vec{j} = \frac{\text{charge flow through } da \text{in time } dt}{(\text{time})(\perp \text{ area})} = \frac{\rho \cdot \text{volume}}{(dt)(da)} = \rho\vec{v}$$

If different charge density ρ_α traveling at different velocities $\vec{v}_\alpha$,

$$\vec{j} = \sum_\alpha \vec{j}_\alpha = \sum_\alpha \rho_\alpha \vec{v}_\alpha$$

Point Particles

We label particles $\alpha = 1, 2, 3, \ldots$ Let $\vec{S}_\alpha(t)$ to be position vector of q_α at time t.

$$\rho(\vec{x}, t) = \sum_\alpha q_\alpha \delta^3(\vec{x} - \vec{S}_\alpha(t))$$

And

$$\vec{j}(\vec{x}, t) = \sum_\alpha \rho_\alpha(\vec{x}, t)\vec{v}_\alpha(t) = \sum_\alpha q_\alpha \vec{v}_\alpha(t) \delta^3(\vec{x} - \vec{S}_\alpha(t))$$

Electric Dipole Moment $\vec{d}$

This is the first moment of charge distribution. We assume $\rho(\vec{x},t)$ is localized, so the integral converges.

$$\vec{d}(t) = \int_{\text{all source}} d^3x\vec{x}\rho(\vec{x},t)$$

Example: system of point particles

$$\vec{d}(\vec{x},t) = \int d^3x\vec{x}\left[\sum q_\alpha\delta^3\left(\vec{x}-\vec{S}_\alpha(t)\right)\right] = \sum q_\alpha\vec{S}_\alpha(t)$$

If we look at non-relativistic particles with same mass

$$\frac{q_\alpha}{m_\alpha} = \frac{e}{m}$$

then

$$\vec{d}(t) = \frac{e}{m}\sum m_\alpha\vec{S}_\alpha(t) = \frac{e}{m}\left(M_{total}\cdot\vec{R}_{cm}(t)\right)$$

where $\vec{R}_{cm}(t)$ denotes center of mass point.

Magnetic Dipole Moment $\vec{\mu}$

This is the first vector moment of $\vec{j}$. We assume $\vec{j}$ is localized

$$\vec{\mu}(t) = \frac{1}{2c}\int_{\text{all source}} d^3x\vec{x}\times\vec{j}$$

Note: both $\vec{d}(t)$, $\vec{\mu}(t)$ depend on choice of origin, unless $\int dx^3\rho(\vec{x}) = 0$ for $\vec{d}$, or $\int d^3x\vec{j}(\vec{x}) = 0$

Example. system of point particles

$$\begin{aligned}\vec{\mu}(t) &= \frac{1}{2c}\int d^3x\vec{x}\times\sum q_\alpha\vec{v}_\alpha(t)\delta^3(\vec{x}-\vec{S}_\alpha(t))\\ &= \sum\frac{q_\alpha}{2c}\vec{S}_\alpha(t)\times\vec{v}_\alpha(t)\end{aligned}$$

If $\frac{q_\alpha}{m_\alpha} = \frac{e}{m}$,

$$\vec{\mu}(t) = \frac{e}{2mc} \sum \vec{S}_\alpha(t) \times m_\alpha \vec{v}_\alpha(t) = \frac{e}{2mc} \vec{L}_{total}$$

where $\vec{L}_{total}$ denotes the total angular momentum.

Elementary Particles

They have definite spin and definite parties. In this particular instant, Wigner Eckart theorem states that

$$\left\langle \vec{d} \right\rangle = C \left\langle \vec{S}_{\text{spin}} \right\rangle$$

And

$$\langle \vec{\mu} \rangle = \tilde{C} \left\langle \vec{S}_{\text{spin}} \right\rangle$$

We know spin is even under parity and odd under time reversal, so is $\vec{\mu}$, but $\vec{d}$ is odd under parity and even under time reversal. This implies that

$$\left\langle \vec{d} \right\rangle_{\text{rest frame}} = 0$$

but $\langle \vec{\mu} \rangle_{\text{rest frame}}$ needs not be zero.

4.4. Multipole Expansion of the Radiation Field

Recall in radiation approximation,

$$\vec{A}(\vec{x}, t) = \frac{1}{r} \frac{1}{c} \int d^3x' \vec{j}(\vec{x}, t' - r/c + \hat{r} \cdot \vec{x}'/c)$$

we assumed $r \gg \lambda, L$. This makes calculation for $\vec{A}$ much easier. If we further have condition that $\lambda > L$, we can use multipole expansion method, which is just Taylor expansion in L/λ.

This is often called "long-wavelength expansion" or "non-relativistic expansion".

- Long-Wavelength Expansion condition $L/\lambda \ll 1$

- Non-relativistic Expansion condition $v/c \ll 1$

They are the same, by simply equating

$$\frac{L}{\lambda} = \frac{L}{cT} = \frac{v}{c}.$$

Now can expand Taylor in $\hat{r} \cdot \vec{x}'/c$

$$\vec{j}(\vec{x}, t' - r/c + \hat{r} \cdot \vec{x}'/c) = \sum \frac{1}{N!} \frac{\partial^N \vec{j}(\vec{x}, t' - r/c)}{\partial t'^N} \left(\frac{\hat{r} \cdot \vec{x}'}{c} \right)^N$$

This is legitimate because $\partial t'^N$ goes like T^N, and $(\hat{r} \cdot \vec{x}'/c)^N$ goes like $(L/c)^N$, thus we have

$$\left(\frac{L}{cT} \right)^N$$

$N = 0$ **term**

$$\vec{A} = \frac{1}{c} \frac{1}{r} \int d^3x' \vec{j}(\vec{x}, t' - \frac{r}{c})$$

It has the same retarded time for all parts of system.

Consider α component of $\vec{A}$. ($\alpha = 1, 2, 3$.)

$$A_\alpha = \int d^3x j_\alpha(\vec{x}, t - \frac{r}{c})$$

Take

$$div\,(x_\alpha \vec{j}) = \partial_\beta (x_\alpha j_\beta) = x_\alpha \partial_\beta j_\beta + \delta_{\alpha\beta} j_\beta = x_\alpha div\,\vec{j} + j_\alpha$$

Hence

$$\begin{aligned} \int d^3x j_\alpha(\vec{x}, t) &= \int d^3x \left(div\,(x_\alpha \vec{j}) - x_\alpha div\,\vec{j} \right) \\ &= \int d^3x x_\alpha \frac{\partial \rho}{\partial t} \\ &= \dot{\vec{d}}(t) \end{aligned}$$

The first term on the right vanishes by Gauss. Therefore

$$\vec{A} = \frac{1}{cr}\left[\frac{d}{dt}\vec{d}(t)\right]_{\text{retarded}}$$

Then $N = 0$ term electric field

$$\vec{E} = \frac{1}{r}\frac{1}{c^2}\hat{r} \times \left(\hat{r} \times \ddot{\vec{d}}\right)_{\text{retarded}}$$

Lecture 15 (10/24/12)

Example 1: one particle in non-relativistic motion
$\vec{r}$ field point (origin to observer). $\vec{S}$ position of particle (origin to particle)

$$\vec{A} = \frac{1}{r}\frac{1}{c}\int d^3x' \frac{\partial \vec{j}}{\partial t'}(t - \frac{r}{c})\frac{\hat{r} \cdot \vec{x}'}{c}$$

From last lecture, we have shown

$$\vec{d}(t) = q\vec{S}(t)$$

$$\ddot{\vec{d}}(t) = q\vec{a}(t)$$

$$\begin{aligned}\vec{E}(\vec{x}, t) &= \vec{E}(r, t) \\ &= \frac{q}{rc^2}\hat{r} \times (\vec{r} \times \vec{a}(t'))_{t'=t-r/c}\end{aligned}$$

This is called Lorentz result for non-relativistic accelerated motion.

$$\begin{aligned}\frac{dP(\theta, \phi, t)}{d\Omega} &= \frac{c}{4\pi}\vec{E} \cdot \vec{E}r^2 \\ &= \frac{q^2}{4\pi c^3}\left|\hat{r} \times \vec{a}\right|^2 \\ &= \frac{q^2}{4\pi c^3}|a|^2 \sin^2\theta\end{aligned}$$

last line is by choosing $\hat{r}$ to be $z-$axis. Recall $\int d\Omega \sin^2\theta = \int \sin^3 d\theta d\phi = 8\pi/3$

$$P(t) = \int \frac{dP}{d\Omega}d\Omega = \frac{2}{3}\frac{q^2}{c^3}|\vec{a}|^2$$

This is Larmor formula.

Example 2: charge under going simple harmonic motion

$$\vec{S}(t) = \hat{z} A \cos wt$$

$$\vec{a} = -\hat{z} A w^2 \cos wt$$

$$\vec{E} = \frac{1}{r}\frac{q}{c^2}\hat{r} \times \left(\hat{r} \times \hat{z}\left(-Aw^2 \cos wt\right)\right)$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\frac{q^2}{c^4} A^2 w^4 \cos^2 wt \sin^2\theta$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{q^2}{8\pi c^3} A^2 w^4 \sin^2\theta$$

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{q^2}{3c^3} A^2 w^4$$

$N = 1$ **term**

$$\vec{A}(\vec{x}, t) = \frac{1}{c}\frac{1}{r}\int d^3x' \frac{\partial \vec{j}(t - r/c)}{\partial t'}\frac{\hat{r}\cdot\vec{x}'}{c}$$

Consider α component of $\vec{A}$

$$A_\alpha(\vec{x}, t) = \frac{1}{c^2}\frac{1}{r}\hat{r}_\beta \frac{d}{dt'}\int d^3x' j_\alpha(\vec{x}', t') x'_\beta$$

We write $j_\alpha(\vec{x}', t')x'_\beta$ into two terms[6].

$$j_\alpha(\vec{x}, t)x_\beta = \frac{1}{2}\left(j_\alpha x_\beta + j_\beta x_\alpha\right) + \frac{1}{2}\left(j_\alpha x_\beta - j_\beta x_\alpha\right)$$

The first term is electric quadrupole, and the second term is magnetic dipole.

Let's take a look this term

$$\begin{aligned} \int d^3x \frac{1}{2}\left(j_\alpha x_\beta - j_\beta x_\alpha\right) &= \frac{1}{2}\epsilon_{\alpha\beta\gamma} \int d^3x \left(\vec{j} \times \vec{x}\right)_\gamma \\ &= -\epsilon_{\alpha\beta\gamma} c \mu_\gamma \end{aligned}$$

Substituting into

$$A_\alpha(\vec{x}, t) = -\frac{1}{rc}\epsilon_{\alpha\beta\gamma}\hat{r}_\beta \dot{\mu}_\gamma$$

Hence $N = 1$ term antisymmetric $\vec{A}$ is

$$\vec{A}(\vec{x}, t) = -\frac{1}{rc}\hat{r} \times \dot{\vec{\mu}}$$

Recall in radiation zone

$$\begin{aligned} \vec{B}(\vec{x}, t) &= curl\, \vec{A} \\ &= -\frac{1}{c}\hat{r} \times \dot{\vec{A}} \\ &= \frac{1}{rc^2}\left[\hat{r} \times \left(\hat{r} \times \ddot{\vec{\mu}}(\vec{x}, t')\right)\right]_{t'=t-r/c} \end{aligned}$$

This is very similar to electric field generated by electric dipole

$$\vec{E}(\vec{x}, t) = \frac{1}{rc^2}\left[\hat{r} \times \left(\hat{r} \times \ddot{\vec{d}}(\vec{x}, t')\right)\right]_{t'=t-r/c}$$

[6] We should write it into three terms (useful in quantum mechanics),

$$j_\alpha(\vec{x}, t)x_\beta = j_\alpha x_\alpha + \frac{1}{2}\left(j_\alpha x_\beta - j_\beta x_\alpha\right) + \frac{1}{2}\left(j_\alpha x_\beta + j_\beta x_\alpha\right)_{\alpha \neq \beta}$$

The first term on the right refers to a diagonal tensor. It represents spin 0 particles. The second term is an antisymmetric tensor. It represents spin 1 particles. The last term is a traceless symmetric tensor. It represents spin 2 particles.

So power radiated from electric or magnetic dipole

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{c}{4\pi} r^2 \left| \left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\} \times \left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\} \right| \\
&= \frac{c}{4\pi} \frac{1}{c^4} \left| \hat{r} \times \left\{ \begin{array}{c} \ddot{\vec{d}} \\ \ddot{\vec{\mu}} \end{array} \right\} \right|^2
\end{aligned}$$

Example. Consider a planar loop (not necessary circular) of current carrying wire.

$$I = I_0 \cos wt$$

Since $\rho = 0$, we have $\vec{d} = 0$.

$$\begin{aligned}
\vec{\mu} &= \frac{1}{2c} \int d^3 x \vec{x} \times \vec{j} \\
&= \frac{I}{2c} \int \vec{x} \times d\vec{l} \\
&= \frac{I\hat{n}}{2c} \int r dl_{\perp} \\
&= \frac{I\hat{n}}{2c} (\text{area of loop})
\end{aligned}$$

choose $\hat{n}$ to be z-axis, so loop lie on x-y plane.

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} \frac{1}{c^4} w^4 (\text{area})^2 \frac{I_0^2 \cos^2 wt}{c^2} \sin^2 \theta$$

power is zero, for $\theta = 0, \pi$.

4.5. Radiation from a Point Charge in Arbitrary Motion

This is not multipole expansion. This is true for long or short wavelength ($\lambda > L$ or $\lambda < L$) and true for relativistic motion. This is called Lienard-Wiechert radiation fields.

Consider a charge particle (q) moving with position vector $\vec{S}(t)$.

$$\vec{A} = \frac{q}{rc} \int d^3 x' \vec{v} \left(t' = t - \frac{r}{c} + \frac{\hat{r} \cdot \vec{x}'}{c} \right) \delta^3 \left(\vec{x}' - \vec{S} \left(t' = t - \frac{r}{c} + \frac{\hat{r} \cdot \vec{x}'}{c} \right) \right)$$

Lecture 16 (11/2/12)

The challenge here is that the argument of the delta function involving an explicit function of $\vec{x'}$, which makes collapsing $\vec{x}'$ difficult. The trick is to integrate $\vec{A}$ in 4 dimensions instead of 3 dimensions, then $\vec{x}'$ and t' become independent. We write

$$\begin{aligned}
\vec{A} &= \frac{q}{rc}\int d^3x' \int dt' \vec{v}\,(t')\, \delta^3\left(\vec{x}' - \vec{S}\,(t')\right) \delta\left(t' - t + \frac{r}{c} - \frac{\hat{r}\cdot\vec{x}'}{c}\right) \\
&= \frac{q}{rc}\int dt' \vec{v}(t') \delta\left(t' - t + \frac{r}{c} - \frac{\hat{r}\cdot\vec{S}(t')}{c}\right)
\end{aligned}$$

Let

$$h(t') = t' - t + \frac{r}{c} - \frac{\hat{r}\cdot\vec{S}(t')}{c}$$

then

$$\frac{dh(t')}{dt'} = 1 - \hat{r}\cdot\frac{\vec{v}(t')}{c}$$

Let $\vec{v}/c = \vec{\beta}(t)$, and $\kappa = 1 - \hat{r}\cdot\vec{\beta}(t)$,

$$\frac{dh(t')}{dt'} = \kappa(t')$$

Notice $\kappa > 0$ for all t', if $|\vec{v}| < c$.

Because $h(t)$ goes from $-\infty$ to ∞, (assume $\vec{S}$ is localized) and $h'(t) > 0$, $h(t)$ can attain one and only one zero. then

$$\vec{A}(\vec{x}, t) = \left(\frac{q}{rc}\frac{\vec{v}(t')}{\kappa(t')}\right)_{t' = t - r/c + \hat{r}\cdot\vec{S}(t')/c}$$

$$\vec{E}(\vec{x}, t) = \frac{1}{c}\hat{r} \times \left(\hat{r} \times \frac{\partial \vec{A}}{\partial t}\right)$$

Consider

$$\begin{aligned}
\frac{\partial \vec{A}}{\partial t} &= \frac{\partial \vec{A}(t')}{\partial t'}\frac{1}{dt/dt'} \\
&= \frac{q}{rc}\left(\frac{\vec{a}}{\kappa} - \frac{\vec{v}}{\kappa^2}\frac{d\kappa}{dt'}\right)\left(\frac{1}{\kappa}\right)
\end{aligned}$$

$d\kappa/dt' = -\hat{r}\cdot\vec{a}/c$

$$\begin{aligned}
\frac{\partial \vec{A}}{\partial t} &= \frac{q}{rc}\left(\frac{\vec{a}}{\kappa} - \frac{\vec{v}}{\kappa^2}\frac{d\kappa}{dt'}\right)_{t'}\left(\frac{1}{\kappa}\right)_{t'} \\
&= \frac{q}{rc\kappa^3}\left(\kappa\vec{a} + \frac{\vec{v}}{c}\hat{r}\cdot\vec{a}\right) \\
&= \frac{q}{rc\kappa^3}\left[\vec{a} - \left(\hat{r}\cdot\frac{\vec{v}}{c}\right)\vec{a} + \frac{\vec{v}}{c}\hat{r}\cdot\vec{a}\right] \\
&= \frac{q}{rc\kappa^3}\left[\vec{a} + \hat{r}\times\left(\frac{\vec{v}}{c}\times\vec{a}\right)\right]
\end{aligned}$$

Therefore

$$\vec{E}(\vec{x},t) = \frac{q}{rc^2\kappa^3}\left[\hat{r}\times(\hat{r}\times\vec{a}) + \hat{r}\times\left(\hat{r}\times\left(\hat{r}\times\left(\vec{\beta}\times\vec{a}\right)\right)\right)\right]$$

$\hat{r}\times\left(\hat{r}\times\left(\vec{\beta}\times\vec{a}\right)\right) = \hat{r}\left(\hat{r}\cdot\left(\vec{\beta}\times\vec{a}\right)\right) - \left(\vec{\beta}\times\vec{a}\right)(\hat{r}\cdot\hat{r})$

$$\begin{aligned}
\vec{E}(\vec{x},t) &= \frac{q}{rc^2\kappa^3}\left[\hat{r}\times(\hat{r}\times\vec{a}) - \hat{r}\times\left(\vec{\beta}\times\vec{a}\right)\right] \\
&= \frac{q}{rc^2\kappa^3}\hat{r}\times\left[\left(\hat{r}-\vec{\beta}\right)\times\vec{a}\right]_{t'=t-r/c+\hat{r}\cdot\vec{S}(t')/c}
\end{aligned}$$

Discussion

(1) If $\vec{a}=0$, then $\vec{E}=0$. No radiation. This will be different situation if we have $|\vec{v}|>c$. For $|\vec{v}|>c$ case, Cherenkov radiation, even $\vec{a}=0$, we can still have $\vec{E}=0$. That is because as the particle moving in the dielectric material with a speed greater than the speed of light in the medium, it causes the material to polarize when the particle passes by, and the polarization generates radiation.

(2) Let's consider the case $|\vec{v}|\ll c$. That is $|\vec{\beta}|\ll 1$, $\kappa\approx 1$.

We expand $\vec{S}(t')$ around $t'=0$,

$$\vec{S}(t') = \vec{S}(0) + \vec{v}(\xi)t'$$

by Taylor reminder theorem there exists some $\xi\in[0,t']$. Because $\vec{v}(\xi)/c\ll 1$, we ignore this term.

Therefore

$$\vec{E}(\vec{x},t) = \frac{q}{rc^2}\hat{r}\times(\hat{r}\times\vec{a})_{t'=t-r/c+\hat{r}\cdot\vec{S}(t'=0)/c}$$

This is same as electric dipole approximation. We could neglect $\vec{S}(t'=0)$ term if we chose the origin to be at $\vec{S}(t'=0)$, but we don't want to do that, for keeping this term enables us to consider several particle system, since only one particle can be at origin at $t'=0$.

(3) Suppose $\vec{S}(t)\sim\hat{z}\cos wt$, which has only one frequency, but $\vec{E}$ will have fundamental frequency w, and overtones $2w$, $3w$,.... This is because algebraically

$$\vec{E}\sim\frac{1}{(1-c\sin wt)^3}\sin wt\cos wt$$

and

$$\frac{1}{1-c\sin wt}\sim 1+c\sin wt+c^2\sin^2 wt+...$$

so multiplying sine & cosine gives all harmonics.

Another way to understand this is to look at $\rho(\vec{x}',t')$ and $\vec{j}(\vec{x}',t')$. If we pick a $\vec{x}'$ where particle would pass by, we see that a series of delta function appear on the graphs of $\rho(t')$ and $\vec{j}(t')$. These peaks are not generally equally spacing (they could be if $\vec{x}'=0$) but they are periodic. Hence w, $2w$, $3w$,... all present in ρ and $\vec{j}$, so they are in $\vec{E}$.

We can compute the power

$$\begin{aligned}\frac{dP(\vec{x},t)}{d\Omega} &= \left\{\frac{q^2}{4\pi c^3}\frac{1}{\kappa^6}\left|\hat{r}\times[(\hat{r}-\beta)\times\vec{a}]\right|^2\right\}_{\text{ret}}\\ &\sim \frac{|\vec{a}|^2}{\left(1-\vec{\beta}\cdot\hat{r}\right)^6}\end{aligned}$$

Note:

(1) $\vec{a}$ or $-\vec{a}$ gives same power. Physically this means power radiated is the same for particle speeding up or particle slowing down.

(2) Examining the denominator $\left(1-\vec{\beta}\cdot\hat{r}\right)^6$, we see that if the angle between $\vec{\beta}$ and $\hat{r}$ is very small, and $|\vec{v}|$ is compatible to c, $1-\vec{\beta}\cdot\hat{r}\ll 1$, very large power in forward direction (called "forward peaking"). At the same time, the power behind

the particle is very small. That is because $\vec{\beta}\cdot\hat{r}<0$, and raising to the power of 6 makes power small. Surprisingly, if the person is in front of the particle, he will not get any radiation, because $\hat{r}$ is parallel to $\vec{a}$, this gives zero numerator.

