General Relativity

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Contents

Introduction

Lecture 1 (1/21/14)

Why do we study general relativity? or why is Newton wrong?

For theoretical reasons: Newton gravity acts instantaneously; Newton gravitational law is not Lorentz invariant.

The best way to think about general relativity is to think the way Einstein thinks. What he called the happiest thought in his life: equivalence principle. The mental process starts from a hidden fact in Newton mechanics. Free fall objects are moving at same rate, or

$$F = m_I a = \frac{GMm_g}{r^2}$$

m_I inertial mass, m_g gravitational mass. If they are proportional, canceling results a is the same regardless how heavy or light the object is. Another fact in Newton mechanics, in non-inertial frame there is an inertial force $-am_I$ should be added to objects in order to make use of Newton mechanics.

Then Einstein concluded (1907):

$$\text{gravitational field} = \text{accelerating frame}$$

At the time he already discovered special relativity, which studies how time and space get transformed. General relativity is an extension to special relativity. It studies how the time-space transformation get distorted (1911), i.e. become curved, due to the gravity. In Newton mechanics

$$\nabla^2\phi = 4\pi G\rho$$

ϕ gravitational potential. ρ mass distribution. In GR, (1915)

$$\text{Operator}\begin{pmatrix}\text{Gravitationalf field}\\ \text{metric}\end{pmatrix} = \begin{matrix}\text{Source}\\ \text{Stress Tensor/energy mom tensor}\end{matrix} \tag{0.1}$$

We will see that not only rest masses cause gravity, but also pressure gravitates. In fact pressure was a dominant factor in early universe.

The second part of GR studies the motion of a test mass. E.g. planetary motion, orbit around black hole. In Newton mechanics

$$\ddot{x} = -\nabla\phi$$

In GR it is geodesic motion. Since gravity is equivalent to some accelerating frame, then it is possible to find some frame so that gravity is canceled out. Hence no force, so called local inertial frame, or freely falling frame. Since gravitational field is not uniform, the freely falling frame is not global. This requires the technique of differential geometry, more specifically tensor analysis in 4D space time (1912).

1 Tensors Analysis

The goal is to write relations between quantities in invariant form so that it's the same in any (accelerating) coordinate system.

Example of tensors we have already seen.

1. 3D vector $d\vec{r}$ has length

$$ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij} dq^i dq^j \tag{1.1}$$

in any curved coordinates q_i. In general we will see that if all indices are contracted, the resulting object is always invariant. Different coordinate systems have different g_{ij}, metric tensor, however the form of expression ds^2 is the same for all systems. For 3D flat space, Cartesian

$$g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

For spherical coordinate

$$g_{ij} = \begin{pmatrix} 1 & & \\ & (q^1)^2 & \\ & & (q^1)^2\sin^2(q^1) \end{pmatrix}$$

note that it is possible to find some straight coordinate system for which flat space has g_{ij} contain off diagonal elements. We will develop some technique to test whether a space is flat or curved when we study manifolds.

Lecture 2 (1/27/14)

2. elastic medium stress over $d\vec{S} = \hat{n}dS$, i.e. the force in i direction is given by

$$dF_i = \sum_{j=1}^{3} f_{ij} n_j dS$$

where f_{ij} is stress tensor. Components of dF_i, f_{ij}, n_j depend on coordinate system, but the form remains the same.

Deformation of elastic medium $\vec{\xi}$, small displacement is described by strain tensor

$$s_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right)$$

Some material obey Hooke's law: stress f_{ij} depend linearly on strain s_{ij}

$$f_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl} s_{kl}$$

This form also remains the same in all coordinates.

3. I_{ij} Inertial of rigid body. Angular momentum

$$\vec{J} = \int \rho[\vec{r} \times (\vec{w} \times \vec{r})]dV = \sum_{i,j=1}^{3} \vec{e}_i I_{ij} w_j$$

or$J_i = \sum_{j=1}^{3} I_{ij} w_j$ Kinetic energy $K = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij} w_i w_j$

1.1 3D Affine Space

Pick a basis $\vec{e}_i$,

$$\vec{r} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = \sum_{i=1}^{3} x^i \vec{e}_i \equiv x^i \vec{e}_i$$

employing Einstein notation.

Pick a new basis $\vec{e}_{i'} = A^i_{i'}\vec{e}_i$. This contains 3 equations for basis change matrix A. This defines $A^i_{i'}$.

For now

$$A^i_{i'} = A_{i'}{}^i = A^i{}_{i'}$$

no distinguish.

$$\vec{r} = x^{i'}\vec{e}_{i'} = x^{i'}A^i_{i'}\vec{e}_i = x^i\vec{e}_i = x^i A^{i'}_i \vec{e}_{i'}$$

so $x^i = x^{i'}A^i_{i'}$. From $x^{i'}\vec{e}_{i'} = x^i A^{i'}_i \vec{e}_{i'}$, so

$$x^{i'} = A^{i'}_i x^i \tag{1.2}$$

Hence $A^{i'}_i$ is the inverse of $A^i_{i'}$, And$\vec{e}_i = A^{i'}_i \vec{e}_{i'}$.

Einstein notation for (1.1) is $ds^2 = g_{ij}dq^i dq^j$ or

$$ds^2 = dq^i dq_i$$

this is reminiscent of dot product, but we have to remind ourselves that $dq_i = g_{ij}dq^j$ hence dq^i and dq_i are two different things. One is called (contra variant) vector; the other is called co vector or 1 form.

More generally we call x^i vector, which has 3 components and is satisfied the linear transformation in (1.2). A co vector (dual vector, or 1-form) is a set of 3 number and the components satisfy

$$\psi_{i'} = A^i_{i'}\psi_i \tag{1.3}$$

Compare (1.2) and (1.3), we say that $A^{i'}_i$ is the inverse of $A^i_{i'}$.

Similarly we can define tensor of higher rank

$$\begin{aligned} a^{i'j'} &= A^{i'}_i A^j_j a^i a^j \text{ rank } \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ a_{i'j'} &= A^i_{i'} A^j_{j'} a_{ij} \text{ rank } \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ a^{i'}_{j'} &= A^{i'}_i A^j_{j'} a^i_j \text{ rank } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

These definitions look simple but they are enough to give rigorous study of tensor analysis and are more useful to us.

The more mathematical way of defining tensors as follows

Start from a vector space. $A \in V$ define 1-form$\psi : V \to \mathbb{R}$ requiring

$$\psi(\vec{a} + \vec{b}) = \psi(\vec{a}) + \psi(\vec{b}) \qquad \psi(\alpha\vec{a}) = \alpha\psi(\vec{a}) \tag{1.4}$$

One then shows all linear functions on V forms a linear vector space V^*, called the dual space, which has the same dimension as V.

For any$\psi \in V^*$let $\psi_i = \psi(\vec{e}_i)$ where $\vec{e}_i$ are basis of V. Then for any $\vec{a} = a^i\vec{e}_i$, by linearity

$$\psi(\vec{a}) = a^i\psi(\vec{e}_i) = a^i\psi_i \tag{1.5}$$

hence ψ_i is basis of V^*.

If we change to a new basis $\vec{e}_{i'}$,

$$\psi_{i'} = \psi(\vec{e}_{i'}) = \psi(A^{i'}_i\vec{e}_i) = A^{i'}_i\psi_i$$

This agrees the physical definition (1.3) of 1 form. Hence there is a 1-1 correspondence between 1 form and linear functions (1.4). Conversely one can define vector from 1 from, because by the same formula (1.5), any vector is a mapping from V^* to $\mathbb{R}$.

By the same token

$$a_{ij} : V \times V \to \mathbb{R} \quad a^{ij} : V^* \times V^* \to \mathbb{R} \quad a^i_j : V \times V^* \to \mathbb{R}$$

And tensor of rank (p, q), $F^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}$ can be associated with a linear function $F(\vec{a}_1, \ldots, \vec{a}_q, \varphi_1, \ldots, \varphi_p) \in \mathbb{R}$

$$F : \underbrace{V \times \ldots \times V}_{q \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{p \text{ times}} \to \mathbb{R}$$

1.2 Tensor Algebra

Lecture 3 (1/29/14)

Algebra has 2 operations: sum, product, and distributive law of the two.

Sum

$$c^i = a^i + b^i \tag{1.6}$$

then the resulting sum must compatible with (1.2), i.e. change basis $c^{i'} = a^{i'} + b^{i'}$
Also check

$$c_{ij} = a_{ij} + b_{ij} \implies c_{i'j'} = a_{i'j'} + b_{i'j'}$$

Multiplication

$$c_{ij} = \alpha a_{ij} \implies c_{i'j'} = \alpha a_{i'j'} \tag{1.7}$$

for $\alpha \in \mathbb{R}$

Tensor Product

$$c^{ij} = a^i b^j \text{ (symbol } c = a \otimes b) \implies c^{i'j'} = a^{i'} b^{j'}$$

More exotic

$$a^{j_1 \ldots j_k}_{i_1 \ldots i_m} b^{l_1 \ldots l_p}_{n_1 \ldots n_q} = c^{j_1 \ldots j_k l_1 \ldots l_p}_{i_1 \ldots i_m n_1 \ldots n_q}$$

Contraction

$$a^i_i \equiv a^1_1 + a^2_2 + a^3_3$$

is contraction of a^i_j. And a^{ijk}_{jl}, which has 27 components, is contraction of a^{ijk}_{ml}, which has 243 components.

Symmetrization

For $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors

$$a_{ij} \text{ is symmetric iff } a_{ij} = a_{ji}$$

$$a_{ij} \text{ is antisymmetric iff } a_{ij} = -a_{ji}$$

Using rules (1.6) and (1.7) we can make a_{ij} to be

$$a_{(ij)} \quad = \quad a_{[ij]}\frac{a_{ij}+a_{ji}}{2} \quad = \frac{a_{ij}-a_{ji}}{2}$$

to symmetric or antisymmetric.

Levi-Civita Symbol

For 3D

$$\tilde{\epsilon}_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ are even permutation of } 123 \\ -1 & \text{if } ijk \text{ are odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

it is antisymmetric. It is easy to show that the Levi-Civita symbol is a pseudo-tensor, i.e. it does not satisfy the tensor transformation law when reflection is involved. One can eliminate reflection by imposing handedness, e.g. right hand rule.

Metric Tensor

Tensor of rank $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, g_{ij}, which is symmetric, not degenerated, i.e. $g = \det g_{ij} \neq 0$.

$$\begin{aligned} g_{ij} : V \times V \quad &\to \quad \mathbb{R} \\ a^i, b^j \quad &\mapsto \quad g_{ij}a^i b^j \text{ (denote } \vec{a}\cdot\vec{b}) \end{aligned}$$

$$g_{ij} = g(\vec{e}_i, \vec{e}_j) = \text{dot product, } g \text{ is the form}$$

g_{ij} is specified by how it behaves on the basis. So

$$g(\vec{a}, \vec{b}) = g(a^i\vec{e}_i, b^j\vec{e}_j) = a^i b^j g_{ij}$$

In orthonormal basis $g_{ij} = \delta_{ij}$. Define Euclidean space is a vector space with a metric tensor g_{ij}, that can be made to δ_{ij} in a special class of orthonormal basis.

Length

$$a=\sqrt{g_{ij}a^i a^i}=\sqrt{g(\vec{a},\vec{a})}$$

Angle

$$\frac{\vec{a}\cdot\vec{b}}{ab}=\cos\theta$$

One can show any bilinear form g_{ij} can be made into diagonal form

$$g_{ij}=\begin{pmatrix}\pm1 & & \\ & \pm1 & \\ & & \pm1\end{pmatrix}$$

by choosing some basis $\{\vec{e}_i\}$. Such basis is called canonical basis, or orthonormal basis, because g_{ij} is symmetric. The number of "1" and "-1" are called the signature, which are invariant under basis transformation. E.g

$$g_{ij}=\begin{pmatrix}-1 & & \\ & 1 & \\ & & 1\end{pmatrix}$$

is a pseudo-Euclidean space.

Lowering Index

$$g_{ij} \text{ covariant metric}$$

because it defines inner product, i.e.

$$\begin{aligned} g_{ij}: V\times V &\rightarrow \mathbb{R} \\ a^i, b^j &\mapsto g(\vec{a},\vec{b}) \end{aligned}$$

$$g_{ij}a^i b^j=a_j b^j$$

The operation

$$\begin{aligned} g_{ij} : V &\to V^* \\ a^i &\mapsto a_j \end{aligned} \tag{1.9}$$

is another way to think about g_{ij}.

Raising Index

$$\begin{aligned} g^{ij} : V^* &\to V \\ a_i &\mapsto a^j \end{aligned}$$

From there one can show

$$g^{ij} \text{ contravariant metric is the inverse of } g_{ij}$$

so g^{ij} defines inner product in $V^* \times V^* \to \mathbb{R}$. That is because by (1.9), if $\vec{a}^*, \vec{b}^* \in V^*$ are the dual vectors of $\vec{a}, \vec{b} \in V$, we define inner product $\vec{a}^* \cdot \vec{b}^* \equiv \vec{a} \cdot \vec{b}$. In components

$$\vec{a}^* \cdot \vec{b}^* = g_{ij}(g^{ik}a_k)(g^{jm}b_m) = \delta^k_j a_k g^{jm} b_m = g^{km} a_k b_m$$

1.3 Invariant Tensors

There are only 3 Lorentz invariant tensors: $\delta_{ij}, \quad \epsilon_{ijk}, \quad g_{ij}$

Metric

We know

$$g_{i'j'} = A^i_{i'} A^j_{j'} g_{ij}$$

or in terms matrix notation$g' = AgA^T$ where A is the basis transformation $\vec{e}_{i'} = A\vec{e}_i$.

Suppose we enforce that

$$g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = g_{i'j'} \tag{1.10}$$

in any base. That will require A to satisfy $AA^T = I$ or $A \in O(3)$

If (1.10) is indeed the case, i.e. Euclidean space in orthonormal basis, then $a^i = a_i$ same thing because $a_i = g_{ij}a^j = \delta_{ij}a^j = a^i$also $a^{ij} = a_{ij}$

In pseudo Euclidean space

$$g_{ij} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$a_i = g_{ij}a^j = (-a^1, a^2, a^3) \neq a^j$$

Vector Derivative

Let $a^i(\vec{r})$ be a vector field. Now change basis $\vec{e}_{i'} = A^i_{i'}\vec{e}_i$ then $a^{i'}(x^{1'}, x^{2'}, x^{3'}) = A^{i'}_i a^i(x^1, x^2, x^3)$

Define differentiation of vector field

$$a^i{}_{,k} = \frac{\partial a^i}{\partial x^k} \tag{1.11}$$

or write

$$a^i{}_{,k} = \partial_k a^i \tag{1.12}$$

Is it a tensor?

$$\begin{aligned} \frac{\partial a^{i'}}{\partial x^{k'}} &= \frac{\partial}{\partial x^{k'}}(A^{i'}_i a^i) = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial}{\partial x^k}(A^{i'}_i a^i) \\ &= \frac{\partial (A^k_{l'}x^{l'})}{\partial x^{k'}}\frac{\partial}{\partial x^k}(A^{i'}_i a^i) = A^k_{k'}A^{i'}_i\frac{\partial a^i}{\partial x^k} \end{aligned} \tag{1.13}$$

Here we pull out $A^{i'}_i$ by assuming it is a constant, hence we consider global transformation independent of x^k. This is not correct when we do local transformation.

So (1.13) shows $\frac{\partial a^i}{\partial x^k}$ is a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor, so derivative $\partial_k a^i$ makes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ tensor into a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor. Similarly derivative $\partial_k a^{ij}$ makes $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor into a $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ tensor.

In general, in a global basis $\vec{e}_i$, the derivative of tensor field a of rank (p, q)

$$b^{i_1,\ldots,i_p}_{j_1,\ldots,j_q,k} = \frac{\partial}{\partial x^k} a^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}$$

is a tensor of rank $(p, q+1)$. The added index k to b indicates the rank has changed. Later when we do covariant derivatives, we will use this notation again.

1.4 Special Relativity

Lecture 4 (2/3/14)

We now study 4D Minkowski space, clock + 3D coordinate system. Experiment shows that there is a special class of frames called inertial which move at a constant velocity relative to each other. There frames satisfy following two postulates:

Postulates of Special Relativity

1st postulate of special relativity: space of event $(t, \vec{x})$ is linear. In other words, the choice of an inertial frame is equivalent to the choice of a basis $\vec{e}_\alpha$ in the linear 4D vector space.

That is we can use vector addition and multiplication by a number, e.g. event 1 $(t_1, \vec{x}_1)$; event 2 $(t_2, \vec{x}_2)$ then we can add them to get event 3 $(t_1 + t_2, \vec{x}_1 + \vec{x}_2)$. If we choose a different coordinate system. event 1 becomes $(t'_1, \vec{x}_1)$; event 2 becomes $(t'_2, \vec{x}'_2)$ and event 3 is $(t'_1 + t'_2, \vec{x}'_1 + \vec{x}'_2)$.

2nd postulate: speed of light is constant in all inertial frame (which exist by experiments)

$$\frac{|x_1 - x_2|}{t_1 - t_2} = c = 3 \times 10^{10} cm/s$$

or put $(t_1, \vec{x}_1) = 0$

$$\frac{|\vec{x}|}{|t|} = c$$

or $-c^2t^2 + x_ix^i = 0$ suggesting to define 4 vector

$$x^\alpha = (ct, x^1, x^2, x^3)$$

For photon $s^2 = -(x^0)^2 + x_ix^i = 0$, write formally

$$s^2 = \eta_{\alpha\beta}x^\alpha x^\beta \text{ where } \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

this looks like a metric in an orthonormal basis.

How does vector transform under changing of base? We use different letters $\eta_{\alpha\beta}$ for g_{ij}, and $\Lambda^{\alpha'}_{\alpha}$ for $A^{i'}_{i}$. That is $a^{\alpha'} = \Lambda^{\alpha'}_{\alpha}a^\alpha$. We impose $\eta_{\alpha\beta}$ is invariant

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta_{\alpha'\beta'} = \vec{e}_{\alpha'} \cdot \vec{e}_{\beta'} \tag{1.14}$$

i.e.

$$\Lambda^T\eta\Lambda = \eta$$

set of all such transformation forms a group, called Lorentz group, denoted $O(1,3)$.

First show that any Lorentz transform is equivalent to 3D rotation (doesn't change x^0) and boost (can choose boost along x_1 direction, then only x_0, x_1 components change values). A boost is described by

$$\begin{aligned} \vec{e}_{0'} &= \Lambda^0_{0'}\vec{e}_0 + \Lambda^1_{0'}\vec{e}_1 \\ \vec{e}_{1'} &= \Lambda^0_{1'}\vec{e}_0 + \Lambda^1_{1'}\vec{e}_1 \end{aligned}$$

There are four unknowns, $\Lambda^{\alpha}_{\alpha'}$ and there are three equations

$$\begin{aligned}
\eta_{0'0'} &= \eta_{00} = \vec{e}_0 \cdot \vec{e}_0 = \vec{e}_{0'} \cdot \vec{e}_{0'} = -1 & (1.15) \\
\eta_{0'1'} &= \eta_{1'0'} = \vec{e}_0 \cdot \vec{e}_1 = \vec{e}_1 \cdot \vec{e}_0 = \vec{e}_{0'} \cdot \vec{e}_{1'} = \vec{e}_{1'} \cdot \vec{e}_{0'} = 0 & (1.16) \\
\eta_{1'1'} &= \eta_{11} = \vec{e}_1 \cdot \vec{e}_1 = \vec{e}_{1'} \cdot \vec{e}_{1'} = 1 & (1.17)
\end{aligned}$$

Thus 3 variables are determined and 1 free parameter, donated β.

By (1.16)

$$-\Lambda^0_{0'}\Lambda^0_{1'} + \Lambda^1_{0'}\Lambda^1_{1'} = 0 \implies \frac{\Lambda^1_{0'}}{\Lambda^0_{0'}} = \frac{\Lambda^0_{1'}}{\Lambda^1_{1'}} = \beta$$

i.e.

$$\Lambda^{\alpha}_{\alpha'} = \begin{pmatrix} a & b\beta & & \\ a\beta & b & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ for some } a, b$$

By (1.15) and (1.17)

$$\begin{aligned}
-a^2 + \beta^2 a^2 &= -1 \\
-\beta^2 b^2 + b^2 &= 1
\end{aligned}$$

because $\det(\Lambda) = 1$

$$a = b = \frac{1}{\sqrt{1-\beta^2}} := \gamma$$

thus

$$\Lambda^{\alpha'}_{\alpha} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

The inverse is

$$\Lambda^{\alpha}_{\alpha'} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

by changing $\beta \to -\beta$.

What is the physical meaning of β?

Suppose a particle moving at $\vec{v}$.

$$(ct', x') \text{ rest frame} \quad (ct, x) \text{ lab frame}$$

Assume at $t = t' = 0$, $x^\alpha = x^{\alpha'}$. After that, as time goes on, the particle still sitting at the origin in the rest frame, $x' = 0$

$$\begin{aligned} x^{\alpha'} &= \Lambda^{\alpha}_{\alpha'} x^\alpha \\ \begin{pmatrix} ct' \\ x' \end{pmatrix} &= \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \end{aligned} \tag{1.18}$$

so

$$x' = -\beta\gamma ct + \gamma x = 0 \implies \beta = \frac{1}{c}\frac{x}{t} = \frac{v}{c}$$

speed of particle moving wrt lab $= v$. So$\beta \le 1$, $\gamma \ge 1$.

What is the physical meaning of γ? Top raw of (1.18) gives

$$ct' = \gamma ct - \beta\gamma x = \gamma ct - \beta^2\gamma ct = ct\gamma(1-\beta^2) = ct\frac{1}{\gamma}$$

so

$$t = \gamma t' \text{ or } t' = \frac{t}{\gamma} \tag{1.19}$$

showing that clock runs slow in rest frame, i.e. γ gives the time dilation, called Lorentz factor.

Define proper time

$$d\tau^2 = -\frac{ds^2}{c^2} = -\frac{1}{c^2}(-c^2dt^2 + dx^2) = dt^2(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2) = dt^2(1-\beta^2) = \frac{dt^2}{\gamma^2}$$

or $cd\tau = \sqrt{-ds^2}$, so by construction proper time is invariant.

Consider a trajectory of a particle, world line $x^\alpha(\tau) : \mathbb{R} \to \mathbb{R}^4$ 4-velocity, tangent vector to the worldline, a tensor of rank $(1,0)$. $U^\alpha = \frac{dx^\alpha}{d\tau}$ then

$$U^0 = c\frac{dt}{d\tau} = c\gamma \qquad U^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma v^i$$

$$U^\alpha = (c\gamma, \gamma\vec{v}) \tag{1.20}$$

so

$$U_\alpha U^\alpha = -\gamma^2 c^2 + \gamma^2 v^2 = \gamma^2 c^2(-1+\beta^2) = -c^2 \tag{1.21}$$

is invariant. If an observer is moving with the particle (co-moving frame), $\vec{v} = 0$, so

$$U^\alpha = (c, 0, 0, 0)$$

This doesn't say that every particle is moving at speed of light. It will become clear when we define 4-momentum and the invariant quantity mc^2 turns out to be the rest energy.

4-acceleration defined on the worldline

$$a^\alpha = \frac{dU^\alpha}{d\tau} = \frac{d^2x^\alpha}{d\tau^2}$$

If we take $U_\alpha a^\alpha = \eta_{\alpha\beta}U^\beta a^\alpha$, this is equal to

$$U_\alpha \frac{dU^\alpha}{d\tau} = \frac{1}{2}\frac{d}{d\tau}(U_\alpha U^\alpha) = 0$$

Same thing in 3D unit vector: the acceleration of unit vector can only be $\perp$ to the unit vector.

Twin Paradox

Consider two twins. One stays at the origin of the lab frame, never moves. The other starts from the origin and goes on a trip, and after time T, comes back to meet the non-moving guy. Hence the world line for non-moving guy is just a straight line along the t axis from the origin to T. The other guy's world line is a curve with some slope, i.e. speeding, going out and then returns to T on the t axis.

It is clear that if we slice the time axis dt and the corresponding $d\tau$. dt is the time elapsed occurred to the non-moving guy and $d\tau$ is the time elapsed to the moving guy. By (1.19)$d\tau = \frac{dt}{\gamma}$, so from $t = 0$ to $t = T$

$$\int_0^T dt > \int d\tau$$

This enplanes twin paradox.

2 Dynamics of Particles & Fields in Flat Spacetime

2.1 Action for Free Particles

Hence in world diagram, the longest proper time is the straight vertical line, which has 0 momentum. This leads to guess the action ($\alpha > 0$)

$$S = -\alpha \int_1^2 d\tau \tag{2.1}$$

because we saw that motion of particle has longest proper time hence minimum action.

If $\beta \ll 1$

$$S \;\; = \;\; -\alpha \int \sqrt{1-\beta^2}dt == -\alpha \int (1-\beta^2/2)dt = \text{const} + \alpha \int_{t_1}^{t_2} \frac{\beta^2}{2}dt$$

We want this to equal$S = \int_{t_1}^{t_2} \frac{mv^2}{2}dt$, so the constant α had better to be mc^2. We now have the action for freely moving particle

$$S = -mc^2 \int d\tau = -mc^2 \int \sqrt{1-\beta^2}dt \tag{2.2}$$

This leads to define relativistic Lagrangian

$$\mathcal{L} = -mc^2\sqrt{1-\beta^2}$$

Using Euler-Lagrangian equation

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x^i} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v^i} \\ 0 &= \frac{d}{dt}p_i \end{aligned}$$

so $\vec{p} = \text{const}$. What is relativistic momentum?

$$p_i = \frac{\partial \mathcal{L}}{\partial v^i} = -mc^2\sqrt{1-\frac{v_i v^i}{c^2}} = -\frac{mc^2}{2\sqrt{1-\beta^2}}\left(-2\frac{v_i}{c^2}\right) = mv_i\gamma = mc\gamma\beta \tag{2.3}$$

From here, we find Hamiltonian

$$\begin{aligned} E = H &= \vec{p}\cdot\vec{v} - \mathcal{L} \qquad (2.4) \\ &= mc^2\gamma\beta^2 + mc^2/\gamma \\ &= \gamma mc^2\left(\beta^2 + 1/\gamma^2\right) = \gamma mc^2 \end{aligned}$$

Hamiltonian-Jacob Theory

We will show energy, momentum are derivatives of Hamiltonian-Jacobian. Suppose we vary the path

$$x^i(t) \to x^i(t) + \delta x^i(t) \quad \dot{x}^i(t) \to \dot{x}^i(t) + \delta\dot{x}^i(t)$$

the change in action is

$$\begin{aligned} \delta S &= \int \delta\mathcal{L}dt = \int\left(\frac{\partial \mathcal{L}}{\partial x^i}\delta x^i + \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\delta\dot{x}^i\right)dt \\ &= \int\left(\frac{\partial \mathcal{L}}{\partial x^i}\delta x^i + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\delta x^i\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right)\delta\dot{x}^i\right)dt \end{aligned}$$

since $\delta\dot{x}^i = d(\delta x)/dt$. Thus

$$\delta S = [p_i\delta x^i]_1^2 + \int\left(\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right)\right)\delta x^i dt \tag{2.5}$$

For true trajectories $\delta x^i(0) = \delta x^i(1) = 0$, use minimum action principle we have

$$\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) = 0 \tag{2.6}$$

However, if we fix end point 1, and allow end point 2 to vary $\delta x^i(1) \neq 0$, and stays on the trajectories for which the integral in (2.5) vanishes, then $\delta S = p_i\, \delta x^i|_2$, thus we get the momentum

$$p_i = \frac{\partial S}{\partial x^i} \tag{2.7}$$

along true trajectories but the destination is moving. Thus by(2.4)

$$\mathcal{L} = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial S}{\partial t} + p_i v^i \implies E = -\frac{\partial S}{\partial t} \tag{2.8}$$

because energy in Lagrangian mechanics is defined as an integral of motion when $\mathcal{L}$ doesn't have explicit t dependence.

Indeed

$$\frac{d\mathcal{L}}{dt} = \underbrace{\frac{\partial \mathcal{L}}{\partial t}}_{0} + \frac{\partial \mathcal{L}}{\partial x^i}\dot{x}^i + \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\ddot{x}^i = \underbrace{\frac{\partial \mathcal{L}}{\partial x^i}\dot{x}^i - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right)\dot{x}^i}_{0\ (2.6)} + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\dot{x}^i\right)$$

so

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\dot{x}^i - \mathcal{L}\right) = 0$$

or

$$E = p_i v^i - \mathcal{L} = \text{const} \tag{2.9}$$

So combining (2.7), (2.8), we define 4 momentum

$$p^\alpha = \frac{\partial S}{\partial x^\alpha} = (\frac{E}{c}, \vec{p})$$

which is also by (1.20), $p^\alpha = U^\alpha m$, so m is in all above the rest mass.

