Introduction to Classical & Quantum Waves

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This is an undergraduate course. Offered in Fall 2013 at Columbia University. Required Course textbooks: Fitzpatrick, *Oscillations and Waves : an introduction;* Harris, *Modern Physics.*

Contents

1 Classical Waves

1.1 Transverse Standing Wave

Lecture 1 (9/4/13)

A beaded string is fixed at two ends. Some vertical motion is going on on the string. The vertical displacements are small so that small angle approximation is valid. Each beat is a apart and the position is labeled

$$x_i = ia$$

First and last beats

$$y(x_0,t) = y_0 = 0 \quad y(x_{N+1},t) = y_{N+1} = 0 \;\forall t$$

T =tension in the string. Consider the forces on ith beat

one T is pulling from the right due to the $i+1$ beat, make an angle θ_{i+1}

another T is pulling from the left due to the $i-1$ beat, make an angle θ_{i-1}

Hence the total vertical force on ith beat is

$$m\ddot{y} = T\sin\theta_{i-1} - T\sin\theta_{i+1} = T(\theta_{i+1} - \theta_{i-1})$$

On the other hand

$$\theta_{i+1} = \tan\theta_{i+1} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{a}$$

and

$$\theta_{i-1} = \frac{y_i - y_{i-1}}{a}$$

Thus

$$\ddot{y}_i = w_0^2(y_{i+1} - 2y_i + y_{i-1}) \quad w_0^2 \equiv \frac{T}{ma} \tag{1.1}$$

Guess solution is a product of space & time factors

$$y_i(x_i,t) = A\sin kx_i\cos(wt - \phi) \tag{1.2}$$

so that 1) for fixed time, see a wave guide in space; 2) for fixed space, see oscillation in time.

$$
\begin{aligned}
A &= \text{amplitude} \\
k &= \frac{2\pi}{\lambda} \text{ wave number} \\
w &= 2\pi\nu \text{ angular freq} \\
\phi &= \text{phase}
\end{aligned}
$$

Substitute (1.2) to (1.1)

$$
\begin{aligned}
-w^2 \sin kx_i &= w_0^2 \left[\sin k(x_i + a) - 2\sin kx_i + \sin k(x_i - a)\right] \\
&= 2w_0^2 \sin kx_i (\cos ka - 1)
\end{aligned}
$$

or

$$w = 2w_0 \sin \frac{ka}{2} \tag{1.3}$$

a relation between w and k is known as dispersion relation. Hence (1.2) is a solution to (1.1) iff (1.3) holds.

What controls k? The boundary.

Boundary can be fixed or free, (freely slide up and down). For the case that 2 ends are fixed. The $\sin kx$ factor in (1.2) automatically satisfies one boundary condition. The other is

$$y_{N+1} = 0$$

so

$$k(N+1)a = n\pi \implies k_n = \frac{n\pi}{(N+1)a} \quad n = 1, 2, 3, ..., N+1$$

As we will see the n label is called the nth mode. When $n = 1$, the standing wave has 1/2 of its full in the clamp. If $n = 2$, one full wave in the clamp. The reason n has to stop at N, because at this point

$$\lambda_{N+1} = \frac{2\pi}{k_{N+1}} = \frac{N+1}{N+1} 2a$$

that is each beat is off phase to its neighbor, and it is the shortest wavelength one can generate on a N beat string.

What controls ϕ? The initial condition. If the initial wave front is $A\sin kx$, $\phi = 0$.

In summary: in our class there is a common routine

1) Guess the functional form of differential equation

2) Plug guess solution and find dispersion relation

3) find condition on k that satisfies boundary condition.

Now study uniform string, so no beats, continuum string.

Change

$$\begin{aligned} y_i(x_i,t) &\rightarrow y(x,t) \\ a &\rightarrow \delta x \\ m &\rightarrow \rho\delta x \end{aligned}$$

ρ =length density

$$y_{i+1} \rightarrow y(x+\delta x, t)$$

so (1.1) becomes

$$\ddot{y}(x,t) = \frac{T}{\rho}\frac{y(x+\delta x,t) - 2y(x,t) + y(x-\delta x,t)}{\delta x^2}$$

hence putting

$$v^2 = \frac{T}{\rho}$$

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{y(x,t) + \frac{\partial y}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}\delta x^2 - 2y(x,t) + y(x,t) - \frac{\partial y}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}\delta x^2}{\delta x^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

Or commonly write

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \tag{1.4}$$

equation of wave, with traveling speed v.