(3) notice the expression $dP(\vec{x},t)/d\Omega$ above is the power detected at time t, is not equivalent to the power emitted

$$\frac{dP(t)}{d\Omega}=\frac{d^2E}{dtd\Omega}=\frac{d^2E}{dt'd\Omega}\frac{dt'}{dt}=\frac{dP'(t')}{d\Omega}\frac{dt'}{dt}=\frac{dP'(t')}{d\Omega}\frac{1}{\kappa}$$

where $dP'(\vec{x},t')/d\Omega$ is the power emitted at the retarded time t'. Therefore power emitted is

$$\frac{dP(\vec{x},t')}{d\Omega}=\left\{\frac{q^2}{4\pi c^3}\frac{1}{\kappa^5}\left|\hat{r}\times\left[\left(\hat{r}-\vec{\beta}\right)\times\vec{a}\right]\right|^2\right\}$$

This is due to relativistic Doppler effect.

Lecture 17 (11/7/12)

Example 1: Relativistically correct

Consider $\vec{v}\parallel\vec{a}$, let $\vec{v}$ on $+\hat{z}$, and $\vec{a}$ is on $\pm\hat{z}$. Using spherical coordinate, we have

$$\vec{\beta}\times\vec{a}=0$$

$$\kappa=1-\frac{v}{c}\cos\theta$$

So

$$\frac{dP'(\vec{x},t')}{d\Omega}=\frac{q^2\left|\vec{a}\right|^2\sin^2\theta}{4\pi c^3\left(1-\frac{v}{c}\cos\theta\right)^5}$$

Note:

(1) $v\ll c$, $dP'/d\Omega\sim\sin^2\theta$. So the polar plot of $dP'/d\Omega$ has two circles on each side of z axis for any fixed ϕ.

(2) $v<c$, and $v/c\to 1$, hence $\gamma=1/\sqrt{1-(v/c)^2}\to\infty$. The polar plot is forward peaking.

One can show

Exercise

(1) Show maximum values of power occur at

$$\cos\theta_m=\frac{1}{3}\left(\sqrt{1+15\beta^2}-1\right)$$

(2) That is

$$\theta_m \approx \frac{1}{2\gamma}$$

(3)

$$P'(t') = \int \frac{dP'(\vec{x}, t')}{d\Omega} d\Omega = \frac{2q^2}{3c^3} |\vec{a}|^2 \gamma^6$$

This is the generalization of Larmor formula.

Example 2: Relativistically correct
Consider $\vec{a} \perp \vec{v}$ at a particular time.
Let $\vec{v}$ on $\hat{z}$, and $\vec{a}$ on $\hat{x}$.

Exercise

One can show

$$\frac{dP'(\vec{x}, t')}{d\Omega} = \frac{q^2}{4\pi c^3} |\vec{a}|^2 \frac{1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}}{(1-\beta\cos\theta)^3}$$

Note:

(1) $v \ll c$,

$$\frac{dP'}{d\Omega} = \frac{q^2}{4\pi c^3} |\vec{a}|^2 \left(1 - \sin^2\theta\cos^2\phi\right)$$

So the polar plot of $dP'/d\Omega$ has two eggs on each side of z axis for fixed ϕ.

(2) $v < c$, and $\beta = v/c \to 1$,

$$\frac{dP'}{d\Omega} \sim \frac{1}{(1-\beta\cos\theta)^3}$$

Power peaks in $\theta = 0$ direction.

(3) Application of this example: Synchrotron Radiation

Radiation from a particle in circular motion. $\vec{v}$ is always perpendicular to $\vec{a}$. Radiation emitted in forward $\vec{v}$ direction as particle moves, creating a flashlight effect. Detector at some field point sees periodic blips as flashlight shines for a brief moment then repeat after each revolution.

4.6. Problem Set 8 (due 11/9/12)

Jackson Ch14: 5, 7(a), 8, 9, 10, 12, 21

For 14-8, we needs this general result. Let α be angle between $\vec{v}$ and $\vec{a}$,

$$P'(t') = \frac{2q^2}{3c^3}|\vec{a}|^2\gamma^6\left[1 - |\vec{\beta}|^2\sin^2\alpha\right]$$

4.7. Problem Set 9 (due 11/14/12)

1) An Undulator

An electron traveling with relativistic speed enters an undulator. The undulator has a periodic series of alternating polarity dipole magnets which cause small transverse oscillations in the electron's motion. Assume v_1 is very small,

$$\vec{v} = \vec{v}_0\hat{z} + v_1\cos(w_0 t)\hat{x}$$

(1) Show the Lienard-Wiechart detected radiation has a frequency $w = w(w_0, v_0, \theta)$.

(2) Find detected radiation per solid angle $dP(t)/d\Omega$.

This problem shows synchrotron light sources use undulators to produce intense electromagnetic radiation over a range of frequencies, and the frequencies observed depend on where the detector is placed.

2)

Consider $\vec{v}$ and $\vec{a}$ have an angle α, find the instantaneous power emitted per solid angle, and show

$$P'(t') = \frac{2q^2}{3c^3}|\vec{a}|^2\gamma^6\left[1 - |\vec{\beta}|^2\sin^2\alpha\right]$$

4.8. Frequency Distribution of Detected Radiation

Consider we have some localized particle motion, detector is r distance from the source, with perpendicular detector area da. We place a frequency filler in front of the detector area that allows detection of angular frequencies range from w to $w + dw$. ($\pm w$ are indistinguishable.)

Fourier analyze in time variable

$$\left\{ \begin{array}{c} \vec{j}(\vec{x},t) \\ \vec{A}(\vec{x},t) \\ \vec{E}(\vec{x},t) \end{array} \right\} = \int_{-\infty}^{\infty} \frac{dw}{\sqrt{2\pi}} e^{-iwt} \left\{ \begin{array}{c} \vec{j}_w(\vec{x}) \\ \vec{A}_w(\vec{x}) \\ \vec{E}_w(\vec{x}) \end{array} \right\}$$

Lecture 18 (11/12/12)

Find $\vec{A}_w(\vec{x})$ in radiation zone,

$$\begin{aligned} \vec{A}(\vec{x},t) &= \frac{1}{r}\frac{1}{c}\int d^3x' \vec{j}(\vec{x}', t' = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{x}'}{c}) \\ &= \frac{1}{r}\frac{1}{c}\int d^3x' \int \frac{dw}{\sqrt{2\pi}} e^{-iw(t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{x}'}{c})} \vec{j}_w(\vec{x}') \\ &= \frac{e^{ikr}}{r}\frac{1}{c}\int \frac{dw}{\sqrt{2\pi}} e^{-iwt} \int d^3x' e^{-ik\hat{r}\cdot\vec{x}'} \vec{j}_w(\vec{x}') \end{aligned}$$

Thus

$$\vec{A}_w(\vec{x}) = \frac{e^{ikr}}{r}\frac{1}{c}\int d^3x' e^{-ik\hat{r}\cdot\vec{x}'} \vec{j}_w(\vec{x}')$$

Note: 1) the factor e^{ikr}/r appears in quantum scattering problem as a spherical wave; 2) $e^{-ik\hat{r}\cdot\vec{x}'}$ takes care of the entire retardation effects.

Because

$$\vec{E}(\vec{x},t) \doteq \frac{1}{c}\hat{r} \times \left(\hat{r} \times \frac{\partial \vec{A}(\vec{x},t)}{\partial t} \right),$$

we have

$$\vec{E}_w(\vec{x}) = -iw\frac{1}{c}\hat{r} \times \left(\hat{r} \times \vec{A}_w(\vec{x}) \right) \sim \frac{e^{ikr}}{r}$$

Detected power per solid angle

$$\frac{d^2\epsilon}{dtd\Omega} = \frac{dP(p)}{d\Omega} = \frac{c}{4\pi}\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t) r^2$$

$$
\begin{aligned}
\frac{d\epsilon}{d\Omega} &= \int_{-\infty}^{\infty} \frac{dP}{d\Omega} dt = \text{total energy detected over all time per solid angle.} \\
&= \int_{-\infty}^{\infty} dt \frac{c}{4\pi} \vec{E} \cdot \vec{E} r^2 \\
&= \int_{-\infty}^{\infty} dw \frac{c}{4\pi} \vec{E}_w^* \cdot \vec{E}_w r^2 \text{ by Parseval} \\
&= \frac{c}{2\pi} r^2 \int_0^{\infty} dw \left| \vec{E}_w \right|^2 \text{ indistinguishable between } \pm w
\end{aligned}
$$

Thus

$$
\frac{\text{energy detected}}{\text{(freq interval)(solid angle)}} = \frac{d^2\epsilon}{dwd\Omega} = \frac{c}{2\pi} r^2 \left| \vec{E}_w \right|^2
$$

Notice: 1) Jackson used $\frac{d^2 I}{dwd\Omega}$, which is less meaningful than our notation, because in his way, I may be interpreted as intensity but it should really be ϵ energy; 2) To determine a frequency accurately and exactly, one needs an infinite time interval for detection.

One can define

$$
\frac{\text{average \# of photons detected}}{\text{(freq interval)(solid angle)}} = \frac{d^2 N}{dwd\Omega} = \frac{1}{\hbar w} \frac{d^2\epsilon}{dwd\Omega}.
$$

Ultraviolet Divergence

Example 1: Tiny antenna at origin

$$
\vec{j}(\vec{x}, t) = J_0 \hat{z} \delta^3(\vec{x}) \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t > 0 \ (\alpha > 0) \end{cases}
$$

$$
\begin{aligned}
\vec{j}_w(\vec{x}) &= \int_0^{\infty} \frac{dt}{\sqrt{2\pi}} e^{iwt} J_0 \hat{z} \delta^3(\vec{x}) \text{ integrate from } t > 0 \\
&= \frac{J_0 \hat{z}}{\sqrt{2\pi}} \delta^3(\vec{x}) \frac{1}{\alpha - iw}
\end{aligned}
$$

So

$$\begin{aligned}\vec{A}_w &= \frac{e^{ikr}}{r}\frac{1}{c}\int d^3x' e^{-ik\hat{r}\cdot\vec{x}'}\frac{J_0\hat{z}}{\sqrt{2\pi}}\delta^3(\vec{x}')\frac{1}{\alpha - iw} \\ &= \frac{e^{ikr}}{r}\frac{1}{c}\frac{J_0\hat{z}}{\sqrt{2\pi}}\frac{1}{\alpha - iw}\end{aligned}$$

$$\begin{aligned}\vec{E}_w(\vec{x}) &= -\frac{iw}{c}\hat{r}\times\left(\hat{r}\times\vec{A}_w\right) \\ &= -\frac{iw}{c^2}\frac{e^{ikr}}{r}\frac{J_0\hat{\theta}\sin\theta}{\sqrt{2\pi}}\frac{1}{\alpha - iw}\end{aligned}$$

Therefore

$$\frac{d^2\epsilon}{dwd\Omega} = \frac{1}{2\pi}\frac{w^2}{c^3}\frac{J_0^2\sin^2\theta}{2\pi}\frac{1}{\alpha^2 + w^2}$$

One can draw the graph of $\frac{w^2}{\alpha^2+w^2}$. The graph starts at the origin and approaches to 1 asymptotically.

If one wants to find the total energy radiated per solid angle,

$$\frac{d\epsilon}{d\Omega} = \int_0^\infty dw \frac{d^2\epsilon}{dwd\Omega} \sim \infty$$

the integral is divergent. But this is understandable. It is due to sudden jump in $\vec{j}$ at $t = 0$, which generates infinite acceleration. That in turn produces infinite amount energy. If we have $\vec{j}$ starts at $t = 0$ gradually build up in times τ, then the graph of $\frac{d^2\epsilon}{dwd\Omega}$ v.s w has the same sharp as before for very small $t > 0$. As t gets larger, it will not approach 1, but delays at time character $1/\tau$. So the integral is convergent.

Infrared Divergence

Example 2: Bremsstrahlung (breaking radiation)

Charge particle moving at constant $\vec{v} = v\hat{z}$ for $t < 0$, a target is placed at origin which suddenly stops the particle at $t = 0$

$$\vec{j}(\vec{x}, t) = \begin{cases} ev\hat{z}\delta^3(\vec{x} - vt\hat{z}) & t < 0 \\ 0 & t \geq 0 \end{cases}$$

$$\begin{aligned}
\vec{A}_w(\vec{x}) &= \frac{e^{ikr}}{r}\frac{1}{c}\int d^3x' e^{-ik\hat{r}\cdot\vec{x}'}\int_{-\infty}^{0}\frac{dt}{\sqrt{2\pi}}ev\hat{z}\delta^3(\vec{x} - vt\hat{z})e^{iwt} \\
&= \frac{e^{ikr}}{r}\frac{1}{c}\int_{-\infty}^{0}\frac{dt}{\sqrt{2\pi}}ev\hat{z}e^{iwt}\int d^3x' e^{-ik\hat{r}\cdot\vec{x}'}\delta^3(\vec{x} - vt\hat{z}) \\
&= \frac{e^{ikr}}{r}\frac{ev\hat{z}}{c\sqrt{2\pi}}\int_{-\infty}^{0} dt e^{iwt}e^{-ikrvt\cos\theta} \\
&= \frac{e^{ikr}}{r}\frac{ev\hat{z}}{c\sqrt{2\pi}}\int_{-\infty}^{0} dt e^{iwt\left(1 - \frac{v}{c}\cos\theta\right)}e^{+\epsilon t} \text{ convergent factor} \\
&= \frac{e^{ikr}}{r}\frac{ev\hat{z}}{c\sqrt{2\pi}}\frac{1}{iw\left(1 - \frac{v}{c}\cos\theta\right)}
\end{aligned}$$

$$\vec{E}_w = -\frac{e^{ikr}}{r}\frac{ev}{c^2\sqrt{2\pi}}\frac{\hat{\theta}\sin\theta}{1 - \frac{v}{c}\cos\theta}$$

Therefore

$$\begin{aligned}
\frac{d^2\epsilon}{dwd\Omega} &= \frac{c}{2\pi}r^2\left|\vec{E}_w\right|^2 \\
&= \frac{e^2v^2}{4\pi^2c^3}\frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^2}
\end{aligned}$$

Note:

1) Polar plot of angular distribution: for $v \ll c$, $\frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^2} \sim \sin^2\theta$. Two circles on two sides of z axis for any fixed ϕ. For $v \approx c$, we have peaking in forward velocity direction.

2) $\frac{d^2\epsilon}{dwd\Omega}$ is independent of w. So clearly total integral of w is divergent, which is due to the same explanation as as example 1. But if one wants to compute the total photons in the small interval from 0 to say w_0, one would see it is divergent.

There are infinitely many photons in the interval

$$\frac{dN}{d\Omega} = \int \frac{d^2N}{dwd\Omega}dw \sim \int_0^{w_0} \frac{1}{w}dw = \ln w|_0^{w_0} = \infty.$$

This is of course not a problem in classical EM, since the concept of quantized photon is meaningless. But in quantum field theory, this runs into a big trouble. Although one can interpret this as infinitely many photons with 0 energy, perturbation theory does not work in that regard. This is called "infrared divergence"

3)

$$\frac{d^2N}{dwd\Omega} = \frac{1}{\hbar w}\frac{d^2\epsilon}{dwd\Omega} = \frac{e^2}{\hbar c}\frac{v^2\sin^2\theta}{4w\pi^2c^2\left(1-\frac{v}{c}\cos\theta\right)^2}$$

The factor $\frac{e^2}{\hbar c}$ is fine structure constant, is $\approx 1/137$.

4) If the acceleration occurs over time interval τ, then $\frac{d^2\epsilon}{dwd\Omega}$ is cut off at $w \gtrsim 1/\tau$.

Radiation of Beta Decay

Beta Decay

$$Z_A \to (Z\pm 1)_A + \left\{\begin{array}{c} e^- \\ e^+ \end{array}\right\} + \left\{\begin{array}{c} \bar{v} \\ v \end{array}\right\}$$

nucleus: atomic number Z number of protons. Atomic mass: A number of neutrons+protons.

e^- electron, e^+positron, v neutrino, $\bar{v}$ anti-neutrino.

Let's consider at $t=0$, $\beta^\pm$ decay takes place. Before decay, Z_A at rest. After decay, emitting an electron or positron at speed v, but the nucleus $(Z\pm 1)_A$ still at rest i.e. essentially zero recoil. In term of charge motion giving $\vec{j}(\vec{x},t)$. Thus $e^\mp$ at rest $t \le 0$, and moves at constant v for $t > 0$,

$$\vec{j} = \begin{cases} \mp ev\hat{z}\delta^3(\vec{x}-vt\hat{z}) & t>0 \\ 0 & t<0 \end{cases}$$

Therefore there must be electromagnetic radiation accompany the $\beta^\pm$ decay, which is the "inner Bremsstrahlung". Same calculation shows sudden start of charge at

$t = 0$ corresponds to a positive infinite acceleration.

$$\frac{d^2\epsilon}{dw d\Omega} \sim \frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^2}$$

forward peaking.

Lecture 19 (11/14/12)

4.9. Problem Set 10 (due 11/21/12)

1)

Consider a tiny antenna at the origin, carrying current density

$$\vec{j}(\vec{x}, t) = J_0 \hat{z} \delta^3(\vec{x}) F(t)$$

where

$$F(t) = e^{-\alpha t} \sin w_0 t \Theta(t)$$

Using $\vec{E}_w(\vec{x})$ find $\frac{d^2\epsilon}{dw d\Omega}$ and find $\frac{d\epsilon}{dw}$, then integrate to find total ϵ. Using $\vec{E}(\vec{x}, t)$ find $\frac{d^2\epsilon}{dt d\Omega}$ and find $P(t) = \frac{d\epsilon}{dt}$, then integrate to find total ϵ. The two methods should give the same ϵ.

2)

Similar to problem 1, change

$$F(t) = \sum_{N=-\infty}^{\infty} \frac{d\delta(t - N\tau)}{dt} \mu^{|N|}$$

$0 < \mu < 1$, current is periodic pulses at $t = 0, \pm\tau, \pm 2\tau, ...$ with decreasing amplitude for $|t|$ large

(a) Find $\frac{d^2\epsilon}{dw d\Omega}$ (notice $\mu = 1$, the series is not convergent.)

(b) Let $\mu \to 1$, show $\frac{d^2\epsilon}{dw d\Omega} = 0$ except at some w. Find those w.

3)

A charge particle is at rest $t < 0$, then it is released under SHO with fictional force. The motion is non-relativistic,

$$m\ddot{\vec{S}} + m2\alpha\dot{\vec{S}} + mw_0^2\vec{S} = 0$$

$0 < \alpha < w_0$. Find $\frac{d^2\epsilon}{dwd\Omega}$.

4)

A particle is at rest $t < 0$, suddenly decays into two particles q_1, q_2 move in opposite direction with the same speed v.

(a) Find $\frac{d^2\epsilon}{dwd\Omega}$.
(b) Consider $v/c \ll 1$, find $\frac{d^2\epsilon}{dwd\Omega}$ for $q_1 = q_2$ and $q_1 = -q_2$.
(c) Consider $v/c \approx 1$, find $\frac{d^2\epsilon}{dwd\Omega}$ for $q_1 = q_2$ and $q_1 = -q_2$.

5)

A hard ball of radius R is at the origin. A point particle moves with speed v in the zoy plane in $+\hat{z}$ parallel direction, collide elastically with the ball at impact parameter b. $|v_{final}| = v$. Assume the ball is transparent, find $\frac{d^2\epsilon}{dwd\Omega}$.

5. Macroscopic Maxwell Equations

Maxwell equations in a material. This involves averaging of the exact Maxwell equations.

We start from microscopic Maxwell equations.

$$div\,\vec{E}'(\vec{x},t) = 4\pi\rho'(\vec{x},t)$$

$$curl\,\vec{E}'(\vec{x},t) = -\frac{1}{c}\frac{\partial\vec{B}'(\vec{x},t)}{\partial t}$$

$$div\,\vec{B}'(\vec{x},t) = 0$$

$$curl\,\vec{B}'(\vec{x},t) = \frac{4\pi}{c}\vec{j}'(\vec{x},t) + \frac{1}{c}\frac{\partial \vec{E}'(\vec{x},t)}{\partial t}$$

They are exact, non-averaging Maxwell equations. And these microscopic quantities $\vec{E}', \vec{B}', \rho'$, and $\vec{j}'$ have enormous fluctuation as one goes from one point to another. This leads to average microscopic quantity $F'(\vec{x},t)$, we take

$$F(\vec{x},t) = \langle F'(\vec{x},t)\rangle$$

only the spatial average not time average.

5.1. Microscopic Units and Weight Function

Microscopic units are the material that stays bounded in as a unit. Examples of microscopic units are bound systems of charges: molecule, atom, ion; point charges: valence electron in a conductor. We let microscopic units have a typical length scale a, and L to be the length scale of the resolution of the measurement instrument that measures charges and fields.

We will average over spatial distance L.

Because

$$\frac{a}{L} \ll 1,$$

we shall expand weight function in a/L.

A weight function $w(\vec{x})$ has 3 general properties:

(1) $w(\vec{x}) > 0$; (2) $\int_{all\,space} d^3x w(\vec{x}) = 1$; (3) $w(\vec{x}) = w(|\vec{x}|)$ so no anisotropic are introduced by the averaging;

(4) support of $w(\vec{x})$ is $|\vec{x}| < L$, and $w(\vec{x})$ is smooth.

We obtain macro (average) F from micro F', by doing $w(\vec{x})$ averaging (or call smearing, or call convoluting) according to

$$F(\vec{x},t) = \langle F'(\vec{x},t)\rangle = \int d^3y w(\vec{x}-\vec{y})F'(\vec{y},t)$$

Properties of averaging:

(1) $\langle \delta^3(\vec{x})\rangle = \int d^3y w(\vec{x}-\vec{y})\delta^3(\vec{y}) = w(\vec{x})$.

(2) $\frac{\partial}{\partial t}\langle F'(\vec{x},t)\rangle = \left\langle \frac{\partial}{\partial t}F'(\vec{x},t)\right\rangle$.

(3) $\frac{\partial}{\partial x_i}\langle F'(\vec{x},t)\rangle = \left\langle \frac{\partial}{\partial x_i}F'(\vec{x},t)\right\rangle$.

Proof: $\frac{\partial}{\partial x_i}\int d^3y w(\vec{x}-\vec{y})F'(\vec{y},t) = \int d^3y \frac{\partial w(\vec{x}-\vec{y})}{\partial x_i}F' = -\int d^3y \frac{\partial w(\vec{x}-\vec{y})}{\partial y_i}F' = \int d^3y w(\vec{x}-\vec{y})\frac{\partial F'(\vec{y})}{\partial y_i} = \left\langle \frac{\partial}{\partial x_i}F'(\vec{x},t)\right\rangle$.