Lecture 5 (2/5/14)

Now take the free particle action (2.2),

$$S = -mc\int \sqrt{-ds^2} = -mc\int \sqrt{-dx^\alpha dx_\alpha}$$

and vary the path

$$x^i(t) \to x^i(t) + \delta x^i(t) \quad dx^i(t) \to dx^i(t) + d\delta x^i(t)$$

so

$$\begin{aligned}
\delta S &= -mc\int_1^2 \delta\sqrt{-dx^\alpha dx_\alpha} = mc\int_1^2 \frac{dx_\alpha d(\delta x^\alpha) + dx^\alpha d(\delta x_\alpha)}{2cd\tau} \\
&= m\int_1^2 \frac{dx^\alpha d(\delta x_\alpha)}{d\tau} = m\int_1^2 U^\alpha d(\delta x_\alpha) \\
&= m[U^\alpha \delta x_\alpha]_1^2 - m\int_1^2 \frac{dU^\alpha}{d\tau}\delta x_\alpha d\tau
\end{aligned} \tag{2.10}$$

Hence true trajectory of free particle

$$\frac{dU^\alpha}{d\tau} = 0$$

2.2 Curvilinear Coordinates

We now generalize a little bit. For simplicity, we consider curvilinear coordinate system (q^1, q^2, q^3) in 3D Euclidean space E^3. In a sense that the space itself is still flat but the coordinate system is not Cartesian, the basis vectors will depend on positions, e.g. spherical coordinate.

Namely a curvilinear coordinate system (q^1, q^2, q^3) defines at any point P of E^3 three coordinate lines along which one coordinate is changing and the other two are constant.

$$\vec{e}_i(P) = \frac{\partial \vec{x}}{\partial q^i} \; i = 1, 2, 3$$

However we still have well defined components of any tensor field in basis $\vec{e}_i$. For example, a vector field

$$\vec{a}(P) = a^i(P)\vec{e}_i(P)$$

at each P. Here we don't assume $\vec{e}_i(P)$ are orthogonal nor normalized. As we will see this implies g_{ij} will be very different.

Consider a small displacement $d\vec{x}$ from point P

$$d\vec{x} = \frac{\partial \vec{x}}{\partial q^i} dq^i = dq^i \vec{e}_i$$

length of $d\vec{x}$ is given by

$$(d\vec{x})^2 = (dq^i \vec{e}_i)(dq^j \vec{e}_j) = \underbrace{\vec{e}_i \cdot \vec{e}_j}_{\equiv g_{ij}} dq^i dq^j$$

And all other thing will be almost the same

Basis Transformation

$$\vec{e}_{i'}(P) = \frac{\partial \vec{x}}{\partial q^{i'}} = \frac{\partial q^i}{\partial q^{i'}} \frac{\partial \vec{x}}{\partial q^i} = \frac{\partial q^i}{\partial q^{i'}} \vec{e}_i(P)$$

with transformation matrix $A^i_{i'}(P) = \frac{\partial q^i}{\partial q^{i'}}$

The inverse is $A^{i'}_i(P) = \frac{\partial q^{i'}}{\partial q^i}$ and for general tensor

$$F^{i'_1,\ldots,i'_p}_{j'_1,\ldots,j'_q} = \frac{\partial q^{i'_1}}{\partial q^{i_1}} \cdots \frac{\partial q^{i'_p}}{\partial q^{i_p}} \frac{\partial q^{j'_1}}{\partial q^{j_1}} \cdots \frac{\partial q^{j'_q}}{\partial q^{j_q}} F^{i_1,\ldots,i_p}_{j_1,\ldots,j_q} \tag{2.11}$$

each $\frac{\partial q^{i'}}{\partial q^i}$ or $\frac{\partial q^{j'}}{\partial q^j}$ is called a Jacobian.

Matrix Notation

The idea of the transformation as matrix multiplication works well for vector and covector transformation. However for higher rank tensor it is done index by index. We will need this idea later e.g. (2.13), the determinant of matrix is the product of the determinant, i.e. product of Jacobian.

We will always put up script as row index that is because

$$V^i = \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

then lower script has to be column index.

Vector transformation

$$A^p_q = \begin{pmatrix} A^1_1 & A^1_2 & A^1_3 \\ A^2_1 & A^2_2 & A^2_3 \\ A^3_1 & A^3_2 & A^3_3 \end{pmatrix}$$

$$V^{i'} = A^{i'}_i V^i$$

so

$$\begin{pmatrix} V^{1'} \\ V^{2'} \\ V^{3'} \end{pmatrix} = \begin{pmatrix} A^{1'}_1 & A^{1'}_2 & A^{1'}_3 \\ A^{2'}_1 & A^{2'}_2 & A^{2'}_3 \\ A^{3'}_1 & A^{3'}_2 & A^{3'}_3 \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

Covector transformation

$$w_{i'} = w_i A^i_{i'}$$

i.e.

$$\begin{pmatrix} w_{1'} & w_{2'} & w_{3'} \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} A^1_{1'} & A^1_{2'} & A^1_{3'} \\ A^2_{1'} & A^2_{2'} & A^2_{3'} \\ A^3_{1'} & A^3_{2'} & A^3_{3'} \end{pmatrix}$$

Delta tensor

It is always

$$\delta^i_j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{2.12}$$

in all coordinate systems. Indeed if (2.12) is true in q^i, then basis transformation

$$\delta^{i'}_{j'} = A^{i'}_i A^j_{j'} \delta^i_j = A^{i'}_i A^i_{j'} = \delta^i_j$$

Metric Tensor

Similarly from (2.11)

$$g_{i'j'} = \frac{\partial q^i}{\partial q^{i'}} \frac{\partial q^j}{\partial q^{j'}} g_{ij}$$

Define $g = \det g_{ij}$, which is different in different curvilinear coordinates

$$|g_{i'j'}| = \left| \frac{\partial q^i}{\partial q^{i'}} \frac{\partial q^j}{\partial q^{j'}} g_{ij} \right| \implies g' = \left| \frac{\partial q^i}{\partial q^{i'}} \right|^2 g \tag{2.13}$$

One can check explicitly in 2D above says

$$g' \quad = g_{1'1'} g_{2'2'} - g_{1'2'} g_{2'1'} = \quad \left(\frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^2}{\partial x^{2'}} - \frac{\partial x^1}{\partial x^{2'}} \frac{\partial x^2}{\partial x^{1'}} \right)^2 (g_{11} g_{22} - g_{12} g_{21})$$

We say $\sqrt{g}$ has the meaning of geometrical volume of the parallelepiped defined by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ (if $\vec{e}$ are unit vectors), because the invariant volume element of $dx^1 dx^2 dx^3$ is given by

$$dV = \sqrt{g} dx^1 dx^2 dx^3 \tag{2.14}$$

Indeed

$$\sqrt{g'} dx^{1'} dx^{2'} dx^{3'} = \left| \frac{\partial x^i}{\partial x^{i'}} \right| \sqrt{g} \left| \frac{\partial x^{i'}}{\partial x^i} \right| dx^1 dx^2 dx^3 = \sqrt{g} dx^1 dx^2 dx^3 = dV$$

Levi-Civita Tensor

Following from (1.8), imposing some handedness, and putting

$$\epsilon_{ijk} = \sqrt{g} \tilde{\epsilon}_{ijk}$$

one can show that ϵ_{ijk} is a tensor. Proof

Recall for any 3×3 matrix, M, the determinant is

$$\tilde{\epsilon}_{i'j'k'} \, |M| = M^i_{i'} M^j_{j'} M^k_{k'} \tilde{\epsilon}_{ijk} \text{ summing over } ijk$$

so

$$\begin{aligned} \epsilon_{i'j'k'} \quad &= \quad \sqrt{g'} \tilde{\epsilon}_{i'j'k'} = \left| \frac{\partial x^i}{\partial x^{i'}} \right| \sqrt{g} \tilde{\epsilon}_{i'j'k'} \\ &= \quad \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} \sqrt{g} \tilde{\epsilon}_{ijk} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} \epsilon_{ijk} \end{aligned}$$

QED

Similarly define the inverse

$$\epsilon^{ijk} = \frac{1}{\sqrt{g}} \tilde{\epsilon}^{ijk} \tag{2.15}$$

2.3 Covariant Derivative

Derivative of a Scalar Field

Following from derivative (1.11), we define derivative in general. Given the components of a tensor field a in a coordinate system x^i one can define a set of components by differentiating with respect to one of the basis, known as partial derivative and it is denoted as

$$\frac{\partial}{\partial x^i} a^{i_1,\ldots,i_p}_{j_1,\ldots,j_q} = a^{i_1,\ldots,i_p}_{j_1,\ldots,j_q,i}$$

This turns out not to be a tensor field unless when a has rank (0,0), scalar field.

Proof: in any curved coordinate system q^i, define

$$h_i = \frac{\partial f}{\partial q^i}$$

at each point P. Then h_i and $h_{i'}$ for two systems q^i and $q^{i'}$ are related by

$$h_{i'} = \frac{\partial f}{\partial q^{i'}} = \frac{\partial q^i}{\partial q^{i'}} \frac{\partial f}{\partial q^i} = \frac{\partial q^i}{\partial q^{i'}} h_i$$

showing h_i is a covector field.

Covariant Derivative of a Vector Field

Consider a vector field

$$\vec{a}(P) = a^i(P)\vec{e}_i(P)$$

The derivative along a coordinate basis x^i is

$$\frac{\partial \vec{a}}{\partial x^i} = \frac{\partial a^k}{\partial x^i} \vec{e}_k + a^j \frac{\partial \vec{e}_j}{\partial x^i} \tag{2.16}$$

Comparing to (1.11), we find the second part on the right is extra. $\frac{\partial \vec{e}_j}{\partial x^i}$ is a vector field which can be written as

$$\frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^k_{ij} \vec{e}_k \tag{2.17}$$

Γ^k_{ij} is called Christoffel symbol. We can show easily Γ^k_{ij} is symmetric wrt ij

$$\Gamma^k_{ij} \vec{e}_k = \frac{\partial \vec{e}_i}{\partial x^j} = \frac{\partial}{\partial x^k} \frac{\partial \vec{r}}{\partial x^i}$$

Then (2.16) becomes

$$\frac{\partial \vec{a}}{\partial x^i} = \left(\frac{\partial a^k}{\partial x^i} + \Gamma^k_{ij} a^j \right) \vec{e}_k$$

First show Γ^k_{ij} is not a tensor, because otherwise $\Gamma^k_{ij} a^j$ will be tensor. Since $\frac{\partial a^k}{\partial x^i} + \Gamma^k_{ij} a^j$ is a tensor (prove later), that implies $\frac{\partial a^k}{\partial x^i}$ would be a tensor, this contradicts remark after (1.13).

Notation we put

$$\frac{\partial a^k}{\partial x^i} + \Gamma^k_{ij} a^j \equiv \nabla_i a^k \equiv a^k{}_{;i}$$

called covariant derivative, evolved from notation (1.12).

Prove $a^k{}_{;i}$ is a tensor of rank $(1, 1)$

$$\frac{\partial \vec{a}}{\partial x^{i'}} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial \vec{a}}{\partial x^i} = \frac{\partial x^i}{\partial x^{i'}} a^k{}_{;i} \vec{e}_k = \frac{\partial x^i}{\partial x^{i'}} a^k{}_{;i} \frac{\partial x^{k'}}{\partial x^k} \vec{e}_{k'}$$

that is

$$a^{k'}{}_{;i'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{k'}}{\partial x^k} a^k{}_{;i}$$

QED.

In Cartesian coordinates $\Gamma^k_{ij} = 0$, we have

$$a^k{}_{;i} = a^k{}_{,i}$$

thus $a^k{}_{;i}$ is the extension of Cartesian tensor $a^k{}_{,i}$ to all curvilinear coordinate systems, so Γ describes property of curvilinear coordinate.

Covariant Derivative of a Covector Field

First consider

$$\vec{a}(P) = \text{const}$$

then $\frac{\partial a^i}{\partial x^k} = 0$, then

$$a^i{}_{,k} = 0 \tag{2.18}$$

in Cartesian coordinates then $a^i{}_{;k} = 0$ in any curvilinear coordinates. That is because by linearity, if a tensor is 0 of all components it is identically 0 in all coordinates.

Then

$$\frac{\partial a^i}{\partial x^k} = -\Gamma^i_{kj} a^j$$

for any constant vector in E^3 in arbitrary curvilinear coordinate.

Now consider a constant covector b_i in E^3, then $b_{i,k} = 0$ in Cartesian coordinates. And we want to say $b_{i;k} = 0$ in any coordinates. Applying constant vector a^i, $b_i a^i = $ constin any coordinates, so

$$\partial_k (b_i a^i) = 0$$

or

$$\frac{\partial b_j}{\partial x^k} a^j + b_i(-\Gamma^i_{kj} a^j) = 0$$

or

$$(b_{j,k} - \Gamma^i_{kj} b_i) a^j = 0 \ \forall \text{ constant } \vec{a}$$

or

$$b_{j,k} - \Gamma^i_{kj} b_i = 0$$

hence we should put

$$b_{i;k} = b_{j,k} - \Gamma^i_{kj} b_i$$

covariant derivative of b_i. It can be shown to be a tensor of rank $(0, 2)$ that reduces to $b_{j,k}$ in Cartesian coordinates.

Notice

$$b_{j;k} = (g_{ji} b^i)_{;k} = g_{ji} b^i{}_{;k}$$

we will prove $g_{ij;k} = 0$

Lecture 6 (2/10/14)

So we can do this for general tensor. Up index gets +, and down index gets −. Each index gets a Γ.

$$\begin{aligned} a^{ij}{}_{;k} &= a^{ij}{}_{,k} + \Gamma^i_{km} a^{mj} + \Gamma^j_{km} a^{im} \\ a_{ij;k} &= a_{ij,k} - \Gamma^m_{ik} a_{mj} - \Gamma^m_{jk} a_{im} \\ a^{ij}{}_{;k} &= a^i{}_{,k} + \Gamma^i_{mk} a^m{}_j - \Gamma^m_{jk} a^i{}_m \end{aligned}$$

we can show covariant derivative of $\delta^i_j \quad g_{ij} \quad \epsilon_{ijk}$ are 0. Because they are constant in Cartesian, so covariant derivative in Cartesian are 0, then using remark after (2.18), we conclude they are 0 in all coordinate systems.

Formula for Christoffel Symbol

We derive an explicit formula for Γ. From (2.17), we put

$$\vec{e}_l \cdot \frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^k_{ij} \vec{e}_k \cdot \vec{e}_l = g_{kl} \Gamma^k_{ij} \equiv [ij, l]$$

so

$$\frac{\partial g_{jl}}{\partial x^i} = \partial_i (\vec{e}_j \cdot \vec{e}_l) = [ij, l] + [il, j]$$

Above contains 27 unknowns of the form $[ij, l]$ and 27 equations, so we can solve

$$[ik, l] = \frac{1}{2}(\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik})$$

therefore

$$\Gamma^m_{ik} = \frac{1}{2} g^{ml} (g_{kl,i} + g_{il,k} - g_{ik,l})$$

so we have Γ even in flat space.

Divergent

$$\nabla_i a^i = a^i{}_{;i} = a^i{}_{,i} + \Gamma^i_{ik} a^k = a^i{}_{,i} + \frac{1}{2} g^{il} g_{il,k} a^k \tag{2.19}$$

We will play with an identity about determinant

$$g = \sum_{k=1}^{3} C_{ik} g_{ik} \text{ for any fixed } i \tag{2.20}$$

where

$$C_{ik} = \text{cofactor matrix}$$

From (2.20) we get two facts

$$\frac{\partial g}{\partial g_{ik}} = C_{ik}$$

and

$$\text{the inverse } g^{ik} = \frac{C_{ik}}{g}$$

so

$$\frac{\partial g}{\partial g_{ik}} = g g^{ik} \tag{2.21}$$

so

$$\frac{\partial g}{\partial x^m} = \frac{\partial g_{ik}}{\partial x^m} \frac{\partial g}{\partial g_{ik}} = g_{ik,m} g g^{ik}$$

so (2.19) becomes

$$\begin{aligned} \nabla_i a^i &= a^i{}_{,i} + \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^k} a^k = a^i{}_{,i} + \frac{\partial \ln \sqrt{g}}{\partial x^k} a^k \\ &= a^i{}_{,i} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} a^k = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g} a^k) \end{aligned} \tag{2.22}$$

this is divergent in any dimension in any coordinate.

Curl

Recall in Cartesian Curl

$$(\nabla \times \vec{a})^i = \tilde{\epsilon}^{ijk} a_{k,j}$$

so we guess in curvilinear

$$(\nabla \times \vec{a})^i = \epsilon^{ijk} a_{k;j} = \epsilon^{ijk} \left(a_{k,j} - \Gamma^m_{kj} a_m \right)$$

Since jk is anti-symmetric in ϵ and symmetric in Γ, $\epsilon\Gamma = 0$, so by (2.15)

$$(\nabla \times \vec{a})^i = \frac{1}{\sqrt{g}} \tilde{\epsilon}^{ijk} a_{k,j}$$

It turns out in spacetime Minkowski Space, the divergent and curl expressions are the same except $\sqrt{g} \to \sqrt{-g}$. Hence divergent

$$a^{\alpha}_{\ ;\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\alpha}} (\sqrt{-g} a^{\alpha}) \tag{2.23}$$

Curl

$$(\nabla \times \vec{a})^{\alpha\beta} = \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\alpha\beta\gamma\delta} a_{\gamma,\delta}$$

this is a pseudo-tensor, not good wrt reflection. If only study proper Lorentz transformation, we are fine.

where

$$\tilde{\epsilon}^{\alpha\beta\gamma\delta} = \tilde{\epsilon}_{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{indices repeat} \end{cases}$$

Levi-Civita symbol in 4D

2.4 Action for Charge in E&M fields

Consider a mass m object of charge e, moving in an external EM field A^{α}, 4D potential

$$A^{\alpha} = (\phi, \vec{A}) \text{ or } A_{\alpha} = \eta_{\alpha\beta} A^{\beta} = (-\phi, \vec{A})$$

the action is combination of free action, S_m =(2.1), interaction with EM field S_{mf},

$$S = S_m + S_{mf}$$

$$S_{mf} = \int_1^2 \frac{e}{c} A_{\alpha} dx^{\alpha} \tag{2.24}$$

the integral is taken along the worldline of the charge. Vary the path, δS_m is given by (2.10).

$$\begin{aligned}
\delta S_{mf} &= \frac{e}{c}\int_1^2 \delta A_\alpha dx^\alpha + A_\alpha d\delta x^\alpha \\
&= \frac{e}{c}\int_1^2 \frac{\partial A_\beta}{\partial x^\alpha}\delta x^\alpha dx^\beta - \frac{\partial A_\alpha}{\partial x^\beta}dx^\beta \delta x^\alpha + d(A_\alpha \delta x^\alpha) \\
&= \frac{e}{c}\int_1^2 \underbrace{(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta})}_{=:F_{\alpha\beta}}\delta x^\alpha \underbrace{dx^\beta}_{U^\beta d\tau} + \left(\frac{e}{c}A_\alpha \delta x^\alpha\right)_1^2
\end{aligned}$$

More generally define in terms of covariant (derivative) form, good in any curvilinear coordinates.

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = A_{\beta;\alpha} - A_{\alpha;\beta}$$

Thus

$$\delta S_m + \delta S_{mf} = \int_1^2 \left(-m\frac{dU_\alpha}{d\tau} + \frac{e}{c}F_{\alpha\beta}U^\beta\right)\delta x^\alpha d\tau + \left((mU_\alpha + \frac{e}{c}A_\alpha)\delta x^\alpha\right)_1^2$$

Following the same argument before, if fix two end points, we get EOM in covariant form

$$m\frac{dU_\alpha}{d\tau} = \frac{e}{c}F_{\alpha\beta}U^\beta \tag{2.25}$$

If we follow the argument in (2.7), allowing the destination to vary, and stay on the trajectory, we get momentum

$$p_\alpha = mU_\alpha + \frac{e}{c}A_\alpha = \frac{\partial S_{m+mf}}{\partial x^\alpha}\bigg|_{\text{along worldline}}$$

or in covariant form

$$p_\alpha = \nabla_\alpha S_{m+mf} \tag{2.26}$$

Lecture 7 (2/12/14)

One check gauge invariance: if $A_\alpha$is changed by a gradient of arbitrary function f

$$A_\alpha \to A_\alpha + \frac{\partial f}{\partial x^\alpha}$$

then $F_{\alpha\beta}$ doesn't change.

$$F_{\alpha\beta} \to \frac{\partial}{\partial x^\alpha}(A_\beta + \frac{\partial f}{\partial x^\beta}) - \frac{\partial}{\partial x^\beta}(A_\alpha + \frac{\partial f}{\partial x^\alpha}) = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$$

We want to show (2.25) gives Lorentz force. $F_{\alpha\beta}$ contains everything about the EM fields.

In Cartesian

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \qquad F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}$$

One can check that

$$\begin{aligned} F_{0i} &= \frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i} \quad &\Longrightarrow \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \\ F_{ij} &= \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \quad &\Longrightarrow \quad \vec{B} = \nabla \times \vec{A} \end{aligned}$$

Therefore we find (2.25) gives

$$\begin{aligned} m\gamma a_1 &= \frac{e}{c}F_{1\beta}U^\beta = \frac{e}{c}(F_{10}U^0 + F_{12}U^2 + F_{13}U^3) \\ m\vec{a} &= e\vec{E} + \vec{v} \times \vec{B} \end{aligned}$$

We now want to get 4 Maxwell equations.

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \end{cases} \tag{2.27}$$

$$\begin{cases} \nabla \cdot \vec{E} = 4\pi\rho \\ \nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} \times \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \end{cases} \tag{2.28}$$

Introduce Hodge duality operation

$$a \to *a \tag{2.29}$$

which acts on

$$\begin{aligned}\text{rank }(p,0) &\to \text{rank }(0,n-p)\\ \text{rank }(0,p) &\to \text{rank }(n-p,0)\end{aligned} \tag{2.30}$$

i.e. this operation works for only all indices up stairs or all indices down stair and works for tensor up to the dimension of the spacetime. For 4D

$$a^\alpha \to *a_{\alpha\beta\gamma} = \frac{1}{1!}\epsilon_{\alpha\beta\gamma\delta}a^\delta$$

$$a_\alpha \to *a^{\alpha\beta\gamma} = \frac{1}{1!}\epsilon^{\alpha\beta\gamma\delta}a_\delta$$

and similarly

$$a^{\alpha\beta} \to *a_{a\beta} = \frac{1}{2!}\epsilon_{\alpha\beta\gamma\delta}a^{\gamma\delta}$$

$$a^{\alpha\beta\gamma} \to *a_\alpha = \frac{1}{3!}\epsilon_{\alpha\beta\gamma\delta}a^{\beta\gamma\delta}$$

Notices

$$**a = (-1)^{s+p(n-p)}$$

where p is defined in (2.30) and s is the number of -1 in the matrix signature.

let us try in Cartesian

$$\begin{aligned}\frac{\partial}{\partial x^\alpha}(*F^{\alpha\beta}) &= \frac{\partial}{\partial x^\alpha}(\frac{1}{2}\tilde{\epsilon}^{\alpha\beta\gamma\delta}F_{\gamma\delta})\\ &= \frac{1}{2}\tilde{\epsilon}^{\alpha\beta\gamma\delta}\partial_\alpha(\partial_\gamma A_\delta - \partial_\delta A_\gamma) = 0\end{aligned} \tag{2.31}$$

because $\tilde{\epsilon}\partial_\alpha\partial_\gamma A = 0$ for $\tilde{\epsilon}$ is antisymmetric in $\alpha\gamma$ and $\partial_\alpha\partial_\gamma A$ is symmetric in $\alpha\gamma$.

In fact

$$(2.31) \implies (2.27)$$

or write (2.31) in covariant form

$$\nabla_\alpha(*F^{\alpha\beta}) = 0 \implies (2.27)$$

In other words, these two Maxwell equations are properties of the electric and magnetic fields, so they are not derived from EOM.

2.5 Action for EM fields Sourced by Charges

To get the other 2 Maxwell equations, we need to put in sources.

4D Current

Distribution of charge in space, charge density, is a frame dependent quantity. $\rho = \frac{dQ}{dV}$ while V is 3D space volume. Current is $\vec{j} = \rho \frac{dx^i}{dt}$, t is lab time, then if we define

$$j^\alpha = \rho \frac{dx^\alpha}{dt} = (\rho c, \vec{j})$$

we get a 4-vector, because

$$j^\alpha = \frac{dQ}{dV} \frac{dx^\alpha}{dt}$$

where Q is invariant scalar, x^α is 4 vector and $dV dt$ is invariant, cf (2.14).

By the way we can write charge conservation in covariant form

$$\nabla_\alpha j^\alpha = 0$$

which is the divergent we discussed in (2.23).

Since $\frac{dQ}{dV dt}$ is invariant scalar, its value is the same in any frame, say we go to rest frame of the charge fluid, then $t \to \tau$, so

$$j^\alpha = \tilde{\rho} \frac{dx^\alpha}{d\tau} = \tilde{\rho} U^\alpha$$

where $\tilde{\rho}$ is proper charge density.

Since Maxwell equations are linear, the action must be quadratic

$$S_f = \int \mathcal{L} dt dV$$

The correct free field action is

$$S_f = -\frac{1}{16\pi} \int_D F_{\alpha\beta} F^{\alpha\beta} dt dV \tag{2.32}$$

D is 4D spacetime. Replace e in (2.24) by $\int_V \rho dV$

$$S_{mf} = \int_D \frac{1}{c} A_\alpha \rho \frac{dx^\alpha}{dt} dtdV = \frac{1}{c} \int_D A_\alpha j^\alpha dtdV$$

Now S_{mf} becomes sources term. In the previous charge-motion problem we varied the path $x^\alpha(\tau)$, now we vary the field A^α, with

$$\delta A^\alpha = 0 \tag{2.33}$$

at the boundary ∂D.

$$A^\alpha(x^\mu) \to A^\alpha + \delta A^\alpha$$

then

$$\delta S_{mf} = \frac{1}{c} \int_D \delta A_\alpha j^\alpha dtdV$$

next

$$\begin{aligned}
\delta S_f &= -\frac{1}{16\pi} \int_D [(\delta F_{\alpha\beta}) F^{\alpha\beta} + F_{\alpha\beta} \delta F^{\alpha\beta}] dtdV \\
&= -\frac{1}{8\pi} \int_D (\delta F_{\alpha\beta}) F^{\alpha\beta} dtdV \\
&= -\frac{1}{8\pi} \int_D (\delta A_{\beta,\alpha} - \delta A_{\alpha,\beta}) F^{\alpha\beta} dtdV
\end{aligned}$$

Change the order of δ and ∂

$$\delta A_{\beta,\alpha} = \partial_\alpha (A_\beta + \delta A_\beta) - \partial_\alpha A_\beta = \partial_\alpha \delta A_\beta$$

continue

$$\begin{aligned}
\delta S_f &= -\frac{1}{4\pi} \int_D \frac{\partial \delta A_\beta}{\partial x^\alpha} F^{\alpha\beta} dtdV \\
&= -\frac{1}{4\pi c} \underbrace{\int_D \partial_\alpha (\delta A_\beta F^{\alpha\beta}) dx^0 dV}_{\underbrace{\int_{\partial D} \delta A_\beta F^{\alpha\beta} d^3 S}_{=0 \ \because (2.33)}} + \frac{1}{4\pi} \int_D \partial_\alpha F^{\alpha\beta} \delta A_\beta dtdV
\end{aligned} \tag{2.34}$$

therefore

$$\delta S_{mf} + \delta S_f = \int_D (\frac{1}{c} j^\beta + \frac{1}{4\pi} F^{\alpha\beta}{}_{,\alpha}) \delta A_\beta dt dV$$

$$\delta S = 0 \implies F^{\alpha\beta}{}_{;\alpha} = -\frac{4\pi}{c} j^\beta$$

or

$$F^{\alpha\beta}{}_{;\beta} = -\frac{4\pi}{c} j^\alpha \tag{2.35}$$

this gives the remaining two Maxwell equations (2.28).

General Field Equation

We can combine the derivation above in one field equation. Consider a set of fields q^A, $A = 1, ..., N$ (or field components) Action in the general form is

$$S = \int_D \mathcal{L} dt dV = \frac{1}{c} \int_D \mathcal{L} d^4x$$

where $\mathcal{L}(q^A, q^A{}_{,\alpha})$ is Lagrangian density. Do variance

$$\delta S = \frac{1}{c} \int_D \left(\frac{\partial \mathcal{L}}{\partial q^A} \delta q^A + \frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} \delta q^A{}_{,\alpha} \right) d^4x$$

$$\frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} \delta q^A{}_{,\alpha} = \frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} \frac{\partial}{\partial x^\alpha} q^A = \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} \delta q^A \right) - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} \right) \delta q^A \tag{2.36}$$

so

$$\delta S = \frac{1}{c} \int_D \left(\frac{\partial \mathcal{L}}{\partial q^A} - \frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} \right) \delta q^A d^4x + \frac{1}{c} \int_{\partial D} \frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} \delta q^A dS^\alpha$$

If $\delta q^A = 0$ at ∂D, we get field equation (EOM)

$$\frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial q^A{}_{,\alpha}} = \frac{\partial \mathcal{L}}{\partial q^A} \tag{2.37}$$

for each $A = 1, ..., N$. The RHS is usually the source term: mechanical potential, electric charge-current or stress-energy tensor; the LHS is usually the derivatives of dynamic variables: mechanical momentum, electromagnetic potential or metrics.

One can show using,

$$\mathcal{L}_{EM} = \frac{1}{c} A_\alpha j^\alpha - \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta}$$

and using A_β for q^A in (2.37), get (2.35).