Lecture 2 (9/9/13)

$$y_n = A_n \sin\frac{n\pi x}{(N+1)a}\cos(w_n t - \phi_n)$$

is known as the nth normal mode solution. Why are they called? Because we will show energy is stored in modes and because modes represent natural frequency of system, which arise to resonance.

The study of the string immediately generalized to other systems of wave equation:

EM wave in vacuum (transverse)

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \quad \frac{\partial^2 B}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0$$

$c = 1/\sqrt{\mu_0 \epsilon_0}$

Sound Wave (longitudinal)

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{\left(\frac{\gamma k T}{M}\right)}\frac{\partial^2 P}{\partial t^2} = 0$$

P pressure, γ =specific heat, M =molecular mass.

Sound wave in solid (or compression wave)

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{\left(\frac{Y}{\rho}\right)}\frac{\partial^2 P}{\partial t^2} = 0$$

Y =Young's modules

Transmission Line

$$\frac{\partial^2 V}{\partial x^2} - LC\frac{\partial^2 V}{\partial t^2} = 0$$

Plug $y = A\sin kx\cos(wt - \phi)$ to wave equation, we get

$$w = kv \tag{1.5}$$

dispersion relation, connecting spatial change with time.

$$2\pi\nu = \frac{2\pi}{\lambda}v \implies v = \nu\lambda = \frac{\lambda}{T}$$

Two assumptions in deriving (1.4): 1) small amplitudes, so the system is linear (small angle approximation) 2) no dissipation. In reality rubber band gets hot, so (1.5) is not exact.

Experiment that products the standing wave on the string is to take an electrical vibrator moving up and down, attack the string to one end (this end is moving with vibrator but we still consider this as a fixed end, because it moves relatively small wrt the anti nodes), and the other end is hanging over a weight on a pulley so that the tension is known. Then adjust the vibrator frequency s.t.

$$L = n\frac{\lambda}{2} \implies \nu = \frac{v}{\lambda} = \frac{\sqrt{T/\rho}}{2l/n}$$

to achieve standing wave.

The mathematics describing longitudinal waves are exactly the same, the displacement y becomes pressure P (or more precisely Gauge pressure, $\tilde{P} = P - P_a =$ ambient pressure $-$ atmospheric pressure). Node maximum y becomes maximum pressure. Anti node becomes 0 gauge pressure. Opposite to string boundaries.

1.2 Traveling Wave

We turn a function $f(x)$ into a traveling wave

$$f(x) \to f(x - vt)$$

so at $t = 0$ for a point x_s on the wave with displacement $f(x_s)$. The same displacement appears at later time t at $x = x_s + vt$. So

$$f(x - vt) \text{ travel to the right}$$

$$f(x + vt) \text{ travel to the left}$$

One can check $y = f(x \pm vt)$ solves the wave equation for any $f(x)$, but we are only interested in certain $f(x)$. Primarily

$$y = y_0 \cos kx \quad k = \frac{2\pi}{\lambda}$$

Periodic function. That is because many interesting physical system is periodic.

So

$$y(x,t) = y_0 \cos k(x - vt) \quad w = kv$$

This is the most common equation for a traveling wave.

Snapshot at $t = t_0$, see y v.s. x a sinusoidal wave. Fix position $x = x_0$, see x_0 moving up and down. If slicing out the t axis, get y v.s. t a sinusoidal oscillation.

1.3 2D Wave

Two ideal models: plane wave (e.g. $\vec{E}$ wave) & sphere wave (e.g. water surface bubble).

One way to illustrate plane wave on a piece of paper is to draw some surface of constant amplitude (or called surface of constant phase), for fixed time.

$$\begin{array}{ccccccccc} & & \uparrow & & & \downarrow & & & \uparrow \\ & \uparrow & & & \downarrow & & & \uparrow & \\ \uparrow & & & \downarrow & & & \uparrow & & \end{array}$$

Similarly draw concentric circles to illustrate sphere wave and the source is at the center of the circles. One can get plant wave from point source of sphere wave by getting far away from the source.

1.4 Complex Notations

Wave can interfere with themselves. From the wave equation, we see that if y_1, y_2 are two solutions, then $y_1 + y_2$ are too solution.