Applying these properties of averaging, we can average Maxwell equations, for example

$$\left\langle div\,\vec{E}'\right\rangle = \langle 4\pi\rho'\rangle$$

implies

$$div\,\left\langle \vec{E}'\right\rangle = 4\pi\,\langle\rho'\rangle$$

That is

$$div\,\vec{E} = 4\pi\rho$$

We define

$$\rho_{total} = \rho = \langle\rho'\rangle$$

$$\vec{j}_{total} = \vec{j} = \left\langle \vec{j}'\right\rangle$$

Conclusion: Macroscopic Maxwell equations are exactly the same as micro equations.

$$div\,\vec{E} = 4\pi\rho_{total}$$

$$curl\,\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$div\,\vec{B} = 0$$

$$curl\,\vec{B} = \frac{4\pi}{c}\vec{j}_{total} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

with

$$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

Macroscopic Maxwell equations

Reference Jackson 6.6

We want to calculate the source ρ_{total}, $\vec{j}_{total}$ in term of properties of the material (polarization, magnetization,..). To do so, we will use a model of the material, classical non-relativistic model of point particle in the material.

Consider a macroscopic size of L consisted of many microscopic units (in the order of 10^{23}). Let $\vec{R}_i(t)$ be the center of mass position of the ith microscopic unit with respect to the origin. (choosing center mass here is not the definite way. Any other representative point also does the job, such as center of charge distribution of the ith unit.) The ith microscopic unit consisted of many charges. Let $\vec{r}_{i\alpha}(t)$ be the position of the α charge in the ith microscopic unit. Thus the position of $i\alpha$ charge is

$$\vec{S}_{i\alpha}(t) = \vec{R}_i(t) + \vec{r}_{i\alpha}(t)$$

and

$$\dot{\vec{S}}_{i\alpha}(t) = \dot{\vec{R}}_i(t) + \dot{\vec{r}}_{i\alpha}(t)$$

so

$$\rho'(\vec{x},t) = \sum_i \sum_{\alpha\in i} q_{i\alpha}\delta^3(\vec{x} - \vec{S}_{i\alpha}(t))$$

$$\rho_{total} = \langle\rho'\rangle = \sum_i \sum_{\alpha\in i} q_{i\alpha}\left\langle \delta^3(\vec{x} - \vec{R}_i(t) - \vec{r}_{i\alpha}(t))\right\rangle = \sum_i \sum_{\alpha\in i} q_{i\alpha} w\left(\vec{x} - \vec{R}_i(t) - \vec{r}_{i\alpha}(t)\right)$$

Recall

$$w(\vec{x}+\vec{x}_0) = w(\vec{x}_0) + \frac{\partial w(\vec{x}_0)}{\partial x_l}\vec{x}_l + \frac{1}{2}\frac{\partial^2 w(\vec{x}_0)}{\partial x_l \partial x_m}x_l x_m + ... = w(\vec{x}_0) + \vec{\nabla}_{x_0} w(\vec{x}_0)\cdot\vec{x}$$

We expand ρ to the first order in a/L,

$$\begin{aligned}
\rho_{total}(\vec{x},t) &= \sum_i \sum_{\alpha\in i} q_{i\alpha}\left[w(\vec{x} - \vec{R}_i(t)) - \vec{\nabla}_x w(\vec{x} - \vec{R}_i(t))\cdot\vec{r}_{i\alpha}(t)\right] \\
&= \sum_i w(\vec{x} - \vec{R}_i(t))\sum_{\alpha\in i} q_{i\alpha} - \sum_i \vec{\nabla}_x w(\vec{x} - \vec{R}_i(t))\cdot\sum_{\alpha\in i} q_{i\alpha}\vec{r}_{i\alpha}(t)
\end{aligned}$$

We define the total charge of the ith microscopic unit,

$$\sum_{\alpha\in i} q_{i\alpha} = Q_i$$

and total electric dipole moment with respect to the center of mass of ith microscopic unit, (notice electric dipole mom depends on the origin.)

$$\sum_{\alpha\in i} q_{i\alpha}\vec{r}_{i\alpha}(t) = p_i$$

then

$$\begin{aligned}
\rho_{total}(\vec{x},t) &= \sum_i w(\vec{x}-\vec{R}_i(t))Q_i - \sum_i \vec{\nabla}_x w(\vec{x}-\vec{R}_i(t))\cdot p_i \\
&= \sum_i Q_i \left\langle \delta^3(\vec{x}-\vec{R}_i(t))\right\rangle - \sum_i \vec{\nabla}_x \left\langle \delta^3(\vec{x}-\vec{R}_i(t))\right\rangle \cdot p_i \\
&= \left\langle \sum_i Q_i\delta^3(\vec{x}-\vec{R}_i(t))\right\rangle - \sum_i \frac{\partial}{\partial x_l}\left\langle \delta^3(\vec{x}-\vec{R}_i(t))\right\rangle p_{il} \\
&= \left\langle \sum_i Q_i\delta^3(\vec{x}-\vec{R}_i(t))\right\rangle - div\left\langle \sum_i \vec{p}_i\delta^3(\vec{x}-\vec{R}_i(t))\right\rangle
\end{aligned}$$

We define

$$\left\langle \sum_i Q_i\delta^3(\vec{x}-\vec{R}_i(t))\right\rangle = \rho_{free}$$

$$-div\left\langle \sum_i \vec{p}_i\delta^3(\vec{x}-\vec{R}_i(t))\right\rangle = -div\,\vec{P}(\vec{x},t) = \rho_{bound}$$

ρ_{free} average charge density where charge density as if all charge on the ith microscopic unit is Q_i concentrated at center of mass. $\vec{P}$ polarization vector, the average electric dipole moment per volume with each $\vec{p}_i$ concentrated at ith center of mass.

$$\rho_{total} = \rho_{free} + \rho_{bound}$$

The reason we include ρ_{bound} is because matter is essentially neutral, ρ_{free} (zero order term) is actually quite small, one needs the next order term (first order in

a/L). Zero order term treats $a = 0$. The bound term is the correction due to non-zero size of microscopic unit length a.

Lecture 20 (11/19/12)

We now do $\vec{j}_{total}$.

$$\begin{aligned}
\vec{j}_{total} &= \left\langle \vec{j}' \right\rangle = \left\langle \sum_i \sum_{\alpha\in i} q_{i\alpha} \dot{\vec{S}}_{i\alpha}(t) \delta^3(\vec{x} - \vec{S}_{i\alpha}(t)) \right\rangle \\
&= \sum_i \sum_{\alpha\in i} q_{i\alpha} \left[\dot{\vec{R}}_i(t) + \dot{\vec{r}}_{i\alpha}(t) \right] w\left(\vec{x} - \vec{R}_i(t) - \vec{r}_{i\alpha}(t) \right) \\
&= \sum_i \sum_{\alpha\in i} q_{i\alpha} \left[\dot{\vec{R}}_i(t) + \dot{\vec{r}}_{i\alpha}(t) \right] \left[w\left(\vec{x} - \vec{R}_i(t) \right) - \vec{r}_{i\alpha} \cdot \vec{\nabla} w\left(\vec{x} - \vec{R}_i(t) \right) \right]
\end{aligned}$$

There are four terms to be calculated.

Before we process, let us make two assumptions

(1) If a microscopic unit moves ($\dot{\vec{R}}_{i\alpha} \neq 0$), then this microcosmic unit has no electric dipole moment ($\vec{p}_i = 0$). This assumption works well for valence electron which do not have electric dipole.

(2) All quadrupole or higher moments are negligibly small for each microscopic unit.

First term

$$\begin{aligned}
\sum_i \sum_{\alpha\in i} q_{i\alpha} \dot{\vec{R}}_i(t) w\left(\vec{x} - \vec{R}_i(t) \right) &= \sum_i w\left(\vec{x} - \vec{R}_i(t) \right) w\left(\vec{x} - \vec{R}_i(t) \right) \sum_{\alpha\in i} q_{i\alpha} \\
&= \left\langle \sum_i Q_i \dot{\vec{R}}_i \delta^3\left(\vec{x} - \vec{R}_i(t) \right) \right\rangle \\
&\equiv \vec{j}_{free}
\end{aligned}$$

$\vec{j}_{free}$ average current density as if each microscopic unit moves at velocity $\dot{\vec{R}}_i$ with all of its charge Q_i at the center of mass.

Second term

$$\begin{aligned}
-\sum_i \sum_{\alpha\in i} q_{i\alpha}\dot{\vec{R}}_i(t)\vec{r}_{i\alpha}\cdot\vec{\nabla}w\left(\vec{x}-\vec{R}_i(t)\right) &= -\sum_i \dot{\vec{R}}_i(t)\vec{\nabla}w\left(\vec{x}-\vec{R}_i(t)\right)\cdot\sum_{\alpha\in i} q_{i\alpha}\vec{r}_{i\alpha} \\
&= -\sum_i \dot{\vec{R}}_i(t)\vec{\nabla}w\left(\vec{x}-\vec{R}_i(t)\right)\cdot\vec{p}_i \\
&= 0
\end{aligned}$$

by assumption (1), one of them, $\dot{\vec{R}}_i$ or $\vec{p}_i$ must be zero.

Third term

$$\begin{aligned}
\sum_i \sum_{\alpha\in i} q_{i\alpha}\dot{\vec{r}}_{i\alpha}(t)w\left(\vec{x}-\vec{R}_i(t)\right) &= \sum_i w\left(\vec{x}-\vec{R}_i(t)\right)\sum_{\alpha\in i} q_{i\alpha}\dot{\vec{r}}_{i\alpha}(t) \\
&= \frac{\partial}{\partial t}\left(\sum_i w\left(\vec{x}-\vec{R}_i(t)\right)\vec{p}_i\right)
\end{aligned}$$

no additional term generated because $\vec{p}_i$ is non-zero only when $\vec{R}_i$ is indep of time by assumption (1).

$$\begin{aligned}
\sum_i \sum_{\alpha\in i} q_{i\alpha}\dot{\vec{r}}_{i\alpha}(t)w\left(\vec{x}-\vec{R}_i(t)\right) &= \frac{\partial}{\partial t}\left\langle\sum_i \vec{p}_i\delta^3\left(\vec{x}-\vec{R}_i(t)\right)\right\rangle \\
&= \frac{\partial}{\partial t}\vec{P}(\vec{x},t)
\end{aligned}$$

Exercise

One may try to combine second and third terms, to see if they cancel without assumption (1). Exercise show they do not cancel, and assumption (1) is necessary.

Four term

$$\begin{aligned}
-\left\{\sum_i \sum_{\alpha\in i} q_{i\alpha}\dot{\vec{r}}_{i\alpha}(t)\vec{r}_{i\alpha}\cdot\vec{\nabla}w\left(\vec{x}-\vec{R}_i(t)\right)\right\}_l &= -\sum_i \sum_{\alpha\in i} q_{i\alpha}\left(\dot{\vec{r}}_{i\alpha}\right)_l\left[(\vec{r}_{i\alpha})_m\frac{\partial}{\partial x_m}w\left(\vec{x}-\vec{R}_i\right)\right] \\
&= -\frac{\partial}{\partial x_m}\sum_i w\left(\vec{x}-\vec{R}_i\right)\sum_{\alpha\in i} q_{i\alpha}\left(\dot{\vec{r}}_{i\alpha}\right)_l(\vec{r}_{i\alpha})_m
\end{aligned}$$

We write

$$\begin{aligned}\sum_{\alpha\in i} q_{i\alpha}\left(\dot{\vec{r}}_{i\alpha}\right)_l(\vec{r}_{i\alpha})_m &= \frac{1}{2}\left[\left(\dot{\vec{r}}_{i\alpha}\right)_l(\vec{r}_{i\alpha})_m + \left(\dot{\vec{r}}_{i\alpha}\right)_m(\vec{r}_{i\alpha})_l\right] + \frac{1}{2}\left[\left(\dot{\vec{r}}_{i\alpha}\right)_l(\vec{r}_{i\alpha})_m - \left(\dot{\vec{r}}_{i\alpha}\right)_m(\vec{r}_{i\alpha})_l\right] \\ &= \sum_{\alpha\in i}\frac{q_{i\alpha}}{2}\frac{d}{dt}\left[(\vec{r}_{i\alpha})_l(\vec{r}_{i\alpha})_m\right] + \sum_{\alpha\in i}\frac{q_{i\alpha}}{2}\epsilon_{lmn}\left(\dot{\vec{r}}_{i\alpha}\times\vec{r}_{i\alpha}\right)_n\end{aligned}$$

as symmetric and antisymmetric parts. And the symmetric part relates to change of electric quadrupole moment, so by assumption (2), that term drops out. The antisymmetric part relates to magnetic dipole moment

$$\vec{\mu}_i = \sum_{\alpha\in i}\frac{q_{i\alpha}}{2c}\vec{r}_{i\alpha}\times\dot{\vec{r}}_{i\alpha}$$

We can simplify fourth term in Cartesian tensor nation becomes to

$$\begin{aligned} c\epsilon_{lmn}\frac{\partial}{\partial x_m}\sum_i w\left(\vec{x}-\vec{R}_i\right)(\mu_i)_n &= c\cdot curl_{\vec{x}}\sum_i w\left(\vec{x}-\vec{R}_i\right)\vec{\mu}_i \\ &= c\cdot curl_{\vec{x}}\left\langle\sum_i\vec{\mu}_i\delta^3\left(\vec{x}-\vec{R}(t)\right)\right\rangle \\ &= c\cdot curl_{\vec{x}}\vec{m}(\vec{x},t)\end{aligned}$$

$\vec{m}(\vec{x},t)$ magnetization vector, average magnetic dipole moment per volume with respect to center of mass of ith microscopic unit.

Summary

$$\rho_{total} = \rho_{free} + \rho_{bound}$$

$\rho_{bound} = -div\,\vec{P}$

$$\vec{j}_{total} = \vec{j}_{free} + \vec{j}_{bound}$$

$\vec{j}_{bound} = \frac{\partial\vec{P}}{\partial t} + c\,curl_{\vec{x}}\vec{m}$

Macroscopic Maxwell equations

$$div\,\vec{E} = 4\pi\rho_{total}$$

gives

$$div\left(\vec{E}+4\pi\vec{P}\right)=4\pi\rho_{free}$$

By defining, electric displacement vector, because of its frequently occurring, not because it has any physical significance, (only $\vec{E}$, $\vec{P}$ have physical significance.)

$$\vec{E}+4\pi\vec{P}=\vec{D}$$

Therefore,

$$div\,\vec{D}=4\pi\rho_{free}$$

Similarly,

$$curl\,\vec{B}=\frac{4\pi}{c}\vec{j}_{total}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

gives

$$curl\left(\vec{B}-4\pi\vec{m}\right)=\frac{4\pi}{c}\vec{j}_{free}+\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E}+4\pi\vec{P}\right)$$

By defining, magnetic field vector, because of its frequently occurring, not because it has any physical significance, (only $\vec{B}$, $\vec{m}$ have physical significance.) i.e. $\vec{E}$ and $\vec{B}$ are fundamental fields, $\vec{D}$, $\vec{H}$ are not.

$$\vec{B}-4\pi\vec{m}=\vec{H}$$

Therefore,

$$curl\,\vec{H}=\frac{4\pi}{c}\vec{j}_{free}+\frac{1}{c}\frac{\partial}{\partial t}\vec{D}$$

Note:

(1) The other two Maxwell equations do not change.

$$div\,\vec{B}=0$$

$$curl\,\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

(2) Units in cgs, $\vec{D}$,$\vec{E}$,$\vec{P}$, $\vec{B}$,$\vec{H}$, $\vec{m}$ are all in gauss.

(3) $\vec{D}$ has ρ_{free} as its source, but $\vec{D}$ is not completely determined by ρ_{free},

one needs $curl\,\vec{D}$. $\vec{H}$ has $\vec{j}_{free}$ as its source, but $\vec{H}$ is not completely determined by $\vec{j}_{free}$, one needs $div\,\vec{H}$.

5.2. Constitutive Relations

They are phenomenological relations, and they are approximately true when $\vec{E}$, $\vec{B}$ are weak.

$\vec{P}$, $\vec{m}$, $\vec{j}_{free}$ are functions of $\vec{E}$ and $\vec{B}$. We do Taylor expansion, and only keep the linear term. Roughly we have

$$\vec{P} = \vec{P}_0 + \chi_{el}\vec{E} + higher\,order(\sim E \cdot E)$$

$$\vec{m} = \vec{m}_0 + \chi_{meg}\vec{H} + higher\,order(\sim H \cdot H)$$

$$\vec{j}_{free} = \vec{j}_{free_0}$$

or

$$\vec{j}_{free} = \sigma_c\vec{E} + higher\,order(\sim E \cdot E)$$

Notes:

(1) anisotropic $\underset{\sim el}{\chi}$, $\underset{\sim mag}{\chi}$, $\underset{\sim c}{\sigma}$ should be second rank tensors.

(2) $\chi_{el} \circ \vec{E}$, $\chi_{meg} \circ \vec{H}$, $\sigma_c \circ \vec{E}$ should be convolutions in space and time, to ensure time lag and space lag. But both (1) & (2) still assume linear relations

(3) $\vec{j}_{free} = \sigma_c\vec{E}$ is the microscopic Ohm's law.

(4) Here we assume fields are weak. For intense field, such as a laser, we have to modify.

Lecture 21 (11/21/12)

(5) $\vec{P}_0$, $\vec{m}_0$ are spontaneous polarization and spontaneous magnetization; $\vec{j}_{free_0}$is permanent superconducting current; χ_{el}, χ_{mag} are electric and magnetic susceptibilities; σ_c is conductivity in a normal conductor.

(6) The three constitutive relations above have inputs $\vec{E}$, $\vec{H}$, and cause output $\vec{P}$, $\vec{m}$, $\vec{j}_{free}$. In general, χ_{el} can depend on $\vec{H}$. χ_{mag} can depend on $\vec{E}$. σ_c can depend on $\vec{H}$. One example of this is in Hall effect, σ_c depends on $\vec{B}$ (hence $\vec{H}$).

(7) Why

$$\vec{P} = \alpha\vec{H} + ...$$

$$\vec{m} = \beta \vec{E} + ...$$

are incorrect? Because they violate parity and time reversal. Parity differentiate polar vectors and axial vectors (pseudovector).

5.3. Linear Input-Output Theory

Consider only time dependence here. ($\vec{x}$ dependence will not be considered, reasons come later.)

Linear Relationship

$$O(t) = \int_{-\infty}^{\infty} dt' G(t-t') I(t') \equiv (G \circ I)(t)$$

G can be a second rank tensor if O, I are vectors.

Fourier transform in t

$$\tilde{O}(w) = \sqrt{2\pi} \tilde{G}(w) \tilde{I}(w)$$

Example: Let $G(t-t') = G_0 \delta(t-t')$. So $O(t) = G_0 I(t)$. Hence no time delay, instantaneous effect of $I(t)$ on $O(t)$. $\tilde{G}(w) = G_0/\sqrt{2\pi}$. Independent of w. Not realistic, material has inertial.

Note:

(1) Unit of $G \sim \frac{[O]}{[I][time]}$. Unit of $\tilde{G} \sim \frac{[\tilde{O}]}{[\tilde{I}]}$.

(2) Causality consideration, when time is used, inputs must precede outputs, but no similar condition for spatial.

$$\therefore G(t-t') = 0$$

for $t < t'$.

5.4. Harmonic Field

Apply to $\vec{P}$, $\vec{m}$, $\vec{j}_{free}$.

$$\vec{P}(\vec{x}, t) = \vec{P}_0 + \int d^3x' \int dt' \underset{\sim el}{\chi} (\vec{x} - \vec{x}', t - t') \vec{E}(\vec{x}', t') + ...$$

$$\vec{m}(\vec{x},t) = \vec{m}_0 + \int d^3x' \int dt' \underset{\sim mag}{\chi}(\vec{x}-\vec{x}', t-t')\vec{H}(\vec{x}', t') + ...$$

$$\vec{j}(\vec{x},t) = \int d^3x' \int dt' \underset{\sim c}{\sigma}(\vec{x}-\vec{x}', t-t')\vec{E}(\vec{x}', t') + ...$$

Note:

(1) use convolution in time due to delay of the output after input; convolution in space due to impact at one spatial point can affect output at another spatial point.

(2) Fourier transform in space and time give $\underset{\sim el}{\tilde{\chi}}$, $\underset{\sim mag}{\tilde{\chi}}$, $\underset{\sim c}{\tilde{\sigma}}$.

(3) For practically all materials, non-locality in space does not occur in first order process. Say external $\vec{E}$ produce a polarization at $\vec{x}'$, then this polarization produce another $\vec{E}'$ at $\vec{x}$. Then this $\vec{E}'$ produces a polarization at $\vec{x}$. So this is higher order effect.

Therefore there are no $\vec{k}$ dependence in $\underset{\sim el}{\tilde{\chi}}$, $\underset{\sim mag}{\tilde{\chi}}$, $\underset{\sim c}{\tilde{\sigma}}$, but there are always w dependence because of mass inertial.

(4) For material whose properties vary in space & time (inhomogeneous time varying material)

$$\vec{P}_0 = \vec{P}_0(\vec{x},t)$$

$$\vec{m}_0 = \vec{m}_0(\vec{x},t)$$

$$\underset{\sim el}{\chi} = \underset{\sim el}{\chi}(\vec{x},t,\vec{x}-\vec{x}',t-t')$$

$$\underset{\sim mag}{\chi} = \underset{\sim mag}{\chi}(\vec{x},t,\vec{x}-\vec{x}',t-t')$$

$$\underset{\sim c}{\sigma} = \underset{\sim c}{\sigma}(\vec{x},t,\vec{x}-\vec{x}',t-t')$$

We only consider homogenous material (no $\vec{x}$ depend) not changing in time (no t depend) and material has no $\vec{k}$ dependence (local in space, so no $\vec{x}-\vec{x}'$ depend). Hence $\underset{\sim el}{\chi}$, $\underset{\sim mag}{\chi}$, $\underset{\sim c}{\sigma}$ are function of $t-t'$ only.

Let's concentrate on monochromatic place wave

$$\vec{E}(\vec{x},t) = e^{ikx-iw_0t}$$

$$\tilde{E}_w(\vec{x}) = \frac{1}{\sqrt{2\pi}} \int dte^{i(w-w_0)t} e^{ikx} = \sqrt{2\pi}\delta(w - w_0)e^{ikx}$$

$$\therefore \vec{E}(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \tilde{E}_w(\vec{x})e^{-iw_0 t}$$

field of definite w_0.