2.6 Calculus of Differential Forms

We would like to justify the step in (2.34) where we used stokes theorem of curvilinear coordinates

$$\int_D dw = \int_{\partial D} w \tag{2.38}$$

Line Integral

$$\int_\lambda \vec{a} \cdot d\vec{\lambda} \equiv \int g_{ik} a^i \frac{dx^k}{dt} dt$$

Since the integration doesn't depend on speed, we may write it simple

$$\int_\lambda a_k dx^k$$

and

$$a_k dx^k \text{ is called 1D differential form or 1-form}$$

This integral is defined for any covector field $a_k(\vec{x})$. The expression $a_k dx^k$ can be viewed as a representation of the covector in components if dx^k is viewed as a basis vector in the dual space

$$dx^k \leftrightarrow \vec{e}^k \in V^*$$

Surface Integral

First consider a tensor fields on surfaces in E^3. Surface

$$S = (x^1(q^1, q^2), x^2(q^1, q^2), x^3(q^1, q^2))$$

parametrized by coordinate (q^1, q^2). At each point P, we find 2 vectors tangent to the surface

$$\vec{e}_\tau = \frac{\partial \vec{x}}{\partial q^\tau} \qquad \tau = 1, 2$$

all linear combinations

$$\alpha \vec{e}_1 + \beta \vec{e}_2$$

form tangent space of dimension 2.

A vector field on S

$$\vec{a}(P) = a^\tau \vec{e}_\tau \quad P \in S$$

We can show $\vec{a}$ is a 2D tensor of rank $(1, 0)$.

$$a^\tau \vec{e}_\tau = a^\tau \frac{\partial \vec{x}}{\partial q^\tau} = a^\tau \frac{\partial q^{\tau'}}{\partial q^\tau} \frac{\partial \vec{x}}{\partial q^{\tau'}} = a^{\tau'} \vec{e}_{\tau'}$$

i.e.

$$a^{\tau'} = \frac{\partial q^{\tau'}}{\partial q^\tau} a^\tau$$

The surface element

$$d\vec{\sigma} \equiv \vec{n} d\sigma \quad |\vec{n}| = 1$$

can be write as

$$d\vec{\sigma} = (dq^1 \vec{e}_1) \times (dq^2 \vec{e}_2) = (\vec{e}_1 \times \vec{e}_2) dq^1 dq^2 = \epsilon_{ijk} \frac{\partial x^j}{\partial q^1} \frac{\partial x^k}{\partial q^2} dq^1 dq^2$$

where

$$\epsilon_{ijk} = \sqrt{g} \tilde{\epsilon}_{ijk}$$

We can write it in a tensor form if we introduce wedge product

$$dq^\tau \wedge dq^\rho \equiv \frac{1}{2} \tilde{\epsilon}^{\tau\rho} dq^1 dq^2$$

and

$$\tilde{\epsilon}^{\tau\rho} = \begin{cases} 1 & \tau\rho = 12 \\ -1 & \tau\rho = 21 \\ 0 & \tau\rho = 11 \text{ or } 22 \end{cases}$$

then

$$
\begin{aligned}
d\vec{\sigma} &= (\vec{e}_1 \times \vec{e}_2)\frac{dq^1 dq^2}{2} - (\vec{e}_2 \times \vec{e}_1)\frac{dq^1 dq^2}{2} \\
&= \vec{e}_\tau \times \vec{e}_\rho dq^\tau \wedge dq^\rho \\
d\sigma_i &= \epsilon_{ijk}\frac{\partial x^j}{\partial q^\tau}\frac{\partial x^k}{\partial q^\rho} dq^\tau \wedge dq^\rho
\end{aligned}
$$

Now define

$$
dx^i \wedge dx^j = \frac{\partial x^j}{\partial q^\tau}\frac{\partial x^k}{\partial q^\rho} dq^\tau \wedge dq^\rho
$$

we have surface integral

$$
\int_S \vec{a} \cdot d\vec{\sigma} \equiv \int_S g_{ik} a^i d\sigma^k = \int_S a^i d\sigma_i = \int \underbrace{a^i \epsilon_{ijk}}_{w_{jk}} dx^j \wedge dx^k
$$

where

$$
w_{jk} = \epsilon_{ijk} a^i
$$

is the Hodge dual of vector a^i, cf (2.29). And

$$
w_{jk} dx^j \wedge dx^k \text{ is called 2D differential form or 2-form}
$$

This integral is defined for any antisymmetric tensor w_{jk} or its Hodge dual $a^i = \frac{1}{2!}\tilde{\epsilon}^{ijk} w_{jk}$. The expression $w_{jk} dx^j \wedge dx^k$ can be viewed as a representation of the antisymmetric tensor w in components if dx^k is viewed as

$$
dx^k \leftrightarrow \vec{e}^k \in V^*
$$

then if $i < j$

$$
dx^i \wedge dx^j = \frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i) = \frac{1}{2}(\vec{e}^i \otimes \vec{e}^j - \vec{e}^j \otimes \vec{e}^i)
$$

In the same logic, one can think line integral as tensor fields on curves, which will be trivial because the tangent space of a curve is 1D.

Volume Integral

Lecture 8 (2/17/14)

Let f be a scalar function.

$$\int_V f dV \equiv \int_V f\sqrt{g} dx^1 dx^2 dx^3$$

Similarly define

$$dx^i \wedge dx^j \wedge dx^k = \frac{1}{3!}\tilde{\epsilon}^{ijk} dx^1 dx^2 dx^3$$

then

$$\int_V f dV \equiv \int_V w_{ijk} dx^i \wedge dx^j \wedge dx^k$$

where

$$w_{ijk} = \frac{1}{1!}\epsilon_{ijk} f$$

is the Hodge dual of f. And

$$w_{ijk} dx^i \wedge dx^j \wedge dx^k \text{ is called 3D differential form or 3-form}$$

this integral is defined for any antisymmetric tensor w_{ijk} and its dual $f = \frac{1}{3!}\epsilon^{ijk} w_{ijk}$.

Generally any antisymmetric tensor

$$w_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \text{ (p form)}$$

can be integrated over a $p-$dimensional region D

$$\int_D w_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

Differentiation

Define operator

$$d : w_{i_1,\ldots,i_p} \to (dw)_{i_1,\ldots,i_{p+1}}$$

by Stokes (2.38).

$$p\text{-form is closed if } dw = 0$$

$$p\text{-form is exact if } w = d\Omega$$

where Ω is a $(p+1)$ form.

HW2 problem 6a Show

$$d(dw) \equiv 0$$

any exact form is closed.

Note that $(dw)_{i_1,\ldots,i_{p+1}}$ can also be defined as

$$(dw)_{i_1,\ldots,i_{p+1}} \equiv (p+1)\partial_{[i_1}w_{i_2,\ldots,i_{p+1}]}$$

It would give the same expression because all terms with Christoffel symbols vanish after antisymmetrization. So our definition was covariant.

3 Riemann Manifolds

There is no global inertial frame in the presence of gravity (no global orthonormal frame). The best hope is local Minkowski $g_{\alpha\beta} = \eta_{\alpha\beta}$.

3.1 Manifolds

An elementary manifold is one that has a bijection from a set M to $\Omega \subset \mathbb{R}^n$, Ω is some connected open region in $\mathbb{R}^n$ union of open balls. n is called dimension of manifold M. The bijection introduces a coordinate system or chart on M.

A non-elementary manifold requires at lease two charts to cover M.

$$M = \bigcup_\alpha U_\alpha \quad U_\alpha \subset M$$

$$\text{charts: } U_\alpha \leftrightarrow \Omega_\alpha \subset \mathbb{R}^n$$

On the overlaps, two charts

$$U_1 \leftrightarrow \Omega_1(x^1, \ldots, x^n)\ U_2 \leftrightarrow \Omega_2(x^1, \ldots, x^n)$$

give two different coordinate system on $U_1 \cap U_2$. The two charts are called consistent if the relation between x^i and y^i on $U_1 \cap U_2$ is described by differentiable functions $x^i(y^i)$ and $y^i(x^i)$.

An atlas is a union of consistent U_α that covers M. Smooth atlas requires $x^i(y^i)$ and $y^i(x^i)$ to be C^∞. Maximal atlas is collection of all possible consistent atlas.

A vector at point P of a manifold M is a set of numbers A^i $i = 1, ..., n$, defined for each chart from the maximal atlas that contains point P such that $A^{i'}$ and A^i for two charts $x^{i'}$ and x^i are related by

$$A^{i'} = \frac{\partial x^{i'}}{\partial x^i} A^i \tag{3.1}$$

that is the reason why we need consistency to be differentiable, i.e. $\frac{\partial x^{i'}}{\partial x^i}$ is smooth.

A vector field on M: vector defined for any point and its components $A^i(x^i)$ are smooth functions for any chart x^i. By (3.1) if A^i is smooth, $A^{i'}$ will be smooth too.

Tensor field of rank (p, q) is described by components that satisfy the transformation law between two charts x^i and $x^{i'}$

$$B^{i'_1,...,i'_p}_{j'_1,...,j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} ... \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} ... \frac{\partial x^{j_q}}{\partial x^{j'_q}} B^{i_1,...,i_p}_{j_1,...,j_q}$$

all the usual tensor operations are defined for tensors on manifold:

sum
multiplication by number
(anti)symmetrization
contraction
tensor product

Tangent Space

There are three equivalent definitions for $T_P M$, tangent space at point P.

Definition 1. $T_P M$ is set all vectors at P.

It forms a vector space of dimension n. For GR, it is no need to embed in higher dimension, 4D is enough.

For any vector A^i at P there exists a curve in M: $\mathbb{R} \to M$ that passes through P and $A^i = \frac{dx^i}{d\lambda}$ in any chart x^i containing P. Here λ is the parameter along the curve.

Definition 2. the set of different $\frac{dx^i}{d\lambda}$ for all possible curves (modules by equivalent class, since ∞ly many curves have the same tangent vectors) that pass through P form linear space T_PM. A basis in T_PM is provided by any chart that contains P. Take the coordinate curves of $x^i : \lambda \to x^k$, $\vec{e}_k$:

$$\frac{dx^i}{d\lambda} = \frac{\partial x^i}{\partial x^k} = \delta^i_k$$

basis vector.

Consider the space of smooth functions on M,

$$f : M \to \mathbb{R}$$

Any tangent vector at point P defines a differential operator

$$d_P : f \to A^i \frac{\partial f}{\partial x^i}$$

basis vectors $\vec{e}_k$ correspond to operators $\frac{\partial}{\partial x^k}$. And a tangent vector looks like $A^i \frac{\partial}{\partial x^k} = A^i \partial_k$.

Definition 3. T_PM is the linear space of differential operators that act of functions f at point P.

Tangent space makes the locally inertial frame in equivalence principle.

Commutator

View two vector fields a and b as differential operators that act on the space $\mathbb{R}(M)$ of all smooth functions f then

$$[a, b](f) = a(b(f)) - b(a(f)) \tag{3.2}$$

defines a new differential operator and correspondingly a new vector field

$$c = [a, b]$$

this field is called commutator of a and b.

In components in chart x^i, (3.2) says

$$\begin{aligned} c^j\partial_j &= a^i\partial_i(b^j\partial_j) - b^i\partial_i(a^j\partial_j) \\ &= a^i(\partial_i b^j)\partial_j - b^i(\partial_i a^j)\partial_j \end{aligned}$$

since commutator of basis vector $[\partial_i, \partial_j] = 0$, then

$$c^j = a^i(\partial_i b^j) - b^i(\partial_i a^j)$$

Tensor as Linear Maps

Any tensor of rank (p, q) at point P is associated with a linear map

$$\underbrace{T_P M \times ... \times T_P M}_{p \text{ times}} \times \underbrace{T_P^* M \times ... \times T_P^* M}_{q \text{ times}} \rightarrow \mathbb{R}$$

A chart x^i defines a basis $\frac{\partial}{\partial x^i}\big|_P$ in $T_P M$ and its dual basis $d_P x^i$ indeed

$$T_P^* M : \; d_P x^i(\frac{\partial}{\partial x^j}) = \delta^i_j$$

In particular

$$d_P f = A^i \frac{\partial f}{\partial x^i}$$

is viewed as a function of A^i gives element of $T_P^* M$.

3.2 Riemann Manifold

Riemann manifold is a manifold with a rank (0,2) g_{ij} metric tensor, symmetric, non-degenerated. Given a metric tensor one can calculate scalar products of vectors, length of curves and volumes:

$$l = \int_\lambda \sqrt{g_{ij} dx^i dx^j} = \int_\lambda \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} d\lambda$$

$d\lambda$ parametrizes the curve.

$$V = \int_D \sqrt{|g|} dx^1 ... dx^n$$

for $n$dimensional space.

Integration of scalar function

$$\int_D f\sqrt{|g|} dx^1 ... dx^n \tag{3.3}$$

This will be useful for actions in curved space.

Lecture 9 (2/19/14)

Spacetime is a Riemann space with pseudo-Euclidean metric of signature $(-+++)$. No global Lorentz frames exists in Riemann spacetime.

Covariant Derivative

Covariant derivative of tensor fields can be defined by introducing $\Gamma^{\alpha}_{\beta\gamma}$ in the same way as it was done for flat spacetime. We don't have now a preferred coordinate system (e.g. Cartesian) where $\Gamma^{\alpha}_{\beta\gamma} = 0$ globally. We do have, however, $g_{\alpha\beta}$ and can impose that locally Minkowski condition (see section 3.4 local inertial coordinates)

$$\nabla_\mu g_{\alpha\beta} = 0 \tag{3.4}$$

that is $g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} - \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} = 0$. This defines

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu}(g_{\beta\nu,\gamma} + g_{\gamma\nu,\beta} - g_{\beta\gamma,\nu}) = \frac{1}{2} g^{a\nu}[\beta\gamma, \nu] \tag{3.5}$$

to be exactly the same as before. Then covariant derivatives of all tensors (p, q) are defined using $\Gamma^{\alpha}_{\beta\gamma}$ in the usual way.

Parallel Transport

Parallel transport of a vector had obvious geometrical meaning in flat spacetime, meaning the vector was moved while its magnitude and direction kept the same, i.e. the vector was kept constant. For curved space we say if

$$\nabla_\alpha a^\beta = 0 \tag{3.6}$$

then the vector is parallel transported.

Directional Derivative

We define the directional derivative of tensor T along a curve $\lambda \to M$, by

$$\frac{D}{D\lambda} T^{\alpha_1,\ldots,\alpha_p}{}_{\beta_1,\ldots,\beta_q} = \frac{dx^\mu}{d\lambda} \nabla_\mu T^{\alpha_1,\ldots,\alpha_p}{}_{\beta_1,\ldots,\beta_q}$$

If $\frac{DT}{D\lambda} = 0$, T is transported parallel along curve λ.

Notice that the definition of direction derivative depends on parametrization of the curve. Choosing different parametrization will result different direction derivative. So in terms of tangent vector, the direction will the same but the magnitude will be different. But if the tangent vector is 0 (parallel transport) the parametrization doesn't matter. But in the case of geodesics equation (see below), because we are talking about mixing second and first derivative of λ, cf (3.9), different parametrization does matter.

Geodesics

It is a curve whose tangent vector $U^\alpha = \frac{dx^\alpha}{d\lambda}$ satisfies

$$\frac{D}{D\lambda} U^\alpha = 0 \tag{3.7}$$

or

$$a^\alpha = U^\beta \nabla_\beta U^\alpha = 0 \tag{3.8}$$

Clearly in flat space if the tangent vector is constant, geodesics is a straight line.

From (3.7), we say geodesic is a curve who has a parametrization that satisfies the geodesic equation

$$a^\alpha = \frac{D}{D\lambda} U^\alpha = \frac{dU^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} U^\beta U^\gamma = 0$$

so a^α is the 4-accelaration or

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{3.9}$$

The geodesic equation limits the choice of parametrization λ within a linear transformation

$$\lambda \to a\lambda + b \tag{3.10}$$

The most important property of a geodesic is that it extremizes the distant for two fixed end points. Let λ be a curve connecting a, b

$$L = \int_a^b \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}d\lambda \quad i, j = 1, ..., n \text{ arbitrary space not necessay spacetime}$$

Consider $x^i(\lambda) \to x^i(\lambda) + \delta x^i(\lambda)$, then

$$\delta L = \int_a^b \frac{\delta(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda})}{2\sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}}d\lambda$$

Rescaling λ so that the length of $u^i = \frac{dx^i}{d\lambda} = 1$, so $g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 1$, then

$$\delta L = \frac{1}{2}\int_a^b \left(\delta g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} + 2g_{ij}\frac{dx^i}{d\lambda}\frac{d\delta x^j}{d\lambda}\right)d\lambda$$

$\delta g_{ij} = g_{ij,k}\delta x^k$ and $g_{ij}\frac{dx^i}{d\lambda}\frac{d\delta x^j}{d\lambda} = \frac{d}{d\lambda}\left(g_{ij}\delta x^j\frac{dx^j}{d\lambda}\right) - \frac{d}{d\lambda}\left(g_{ij}\frac{dx^j}{d\lambda}\right)\delta x^i$, so

$$\delta L = \frac{1}{2}\int_a^b \underbrace{(g_{ij,k}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} - 2\frac{d}{d\lambda}\left(g_{ij}\frac{dx^j}{d\lambda}\right))}_{=II}\delta x^k d\lambda + \underbrace{\left(g_{ij}\delta x^j\frac{dx^j}{d\lambda}\right)_a^b}_{0}$$

To get the II =geodesic, we first replace some $i \to k$

$$2\frac{d}{d\lambda}\left(g_{kj}\frac{dx^j}{d\lambda}\right) = 2g_{kj,m}\frac{dx^m}{d\lambda}\frac{dx^j}{d\lambda} + 2g_{kj}\frac{d^2x^j}{d\lambda^2}$$

then change $m \to i$, then switch i and j

$$2g_{kj,m}\frac{dx^m}{d\lambda}\frac{dx^j}{d\lambda} = 2g_{kj,i}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = (g_{kj,i} + g_{ki,j})\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}$$

therefore

$$II = \underbrace{(g_{ij,k} - g_{kj,i} - g_{ki,j})}_{=-[ij,k]}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} - 2g_{kj}\frac{d^2x^j}{d\lambda^2}$$

Multiplying by $-\frac{1}{2}g^{km}$ and by (3.5), we get

$$-\frac{1}{2}g^{km}(II) = \text{geodesic}$$

3.3 Riemann Tensor

Lecture 10 (2/24/14)

As we discussed before neither $g_{\alpha\beta}$ that has off diagonal elements or non-zero $\Gamma^{\mu}_{\alpha\beta}$, (it contains first derivative of $g_{\alpha\beta}$), implies whether the space is flat or curved, so what describes the curveness? The answer is ${R^{\mu}}_{\sigma\alpha\beta}$, Riemann tensor, which is also called curvature tensor. ${R^{\mu}}_{\sigma\alpha\beta}$ has second derivatives of $g_{\alpha\beta}$.

Consider the second covariant derivative of a vector field A^{μ} , then

$$\nabla_{\alpha}(\nabla_{\beta}A^{\mu}) = {c_{\alpha\beta}}^{\mu}$$

is a rank $(1, 2)$ tensor. In flat space we can choose a global Cartesian coordinate, in which $\Gamma = 0$, then

$${c_{\alpha\beta}}^{\mu} = \partial_{\alpha}\partial_{\beta}A^{\mu}$$

so it is symmetric wrt α, β. Since symmetry is an invariant property, we say ${c_{\alpha\beta}}^{\mu}$ is symmetric wrt α, β in any coordinates. That is

$$[\nabla_{\alpha}, \nabla_{\beta}] = 0$$

However in a curved manifold

$$[\nabla_{\alpha}, \nabla_{\beta}] \neq 0$$

This quantity turns out to give the Riemann tensor. Let's compute

$$\nabla_\alpha \nabla_\beta A^\mu = \partial_\alpha(\nabla_\beta A^\mu) + \Gamma^\mu_{\alpha\beta}\nabla_\beta A^\sigma - \Gamma^\sigma_{\alpha\beta}\nabla_\sigma A^\mu$$

the last term is symmetric in $\alpha\beta$, so it will be canceled in the commutator, so we drop it.

Then

$$\begin{aligned}
\nabla_\alpha \nabla_\beta A^\mu &\to \partial_\alpha(\partial_\beta A^\mu + \Gamma^\mu_{\beta\sigma}A^\sigma) + \Gamma^\mu_{\alpha\sigma}(\partial_\beta A^\sigma + \Gamma^\sigma_{\beta\lambda}A^\lambda) \\
&\to \partial_\alpha \Gamma^\mu_{\beta\sigma}A^\sigma + \underbrace{\Gamma^\mu_{\beta\sigma}\partial_\alpha A^\sigma + \Gamma^\mu_{\alpha\sigma}\partial_\beta A^\sigma}_{\text{sym in } \alpha\beta} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\lambda}A^\lambda \\
&\to \partial_\alpha \Gamma^\mu_{\beta\sigma}A^\sigma + \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\beta\sigma}A^\sigma
\end{aligned}$$

thus

$$[\nabla_\alpha, \nabla_\beta]A^\mu = (\underbrace{\partial_\alpha \Gamma^\mu_{\beta\sigma} - \partial_\beta \Gamma^\mu_{\alpha\sigma} + \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\beta\sigma} - \Gamma^\mu_{\beta\lambda}\Gamma^\lambda_{\alpha\sigma}}_{\equiv R^\mu{}_{\sigma\alpha\beta}})A^\sigma \tag{3.11}$$

Geometrical Meaning of Riemann Tensor

Consider a point A of the spacetime and a chart x^α around point A. Define an infinitesimal contour $ABCD$ as the image of rectangular δx^1,δx^2, with x^0, x^3 constant. The image is distorted shape on the manifold. Suppose a vector v^μ is defined at point A. Let's transport parallel v^μ along the contour $ABCD$. Consider the change in v^μ during this transport.

By (3.6)

$$\begin{aligned}
\delta v^\mu_{A\to B} &= -(\Gamma^\mu_{\lambda 1}v^\lambda)_{x^2}\delta x^1 \\
\delta v^\mu_{C\to D} &= -(\Gamma^\mu_{\lambda 1}v^\lambda)_{x^2+\delta x^2}(-\delta x^1)
\end{aligned}$$

so

$$\delta v^\mu_{A\to B} + \delta v^\mu_{C\to D} = \frac{\partial}{\partial x^2}(\Gamma^\mu_{\lambda 1}v^\lambda)\delta x^1 \delta x^2$$

Similarly

$$\delta v^\mu_{B\to C} + \delta v^\mu_{D\to A} = -\frac{\partial}{\partial x^1}(\Gamma^\mu_{\lambda 2}v^\lambda)\delta x^1 \delta x^2$$

thus the net change is

$$\begin{aligned}
\delta v^\mu &= \left(\frac{\partial}{\partial x^2}(\Gamma^\mu_{\lambda 1} v^\lambda) - \frac{\partial}{\partial x^1}(\Gamma^\mu_{\lambda 2} v^\lambda)\right)\delta x^1 \delta x^2 \\
&= [\partial_2(\Gamma^\mu_{\lambda 1})v^\lambda + \Gamma^\mu_{\lambda 1}\underbrace{\partial_2 v^\lambda}_{-\Gamma^\lambda_{2\sigma}v^\sigma}) - (1 \leftrightarrow 2)]\delta x^1 \delta x^2 \\
&= R^\mu{}_{\sigma 21} v^\sigma \delta x^1 \delta x^2
\end{aligned}$$

Instead of using rectangular δx^1,δx^2, we could use δx^0,δx^3, or any other direction, so

$$\delta v^\mu = R^\mu{}_{\sigma\alpha\beta} v^\sigma \delta x^\alpha \delta x^\beta \text{ no sum over } \alpha\beta$$

Symmetries of Riemann Tensor

Commonly people define Riemann tensor

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^\mu{}_{\beta\gamma\delta}$$

From (3.11), one find

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{3.12}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{3.13}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{3.14}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \tag{3.15}$$

then by the first 3 identities, one can show (3.15) is true for cyclic permutation for fixing any other 3 indices not just α. For this reason we only list one identity for (3.15), and it only produces one constraint as it will become clear soon.

There is one more symmetry involving derivative (4.22), we will discuss it later.

Let us see what these 4 equations say about number of independent components of $R_{\alpha\beta\gamma\delta}$.

By (3.12), (3.13), if $\alpha = \beta$ or $\gamma = \delta$, $R_{\alpha\beta\gamma\delta} = 0$, and by (3.12), (3.13) there are 36 independent components.

$\alpha\beta\backslash\gamma\delta$	01	02	03	12	13	23
01	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
02		$\times$	$\times$	$\times$	$\times$	$\times$
03			$\times$	$\times$	$\times$	$\times$
12				$\times$	$\times$	$\times$
13					$\times$	$\times$
23						$\times$

By (3.14), among them only $6(6+1)/2 = 21$ independent components. By (3.15), among them only $21 - 1 = 20$ independent components.

3.4 Local Inertial Coordinates

Theorem. *In any Riemann manifold (i.e. $g_{\alpha\beta}$ is given in a chart x^α around P), for any point P, one can find a chart $x^{\alpha'}$ in a neighborhood of P such that*

$$g_{\alpha'\beta'} = \eta_{\alpha'\beta'}$$

and

$$g_{\alpha'\beta',\gamma'} = 0 \tag{3.16}$$

i.e. $\Gamma = 0$.

Proof. Use transformation at P

$$g_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}} g_{\alpha\beta} \tag{3.17}$$

this consists of 10 linear equations (because the independent components of $g_{\alpha'\beta'}$ are 10) and there are 16 unknowns $\frac{\partial x}{\partial x'}$, so it can be solved for $\frac{\partial x}{\partial x'}$ in a neighborhood of P for $g_{\alpha'\beta'} = \eta_{\alpha'\beta'}$ also require $\frac{\partial x}{\partial x'}$ to be C^∞ and locally invertible. This can all be done e.g. use $16 - 10 = 6$ dimensional group of rotations, cf (3.31).

Next differentiate (3.17) at P

$$\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu'}} = 2\frac{\partial^2 x^\alpha}{\partial x^{\mu'}\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}} g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} \tag{3.18}$$

This consists of 40 linear equations (because the independent components of $\frac{\partial g_{\alpha'\beta'}}{\partial x^{\mu'}}$

are 10×4, 4 for μ' and 10 for $\alpha' \otimes \beta'$ since they are symmetric) and there are 40 unknowns $\frac{\partial^2 x^\alpha}{\partial x^{\mu'} \partial x^{\alpha'}}$, (because the independent components of $\frac{\partial^2 x^\alpha}{\partial x^{\mu'} \partial x^{\alpha'}}$ are $4 \times 10 = 40$, 4 for α and 10 for $\mu' \otimes \alpha'$ since they are symmetric), and $\frac{\partial x^\alpha}{\partial x^{\alpha'}}$, $g_{\alpha\beta}$, $\frac{\partial g_{\alpha\beta}}{\partial x^\mu}$ are known. So it can be solved for $\frac{\partial^2 x^\alpha}{\partial x^{\mu'} \partial x^{\alpha'}}$ in a neighborhood of P for $g_{\alpha'\beta',\gamma'} = 0$ since $\det \frac{\partial x^\beta}{\partial x^{\beta'}} g_{\alpha\beta} \neq 0$. QED

What if we continue the process: next imposing $g_{\alpha'\beta',\gamma'\mu'} = 0$, do we get Riemann tensor $\equiv 0$, flat space? The answer is No.

Differentiate (3.18) at P

$$\frac{\partial g_{\alpha'\beta'}}{\partial x^{\lambda'} \partial x^{\mu'}} = (\ldots) \frac{\partial^2 x^\alpha}{\partial x^{\lambda'} \partial x^{\mu'} \partial x^{\alpha'}} + (\ldots)$$

This consists of 100 linear equations (because the independent components of $\frac{\partial g_{\alpha'\beta'}}{\partial x^{\lambda'} \partial x^{\mu'}}$ are $10 \times 10 = 100$) and there are 80 unknowns $\frac{\partial^2 x^\alpha}{\partial x^{\lambda'} \partial x^{\mu'} \partial x^{\alpha'}}$, (because the independent components of $\frac{\partial^2 x^\alpha}{\partial x^{\lambda'} \partial x^{\mu'} \partial x^{\alpha'}}$ are $20 \times 4 = 80$, 4 for α and $20 = \begin{pmatrix} 4+3-1 \\ 3 \end{pmatrix}$ for $\lambda' \otimes \mu' \otimes \alpha'$ since they are totally symmetric), so there are more equations than unknowns. The excess number is

$$100 - 80 = 20$$

is the exact independent components in the Riemann tensor, which contains all information about the curvature.

Exponential Map

Lecture 11 (2/26/14)

How to construct a local inertial coordinates?

Since the curvature is smooth, it is possible to find a neighborhood of P so that all geodesics passing through P with no intersection (no coming back) so we can define a chart that maps the tangent space of P, T_pM, to the geodesic curves within the neighborhood with parametrization λ, x^α s.t. $\lambda \in [0, 1]$ and $x^\alpha(\lambda)$ lie within the neighborhood.

Define exponential map

$$
\begin{aligned}
T_P M &\rightarrow M \\
u^\alpha &\mapsto x^\alpha(1)
\end{aligned}
$$

by shooting geodesics from P for each tangent vector u^μ at P. And choosing the point $\lambda = 1$ on the geodesic.