Consider superposition of two waves traveling to the left and right,

$$y = A\cos(wt - kx) \pm A\cos(wt + kx) = \begin{cases} 2A\cos kx \cos wt & \text{for } + \text{ constructive} \\ 2A\sin kx \sin wt & \text{for } - \text{ destructive} \end{cases} \tag{1.6}$$

showing that one can construct standing wave solution from traveling waves.

Physically it means, refereed to experimental setup in the end of 1.1, that the vibrator sends a wave to the right and gets reflected by the weight over the pulley, whether should take $\pm$ depends whether it has a π phase change. We will discuss them later.

Schrodinger equation has only complex traveling solution, so we need to study complex notation

$$\begin{aligned} z &= x + iy \\ z^* &= x - iy \end{aligned}$$

$$\Re z = x \quad \Im z = y$$

$$z^* z = x^2 + y^2 = |z|^2 \text{ modules}$$

Second way writing $z \in \mathbb{C}$

$$z = re^{i\theta} \qquad r = \sqrt{|z|^2} \qquad \tan\theta = \frac{y}{x} \qquad z^* = r^{-1}e^{-i\theta}$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$, write

$$\tilde{y} = Ae^{i(wt-kx)} \quad w = kv$$

$\tilde{y}$ denotes complex solution. For classical problem take the real part at the end of calculation, but for QM solutions are intrinsic complex, need to figure out a rule to connect complex solution to the real world.

Back to 1.6, we use complex notation to manipulate. First complexize

$$\tilde{y} = Ae^{i(wt-kx)} + Ae^{i(wt+kx)} = 2Ae^{iwt}\frac{e^{-ikx} + e^{ikx}}{2} = 2Ae^{iwt}\cos kx$$

then take real part

$$y = \Re\tilde{y} = 2A\cos wt\cos kx$$

The other one

$$\tilde{y} = Ae^{i(wt-kx)} - Ae^{i(wt+kx)} = 2iAe^{iwt}\frac{e^{-ikx} - e^{ikx}}{2i} = -2iAe^{iwt}\sin kx$$

take real part

$$y = \Re\tilde{y} = 2A \sin wt \sin kx$$

Lecture 3 (9/11/13)

We will see that complex form is more useful for dissipation. Then dispersion relation comes simply from complex form.

1.5 Energy Conservation

As wave moves down the string, it takes energy with it. Oscillation up and down gets KE

$$KE = \int_0^l \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 dx \ (dm = \rho dx)$$

Tension, T, $\leftrightarrow$ Potential energy. dx is the natural length of the string between x and $x + dx$. ds is arc length of that segment of string from x to $x + dx$

$$\begin{aligned} UE &= \int_0^l T(ds - dx) \\ &= \int T(\sqrt{dx^2 + dy^2} - dx) \\ &= \int T[dx(1 + \frac{1}{2}(\frac{\partial y}{\partial x})^2) - dx \\ &= \int T\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx \end{aligned}$$

$$E = \int \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 dx + \int \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \int \epsilon dx \tag{1.7}$$

ϵ energy density

$$\epsilon = \frac{1}{2}\left[\rho\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]$$

Figure out rate of energy going down the string

$$I \equiv \underbrace{-T\frac{\partial y}{\partial x}}_{F_y}\frac{\partial y}{\partial t}$$

By the small angle approximation $F_y = T\tan\theta = T\sin\theta$ =force of tension in y

direction, so I has unit of power.

Then we get continuity equation for energy

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial I}{\partial x} = 0$$

Indeed

$$\rho y_t y_{tt} + T y_x y_{xt} - T y_{xx} y_t - T y_x y_{xt} = y_t T(\frac{\rho}{T} y_{tt} - y_{xx}) = y_t T(\frac{1}{v^2} y_{tt} - y_{xx}) = 0$$

Then

$$\frac{d}{dt} \underbrace{\int_0^l \epsilon dx}_{E} + \underbrace{\int_0^l dx \frac{\partial I}{\partial x}}_{I(l)-I(0)} = 0$$

Or

$$\dot{E} = \text{something flow in at } x = 0 - \text{st flow out at } x = l$$

Hence I is the rate of energy stored in the string.