Fourier transform in times of

$$\vec{P}(\vec{x}, t) = \vec{P}_0 + \int d^3x' \int dt' \underset{\sim el}{\chi} (t - t')\vec{E}(\vec{x}', t') + ...$$

gives

$$\begin{aligned} \vec{P}_w(\vec{x}) &= 2\pi\delta(w)\vec{P}_0 + \sqrt{2\pi}\underset{\sim el}{\tilde{\chi}} (w)\tilde{E}_w(\vec{x}) \\ &= \sqrt{2\pi}\underset{\sim el}{\tilde{\chi}} (w)\tilde{E}_w(\vec{x}) \end{aligned}$$

we drop $\delta(w)$, because $w \neq 0$. This tells that $\vec{P}$ is also harmonic field,

$$\therefore \vec{P}(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \tilde{P}_w(\vec{x})e^{-iw_0 t}$$

Therefore, if we absorb the factor $\sqrt{2\pi}$ into $\underset{\sim el}{\tilde{\chi}} (w)$ and write it as $\underset{\sim el}{\chi} (w)$, we will have

$$\vec{P}(\vec{x}, t) = \underset{\sim el}{\chi} (w)\vec{E}(\vec{x}, t)$$

Similarly

$$\vec{m}(\vec{x}, t) = \underset{\sim mag}{\chi} (w)\vec{H}(\vec{x}, t)$$

$$\vec{j}_{free}(\vec{x}, t) = \underset{\sim c}{\sigma} (w)\vec{E}(\vec{x}, t)$$

They are so called harmonic field at w_0.

Furthermore,

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \left(I + 4\pi\underset{\sim el}{\chi} (w)\right) \vec{E}(\vec{x}, t) = \underset{\sim}{\epsilon}(w)\vec{E}(\vec{x}, t)$$

$\underset{\sim}{\epsilon}(w) = I + 4\pi \underset{\sim el}{\chi}(w)$ permittivity, it is complex and not hermitian.

$$\vec{B} = \vec{H} + 4\pi\vec{m} = \left(I + 4\pi \underset{\sim mag}{\chi}(w)\right)\vec{H}(\vec{x},t) = \underset{\sim}{\mu}(w)\vec{H}(\vec{x},t)$$

$\underset{\sim}{\mu}(w) = I + 4\pi \underset{\sim mag}{\chi}(w)$ permeability, it is complex and not hermitian.

Summary of harmonic field

$$\begin{cases} \vec{D}(\vec{x},t) = \underset{\sim}{\epsilon}(w)\vec{E}(\vec{x},t) \\ \vec{B}(\vec{x},t) = \underset{\sim}{\mu}(w)\vec{H}(\vec{x},t) \\ \vec{j}_{free}(\vec{x},t) = \underset{\sim c}{\sigma}(w)\vec{E}(\vec{x},t) \end{cases}$$

$\underset{\sim c}{\sigma}(w)$ conductivity, it is complex and not hermitian.

Note:

(1)$\underset{\sim}{\epsilon}$, $\underset{\sim}{\mu}$, $\underset{\sim c}{\sigma}$ in time convolution are real, and they correspond to real field. Reality condition on Fourier transform in w implies that $\underset{\sim}{\epsilon}^*(w) = \underset{\sim}{\epsilon}(-w)$, $\underset{\sim}{\mu}^*(w) = \underset{\sim}{\mu}(-w)$, and $\underset{\sim c}{\sigma}^*(w) = \underset{\sim c}{\sigma}(-w)$.

(2) unit of$\underset{\sim}{\epsilon}$, $\underset{\sim}{\mu}$, $\underset{\sim el}{\chi}$, $\underset{\sim mag}{\chi}$, are dimensionless, unit $\underset{\sim c}{\sigma}$ is $1/time$ = frequency.

5.5. Monochromatic Plane Wave in a Material

Reference Principle of Optics, by Born and Wolf.

We still consider material that is spatially homogenous and non time averaging.

Consider solution in which $\rho_{free} = 0$ and in which $\vec{j}_{free}$ is zero except due to conductivity $\vec{j}_{free} = \underset{\sim c}{\sigma}\vec{E}_{allowed}$. i.e. $\vec{j}_{free_0} = 0$. We look for solution to macroscopic Maxwell equations of the form

$$\left\{\begin{array}{c} \vec{E}(\vec{x},t) \\ \vec{B}(\vec{x},t) \\ \vec{D}(\vec{x},t) \\ \vec{H}(\vec{x},t) \end{array}\right\} = \left\{\begin{array}{c} \vec{E}_0 \\ \vec{B}_0 \\ \vec{D}_0 \\ \vec{H}_0 \end{array}\right\} e^{i(k\hat{n}\cdot\vec{x} - wt)}$$

Note:

(1) k here is allowed to be complex, because this is not the k in Fourier transform.

(2) w, $\hat{n}$ are real. w frequency of the wave produced by antenna. w is positive. $\hat{n}$ unit vector in direction of wave propagation, determined by the way antenna is focused.

(3) Physical fields are the real parts of these complex fields.

(4) k, $\vec{E}_0$, $\vec{B}_0$, $\vec{D}_0$,$\vec{H}_0$ are constant to be determined for the material properties. First one finds the allowed values of k, then for each allowed k, one finds the corresponding $\vec{E}_0$, $\vec{B}_0$, $\vec{D}_0$,$\vec{H}_0$.

Index of refraction of a material

$$N = \frac{kc}{w}$$

$k = k(w, \hat{n})$, so

$$N = N(w, \hat{n})$$

N is generally complex. Because

$$e^{i(k\hat{n}\cdot\vec{x}-wt)} = e^{-Im\{k\}\hat{n}\cdot\vec{x}}e^{i(Rp\{k\}\hat{n}\cdot\vec{x}-wt)}$$

$RpN = \frac{Rp\{k\}\cdot c}{w}$ associated with propagation of phase. $Rp\{k\}$ is related to phase velocity. Choose $Rp\{k\} > 0$ (or $Rp\{N\} > 0$) so that propagation is in the $+\hat{n}$ direction. If wave repeat in time τ, $w\tau = 2\pi$, $w = 2\pi/\tau$. $f = 1/\tau$ (Hertz=cycle/second), $w = 2\pi f$. If wave repeat in distance λ(wave length) along propagation direction, $Rp\{k\}\lambda = 2\pi$.

$ImN = \frac{Im\{k\}\cdot c}{w}$ associated with wave attenuation. $Im\{k\}$ related to attenuation length. If $Im\{k\} > 0$, passive material. Wave amplitude goes to $1/e$ of the initial value in distance

$$\delta = \frac{1}{Im\{k\}} = \frac{c}{w}\frac{1}{Im\{N\}}$$

(skin depth). If $Im\{k\} < 0$, amplification, for active material.

Lecture 22
(11/26/12)

Surface of constant phase $Rp\{k\}\hat{n}\cdot\vec{r}-wt=const.$ let $\hat{n}=\hat{r}$, then

$$Rp\{k\}r-wt=const$$

or

$$r=\frac{w}{Rp\{k\}}t+const$$

this give

$$\begin{aligned} v_{phase} &= \frac{w}{Rp\{k\}}=\frac{2\pi f}{2\pi/\lambda}=f\lambda \\ &= \frac{w}{Rp\{N\frac{w}{c}\}}=\frac{c}{Rp\{N\}} \end{aligned}$$

as a given t, surface of constant phase moves at v_{phase}.

Method for finding allowed N

1. Use Maxwell equation, but only *curl* equations are useful. (*div* equation is contained in the *curl* equation for $w\neq 0$.)

$$curl\,\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=-\frac{1}{c}\left(-iw\vec{B}\right)$$

Recall$\vec{B}=\vec{B}_0e^{i(k\hat{n}\cdot\vec{x}-wt)}$, then take *div*

$$0=div\,curl\,\vec{E}=\frac{iw}{c}div\,\vec{B}$$

This shows that $div\,\vec{B}=0$.

$$curl\,\vec{H}=\frac{4\pi}{c}\vec{j}_{free}+\frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

take *div* to both sides

$$0=\frac{4\pi}{c}\left(-\frac{\partial\rho_{free}}{\partial t}\right)+\frac{1}{c}\left(-iw\right)div\,\vec{D}$$

this gives

$$div\,\vec{D} = 4\pi\rho_{free}$$

2. Recall

$$div\,\vec{F}_0 e^{i(k\hat{n}\cdot\vec{x}-wt)} = ik\hat{n}\cdot\vec{F}_0 e^{i(k\hat{n}\cdot\vec{x}-wt)}$$

$$curl\,\vec{F}_0 e^{i(k\hat{n}\cdot\vec{x}-wt)} = ik\hat{n}\times\vec{F}_0 e^{i(k\hat{n}\cdot\vec{x}-wt)}$$

They can be derived from Cartesian tensor notation.

$$curl\,\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

is

$$ik\hat{n}\times\vec{E} = \frac{iw}{c}\vec{B}$$

That is

$$\vec{B} = N\hat{n}\times\vec{E}$$

This shows (1) $\vec{B}$ is always transverse. (2) If N is complex, $\vec{B}$, $\vec{E}$ are not necessarily in phase.

3. From

$$\vec{B} = \mu\vec{H}$$

we will assume that $\vec{H}(\vec{B})$ is invertible, hence it is single valued (assume no hysteresis), so

$$\vec{H} = \underset{\sim}{\mu}^{-1}\vec{B}$$

Use

$$curl\,\vec{H} = \frac{4\pi}{c}\vec{j}_{free} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

we got

$$ik\hat{n}\times\left(\underset{\sim}{\mu}^{-1}N\hat{n}\times\vec{E}\right) = \frac{4\pi}{c}\underset{\sim c}{\sigma}\,\vec{E} - \frac{iw}{c}\underset{\sim el}{\epsilon}\,\vec{E}$$

Multiply iw/c to both sides,

$$N\hat{n} \times \left(\underset{\sim}{\mu}^{-1} N\hat{n} \times \vec{E} \right) = \frac{4\pi}{iw} \underset{\sim c}{\sigma} \vec{E} - \underset{\sim el}{\epsilon} \vec{E}$$

This is a linear homogeneous matrix equation for $\vec{E}$.

$$\underset{\sim}{M}(\hat{n}, w, N^2, \underset{\sim c}{\sigma}, \underset{\sim el}{\epsilon})\vec{E} = 0$$

For non-trial solution,

$$det\, M \neq 0$$

Expand this secular equation, one gets

$$(\quad)N^6 + (\quad)N^4 + (\quad)N^2 + (\quad) = 0$$

Miracle the leading coefficient of the 6 power is always 0, so one has a quadratic equation in N^2, that gives 2 possible N^2. One should also choose $Rp\{N\} > 0$. So that leaves 2 possible N. They are called "bi refraction" and "double refraction". In plasma, one gets third index of refraction.

Finally, for each allowed N^2, find corresponding $\vec{E}$ for

$$\underset{\sim}{M}(\hat{n}, w, N^2, \underset{\sim c}{\sigma}, \underset{\sim el}{\epsilon})\vec{E} = 0$$

5.6. Example: Uniaxial Crystal

ϵ real symmetric, so it can be diagonalized with respect to orthogonal principle axis, with eigenvalues $\epsilon_\perp$, $\epsilon_\perp$, $\epsilon_\parallel$. Let $\hat{x}$, $\hat{y}$, $\hat{z}$ along the orthogonal axes, with two fold degeneracy in $x-y$ plane. Assume $\epsilon_\perp$, $\epsilon_\parallel$ are real and positive. Choose $\hat{n}$ in the $x-z$ plane, $\hat{n}$ has angle θ about the x axis, i.e. $\hat{n} = (\sin\theta, 0, \cos\theta)$. Assume $\mu = 1$ (hence $\vec{B} = \vec{H}$, not magnetizing crystal) and $\sigma_c = 0$ (not conducting crystal).

$$\epsilon = \begin{pmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}$$

So the equation involving N^2 is

$$N^2\hat{n}\times\left(\hat{n}\times\vec{E}\right)=-\epsilon\vec{E}$$

Use BCA-CAB rule,

$$\left(\epsilon-N^2I\right)\vec{E}+\hat{n}(\hat{n}\cdot\vec{E})N^2=0$$

Take $i$the component of this equation

$$\left(\epsilon_{ij}-N^2\delta_{ij}\right)E_j+N^2n_i\left(n_jE_j\right)=0$$

That is

$$\left(\epsilon_{ij}-N^2\delta_{ij}+N^2n_in_j\right)E_j=0$$

Hence

$$M_{ij}=\left(\epsilon_{ij}-N^2\delta_{ij}+N^2n_in_j\right)$$

Or by $n_1=\sin\theta$, $n_2=0$, $n_3=\cos\theta$, we get

$$M=\begin{pmatrix}\epsilon_\perp-N^2+N^2\sin^2\theta & 0 & N^2\sin\theta\cos\theta\\ 0 & \epsilon_\perp-N^2 & 0\\ N^2\sin\theta\cos\theta & 0 & \epsilon_\parallel-N^2+N^2\cos^2\theta\end{pmatrix}$$

$$\begin{aligned}det\,M &= \left(\epsilon_\perp-N^2\right)\left[\left(\epsilon_\perp-N^2\cos^2\theta\right)\left(\epsilon_\parallel-N^2\sin^2\theta\right)-N^4\sin^2\theta\cos^2\theta\right]\\ &= \left(\epsilon_\perp-N^2\right)\left(\epsilon_\perp\epsilon_\parallel-N^2\epsilon_\parallel\cos^2\theta-N^2\epsilon_\perp\sin^2\theta\right)\\ &= 0\end{aligned}$$

One solution is

$$N^2=\epsilon_\perp$$

The other solution is

$$N^2=\frac{\epsilon_\perp\epsilon_\parallel}{\epsilon_\parallel\cos^2\theta+\epsilon_\perp\sin^2\theta}$$

Therefore the uniaxial crystal has two indices of refraction for any give θ, $O-$wave (ordinary wave) has N^2 independent of $\hat{n}$, which is the first solution.

$E-$wave (extraordinary wave) has N^2 dependent of $\hat{n}$, which is the second solution.

The optic axis of a crystal is the direction ($\pm\hat{n}$) for which both indices of refraction are the same. For this case, it is when $\theta = 0$ or π, then

$$N^2_{o-\text{wave}} = N^2_{E-\text{wave}} = \epsilon_\perp$$

Hence the optic axis is the $\hat{z}$ axis.

Lecture 23 (11/28/12)

One can find the corresponding $\vec{E}$ for these 2 waves.

Exercise

One should find that $\vec{E}_{o-\text{wave}}$is in $\hat{y}$ direction. Thus $\vec{E}_{o-\text{wave}}$ is linearly polarized in $\hat{y}$ and it is transverse (because $\vec{E}_{o-\text{wave}} \perp \hat{n}$). $\vec{E}_{E-\text{wave}}$is in the $x-z$ plane, and it makes an angle

$$\arctan\left(\frac{\epsilon_\perp}{\epsilon_\parallel}\tan\theta\right)$$

with the x axis. So $\vec{E}_{E-\text{wave}}$ is not perpendicular to $\hat{n}$ unless $\epsilon_\perp = \epsilon_\parallel$.

Although $\vec{E}_{E-\text{wave}}$ is not transverse, the corresponding $\vec{D}_{E-\text{wave}}$ is transverse.

Proof.

$$div\,\vec{D} = 4\pi\rho_{free} = 0$$

and

$$div\,\vec{D} = ik\hat{n}\cdot\vec{D} = 0$$

So $\vec{D}$ is transverse.

Application: 1/4 Waveplate

Use the uniaxial crystal above, take $\hat{n} = \hat{x}$. This changes linear polarized wave into circular polarized wave.

Incident

$$\vec{E}(x=0) = \frac{E_0}{\sqrt{2}}(\hat{y}+\hat{z})$$

Then $\vec{E}_{o-\text{wave}}$is in $\hat{y}$ direction, and $N^2_{o-\text{wave}} = \epsilon_\perp$; $\vec{E}_{E-\text{wave}}$is in $\hat{z}$ direction, and $N^2_{E-\text{wave}} = \epsilon_\parallel$.

The refraction wave coming out of the crystal, let crystal have size L in y and z direction

$$\begin{aligned}\vec{E}(x=L) &= \frac{E_0}{\sqrt{2}}\left(\hat{y}e^{i\left(k_{ordinary}x-wt\right)} + \hat{z}e^{i\left(k_{extraordinary}x-wt\right)}\right) \\ &= \frac{E_0}{\sqrt{2}}e^{iwt}e^{ik_0L}\left(\hat{y}+\hat{z}e^{i(k_e-k_o)L}\right)\end{aligned}$$

$k_0 = wN/c = w\sqrt{\epsilon_\perp}/c$ and $k_e = w\sqrt{\epsilon_\parallel}/c$. If L is such that

$$(k_e - k_o)\,L = \pi/2$$

then

$$= \vec{E}(x=L)\frac{E_0}{\sqrt{2}}e^{iwt}e^{ik_0L}\left(\hat{y}+\hat{z}i\right)$$

we have circular polarized wave with positive helicity.

5.7. Isotropic Conductive Material

Isotropic with σ_c, ϵ, μ real + positive

$$\left\{\begin{array}{c}\underset{\sim}{\sigma} = \sigma\underset{\sim}{I} \\ \underset{\sim}{\epsilon} = \epsilon\underset{\sim}{I} \\ \underset{\sim}{\mu} = \mu\underset{\sim}{I}\end{array}\right\}$$

let's choose $\hat{n}$ along $\hat{z}$.

Find allowed N values.

$$\frac{N^2}{\mu}\hat{z}\times\left(\hat{z}\times\vec{E}\right) = \frac{4\pi}{iw}\sigma\vec{E} - \epsilon\vec{E}$$

That gives

$$N^2\left(\hat{z}E_z - \vec{E}\right) + \frac{4\pi i}{w}\sigma\vec{E} + \mu\epsilon\vec{E} = 0$$

Or

$$M = \begin{pmatrix} -N^2 + \frac{4\pi i}{w}\sigma + \mu\epsilon & 0 & 0 \\ 0 & -N^2 + \frac{4\pi i}{w}\sigma + \mu\epsilon & 0 \\ 0 & 0 & \frac{4\pi i}{w}\sigma + \mu\epsilon \end{pmatrix}$$

$$det\, M = 0$$

implies

$$N^2 = \mu\epsilon + \frac{4\pi i}{w}\sigma$$

$$N = \left[(\mu\epsilon)^2 + \left(\frac{4\pi\sigma}{w}\right)^2 \right]^{1/4} e^{\frac{i}{2}\arctan\frac{4\pi\sigma}{w\mu\epsilon}}$$

The eigenvector $\vec{E}$ has 0 z-component, so $\vec{E}$ is transverse.

Because

$$\vec{B} = N\hat{n} \times \vec{E}$$

N complex, $\vec{B}$, $\vec{E}$ are not in phase and $\left|\vec{B}\right| \neq \left|\vec{E}\right|$.

Consider a very good conductor. σ very large ($4\pi\sigma \gg w\mu\epsilon$). This says w should be small, low frequency.

$$\therefore N = \sqrt{\frac{2\pi\sigma}{w}}(1+i)$$

Note:

(1) $N_{good\, conductor}$ is very big

(2) $\left|\vec{B}\right| \gg \left|\vec{E}\right|$, $\left|\vec{B}\right|$, $\left|\vec{E}\right|$ are $45^{\rm O}$ out of phase

(3) Attenuation length

$$\delta = \frac{1}{Im\{k\}} = \frac{c}{wIm\{N\}} = \frac{c}{\sqrt{2\pi\sigma w}}$$

Application: Sea water is very good conductor. If one wants to send signal to a submarine, one needs to a large δ, which in turn requires very small w, and small w will require long time. Therefore communication with submarine takes long time.

5.8. Problem Set 11 (due 12/3/12)

1)

A non magnetic, non-conducting biaxial crystal. $\mu = 1$, $\sigma_c = 0$, and $0 < \epsilon_1 < \epsilon_2 < \epsilon_3$,

$$\underset{\sim}{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$$

Given w, $\hat{n}$ of monochromatic plane wave.

Find allowed N^2. There are two indices of refraction (birefrigence). Both corresponding waves are extraordinary.

Find the optic axes. There are two optic axes. That is why the material is called biaxial crystal.

2)

Jackson 7.16

3)

A dextrose solution, $\sigma_c = 0$, has polarization $\vec{P}$ and magnetization $\vec{m}$ given by the parity violating relations

$$\vec{P} = \alpha \frac{\partial \vec{B}}{\partial t} \qquad \vec{m} = \gamma \frac{\partial \vec{E}}{\partial t}$$

α, γ are real, positive not equal constants. Consider a monochromatic plane wave w entering the solution $\hat{n} = \hat{z}$.

Find allowed N, find the corresponding $\vec{E}$ and its polarization. Suppose the entering $\vec{E}$ is linearly polarized in x direction, and exits the solution at $z = L$. You should find the exiting $\vec{E}$ is polarized in the $x - y$ plane with angle ψ to x axis. Find ψ. ψ is related to α, γ. This problem is used in beer making industry to test the sugar concentration.

5.9. Drude-type Models

Classical model for σ. (this can be modified for ϵ as well)

Consider valance electrons. Free to move through material. Consider non-relativistic electron motions. We will calculate σ.

Electron equation of motion

$$m\frac{d\vec{v}}{dt} = -m\nu\vec{v} + (-e)\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

$-m\nu\vec{v}$ term is due to collision with positive ions, which gives fictional drag. ν, collisional frequency and each collision has momentum change by about $m\vec{v}$. This term can be regarded as damping. The second term on the right is magnetic force, where $\vec{E}$, $\vec{B}$ are the incident harmonic fields at the position of the electron. Since $v \ll c$, we drop the $\vec{B}$ term. (See HW12 1) where $\vec{B}$ term is kept.) Therefore we have

$$m\frac{d\vec{v}}{dt} + m\nu\vec{v} = -e\vec{E}$$

where $\vec{E} = \vec{E}_0 e^{-iwt}$.

We look for steady state solution, (initial condition damp out), $\vec{v} = \vec{v}_0 e^{-iwt}$, we get

$$m(-iw)\vec{v} + m\nu\vec{v} = -e\vec{E}$$

$$\therefore \vec{v} = \frac{-e/m}{\nu - iw}\vec{E}$$

Since $\vec{j}_{free} = \rho\vec{v} = (-en)\vec{v}$, where n =density of valance electrons, and $\vec{j}_{free} = \sigma_c\vec{E}$,

$$\sigma_c = \frac{ne^2/m}{\nu - iw}$$

- Good Conductor

ν is very large. Collision dominates applied $\vec{E}$'s frequency w.

$$\sigma_c = \frac{ne^2}{m\nu}$$

has no w dependence.