It is done by solving geodesic equation (3.9), solve for $x^\alpha(\lambda)$ with initial conditions

$$x^\alpha(0) = P$$

and

$$\left.\frac{dx^\alpha}{d\lambda}\right|_{\lambda=0} = u^\alpha \in T_P M$$

then associate symbolically

$$x^\alpha|_{\lambda=1} = \exp_P(u^\alpha) \tag{3.19}$$

If at the beginning we choose coordinate x^α such that $g_{\alpha\beta} = \eta_{\alpha\beta}$ (Cartesian) at P, then (3.19) produces Riemann normal coordinates (also called a tetrad)

$$u^\alpha = x^{\hat{\alpha}}$$

Geodesics through P maps to straight lines in $x^{\hat{\alpha}}$, because $x^{\hat{\alpha}}$ is Cartesian, the basis vectors are constant. So geodesic equation is simply

$$\frac{d^2 x^{\hat{\alpha}}}{d\lambda^2} = 0$$

so $\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} = 0$.

3.5 "Global" Coordinates

Geodesic Deviation

After studying local inertial frame, we now study how to connect frames. Consider a 2D surface $x^\alpha(t,s)$ formed by 1 parameter family of geodesics in 4D spacetime (or any other $n$dimensional Riemann space). t is an affine parameter along the geodesics, and s labels the geodesics.

Define two tangent vector fields on the 2D surface for any chart x^α and transforms as vectors from x^α to a new chart $x^{\alpha'}$, and (t,s) are "global" coordinates.

$$S^\alpha = \frac{\partial x^\alpha}{\partial s} \qquad T^\alpha = \frac{\partial x^\alpha}{\partial t}$$

To understand the structure, let's consider two very close ("parallel") geodesic. Assume that within the "global" region under investigation the two geodesics don't intersect, so (t,s) is well-defined. Then the separation between the two geodesic is

$$\delta x^\alpha = S^\alpha\big|_{\delta t=0}\,\delta s$$

then allow t to move,

$$\left.\frac{D}{Dt}\right|_{\text{fix }\delta s} \delta x^\alpha = \frac{\partial x^\beta}{\partial t}\frac{\partial}{\partial x^\beta}(\delta x^\alpha) = T^\beta \nabla_\beta (S^\alpha \delta s) = V^\alpha \delta s$$

Naturally we call it relative velocity, because it measures the speed of two geodesics getting closer or apart to each other,

$$V^\alpha = T^\beta \nabla_\beta S^\alpha$$

Next we do

$$\left.\frac{D}{Dt}\frac{D}{Dt}\right|_{\text{fix }\delta s} \delta x^\alpha = \frac{D}{Dt} V^\alpha \delta s = T^\beta \nabla_\beta (V^\alpha \delta s) = A^\alpha \delta s$$

We call

$$A^\alpha = T^\beta \nabla_\beta V^\alpha \tag{3.20}$$

tidal acceleration. It is directly related to Riemann tensor. To show that, we need two identities.

The geodesic (3.7) for T^α becomes

$$\frac{d}{dt}T^\alpha = T^\beta \nabla_\beta T^\alpha = 0$$

Later we will use the following by identifying $T^\alpha \to U^\alpha = \frac{dx^\alpha}{d\tau}$, τ = proper time, and we get

$$U^\beta \nabla_\beta U^\alpha = 0 \tag{3.21}$$

Second identity

$$T^\beta \nabla_\beta S^\alpha = S^\beta \nabla_\beta T^\alpha$$

It's enough to show

$$T^\beta(\partial_\beta S^\alpha + \Gamma^\alpha_{\beta\mu} S^\mu) = S^\beta(\partial_\beta T^\alpha + \Gamma^\alpha_{\beta\mu} T^\mu)$$

which is indeed true.

$$\frac{\partial x^\beta}{\partial t}\frac{\partial x^\alpha}{\partial x^\beta \partial s} + \Gamma^\alpha_{\beta\mu} S^\mu T^\beta = \frac{\partial x^\beta}{\partial s}\frac{\partial x^\alpha}{\partial x^\beta \partial t} + \Gamma^\alpha_{\beta\mu} T^\mu S^\beta$$

QED

Back to (3.20)

$$\begin{aligned}
A^\alpha &= T^\beta \nabla_\beta (S^\mu \nabla_\mu T^\alpha) = (\underbrace{T^\beta \nabla_\beta S^\mu}_{S^\beta \nabla_\beta T^\mu}) \nabla_\mu T^\alpha + T^\beta S^\mu \nabla_\beta \nabla_\mu T^\alpha \\
&= S^\beta [\nabla_\beta (\underbrace{T^\mu \nabla_\mu T^\alpha}_{0}) - T^\mu \nabla_\beta \nabla_\mu T^\alpha] + T^\beta S^\mu \nabla_\beta \nabla_\mu T^\alpha \\
&= T^\beta S^\mu \underbrace{(\nabla_\beta \nabla_\mu - \nabla_\mu \nabla_\beta) T^\alpha}_{R^\alpha{}_{\sigma\beta\mu} T^\sigma} \\
&= R^\alpha{}_{\sigma\beta\mu} T^\sigma T^\beta S^\mu
\end{aligned} \tag{3.22}$$

Symmetries of metric and Conservation Laws

Consider a free particle in curved space. Since (3.21), (3.4)

$$0 = g_{\alpha\sigma} U^\beta \nabla_\beta U^\alpha = U^\beta \nabla_\beta g_{\alpha\sigma} U^\alpha = U^\beta \nabla_\beta U_\sigma$$

so $U^\beta(\partial_\beta U_\sigma - \Gamma^\lambda_{\beta\sigma} U_\lambda) = 0$,

$$\frac{dU_\sigma}{d\tau} = \Gamma^\lambda_{\beta\sigma} U^\beta U_\lambda = \frac{1}{2}(g_{\nu\beta,\sigma} + g_{\nu\sigma,\beta} - g_{\beta\sigma,\nu}) U^\beta U^\nu = \frac{1}{2} g_{\nu\beta,\sigma} U^\beta U^\nu \tag{3.23}$$

drop the other terms because g is antisymmetric in $\beta\nu$ while UU is symmetric in $\beta\nu$.

So if $\frac{\partial g_{\alpha\beta}}{\partial t} = 0$,

$$\frac{dU_0}{d\tau} = 0 \implies \frac{dU_0}{dt} = 0 \implies \gamma = \text{const} \implies v = \text{const}$$

hence

Theorem. *Kinetic energy of a free particle is conserved in coordinates x^α if*

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0 \tag{3.24}$$

Following the same logic, using (3.23), if $\frac{\partial g_{\alpha\beta}}{\partial x^i} = 0$, momentum

$$p_i = mU_i \tag{3.25}$$

is conserved.

3.6 Covariant Form of Conservation Laws

Killing Vectors

Suppose

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0 \tag{3.26}$$

in a coordinate system x^α. How can this fact be expressed in a covariant way, in other coordinate system x^{α^*}?

The answer is to write (3.26) as

$$\begin{aligned}\frac{\partial g_{\alpha\beta}}{\partial x^0} &= K^\mu \partial_\mu g_{\alpha\beta} + K^\mu{}_{,\alpha} g_{\mu\beta} + K^\mu{}_{,\beta} g_{\mu\alpha} \\ &= 0\end{aligned} \tag{3.27}$$

with

$$K^\mu = (1,0,0,0) \tag{3.28}$$

K^μ is the vector field that is tangent to the coordinate curves of x^0. Or say K^μ is $\frac{\partial}{\partial x^0}$.

Then $\frac{\partial g_{\alpha^*\beta^*}}{\partial x^{0^*}}$ is defined by using the tensor transformation law on (3.27) and (3.28). So first show (3.27) is covariant.

$$\begin{aligned}K^\mu{}_{,\alpha} &= \nabla_\alpha K^\mu - \Gamma^\mu_{\sigma\alpha} K^\sigma \\ K^\mu{}_{,\beta} &= \nabla_\beta K^\mu - \Gamma^\mu_{\sigma\beta} K^\sigma\end{aligned}$$

so

$$\begin{aligned}(3.27) &= K^\mu \partial_\mu g_{\alpha\beta} + \nabla_\alpha K_\beta - \Gamma^\mu_{\sigma\alpha} g_{\mu\beta} K^\sigma + \nabla_\beta K_\alpha - \Gamma^\mu_{\sigma\beta} g_{\mu\alpha} K^\sigma \\ &= K^\sigma \underbrace{(\partial_\sigma g_{\alpha\beta} - \Gamma^\mu_{\sigma\alpha} g_{\mu\beta} - \Gamma^\mu_{\sigma\beta} g_{\mu\alpha})}_{\nabla_\sigma g_{\alpha\beta}=0} + \nabla_\alpha K_\beta + \nabla_\beta K_\alpha\end{aligned}$$

showing (3.27) is a tensor. And

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 2\nabla_{(\alpha} K_{\beta)} = 0 \tag{3.29}$$

is called Killing equation. After knowing (3.27) is covariant and knowing (3.27) gives (3.26) when $K^\mu = (1,0,0,0)$, then we know (3.27) is the only tensor expression. That is because tensor with given components in one coordinate system is uniquely defined (clearly change component of a vector will not be the same vector any more.) then apply tensor transformation law (which is linear), so the extension of a tensor to all system is unique.

Conversely, if we find a Killing vector K^α, satisfying (3.29), we can construct its integral curves $x^\alpha(\lambda)$ defined by $\frac{dx^\alpha}{d\lambda} = K^\alpha$ and choose $\lambda = x^0$ as one of the coordinates, then we get a coordinate system where $K^\alpha =$(3.28), and $\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0 \implies p_0 = \text{const}$. Similarly choose $\lambda = x^i$ as one of the coordinates, we get $p_i = \text{const}$. The covariant form of conservation law is

$$U^{\beta^*}\nabla_{\beta^*}(K^{\alpha^*}p_{\alpha^*}) = 0$$

i.e. p_{α^*} is conserved along K^{α^*}.

Lie Derivatives

Lecture 12 (3/3/14)

Back to (3.27). It is called Lie derivative, $L_K g_{\alpha\beta}$ of $g_{\alpha\beta}$ along K^μ.

Lie derivative along a vector field X can be defined for any tensor $a_{\alpha\beta}$ using same formula

$$\begin{aligned} L_X a_{\alpha\beta} &= X^\mu a_{\alpha\beta,\mu} + X^\mu{}_{,\alpha} a_{\mu\beta} + X^\mu{}_{,\beta} a_{\mu\alpha} \\ &= X^\mu a_{\alpha\beta;\mu} + X^\mu{}_{;\alpha} a_{\mu\beta} + X^\mu{}_{;\beta} a_{\mu\alpha} \end{aligned}$$

terms with Christoffel symbols cancel. A similar definition can be designed for covariant tensors $a^{\alpha\beta}$

$$\begin{aligned} L_X a^{\alpha\beta} &= X^\mu a^{\alpha\beta}{}_{,\mu} - X^\alpha{}_{,\mu} a^{\mu\beta} - X^\beta{}_{,\mu} a^{\mu\alpha} \\ &= X^\mu a^{\alpha\beta}{}_{;\mu} - X^\alpha{}_{;\mu} a^{\mu\beta} - X^\beta{}_{;\mu} a^{\mu\alpha} \end{aligned} \tag{3.30}$$

the minus signs are chosen to achieve cancellation of Christoffel symbols. Recall commutator (3.2),

$$L_X A^\alpha = [X, A^\mu]^\alpha \text{ and } L_X A = -L_A X$$

We see that Lie derivative resembles the covariant derivative, instead of $\Gamma^\alpha_{\beta\mu}$ use $X^\alpha{}_{,\mu}$, (but unlike covariant derivative, Lie derivative does't increase rank and only defined along a specific X), so we can define Lie derivative of any tensor $L_X A^{\alpha_1,\ldots,\alpha_p}_{\beta_1,\ldots,\beta_q}$. In this way Lie derivative reduces to directional derivative when $X^\mu = \text{const}$.

Maximally Symmetric Space

A Killing vector field is associated with isometry: a transformation $x^\alpha \to x^{\alpha'}$ that leaves the inner product (metric tensor or $R_{\rho\sigma\mu\nu}$) invariant. For example, if $\frac{\partial}{\partial x^0}$ is Killing vector then transformation $(x^0, x^i) \to (x^0 + a, x^i)$ is isometry.

Different isometrics are represented by linearly independent Killing vector fields. A maximally symmetric n dimensional space has $\frac{n(n+1)}{2}$ linearly independent Killing vector fields. [For proof see Kobayashi, *Transformation Groups in Differential Geometry*, page 46, theorem 3.1]

There are

$$n \text{ translations}$$

and

$$\frac{n(n-1)}{2} \text{ rotations} \tag{3.31}$$

Physically a maximal symmetric Riemann space means $g_{\alpha\beta}$ is homogeneous (translational invariant) and isotropic (rotational invariant), then dropping all derivative terms (3.11),

$$R_{\rho\sigma\mu\nu} = C(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{3.32}$$

and $C = \frac{R}{n(n-1)} = \text{const}$, where

$$\text{Ricci scalar } R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^\lambda{}_{\mu\lambda\nu} = R^{\mu\lambda}{}_{\mu\lambda} \tag{3.33}$$

For those who know more about differential geometry, (3.32) implies Weyl tensor is 0.

Three cases all controlled by one parameter

$$\begin{aligned} R &= 0 \qquad \text{flat space } R_{\rho\sigma\mu\nu} = 0 \\ R &> 0 \qquad \text{sphere of dimen } n \\ R &< 0 \qquad \text{hyperboloid of dimen } n \end{aligned} \tag{3.34}$$

4 Dynamics of Particles & Fields in Curved Spacetime

The equivalence principle implies that equations of motion look the same as in the absence of curvature (gravity) if they are written in the local inertial coordinate (with $\Gamma^{\alpha}_{\beta\mu} = 0$) at a given point P. The equations state a relation between tensors and may by written in any other coordinate system using the tensor transformation law. In particular, partial derivatives become covariant derivatives around P.

	local inertial coordinates	general coordinate
free particle geodeics (3.21)	$\frac{dU^\alpha}{d\tau} = U^\beta \partial_\beta U^\alpha = 0$	$\frac{DU}{D\tau} = U^\beta \nabla_\beta U^\alpha = 0$
particle interacts EM field (2.25)	$mU^\beta \partial_\beta U^\alpha = \frac{e}{c} F^{\alpha\beta} U_\beta$	$mU^\beta \nabla_\beta U^\alpha = \frac{e}{c} F^{\alpha\beta} U_\beta$
EM field sourced by charges (2.35)	$\partial_\beta F^{\beta\alpha} = \frac{4\pi}{c} j^\alpha$	$\nabla_\beta F^{\beta\alpha} = \frac{4\pi}{c} j^\alpha$
energy-mom conservation (4.15)	$\partial_\beta T^{\alpha\beta} = 0$	$\nabla_\beta T^{\alpha\beta} = 0$

4.1 Stationary & Static Gravitational Fields

If there is a timelike Killing vector field, one can choose a coordinate system (ct, x^i) so that

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = \frac{1}{c} \frac{\partial g_{\alpha\beta}}{\partial t} = 0$$

such a gravitational field is called *stationary*. Suppose x^i are Lagrangian coordinates (co-moving frame) of observers everywhere in the 3D space (hypersurface for fixed t), And for each observer $dx^i = 0$ because of co-moving frame.

Proper time of observer

$$c^2 d\tau^2 = -ds^2 = -g_{\alpha\beta} dx^\alpha dx^\beta = -g_{00} dx^0 dx^0$$

therefore

$$d\tau = \sqrt{-g_{00}} dt \tag{4.1}$$

To understand the structure (the metric γ_{ij}) of the 3D space, put A, B two observers infinity close (so speed of light is constant), separation $dx^i_{A\to B}$ is fixed for some fixed i. A sends light to B, and B has a mirror that reflects light back to A. $(c\tau, x^i)$ refers to A's coordinate. In $A's$ view, light is constant speed

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = 0$$

or

$$g_{00}(dx^0)^2 + 2g_{0i}dx^0 dx^i + g_{ij}dx^i dx^j = 0$$

solve for dx^0

$$dx^0 = \frac{-g_{0i}dx^i \pm \sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ij}dx^i dx^j}}{g_{00}} \tag{4.2}$$

drop the + solution, because $g_{00} < 0$, + solution gives $dx^0 < 0$ time goes backwards.

Putting $dx^i_{A\to B} = dx^i$ and $dx^i_{B\to A} = -dx^i$, A finds the total time from $A \to B \to A$ is

$$dx^0 = \frac{-2\sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ij}dx^i dx^j}}{g_{00}} \tag{4.3}$$

so the physical distant between them is by (4.1)

$$dl = \sqrt{-g_{00}}\frac{dx^0}{2} = \frac{\sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ij}dx^i dx^j}}{\sqrt{-g_{00}}}$$

Since $dl^2 = \gamma_{ij}dx^i dx^j$, A finds

$$\gamma_{ij} = -\frac{g_{0i}g_{0j}}{g_{00}} + g_{ij} \tag{4.4}$$

which is g_{ij} the inverse of g^{ij}, $i, j = 1, 2, 3$. And

$$g = -g_{00}\gamma$$

where $g = \det g_{\alpha\beta}$ and $\gamma = \det \gamma_{ij} = \det g_{ij}$.

Initially A and B are on the same hypersurface, during the light travels from $A \to B \to A$, will A and B still on a same hypersurface? i.e. are the events light traveling from $A \to B \to A$ simultaneous to A and B?

If A and B are progressing at the same pace, from A's view the time takes for light from $A \to B$ should equal to the time takes from $B \to A$. So by (4.2) and (4.3)

$$\frac{-g_{0i}dx^i - \sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ij}dx^i dx^j}}{g_{00}} = \frac{-\sqrt{(g_{0i}dx^i)^2 - g_{00}g_{ij}dx^i dx^j}}{g_{00}}$$

or

$$g_{0i} = 0$$

Since i was initially chosen arbitrary, it should be true for all i.

We say if $g_{0i} = 0\ \forall i$ then $t =$ const hypersurface corresponds to simultaneous events. Spacetime in which one can choose x^α so that $\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0$ and $g_{0i} = 0$ are called static. And by (4.4)

$$\gamma_{ij} = g_{ij}$$

so the metric in such static spacetime is

$$ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^i dx^j \tag{4.5}$$

and there is a universal time x^0. Later we will say this x^0 is measured by a standard clock which all observers can compare to by (4.1).

HW4 problem 3 gives an example of stationary, non-static metric. Consider in flat 4D Minkowski spacetime, there is a rotating cylindrical 3D space with radius $R < c/\Omega$, Ω is angular speed. The static inertial observers has the usual coordinate (t, r, ϕ, z) and metric $ds^2 = -c^2dt^2 + dr^2 + r^2d\phi^2 + dz^2$. What about a co-moving observe rotating with the 3D space at $r < R$. His coordinate (t', r', ϕ', z')

$$t' = t \qquad r' = r \qquad \phi' = \phi + \Omega t \qquad z' = z$$

then

$$\begin{aligned} ds'^2 &= -c^2dt^2 + dr^2 + r^2(d\phi + \Omega dt)^2 + dz^2 \\ &= -c^2(1 - \frac{\Omega^2 r^2}{c^2})dt^2 + 2r^2\Omega d\phi dt + dr^2 + r^2d\phi^2 + dz^2 \end{aligned} \tag{4.6}$$

Conclusion:

1. It is stationary $\frac{\partial g_{\alpha\beta}}{\partial t} = 0$, we can follow the same logic after (4.1), we can get γ_{ij}.
2. It is non-static, so no universal clock, i.e. for two observes on the same radius r cannot compare their clocks, it could depends on not only r but also ϕ see (4.6).
3. In this space, $\Gamma^{\alpha}_{\beta\gamma} \neq 0$, because of outward acceleration/gravity
4. Although the 4D space is flat $R^{\alpha}{}_{\beta\gamma\delta} = 0$, the 3D space is curved.

Newtonian Limit of GR

Assume stationary, static gravitation field, and

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \underbrace{h_{\alpha\beta}}_{\ll 1}$$

almost flat. Consider a test particle moves with non-relativistic speed, $u^i \ll u^0$, or $u^\alpha = (c, 0, 0, 0)$ with $\gamma = 1$. $dt \approx d\tau$, geodesics becomes

$$\frac{d^2 x^\alpha}{dt^2} + \Gamma^{\alpha}_{00} c^2 = 0$$

and

$$\Gamma^{\alpha}_{00} = \frac{1}{2} \underbrace{g^{\alpha\beta}}_{\eta^{\alpha\beta}} (\underbrace{g_{0\beta,0}}_{0} + \underbrace{g_{\beta 0,0}}_{0} - g_{00,\beta}) \implies \Gamma^{0}_{00} = 0 \quad \Gamma^{i}_{00} = -\frac{1}{2} g_{00,i}$$

since $g_{00,\beta}$ is small, replace $g^{\alpha\beta}$ by $\eta^{\alpha\beta}$. Define

$$g_{00} = -1 - \frac{2\phi}{c^2} \qquad \left|\phi/c^2\right| \ll 1 \tag{4.7}$$

Then

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i}$$

equation of motion in the Newtonian gravitational field ϕ.

Lecture 13 (3/5/14)

In (3.24), we showed

$$\text{stationary} \implies \frac{dU^0}{dt} = 0 \implies \text{conservation of Kinetic energy}$$

Since we have static, we can go further by (4.5)

$$\begin{aligned} U^0 &= \frac{dt}{d\tau}c = \frac{dtc^2}{\sqrt{-ds^2}} = \frac{dtc^2}{\sqrt{-g_{00}c^2dt^2 + dl^2}} \\ &= \frac{c}{\sqrt{-g_{00}}\sqrt{1 - \frac{dl^2}{c^2(-g_{00}dt^2)}}} = \frac{c}{\sqrt{-g_{00}}\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c\gamma}{\sqrt{-g_{00}}} \end{aligned}$$

$$p^0 = mU^0 = \frac{mc\gamma}{\sqrt{-g_{00}}} = \text{const}$$

so is

$$p_0 = g_{0\alpha}p^\alpha = g_{00}p^0 = \text{const}$$

because of static.

Applying Newtonian limit $|\phi/c^2| \ll 1$, $v/c \ll 1$, by (4.7)

$$\begin{aligned} p_0 &= -\sqrt{-g_{00}}\gamma mc \\ &= -(1+\frac{\phi}{c^2})(1+\frac{v^2}{2c^2})mc = -(1+\frac{\phi}{c^2}+\frac{v^2}{2c^2})mc = \text{const} \end{aligned} \tag{4.8}$$

therefore we get energy conservation, combining with gravitational energy.

$$\phi + \frac{v^2}{2} = \text{const}$$

Gravitational Redshift

By (4.1)

$$d\tau = (1+\frac{\phi}{c^2})dt \tag{4.9}$$

At ∞, $\phi = 0$, that gives the reference clock, for $g_{00} = 1$ good orthonormal basis. Clock in gravitational field $\phi < 0$, $d\tau \downarrow$, clock goes slower. Go to a gravitating object to live longer or in the case of twin paradox going to an accelerating trip live longer.

If the test particle above is photon, we can modify (4.8)

$$-\underbrace{\sqrt{-g_{00}}\gamma mc^2}_{E} = \text{const along the geodesics} \tag{4.10}$$

so

$$E = \frac{E_\infty}{\sqrt{-g_{00}}} = (1 - \frac{\phi}{c^2})E_\infty = \hbar w$$

with E_∞ photon energy at ∞ hence the frequency measured by the reference clock at ∞. $E \uparrow$ blue shift when photon moves to stronger gravitational field, and redshift when it moves away from gravity field.

4.2 General Stress-Energy Tensor

As mentioned in the beginning of the lecture (0.1), we want to find the source, RHS of (0.1). The expression has to be tensorial. The source must be part of mass and energy, so it turns out to be the stress-energy tensor. Recall how energy was defined (2.9). We now follow the same derivation led to (2.9) but in a setting of general field. Use Minkowski, use $\mathcal{L}(q^A(x^\beta), {q^A}_{,\alpha}(x^\beta))$ and general field equation (2.37).

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\beta} &= \underbrace{\frac{\partial \mathcal{L}}{\partial x^\beta}}_{0} + \frac{\partial \mathcal{L}}{\partial q^A}\frac{\partial q^A}{\partial x^\beta} + \frac{\partial \mathcal{L}}{\partial {q^A}_{,\alpha}}\underbrace{\frac{\partial {q^A}_{,\alpha}}{\partial x^\beta}}_{\frac{\partial {q^A}_{,\beta}}{\partial x^\alpha}} \\
&= \underbrace{\frac{\partial \mathcal{L}}{\partial q^A}\frac{\partial q^A}{\partial x^\beta} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial {q^A}_{,\alpha}}\right)\frac{\partial q^A}{\partial x^\beta}}_{0\ (2.37)} + \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial {q^A}_{,\alpha}}\frac{\partial q^A}{\partial x^\beta}\right)
\end{aligned}$$

so

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial {q^A}_{,\alpha}}\frac{\partial q^A}{\partial x^\beta} - \mathcal{L}\delta^\alpha_\beta\right) = 0$$

So the generalized energy: stress-energy tensor

$$T^\alpha{}_\beta \equiv -\frac{\partial \mathcal{L}}{\partial {q^A}_{,\alpha}}\frac{\partial q^A}{\partial x^\beta} + \mathcal{L}\delta^\alpha_\beta + \frac{\partial {\psi^{\alpha\gamma}}_\beta}{\partial x^\gamma} \tag{4.11}$$

and

$$T^{\alpha}{}_{\beta,\alpha} = 0 \tag{4.12}$$

or

$$T^{\alpha\beta}{}_{,\alpha} = 0 \tag{4.13}$$

is conservation law. Gauge freedom: Arbitrary term of the form

$$\frac{\partial \psi^{\alpha\beta\gamma}}{\partial x^{\gamma}} \tag{4.14}$$

with $\psi^{\alpha\beta\gamma} = -\psi^{\gamma\beta\alpha}$ to be added to $T^{\alpha\beta}$ to make $T^{\alpha\beta}$ symmetric and still hold (4.13) be true.

Generalize to any coordinates

$$\nabla_{\alpha} T^{\alpha\beta} = 0 \tag{4.15}$$

Conservation of Energy

Note

$$T^{00} = -T^{0}{}_{0} = \frac{\partial \mathcal{L}}{\partial \dot{q}^{A}} \dot{q}^{A} - \mathcal{L} = E$$

gives the usual energy density.

From (4.13)

$$\frac{\partial T^{0\alpha}}{\partial x^{\alpha}} = \frac{1}{c} \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^{i}} = 0$$

define

$$S^{i} = cT^{0i}$$

$\vec{S}$ is energy flux, then

$$\frac{\partial T^{00}}{\partial t} + \nabla \cdot \vec{S} = 0$$

General form of a conservation law

$$\frac{\partial}{\partial t} (\text{density of } X) + \nabla \cdot (\text{flux of } X) = 0 \tag{4.16}$$

$$\text{flux of } X = \text{density of } X \times \text{velocity of the flow } \vec{v}$$

Conservation of Momentum

From (4.13)

$$\frac{\partial T^{i\alpha}}{\partial x^\alpha} = \frac{\partial}{\partial t}\left(\frac{T^{i0}}{c}\right) + \frac{\partial T^{ij}}{\partial x^j} = 0$$

define

$$\frac{T^{i0}}{c} = \text{momentum density}$$

dimensionally correct $pc = E$. Then by stokes we say

$$T^{ij} = \text{flux of } i\text{momentum in the } j\text{th dirction}$$

In sum

$$T^{\alpha\beta} = \begin{pmatrix} \text{energy density} & \begin{matrix}\text{energy flux}\\ \text{along } x\text{axis}/c\end{matrix} & \begin{matrix}\text{energy flux}\\ \text{along } y\text{axis}/c\end{matrix} & \begin{matrix}\text{energy flux}\\ \text{along } z\text{axis}/c\end{matrix} \\ \begin{matrix}\text{density}\\ x\text{ mom}\end{matrix} \cdot c & \begin{matrix}x\text{ mom flux}\\ \text{along } x\text{ axis}\end{matrix} & \begin{matrix}x\text{ mom flux}\\ \text{along } y\text{ axis}\end{matrix} & \begin{matrix}x\text{ mom flux}\\ \text{along } z\text{ axis}\end{matrix} \\ \begin{matrix}\text{density}\\ y\text{ mom}\end{matrix} \cdot c & \begin{matrix}y\text{ mom flux}\\ \text{along } x\text{ axis}\end{matrix} & \begin{matrix}y\text{ mom flux}\\ \text{along } y\text{ axis}\end{matrix} & \begin{matrix}y\text{ mom flux}\\ \text{along } z\text{ axis}\end{matrix} \\ \begin{matrix}\text{density}\\ z\text{ mom}\end{matrix} \cdot c & \begin{matrix}z\text{ mom flux}\\ \text{along } x\text{ axis}\end{matrix} & \begin{matrix}z\text{ mom flux}\\ \text{along } y\text{ axis}\end{matrix} & \begin{matrix}z\text{ mom flux}\\ \text{along } z\text{ axis}\end{matrix} \end{pmatrix} \tag{4.17}$$

Stress-Energy Tensor of EM Field in Vacuum

$$\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta}$$

By (4.11)

$$T^\alpha{}_\beta = \frac{1}{4\pi} F^{\alpha\mu} A_{\mu,\beta} - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \delta^\alpha_\beta$$

so

$$T^{\alpha\beta} = \frac{1}{4\pi} F^{\alpha\mu} A_{\mu,\nu} \eta^{\beta\nu} - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \eta^{\alpha\beta}$$

The first term on the right is not symmetric, to make it symmetric, we add

$$-\frac{1}{4\pi}F^{\alpha\mu}A_{\nu,\mu}\eta^{\beta\nu} = -\frac{1}{4\pi}\frac{\partial}{\partial x^\mu}(F^{\alpha\mu}A_\nu\eta^{\beta\nu}) + \frac{1}{4\pi}A_\nu\eta^{\beta\nu}\underbrace{\frac{\partial}{\partial x^\mu}(F^{\alpha\mu})}_{0\ \because \text{vacuum}}$$

and $\frac{\partial}{\partial x^\mu}(A_\nu\eta^{\beta\nu}F^{\alpha\mu})$ has the correct form as (4.14), so

$$T^{\alpha\beta} \to \frac{1}{4\pi}F^{\alpha\mu}F_{\nu\mu}\eta^{\beta\nu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\eta^{\alpha\beta}$$

One can also check it is traceless

$$T^\alpha{}_\alpha = \frac{1}{4\pi}F^{\alpha\mu}F_{\alpha\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\delta^\alpha_\alpha = 0$$

4.3 Stress-Energy Tensor for Matter

Dust

Consider the simplest form of matter dust in Minkowski space. Dust is a cold (no thermal motion) flow of particles with same 4-velocity $U^\alpha(x^\beta) = (c\gamma, \gamma\vec{v})$. m rest mass of each particle, n_{rest} proper number density of particles. In a given lab frame each particle has energy, cf (4.10)

$$E_{lab} = U^0 cm = \gamma mc^2$$

number density in lab frame

$$n_{lab} = \gamma n_{rest} = \frac{U^0}{c}n_{rest}$$

and

$$\rho_{lab} = \gamma\rho_{rest}$$

due to length contraction in the direction of motion. Energy density

$$w_{lab} = n_{lab}E_{lab} = \underbrace{n_{rest}m}_{\rho_{rest}}U^0U^0$$

Generalize to

$$T^{\alpha\beta} = \rho_{rest} U^\alpha U^\beta \tag{4.18}$$

This is a natural candidate for the source of gravity in GR. So

$$T^{00} = w_{lab} \qquad T^{0i} = w_{lab} v^i / c = T^{i0} = \gamma(\rho_{lab} v^i) \cdot c \qquad T^{ij} = \gamma(\rho_{lab} v^i v^j)$$

they have the same meaning in (4.17).