For $y = A\cos(wt - kx)$

$$I = TkA^2 w \sin^2(wt - kx)$$

time average

$$\langle I \rangle = \frac{1}{2} TkA^2 w \tag{1.8}$$

Recall

$$\left\langle \sin^2(wt - kx) \right\rangle = \frac{1}{T} \int_t^{t+T} \sin^2(wt' - kx) dt' = \left\langle \cos^2(wt - kx) \right\rangle = \frac{1}{2}$$

Using the natural characters of the string

$$\langle I \rangle = \frac{1}{2} T \frac{w}{v} A^2 w = \frac{1}{2} \sqrt{\rho T} w^2 A^2 = \frac{1}{2} Z w^2 A^2 \tag{1.9}$$

Define impedance

$$Z = \sqrt{\rho T} = \frac{T}{v} = \rho v$$

Later we will do two strings transmission and reflection, Z will be a deciding factor, since w and A in (1.9) are independent of medium. What is impedance?

$$Z = \frac{T}{v} = \frac{T}{\frac{\partial x}{\partial t}} = \frac{T\frac{\partial y}{\partial x}}{\frac{\partial y}{\partial t}} = \frac{F_y}{v_y}$$

$$v_y = \frac{F_y}{Z}$$

Hence high Z means need more force to vibrate the string, so the string is stiffer.

Here we consider $y = A\cos(wt - kx)$ wave traveling to the right. If we do $y = A\cos(wt + kx)$, we will get

$$\langle I \rangle = -\frac{1}{2} Z w^2 A^2$$

Now plug $y = A\cos(wt - kx)$ into(1.7)

$$\langle E \rangle = \frac{1}{2}\frac{1}{2}\rho w^2 A^2 l + \frac{1}{2}\frac{1}{2} k^2 A^2 T l \tag{1.10}$$

The two terms on the right are the same, agrees what we know about harmonics $\langle KE \rangle = \langle UE \rangle$.

One can define the average energy flux

$$\langle \tau \rangle = \langle \epsilon \rangle v = \frac{\langle E \rangle}{l} v$$

E.g. $\vec{E}$ wave

$$\text{flux} = \frac{1}{2}\epsilon_0 E^2 c$$

Later we will do mode decomposition, which has many applications including Hawking radiation by the black holes. We will revisit the energy of the wave.

1.6 Transmission & Reflection Waves

Feynman said that if our civilization is about to end and there is only one piece information can be stored and passed on to future ETs that should be "particles are waves". Because QM was developed based on the idea of wave mechanics. Particularly we will see the transmission and reflection waves go hand to hand in

QM. In quantum scattering problem

$$\begin{aligned} w &\sim \text{energy} \\ k &\sim \text{momentum} \\ |A|^2 &\sim \text{probability} \end{aligned}$$

Suppose two strings with different impedance ($\rho_1 > \rho_2$ impedance mismatch) are connected together. A wave, incident wave y_i, is traveling on string 1 and passing to string 2. What is reflected wave y_r? transmitted wave y_t?

$$y_i = A_i e^{i(k_1 x - wt - \phi_i)}$$

or

$$y_i = A_i e^{i(k_1 x - wt)}$$

allow A_i to be complex. Many statements proved above will not change much, like (1.8) becomes

$$\langle I \rangle = \frac{1}{2} T k |A|^2 w$$

Similarly use complex amplitude

$$\begin{aligned} y_r &= A_r e^{i(-k_1 x - wt)} \\ y_t &= A_t e^{i(k_2 x - wt)} \end{aligned}$$

Frequencies are the same. It is equal to the vibrator's frequency regardless of the stiffness of the string. $k_r = k_i$ because $w = kv$, since w, v are the same for string 1.

At the interface $x = 0$, there are two matching conditions: y displacement must be continuous, in QM it means probability is continuous; energy flux $T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}$ must be continuous, in QM it means flux of particles go in out should be the same, or

$$F_y = T \frac{\partial y}{\partial x} \text{ must be continuous}$$

if not, then at x near 0, $dm \to 0$ so there are ∞ acceleration.

From the two conditions above

$$\text{At } x = 0, \quad y_i + y_r = y_t \implies A_i + A_r = A_t$$

$$\left.\frac{\partial(y_i + y_r)}{\partial x}\right|_{x \to 0^-} = \left.\frac{\partial y_t}{\partial x}\right|_{x \to 0^+} \implies -k_1 A_i + k_1 A_r = -k_2 A_t$$

Since $Z \sim 1/v \sim k$

$$(A_i - A_r) Z_1 = A_t Z_2$$

Therefore

$$\frac{A_r}{A_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \text{ and } \frac{A_t}{A_i} = \frac{2Z_1}{Z_1 + Z_2} \tag{1.11}$$

Take $Z_2 \to \infty$

$$v_y = \frac{F_{y2}}{Z_2} \to 0 \quad \text{or} \quad \frac{A_t}{A_i} \to 0$$

no much oscillation transmitted to heavy cable.

$$\frac{A_r}{A_i} = -1$$

total reflection with a π phase shift.