- Tenuous Plasma

Essentially no collision. Positive ions can move but move very little at reasonable applied frequency (large masses respond mainly to very low freq applied)

$\nu = 0$, $\epsilon = 1$, $\mu = 1$ no collision, no permittivity, no permeability.

$$\sigma = i\frac{ne^2}{mw}$$

$$\begin{aligned} N^2 &= \mu\epsilon + \frac{4\pi i}{w}\sigma \\ &= 1 - \frac{\left(\frac{4\pi ne^2}{m}\right)}{w^2} \end{aligned}$$

where $w_{plasma} = \sqrt{4\pi ne^2/m}$ is characteristic frequency of plasma, depended on material

$$N^2 = 1 - \frac{w_{plasma}^2}{w^2}$$

If $w > w_{plasma}$, then $N^2 > 0$, N is pure real. $Im\{N\} = 0$ no attenuation, only propagation.

If $w < w_{plasma}$, then $N^2 < 0$, N is pure imaginary. No propagation, only attenuation. Application: Short wave radio (very low frequency) can travel from New York to Paris because waves go to the ionosphere (tenuous plasma), it can not penetrate the ionosphere, so the amount that is not absorbed by the ionosphere will be reflected back to the earth and continue its way to Paris.

5.10. Problem Set 12 (not to be handed in)

1)

Consider a magnetically biased collisionless one-component electronic plasma. The non-relativistic equation of motion for an electron in this plasma is

$$m\frac{d\vec{v}}{dt} = -e\vec{E}_{wave}(\vec{x}=0,t) - e\frac{\vec{v}}{c} \times \vec{B}_{external}$$

where $\vec{B}_{external} = a\hat{z}$, a is real positive constant. Assume the magnetic force $\vec{F} = -e\vec{v}/c \times \vec{B}_{wave}$ can be neglected. And the fluctuation of $\vec{E}$ due to electron motion can be neglected, hence, $\vec{E}_{wave}(\vec{x}, t) = \vec{E}_{wave}(\vec{x} = 0, t) = Rp\vec{E}_0 e^{-iwt}$.

(1) Find $\underset{\sim}{\sigma}_c$ in terms of w, $w_{plasma} \equiv \sqrt{4\pi ne^2/m}$, and $\Omega_{cyclotron} \equiv eB_{ext}/mc$.

(2) For parts (2) and (3), assume $w_{plasma} > \Omega_{cyclotron}$. A monochromatic plane wave propagating in $\hat{z}$ direction. Find allowed N^2. Plot N^2 v.s. w and find the corresponding electric polarization $\vec{E}_{wave}$.

(3) Do the same for a monochromatic plane wave propagating in $\hat{x}$ direction.

ANS: (2)

$$N^2 = \pm \frac{\Omega_{cyc} w_{pl}^2}{\left(\Omega_{cyc}^2 - w^2\right) w^2} + 1 + \frac{w_{pl}^2}{\left(\Omega_{cyc}^2 - w^2\right)}$$

one sees that the one with "+" even below the plasma frequency, $N^2 > 0$, wave still propagates.

(3)

$$N^2 = 1 - \frac{w_{pl}^2}{w^2}$$

and

$$N^2 = 1 - \frac{\Omega_{cyc}^2 \left(\frac{w_{pl}^2}{\Omega_{cyc}^2 - w^2}\right)^2}{\left(\frac{w_{pl}^2}{\Omega_{cyc}^2 - w^2} + 1\right) w^2}$$

The second N^2 would greater than 0, even below the plasma frequency. This problem discusses an important phenomenon that the effect of earth's magnetic field to radio transmission.

6. Theory of Special Relativity

Lecture 24 (12/3/12)

6.1. Relativistic Covariance

We will describe event that occurs at a given spatial point at a give time.

Notation: contravariant index upper index; covariant index lower index.

$$x^0 = ct \qquad x_0 = ct$$

$$x^1 = x \qquad x_1 = -x$$

$$x^2 = y \qquad x_2 = -y$$

$$x^3 = z \qquad x_3 = -z$$

$$x^\mu = (ct, x, y, z) = (ct, \vec{x}) \qquad x_\mu = (ct, -x, -y, -z) = (ct, -\vec{x})$$

This is east coast notations. On west coast people use $x_\mu = (-ct, x, y, z) = (-ct, \vec{x})$. Old notation $(ict, \vec{x})$, old notation is not good, they cannot be generalized to general relativity.

Summation convention: repeated Greek index summed from 0 to 3. One upper + one lower index. Latin indices will refer to 1,2,3. One cannot sum repeated indices if both are upper or both are lower indices.

Examples:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

$$\partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box$$

Two Infinitesimally Separated Events

$$dx^\mu = (cdt, d\vec{x})$$

We defined space time "interval"

$$(ds)^2 \equiv dx^\mu dx_\mu = c^2(dt)^2 - d\vec{x} \cdot d\vec{x} = c^2(dt)^2 - |d\vec{x}|^2$$

- Space-like separation $(ds)^2 < 0$

$$c^2(dt)^2 < |d\vec{x}|^2$$

$$\therefore \frac{|d\vec{x}|}{dt} > c$$

would require $v > c$ to get from one point to other. (not allow)

- Time-like separation $(ds)^2 > 0$

$$\therefore \frac{|d\vec{x}|}{dt} < c$$

a real particle with $v < c$ can connect point to the other.

- Light-like separation $(ds)^2 = 0$

$$\therefore \frac{|d\vec{x}|}{dt} = c$$

light connecting the two points.

Coordinate Transformation

Define matrix tensor

$$g^{\mu\nu} \equiv g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\left\{ \begin{array}{c} x^\mu = g^{\mu\nu} x_\nu \\ x_\mu = g_{\mu\nu} x^\nu \end{array} \right\}$$

agree with our definition of covariant and contravariant vectors.

General rules: (1) Move 0 index from upper to lower or from lower to upper, no change of sign. (2) Move $i = 1, 2, 3$ index from upper to lower or from lower to upper gives a minus sign.

Use the general rules one can prove the following theorem.

Theorem

$$g^\mu{}_\nu = \delta^\mu{}_\nu$$

Proof. $g^0{}_0 = +1$, $g^1{}_1 = -g^{11} = +1$, $g^2{}_2 = -g^{22} = +1$, $g^3{}_3 = -g^{33} = +1$.

Consider the following coordinate transformation

$$x^\mu \to (x^\mu)'$$

$$(x^\mu)' = f^\mu(x^0, x^1, x^2, x^3, \lambda)$$

f^μdepends smoothly on continuous parameters λ. λ is a real parameters characterizing the transformation. $\lambda = 0$ corresponds to identity transformation $(x^\mu)' = x^\mu$.

Theorem

$$(ds')^2 = 0 \iff (ds)^2 = 0$$

This is

$$dx^{\mu'} dx'_\mu = dx^\mu dx^\mu$$

light propagates in $x^\mu$is the same as in $x^{\mu'}$ with respect to both coordinate systems. c is the same.

Theorem. *The only smooth transformation connecting to the identity which maps light into light are the transformation of the 15 parameters conformal group.*

We characterize them into 4 kinds.

(1) Translation (4 parameters)

Shift of the origin

$$x^{\mu'} = x^\mu + b^\mu$$

$b^\mu = (b^0, b^1, b^2, b^3)$ fixed vector. This is an abelian subgroup of conformal group.

(2) Lorentz transformation (6 parameters)

$$x^{\mu'} = a^\mu{}_\nu x^\nu$$

summing ν. $a^\mu{}_\nu$ are orthogonal, i.e.

$$a_\mu{}^\sigma a^\mu{}_\nu = \delta^\sigma{}_\nu, \qquad a^\sigma{}_\mu a_\nu{}^\mu = \delta^\sigma{}_\nu$$

This is equivalent to conditions $a^{tr} a = I$ and $a a^{tr} = I$ or $a^{-1} = a^{tr}$.

Because taking transpose of $x^\mu = a^\mu{}_\sigma x^\sigma$ gives

$$x_\mu = x_\sigma (a^{tr})^\sigma{}_\mu$$

which is normally written as

$$x_\mu = a_\mu{}^\sigma x_\sigma$$

This shows that we should define

$$a_\mu{}^\sigma \equiv (a^{tr})^\sigma{}_\mu$$

Then

$$a_\mu{}^\sigma a^\mu{}_\nu = (a^{tr})^\sigma{}_\mu a^\mu{}_\nu = (a^{tr} a)^\sigma{}_\nu = I^\sigma{}_\nu = \delta^\sigma{}_\nu$$

We can think the interchange of upper and lower indices when taking transpose is like for matrix, so upper index as index of rows and lower index as index of column. And Lorentz transformations are orthogonal 4×4 matrices. Other way to think this is to think $a \in V \otimes W^*$, and $a^{tr} \in W \otimes V^*$.

The Lorentz transformations form non-abelian subgroup of conformal group. There are two kinds of Lorentz transformations.

a) Spatial rotation.

Rotation about a fixed spatial axis. This is a non-abelian subgroup.

$$x^{0'} = x^0$$

$$\vec{x}' = R\vec{x}$$

This uses 3 parameters, because R has 3 parameters. θ, ψ are for the direction of rotation axis, and ϕ is angle of rotation about rotational axis.

b) Lorentz boosts

Constant velocity $\vec{v}$ spatial coordinates of O' are parallel to spatial coordinates of O. This transformation has 3 parameters v_x, v_y, v_z. This is not subgroup, because it is not closed. The composition of two boosts gives a rotation followed by a boost when $\vec{v}$ is not parallel to one of the spatial coordinates of O. This effect is called "Thomas precession".

Translations (4 parameters) + Lorentz transformations (6 parameters) = Poincare groups (10 parameters).

(3) Scale transformation (1 parameter)

$$x^{\mu'} = \lambda x^{\mu}$$

where λ is the same scale factor for $\mu = 0, 1, 2, 3$. This is a subgroup of conformal group. But since this leaves

$$\frac{d\vec{x}}{dt} = \frac{d\vec{x}'}{dt'}$$

invariant. This implies all speeds, not only speed of light are invariant. This is not correct invariant unless there is no particles with non-zero mass under consideration.

(4) Special conformal transformation

This is abelian 4 parameters subgroup of conformal group

$$x^{\mu'} = \frac{x^{\mu} + (x_{\nu}x^{\nu})\alpha^{\mu}}{1 - 2(\alpha_{\nu}x^{\nu}) + (\alpha_{\nu}\alpha^{\nu})(x_{\nu}x^{\nu})}$$

$\alpha^{\mu} = \alpha^0$,α^1, α^2, α^3 4 parameters. This transformation is non-linear, it has minimum or maximum inflection points. Because in special relativity space + time are homogenous, the transformation violates homogeneity.

In summary Maxwell equations give

$$(ds')^2 = 0 \iff (ds)^2 = 0$$

and that gives all 15 parameters conformal group and only the Poincare group has

$$(ds')^2 = (ds)^2$$

for any separations. Scale transformation and special conformal transformation suggest interval is not invariant, so they are not very useful in special relativity. We will not consider them.

Lecture 25 (12/5/12)

6.2. Relativistic Scale, Vector, Tensor

Consider a Lorentz boost. axes of O and O' are parallel, Assume origins O and O' coincide at $t = 0 = t'$. $xyzO$ and $x'y'z'O'$ are two inertial observers. They describe event by using

$$(t, x, y, z)$$

and other other

$$(t', x', y', z')$$

let $\vec{v}$ velocity of O' with respect to O. Let $\vec{x}_\parallel$, $\vec{x}_\perp$ be spatial coordinates that is parallel or perpendicular to $\vec{v}$. Then the Lorentz boost is given by

$$\begin{cases} \vec{x}'_\perp = \vec{x}_\perp \\ \vec{x}'_\parallel = \gamma(\vec{x}_\parallel - vt) \\ t' = \gamma(t - \frac{\vec{v}\cdot\vec{x}}{c^2}) \end{cases}$$

The inverse Lorentz boost is

$$\begin{cases} \vec{x}_\perp = \vec{x}'_\perp \\ \vec{x}_\parallel = \gamma(\vec{x}'_\parallel + vt') \\ t' = \gamma(t' + \frac{\vec{v}\cdot\vec{x}'}{c^2}) \end{cases}$$

The approach we used here is passive point of view, i.e. same event 2 observers.

Lorentz Scalar

Lorentz invariant scalar has,

$$\psi' = \psi$$

ψ is in O,ψ' is in O'

E.g. Interval

$$(ds)^2 = c^2(dt)^2 - |d\vec{x}|^2$$

satisfies

$$(ds)^2 = (ds')^2$$

E.g. Space-time differential

$$d^4x = (cdt)(d^3x)$$

satisfies

$$d^4x = d^4x'$$

Proof. $d^4x' = |\text{Jacobian}|\, d^4x = det(a^\mu{}_\nu)d^4x = d^4x$, for $a^\mu{}_\nu$ is orthogonal.

4 vectors

$$A^\mu = (A^0, \vec{A}) = (A^0, A^1, A^2, A^3) = (A^0, A_x, A_y, A_z)$$

note A^0 and $\vec{A}$ have same units.

$$A_\mu = (A^0, -\vec{A}) = (A^0, A_1, A_2, A_3) = (A^0, -A_x, -A_y, -A_z)$$

A^μ is a 4 vector iff it transform as x^μ.

$$A^{\mu'} = a^\mu{}_\nu A^\nu$$

$$A'_\mu = a^\mu{}_\nu A_\nu$$

For example, boost along $\hat{x}$.

$$\begin{cases} A^{0'} = \gamma(A^0 - \frac{v}{c}A^1) \\ A'_x = \gamma(A^1 - \frac{v}{c}A^0) \\ A'_y = A_y \\ A'_z = A_z \end{cases}$$

inverse transformation
$$\begin{cases} A^0 = \gamma(A^{0'} + \frac{v}{c}A^{1'}) \\ A_x = \gamma(A^{1'} + \frac{v}{c}A^{0'}) \\ A_y = A_y' \\ A_z = A_z' \end{cases}$$

The only difference between this and the one for $x^\mu$is c here and c^2 there. The reason is to make dimension correct.

Recall

$$A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 = A^0 B_0 - \vec{A} \cdot \vec{B}$$

Theorem. *A^μ, B^μ are two 4-vectors , then $A^\mu B_\mu$ is Lorentz scalar.*

Proof. $A^{\mu'} B_\mu' = a^\mu{}_\sigma a_\mu{}^\lambda A^\sigma B_\lambda = \delta_\sigma{}^\lambda A^\sigma B_\lambda = A^\sigma B_\sigma$. □

Recall the terminology:

A^μ is time-like iff $A^\mu A_\mu > 0$; A^μ is space-like iff $A^\mu A_\mu < 0$; A^μ is light-like iff $A^\mu A_\mu = 0$.

Theorem. *(1) If A^μ is space-like, then there exists a Lorentz frame (inertial observer) s.t. $(A^0)' = 0$. (2) If A^μ is time-like, then there exists a Lorentz frame s.t. $(\vec{A})' = 0$.*

Proof. Suppose A^μ is space-like, $A^\mu A_\mu < 0$. Just use Lorentz boost in direction of $\vec{A}$. In O': $(A^0)' = \gamma(A^0 - \frac{\vec{v} \cdot \vec{A}}{c})$. Choose $\vec{v} = \frac{A^0 c}{\left|\vec{A}\right|^2}\vec{A}$ This is possible because $(A^0)^2 < \left|\vec{A}\right|^2$. Suppose A^μ is time-like, $A^\mu A_\mu > 0$. We use rotation, so that in O', $\vec{A}$ lies on $\hat{x}'$, so A^μ becomes $(A^0, A^{1'}, 0, 0)$. Then we apply Lorentz boost in $\hat{x}'$ direction

$$A_x'' = \gamma(A_x' - \frac{v}{c}A^0)$$

choose $v = cA_x'/A^0$. This is possible, because $(A^0)^2 > \left|\vec{A}\right|^2$. □

4 tensors

$T^{\mu\nu\dots}{}_{\lambda\delta\dots}$ is a 4 tensor iff it transforms as a 4 vector on each index.

$$(T^{\mu\nu\dots}{}_{\lambda\delta\dots})' = a^{\mu}{}_{\alpha} a^{\nu}{}_{\beta} \cdots a^{\gamma}{}_{\lambda} a^{\kappa}{}_{\delta} \cdots T^{\alpha\beta\dots}{}_{\gamma\kappa\dots}$$

E.g. A^μ, $B^\nu$are two vectors, and then

$$A^\mu B^\nu$$

is a second rank tensor.

E.g. 4 dimensional Levi-Civita symbol

$$\epsilon^{\alpha\beta\gamma\delta} \equiv \begin{cases} \text{completely anti-sym when only 2 indicies interchanged} \\ \epsilon^{0123} = +1 \end{cases}$$

It's easy to check the following.

Cyclic change $\epsilon^{1230} = -\epsilon^{0123}$. $\epsilon^{\alpha\beta\gamma\delta} = 0$ if any 2 indices equal. One can get mixed co/contra-variant tensor by using the rules given before, raising/lowering spatial index becomes negative, raising/lowering temporal index no change of sign. E.g.

$$\epsilon^{13}{}_{20} = -\epsilon^{3}_{1}{}_{20} = -\epsilon^{30}_{12}$$

Theorem. *$\epsilon^{\alpha\beta\gamma\delta}$ is 4th rank tensor whose value is same in all Lorentz frames.*

Proof. By the definition of determinant of a,

$$det(a) = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} a^{i}{}_{\sigma_i}$$

and σ permutation on S_4 is one of the $\epsilon^{\alpha\beta\gamma\delta}$.

Using Einstein summation notation

$$det(a) = \epsilon^{\alpha\beta\gamma\delta} a^{1}{}_{\alpha} a^{2}{}_{\beta} a^{3}{}_{\gamma} a^{4}{}_{\delta}$$

or a little bit more general

$$det(a)\epsilon^{\lambda\kappa\mu\nu} = \epsilon^{\alpha\beta\gamma\delta} a^{\lambda}{}_{\alpha} a^{\kappa}{}_{\beta} a^{\mu}{}_{\gamma} a^{\nu}{}_{\delta}$$

but the right hand side is

$$\begin{aligned} \left(\epsilon^{\lambda\kappa\mu\nu}\right)' &= a^{\lambda}{}_{\alpha} a^{\kappa}{}_{\beta} a^{\mu}{}_{\gamma} a^{\nu}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} \\ &= det(a)\epsilon^{\lambda\kappa\mu\nu} \\ &= \epsilon^{\lambda\kappa\mu\nu} \end{aligned}$$

because $det(a) = 1$ proper Lorentz transformation. □

6.3. Relativistic Postulates

Definition. An inertial observer is one who sees Newton's first law as true. That is an isolated object that has no force act on travels at constant velocity with respect to the observer.

Experimental result

Note: the definition above sometime to be mistakenly considered as circle argument. The circle argument goes like inertial observer is one who see Newton's first law is true, and Newton's first law is true when it is applied to inertial observers. The experiment results break the circle argument and grand the existence of such observer.

This is known as "Mach's Principle", says inertial observers exist. They are at rest with respect to the "fixed distance starts" or they travel at constant velocity with respect to the "fixed distanced stars". The fixed distanced stars can be thought as referring to the average distribution of matter in universe.

Examples of inertial observes: atomic processes are used to define standard of length + of time.

Postulates

(1) The law physics are the same with respect to all inertial observers. The statement also says all inertial observers are equivalent or no closed box experiment can distinguish one initial observer from another.

(2) Speed of light in vacuum is same with respect to all inertial observers. The statement is incorrect phrased by many textbooks, saying that light speed is independent of speed of source. But this is true for all classical waves also.

(3) Clock Hypothesis: An ideal clock in arbitrary motion (possibly accelerating) ticks at same rate as a second clock in a co-moving inertial frame, the instantaneous rest frame of first clock. This additional postulate in special and general relativity is often assumed without stating explicitly. Note: it is possible to do accelerating problem in special relativity as suggested by the clock hypothesis and the problem of twin paradox. The difference between special and general relativity is not on deals with no acceleration and the other has, but the difference is that one has gravity and the other doesn't.

6.4. Time Dilation, Length Contraction

Space-time Diagram

One can draw a 2 dimensional plot of space-time diagram. Horizontal axis is $\vec{x}$ and vertical axis is ct. We set the origin to be the position where the massive particle is when $t = 0$. As the particle moves, one can trace $(ct, \vec{x})$ and draw a curve on the space-time diagram, and the curve is called "world line" (generalized from trajectory).

Consider

$$(ds)^2 = c^2(dt)^2 - |d\vec{x}|^2$$

For massive particle, $(ds)^2 > 0$ (time-like).

The slope of massive particle world-line is

$$\left|\frac{cdt}{d\,|\vec{x}|}\right| = \left|\frac{c}{v}\right| > 1$$

so the particle world-line is confined in the light cone bounded by $x = ct$ and

$x = -ct$.

Time Dilation

Recall we have a theorem saying that given a time-like vector $ds = (cdt, d\vec{x})$, there is an Lorentz frame in which $ds' = (cdt', d\vec{x}')$, and $d\vec{x}' = 0$. This tells us that this Lorentz frame is the instantaneous rest frame and we should define proper time of particle

$$d\tau \equiv \frac{ds}{c}$$

then

$$(d\tau)^2 \quad = \quad (dt)^2 - \frac{|d\vec{x}|^2}{c^2} = (dt)^2 \left[1 - \frac{|d\vec{x}|^2}{(dt)^2 c^2}\right] = \frac{(dt)^2}{\gamma^2}$$

or

$$d\tau = \frac{dt}{\gamma}$$

or

$$dt = \gamma d\tau$$

Because in the laboratory frame, the particle is moving at $\vec{v}$, then time in rest frame $d\tau$ and time in laboratory frame dt is related by the equation above. This works even when the particle is accelerating. By the clock hypothesis, the moving (accelerating) clock has same $d\tau$ as the clock in its rest frame (proper time).

Therefore people say that

$$dt = \gamma d\tau$$

gives the time dilation of a moving clock.

Length Contraction

Consider an object with L_0 proper length measured in rest frame, and it is moving the direction of L_0, which is parallel to $\hat{x}$. Suppose at t, the two ends x'_a and x'_b are found to be x_a, x_b with respect to the lab frame. Since

$$x'_a = \gamma(x_a + vt')$$

$$x'_b = \gamma(x_b + vt')$$

Therefore

$$x'_a - x'_b = \gamma(x_a - x_b) \implies L_0 = \gamma L$$

or

$$L = \frac{L_0}{\gamma}$$

This effect is called length contraction.