Lecture 14 (3/10/14)

First show it satisfies (4.16), i.e. we want to show

Conservation of energy

$$\frac{\partial}{\partial t}(\underbrace{w_{lab}}_{T^{00}}) + \partial_i(\underbrace{w_{lab} v^i}_{cT^{0i}}) = 0$$

so

$$\partial_0 T^{00} + \partial_i T^{0i} = 0$$

Conservation of momentum

$$\frac{\partial}{\partial t}(\underbrace{\text{density of } p^i}_{n_{lab} p^i = \underbrace{\rho_{lab} v^i}_{T^{i0}/c\gamma}}) + \partial_k(\underbrace{n_{lab} v^i v^k}_{T^{ij}/\gamma}) = 0$$

so

$$\partial_0 T^{i0} + \partial_k T^{ik} = 0$$

thus

$$\partial_\beta T^{\alpha\beta} = 0$$

One can also get conservation of number of particles, known as continuity equation

$$\frac{\partial}{\partial t}(\underbrace{n_{lab}}_{F^0/c}) + \partial_i(\underbrace{n_{lab} v^i}_{F^i}) = 0$$

with the definition of flux of flow $F^\alpha \equiv n_{rest} U^\alpha$, thus

$$\partial_\alpha F^\alpha = 0$$

Ideal Fluid with Non-zero Pressure

In the rest frame of the ideal fluid

$$T^{\alpha\beta} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \tag{4.19}$$

Ideal fluid means no viscosity, no conductivity, so no off diagonal elements. If $p = 0$, we get dust in its rest frame

$$T^{\alpha\beta} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Why is p pressure? Because of (4.17), flux of momentum is dimension of pressure. And it is the internal pressure from random thermal velocity.

$T^{\alpha\beta}$ in any other frame can be found by applying tensor transformation to (4.19)

$$T^{\alpha\beta} = (\rho + p)\frac{U^\alpha U^\beta}{c^2} + p g^{\alpha\beta} \tag{4.20}$$

check: 1) when $p = 0$, we get back (4.18). 2) in locally co-moving frame we have $U^\alpha = (c, 0, 0, 0)$, and $g^{\alpha\beta} = \eta^{\alpha\beta}$, then $T^{\alpha\beta}$ reduces (4.19). Following the remark after (3.29), we say (4.20) is the unique extension of (4.19) to all coordinate system.

For a more convincing argument why p is pressure, see HW4 problem 8, in the Newton limit, (4.20) gives Euler equation.

4.4 Einstein's Equation

Let $T_{\alpha\beta}$ be the source of Einstein equation, compare to Newtonian's gravity

$$\nabla^2\phi = 4\pi G\rho \tag{4.21}$$

what to be on the left? We know one tensor of rank $(0,2)$ that is made of second derivatives of $g_{\alpha\beta}$. Ricci tensor

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$$

However unlike the RHS, $\nabla^{\mu} R_{\mu\nu} \neq 0$. What is $\nabla^{\mu} R_{\mu\nu}$? We need Bianchi identity:

$$\nabla_{\mu} R_{\alpha\beta\gamma\delta} + \nabla_{\alpha} R_{\beta\mu\gamma\delta} + \nabla_{\beta} R_{\mu\alpha\gamma\delta} = 0 \tag{4.22}$$

Proof, using local inertial coordinates, $\Gamma = 0$ cf (3.16),

$$\begin{aligned}
\nabla_{\mu} R^{\alpha}{}_{\beta\gamma\delta} &= \partial_{\mu} R^{\alpha}{}_{\beta\gamma\delta} \\
&= \partial_{\mu}(\partial_{\gamma}\Gamma^{\alpha}_{\delta\beta} - \partial_{\delta}\Gamma^{\alpha}_{\delta\beta} + \Gamma^{\alpha}_{\gamma\lambda}\Gamma^{\lambda}_{\beta\delta} - \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\beta\gamma}) \\
&= \partial_{\mu}\partial_{\gamma}\Gamma^{\alpha}_{\delta\beta} - \partial_{\mu}\partial_{\delta}\Gamma^{\alpha}_{\delta\beta}
\end{aligned}$$

so

$$\begin{aligned}
\nabla_{\mu} R_{\alpha\beta\gamma\delta} &= g_{\alpha\nu}\nabla_{\mu} R^{\nu}{}_{\beta\gamma\delta} \\
&= \partial_{\mu}\partial_{\gamma}(\underbrace{g_{\alpha\nu}\Gamma^{\nu}_{\delta\beta}}_{[\delta\beta,\alpha]}) - \partial_{\mu}\partial_{\delta}(\underbrace{g_{\alpha\nu}\Gamma^{\nu}_{\delta\beta}}_{[\gamma\beta,\alpha]}) \\
&= \partial_{\mu}\partial_{\gamma}\frac{1}{2}(g_{\alpha\delta,\beta} + g_{\alpha\beta,\delta} - g_{\delta\beta,\alpha}) - \partial_{\mu}\partial_{\delta}\frac{1}{2}(g_{\gamma\alpha,\beta} + g_{\beta\alpha,\gamma} - g_{\gamma\beta,\alpha}) \\
&= \frac{1}{2}(g_{\alpha\delta,\beta\mu\gamma} - g_{\delta\beta,\alpha\mu\gamma} - g_{\gamma\alpha,\beta\mu\delta} + g_{\gamma\beta,\alpha\mu\delta})
\end{aligned}$$

Similarly find $\nabla_{\alpha} R_{\beta\mu\gamma\delta}$, $\nabla_{\beta} R_{\mu\alpha\gamma\delta}$, and they cancel. QED

Compute

$$\begin{aligned}
\nabla^\mu R_{\mu\nu} &= \nabla^\mu R^\alpha{}_{\mu\alpha\nu} \\
&= \nabla_\mu R^{\alpha\mu}{}_{\alpha\nu} \\
&= \nabla_\mu R_{\alpha\nu}{}^{\alpha\mu} \\
&= -\underbrace{\underbrace{\underbrace{\nabla_\alpha R_{\nu\mu}{}^{\alpha\mu}}_{\nabla^\alpha R_{\nu\mu\alpha}{}^{\mu}}}_{\nabla^\alpha R^{\mu\alpha}{}_{\mu\nu}}}_{\nabla^\alpha R^\alpha{}_\nu} - \nabla_\nu \underbrace{R_{\mu\alpha}{}^{\alpha\mu}}_{-R}
\end{aligned}$$

using (3.12), (3.13), (3.14) and (4.22).

So

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R = \frac{1}{2}\nabla^\mu (R g_{\mu\nu})$$

or

$$\nabla^\mu \underbrace{(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu})}_{G_{\mu\nu}} = 0$$

$G_{\mu\nu}$ called Einstein tensor. Einstein equation reads

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

κ is a proportionality constant. This turns out to be correct description of metric that corresponds to motion of planets, lights and expansion of universe.

To find κ we go to the simplest system, dust, in Newton limit:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \qquad h_{\alpha\beta} \ll \eta_{\alpha\beta},\ h_{00} = -\frac{2\phi}{c^2} \qquad v/c \ll 1,\ \gamma = 1$$

and assume stationary metric

$$\frac{\partial g_{\alpha\beta}}{\partial t} = 0 \tag{4.23}$$

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

First take trace of Einstein equation

$$R^{\alpha}{}_{\alpha} - \frac{1}{2}\delta^{\alpha}_{\alpha}R = \kappa T^{\alpha}{}_{\alpha} \implies -R = \kappa T \tag{4.24}$$

substituting back to Einstein equation

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}) \tag{4.25}$$

and0

$$T = T^{0}{}_{0} = \underbrace{g_{00}}_{\approx -1} T^{00} = -\rho c^2$$

so (4.25) gives

$$R_{00} = \frac{1}{2}\kappa\rho c^2$$

On the other hand

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = \partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\beta \Gamma^{\mu}_{\alpha\mu} + \Gamma\Gamma - \Gamma\Gamma$$

The last two terms on the right are second order in $h_{\alpha\beta}$, so drop them.

$$\begin{aligned} R_{00} &= \partial_\mu \Gamma^{\mu}_{00} - \underbrace{\partial_0 \Gamma^{\mu}_{0\mu}}_{0 \because (4.23)} \\ &= \partial_\alpha (\frac{1}{2} g^{\alpha\beta} (\underbrace{g_{0\beta,0} + g_{0\beta,0}}_{0 \because (4.23)} - g_{00,\beta})) \\ &= \frac{1}{2}\eta^{ij}\partial_i\partial_j(-g_{00}) = \nabla^2 \frac{\phi}{c^2} \end{aligned}$$

thus

$$\frac{1}{2}\kappa\rho c^2 = \nabla^2 \frac{\phi}{c^2}$$

so compare to (4.21),

$$\kappa = \frac{8\pi G}{c^4}$$

4.5 Hilbert Action

Lecture 15 (3/12/14)

Normally the scalar Lagrangian is constructed from the first derivatives of the field, which gives second order field equation after variation. For $g_{\alpha\beta}$ however it's not possible to construct a scalar from $g_{\alpha\beta,\mu}$. because $g_{\alpha\beta,\mu} = 0$ in locally inertial frame. Let's try a scalar Lagrangian made linearly from $g_{\alpha\beta,\mu\nu}$ and show it returns the correct field equation.

Try Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$, cf (3.33), (3.3)

$$S_g = \alpha \int_D R\sqrt{-g}d^4x \tag{4.26}$$

α is a constant. Vary

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \tag{4.27}$$

then

$$\begin{aligned}\delta S_g &= \alpha \int_D \left(\delta g^{\mu\gamma}R_{\mu\gamma}\sqrt{-g} + R\delta\sqrt{-g} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}\right)d^4x \\ &= \alpha \int_D (R^{\nu}{}_{\gamma}\underbrace{g_{\mu\nu}\delta g^{\mu\gamma}}_{A} + R\underbrace{\frac{\delta\sqrt{-g}}{\sqrt{-g}}}_{B} + g^{\mu\nu}\delta R_{\mu\nu})\sqrt{-g}d^4x \end{aligned} \tag{4.28}$$

Since $g_{\mu\nu}g^{\mu\gamma} = \delta^{\gamma}_{\nu} \implies g_{\mu\nu}\delta g^{\mu\gamma} + \delta g_{\mu\nu}g^{\mu\gamma} = 0$

$$A = g_{\mu\nu}\delta g^{\mu\gamma} = -\delta g_{\mu\nu}g^{\mu\gamma} \tag{4.29}$$

By (2.21)

$$B = \frac{\delta\sqrt{-g}}{\sqrt{-g}} = \frac{1}{2}\frac{\delta g}{g} = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu} \tag{4.30}$$

so

$$\delta S_g = \alpha \int_D (\underbrace{-R^{\nu\mu}\delta g_{\mu\nu} + \frac{1}{2}Rg^{\mu\nu}\delta g_{\mu\nu}}_{-G^{\nu\mu}\delta g_{\mu\nu}} + g^{\mu\nu}\delta R_{\mu\nu})\sqrt{-g}d^4x$$

It turns out that $g^{\mu\nu}\delta R_{\mu\nu}$ can be written as divergence of a vector field. Using locally inertial coordinates

$$g^{\mu\nu}\delta R^{\gamma}{}_{\mu\gamma\nu} = g^{\mu\nu}\delta(\partial_\gamma \Gamma^{\gamma}_{\mu\nu} - \partial_\nu \Gamma^{\gamma}_{\mu\gamma} + \Gamma\Gamma - \Gamma\Gamma)$$

The last two terms go to 0, since $\delta\Gamma\Gamma = \Gamma\delta\Gamma$, because $\Gamma = 0$. So

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= \partial_\gamma(g^{\mu\nu}\delta\Gamma^{\gamma}_{\mu\nu}) - \partial_\nu(g^{\mu\nu}\delta\Gamma^{\gamma}_{\mu\gamma}) \\ &= \partial_\gamma(g^{\mu\nu}\delta\Gamma^{\gamma}_{\mu\nu} - g^{\mu\gamma}\delta\Gamma^{\nu}_{\mu\nu}) \end{aligned}$$

That $g^{\mu\nu}\delta\Gamma^{\gamma}_{\mu\nu} - g^{\mu\gamma}\delta\Gamma^{\nu}_{\mu\nu}$ is a tensor, although Christoffel symbol is not a tensor.

Consider varying (4.27), covariant derivatives of a vector field A^α in the old and new metrics are

$$\nabla_\beta A^\alpha = A^\alpha{}_{,\beta} + \Gamma^{\alpha}_{\beta\gamma}A^\gamma$$

$$\tilde{\nabla}_\beta A^\alpha = A^\alpha{}_{,\beta} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}A^\gamma$$

then

$$\nabla_\beta A^\alpha - \tilde{\nabla}_\beta A^\alpha = (\underbrace{\Gamma^{\alpha}_{\beta\gamma} - \tilde{\Gamma}^{\alpha}_{\beta\gamma}}_{\delta\Gamma^{\alpha}_{\beta\gamma}})A^\gamma$$

showing $\delta\Gamma^{\alpha}_{\beta\gamma}$ is a tensor, so $g^{\mu\nu}\delta\Gamma^{\gamma}_{\mu\nu} - g^{\mu\gamma}\delta\Gamma^{\nu}_{\mu\nu} = V^\gamma$ is a tensor (vector).

This was proved in locally inertial frame, in other frame, cf (2.23)

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\gamma(\sqrt{-g}V^\gamma)$$

so

$$\int_D (g^{\mu\nu}\delta R_{\mu\nu})\sqrt{-g}d^4x = \int_{\partial D} \sqrt{-g}V^\gamma d^3S = 0$$

for $\delta g_{\mu\nu} = 0$, $\delta\Gamma = 0$ at ∂D. Therefore

$$\delta S_g = \alpha\int_D (-G^{\nu\mu}\delta g_{\mu\nu})\sqrt{-g}d^4x$$

EOM

$$G_{\mu\nu} = 0 \tag{4.31}$$

Actually the derivation will be simpler if instead of (4.27), we vary (although the field dynamics variable is $g_{\mu\nu}$)

$$g^{\mu\nu} \to \tilde{g}^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu}$$

then in (4.28), we only have to simplify B, (4.30)

$$B = \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu}$$

then

$$\delta S_g = \alpha \int_D [\underbrace{(R_{\mu\nu} - \frac{R}{2} g_{\mu\nu})}_{G_{\mu\nu}} \delta g^{\mu\nu} + \underbrace{g^{\mu\nu} \delta R_{\mu\nu}}_{\partial_\gamma V^\gamma}] \sqrt{-g} d^4x \tag{4.32}$$

We get immediately

$$G_{\mu\nu} = 0$$

4.6 Action with Matter

Put in matter

$$S = S_g + S_m = \alpha \int_D R \sqrt{-g} d^4x + \int_D \mathcal{L} \sqrt{-g} \frac{d^4x}{c}$$

Following the remark after (2.37), to find the source term, we compute $\frac{\delta \mathcal{L}}{\delta g^{\alpha\beta}}$, here we use $g^{\alpha\beta}$ instead of $g_{\alpha\beta}$ because it is simpler see remark after (4.31)

$$\begin{aligned} \delta S_m &= \int_D \left(\frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{\alpha\beta}} \delta g^{\alpha\beta} - \frac{\partial \mathcal{L}\sqrt{-g}}{\partial {g^{\alpha\beta}}_{,\gamma}} \delta {g^{\alpha\beta}}_{,\gamma} \right) \frac{d^4x}{c} \\ &= \int_D \left(\frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{\alpha\beta}} - \frac{\partial}{\partial x^\gamma} \frac{\partial \mathcal{L}\sqrt{-g}}{\partial {g^{\alpha\beta}}_{,\gamma}} \right) \delta g^{\alpha\beta} \frac{d^4x}{c} \end{aligned}$$

by (2.36). Thus by (4.32)

$$\delta S_m + \delta S_g = \int_D \left(\alpha c G_{\alpha\beta} + \frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{\alpha\beta}} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\gamma} \frac{\partial \mathcal{L}\sqrt{-g}}{\partial {g^{\alpha\beta}}_{,\gamma}} \right) \delta g^{\alpha\beta} \sqrt{-g} \frac{d^4x}{c} \tag{4.33}$$

It turns out defining

$$T_{\alpha\beta} = -2\left(\frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}\sqrt{-g}}{\partial g^{\alpha\beta}} - \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\gamma}\frac{\partial\mathcal{L}\sqrt{-g}}{\partial {g^{\alpha\beta}}_{,\gamma}}\right) \tag{4.34}$$

agrees with the old definition (4.11), and it is already symmetric. Then Einstein equation

$$\alpha c G_{\alpha\beta} = \frac{1}{2}T_{\alpha\beta}$$

so

$$\alpha = \frac{c^3}{16\pi G}$$

Lecture 16 (3/31/14)

Does (4.34) satisfy conservation of stress tensor, (4.15)? This is a more advanced exercise of applying Noether in GR than the one we did for (3.24).

Since S_m is a scalar, it is invariant under coordinate transformations. Consider two coordinate systems x^α (locally inertial) and $\tilde{x}^\alpha$ such that

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha$$

where ξ^α is some vector field, and $\xi^\alpha \equiv 0$ at ∂D. Let $g^{\alpha\beta}$, $\tilde{g}^{\alpha\beta}$ be the metrics in x^α and $\tilde{x}^\alpha$, and let

$$\hat{g}^{\alpha\beta}(x) = \tilde{g}^{\alpha\beta}(\tilde{x}-\xi)$$

then change of variables

$$\begin{aligned} S_m &= \int_D \mathcal{L}(\tilde{x})\sqrt{-\tilde{g}}\frac{d^4\tilde{x}}{c} = \int_D \mathcal{L}(\tilde{x}-\xi)\sqrt{-\tilde{g}(\tilde{x}-\xi)}\frac{d^4(\tilde{x}-\xi)}{c} \\ &= \int_D \mathcal{L}(x)\sqrt{-\hat{g}}\frac{d^4x}{c} \end{aligned}$$

This means mathematically changing

$$g^{\alpha\beta}(x) \to \hat{g}^{\alpha\beta}(x) = g^{\alpha\beta} + \delta g^{\alpha\beta}$$

S_m stays the same. And we know how δS_m responds to $\delta g^{\alpha\beta}$, cf (4.33)

$$\delta S_m = \int_D -\frac{1}{2}T_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}\frac{d^4x}{c}$$

Does this mean that $\delta S_m = 0 \implies T_{\alpha\beta} \equiv 0$? No, because here $\delta g^{\alpha\beta}$ is not completely arbitrary (except fixing at boundary), if it were, it would have 10 independent functions. Here $\delta g^{\alpha\beta}$ is induced by 4 arbitrary functions ξ^α only.

$$\begin{aligned}\tilde{g}^{\alpha\beta}(\tilde{x}) &= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu} g^{\mu\nu}(x) = (\delta^\alpha_\mu + \xi^\alpha{}_{,\mu})(\delta^\beta_\nu + \xi^\beta{}_{,\nu}) g^{\mu\nu}(x) \\ &= g^{\alpha\beta}(x) + \xi^\alpha{}_{,\mu} g^{\mu\beta}(x) + \xi^\beta{}_{,\nu} g^{\alpha\nu}(x) \end{aligned} \tag{4.35}$$

So the trick is to somehow replacing $\delta g^{\alpha\beta}$ by ξ^4.

$$\delta g^{\alpha\beta} = \tilde{g}^{\alpha\beta}(\underbrace{\tilde{x} - \xi}_{x}) - g^{\alpha\beta}(x) = \tilde{g}^{\alpha\beta}(\tilde{x}) - \underbrace{\tilde{g}^{\alpha\beta}{}_{,\mu}(\tilde{x})}_{g^{\alpha\beta}{}_{,\mu}(x)}\xi^\mu - g^{\alpha\beta}(x) \tag{4.36}$$

using (4.35), in 1st order $\left.\frac{\partial \tilde{g}^{\alpha\beta}}{\partial \tilde{x}}\right|_{\tilde{x}} \approx \left.\frac{\partial g^{\alpha\beta}}{\partial x}\right|_{x}$.

Next use geodesics for $g_{\alpha\beta}$,

$$\xi^\sigma g^{\mu\nu}{}_{;\sigma} = 0$$

that is

$$\xi^\sigma g^{\mu\nu}{}_{,\sigma} + \underbrace{\Gamma^\mu_{\sigma\lambda}\xi^\sigma}_{\xi^\mu{}_{;\lambda} - \xi^\mu{}_{,\lambda}} g^{\lambda\nu} + \underbrace{\Gamma^\nu_{\sigma\lambda}\xi^\sigma}_{\xi^\nu{}_{;\lambda} - \xi^\nu{}_{,\lambda}} g^{\mu\lambda} = 0$$

thus we get

$$\begin{aligned}\delta g^{\alpha\beta} &= \xi^\alpha{}_{,\mu} g^{\mu\beta}(x) + \xi^\beta{}_{,\nu} g^{\alpha\nu}(x) - g^{\alpha\beta}{}_{,\mu}(x)\xi^\mu \\ &= -(\underbrace{\xi^{\alpha;\beta} + \xi^{\beta;\alpha}}_{L_\xi g^{\alpha\beta}}) = -2\xi^{(\alpha;\beta)} \end{aligned} \tag{4.37}$$

cf Lie derivative (3.30).

Since $T_{\alpha\beta}$ is too symmetric wrt $\alpha\beta$

$$\begin{aligned}
\delta S_m &= \int_D T_{\alpha\beta}\xi^{(\alpha;\beta)}\sqrt{-g}\frac{d^4x}{c} \\
&= \int_D T_{\alpha\beta}\xi^{\alpha;\beta}\sqrt{-g}\frac{d^4x}{c} \\
&= \int_D [(T_{\alpha\beta}\xi^{\alpha})^{;\beta} - T_{\alpha\beta}{}^{;\beta}\xi^{\alpha}]\sqrt{-g}\frac{d^4x}{c} \\
&= \int_D T_{\alpha\beta}{}^{;\beta}\xi^{\alpha}\sqrt{-g}\frac{d^4x}{c} \\
&= \int_D T^{\alpha\beta}{}_{;\beta}\xi_{\alpha}\sqrt{-g}\frac{d^4x}{c}
\end{aligned}$$

The third line uses the Gauss theorem and $\xi^{\alpha} = 0$ at ∂D. Therefore

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{4.38}$$

In other words not only conservation of stress tensor was built in Einstein equation, but also it was built in the formalism of principle of least action.

5 Schwarzschild Solutions & Black Holes

5.1 Spherically Symmetric Spacetime

The simplest situation to study Einstein equation is to consider a spherical massive M object sitting in an empty space. Then solve $g_{\alpha\beta}$ from Einstein equation. $g_{\alpha\beta}$ got to be spherical symmetric. This problem approximates our solar system.

Although it says spherically symmetric spacetime, it doesn't mean a S^3 topological sphere. It means $g_{\alpha\beta}$ is a foliation of S^2 when fixing x^0, x^1. Its Killing vector fields correspond to rotations in xy, yz, zy planes in Euclidean space. Such metric can be denoted by 3D spherical coordinates writing $x^2 = \theta$ and $x^3 = \phi$ so

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01}dx^0dx^1 + 2g_{0\psi}dx^0dx^{\psi} + g_{11}(dx^1)^2 + 2g_{1\psi}dx^1dx^{\psi} + K(x^0, x^1)d\Omega^2 \tag{5.1}$$

$\psi = 2, 3.$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Next show in above

$$g_{0\psi} = g_{1\psi} = 0 \tag{5.2}$$

Because by definition fixing x^0, x^1, varying Ω, we get a 2D tangent space which is a S^2, and moving to a different pair of x^0, x^1, we get another S^2. All S^2 are concentric, meaning that $\vec{e}_0$, $\vec{e}_1$ will always perpendicular to all tangent space S^2. This proves (5.2).

Next show

$$g_{01} = 0 \quad K(x^0, x^1) = (x^1)^2 \tag{5.3}$$

That is because at this point x^0, x^1 are living in an arbitrary 2D space tangent to S^2, we can choose $\vec{e}_0$, $\vec{e}_1$ to satisfy (5.3), and denote $x^0 \to t, \quad x^1 = r$. Since x^0, x^1 are living on a space tangent to S^2 and S^2 is spherical symmetric, g_{tt}, g_{rr} cannot depend on Ω.

Therefore

$$ds^2 = -e^{2\alpha(r,t)} dt^2 c^2 + e^{2\beta(r,t)} dr^2 + r^2 d\Omega^2 \tag{5.4}$$

The reason we chose $g_{01} = 0$ is to make the metric static, so of course t here we mean time. The exponential emphasizes $g_{tt} < 0$, $g_{rr} > 0$ as usual time like/null like separation ds.

5.2 Schwarzschild Metric

$$g_{\alpha\beta} = \begin{pmatrix} -e^{2\alpha} & & & \\ & e^{2\beta} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

Solve for α,β in (5.4) for space outside of M, so by(4.24)

$$G_{\alpha\beta} = R_{\alpha\beta} = 0 \tag{5.5}$$

Calculate $\Gamma^{\alpha}_{\beta\gamma}$, $R^{\mu}{}_{\nu}$. Non-zero $\Gamma^{\alpha}_{\beta\gamma}$.

Since $g_{\alpha\beta}$ is diagonal, $\Gamma^{\alpha}_{\beta\gamma}$ may be non-zero only if it has two equal indices.

$$\Gamma^t_{tt} = \dot{\alpha} \quad \Gamma^t_{tr} = \alpha' \quad \Gamma^t_{rr} = e^{2(\beta-\alpha)}\dot{\beta} \quad \Gamma^t_{tt} = \dot{\alpha}$$

$$\Gamma^t_{tr} = \alpha' \quad \Gamma^t_{rr} = e^{2(\beta-\alpha)}\dot{\beta}$$

$$\Gamma^r_{\phi\phi} = -e^{2\beta} r \sin^2\theta \quad \Gamma^r_{tt} = \dot{\alpha} e^{2(\alpha-\beta)} \quad \Gamma^r_{rr} = \beta' \quad \Gamma^r_{\theta\theta} = -e^{2\beta} r \quad \Gamma^r_{rt} = \dot{\beta}$$

$$\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \quad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{5.6}$$

denote $\dot{\alpha} = \partial_t \alpha$, $\alpha' = \partial_r \alpha$.