Lecture 4 (9/16/13)

So for this case, on string 1 we have superposition of two opposite moving waves, assume $\phi = 0$

$$\tilde{y}_1 = A_i e^{i(wt - kx)} - A_i e^{i(wt + kx)} = 2i A_i e^{iwt} \sin kx$$

Real part

$$y_1 = 2A_i \sin wt \sin kx$$

$k_n l = n\pi$ due to fixed 2 ends boundary.

By (1.9), we compute ratio energy reflected

$$R = \frac{\frac{1}{2} Z_1 w^2 |A_r|^2}{\frac{1}{2} Z_1 w^2 |A_i|^2} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \tag{1.12}$$

ratio of energy transmitted

$$T = \frac{\frac{1}{2} Z_2 w^2 \left|A_t\right|^2}{\frac{1}{2} Z_1 w^2 \left|A_i\right|^2} = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2} \tag{1.13}$$

Check

$$R + T = 1$$

1.7 Time Evolutions of Strings

Powerful technique: modal decomposition. You will see it is used in QFT, excitation of black holes, and black body radiation, etc.

General solution

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \left(\frac{n\pi v}{l} t - \phi_n \right)$$

$$y(x,0) = y_0(x) \text{ can be any given shape}$$

$$y_0(x) = \begin{cases} \frac{2A}{l} x & x < \frac{l}{2} \\ \frac{2A}{l}(1-x) & x > \frac{l}{2} \end{cases}$$

need another initial condition

$$v_y(x,0) = v_0(x)$$

Suppose $v_0(x) = 0$ $\forall x$ string start out at rest

$$y_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \phi_n$$

$\exists$ ∞ many constants A_n, ϕ_n to compute. Recall orthogonality

$$\frac{2}{l} \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \delta_{nm}$$

Consider

$$\frac{2}{l}\int_0^l y_0(x)\sin\frac{N\pi x}{l}dx = \frac{2}{l}\int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\phi_n \sin\frac{N\pi x}{l}dx = \sum_{n=1}^{\infty} A_n\delta_{nN}\cos\phi_n = A_N\cos\phi_N \tag{1.14}$$

We will show $\phi_N = 0$ for any N, so

$$\begin{aligned} A_N &= \frac{2}{l}\int_0^l y_0(x)\sin\frac{N\pi x}{l}dx \\ &= 2A\frac{\sin\frac{N\pi}{2}}{\left(\frac{N\pi}{2}\right)^2} \end{aligned}$$

So $A_n = 0$ if n is even. One can see that higher mode have smaller amplitude due to convergent requirement. The lowest frequency, i.e. lowest n gives highest amplitude.

Why $\phi_N = 0$?

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = v_0$$

$$\sum_{n=1}^{\infty} -A_n\frac{n\pi v}{l}\sin\frac{n\pi x}{l}\sin\phi_n = v_0$$

Similarly

$$0 = \frac{2}{l}\int_0^l v_0(x)\sin\frac{N\pi x}{l}dx = \frac{2}{l}\int_0^l \sum_{n=1}^{\infty} -A_n\frac{n\pi v}{l}\sin\frac{n\pi x}{l}\sin\phi_n\sin\frac{N\pi x}{l}dx = \frac{A_N N\pi v}{l}\sin\phi_N \tag{1.15}$$

since $A_N = 0$ is not good, $\phi_N = 0$ for all N.