6.5. Relativistic Kinematics

4 Momentum of a particle

We define 4 momentum (it is a 4 vector)

$$p^\mu = m\frac{dx^\mu}{d\tau}$$

m is rest mass (parameter property of the particle). $d\tau = dt/\gamma$.

$$p^\mu = m\frac{dx^\mu}{dt}\gamma = m\gamma\left[\frac{dct}{dt}, \frac{d\vec{x}}{dt}\right] = m\gamma\,(c, \vec{v})$$

$$p^\mu p_\mu = m^2\gamma^2\left(c^2 - v^2\right) > 0$$

$\therefore p^\mu$ is time-like. Therefore there exists an inertial frame (rest frame), $\vec{p} = 0$.

Since $p^\mu p_\mu$ is Lorentz scalar, we can take its value in the rest frame, where $v = 0$, and $\gamma = 1$. Thus

$$p^\mu p_\mu = m^2 c^2$$

We define kinematic energy (not the same thing of kinetic energy), ε,

$$p^\mu = \left(\frac{\varepsilon}{c}, \vec{p}\right)$$

or

$$p_\mu = \left(\frac{\varepsilon}{c}, -\vec{p}\right)$$

This implies

$$\varepsilon = m\gamma c^2 \qquad \vec{p} = m\gamma\vec{v}$$

Since

$$p^\mu p_\mu = \left(\frac{\varepsilon}{c}\right)^2 - |\vec{p}|^2 = m^2c^2$$

$$\therefore \varepsilon = \sqrt{\left(c\,|\vec{p}|\right)^2 + \left(mc^2\right)^2}$$

This is the relativistic energy-momentum equation. ε kinematic energy is the sum of rest mass energy mc^2 and kinetic energy . Note: $\varepsilon_{photon} = cp$.

4 Momentum Conservation

Consider an isolated system and an inertial observer, e.g. collision, decay, or some chemical reaction (where the observer can be the catalysis).

4 momentum conservation says that total 3-monoutum $\sum \vec{p}$ conserved and total kinematic energy $\sum \varepsilon$ conversed.

For this kind of problem, one should follow these two rules (fatherly hints):

(1) Use "well chosen" Lorentz invariants ($A_\mu B^\mu$) to get answers

(2) Let $c = 1$ in all equations and then resurrect factors of c in the final answers so that they are dimensional correct.

Example: Compton Scattering

Photon $(\varepsilon_a, \vec{p_a})$ collides an electron $(\varepsilon_b, \vec{p_b}) = (m, 0)$ at rest, then photon $(\varepsilon_c, \vec{p_c})$ scattered at angle θ, and electron recoils $(\varepsilon_d, \vec{p_d})$. Problem states given m, ε_a, θ. Find ε_c.

Use 4 momentum conservation

$$(p_a)^\mu + (p_b)^\mu = (p_c)^\mu + (p_d)^\mu$$

then we form a useful Lorentz invariant. Because we are not interested in $\vec{p_d}$ and

ε_d, we can eliminate them

$$\begin{aligned}
\underbrace{(p_d)^\mu (p_d)_\mu}_{m^2} &= [(p_a)^\mu + (p_b)^\mu - (p_c)^\mu]\left[(p_a)_\mu + (p_b)_\mu - (p_c)_\mu\right] \\
&= \underbrace{(p_a)^\mu (p_a)_\mu}_{m^2_{photon}=0} + \underbrace{(p_b)^\mu (p_b)_\mu}_{m^2} + \underbrace{(p_c)^\mu (p_c)_\mu}_{0} \\
&\quad +2\underbrace{(p_a)^\mu (p_b)_\mu}_{(\varepsilon_a,\varepsilon_a\hat{n}_a)(m,0)} - 2\underbrace{(p_a)^\mu (p_c)_\mu}_{(\varepsilon_a,\varepsilon_a\hat{n}_a)(\varepsilon_c,\varepsilon_c\hat{n}_c)} - 2\underbrace{(p_b)^\mu (p_c)_\mu}_{(m,0)(\varepsilon_c,\varepsilon_c\hat{n}_c)}
\end{aligned}$$

note: (1) 4 vector dot products commute. i.e. $(p_a)^\mu (p_b)_\mu = (p_b)^\mu (p_a)_\mu$. (2) We evaluate the product in Lorentz frame that simplifies the form.

Thus we obtain

$$\varepsilon_a m = \varepsilon_a\varepsilon_c - \varepsilon_a\varepsilon_c\hat{n}_a\cdot\hat{n}_c + m\varepsilon_c$$

$$\therefore \varepsilon_a\varepsilon_c\left(1-\cos\theta\right) = m\left(\varepsilon_a - \varepsilon_c\right)$$

Finally

$$\frac{1}{\varepsilon_c} - \frac{1}{\varepsilon_a} = \frac{1-\cos\theta}{mc^2}$$

we add c^2 to complete dimension. Hence $\varepsilon_c < \varepsilon_a$ after collision, photon losses energy.

Lecture 26 -Last Lec- (12/10/12)

Center of Momentum Frame

We can do this for a system of many particles labeled $a, b, c...$, then

$$(p_{total})^\mu = (p_a)^\mu + (p_b)^\mu + (p_c)^\mu + ...$$

so

$$(p_{total})^\mu (p_{total})_\mu = \underbrace{(p_a)^\mu (p_a)_\mu}_{m^2} + \underbrace{(p_b)^\mu (p_b)_\mu}_{m^2} + ... + 2\underbrace{(p_a)^\mu (p_b)_\mu}_{(m_a,0)(\varepsilon_b,\vec{p}_b)} + 2\underbrace{(p_a)^\mu (p_c)_\mu}_{(m_a,0)(\varepsilon_c,\vec{p}_c)} + ...$$

This slows all terms on the right are positive, so $(p_{total})^\mu (p_{total})_\mu > 0$ time-like.

Therefore there is a Lorentz frame s.t. $(p_{total})^{\mu} = (\varepsilon_{total}, 0)$. This frame is called the center of momentum frame.

For example: Particle a collides on particle b, which is at rest. Given ε_a with respect to lab, find ε_{total} with respect to the center of momentum frame.

$$\underbrace{(p_{total})^{\mu}(p_{total})_{\mu}}_{\left(\frac{\varepsilon_{total\,cm}}{c},0\right)\left(\frac{\varepsilon_{total\,cm}}{c},0\right)} = \underbrace{(p_a)^{\mu}(p_a)_{\mu}}_{m_a^2c^2} + \underbrace{(p_b)^{\mu}(p_b)_{\mu}}_{m_b^2c^2} + 2\underbrace{(p_a)^{\mu}(p_b)_{\mu}}_{(\frac{\varepsilon_{a\,lab}}{c},\vec{p}_{a\,lab})(m_bc,0)}$$

$$\therefore \varepsilon_{total\,cm} = \sqrt{(m_ac^2)^2 + (m_bc^2)^2 + 2\varepsilon_{a\,lab}m_bc^2}$$

Thus for very high energy collision (large $\varepsilon_{a\,lab}$), $\varepsilon_{total\,cm} \sim \sqrt{\varepsilon_{a\,lab}}$. This was at a time, physicists had to testify in front of congress and to teach them about this square relation in other to justify the needs of building higher energy collider.

6.6. Problem Set 12 (continued)

Jackson 11.19, 11.20, 11.23

6.7. Covariant Form of E&M

Recall postulate (1): Laws of physics (specifically microscopic reversible processes) have same form in all Lorentz frames. i.e. all Lorentz frames are equivalent. Another way of saying is that law of physics are covariant. ("covariant" here just means has same form, this makes no connection to covariant or contravariant indices.)

We will only concentrate on Maxwell equations and Lorentz force law.

If O and O' are two Lorentz frames, and x^{μ}, $\vec{E}$, $\vec{B}$ are in O and the corresponding $x^{\mu'}$, $\vec{E}'$, $\vec{B}'$ are in O', we would like to have Maxwell equations and L.F.L in terms of x^{μ}, $\vec{E}$, $\vec{B}$ in O and Maxwell equations and L.F.L in terms of $x^{\mu'}$, $\vec{E}'$, $\vec{B}'$ in O' to have the same forms.

To show covariance, one writes the law of physics in manifestly covariant form; (here "manifestly" just means obviously) that is, one writes law of physics as 4-tensor equations

$$T^{\mu\nu\cdots} = S^{\mu\nu\cdots}$$

If $T^{\mu\nu\cdots} = S^{\mu\nu\cdots}$ is true in O, then

$$a^{\alpha}{}_{\mu} a^{\beta}{}_{\nu} \cdots T^{\mu\nu\cdots} = a^{\alpha}{}_{\mu} a^{\beta}{}_{\nu} \cdots S^{\mu\nu\cdots}$$

and that is

$$\left(T^{\mu\nu\cdots}\right)' = \left(S^{\mu\nu\cdots}\right)'$$

is true in O'.

Four-vector Current

Charge is Lorentz scalar.

$$q = q'$$

$$\because q = \left(\rho d^3 x\right)$$

$$\therefore \left(\rho d^3 x\right) dx^{\mu}$$

is 4-vector. So is

$$\rho \underbrace{d^3 x dt}_{\frac{1}{c} d^4 x} \frac{dx^{\mu}}{dt}$$

Recall $d^4 x$ is Lorentz scalar, so

$$j^{\mu} \equiv \rho \frac{dx^{\mu}}{dt}$$

is 4-vector. $j^{\mu}$is 4-vector current density.

$$j^{\mu} = \left(\rho \frac{dct}{dt}, \rho \vec{v}\right) = \left(\rho c, \vec{j}\right)$$

so $j^{\mu'} = a^{\mu}{}_{\nu} j^{\nu}$.

Four-vector Potential

Recall Maxwell equations in Lorentz gauge $div\,\vec{A} + \frac{1}{c}\frac{\partial\Phi}{dt} = 0$ are

$$\left\{ \begin{array}{c} \Box\vec{A} = \frac{4\pi}{c}\vec{j} \\ \Box\Phi = 4\pi\rho \end{array} \right\}$$

We define

$$A^\mu = (\Phi, \vec{A})$$

Is this a 4-vector? Yes. Recall $\Box = \partial_\nu\partial^\nu$, so

$$\Box \left\{ \begin{array}{c} \vec{A} \\ \Phi \end{array} \right\} = \frac{4\pi}{c} \left\{ \begin{array}{c} \vec{j} \\ c\rho \end{array} \right\}$$

Thus

$$\partial_\nu\partial^\nu A^\mu = \frac{4\pi}{c} j^\mu$$

So A^μ should be a 4-vector in Lorentz gauge.

Gauge Transformation

What is about Lorentz gauge condition?

$$\partial_\mu A^\mu = 0$$

That is

$$\partial_\mu A^\mu = \partial_0 A^0 + \partial_i A^i = \frac{1}{c}\frac{\partial\Phi}{\partial t} + div\,\vec{A} = 0$$

Field Strength Tensor

We define

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

then $F^{\mu\nu}$ is a 4-tensor in Lorentz gauge, for A^μ is a 4-vector.

We claim that $F^{\mu\nu}$ is gauge invariant.

Recall gauge transformation that leaves $\vec{E}$, $\vec{B}$ unchanged

$$\begin{cases} \vec{A}_G = \vec{A} + \nabla\Lambda \\ \Phi_G = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \end{cases}$$

This can be written

$$\left(A^\mu\right)_G = A^\mu - \partial^\mu\Lambda$$

Proof of equation above. If $\mu = 0$, $\left(A^0\right)_G = A^0 - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$. If $\mu = i$, $-\partial^i = \partial_i = grad$. q.e.d.

Suppose we do a gauge transformation.

$$\begin{aligned} \left(F^{\mu\nu}\right)_G &= \partial^\mu\left(A^\nu\right)_G - \partial^\nu\left(A^\mu\right)_G \\ &= \partial^\mu\left(A^\nu - \partial^\nu\Lambda\right) - \partial^\nu\left(A^\mu - \partial^\mu\Lambda\right) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ &= F^{\mu\nu} \end{aligned}$$

This shows $F^{\mu\nu}$ is gauge invariant, so $F^{\mu\nu}$ is a 4-tensor in any gauge. And it's called field strength tensor. In fact we will show $F^{\mu\nu}$ is 2-rank antisymmetric 4 tensor.

6.8. Maxwell Equations in Covariant Form

Consider summing over μ

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= \Box A^\nu - \partial^\nu\partial_\mu A^\mu \\ &= \frac{4\pi}{c}j^\nu \end{aligned}$$

$\partial_\mu A^\mu = 0$ in Lorentz gauge. But $F^{\mu\nu}$ is gauge invariant, so we get 2 of the Maxwell equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu$$

with sources $(\rho, \vec{j})$ in any gauge.

Let us find the element of $F^{\mu\nu}$. E.g. $i = x, y, z$

$$
\begin{aligned}
F^{0i} &= \partial^0 A^i - \partial^i A^0 \\
&= \frac{1}{c}\frac{\partial}{\partial t} A^i + \partial_i A^0 \\
&= \frac{1}{c}\frac{\partial}{\partial t} A^i + \frac{\partial \Phi}{\partial x^i} \\
&= -E_i
\end{aligned}
$$

Similarly for all other elements of $F^{\mu\nu}$,

$$
F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_y & +B_z \\ E_y & B_y & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}
$$

μ gives rows and ν gives columns.

Lorentz Transformation of the Fields

Using this matrix, one can see how $\vec{E}$, $\vec{B}$ transform from one Lorentz frame to another, O to O'.

Exercise

Show use Lorentz transformations

$$
\begin{cases}
x^{\mu'} = a^{\mu}{}_{\nu} x^{\nu} \\
\vec{x}'_{\perp} = \vec{x}_{\perp} \\
\vec{x}'_{\parallel} = \gamma\left(\vec{x}_{\parallel} - \vec{v}t\right) \\
t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{x}_{\parallel}}{c^2}\right)
\end{cases}
$$

and

$$
\left(F^{\mu\nu}\right)' = a^{\mu}{}_{\alpha} a^{\nu}{}_{\beta} F^{\alpha\beta}
$$

one can get

$$\begin{cases} \vec{E}'_{\|} = \vec{E}_{\|} \\ \vec{B}'_{\|} = \vec{B}_{\|} \\ \vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \frac{\vec{v}}{c} \times \vec{B}_{\perp}\right) \\ \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} + \frac{\vec{v}}{c} \times \vec{E}_{\perp}\right) \end{cases}$$

Dual Field Strength Tensor

Define

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$$

Exercise

Show elements of $\tilde{F}^{\alpha\beta}$ are the same as of $F^{\alpha\beta}$, except $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$.

Consider summing over α,

$$\begin{aligned} \partial_\alpha \tilde{F}^{\alpha\beta} &= \partial_\alpha \left[\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\left(\partial_\gamma A_\delta - \partial_\delta A_\gamma\right)\right] \\ &= \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\gamma A_\delta - \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\delta A_\gamma \end{aligned}$$

The first term on the right is zero, because in the first term ϵ is antisymmetric with respect to exchange α and γ and $\partial\partial$ is symmetric w.r.t. exchange α and γ. Similarly the second term on the right is zero too. Hence

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0$$

This gives 2 of the homogeneous Maxwell equations.

Summary:

Manifestly covariant form of Maxwell equations

$$\begin{aligned} \partial_\alpha F^{\alpha\beta} &= \frac{4\pi}{c}j^\beta \\ \partial_\alpha \tilde{F}^{\alpha\beta} &= 0 \end{aligned}$$

Exercise

Show the two equations above is equivalent to "Bianchi identity"

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

Note:

(1) One wants to put in j_{mag} to the two equations symmetric, then the logic of derivation does not follow because $\partial_\mu A^\mu$ is no longer 0, but nevertheless one can still do so and get the correct equations

$$\begin{aligned}\partial_\alpha F^{\alpha\beta} &= \frac{4\pi}{c} j^\beta \\ \partial_\alpha \tilde{F}^{\alpha\beta} &= \frac{4\pi}{c} j^\beta_{mag}\end{aligned}$$

(2) Lorentz Invariant

Exercise

show the following

$$F^{\alpha\beta} F_{\alpha\beta} = 2\left(\left|\vec{B}\right|^2 - \left|\vec{E}\right|^2\right)$$

$$\tilde{F}^{\alpha\beta} \tilde{F}_{\alpha\beta} = 2\left(\left|\vec{E}\right|^2 - \left|\vec{B}\right|^2\right)$$

$$\tilde{F}^{\alpha\beta} F_{\alpha\beta} = -4\vec{E} \cdot \vec{B}$$

They imply that if $\left|\vec{E}\right| = \left|\vec{B}\right|$ in one frame, $\left|\vec{E}\right| = \left|\vec{B}\right|$ in all frames; if $\left|\vec{E}\right| \perp \left|\vec{B}\right|$ in one frame, $\left|\vec{E}\right| \perp \left|\vec{B}\right|$ in all frames.

(3) If the field is not homogenous, one doesn't find one Lorentz frame work globally, but to use technique like adiabatic approximation in studying plasma or magnetic mirrors.

6.9. Covariant Form of Lorentz Force Law

Recall $\vec{p} = m\gamma\vec{v}$ and $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$, we guess of manifestly covariant form of L.F.L

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} \frac{dx_\nu}{d\tau}$$

Exercise

Check this gives L.F.L. let $\mu = i$, $\frac{dp^i}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$.

For $\mu = 0$, $\frac{dp^0}{d\tau} = \frac{d\varepsilon/c}{d\tau} = \frac{q}{c}\vec{E}\cdot\vec{v}$. That is

$$\frac{d\varepsilon}{d\tau} = q\vec{E}\cdot\vec{v} = (force)(\vec{v})$$

This tells us that the rate of change of kinematic energy is equal to rate at which force does work and this also tells us magnetic force does no work.

6.10. Problem Set 12 (continued)

Jackson 11.13, 11.15, 11.30

Appendix

A. Course Review Problem Session

Review (12/12/12)

A.1. Types of Problems on Final

Propagation of Plane Wave in Anisotropic Material

$$\vec{D} = \underset{\sim}{\epsilon}\vec{E}$$

$$\vec{B} = \underset{\sim}{\mu}\vec{H}$$

$$\vec{j} = \underset{\sim}{\sigma}\vec{E}$$

find allowed N^2 corresponding $\vec{E}$ polarization.

Antenna Radiation

$$\vec{A}(\vec{x},t) = \frac{1}{rc}\int d^3x'\vec{j}(\vec{x}', t' = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{x}'}{c})$$

$$\vec{E}(\vec{x},t) = \frac{1}{c}\hat{r} \times \left(\hat{r} \times \dot{\vec{A}}(\vec{x},t)\right)$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\left(\vec{E} \cdot \vec{E}\right) r^2$$

Use complex field for harmonic time dependence field and then find $\left\langle \frac{dP}{d\Omega} \right\rangle$.

Radiation from Moving Particles (L-W field)

$$\vec{E}(\vec{x},t) = \left\{ \frac{q}{rc^2} \frac{\hat{r} \times \left[\left(\hat{r} - \vec{\beta}\right) \times \vec{a}\right]}{\kappa^3} \right\}_{t'=t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{S}(t')}{c}}$$

$\kappa = 1 - \vec{\beta} \cdot \hat{r}$. Power detected at time t

$$\frac{dP(t)}{d\Omega} = \frac{c}{4\pi}\left(\vec{E} \cdot \vec{E}\right) r^2$$

Power emitted at time t'

$$\frac{dP'(t')}{d\Omega} = \frac{c}{4\pi}\left(\vec{E} \cdot \vec{E}\right) r^2 \kappa$$

Frequency Distribution of Radiation of antennas or particles

$$\begin{array}{ccccc} \vec{j}_w(\vec{x}) & \rightarrow & \vec{A}_w(\vec{x}) & \rightarrow & \vec{E}_w(\vec{x}) \\ \uparrow & & \uparrow & & \uparrow \\ \vec{j}(\vec{x},t) & & \vec{A}(\vec{x},t) & & \vec{E}(\vec{x},t) \end{array}$$

There are three ways to get $\vec{E}_w(\vec{x})$ which is needed for frequency distribution

$$\frac{d^2\epsilon}{dwd\Omega} = \frac{c}{2\pi}\vec{E}^*_w(\vec{x})\vec{E}_w r^2$$

the factor $1/2$ instead of $1/4$ is due to indistinguish $\pm w$. and the integration from 0 to infinity,

$$\frac{d\epsilon}{d\Omega} = \int_0^\infty dw \frac{d^2\epsilon}{dwd\Omega}$$

Among these ways $\vec{E}(\vec{x},t) \rightarrow \vec{E}_w(\vec{x})$ is most algebraically challenging.

Solving Linear PDE w/wo inhomo source

Choose a contour s.t. one gets initial condition satisfied (if possible)

Special Relativity

(1) Kinematics of decays +/or collision

$$P_a P_b = (P_a)^\mu (P_b)_\mu = \frac{\varepsilon_a \varepsilon_b}{c^2} - \vec{p}_a \cdot \vec{p}_b$$

(2) Manipulation of covariant form of M. eq + L.F.L (use cartesian tensor notation)

A.2. Examples of Types of Problems

F.T. Solving Linear PDE

1 spatial dimension wave equation of inhomo source term $f(x,t)$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\psi(x,t) = 4\pi f(x,t)$$

Expand

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)$$

first term on the right corresponds to wave propagating in $+x$ direction, whiles the second term corresponds to move in -x direction.

Consider

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\psi(x,t) = 4\pi f(x,t)$$

s.t. $\psi(x, t = -\infty) = 0$, find $\psi(x,t)$.