Next compute some components of Ricci tensor

$$R_{tr} = \frac{2}{r}\dot{\beta} = 0 \implies \beta = \beta(r)$$

$$R_{\theta\theta} = e^{-2\beta}[(\beta' - \alpha')r - 1] + 1 = 0 \tag{5.7}$$

Taking $\frac{\partial}{\partial t}$ of above

$$e^{-2\beta}(-\dot{\alpha}' r) = 0 \implies \dot{\alpha}' = 0$$

so

$$\alpha = f(r) + g(t)$$

Rescale time

$$t' = \int e^{g(t)} dt$$

so $dt' = e^g dt$ or $(dt')^2 = e^{2g}(dt)^2$, so we absorb t dependence of α to t' then

$$ds^2 = -e^{2\alpha(r)} dt^2 c^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \tag{5.8}$$

Hence the metric is time-independent, in other words, $\frac{\partial}{\partial t}$ is a Killing vector. We prove this fact from a sole condition spherical symmetric gravitation field with source M. Following the logic above we may conclude that the spacetime is static even if the source is a pulsating star (central sphere of oscillating radius), because in the Newton's limit it doesn't care as long as the center of mass is static at the origin. However later cf (7.8) we will show that is not true.

Next

$$\begin{aligned} R_{tt} &= e^{2(\alpha-\beta)}\left((\alpha')^2 - \alpha'\beta' + \alpha'' + \frac{2}{r}\alpha'\right) = 0 \\ R_{rr} &= -\alpha'' + \alpha'\beta' - (\alpha')^2 + \frac{2}{r}\beta' = 0 \end{aligned}$$

combine above

$$R_{tt} = e^{2(\alpha-\beta)}\left(\frac{2}{r}\beta' + \frac{2}{r}\alpha'\right) = 0$$

so

$$\alpha + \beta = \text{const}$$

setting the constant to be 0 by rescaling t or r.

$$\alpha = -\beta$$

Substituting to (5.7)

$$R_{\theta\theta} = e^{2\alpha}(-2\alpha' r - 1) + 1 = 0$$

or

$$\partial_r(re^{2\alpha}) = 1 \implies re^{2\alpha} = r - C$$

or

$$e^{2\alpha} = 1 - \frac{C}{r}$$

We know at large r, gravitational field is weak so

$$g_{tt} = -1 - \frac{2\phi}{c^2} = -1 + \frac{2GM}{rc^2} \implies C = \frac{2GM}{c^2} \equiv r_g \tag{5.9}$$

known as Schwarzschild radius. Sun has a Schwarzschild radius of 3 km.

Therefore we get the Schwarzschild solution outside the central mass in vacuum. As we have proven, this solution is unique. This is known as Birkhoff's theorem.

$$g_{\alpha\beta} = \begin{pmatrix} -1+\frac{r_g}{r} & & & \\ & \frac{1}{1-\frac{r_g}{r}} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \tag{5.10}$$

Note that (5.10) is an orthogonal but not normalized coordinate. If it were normalized it would be $diag(-1, 1, r^2, r^2 \sin\theta)$, which is what looks like for observer at ∞. So for example, a local static observer at r measuring radial displacement Δr (coordinate displacement or physical displacement wrt observer at ∞). Because

$$g^{rr}\vec{e}_r\vec{e}_r = \vec{e}_{\hat{r}}\vec{e}_{\hat{r}} = 1$$

$\vec{e}_{\hat{r}}$ is an actually unit basis vector should be to the local observer,

$$\vec{e}_{\hat{r}} = \frac{\vec{e}_r}{\sqrt{g_{rr}}}$$

so the actual measured value obtained by the local observer

$$\Delta\hat{r} = \sqrt{g_{rr}}\Delta r \tag{5.11}$$

similarly if the local observer measures time, cf (4.1).

Another thing about (5.10)is that if the central mass is smaller than r_g, then the Schwarzschild metric can go to region $r < r_g$, interesting thing will happen.

Also note that (5.5) doesn't imply the space is flat, for Riemann tensor isn't 0. (HW5 problems 1 & 5) For example

$$R^{\phi}{}_{rr\phi} = \partial_r\Gamma^{\phi}_{\phi r} - \partial_{\phi}\Gamma^{\phi}_{rr} + \Gamma^{\phi}_{r\lambda}\Gamma^{\lambda}_{\phi r} - \Gamma^{\phi}_{\phi\lambda}\Gamma^{\lambda}_{rr} = -\frac{\beta'}{r} = \frac{1}{1-\frac{r_g}{r}}\frac{r_g}{2r^3}$$

We can use (3.22) to compute the tidal force. Suppose one is falling radially $T = (0, 1, 0, 0)$, measured by local orthonormal inertial frame, and we are interested in tidal force in tangential direction $S = (0, 0, 0, 1)$. Since A^α, S^α appear in both sides of (3.22) the normalization will cancel (in other words one is covariant, the other is contravariant), but we need to normalize T, so

$$R^{\phi}{}_{rr\phi} \to R^{\phi}{}_{\hat{r}\hat{r}\phi} = \frac{R^{\phi}{}_{rr\phi}}{(\sqrt{g_{rr}})^2} = \frac{r_g}{2r^3} \tag{5.12}$$

which gives the tidal acceleration in tangential direction, measured in the local orthonormal inertial frame.

Furthermore (5.5) doesn't contradict (3.34). Because in (3.34) not only $R = 0$, but also the space is homogeneous, hence black hole is placed at the origin so not homogeneous.

5.3 Dynamics of Particles in Schwarzschild

Lecture 17 (4/2/14)

Motion of planets and propagation of light in the solar system follow the geodesics (3.9) of Schwarzschild spacetime if we ignore the interaction between planets. It has 4 Killing vectors: $\frac{\partial}{\partial t}$ gives energy conservation and the three Killing vectors of sphere S^2, cf remark above (5.1), give conservation of momentum.

From one of the angular momentum symmetric any geodesic lies in a plane passing the origin, we choose test particle ($m \neq 0$) initial conditions $U^\theta = 0$, $\theta = \pi/2$. Then by

$$\frac{dU^\theta}{d\lambda} = -\Gamma^\theta_{\mu\nu} U^\mu U^\nu = -\frac{2}{r} U^r \underbrace{U^\theta}_{0} - \sin\theta \underbrace{\cos\theta}_{0} U^\phi U^\phi = 0$$

showing staying in the plane.

Moreover by (3.25)

$$U_t = \text{const} \equiv -\frac{E}{c} = -\text{some constant energy}/c \tag{5.13}$$

$$U_\phi = \text{const} \equiv l = \text{some constant angular momentum} \tag{5.14}$$

Of course U_r is not a constant of time, but we can get a relation between U_r, E and l.

Since $U_\alpha U^\alpha = -c^2$ massive particle, and

$$U^t = \frac{U_t}{g_{tt}} = \frac{E/c}{1-\frac{r_g}{r}} \quad U^\phi = \frac{U_\phi}{g_{\phi\phi}} = \frac{l}{r^2}, \tag{5.15}$$

showing E is the energy wrt its value at $r = \infty$, because $g_{tt} \to 1$ at ∞.

So

$$-\frac{(E/c)^2}{1-\frac{r_g}{r}} + \frac{l^2}{r^2} + \frac{(U^r)^2}{1-\frac{r_g}{r}} = -c^2$$

we get

$$\frac{1}{2}(U^r)^2 + V(r) = \frac{1}{2}(\frac{E}{c})^2 \tag{5.16}$$

where

$$V(r) = \frac{1}{2}(1-\frac{r_g}{r})(\frac{l^2}{r^2}+c^2) \tag{5.17}$$

This is kind of Newton's law of 1D motion, because take $\frac{d}{dr}$ of (5.16)

$$\frac{dU^r}{d\tau} = U^r\frac{dU^r}{dr} = -\frac{dV}{dr}$$

so $V(r)$ is called effective potential for radial motion.

Newtonian Limit

Keep first order in r_g/r and v^2/c^2, then by (5.9)

$$V(r) = \frac{1}{2}\left((1-\frac{r_g}{r})c^2 + \frac{l^2}{r^2}\right) = \frac{c^2}{2}\underbrace{-\frac{GM}{r} + \frac{(v_\phi)^2}{2}}_{V_{newton}}$$

where $v_\phi = l/r$. And (5.16) becomes

$$\frac{1}{2}(U^r)^2 + V_{newton}(r) = \frac{1}{2}(\frac{E}{c})^2 - \frac{c^2}{2} = E_{newton} \tag{5.18}$$

and

$$V_{newton}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2}$$

is well-known curve that $V \to 0$ at $r \to \infty$, and $V \to \infty$ at $r \to 0$, and has one stable minimum at $r = \frac{l^2}{GM}$, at which the orbit is circular.

Parabolic Orbit

Now back to relativistic $V(r)$, (5.17). It is valid for $r_g < r < \infty$. $V \to c^2/2$ at $r \to \infty$, and for large enough l, it has clearly two extremes. One unstable maximum, and one stable minimum. For energy in between $c^2/2$ and V_{max}, the orbit is not closed, "hyperbolic" if it were purely V_{newton}. For energy in between $c^2/2$ and V_{min} the orbit is closed, "ellipse" if it were purely $V_{newton.}$ The critical point when

$$\frac{1}{2}(\frac{E}{c})^2 = \frac{c^2}{2} \implies E = c^2 \tag{5.19}$$

is called parabolic orbit. And at which by (5.18) $E_{newton} = 0$, consist with the definition of parabolic orbit in Newton gravity, and that is the reason it is still called. The turning point, called pericenter, of the orbit where $U^r = 0$, can be solved by

$$V(r_{pericenter}) = \frac{1}{2}(1 - \frac{r_g}{r})(\frac{l^2}{r^2} + c^2) = c^2$$

Putting dimensionless $\tilde{x} = r/r_g$, $\tilde{l} = l/r_g c$, we get

$$\tilde{x} = \frac{\tilde{l}^2}{2}(1 \pm \sqrt{1 - \frac{4}{\tilde{l}^2}})$$

From the graph (5.17), only the + makes sense. And from above we see why we need l to be large enough for such orbit to exist.

Circular Orbit

From (5.17), compute $dV/dr = 0$ we can find the two extremes

$$\tilde{x} = \tilde{l}^2 \pm \sqrt{\tilde{l}^4 - 3\tilde{l}^2}$$

or implicitly

$$\tilde{l} = \frac{\tilde{x}}{\sqrt{2\tilde{x} - 3}}$$

showing that no solution for $\tilde{x} < 3/2$ or no $r < 3r_g/2$ circular orbit.

5.4 Procession of Mercury Orbit

Lecture 18 (4/7/14)

Let us see ellipse orbits are not perfect ellipses. Deriving the equation of orbit $r(\phi)$. Since $dr/d\phi = U^r/U^\phi$, by (5.15), (5.16)

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{l^2}\left(\frac{E^2}{c^2} - (1 - \frac{r_g}{r})(\frac{l^2}{r^2} + c^2)\right)$$

It turns out to be simpler to use

$$\left(\frac{d}{d\phi}\frac{1}{r}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2$$

Putting dimensionless $\tilde{u} = r_g/r$, $\tilde{E} = E/c^2$,

$$\left(\frac{d\tilde{u}}{d\phi}\right)^2 = \frac{\tilde{E}}{\tilde{l}^2} - (1 - \tilde{u})(\tilde{u}^2 + \frac{1}{\tilde{l}^2})$$

take $d/d\phi$ of above, put $\tilde{u}' = d\tilde{u}/d\phi$

$$2\tilde{u}'\tilde{u}'' = 3\tilde{u}^2\tilde{u}' - 2\tilde{u}\tilde{u}' + \frac{\tilde{u}'}{\tilde{l}^2}$$

or

$$\tilde{u}'' + \tilde{u} - \frac{1}{2\tilde{l}^2} = \frac{3}{2}\tilde{u}^2 \tag{5.20}$$

First consider an orbit of radius $r \gg r_g$ ($\tilde{u} \ll 1$), mercury has such an orbit, see remark after (5.9). So ignoring RHS of (5.20), we get SHO with a shift origin,

$$\tilde{u} = \tilde{u}_0(1 + \epsilon\cos\phi) = \tilde{u}_{newton} \qquad \tilde{u}_0 = \frac{1}{2\tilde{l}^2}$$

or

$$r = \frac{r_0}{1 + \epsilon\cos\phi} \tag{5.21}$$

elliptical orbit. Now solve full (5.20) with RHS use $\tilde{u}_{newton}$. And assume $\epsilon \ll 1$, (almost circular with radius r_0) again mercury has such an orbit. We get

$$\tilde{u}'' + \tilde{u} - \tilde{u}_0 = \frac{3}{2}\tilde{u}_0^2(1 + 2\epsilon\cos\phi) \tag{5.22}$$

Let $\widetilde{u} = \tilde{u} - \tilde{u}_{newton}$, then (5.22) becomes

$$\widetilde{u}'' + \widetilde{u} = \frac{3}{2}\tilde{u}_0^2(1 + 2\epsilon\cos\phi)$$

the solution is

$$\widetilde{u} = \frac{3}{2}\tilde{u}_0^2(1 + \epsilon\phi\cos\phi)$$

so

$$\begin{aligned}
\tilde{u} &= \tilde{u}_0(1 + \epsilon\cos\phi) + \frac{3}{2}\tilde{u}_0^2(1 + \epsilon\phi\sin\phi) \\
&\approx \underbrace{\tilde{u}_0 + \frac{3}{2}\tilde{u}_0^2}_{\text{const}} + \tilde{u}_0\epsilon\cos(\phi - \delta)
\end{aligned}$$

with $\delta = \frac{3}{2}\tilde{u}_0\phi$. So without δ, we would get ellipse, and δ gives the precession, rotation of perihelia

$$\text{precession rate} = \frac{\delta}{\text{one orbit}} = \frac{3}{2}\tilde{u}_0 2\pi = \frac{6\pi GM}{c^2 r_0}$$

r_0 is in (5.21). For Mercury this gives $43''$ per century. The observed precession rate after subtraction of the effects of inter-planet interaction is $(43.1 \pm 0.5)''$.

5.5 Bending of Light

Consider a photon coming from $y = -\infty$ parallel to y axis with $x = b$, impact parameter. The trajectory is bent and goes to ∞. It reaches a minimum radius, r_p, pericenter. Redefine axes, putting pericenter on the $+x$ axis, and the angle between x axis and the out going to ∞ trajectory is ϕ, so the total angle bent is

$$2\phi - \pi \tag{5.23}$$

We want to follow the argument in section 5.3 with $m = 0$ particles. Still we have chosen the trajectory remaining in a plane with $U_\theta = 0$, and

$$U_t = \text{const} \equiv -\frac{E}{c} = -c$$

$$U_\phi = \text{const} \equiv l = bc$$

choosing $E = c^2$ cf (5.19) for the parabolic trajectory. More specifically it is because we need E and l to be consistent. Since we know

$$U_t = g_{tt}U^t = g_{tt}\frac{dt}{d\lambda}c = \text{const}$$

Unlike for massive particle $\lambda = \tau$, for photon λ can be any affine parameter cf (3.10). We want to impose

$$\left.\frac{dt}{d\lambda}\right|_{r=\infty} = 1 \tag{5.24}$$

since also $g_{tt} \to -1$ at ∞, so $U_t = -c$. The reason we impose (5.24) is the following we have

$$U_\phi = g_{\phi\phi}\frac{d\phi}{d\lambda} = \text{const}$$

Similarly evaluating at $r \to \infty$,

$$U_\phi = r^2\frac{d\phi}{dt}\frac{dt}{d\lambda} = r^2\frac{d\phi}{dt} = l$$

which agrees the usual definition of angular momentum, so clearly evaluating at ∞, $l = \vec{r}\cdot\vec{v} = bc$.

U_r is found from $U_\alpha U^\alpha = 0$,

$$U^t = \frac{U_t}{g_{tt}} = \frac{c}{1-\frac{r_g}{r}} \quad U^\phi = \frac{U_\phi}{g_{\phi\phi}} = \frac{bc}{r^2},$$

So

$$(U^r)^2 = (1-\frac{r_g}{r})(\frac{c^2}{1-\frac{r_g}{r}} - \frac{b^2c^2}{r^2}) \tag{5.25}$$

so

$$\frac{d\phi}{dr} = \frac{U^\phi}{U^r} = \frac{b}{r^2\sqrt{1-z^2}}$$

where

$$z^2 = \frac{b^2}{r^2}(1-\frac{r_g}{r}) \tag{5.26}$$

Since $\phi = 0$ at $r = r_p$

$$\phi(\infty) = \int_{r_p}^{\infty} \frac{d\phi}{dr} dr$$

By (5.25), $r = r_p$, $U^r = 0$, $z = 1$, at $r = \infty$, $z = 0$, and simplify and change of variable $dr = \frac{-zdz}{b^2(\frac{1}{r^3} - \frac{3}{2}\frac{r_g}{r^4})}$,

$$\phi = \int_0^1 \frac{\sqrt{1 - \frac{r_g}{r}}}{1 - \frac{3}{2}\frac{r_g}{r}} \frac{dz}{\sqrt{1 - z^2}}$$

Consider trajectories with $r_p \gg r_g$, then

$$\frac{\sqrt{1 - \frac{r_g}{r}}}{1 - \frac{3}{2}\frac{r_g}{r}} \approx (1 - \frac{1}{2}\frac{r_g}{r})(1 + \frac{3}{2}\frac{r_g}{r}) \approx 1 - \frac{1}{2}\frac{r_g}{r} + \frac{3}{2}\frac{r_g}{r}$$

and (5.26) becomes $z = b/r$, so

$$\begin{aligned} \phi &= \int_0^1 (1 + \frac{r_g}{r}) \frac{dz}{\sqrt{1 - z^2}} \\ &= \int_0^1 (1 + \frac{r_g}{b} z) \frac{dz}{\sqrt{1 - z^2}} = \frac{\pi}{2} + \frac{r_g}{b} \end{aligned}$$

so the total deflection angle (5.23) is

$$\frac{4GM}{bc^2}$$

For a photon passing behind the sun $b \approx R_\odot = 7 \times 10^{10}$cm, $M_\odot = 2 \times 10^{33}$g, we get 1.75” perfect agreement with observations.

5.6 Schwarzschild Black Holes

Lecture 19 (4/9/14)

Good reference: Chandrasekhar, The Mathematical Theory of Black Holes

Now we study the interesting things happen as $r \to r_g^+$ in Schwarzschild spacetime.

1. Time of a static observer slows down with respect to coordinate time t and stops as $r \to r_g^+$, because by (4.1) $g_{tt} \to 0$, as $r \to r_g^+$. In (4.1), t is standard

reference clock: distant static observer at $r = \infty$ cf remark after (4.9) and τ is static observer at r.

2. Infinite force must be applied to a particle $m \neq 0$ to keep it static at $r = r_g$ because by (3.8)
$$a^\alpha = \frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} U^\beta U^\gamma$$
Using static frame $U^\alpha = (\frac{dtc}{d\tau}, 0, 0, 0)$, by remark after (5.8), $dU^\alpha/dt = 0$ so is $dU^\alpha/d\tau = 0$. Then
$$a^\alpha = \Gamma^\alpha_{tt} U^t U^t$$
$$a^r = \frac{1}{2} c^2 g^{rr} g_{tt,r} \left(\frac{dt}{d\tau}\right)^2 = \frac{c^2}{2} \frac{r_g}{r^2} \underbrace{g^{rr} \underbrace{\left(\frac{dt}{d\tau}\right)^2}_{g_{tt}}}_{1}$$
the measured value wrt to local static observer is cf (5.11)
$$a^{\hat{r}} = \sqrt{g_{rr}} a^r = \frac{c^2}{2} \frac{r_g}{r^2 \sqrt{1 - \frac{r_g}{r}}} \to \infty \text{ as } r \to r_g \tag{5.27}$$

3. A photon emitted radially outward at $r = r_g$ doesn't propagate in r, it is frozen at $r = r_g$ seen from observer at ∞. Because speed of photon measured by local static observer is still c, but for observer at ∞, cf (5.11)
$$dr = \frac{d\hat{r}}{\sqrt{g_{rr}}} = \frac{c d\tau}{\sqrt{g_{rr}}} = 0$$

Radial Infall

Consider a $m \neq 0$ particle falls radially $U^\theta = U^\phi = 0$, because $U_\alpha U^\alpha = -c^2$, and choosing $E = -U_t c = c^2$ cf (5.13), (5.19) for the parabolic trajectory i.e. $U^r = 0$ at $r = \infty$,
$$(U^r)^2 = -(1 - \frac{r_g}{r})c^2 + U_t^2 = \frac{r_g}{r} c^2 \implies \frac{dr}{d\tau} = -c\sqrt{\frac{r_g}{r}}$$

$-$ sign for infalling.

$$U^t = g^{tt}U_t \implies \frac{dt}{d\tau} = \frac{1}{1-\frac{r_g}{r}}$$

so

$$\frac{dr}{dt} = -c(1-\frac{r_g}{r})\sqrt{\frac{r_g}{r}}$$

Let us calculate the time (measured by an observer at ∞) of free fall from a given $r > r_g$ to r_g,

$$ct_{r\to r_g} = \int_{r_g}^{r} \frac{cdt}{dr}dr = \int_{r_g}^{r} \frac{dr}{(1-\frac{r_g}{r})\sqrt{\frac{r_g}{r}}} = \infty$$

However if the time is measured by co-moving local observer

$$c\tau_{r\to r_g} = \int_{r_g}^{r} \frac{cd\tau}{dr}dr = \int_{r_g}^{r} \sqrt{\frac{r}{r_g}}dr = \frac{2}{3}r_g\left((\frac{r}{r_g})^{\frac{3}{2}} - 1\right)$$

is finite. This slows the coordinate system (refer to standard clock and ruler at ∞) is not good. There is no problem for particle to go into r_g, it is a problem of the coordinate.

The calculation is slightly more involved if we instead of choosing parabolic trajectory, we consider the object (e.g. a star) radially free falls at r_{max}, i.e. $U^r = 0$ at r_{max}. To compute the infall time, we need $r(t)$ and $r(\tau)$. They are given in a parametric form (cycloid)

$$\begin{aligned} r &= \frac{1}{2}r_{max}(1+\cos\eta) \\ \tau &= \frac{r_{max}}{2c}\sqrt{\frac{r_{max}}{r_g}}(\eta+\sin\eta) \qquad (5.28) \\ t &= \frac{r_g}{c}\ln\left|\frac{\sqrt{r_{max}/r_g - 1}+\tan(\eta/2)}{\sqrt{r_{max}/r_g - 1}-\tan(\eta/2)}\right| \qquad (5.29) \end{aligned}$$

For $r(\tau)$, solve η using (5.28); for $r(t)$, solve η using (5.29). From $r > r_g$ it reaches the block hole in

$$\tau_{total} = \frac{\pi}{2}\frac{r_{max}}{c}\sqrt{\frac{r_{max}}{r_g}}$$

and

$$t_{total} = \infty$$

Same for photons infalling:

$$ds^2 = -(1 - \frac{r_g}{r})c^2 dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} = 0$$

or

$$\frac{rdr}{r - r_g} = cdt \tag{5.30}$$

so

$$ct_{r \to r_g} = \int_{r_g}^{r} \frac{rdr}{r - r_g} = \infty$$

if measured by a local observer

$$c\tau_{r \to r_g} = r - r_g$$

is finite. To be more precious, use affine parameter λ

$$\frac{dr}{d\lambda} = U^r = \sqrt{-g^{rr} g^{tt} U_t^2} = \sqrt{U_t^2} = c \implies \lambda = \frac{r}{c} + \text{const}$$

is finite.

Kruskal Coordinates

$r = r_g$ appears as a problem in Schwarzschild solution $g_{tt} = 0$, $g_{rr} = \infty$. However all invariants that characterize the spacetime (e.g. curvature) remain finite at r_g. So the bad behavior of $g_{\alpha\beta}$ is related to the choice of coordinate system not singularity of the spacetime. Actually if one is filling into a black hole, he won't feel anything cross r_g. Because tidal force (5.12) is

$$\left(\frac{r_g}{r^3}\right)_{r=r_g} =\sim \frac{1}{M^2}$$

for a massive black hole, $10^9 M_{\odot}$, is tiny compared to falling to a planet.

The fact that $\frac{\partial}{\partial t}$ is not timelike $r \leq r_g$ and $\frac{\partial}{\partial r}$ is not spacelike just mean that physically if one is at $r \leq r_g$ it is not possible stay at that r, r must change and it will reach $r = 0$ in finite time $\sim r_g/c$. The sphere $r = r_g$ is called event horizon because a distant observer cannot see the region $r < r_g$: no null or timelike geodesic come emerge with $U^r > 0$ from this region.

The problem with $t = \infty$ at r_g reflects the fact that the surface of simultaneity that connects to infinity doesn't continuous to $r < r_g$. (two observers at $r < r_g$ and $r > r_g$ cannot exchange signals to establish simultaneity.) If we give up trying to describe the whole spacetime as slices of constant t, we can find better coordinates where $g_{\alpha\beta}$ is finite at r_g. The true singularity appears only when the observer reaches $r = 0$ then tidal forces become infinite.

Lecture 20 (4/14/14)

It is more convenient to use the radial null geodesics to define the new coordinates, then our coordinates transformation will involve t and r only; θ and ϕ will remain untouched and the part $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ will be the same.

Radial null geodesics from (5.30)

$$\frac{rdr}{r - r_g} = \pm cdt$$

$+$ for outgoing light; $-$ for ingoing light. Integrating both sides (this removes the singularity in space at r_g)

$$r_* = r + r_g \ln\left|\frac{r}{r_g} - 1\right| = \pm cdt + \text{const}$$

define for $\forall\ r > 0$ and

$$dr_* = \frac{rdr}{r - r_g} \tag{5.31}$$

and define

$$v = ct + r_* \qquad u = ct - r_* \tag{5.32}$$

then since $t = \pm\infty$ at r_g, $\pm$ depends past or future

$$\begin{aligned} \text{For ingoing light } v &= \text{const} \\ u &= ct - (v - ct) \to \infty \text{ at } r_g \text{ in future} \end{aligned}$$

$$\begin{aligned}\text{For outgoing light } u &= \text{const}\\ v &= ct + (ct-u) \to -\infty \text{ at } r_g \text{ in past}\end{aligned}$$

then define

$$v'^2 = \exp(\frac{v}{r_g}) \qquad u'^2 = \exp(-\frac{u}{r_g}) \tag{5.33}$$

$$\text{choose } v',\ u' \text{to be opposite sige for } r > r_g,\ \text{same sign for } r \le r_g \tag{5.34}$$

(this removes the infinity in time at r_g), then $2v'dv' = \frac{1}{r_g}e^{v/r_g}dv$, $2u'du' = -\frac{1}{r_g}e^{-u/r_g}du$, we get

$$\frac{dv'}{v'} = \frac{dv}{2r_g} \qquad \frac{du'}{u'} = -\frac{du}{2r_g} \tag{5.35}$$

Now we can write down the metric in the new coordinate, by (5.31)

$$\begin{cases} dv = cdt + dr_* = cdt + \frac{rdr}{r-r_g} \\ du = cdt - dr_* = cdt - \frac{rdr}{r-r_g} \end{cases}$$

solve for dt, dr, and by (5.35)

$$\begin{aligned} cdt &= \frac{du+dv}{2} = r_g\left(\frac{dv'}{v'} - \frac{du'}{u'}\right) \\ dr_* &= \frac{dr}{1-\frac{r_g}{r}} = \frac{du-dv}{2} = r_g\left(\frac{dv'}{v'} + \frac{du'}{u'}\right) \end{aligned}$$

thus

$$\begin{aligned} ds^2 &= (1-\frac{r_g}{r})r_g^2[\left(\frac{dv'}{v'} - \frac{du'}{u'}\right)^2 - \left(\frac{dv'}{v'} + \frac{du'}{u'}\right)^2] + r^2 d\Omega^2 \\ &= (1-\frac{r_g}{r})r_g^2 4[\frac{dv'}{v'}\frac{du'}{u'}] + r^2 d\Omega^2 \end{aligned}$$

and r should be written in terms of v', u'.

From (5.33), (5.34), and (5.32),

$$u'v' = \mp\exp(\frac{r_*}{r_g}) = \mp\left|\frac{r}{r_g} - 1\right|\exp\frac{r}{r_g} = (1-\frac{r}{r_g})\exp\frac{r}{r_g} \tag{5.36}$$

therefore

$$ds^2 = -\frac{4r_g^3}{r}\exp(-\frac{r}{r_g})du'dv' + r^2 d\Omega^2$$

Finally define

$$u' = T - R \quad v' = T + R$$

we get

$$ds^2 = -\frac{4r_g^3}{r}\exp(-\frac{r}{r_g})(dT^2 - dR^2) + r^2 d\Omega^2 \tag{5.37}$$

(T, R, θ, ϕ) are called Krushal coordinates. From (5.36)

$$r = \text{const} \implies u'v' = T^2 - R^2 = \text{const} \begin{cases} > 0 & r < r_g \\ < 0 & r > r_g \end{cases}$$

and $u'v'$ flips signs cross r_g, so the axis of hyperbolic $T^2 - R^2$ change from x axis (R axis) to y axis (T axis).

$$r = r_g \text{ horizon} \implies u'v' = T^2 - R^2 = 0 \implies T = \pm R$$

$$r = 0 \implies T^2 - R^2 = 1$$

This gives 2 branches: black hole & white hole. Both are legitimate solutions to Einstein equation in vacuum for the maximally symmetric spacetime. White hole is just opposite (or time reverse) to black hole, repelling everything within radius r_g.

Wormhole

Lecture 21 (4/16/14)

$$T = \pm R$$

separates (T, R) space into 4 regions $\varphi \in (-\pi/4, \pi/4) = I$, $(\pi/4, 3\pi/4) = II$, $(3\pi/4, 5\pi/4) = VI$, and $(5\pi/4, 7\pi/4) = III$. Regions I and II give black holes type of solutions with I being $r > r_g$. Regions III and IV give white holes type of solutions with IV being $r > r_g$.