1.8 Energy Stored in the Mode

From (1.10), we see that energy stored in each mode is

$$\langle E_n\rangle \propto A_n^2$$

and from (1.15), (1.15) one can define

$$
\begin{aligned}
c_n &= \frac{2}{l}\int_0^l y_0(x)\sin\frac{n\pi x}{l}dx = A_n\cos\phi_n \\
s_n &= \frac{2}{n\pi v}\int_0^l v_0(x)\sin\frac{N\pi x}{l}dx = A_n\sin\phi_n
\end{aligned}
$$

For the case considered above

$$A_n^2 = s_n^2 + c_n^2$$

The total energy

$$\langle E\rangle = \sum_{n=1}^{\infty}\frac{1}{2}\rho l w_n^2 A_n^2$$

2 Wave Applications

2.1 Transmission lines

The most common type of transmission line is coaxial cable that is made of two conductors, the code grounded and the shell connecting to power line, or the other way. Below is a structure diagram of transmission line, the bottom line is grounded.

$$
\begin{array}{ccccccccccc}
 & \underline{\qquad\qquad} & \overset{L}{\bowtie} & \underline{\qquad\qquad} & & \bowtie & \underline{\qquad\qquad} & \bowtie & \underline{\qquad\qquad} & & \\
 & | & & | & & & | & & | & & \\
\cdots & = & & = & C & & = & & = & & \cdots \\
 & | & & | & & & | & & | & & \\
 & \underline{\qquad\qquad} & \underline{\quad} & \underline{\qquad\qquad} & & \underline{\quad} & \underline{\qquad\qquad} & \underline{\quad} & \underline{\qquad\qquad} & &
\end{array}
$$

If the code has radius a and shell has radius b, one can show the capacitance

$$C = \frac{2\pi\epsilon_r\epsilon_0}{\ln\frac{b}{a}} = \frac{\text{capacitance}}{\text{unit length}}$$

Let L be inductance per unit length. Consider one unit of cable, assuming current flowing from left to right is positive direction,

$$
\begin{array}{cccc}
\underline{\scriptstyle \to I(x,t)} & \bowtie & \underline{\qquad\qquad} & \underline{\scriptstyle \to I(x+\delta x,t)} \\
\uparrow & & | & \uparrow \\
V(x,t) & & = & V(x+\delta x,t) \\
\downarrow & & | & \downarrow \\
\underline{\qquad\qquad} & - & \underline{\qquad\qquad} & \underline{\qquad\qquad}
\end{array}
$$

Then V across inductor is

$$V_L = L\frac{\partial I}{\partial t}$$

so

$$V(x,t) - V(x+\delta x,t) = L\frac{\partial I(x,t)}{\partial t}$$

or

$$\frac{V(x+\delta x,t) - V(x,t)}{\delta x} = -\frac{L}{\delta x}\frac{\partial I}{\partial t}$$

denote $L/\delta x = L$ be the unit inductance, hence

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}$$

On the other hand current across the capacitor is

$$I_C = \frac{\partial}{\partial t}Q = \frac{\partial}{\partial t}CV(x+\delta x,t)$$

similarly denote $C/\delta x = C$ be the unit capacitance,

$$I(x,t) - I(x+\delta x,t) = C\frac{\partial V(x,t)}{\partial t}$$

so

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \tag{2.1}$$

Therefore

$$\frac{\partial^2 V}{\partial x^2} - LC\frac{\partial^2 V}{\partial t^2} = 0$$

and same wave equation for I. So $v = 1/\sqrt{LC}$. One solution is

$$V = V_0 \cos(kx - wt) \quad w = kv$$

By (2.1)

$$I = V_0 \frac{w}{k} C \cos(kx - wt)$$

suggests

$$Z = \frac{V}{I} = \frac{1}{Cv} = \sqrt{\frac{L}{C}} \tag{2.2}$$

One can also get above expression from the analogy RLC oscillator with a Mass-Spring oscillator

$$\begin{aligned}
\text{mass } m &\leftrightarrow \text{ inductance} \\
\text{stiffness } k &\leftrightarrow 1/C \text{ capacitance} \\
\text{damping factor} &\leftrightarrow R \\
\text{KE } \frac{1}{2}mv^2 &\leftrightarrow \text{ magnetic energy } \frac{1}{2}LI^2 \\
\text{UE } \frac{1}{2}kx^2 &\leftrightarrow \text{ electric energy } \frac{Q^2}{2C^2}
\end{aligned}$$

In RHS of (2.2) both quantities are per unit length, and they cancel out, so later when we do resistors, we put

$$Z = R$$

make no reference to unit length.

One can get impedance of free space

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$$

and impedance of dielectric medium by arguing that

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0 \epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{n}$$

define index of refraction

$$n = \sqrt{\epsilon_r}$$

then

$$Z = \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}} = \frac{Z_0}{n} \tag{2.3}$$

For completeness what can we say about reflection ratio R and transmitted ratio T when light is passing through interface of two mediums, n_1, n_2 normally (require normality because we derived R and T from 1D wave.)