Use F.T.

$$\left\{\begin{array}{c} \psi(x,t) \\ f(x,t) \end{array}\right\} = \int \frac{dx}{\sqrt{2\pi}} \int \frac{dw}{\sqrt{2\pi}} e^{ikx} e^{-iwt} \left\{\begin{array}{c} \tilde{\psi}(k,w) \\ \tilde{f}(k,w) \end{array}\right\}$$

then F.T pde

$$\frac{1}{c}(-iw)\tilde{\psi} + (ik)\tilde{\psi} = 4\pi\tilde{f}$$

so

$$\tilde{\psi} = \frac{4\pi\tilde{f}}{-iw\frac{1}{c} + ik} = \frac{4\pi\tilde{f}ic}{w - kc}$$

one should retain the temptation to write $kc = w$, for here k is the just the fourier variable and we are not dealing with plane wave.

$$\begin{aligned}
\psi(x,t) &= \int \frac{dk}{\sqrt{2\pi}} \int \frac{dw}{\sqrt{2\pi}} e^{ikx} e^{-iwt} \frac{4\pi ic}{w - kc} \int \frac{dx'}{\sqrt{2\pi}} \int \frac{dt'}{\sqrt{2\pi}} e^{-ikx'} e^{iwt'} f(x',t') \\
&= \frac{4\pi ic}{(2\pi)^2} \int dx' \int dt' f(x',t') \int dk e^{ik(x-x')} \int dw e^{-iw(t-t')} \frac{1}{w - kc}
\end{aligned}$$

To do $\int dw e^{-iw(t-t')} \frac{1}{w-kc}$, we have to choose a contour. For retarded solution we need $t > t'$ for the source is evaluated at t'. So by Jordan's lemma we will close the lower half circle, and to get non-trivial solution, we need to move the pole to lower half circle. Thus

$$\oint dw e^{-iw(t-t')} \frac{1}{w - (kc - \epsilon i)} = -2\pi i e^{-i(kc-\epsilon i)(t-t')} = -2\pi i e^{-ikc(t-t')}$$

for some $\epsilon \to 0^+$, and $t > t'$.

$$\begin{aligned}
\psi(x,t) &= \frac{4\pi ic}{(2\pi)^2} \int dx' \int dt' f(x',t') \int dk e^{ik(x-x')} (-2\pi i)\Theta(t-t') e^{-ikc(t-t')} \\
&= \frac{4\pi ic}{(2\pi)^2} \int dx' \int dt' f(x',t')(-2\pi i)\Theta(t-t')(2\pi)\delta\left[x - x' - c\,(t-t')\right] \\
&= 4\pi c \int dt' \Theta(t-t') f(x - c(t-t'), t')
\end{aligned}$$

e.g. $f(x,t) = g(x)\delta(t)$ blip, then

$$\psi(x,t) = A\Theta(t)g(x - ct)$$

this says $g(x-ct)$ =cost for $x - ct = x_0$ some const

So

$$x = x_0 + ct$$

right moving at speed c.

Propagation of Plane Wave in Anisotropic Material

material $\epsilon = 1$, $\mu = 1$, and conductivity in $x - y$ plane

$$\underset{\sim}{\sigma_c} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

w.r.t principle axes. σ real positive. Find allowed N^2 ($N \equiv kc/w$), consider plane wave propagate in $\hat{x}$ direction.

Plane wave

$$\vec{F} = \vec{F}_0 e^{i(k\hat{n}\cdot\vec{x} - wt)}$$

$$\underbrace{curl\,\vec{H}}_{\underbrace{curl\,\vec{B}}_{ik\hat{n}\times\underbrace{\vec{B}}_{N\hat{n}\times\vec{E}}}} = \frac{4\pi}{c}\underbrace{\vec{j}_{free}}_{\underset{\sim}{\sigma_c}\cdot\vec{E}} + \underbrace{\frac{1}{c}\frac{\partial\vec{D}}{\partial t}}_{\underbrace{\frac{1}{c}\frac{\partial\vec{E}}{\partial t}}_{-\frac{iw}{c}\vec{E}}}$$

That is

$$ik\hat{n} \times \left(N\hat{n} \times \vec{E}\right) = \frac{4\pi}{c}\underset{\sim}{\sigma_c} \cdot \vec{E} - \frac{iw}{c}\vec{E}$$

multiply c/iw,

$$N^2 \underbrace{\hat{n} \times \left(\hat{n} \times \vec{E}\right)}_{\hat{x}(E_x) - \vec{E}} = \frac{4\pi}{iw}\underset{\sim}{\sigma_c} \cdot \vec{E} - \vec{E}$$

$$N^2\left(-\hat{y}E_y - \hat{z}E_z\right) + \frac{4\pi i}{w}\underset{\sim}{\sigma_c} \cdot \vec{E} + \vec{E} = 0$$

This gives

$$\begin{pmatrix} 1 + \frac{4\pi i}{w}\sigma & 0 & 0 \\ 0 & 1 + \frac{4\pi i}{w}\sigma - N^2 & 0 \\ 0 & 0 & 1 - N^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

determinant is zero.

$$N^2 = 1$$

Polarization $E_x = E_y = 0$ and E_z arbitrary. This makes sense, because in z direction no conductivity, the polarization in z direction does not affect anything.

or

$$N^2 = 1 + \frac{4\pi i}{w}\sigma$$

Polarization $E_x = E_z = 0$ and E_y arbitrary.

What if σ is pure imaginary? Then for the second eigenvalue N can turn out to be pure imaginary. So the wave is not propagate at all? ANS: No. In this case, the some additional boundary condition would have to take into account. such as damping, because if σ is pure imaginary, the whole material (plasma) is oscillating.

Special Relativity

(1) Show current charge conservation.

Solve:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta$$

take ∂_β,

$$\partial_\beta \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} \partial_\beta j^\beta$$

On left hand side, $\partial_\beta \partial_\alpha$ is symmetric w.r.t exchange α, β and $F^{\alpha\beta}$ is antisymmetric w.r.t. α, β. So

$$\partial_\beta j^\beta = 0$$

That is

$$div\, \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

(2) Show that $F^{\alpha\beta}$ obey inhomo wave equation $\Box F^{\alpha\beta} = 4\pi T^{\alpha\beta}$.

Solve: take

$$\partial^\gamma F^{\alpha\beta} + \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} = 0$$

apply ∂_γ,

$$\underbrace{\partial_\gamma \partial^\gamma}_{\Box} F^{\alpha\beta} + \partial^\alpha \underbrace{\underbrace{\partial_\gamma F^{\beta\gamma}}_{-\partial_\gamma F^{\gamma\beta}}}_{-\frac{4\pi}{c} j^\beta} + \partial^\beta \underbrace{\partial_\gamma F^{\gamma\alpha}}_{\frac{4\pi}{c} j^\alpha} = 0$$

Hence

$$\Box F^{\alpha\beta} = \frac{4\pi}{c} \left(\partial^\alpha j^\beta - \partial^\beta j^\alpha \right)$$

(3) Consider an electron-positron annihilation. e^+(particle a) collides e^-(particle b) at rest and produce one photon (particle c) This is called one photon decay. Show this is not possible.

Solve:

$$(p_a)^\mu + (p_b)^\mu = (p_c)^\mu$$

$$(p_c)^\mu (p_c)_\mu = (p_a)^\mu (p_a)_\mu + (p_b)^\mu (p_b)_\mu + 2 (p_a)^\mu (p_b)_\mu$$

That gives

$$0 = m^2 + m^2 + 2 \left(\varepsilon_a \varepsilon_b - \vec{p}_a \cdot \vec{p}_b \right) = 2m^2 + 2(\varepsilon_a m - 0)$$

Not possible.

(4) Consider 2 photons decay. e^+(particle a) collides e^-(particle b) at rest and produce two photons (particles c, d). and c has scattering angle θ. Given θ, ε_a, and m rest mass of electron and positron. Find ε_c.

Solve:

$$P_d = P_a + P_b - P_c$$

then

$$\begin{aligned} 0 = P_d P_d &= P_a P_a + P_b P_b + P_c P_c + 2 \left(P_a P_b - P_a P_c - P_b P_c \right) \\ &= m^2 + m^2 + 0 + 2 \left[\left(\varepsilon_a \varepsilon_b - \vec{p}_a \vec{p}_b \right) - \left(\varepsilon_a \varepsilon_c - \vec{p}_a \vec{p}_c \right) - \left(\varepsilon_b \varepsilon_c - \vec{p}_b \vec{p}_c \right) \right] \\ &= 2m^2 + 2 \left(\varepsilon_a m - \varepsilon_a \varepsilon_c + p_a p_c \cos\theta - m \varepsilon_c \right) \end{aligned}$$

$p_a = \sqrt{\varepsilon_a^2 - m^2}$, $p_c = \varepsilon_c$

$$\therefore \varepsilon_c = \frac{m\left(\varepsilon_a + m\right)}{\varepsilon_a + m - \sqrt{\varepsilon_a^2 - m^2}\cos\theta}$$

If both electron and positron are at rest, annihilation at rest, then $\varepsilon_a = m$

$$\varepsilon_c = \frac{m2m}{2m} = m$$

Antenna Radiation

Consider 3 tiny antennas. This problem leads to linear quadrupole.

Given

$$\vec{j}(\vec{x},t) = \hat{z}J_0\left[2\delta^3(\vec{x}) - \delta^3(\vec{x} - L\hat{y}) - \delta^3(\vec{x} + L\hat{y})\right]e^{-iwt}$$

Find $\left\langle \frac{dP}{d\Omega}\right\rangle$.

Solve:

$$\begin{aligned}
\vec{A}(\vec{x},t) &= \frac{1}{rc}\int d^3x'\vec{j}(\vec{x}', t' = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{x}'}{c}) \\
&= \frac{\hat{z}J_0}{rc}\left[2e^{-iw(t-\frac{r}{c})} - e^{-iw(t-\frac{r}{c}+\frac{\hat{r}\cdot\hat{y}L}{c})} - e^{-iw(t-\frac{r}{c}-\frac{\hat{r}\cdot\hat{y}L}{c})}\right] \\
&= \frac{\hat{z}J_0}{rc}e^{-iw(t-\frac{r}{c})}\left[2 - e^{-iw\frac{\hat{r}\cdot\hat{y}L}{c}} - e^{iw\frac{\hat{r}\cdot\hat{y}L}{c}}\right] \\
&= \frac{\hat{z}J_0}{rc}e^{-iw(t-\frac{r}{c})}\left[2 - 2\cos\left(kL\sin\theta\sin\phi\right)\right] \\
&= \frac{4\hat{z}J_0}{rc}e^{-iw(t-\frac{r}{c})}\sin^2\left(\frac{kL}{2}\sin\theta\sin\phi\right)
\end{aligned}$$

$$\begin{aligned}
\vec{E} &= \frac{1}{c}\hat{r}\times\left(\hat{r}\times\dot{\vec{A}}\right) \\
&= \frac{-iw}{c}\frac{4J_0}{rc}e^{-iw(t-\frac{r}{c})}\sin^2\left(\frac{kL}{2}\sin\theta\sin\phi\right)\hat{r}\times\left(\hat{r}\times\hat{z}\right)
\end{aligned}$$

$$
\begin{aligned}
\left\langle \frac{dP}{d\Omega} \right\rangle &= \frac{1}{2}\frac{c}{4\pi}\left(\vec{E}^{*}\vec{E}\right)r^{2} \\
&= \frac{cw^{2}}{8\pi c^{2}}\left(\frac{16J_0^2}{c^2}\right)\sin^4\left(\frac{kL}{2}\sin\theta\sin\phi\right)\sin^2\theta \\
&= \frac{2w^2}{\pi c^3}J_0^2\sin^4\left(\frac{kL}{2}\sin\theta\sin\phi\right)\sin^2\theta
\end{aligned}
$$

Frequency Distribution of Radiation

This problem can be used to detect underground nuclear exploration.

Given a nuclear exploration at $t = 0$, and $\alpha > 0$ cooling time scale. That expels earth magnetic field in the region of the nuclear exploration, thus magnetic dipole is formed

$$
\vec{\mu}(t) = \Theta(t)\mu_0\hat{z}e^{-\alpha t}
$$

This also leads to form time dependent $\vec{E}$ field, but we won't consider that here.

Find $\frac{d^2\epsilon}{dwd\Omega}$.

Solve: recall electric/magnetic field due to electric/magnetic dipole

$$
\vec{E} = \frac{1}{rc^2}\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{d}}\right)\right]_{t'=t-\frac{r}{c}}
$$

$$
\vec{B} = \frac{1}{rc^2}\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{\mu}}\right)\right]_{t'=t-\frac{r}{c}}
$$

Since

$$
\frac{d^2\epsilon}{dwd\Omega} = \frac{c}{2\pi}\left[\left\{\begin{array}{c}\vec{E}_w^{*}\vec{E}_w \\ \vec{B}_w^{*}\vec{B}_w\end{array}\right\}\right]r^2
$$

We can get $\vec{B}_w$ from $\vec{\mu}_w$

$$\begin{aligned}
\vec{\mu}_w &= \int \frac{dt}{\sqrt{2\pi}} e^{iwt} \vec{\mu}(t - \frac{r}{c}) \\
&= \int \frac{dt}{\sqrt{2\pi}} e^{iwt} \Theta(t - \frac{r}{c}) \mu_0 \hat{z} e^{-\alpha\left(t - \frac{r}{c}\right)} \\
&= \int \frac{dt'}{\sqrt{2\pi}} e^{iw\left(t' + \frac{r}{c}\right)} \Theta(t') \mu_0 \hat{z} e^{-\alpha t'} \\
&= \frac{\mu_0}{\sqrt{2\pi}} e^{ikr} \hat{z} \int_0^\infty dt' e^{t'(iw-\alpha)} \\
&= \frac{\mu_0}{\sqrt{2\pi}} e^{ikr} \hat{z} \frac{-1}{iw - \alpha}
\end{aligned}$$

$$\ddot{\vec{\mu}} = -w^2 \vec{\mu} = \frac{\mu_0}{\sqrt{2\pi}} e^{ikr} \hat{z} \frac{w^2}{iw - \alpha}$$

$$\vec{B}_w(\vec{x}) = \frac{1}{rc^2} \frac{\mu_0}{\sqrt{2\pi}} e^{ikr} \frac{w^2}{iw - \alpha} \hat{r} \times (\hat{r} \times \hat{z})$$

Thus

$$\frac{d^2\epsilon}{dw d\Omega} = \frac{c}{2\pi} \frac{\mu_0^2}{2\pi c^4} \frac{w^4}{w^2 + \alpha^2} \sin^2\theta$$

Question: How can people at the opposite side of earth detect this wave? Because earth is transparent to high frequency waves.

Radiation from Moving Particles

Consider a particle moving in a circle in $x - y$ plane relativistically. Find $\frac{dP'(t')}{d\Omega}$ and $\frac{dP(t)}{d\Omega}$ on z axis.

Solve:

$$\vec{E}(\vec{x}, t) = \frac{q}{rc^2} \left\{ \frac{\hat{r} \times [(\hat{r} - \beta) \times \vec{a}]}{\kappa^3} \right\}_{t' = t - \frac{r}{c} + \frac{\hat{r} \cdot \vec{S}(t')}{c}}$$

$\hat{r} \cdot \vec{S} = 0$, $\kappa = 1$ for $\hat{r} \perp \vec{v}$

$$\begin{aligned}
\vec{E}(\vec{x}, t) &= \frac{q}{rc^2} \left\{ (\hat{r} - \beta)(0) - \vec{a}\kappa \right\}_{t' = t - \frac{r}{c}} \\
&= -\frac{q}{rc^2} \vec{a}(t' = t - \frac{r}{c}) \\
&= -\frac{q}{rc^2} \left(\frac{v^2}{R} \right) \hat{R}_\perp
\end{aligned}$$

$$\begin{aligned}\frac{dP'(t')}{d\Omega} &= \frac{c}{4\pi}\left|\vec{E}\cdot\vec{E}\right|r^2 \\ &= \frac{c}{4\pi}\frac{q^2}{c^4}\left(\frac{v^2}{R}\right)^2 = \frac{dP(t)}{d\Omega}\end{aligned}$$

B. Qualifying Exam E&M Review

Session 1 (11/30/12)

B.1. Time independent E&M Phenomena

Electrostatics

$$\left|\begin{array}{c} curl\,\vec{E} = 0 \\ div\,\vec{D} = 4\pi\rho_{free} \end{array}\right| \begin{array}{c} div\,\vec{E} = 4\pi\rho_{total} \\ div\,\vec{P} = -\rho_{bound} \end{array}\left| \begin{array}{c} \vec{D} = \vec{E} + 4\pi\vec{P} \\ \vec{P} = \vec{P}_0 + \chi_{el}\vec{E} \end{array}\right|$$

$curl\,\vec{E} = 0$ implies $\vec{E} = -\nabla\phi$

$\epsilon = 1 + 4\pi\chi_{el}$ then $\vec{D} = 4\pi\vec{P}_0 + \epsilon\vec{E}$. For problems on qualification exams, you will be given either $\vec{P}_0$ permanent polarization, or ϵ. (see B.5. Five famous problems)

Magnetostatics

$$\left|\begin{array}{c} div\,\vec{B} = 0 \\ curl\,\vec{H} = \frac{4\pi}{c}\vec{j}_{free} \end{array}\right| \begin{array}{c} curl\,\vec{B} = \frac{4\pi}{c}\vec{j}_{free} \\ curl\,\vec{M} = \frac{1}{c}\vec{j}_{bound} \end{array}\left| \begin{array}{c} \vec{B} = \vec{H} + 4\pi\vec{M} \\ \vec{M} = \vec{M}_0 + \chi_{mag}\vec{H} \end{array}\right|$$

$div\,\vec{B} = 0$ implies $\vec{B} = curl\,\vec{A}$. undergraduate EM try to avoid using $\vec{A}$.

$curl\,\vec{H} = \frac{4\pi}{c}\vec{j}_{free}$ implies $div\,\vec{j}_{free} = 0$. This is an important condition when one tries to use Stoke Theorem.

Suppose one wants to calculate the $\vec{B}$ field at the center of a square current loop. If one uses result from a infinitely long wire, he will get

$$\frac{4\mu_0 I}{\pi a}$$

but the correct one is

$$\frac{2\sqrt{2}\mu_0 I}{\pi a}$$

That is because due to charge build up at the corners, Stoke Theorem doesn't apply.

$\mu = 1 + 4\pi\chi_{mag}$ then $\vec{B} = 4\pi\vec{M}_0 + \mu\vec{H}$. For problems on qualification exams, you will be given either $\vec{M}_0$ permanent magnetization, or μ.

QM often uses transverse gauge ($div\,\vec{A} = 0$) to make $\vec{p} = \frac{\hbar}{i}\nabla$ commute with $\vec{A}$, so recall non-relativistic Hamiltonian for a point charge in quantum mechanics

$$H = \frac{(\vec{p} - \frac{q}{c}\vec{A})^2}{2m} + q\phi$$

then

$$(\vec{p} - \frac{q}{c}\vec{A})^2 = \vec{p}^2 - 2\frac{q}{c}\vec{A}\vec{p} + \frac{q^2}{c^2}\vec{A}^2$$

Steady State Current Flow

$$\left| \begin{array}{c} div\,\vec{j}_{free} = 0 \\ curl\,\vec{E} = 0 \end{array} \right| \begin{array}{c} \vec{j}_{free} = \sigma_c \vec{E} \end{array} \Bigg|$$

$div\,\vec{j}_{free} = 0$ is same as $\frac{\partial \rho_{free}}{\partial t} = 0$ (definition of steady-state).

General Solutions

Assume field goes to 0 as $r \to \infty$. In vacuum $\epsilon = 1$,$\mu = 1$, $\sigma_c = 0$. Known ρ and $\vec{j}$ everywhere in space, then

$$\vec{E}(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')\,(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \to \sum_\alpha \frac{q_\alpha(\vec{x} - \vec{S}_\alpha)}{\left|\vec{x} - \vec{S}_\alpha\right|^3}$$

$$\phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \to \sum_\alpha \frac{q_\alpha}{\left|\vec{x} - \vec{S}_\alpha\right|}$$

$$\vec{B}(\vec{x}) = \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \nrightarrow$$

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} \nrightarrow$$

The last two equations has no steady state forms for separated moving point charges.

Notations

$$\rho d^3x \to \sigma da \to \lambda dl$$

$\rho = \frac{charge}{vol}$, $\sigma = \frac{charge}{area}$, $\lambda = \frac{charges}{length}$.

$$\vec{j}d^3x \to \vec{K}da \to \vec{I}dl$$

$\vec{j} = \frac{current}{\perp area}$, $\vec{K} = \frac{current}{\perp length}$, $\vec{I} = line\, current.$

B.2. Use Symmetry

Problems with symmetry, one can often use

- $div\, \vec{D}$ equation + Gauss's Theorem
- $curl\, \vec{H}$ equation + Stokes Theorem

Example: Electrostatics

Consider a charge Q embedded in uniform medium ϵ. Then

$$div\, \vec{D} = 4\pi\rho_{free}$$

Apply Gauss

$$\oint div\, \vec{D} da = 4\pi \int dv \rho_{free}$$

so $D4\pi r^2 = 4\pi Q$, that gives

$$D = \frac{Q}{r^2}$$

and $D = \epsilon E$ and $E = \frac{Q_{total}}{r^2}$, therefore

$$Q_{total} = Q/\epsilon$$

Since $\epsilon > 1$, $Q_{total} < Q_{free}$. Hence medium generates negative charges (bound charges) that shields the Q.

Magnetostatics

Consider a line current I_{free} embedded in uniform μ. Then

$$curl\,\vec{H} = \frac{4\pi}{c}\vec{j}_{free}$$

Apply Stoke

$$\oint \vec{H}dl = \frac{4\pi}{c}\int \vec{j}_{free}da$$

so $H2\pi r = \frac{4\pi}{c}I$, that gives

$$H = \frac{2I}{cr}$$

and $B = \mu H$ and $B = \frac{2I_{total}}{cr}$, therefore

$$I_{total} = \mu I_{free}$$

Since $\mu > 1$, $I_{total} > I_{free}$. Hence I_{free} generates magnetic field in the medium that in turns generates magnetic dipoles (bound current) that increase the total current.

B.3. Boundary Values Problems

Electrostatics, magnetostatics, steady-state current flow can all be reduced to solving Poisson's equation

$$\nabla^2\psi(\vec{x}) = -4\pi f(\vec{x})$$

But there are accessible regions S where sources ρ, $\vec{j}$ are known, and there are inaccessible regions where ρ, $\vec{j}$ are not known. This leads to boundary values problems. And combined with some techniques: separation of variables and image sources in inaccessible regions.