In Schwarzschild coordinate metric $r < r_g$, ∂_t is spacelike, what about Krushal coordinate (5.37)? One can show $r < r_g$ regions II & III geometry is not static, any timelike coordinate (e.g. T) will give spacelike slices T =const with evolving geometry. From $r = 0$ in the past to $r = 0$ in the future. For a finite time ΔT there is a spacelike bridge (Einstein-Rosen bridge or wormhole) between regions I and IV, two asymptotically flat, causally disconnected universes.

5.7 Formation of Black Holes

Kerr Black Holes

Black hole is formed from collapsing massive star. A clock on the surface will show that it reaches singularity at $r = 0$ in a finite time. However for a distant observer the star remains forever frozen at r_g ($t \to \infty$). The observed luminosity of the star will decay as $\exp(-\frac{2}{3\sqrt{3}}\frac{ct}{r_g})$ and the arriving photons at ∞ will be red shifted as $z = \frac{\Delta\lambda}{\lambda} \propto \exp(\frac{ct}{2r_g})$.

Real stars have non-zero angular momentum (spin, itself is so massive that forget about orbital motion), which is conserved in collapse. Then the spacetime is not spherically symmetric and a new axially-symmetric solution of Einstein equation is needed, it is Kerr metric.

Let

$$\tilde{a} = \frac{J}{(r_g/2)cM} \tag{5.38}$$

then $\tilde{a} = 1$ is the maximally rotating black hole, because for a rotating disc of radius r_g, the maximum angular speed is c/r_g, the moment of inertia of a circular disk is $\frac{1}{2}Mr_g^2$, so maximal

$$J = \frac{1}{2}Mr_g^2 c/r_g \implies \tilde{a} = 1$$

Axially symmetric (spinning) source of gravity: the vacuum solution outside of the object has two Killing vectors: $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$, the Kerr solution in Boyer-

Lindquist coordinates (t, r, θ, ϕ),

$$g_{\alpha\beta} = \begin{pmatrix} -(1-\frac{rr_g}{\varrho^2}) & & & -\frac{r_g ar\sin^2\theta}{\varrho^2} \\ & \frac{\varrho^2}{\Delta} & & \\ & & \varrho^2 & \\ -\frac{r_g ar\sin^2\theta}{\varrho^2} & & & \frac{\sin^2\theta}{\varrho^2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta] \end{pmatrix}$$

where

$$a = \tilde{a}\frac{r_g}{2} = \frac{J}{cM} = \text{conserved} \quad \Delta = r^2 - r_g r + a^2 \quad \varrho^2 = r^2 + a^2\cos^2\theta$$

$0 \le a \le r_g/2$. Clearly when $J = 0 \implies a = 0$, Kerr is Schwarzschild. The Kerr metric is stationary but not static, recall (4.6). Just like before, when g_{tt} become > 0, a static position: $r = \text{const}$, $\theta = \text{const}$, $\phi = \text{const}$ is not possible, any particle must move. $g_{tt} = 0$ is called ergosphere, which is at

$$r^2 + r_g r + a^2\cos^2\theta = 0 \implies r(\theta) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2}) - a^2\cos^2\theta}$$

choosing + sign. The place where g_{rr} changes sign is called Horizon, which is at

$$r^2 + r_g r + a^2 = 0 \implies r = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2}) - a^2} = \text{const} \tag{5.39}$$

choosing + sign. So Horizon is a sphere inside of ergosphere. Inside ergosphere, $\partial/\partial t$ is spacelike; inside horizon $\partial/\partial r$ is timelike.

Penrose's Process

There is a classical process to extract energy from black holes. (Hawking radiation is quantum mechanical. Treat black hole like a black body

$$kT \sim \frac{\hbar}{r_g/c}$$

hence temperature is inversely proportional to mass. Losing mass, $t \uparrow$, evaporation becomes faster, eventually causes explosion).

Following similar argument in (5.27), we conclude that nothings can come out of horizon. But what if things are outside of horizon and inside of ergoshpere? Similar to section 5.3, we need to solve, assume $U^\theta = 0$,

$$-c^2 = \underbrace{g_{tt}}_{>0}(U^t)^2 + \underbrace{g_{rr}}_{>0}(U^r)^2 + \underbrace{g_{t\phi}}_{<0}U^tU^\phi$$

so this cannot happen unless $U^\phi \neq 0$.

Consider a star, A in the region outside of horizon and inside of ergoshpere, disintegrates into two stars B, C. The two stars rotate in opposite directions wrt the co-rotating observer of the black hole.

By (5.13), before A disintegrates

$$E_A = -c(U_t)_A = -cg_{tt}(U^t)_A$$

after A disintegrates

$$E_B = -c(U_t)_B = -c[g_{tt}(U^t)_B + \underbrace{g_{t\phi}}_{<0}(-U^\phi)_C]$$

$$E_C = -c(U_t)_C = -c[g_{tt}(U^t)_C + g_{t\phi}(U^\phi)_C]$$

where $(U^t)_A = (U^t)_B = (U^t)_C = c$ (in co-moving frames). Suppose $(U^\phi)_C > 0$ counterclockwise for example. Thus the orbit of B has $E_B < E_A$, and $E_C > E_A$. This energy by definition is total energy, including rest energy. If C flies out of the black hole, it brings out more energy than A brings in. By conversation of angular momentum, B will fall in to the horizon.

Frame Dragging

In above we mentioned the co-rotating observer. In more general term, it is called frame dragging effect. That is because J in (5.38) is of course refereed to observer at ∞. Let's say there is an observer at r with $l = 0$ wrt the observer at ∞, then by (5.14)

$$l = U_\phi = g_{t\phi}U^t + g_{\phi\phi}U^\phi = 0$$

so the angular speed wrt to observer at ∞ is

$$\Omega = \frac{d\phi}{dt} = \frac{U^\phi}{U^t} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{ar_g rc}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}$$

In particular the angular speed of the horizon is

$$\Omega_H = \frac{ac}{r_H^2 + a^2}$$

by (5.39).

Radiation from a rotating accretion disc black hole disc happens all the time, because of the dissipation due to rotation friction, the rate of energy is huge

$$\dot{E} \sim \underbrace{\frac{GM}{r_g}}_{=c^2}\dot{M}$$

it can be calculated from the area, one finds the temperature is about 1keV.

6 Cosmology

6.1 Copernican Principle

Lecture 22 (4/21/14)

$$1\text{ parsecs(pc)} = 3.1\times 10^{18}\text{cm} \approx 3\text{ light years} \tag{6.1}$$

If the density of the observed universe is averaged on large scales ($\gg$ 10Mpc, size of milky way), it will be approximately constant (with averaged energy density ρ, defined in (4.18)) in all parts of the observed universe, i.e. the universe is homogeneous and isotropic on large scales. E.g. the cosmic microwave background is isotropic to a high accuracy $\sim 10^5$. The spacetime of the universe can be idealized as a foliation of slices $t = \text{const}$ that are maximally symmetric $3D$ spaces (homogenous & isotropic). The metric then takes the form

$$ds^2 = g_{tt}dt^2c + \gamma_{ij}dx^i dx^j$$

where g_{tt} is only function of t. One may therefore absorb g_{tt} into t and call the new t as proper time of the co-moving observes (co-moving means γ_{ij} may depend on t, but must be separable, so we should fact out the t dependence from γ_{ij}), later we will show that the metric is

$$ds^2 = -dt^2c + a^2(t)\gamma_{ij}dx^i dx^j \tag{6.2}$$

(t, x) is co-moving coordinates and t is a universal time, cf static gravitation field (4.5). $\gamma_{ij}dx^i dx^j$ is independent of t and it is the 3D maximum symmetric space. For now forget about $a^2(t)$, or absorb it into γ_{ij} and apply section maximally symmetric space on page 60 to $\gamma_{ij}dx^i dx^j$ with $n = 3$. There are three types of maximally symmetric spaces: sphere, flat and hyperboloid, dependent on whether

$$^{(3)}R \begin{array}{c} > \\ = \\ < \end{array} 0$$

The superscript (3) indicates it is R on S^3 space. And (3.32) becomes

$$^{(3)}R_{ijkm} = \frac{^{(3)}R}{n(n-1)}(\gamma_{ik}\gamma_{jm} - \gamma_{im}\gamma_{jk})$$

6.2 Robertson-Walker Metric

All of the three types can be embedded in Euclidean (or pseudo-Euclidean) space of $n+1 = 4$ dimension. Example: sphere S^3 is embedded in $\mathbb{E}^4$. It is described by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2 \tag{6.3}$$

where (x_1, x_2, x_3, x_4) are coordinate of $\mathbb{E}^4$, and a is radius of S^3. Let's compute metric γ_{ij} on S^3, using similar technique in section 4.1 about stationary gravitational field. Use x_1, x_2, x_3 as independent coordinates (chart on S^3) and define

$r^2 = x_1^2 + x_2^2 + x_3^2$, with $r \leq a$. Then $x_4^2 = a^2 - r^2$, $dx_4 = -\frac{rdr}{x_4}$. The metric γ_{ij} on S^3 is induced by the Euclidean $4D$ distance dl^2 between two points $\vec{x}$ and $\vec{x} + d\vec{x}$ on the sphere

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \underbrace{\frac{r^2 dr^2}{a^2 - r^2}}_{dx_4^2} \tag{6.4}$$

Then introduce new coordinates $(x_1, x_2, x_3, x_4) \to (r, \theta, \phi, x_4)$, with $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$. Then (6.4) becomes

$$\begin{aligned} dl^2 &= dr^2 + r^2(\underbrace{d\theta^2 + \sin^2\theta d\phi^2}_{d\Omega^2}) + \frac{r^2 dr^2}{a^2 - r^2} = r^2 d\Omega^2 + \frac{a^2 dr^2}{a^2 - r^2} \\ &= \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 d\Omega^2 \end{aligned} \tag{6.5}$$

For flat space, replace a^2 in (6.4) by ∞; for hyperbolic space, replace $a^2 \to -a^2$ and $x_4^2 \to -x_4^2$, so we get the general form of metric for maximally symmetric $3D$ space

$$dl^2 = \frac{dr^2}{1 - k\frac{r^2}{a^2}} + r^2 d\Omega^2 \qquad \text{with} \begin{cases} k = 0 & \text{flat } \mathbb{E}^3 \\ k = 1 & \text{sphere } S^3 \\ k = -1 & \text{hyperboloid } H^3 \end{cases}$$

For S^3 since $r \leq a$, it is called closed universe, for H^3 not requirement for r, so called open universe.

It will be clear soon that r is not a good coordinate because it depends on t. One can define a new coordinate $(r, \theta, \phi, x_4) \to (R, \theta, \phi, x_4)$ by dimensionless $R = r/a$, $0 \leq R \leq 1$, then above becomes

$$dl^2 = a^2 \left(\frac{dR^2}{1 - kR^2} + R^2 d\Omega^2 \right) \quad \text{with} \begin{cases} k = 0 & \text{flat } \mathbb{E}^3 \\ k = 1 & \text{sphere } S^3 \\ k = -1 & \text{hyperboloid } H^3 \end{cases} \tag{6.6}$$

compare this to (6.2), we would like to have $a = a(t)$ and R to be independent of t. So R is a good coordinate.

Or for S^3, one can define a new coordinate $(r, \theta, \phi, x_4) \to (\chi, \theta, \phi, x_4)$, by

$r = a\sin\chi$, $dr = a\cos\chi d\chi$, (for the same reason χ is independent of t) then above becomes

$$dl^2 = a^2\left(d\chi^2 + \sin^2\chi d\Omega^2\right) \quad 0 \le \chi \le \pi$$

For H^3, one can define $r = a\sinh\chi$, $dr = a\cosh\chi d\chi$, then above becomes

$$dl^2 = a^2\left(d\chi^2 + \sinh^2\chi d\Omega^2\right)$$

Relation between a and curvature scalar ${}^{(3)}R$ of S^3

From (6.4), $dl^2 = \gamma_{ij}dx^i dx^j$ with

$$\gamma_{ij} = \delta_{ij} + \frac{x_i x_j}{a^2 - x_k x^k}$$

because $x_i x^i = x_1^2 + x_2^2 + x_3^2 = r^2$, $x_i dx^i = \frac{1}{2}d(x_i x^i) = rdr$.

Using the homogeneity of S^3, one can calculate its curvature scalar at any point. For convenience, choose point $x_1 = x_2 = x_3 = 0$, so in the neighborhood of this point, we have

$$\gamma_{ij} = \delta_{ij} + \frac{x_i x_j}{a^2} + O((x_i x_j)^2)$$

so first derivative of γ_{ij} is 0 at $x_1 = x_2 = x_3 = 0$, but second derivative is not 0. It is

$$\gamma_{ij,kl} = \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{a^2}$$

Therefore

$$\begin{aligned} {}^{(3)}\Gamma^i_{kl} &= \frac{1}{2}\delta^{ij}(\gamma_{jk,l} + \gamma_{jl,k} - \gamma_{kl,j}) + O(x_i^2) \\ &= \frac{1}{2}\sum_i(\gamma_{ik,l} + \gamma_{il,k} - \gamma_{kl,i}) \end{aligned}$$

which is 0 at $x_1 = x_2 = x_3 = 0$, so

$$\begin{aligned}
{}^{(3)}R^i{}_{kil} &= \partial_i {}^{(3)}\Gamma^i_{kl} - \partial_l {}^{(3)}\Gamma^i_{ki} \\
&= \frac{1}{2}\sum_i \left(\partial_i(\gamma_{ik,l} + \gamma_{il,k} - \gamma_{kl,i}) - \partial_l(\gamma_{ik,i} + \gamma_{ii,k} - \gamma_{ki,i})\right) \\
&= \frac{1}{2}\sum_i \left(\gamma_{ik,il} + \gamma_{il,ik} - \gamma_{kl,ii} - \gamma_{ii,lk}\right) \\
&= \frac{1}{2a^2}\sum_i \left(\delta_{ii}\delta_{kl} + \delta_{il}\delta_{ik} + \delta_{ii}\delta_{lk} + \delta_{ik}\delta_{il} - 2\delta_{li}\delta_{ki} - 2\delta_{li}\delta_{ik}\right) \\
&= \frac{1}{2a^2} 2\sum_i (\delta_{ii}\delta_{kl} - \delta_{li}\delta_{ki}) = \frac{(n-1)\delta_{kl}}{a^2}
\end{aligned}$$

$$ {}^{(3)}R = \gamma^{kl}\frac{(n-1)\delta_{kl}}{a^2} = \underbrace{\delta^{kl}\delta_{kl}}_{\delta_{kk}}\frac{n-1}{a^2} = \frac{n(n-1)}{a^2} = \frac{6}{a^2} \tag{6.7}$$

which agrees what we have done in remark after (6.5), i.e. a^2 gives $R > 0$ sphere, $-a^2$ gives $R < 0$ hyperboloid. $a^2 = \infty$ gives $R = 0$ flat.

6.3 Friedman Equation

From (6.6), we have Robertson-Walker metric

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dR^2}{1-kR^2} + R^2d\Omega^2\right)$$

with

$$g_{\alpha\beta}(t, R, \theta) = \begin{pmatrix} -1 & & & \\ & \frac{a^2}{1-kR^2} & & \\ & & a^2R^2 & \\ & & & a^2R^2\sin^2\theta \end{pmatrix}$$

Similar to what we did in section 5.2, we want to apply Einstein equation to this metric and get some relationship between a and some physical properties like energy density ρ.

Compute $\Gamma^{\alpha}_{\beta\gamma}$, e.g. $\Gamma^{t}_{t\alpha} = \frac{1}{2}g^{tt}g_{tt,\alpha} = 0$,

$$\Gamma^{t}_{ii} = -\frac{1}{2}g^{tt}g_{ii,t} = \frac{1}{2}\frac{\partial}{c\partial t}g_{ii} = \frac{1}{2}\frac{\partial}{c\partial t}(a^2\frac{g_{ii}}{a^2}) = \frac{\dot{a}}{a}g_{ii} \tag{6.8}$$

use trick to write $g_{ii} = a^2\frac{g_{ii}}{a^2}$, then $\frac{g_{ii}}{a^2}$ is independent of t. Denote $\dot{a} = \frac{da}{cdt} = \frac{da}{dx^0}$. Use the trick again

$$\Gamma^{i}_{ti} = \frac{1}{2}g^{ii}g_{ii,t} = \frac{1}{2}g^{ii}\frac{g_{ii}}{a^2}\frac{\partial}{c\partial t}(a^2) = \frac{\dot{a}}{a}$$

In sum, non-zero $\Gamma^{\alpha}_{\beta\gamma}$,

$$\Gamma^{t}_{RR} = \frac{a\dot{a}}{1-kR^2} \quad \Gamma^{R}_{RR} = \frac{kR}{1-kR^2} \quad \Gamma^{R}_{Rt} = \frac{\dot{a}}{a}$$

$$\Gamma^{t}_{\theta\theta} = a\dot{a}R^2 \quad \Gamma^{R}_{\theta\theta} = -R(1-kR^2) \quad \Gamma^{\theta}_{\theta t} = \frac{\dot{a}}{a} \quad \Gamma^{\theta}_{\theta R} = \frac{1}{R}$$

$$\Gamma^{t}_{\phi\phi} = a\dot{a}R^2\sin^2\theta \quad \Gamma^{R}_{\phi\phi} = -R(1-kR^2)\sin^2\theta \quad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\phi t} = \frac{\dot{a}}{a} \quad \Gamma^{\phi}_{\phi R} = \frac{1}{R} \quad \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

Compute Ricci tensor, non-zero components

$$\begin{aligned}
R_{tt} &= -3\frac{\ddot{a}}{a} \\
R_{RR} &= \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kR^2} \\
R_{\theta\theta} &= R^2(a\ddot{a}+2\dot{a}^2+2k) \\
R_{\phi\phi} &= R^2\sin^2\theta(a\ddot{a}+2\dot{a}^2+2k)
\end{aligned}$$

that is

$$R^{R}{}_{R} = R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = \frac{a\ddot{a}+2\dot{a}^2+2k}{a^2}$$

one can easily see why $R^{R}{}_{R} = R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi}$ because homogeneity and isotropic imply

$$R_{\hat{R}\hat{R}} = R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} \tag{6.9}$$

and by (5.11)

$$R^R{}_R = g^{RR}R_{RR} = R_{\hat{R}\hat{R}}$$

Lecture 23 (4/23/14)

To take advantage of (6.9), we will evaluate Einstein equation in local orthonormal inertial frame, (4.25)

$$R_{\hat{\alpha}\hat{\beta}} = \kappa(T_{\hat{\alpha}\hat{\beta}} - \frac{1}{2}T^\mu{}_\mu g_{\hat{\alpha}\hat{\beta}})$$

with $g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$. The universe is filled with mater with an averaged energy density ρ. In the co-moving coordinates the stress-energy tensor must be isotropic: no fluxes of energy or momentum and pressure is the same in all three direction, that turns out to be exactly (4.19), ideal fluid in local frame

$$T_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \text{ or } T^{\hat{\alpha}}{}_{\hat{\beta}} = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \tag{6.10}$$

then

$$T^\mu{}_\mu = 3p - \rho \tag{6.11}$$

In (t,t) component

$$-3\frac{\ddot{a}}{a} = \kappa(\rho + \frac{3p-\rho}{2}) \tag{6.12}$$

In (i,i) component

$$\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{a^2} = \kappa(p + \frac{3p-\rho}{2}) \tag{6.13}$$

Combining the two above and eliminate $\ddot{a}$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa\rho - \frac{k}{a^2} \tag{6.14}$$

the most famous equation in cosmology–Friedman equation. Before people knew about dark matter, dark energy, Friedmann equation gives the condition for the universe to be open, close, flat.

Define Hubble constant

$$H = c\frac{\dot{a}}{a}$$

gives the rate of expansion universe. The observed expansion rate is

$$H(\text{now}) \approx 75\frac{\text{km/s}}{\text{Mpc}} = 2.5 \times 10^{-18}/\text{s}$$

the funny unit is because historically Hubble constant is wrt redshift. One can immediately estimate the age of universe as by (6.1)

$$t_{\text{age of universe}} = H^{-1} \sim 4 \times 10^{17}\text{s} \sim 1.3 \times 10^{10}\text{yr}$$

note 1 yr $\approx \pi \times 10^7$s. More exact age can be found by detailed model and observation is 1.4×10^{10}yr.

Putting H in (6.14), we see that

$$\frac{H^2}{c^2} = \frac{\kappa}{3}\rho - \frac{k}{a^2}$$

Even without knowing exact a, since we can measure H. There is a critical ρ_{crit} at which $k = 0$, flat universe

$$\rho_{crit} = \frac{3H^2}{c^2\kappa}$$

which is very close to observed value. If $\rho > \rho_{crit}$, $k = 1$ and if $\rho < \rho_{crit}$, $k = -1$. More commonly people define

$$\Omega = \frac{\rho}{\rho_{crit}} \tag{6.15}$$

then

$$\Omega - 1 = \frac{kc^2}{a^2H^2} \tag{6.16}$$

The current universe has $\Omega - 1 < 0.1$, so very flat.

Since (6.12), (6.13) are two independent equations, we should extrapolate 2 equations: one is Friedmann which doesn't have p. The other one is found from energy conservation, cf remark after (4.38)

$$\begin{aligned} 0 &= T^{\alpha}_{t;\alpha} = T^{\alpha}_{t,\alpha} + \Gamma^{\alpha}_{\alpha\mu}T^{\mu}_{t} - \Gamma^{\mu}_{\alpha t}T^{\alpha}_{\mu} \\ &= T^{t}_{t,t} + \Gamma^{\alpha}_{\alpha t}T^{t}_{t} - \left(\Gamma^{R}_{Rt}T^{R}_{R} + \Gamma^{\theta}_{\theta t}T^{\theta}_{\theta} + \Gamma^{\phi}_{\phi t}T^{\phi}_{\phi}\right) \\ &= -\dot{\rho} + 3\frac{\dot{a}}{a}(-\rho) - 3\frac{\dot{a}}{a}p \end{aligned}$$

thus

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \tag{6.17}$$

From thermodynamics we know depending on the type of material, p is related to ρ

$$p = w\rho \tag{6.18}$$

called equation of the state, where w is constant.

6.4 Big Bang

We now have 3 equations: (6.14), (6.17) and (6.18). We can solve for 3 unknowns: $a(t)$, $\rho(t)$ and $p(t)$.

From (6.17) and (6.18), we get

$$\rho \propto a^{-3(1+w)} \tag{6.19}$$

this shows if

1) $w = 0$, $\rho \propto a^{-3}$ ordinary matter, $p = 0$, dust, no thermos energy, close to current universe.

2) $w = 1/3$, $\rho \propto a^{-1/4}$, radiation, see section on CBS 6.4. This describes early universe. By (6.11), $T = 0$.

Putting initial condition to (6.19), we get

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

substituting into (6.14), we get an equation for $a(t)$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} - \frac{k}{a^2}$$

If $k = 0$,

$$\dot{a}^2 \propto a^{-1-3w} \implies a^{\frac{1+3w}{2}} da \propto dt \implies a \propto t^{\frac{2}{3(1+w)}}$$

1) $w = 0$, $a \propto t^{2/3}$.
2) $w = 1/3$,

$$a \propto t^{1/2} \tag{6.20}$$

As mentioned before (6.14), (6.17) are equivalent to (6.12), (6.13). In the same token, we can take (6.14) and (6.12), then we get a condition for static universe $\dot{a} = 0$ and $\ddot{a} = 0$

$$\rho = \frac{3k}{\kappa a^2} \text{ and } 3p + \rho = 0 \tag{6.21}$$

Cosmological Constant

Einstein thought the universe was static, but (6.21) cannot be satisfied for matter (dust $p = 0$) dominated universe, so he added another term to

$$G_{\alpha\beta} \to R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta}$$

with constant Λ. This will still satisfy $G_{\alpha\beta;\beta} = 0$ because of (3.4). Then Einstein's equation reads

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \kappa T^{matter}_{\alpha\beta}$$

This is equivalent to the old Einstein equation with

$$T^{eff}_{\alpha\beta} = T^{matter}_{\alpha\beta} + T^{Vac}_{\alpha\beta} \tag{6.22}$$

with

$$T^{Vac}_{\alpha\beta} = -\frac{\Lambda}{\kappa}g_{\alpha\beta}$$

The homogeneous, isotropic universe vacuum stress-energy tensor in local frame should have the same form of (6.10),

$$T^{Vac}_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \rho^{Vac} & & & \\ & p^{Vac} & & \\ & & p^{Vac} & \\ & & & p^{Vac} \end{pmatrix}$$

to be consistent with $g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$, we have to have

$$\rho^{Vac} = -p^{Vac} = \frac{\Lambda}{\kappa}$$

this is a unusual equation of state. In words vacuum has ordinary energy, but negative pressure.

Does κ have the same value as before? Before answering that question, let's go back to Friedmann equation where we didn't use a specific value for κ. Now our universe is filled with matter (dust) and vacuum energy, by (6.22), we can put

$$\begin{aligned} \rho &= \rho^{matter} + \rho^{Vac} \\ p &= \underbrace{p^{matter}}_{0} + p^{Vac} \end{aligned}$$

then Einstein static model condition (6.21) says

$$\rho = -3p = -3p^{Vac} = 3\rho^{Vac}$$

so

$$\rho^{Vac} = \frac{\rho^{matter}}{2} \text{ or } \Lambda = \frac{\kappa\rho^{matter}}{2} \tag{6.23}$$

Hence picking such Λ gives static universe. Then Einstein went back to (4.25) to find value of κ. (4.25) becomes

$$R_{\alpha\beta} = \kappa(T^{matter}_{\alpha\beta} - \frac{1}{2}T^{matter}g_{\alpha\beta} + \rho^{Vac}g_{\alpha\beta})$$

then follow the same step after (4.25)

$$\nabla^2 \frac{\phi}{c^2} = \frac{1}{2}\kappa(\rho^{matter} + 2\rho^{Vac})c^2$$

so Einstein concluded that κ was half as before.

Of course He was wrong, (but his idea will be important when we study inflation) because (6.23) was wrong, the universe is not static. The average density of universe

$$10^{-29}\frac{\text{g}}{cm^3}$$

is much lower than the average density of the solar system

$$\frac{M_\odot}{(10^{15}\text{cm})^3} = 10^{-12}\frac{\text{g}}{cm^3}$$

The presence of vacuum energy is not detectable on scales of our solar system or our galaxy, that is called dark energy. It can affect the evolution of the Universe a global scale $\sim cH^{-1}$.

Studies of the global expansion history revealed the presence of a vacuum-like $T_{\mu\nu}$ with $p < 0$ that make the Universe expand with acceleration. If this $T_{\mu\nu}$ is interpreted as vacuum energy $\rho^{Vac}g_{\alpha\beta}$ or cosmological constant $\Lambda = \rho^{Vac}\kappa$ then the observed parameters are, cf (6.15),

$$\begin{aligned} \rho^{Vac} &\approx 0.7\rho_{crit} \text{ or } \Omega_{vacuum} = 0.7 \\ \rho^{matter} &\approx 0.3\rho_{crit} \text{ or } \Omega_{matter} = 0.3 \end{aligned}$$

so (6.23) was very wrong. Among ρ^{matter}, most of which is dark matter of unknown origin.

Cosmological Redshift

Lecture 24 (4/28/14)

In section 4.1 we discussed redshift due to photons move from strong gravity field to weak field. Now we discuss another type of redshift. This happens due to expansion of the universe, so it can occur in the uniformly homogenous universe.

As photon propagates in the expanding universe its energy E, measured by the local observer, changes with cosmological time t. That we emphasize cosmological time instead of local inertial time is because we know there is a universal clock as we have assume that the universe is purely homogenous, cf (6.2) with $g_{00} = -1$. However e.g. if we have black holes (e.g. Schwarzschild metric) in the expanding universe, the local time $g_{00} \neq$ const. Thus both types of redshift should be added up together.

For this type of redshift, we assume $g_{tt} = -1$, i.e. cosmological time=local time. Consider a photon moving in R direction, $U^\alpha = (U^t, U^R, 0, 0)$. The geodesic equation for photon,

$$\frac{dU^t}{d\lambda} + \Gamma^t_{\alpha\beta} U^\alpha U^\beta = 0 \tag{6.24}$$

By (6.8)

$$\frac{dU^t}{d\lambda} + \Gamma^t_{RR}(U^R)^2 = 0 \qquad \Gamma^t_{RR} = \frac{\dot{a}}{a} g_{RR}$$

we also have

$$U_\alpha U^\alpha = g_{tt}(U^t)^2 + g_{RR}(U^R)^2 = 0 \implies g_{RR}(U^R)^2 = (U^t)^2$$

thus

$$\frac{dU^t}{d\lambda} = -\frac{\dot{a}}{a}(U^t)^2$$

that is

$$\frac{dU^t}{U^t \cdot \underbrace{U^t d\lambda}_{dt}} = -\frac{\dot{a}}{a} \text{ or } \frac{\dot{U}^t}{U^t} = -\frac{\dot{a}}{a}$$

therefore

$$U^t \propto a^{-1}$$

Since $E \propto U^t$, we get

$$E \propto a^{-1} \tag{6.25}$$

(HW6 problem6) Similarly one can derives for momentum of particle $m \neq 0$, by first modifying (6.24)$\frac{dU^R}{d\lambda} + \Gamma^R_{\alpha\beta} U^\alpha U^\beta = 0$, one obtains

$$p \propto a^{-1}$$

Hubble Law

Let $a_0 = a(t = t_0) = a(\text{current})$, $a(t) = a_0 + \dot{a}(t - t_0)c$ for $|t - t_0| \ll t_0 \approx 10^{10}$yr. Then

$$\frac{a_0}{a} = 1 + \left.\frac{\dot{a}}{a}\right|_{t_0} (t_0 - t)c = 1 + H(\underbrace{t_0 - t}_{\frac{d}{c}})$$

for two time events separated by d. The photon wavelength

$$\frac{\lambda_0}{\lambda} = \frac{a_0}{a} = 1 + \underbrace{H\frac{d}{c}}_{z}$$

when $z > 0$, called redshift.