Boundary Values

- Dirichlet problem: $\psi(\vec{x})$ is given everywhere on boundary of S of the accessible regions. The solution for ψ in the accessible region exists and it is unique. Dirichlet problem is used electrostatics.
- Neumann problem: $\frac{\partial\psi(\vec{x})}{\partial\hat{n}} = \left(\hat{n}\cdot\vec{\nabla}\right)\psi(\vec{x})$ is given everywhere on boundary of S of the accessible regions. $\hat{n}$ is the unit vector out of accessible region into inaccessible region. Neumann problem is used in magnetostatics. Neumann condition has a mild constraint: $\oint_S \frac{\partial\psi(\vec{x})}{\partial\hat{n}}da = -4\pi\int d^3x f(\vec{x})$. This is because

  $$\nabla^2\psi = div(grad\,\psi) = -4\pi f(\vec{x})$$

  apply Gauss,

  $$\oint_S grad\,\psi\cdot d\vec{a} = \oint_S \left(\hat{n}\cdot\vec{\nabla}\right)\psi(\vec{x})da = -4\pi\int d^3x f(\vec{x})$$

Most common problems would be like. Given ϵ, μ, σ_c of different medium, and $\rho_{free} = 0$, $\vec{j}_{free} = 0$ except possible on surface within the accessible regions, to solve Laplace equation. Very often problems involve boundary conditions at conductors.

Conductors

Consider an ordinary conductor ($\sigma_c \neq 0$, but finite) and a perfect conductor ($\sigma_c = \infty$).

- Ordinary Conductor. $\vec{j}_{free}$, $\vec{E}$, and $\vec{B}$ need not to be zero inside.
- Perfect Conductor. $\vec{j}_{free} = 0$ ($\rho_{total} = 0$ no rest charge inside) $\vec{E} = 0$ and $\vec{B} = 0$. $\vec{E}$ has to be 0 otherwise $\vec{j}_{free} = \infty$ that gives infinite energy. Since $curl\,\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$, $\frac{\partial\vec{B}}{\partial t} = 0$. Then $curl\,\vec{B} = \frac{4\pi}{c}\vec{j}_{free} = 0$, and $\vec{B} = 0$.

Electrostatics

Consider a donuts sharped accessible region. Outside the donuts and the hole are inaccessible regions. And we cut the donuts in half, left half filled with ϵ_1 (region1) and left half filled with ϵ_2 (regoin2). $\rho_{free} = 0$ within regions 1 & 2. But ϵ changes from region 1 to 2, so σ_{free} (surface charge density) maybe present on the surface between regions 1 & 2 (We call it internal surface).

To solve this kind of problem, we know

$$curl\,\vec{E} = 0 \implies \vec{E} = -\vec{\nabla}\psi$$

$$div\,\vec{D} = 4\pi\rho_{free} = 0$$

since ϵ_1, ϵ_2 are constant within regions 1 & 2, we have $\vec{D} = \epsilon_{1,2}\vec{E}$. So

$$div\,\vec{E} = 0$$

Therefore we end up solving

$$\nabla^2\psi_1(\vec{x}) = 0 \qquad \text{in region 1}$$

$$\nabla^2\psi_2(\vec{x}) = 0 \qquad \text{in region 2}$$

and both should satisfy boundary condition on the inaccessible surface (Dirichlet usually) and match solutions ψ_1, ψ_2 across the internal surface.

Matching condition:

$$curl\,\vec{E} = 0$$

This gives

- $E_{tangent}$ is continuous (this is not convenient to use), so we use
- ψ is continuous, because $\vec{E} = -\vec{\nabla}\psi \implies$ change in $\psi, \Delta\psi = -\vec{E}\cdot d\vec{l} \to 0$ as $d\vec{l} \to 0$ ($\vec{E} \neq \infty$), i.e. ψ is continuous across internal surface.

Second matching condition:

$$div\,\vec{D} = 4\pi\rho_{free}$$

This gives

- D_{normal} is discontinuous by $4\pi\sigma_{free}$ or $(\epsilon E)_{normal}$ is discontinuous by $4\pi\sigma_{free}$. Only when $\sigma_{free} = 0$ on the internal surface, then D_{normal} is continuous.

Magnetostatics

Consider the same donuts sharped accessible region. Outside the donuts and the hole are inaccessible regions. And we cut the donuts in half, left half filled with μ_1 (region1) and left half filled with μ_2 (regoin2). $\vec{j}_{free} = 0$ within regions 1 & 2. But μ changes from region 1 to 2, so $\vec{K}_{free}$ (surface current density) maybe present on the surface between regions 1 & 2.

To solve this kind of problem, we know

$$curl\,\vec{H} = \frac{4\pi}{c}\vec{j}_{free} = 0 \implies \vec{H} = -\vec{\nabla}\psi$$

$$div\,\vec{B} = 0$$

since μ_1, μ_2 are constant within regions 1 & 2, we have $\vec{B} = \mu_{1,2}\vec{H}$. So

$$div\,\vec{H} = 0$$

Therefore we end up solving

$$\nabla^2\psi_1(\vec{x}) = 0 \qquad \text{in region 1}$$

$$\nabla^2\psi_2(\vec{x}) = 0 \qquad \text{in region 2}$$

and both should satisfy boundary condition on the inaccessible surface (usually Neumann) and match solutions ψ_1, ψ_2 across the internal surface.

Matching condition:

$$curl\,\vec{H} = \frac{4\pi}{c}\vec{j}_{free}$$

This gives

- $H_{tangent}$ is discontinuous by $\frac{4\pi}{c}\vec{K}_{free}$. When $K_{free} = 0$ (surface currents can exist only on a perfect conductor surface), $H_{tangent}$ is continuous and ψ is continuous.

Second matching condition:

$$div\,\vec{B} = 0$$

B_{normal} is continuous across surface, i.e. $(\mu_1 H_1)_{normal} = (\mu_2 H_2)_{normal}$ at boundary.

Very often problems involve boundary conditions at perfect conductors. At the surfuce of a perfect conductors, use exterior normal.

Tangential $E = 0$ (b/c fields are 0 inside perfect conductors) ; normal $D = 4\pi\rho_{free}$

Tangential $H = 4\pi K_{free}/c$; normal $B = 0$.

Steady-State Current Flow

Consider the same donuts sharped accessible region. Outside the donuts and the hole are inaccessible regions. And we cut the donuts in half, left half filled with σ_1 (region1) and left half filled with σ_2 (regoin2). $\partial\rho_{free}/\partial t = 0$.

To solve this kind of problem, we know

$$curl\,\vec{E} = 0 \implies \vec{E} = -\vec{\nabla}\psi$$

$$div\,\vec{j}_{free} = -\frac{\partial\rho_{free}}{\partial t}0$$

since σ_1, σ_2 are constant within regions 1 & 2, we have $\vec{j}_{free} = \sigma_{1,2}\vec{E}$. So

$$div\,\vec{E} = 0$$

Therefore we end up solving

$$\nabla^2\psi_1(\vec{x}) = 0 \qquad \text{in region 1}$$

$$\nabla^2\psi_2(\vec{x}) = 0 \qquad \text{in region 2}$$

and both should satisfy boundary condition on the inaccessible surface (usually Neumann) and match solutions ψ_1, ψ_2 across the internal surface.

Matching condition:

$$curl\,\vec{E} = 0$$

This gives

- $E_{tangent}$ is continuous. This is not convenient to use, so we use

- ψ is continuous

Second matching condition:

$$div\, \vec{j}_{free} = 0$$

- $\vec{j}_{normal}$ is continuous across surface, i.e. $(\sigma_1 E_1)_{normal} = (\sigma_2 E_2)_{normal}$ at boundary.

Session 2 (12/7/12)

B.4. Separation of Variables

Laplace equation is separable in 11 coordinate systems.

Cartesian Coordinates

Cartesian coordinate is very hard to use, because it doesn't have a single general formula for all problems. Solutions in Cartesian coordinate lead to sin, cos, sinh, and cosh. (exponential, sinusoidal, hyperbolic functions).

Example
Solve

$$\nabla^2 \psi = 0$$

inside a rectangular cuboid sitting in the first quadrant, and subject to boundary conditions

$$\psi(x = 0, y, z) = V_1(y, z)\ \psi(x = L_1, y, z) = V_2(y, z)$$

$$\psi(x, y = 0, z) = V_3(x, z)\ \psi(x, y = L_2, z) = V_4(x, z)$$

$$\psi(x, y, z = 0) = V_5(x, y)\ \psi(x, y, z = L_3) = V_6(x, y)$$

Trick (a) solve for ψ inside when $\Phi \equiv 0$ on 5 surfaces, and $\Phi =$the given function on the 6th surface. (b) solve 6 such problems (c) ψ is sum of the 6 Φ.

Consider $\Phi(x, y, z = L_3) = V_6(x, y)$ and $\Phi = 0$ on the other 5 surfaces.

Try

$$\Phi(x, y, z) = X(x)Y(y)Z(z)$$

then $\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$, substitute and divide by XYZ,

$$\frac{\frac{d^2X}{dx^2}}{X} + \frac{\frac{d^2X}{dx^2}}{y} + \frac{\frac{d^2X}{dx^2}}{X} = 0$$

$$\therefore \frac{d^2X}{dx^2} = \alpha X \; \frac{d^2Y}{dy^2} = \beta Y \; \frac{d^2Z}{dz^2} = \gamma Z$$

α, β, γ are separation constants, and $\gamma = -\alpha - \beta$.

Periodic boundary conditions for X, Y, give

$$\alpha = -k_a^2, \; \beta = -k_b^2$$

k_a, k_b are positive, and X, Y are linear combinations of $\sin(k_a x)$, $\cos(k_a x)$, or $\sin(k_b y)$, $\cos(k_b y)$.

$X(x = 0) = 0 \implies$ no cosine term; $X(x = L_1) = 0 \implies k_a L_1 = n_1\pi$, $n_1 = 1, 2, ...$

So

$$\gamma = k_a^2 + k_b^2 = \left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2$$

$\frac{d^2Z}{dz^2} = \gamma Z$ ($\gamma > 0$) gives Z is linear combination of $\sinh\sqrt{\gamma}z$, $\cosh\sqrt{\gamma}z$.

$Z(z = 0) = 0 \implies$ no cosh term.

Therefore

$$\Phi(x, y, z) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} A_{n_1n_2} \sin\frac{n_1\pi x}{L_1} \sin\frac{n_2\pi y}{L_2} \sinh\sqrt{\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2} z$$

find $A_{n_1n_2}$ by

$$A_{n_1n_2} = \frac{\int_0^{L_1} dx \int_0^{L_2} dy V_6(x, y) \sin\frac{n_1\pi x}{L_1} \sin\frac{n_2\pi y}{L_2}}{\left(\sinh\sqrt{\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2} z\right)\left(\frac{L_1}{2}\right)\left(\frac{L_2}{2}\right)}$$

that is because $\int_0^L dx \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} = \delta_{mn}\frac{L}{2}$, i.e. $\{\sin\frac{n_1\pi x}{L_1} \sin\frac{n_2\pi y}{L_2}\}_{n_{1,2}=1,2,...}$complete set.

Remark

(1) If all 6 surfaces $V = 0$, then $\psi = 0$ and this solution is the only solution, because Dirichlet solution is unique. This has an analogy in electrostatic shielding. Given a metal cavity, no charge inside, what is the electric field inside? Because V is the same on the shell of the cavity, $\psi = V$ is the unique solution of Laplace equation. So $\vec{E} = -\nabla\psi = 0$ inside.

(2) The method used here gives only ψ inside, the solution does not work outside. Unlike solution obtained from spherical polar coordinate, solution inside of a sphere can be easily modified to be the solution outside of the sphere.

Spherical Polar Coordinate

(a) With azimuthal symmetry (no ϕ dependence)

$$\begin{aligned}\Phi &= \Phi(r,\theta)\\ &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)\end{aligned}$$

$l \in$Integer, because Φ must be regular at $\theta = 0,\ \pi$. If the accessible region is a cone so $\theta = \pi$ is in the inaccessible region, then $l \in$Integer condition is dropped and one gets hypergeometric functions.

Here we have

$$P_l(\cos\theta) = P_l(u)$$

Legendre polynomial.

Property:

1. $\{P_l(\cos\theta)\}$ linear independent for $0 \le \theta \le \pi$ $(-1 \le u \le 1)$. complete set

$$\int_{-1}^{1} du P_l(u) P_{l'}(u) = \frac{2\delta_{ll'}}{2l+1}$$

note P_l has not been normalized.

2. $P_l(1) = 1$

E.g. $P_0(u) = 1$, $P_1(u) = u$, $P_2(u) = \frac{3}{2}u^2 - \frac{1}{2}$.

(b) Spherical polar coordinate in general

$$\begin{aligned}\Phi &= \Phi(r,\theta,\phi) \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{2^{l+1}}\right)Y_{lm}(\theta,\phi)\end{aligned}$$

Similarly l $\in$Integer, because Φ must be regular at $\theta = 0,\ \pi$. $m \in [-l,\ l]$ m is an integer, because Φ must be single valued in ϕ. If the accessible region is a wedge in ϕ, then m needs not be an integer.

Here we have

$$Y_{lm}(\theta,\phi)$$

spherical harmonics.

Property:

$\{Y_{lm}(\theta,\phi)\}$ linear independent, complete set

$$\int d\Omega Y_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

note $Y_{lm}(\theta,\phi)$ is normalized.

E.g. $Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$,

$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$, $Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$.

Cylindrical Coordinate

We only consider cylindrical coordinate no z dependence

$$\begin{aligned}\Phi &= \Phi(r,\phi) \\ &= A + B\ln r + \sum_{N=1}^{\infty}\left(A_N r^N + \frac{B_N}{r^N}\right)\sin N\phi + \sum_{N=1}^{\infty}\left(C_N r^N + \frac{D_N}{r^N}\right)\cos N\phi\end{aligned}$$

here r is the projection of ρ to xy plane. N is integer because Φ is single valued for $0 \le \phi \le 2\pi$

If Φ has z dependence, Φ involves Bessel functions.

Example: Electrostatics

A conducting sphere of radius R is placed in an uniform electric field. So far away from the sphere the field is unaltered $\vec{E} = E_0\hat{z}$, the sphere is kept at potential V_0. Solve for $\nabla^2\Phi = 0$ outside sphere.

$$\Phi = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l\left(\cos\theta\right)$$

we have two boundary conditions

At $r = R$, $\Phi = V_0 = \sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}} \right) P_l\left(\cos\theta\right)$. Since $P_l\left(\cos\theta\right)$ has θ dependence for $l \neq 0$, we should deduce that

$$\begin{cases} V_0 = A_0 + \frac{B_0}{R} \\ A_l R^l + \frac{B_l}{R^{l+1}} = 0 \quad l > 0 \end{cases}$$

At $r = \infty$, $\vec{E} = E_0\hat{z}$ and $\vec{E} = -\nabla\Phi$. So $\Phi(r \to \infty) = -E_0 z + C = -E_0 r\cos\theta + C = -E_0 r P_1\left(\cos\theta\right) + C$, we deduce that

$$\begin{cases} A_0, B_0 \text{ arbitrary} \\ A_1 = -E_0 \\ A_{l>1} = 0 \end{cases}$$

Combining, we get A_0 arbitrary, $B_0 = R\left(V_0 - A_0\right)$, $A_1 = -E_0$, $B_1 = E_0 R^3$, $A_{l>1} = B_{l>1} = 0$, thus

$$\Phi(r,\theta) = A_0 + \frac{R\left(V_0 - A_0\right)}{r} + \left(-E_0 r + \frac{E_0 R^3}{r^2} \right)\cos\theta$$

A_0 makes no contribution to the physical field, let $A_0 = 0$.

$$\Phi(r,\theta) = -E_0 r\cos\theta + \frac{RV_0}{r} + \frac{E_0 R^3}{r^2}\cos\theta$$

The first term on the right is the external applied field at infinity.

Second term is

$$\frac{Q}{r}$$

where $Q \equiv RV_0$, Q total charge on sphere, R here becomes the capacitance of the sphere. This is in CGS unit; whiles in SI unit, capacitance of a sphere of radius R is $4\pi\varepsilon_0 R$.

The third term on the right is

$$\frac{\vec{D} \cdot \hat{r}}{r^2}$$

where $\vec{D} \equiv E_0 R^3 \hat{z}$, because $\cos\theta = \hat{z} \cdot \hat{r}$. $\vec{D}$ is total electric dipole moment.

Example: Magnetostatics

A perfect conducting sphere of radius R is placed in an uniform magnetic field. So far away from the sphere the field is unaltered $\vec{B} = B_0 \hat{z}$. Solve for $\nabla^2 \Phi = 0$ outside sphere.

$$curl\, \vec{B} = 0 \implies \vec{B} = -\nabla\Phi$$

$$div\, \vec{B} = 0 \implies \nabla^2\Phi = 0$$

$$\Phi = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{C_l}{r^{l+1}} \right) P_l\left(\cos\theta\right)$$

we have two boundary conditions

Similarly at $r = \infty$, $\vec{B} = B_0 \hat{z}$, we deduce that

$$\begin{cases} A_0, C_0 \text{ arbitrary} \\ A_1 = -B_0 \\ A_{l>1} = 0 \end{cases}$$

Inside conductor, $\vec{B} = 0$, at $r = R$, $B_{normal} = 0$, and B_{normal} is continuous. Since $\vec{B} = -\nabla\Phi$, this gives

$$-\frac{\partial \Phi(r = R)}{\partial r} = 0$$

Neumann boundary condition. This implies

$$\sum_{l=0}^{\infty}\left(lR^{l-1}A_l - \frac{(l+1)C_l}{R^{l+2}}\right)P_l\left(\cos\theta\right) = 0$$

[note: free surface current does present, so *curl* $\vec{H}$, *curl* $\vec{B}$ not 0, so continuity of Φ is not appropriate.]

Combining, we get A_0 arbitrary, $C_0 = 0$, $A_1 = -B_0$, $C_1 = -B_0R^3/2$, $A_{l>1} = C_{l>1} = 0$, thus

$$\Phi(r,\theta) = A_0 + \left(-B_0 r - \frac{B_0R^3/2}{r^2}\right)\cos\theta$$

A_0 makes no contribution to the physical field, let $A_0 = 0$.

$$\Phi(r,\theta) = -B_0 r\cos\theta - \frac{B_0R^3/2}{r^2}\cos\theta$$

The first term on the right is the external applied field at infinity.

Second term is

$$\frac{\vec{\mu}}{r^2}$$

where $\vec{\mu} \equiv -B_0R^3\hat{z}$, because $\cos\theta = \hat{z}\cdot\hat{r}$. $\vec{\mu}$ total magnetic dipole moment. Compare to result before $\vec{D} = +E_0R^3\hat{z}$, we see prefect conductor acts like diamagntic, which produces a magnetic field opposite to external applied magnetic field.

Review Multipole Expansion

Consider a localized source (charge, or current) confined in a sphere of radius a. At a field point, $a \ll r$, we make expansion in a/r.

Electrostatics

$$\Phi = \frac{Q}{r} + \frac{\vec{D}\cdot\hat{r}}{r^2} + \frac{\text{el quadrupole mom}}{r^3} + \frac{\text{octupole}}{r^4} + ...$$

$$\vec{E} = -\nabla\Phi = \frac{Q}{r^2} + \frac{\vec{D}\cdot\hat{r}}{r^3} + \frac{\text{el quadrupole mom}}{r^4} + ...$$

Magnetostatics

$$\Phi = \frac{0}{r} + \frac{\vec{\mu} \cdot \hat{r}}{r^2} + \frac{\text{mag quadrupole mom}}{r^3} + \frac{\text{octupole}}{r^4} + ...$$

no magnetic monopole

$$\vec{E} = -\nabla\Phi = \frac{\vec{\mu} \cdot \hat{r}}{r^3} + \frac{\text{mag quadrupole mom}}{r^4} + ...$$

Let us compare the force exert on them.

Suppose we have applied electric/magnetic fields $\vec{E}$ and $\vec{B}$. Then

$$\vec{F}_{\text{on } Q} = Q\vec{E} \sim E$$

$$\vec{F}_{\text{on } \vec{D}} = \left(\vec{D} \cdot \nabla\right) \vec{E} \sim \frac{\partial E}{\partial \vec{x}}$$

$$\vec{F}_{\text{on } \vec{\mu}} = \left(\vec{\mu} \cdot \nabla\right) \vec{E} \sim \frac{\partial B}{\partial \vec{x}}$$

This is the based of Stern-Gerlach experiment in Quantum Mechanics

$$\vec{F}_{\text{on quadrupole}} \sim \frac{\partial^2 E}{\partial \vec{x}^2}, \sim \frac{\partial^2 B}{\partial \vec{x}^2}$$

Example: Steady-State Current Flow

A large piece of conducting material has a small sphere of hole of radius R. There is an uniform current flow in the material. So far away from the hole $\vec{j} = j_0\hat{z}$. Solve for $\nabla^2\Phi = 0$ outside hole.

$$curl\,\vec{E} = 0 \implies curl\,\frac{\vec{j}}{\sigma_c} = 0 \implies curl\,\vec{j} = 0 \implies \vec{j} = -\nabla\Phi$$

$$div\,\vec{j} = 0 \implies \nabla^2\Phi = 0$$

$$\Phi = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{C_l}{r^{l+1}}\right) P_l\left(\cos\theta\right)$$

we have two boundary conditions

Similarly at $r = \infty$, $\vec{j} = j_0\hat{z}$, Inside hole, $\vec{j} = 0$, at $r = R$, $j_{normal} = 0$, and j_{normal} is continuous ($\because div\,\vec{j} = 0$), this gives

$$-\frac{\partial \Phi(r = R)}{\partial r} = 0$$

Neumann boundary condition. So the result is exactly the same

$$\Phi(r, \theta) = -j_0 r \cos\theta - \frac{j_0 R^3/2}{r^2} \cos\theta$$

This problem is the same in fluid dynamics.

B.5. Five Famous Problems

1)

A sphere filled with ϵ_2 in a uniform medium ϵ_1, a $\vec{E} = E_0 \hat{z}$ is applied to the space. Find $\vec{E}$ everywhere.

Do the same problem, change sphere to a infinite long cylinder.

2)

A sphere filled with μ_2 in a uniform medium μ_1, a $\vec{B} = B_0 \hat{z}$ is applied to the space. Find $\vec{B}$ everywhere.

Do the same problem, change sphere to a infinite long cylinder.

3)

A sphere filled with σ_2 in a uniform medium σ_1, a $\vec{j} = j_0 \hat{z}$ is applied to the space. Find $\vec{j}$ everywhere.

Do the same problem, change sphere to a infinite long cylinder.

4)

A sphere inside has polarization $\vec{P} = P_0 \hat{z}$, no $\vec{E}$ is applied outside. Find $\vec{E}$ everywhere.

Do the same problem, change sphere to a infinite long cylinder.

5)

A sphere inside has magnetization $\vec{M} = M_0\hat{z}$, no $\vec{B}$ is applied outside. Find $\vec{B}$ everywhere.

Do the same problem, change sphere to a infinite long cylinder.