We can then apply relativistic Doppler effect $\frac{\lambda_0}{\lambda} = \frac{1}{\gamma} \approx 1 + \frac{v}{c}$ then

$$v = Hd \tag{6.26}$$

we can Hubble law, the relative speed of the two planets moving away from each other is proportional to their distance d.

Cosmic Microwave Background (CMB)

The observed current universe with thermal radiation

$$T = 2.74K$$

blackbody spectrum. It was predicted by Gamow, and discovered in 60's. The current energy density of radiation

$$\rho_{rad} \sim 10^{-12}\frac{erg}{cm^3} \ll \rho_{matter} \sim 10^{-8}\frac{erg}{cm^3}$$

much smaller than rest energy density of matter.

$$\frac{\rho_{rad}}{\rho_{matter}} \sim 10^{-4} \tag{6.27}$$

Clearly

$$\rho_{matter} \propto a^{-3}$$

and why

$$\rho_{rad} \propto a^{-4}?$$

because of (6.25), $n_{photons} \propto a^{-3}$ and $E_{photons} \propto a^{-1}$.

Therefore

$$\frac{\rho_{rad}}{\rho_{matter}} \propto a^{-1}$$

When the universe was small by (6.27), go down by 10^{-4}, i.e. $a/a_0 = 10^{-4}$, $\frac{\rho_{rad}}{\rho_{matter}} \sim 1$, radiation dominated the energy density. The equation of state was $p = \rho/3$ rather than $p = 0$.

Nucleosynthesis

So a was very small, hot universe (Stefan, $\rho_{rad} \propto T^4$) dominated by radiation and relativistic particles (e.g. $e^{\pm}$), over the time it expanded and cooled down.

$$\text{matter-radiation equality at } z \approx 3 \times 10^3, \text{ at } T \approx 2.7 \times 3 \times 10^3 \text{K}$$

$$\text{recombination of hydrogen at } z \approx 1.2 \times 10^3$$

Going Back in Time

By Stefan and (6.20)

$$T \propto a^{-1} \propto t^{-1/2}$$

Big bang model says 14 billion years ago

$$kT = 1\text{MeV at } t = 1\text{s}$$

$$kT = 1\text{GeV at } t = 10^{-6}\text{s}$$

$$kT = 1\text{TeV at } t = 10^{-12}\text{s}$$

$T \to \infty$ at $t \to 0$ at which curvature diverges, this only happens at $t = 0$ and this doesn't contradict flatness problem stating that the "universe" was even flatter at early time. That "universe" was the part of the universe that has casual contact with our observable universe and that "universe" was very small part of the "whole" universe.

Important events at $kT = 300\text{MeV}$ quarks become free particles: deconfinement (QCD phase transition). At $kT = 10^{16}\text{GeV}$ grand unification supersymmetry(?)

Planck Time

Classical GR is not applicable when the gravitational action $S_g \sim \hbar$, cf (4.26), then gravity has to be quantized. The time is called Planck time.

$$S_g = \frac{c^3}{16\pi G}\int_D R\sqrt{-g}d^4x$$

and $R \sim \frac{d^2g}{dx^2}$, $g_{\alpha\beta}$ is unit less, at such time the universe is so small that differential dx becomes ct, see inflation below,

$$S_g \sim \frac{c^3}{G}(ct)^4\frac{1}{(ct)^2} = \frac{c^5t^2}{G} \sim \hbar$$

thus

$$t_{pl} \sim \left(\frac{\hbar G}{c^5}\right)^{1/2} = 10^{-43}\text{s}$$

6.5 Inflation

Two Problems of "Fine-Tuning"

1. (flatness problem) The fact that the universe is almost flat now implies that it was flat with huge accuracy at small times: because from Friedmann

$$\left(\frac{H}{c}\right)^2 = \underbrace{\frac{1}{3}\kappa\rho}_{a^{-3}/a^{-4}} - \frac{k}{a^2} \tag{6.28}$$

In either case, matter and radiation completely dominated over curvature term k/a^2. So $H^2 \propto a^{-3}$ or a^{-4} then by (6.16), if $\Omega - 1 \propto a^2$ is small now, it should be even smaller at early time.

2. (horizon problem) CMB is observed to be isotropic with high accuracy. It comes from the last-scattering surface at $z \sim 10^3$ with uniform T. However different parts of this surface were out of causal contact given the age of the universe at that moment. Causally connected region is, cf (6.26),

$$\sim c/H(t) \ll \text{size of last-scattering sphere}$$

These two problems in big bang model were solved if it was proceeded by a stage of exponential expansion called "inflation". Suppose

$$p = -\rho = \text{const} \tag{6.29}$$

then (6.28) says as $a \to 0$, $H = \text{const} \implies \dot{a} \propto a$, so

$$a \propto e^{Ht}$$

The existence of such a stage in the early history of the universe implies:

1. curvature term (now)≈ 0, it was killed exponentially by inflation, cf (6.16).
2. the whole observed universe expanded from one causally connected region if inflation lasted ~ 60 e-folding times.

The physical horizon distance is $d_H \approx \frac{c}{H}$, the distance to the observer that information can be exchanged during the age of the universe. Because $ds^2 = -c^2dt^2 + a^2(t)dl^2$ we put

$$l_H = \frac{c}{aH} \tag{6.30}$$

to the horizon distance in co-moving coordinates, that is Δl is fixed for two distant static observers during expansion, cf (6.2). However from (6.30) we see that the causally connected observers wrt co-moving frames are changing. There are two stages:

1. Inflation, $H = \text{const}, \quad a \propto e^{Ht}$, so

$$l_H \propto e^{-Ht}$$

hence during inflation l_H decreases, so the source of what causally connect from big bang to end of inflation decreases.

2. Radiation-dominated expansion: $H \propto 1/t$, $a \propto t^{1/2}$, so

$$l_H \propto t^{1/2}$$

hence this is an opposite process more regions become causally connected. But 60 e-folds times of inflation is enough for what's needed.

Scalar-Field Inflation

Now we need to justify what produce the inflation condition (6.29). Consider a universe that is filled with scalar field Φ in Minkowski with Lagrangian

$$\mathcal{L} = \text{"kinetic"-potential} = -\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi - V(\Phi) \tag{6.31}$$

note kinetic term has a strange $-$ sign, but it will be gone,

By (4.11)

$$\begin{aligned} T^\alpha{}_\beta &= -\frac{\partial\mathcal{L}}{\partial\Phi_{,\alpha}}\Phi_{,\beta} + \mathcal{L}\delta^\alpha_\beta \\ &= -\Phi_{,\beta}\frac{\partial\mathcal{L}}{\partial\Phi_{,\alpha}}(-\frac{1}{2}\eta^{\alpha\alpha}\Phi_{,\alpha}\Phi_{,\alpha} - V) + \mathcal{L}\delta^\alpha_\beta \\ &= \Phi_{,\beta}\Phi^{,\alpha} + \mathcal{L}\delta^\alpha_\beta \\ T_{\alpha\beta} &= \Phi_{;\alpha}\Phi_{;\beta} + \mathcal{L}g_{\alpha\beta} \end{aligned} \tag{6.32}$$

Lecture 25 (4/30/14)

One can derive (6.32) without resolving to Minkowski first.

$$\mathcal{L} = -\frac{1}{2}g^{\alpha\beta}\Phi_{;\alpha}\Phi_{;\beta} - V(\Phi)$$

From (4.34)

$$\begin{aligned}
T_{\alpha\beta} &= -\frac{2}{\sqrt{-g}}(\frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{\alpha\beta}} - \underbrace{\frac{\partial}{\partial x^\gamma}\frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}}_{0}) \\
&= -2\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - 2\frac{1}{\sqrt{-g}}\mathcal{L}\frac{\partial\sqrt{-g}}{\partial g^{\alpha\beta}} \\
&= \Phi_{;\alpha}\Phi_{;\beta} - \frac{\mathcal{L}}{g}\frac{\partial g}{\partial g^{\alpha\beta}} \\
&= \Phi_{;\alpha}\Phi_{;\beta} + \mathcal{L}g_{\alpha\beta}
\end{aligned}$$

because (4.29), (4.30)

$$\frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\gamma} \implies \frac{1}{g}\frac{\partial g}{\partial g^{\alpha\beta}} = -g_{\alpha\beta}$$

QED

For homogenous expanding universe (maximum symmetric 3D slice) in co-moving frame, we have

$$\Phi_{,\alpha} = (\dot{\Phi}, 0, 0, 0) \text{ or } \Phi^{,\alpha} = (-\dot{\Phi}, 0, 0, 0) \tag{6.33}$$

then by (6.31)

$$\mathcal{L} = \frac{\dot{\Phi}^2}{2} - V(\Phi)$$

and

$$\begin{aligned}
T^0{}_0 &= -\dot{\Phi}^2 + \mathcal{L} = -\frac{\dot{\Phi}^2}{2} - V(\Phi) \\
T^i{}_j &= \mathcal{L}\delta^i_j = [\frac{\dot{\Phi}^2}{2} - V(\Phi)]\delta^i_j
\end{aligned}$$

We want this to agree with (6.10). So

$$\rho = \frac{\dot{\Phi}^2}{2} + V(\Phi) \qquad p = \frac{\dot{\Phi}^2}{2} - V(\Phi) \tag{6.34}$$

If

$$\frac{\dot{\Phi}^2}{2} \ll V \tag{6.35}$$

then $\rho = -P$ we get the condition we want.

Equation of Motion for Scalar Field

To satisfy (6.35), we look for dynamics equation. From $T^{\alpha\beta}{}_{;\beta} = 0$,

$$T^{t\beta}{}_{;\beta} = \dot{T}^{tt} + \Gamma^t_{ii}T^{ii} + \Gamma^\beta_{\beta t}T^{tt} = \dot{\rho} + 3H(\rho + p) = 0$$

substituting (6.34),

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{dV}{d\Phi} = 0 \tag{6.36}$$

EOM describes the evolution of Φ in Robertson-Walker universe: $\Phi = \Phi(t)$ uniform in space. $3H\dot{\Phi}$ is friction term.

One can derive (6.36) from EL equation (2.37) of Φ first in Minkowski space

$$\mathcal{L} = -\eta^{\alpha\alpha}\Phi_{,\alpha}\Phi_{,\alpha} - V(\Phi)$$

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial \Phi_{,\alpha}} = \frac{\partial \mathcal{L}}{\partial \Phi}$$

gives

$$\eta^{\alpha\alpha}\partial_\alpha\partial_\alpha\Phi = \frac{\partial V}{\partial \Phi}$$

or

$$\Box\Phi = \frac{\partial V}{\partial \Phi}$$

where

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta \text{ d'Alambertian} \tag{6.37}$$

Then go to curved space by equivalence principle

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\Phi = \frac{\partial V}{\partial \Phi} \text{ or } \nabla_\alpha\nabla^\alpha\Phi = \frac{\partial V}{\partial \Phi}$$

Then by (6.33), $\nabla^\alpha \Phi = (-\dot{\Phi}, 0, 0, 0)$ is a vector so Lorentz invariant,

$$\begin{aligned}\nabla_\alpha \nabla^\alpha \Phi &= \partial_\alpha (\nabla^\alpha \Phi) + \Gamma^\alpha_{\alpha\mu} (\nabla^\alpha \Phi) \\ &= -\ddot{\Phi} + \Gamma^\alpha_{\alpha t}(-\dot{\Phi}) = -\ddot{\Phi} - 3H\dot{\Phi}\end{aligned}$$

QED

What kind of $V(\Phi)$ gives (6.35)–slow-roll inflation:

$$\text{strong friction } 3H\dot{\Phi} \approx -V' \tag{6.38}$$

$$\text{flat potential } \frac{V'^2}{3H^2} \ll V \tag{6.39}$$

(6.38) is because (6.35) means over damped, so $\ddot{\Phi}$ is small. (6.39) is because the characteristic time of expansion is $1/H$, so

$$\Delta\Phi \sim \frac{\dot{\Phi}}{H}$$

(6.34) $\Longrightarrow$ $V \approx$ constant, so

$$\Delta V \ll V \implies V'\Delta\Phi \sim \frac{V'\dot{\Phi}}{H} \sim \frac{V'^2}{3H^2} \ll V$$

The study of scale field inflation is yet finished. One can ask when Φ rolls at the bottom of V, oscillation/quantization create particles then decay to another field or convert to thermos energy. This opens a whole new area of fluctuation cosmology.

7 Gravitational Wave

7.1 Linearize Gravitational Wave

We only consider weak gravity field. Let $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $h_{\alpha\beta} \ll 1$

$$\begin{aligned} R^{\alpha}{}_{\beta\mu\nu} &= \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \underbrace{O(\Gamma^2)}_{O(h^2)} \\ &= \partial_\mu [\frac{1}{2}\underbrace{g^{\alpha\lambda}}_{\eta^{\alpha\lambda}}(h_{\lambda\beta,\nu} + h_{\lambda\nu,\beta} - h_{\beta\nu,\lambda})] - (\mu \leftrightarrow \nu) \end{aligned}$$

so we get a linear equation for $h_{\alpha\beta}$. We can simplify further,

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \underbrace{g_{\alpha\sigma}}_{\eta_{\alpha\sigma}} R^{\sigma}{}_{\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} - h_{\beta\nu,\alpha\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu}) && (7.1) \\ R^{\alpha}{}_{\beta\mu\nu} &= \frac{1}{2}(h^{\alpha}{}_{\nu,\beta\mu} - h_{\beta\nu,}{}^{a}{}_{\mu} - h^{\alpha}{}_{\mu,\beta\nu} + h_{\beta\mu,}{}^{\alpha}{}_{\nu}) \end{aligned}$$

$$\begin{aligned} R_{\beta\nu} &= R^{\alpha}{}_{\beta\alpha\nu} = \frac{1}{2}(h^{\alpha}{}_{\nu,\beta\alpha} - \underbrace{h_{\beta\nu,}{}^{a}{}_{\alpha}}_{\Box h_{\beta\nu}} - \underbrace{h^{\alpha}{}_{\alpha,\beta\nu}}_{h_{,\beta\nu}} + \underbrace{h_{\beta\alpha,}{}^{,\alpha}{}_{\nu}}_{h^{\alpha}{}_{\beta,\nu\alpha}}) \\ &= \frac{1}{2}(h^{\alpha}{}_{\nu,\beta\alpha} + h^{\alpha}{}_{\beta,\nu\alpha} - \Box h_{\beta\nu} - h_{,\beta\nu}) \\ &= \frac{1}{2}[\underbrace{h^{\alpha}{}_{\nu,\beta\alpha} - \frac{1}{2}(h\delta^{\alpha}_{\nu})_{,\beta\alpha}}_{\bar{h}^{\alpha}{}_{\nu,\alpha\beta}} + \underbrace{h^{\alpha}{}_{\beta,\nu\alpha} - \frac{1}{2}(h\delta^{\alpha}_{\beta})_{,\alpha\nu}}_{\bar{h}^{\alpha}{}_{\beta,\alpha\nu}} - \Box h_{\beta\nu}] && (7.2) \end{aligned}$$

where we define $\bar{h}^{\alpha}{}_{\beta} = h^{\alpha}{}_{\beta} - \frac{1}{2}h\delta^{\alpha}_{\beta}$.

7.2 Lorentz Gauge

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It turns out there is gauge freedom we can make (7.2) simpler. Consider coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$$

and impose $\xi^\mu(x^\alpha)$ to be a small vector of the same order as $h_{\alpha\beta}$. Then we can show that in new coordinates $\tilde{x}^\mu$ metric tensor is given by

$$\tilde{g}_{\alpha\beta} = \eta_{\alpha\beta} + \tilde{h}_{\alpha\beta}$$

where $\tilde{h}_{\alpha\beta}$ is of the same $h_{\alpha\beta}$. Indeed, from (4.36) and (4.37)

$$\begin{aligned} \tilde{h}^{\alpha\beta}(x) &= h^{\alpha\beta}(x) + \xi^{\alpha;\beta} + \xi^{\beta;\alpha} \\ &= h^{\alpha\beta}(x) + \xi^{\alpha,\beta} + \xi^{\beta,\alpha} \end{aligned}$$

replacing $\xi^{\alpha;\beta}$ by $\xi^{\alpha,\beta}$ because in the linear order of ξ^μ and $h_{\alpha\beta}$.

This shows $h_{\alpha\beta}$ is free to shift by any 4 arbitrary functions of the same order of $h_{\alpha\beta}$, so we can pick 4 functions so that we have

$$\bar{h}^{\alpha}{}_{\beta,a} = 0$$

called Lorentz gauge. Then (7.2) becomes

$$R_{\beta\nu} = -\frac{1}{2}\Box h_{\beta\nu} \tag{7.3}$$

How to pick ξ^μ? Suppose $\bar{h}^{\alpha\beta}{}_{,a} = V^\beta \neq 0$, we want

$$\tilde{h}^{\alpha\beta} = h^{\alpha\beta} + \xi^{\alpha,\beta} + \xi^{\beta,\alpha} \text{ s.t. } \bar{\tilde{h}}^{\alpha\beta}{}_{,a} = 0$$

that is $(\tilde{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\tilde{h})_{,\alpha} = 0$, so

$$\tilde{h}^{\alpha\beta} = h^{\alpha\beta}{}_{,\alpha} + \xi^{\alpha,\beta}{}_{,\alpha} + \xi^{\beta,\alpha}{}_{,\alpha} - \frac{1}{2}\eta^{\alpha\beta}(h + \xi^\mu{}_{,\mu} + \xi^\nu{}_{,\nu})_{,\alpha} = 0$$

so

$$\tilde{h}^{\alpha\beta} = \underbrace{(h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h)_{,\alpha}}_{\bar{h}^{\alpha\beta}{}_{,a}} + \underbrace{\xi^{\alpha,\beta}{}_{,\alpha} - \eta^{\alpha\beta}\xi^\mu{}_{,\mu\alpha}}_{0} + \underbrace{\xi^{\beta,\alpha}{}_{,\alpha}}_{\Box\xi^\beta} = 0 \tag{7.4}$$

so

$$\Box\xi^\beta = -\bar{h}^{\alpha\beta}{}_{,a}$$

7.3 Gravitational Wave in Vacuum

In vacuum $R = 0$, $R_{\alpha\beta} = 0$, in Lorentz gauge by (7.3)

$$\Box h_{\alpha\beta} = 0$$

that is

$$-\frac{1}{c^2}\frac{\partial^2 h_{\alpha\beta}}{\partial t^2} + \nabla^2 h_{\alpha\beta} = 0$$

each component of $h_{\alpha\beta}$ satisfies the wave equation with propagating speed c.

Suppose it is plane wave propagating in $+z$ direction, so no dependence on x, y

$$h_{\alpha\beta} = f_{\alpha\beta}(\underbrace{t - \frac{z}{c}}_{w})$$

for any symmetric functions $f_{\alpha\beta}(w)$. Now incorporate gauge condition.

$$\bar{h}^{\alpha\beta}{}_{,a} = 0 \implies \underbrace{\underbrace{\bar{h}^{0\beta}{}_{,0}}_{\frac{\partial \bar{h}^{0\beta}}{c\partial t}}}_{\frac{d\bar{h}^{0\beta}}{cdw}} + \underbrace{\bar{h}^{1\beta}{}_{,1}}_{0} + \underbrace{\bar{h}^{2\beta}{}_{,2}}_{0} + \underbrace{\underbrace{\bar{h}^{3\beta}{}_{,3}}_{\frac{\partial \bar{h}^{3\beta}}{\partial z}}}_{-\frac{d\bar{h}^{3\beta}}{cdw}} = 0$$

because no x, y dependence. So

$$\frac{d\bar{h}^{0\beta}}{dw} = \frac{d\bar{h}^{3\beta}}{dw} \implies \bar{h}^{0\beta} = \bar{h}^{3\beta} + \text{const}$$

redefine the origin so that const$= 0$. Still Lorentz gauge doesn't complete get rid of extra freedom, from (7.4) we see that even if $\bar{h}^{\alpha\beta}{}_{,a} = 0$, we can still add ξ^β s.t. $\Box\xi^\beta = 0$ and the resulting $\bar{\bar{h}}^{\alpha\beta}{}_{,a}$ will be 0 too. So we have 4 more freedom, we impose

$$\bar{h} = 0 \text{ (traceless) and } h^{3\beta} = 0 \text{ (transverse)}$$

called TT gauge (TT for traceless, transverse). Transverse is because z component is constant or oscillation is in xy plane. Therefore the number of degree of freedom of $f_{\alpha\beta}$ is $10-4-4=2$.

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{12} & -h_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.5}$$

That $h_{\alpha\beta} = \bar{h}_{\alpha\beta}$ is because $h = -\bar{h} = 0$.

7.4 Detection of Gravitational Wave

Just like EM, we can pick any convenient gauges to work with since they don't change physical quantities like $\vec{E}$, $\vec{B}$. What kind of physical quantities we can measure to detect gravitational wave?

Consider two stationary test particles separated by a vector $\vec{S}$. As the wave passes through the metric and the distance between the particles changes. This effect is observed as a tidal acceleration in local orthonormal inertial frame. By (3.22), two stationary test particles $T^\alpha = (1,0,0,0)$

$$\frac{\partial^2 S^\mu}{\partial t^2} = R^\mu{}_{00\nu} S^\nu \tag{7.6}$$

and by (7.1)

$$R_{\mu 00\nu} = \frac{1}{2}(h_{\mu\nu,00} - \underbrace{h_{0\nu,0\mu}}_{0} - \underbrace{h_{\mu 0,0\nu}}_{0} + \underbrace{h_{00,\mu\nu}}_{0})$$

0 is because of (7.5). So (7.6) becomes

$$\frac{\partial^2 S_\mu}{\partial t^2} = \frac{1}{2}\frac{\partial^2 h_{\mu\nu}}{\partial t^2} S^\nu \tag{7.7}$$

Consider a periodic (wave) fluctuation of S_μ of the form

$$\delta S_\mu = \tilde{S}_\mu e^{-iwt+ikz} = \tilde{S}_\mu e^{ik_\alpha z^\alpha}$$

then (7.7) gives

$$\begin{aligned} -w^2 \delta S_\mu &= \frac{1}{2}(-w^2) h_{\mu\nu}(S_\mu + \delta S_\mu) \\ &= \frac{1}{2}(-w^2) h_{\mu\nu} S_\mu \end{aligned}$$

neglect the δS_μ in the leading order. Thus

$$\delta S_\mu = \frac{h_{\mu\nu}}{2} S_\mu$$

In (7.5), we find two modes

1. $h_\times = 0$, $h_+ \neq 0$. Plus mode

$$\delta S_1 = \frac{h_+}{2} S_1 \qquad \delta S_2 = -\frac{h_+}{2} S_2$$

2. $h_+ = 0$, $h_{+\times} \neq 0$. Cross mode

$$\delta S_1 = \frac{h_\times}{2} S_2 \qquad \delta S_2 = \frac{h_\times}{2} S_1$$

The distinction between the two modes is not absolute. Suppose we are at $\times$ mode, now rotate coordinates xy by 45^0,

$$\begin{aligned} \delta \hat{S}_1 &= \frac{\delta S_1}{\sqrt{2}} + \frac{\delta S_2}{\sqrt{2}} = \frac{h_\times}{2\sqrt{2}}(S_2 + S_1) = \frac{h_\times}{2} \hat{S}_1 \\ \delta \hat{S}_2 &= -\frac{\delta S_1}{\sqrt{2}} + \frac{\delta S_2}{\sqrt{2}} = \frac{h_\times}{2\sqrt{2}}(-S_2 + S_1) = -\frac{h_\times}{2} \hat{S}_2 \end{aligned}$$

We arrive + mode.

7.5 Source of Gravitational Wave

Following the same derivation in (7.2), one can get Einstein tensor

$$G_{\alpha\beta} = -\frac{1}{2}(\bar{h}_{\alpha\beta,\mu}{}^{\mu} + \eta_{\alpha\beta} \bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{\mu} - \bar{h}_{\beta\mu,\alpha}{}^{\mu})$$

then using Lorentz gauge, one gets

$$G_{\alpha\beta} = -\frac{1}{2}\Box\bar{h}_{\alpha\beta}$$

so the Einstein equation

$$\Box\bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}$$

wave equation with source $T_{\alpha\beta}$.

Using the Green function

$$-\frac{\delta(\frac{1}{c}\left|\vec{r}-\vec{r}'\right| - (t-t'))\theta(t-t')}{4\pi\left|\vec{r}-\vec{r}'\right|}$$

which solves $\Box\bar{h}_{\alpha\beta} = \delta(\vec{r}')$. We get retarded solution

$$\bar{h}_{\alpha\beta}(t,\vec{r}) = \frac{4G}{c^4}\int\frac{T_{\alpha\beta}(t-\frac{|\vec{r}-\vec{r}'|}{c},\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}' \tag{7.8}$$

This clarifies a common mistake (see Carroll page 159) People think if we are outside a star then the RHS of Einstein equation is 0, but because of the retarded effect. That is not true. It is obvious if we think this in term of EM, outside of an electric charge in vacuum

$$\Box\phi(\vec{x}) = \delta(\vec{x}) \neq 0$$

Now consider a specific example: an oscillating $T_{\alpha\beta}$

$$T_{\alpha\beta} = a_{\alpha\beta}(x^\mu)e^{-iwt} \tag{7.9}$$

assume $T_{\alpha\beta} \neq 0$ in a region $r' \ll c/w$ so the motion of source matter is $v \ll c$ to ensure weak field.

$$\bar{h}_{\alpha\beta}(t,\vec{r}) = \frac{4G}{c^4}\int\frac{a_{\alpha\beta}(\vec{r}')e^{-iw(t-\frac{|\vec{r}-\vec{r}'|}{c})}}{|\vec{r}-\vec{r}'|}\underbrace{d^3\vec{r}'}_{dV}$$

If $r \gg r'$, far field approximation

$$\bar{h}_{\alpha\beta}(t,\vec{r}) \approx \frac{4G}{c^4}\frac{e^{iwr}}{r}\int T_{\alpha\beta}dV$$

It is possible to express the oscillating $\int T_{ik}dV$ in terms of integrals of T_{00}, energy density only. Assume small $T^{\mu\nu}$ ($a_{\alpha\beta}(x^\mu) \approx$ const), weak gravitational field, and approximate the spacetime as flat.

$$T^{\mu\nu}{}_{,\mu} = 0 \implies \frac{iw}{c}T^{0\mu} + \partial_l T^{l\mu} = 0$$

then

$$\int T^{lk}dV = \int \underbrace{\delta^l_m}_{\partial_m x^l} T^{mk}dV = \underbrace{\int \partial_m(x^l T^{mk})dV}_{0} - \int x^l \underbrace{\partial_m T^{mk}}_{-\frac{iw}{c}T^{0k}}dV = \frac{iw}{c}\int x^l T^{0k}dV$$

Note that T^{lk} is symmetric, so

$$\int x^l T^{0k}dV = \int x^k T^{0l}dV$$

Do IBP again

$$\begin{aligned}
\int x^l T^{0k}dV &= \int x^l(\partial_m x^l)T^{0m}dV \\
&= \underbrace{\int \partial_m(x^l x^k T^{0m})dV}_{0} - \int x^k \underbrace{\partial_m(x^l T^{0m})}_{\delta^l_m T^{0m} + x^l\partial_m T^{0m}} dV \\
&= -\underbrace{\int x^k T^{0l}dV}_{\int x^l T^{0k}dV} - \int x^k x^l \underbrace{\partial_m T^{0m}}_{-\frac{iw}{c}T^{00}}dV
\end{aligned}$$

thus

$$\int x^l T^{0k}dV = -\frac{iw}{2c}\underbrace{\int x^k x^l T^{00}dV}_{\equiv I^{kl}}$$

define quadrupole. Therefore

$$\bar{h}^{lk} = -\frac{2G}{c^6}\frac{e^{iwr}}{r}w^2 I^{lk}$$

This is true for any Fourier mode (7.9), so for general $T_{\alpha\beta}$,

$$\bar{h}_{lk} = \frac{2G}{c^6 r}\frac{d^2 I_{lk}}{dt^2} \qquad lk = 1, 2, 3$$

$\bar{h}^{0\alpha}$ can be found from the gauge conditions

$$\bar{h}^{0\alpha} = -\frac{ic}{w}\partial_l \bar{h}^{lk}$$

In TT gauge I^{lk} should be replaced by its TT part

$$I^{TT}_{lk}(P^m_l P^i_k - \frac{1}{2}P_{lk}P^{mi})I_{mi}$$

where $P_{ij} = \delta_{ij} - n_i n_j$ and n_i unit vector in the wave direction.