From (2.3), (1.12) and (1.13)

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 \qquad T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

2.2 Impedance Matching

Given two transmission lines with impedance Z_1, Z_3. Is it possible to insert a line l, Z_2 in between Z_1, Z_3 so that 100% energy transfer from $Z_1$to Z_3? Yes. The idea is to have destructive interference.

Let the interface of $Z_1$and Z_2 be $x = 0$

	in line Z_1	in line Z_2	in line Z_3
moving to right	$A_1 e^{i(k_1 x - wt)}$	$A_2 e^{i(k_2 x - wt)}$	$A_3 e^{i(k_3(x-l) - wt)}$
moving to left	$B_1 e^{i(-k_1 x - wt)}$	$B_2 e^{i(-k_2 x - wt)}$	

A_1, A_2, A_3, B_1 and B_2 can be complex. Factor out extra term $e^{ik_3 l}$ from A_3 makes calculation easier.

Lecture 5 (9/18/13)

$x = 0$

$$A_1 + B_1 = A_2 + B_2$$

$T\frac{\partial y}{\partial x}$ is continuous

$$Z_1(A_1 - B_1) = Z_2(A_2 - B_2)$$

$x = l$

$$A_2 e^{ik_2 l} + B_2 e^{-ik_2 l} = A_3$$

$$Z_2(A_2 e^{ik_2 l} - B_2 e^{-ik_2 l}) = Z_3 A_3$$

Four equations for A_1, A_2, A_3, B_1 and B_2, 5 unknowns, want

$$T = \frac{\frac{1}{2} Z_3 w^2 |A_3|^2}{\frac{1}{2} Z_1 w^2 |A_1|^2} = 1 \tag{2.4}$$

One possible solution is to choose

$$k_2 l = \frac{\pi}{2} \iff l = \frac{\lambda_2}{4}$$

absorb i into A_3, the last two of the four equations simplify to

$$\begin{cases} A_2 - B_2 = A_3 \\ Z_2(A_2 + B_2) = Z_3 A_3 \end{cases}$$

then

$$\begin{aligned} 2A_2 &= (1 + \frac{Z_3}{Z_2}) A_3 \\ 2B_2 &= (\frac{Z_3}{Z_2} - 1) A_3 \end{aligned}$$

$$2A_1 = (1 + \frac{Z_2}{Z_1}) A_2 + (1 - \frac{Z_2}{Z_1}) B_3$$

$$4A_1 = \left((1 + \frac{Z_2}{Z_1})(1 + \frac{Z_3}{Z_2}) + (1 - \frac{Z_2}{Z_1})(\frac{Z_3}{Z_2} - 1) \right) A_3 = 2 \left(\frac{Z_2}{Z_1} + \frac{Z_3}{Z_2} \right) A_3$$

By (2.4)

$$\frac{Z_3}{Z_1} = \frac{1}{4} \left(\frac{Z_2}{Z_1} + \frac{Z_3}{Z_2} \right)^2 \implies 2\sqrt{Z_1 Z_3} = \frac{Z_2^2 + Z_1 Z_3}{Z_2} \implies Z_2^2 - 2Z_2\sqrt{Z_1 Z_3} + Z_1 Z_3 = 0$$

Therefore

$$Z_2 = \sqrt{Z_1 Z_3} \tag{2.5}$$

This is known as matching impedance by a quarter wave transformer. There is a analogy in QM, called Ramsauer-Townsend effect, that particles passing through some inert gas without any reflection.

Clearly (2.5) works for strings too

$$\rho_2 = \sqrt{\rho_1 \rho_3}$$

2.3 Loaded Transmission Line

Suppose a coaxial cable Z is connected to a source at $x = -\infty$ with frequency w and it is loaded to a resistor $R = Z_L$ at $x = 0$. So the voltage pulse traveling down the cable to the device. Let

$$\begin{aligned} V_+ &= \text{voltage pulse traveling to the right} \\ V_- &= \text{voltage pulse reflected back to the left} \\ I_+ &= \text{current pulse traveling to the right} \\ I_- &= \text{current pulse reflected back to the left} \end{aligned}$$

Suppose $I_+ = I_{0+} \cos(wt - kx)$, $I_- = I_{0-} \cos(wt + kx)$, we know

$$V_+ = ZI_+ \qquad V_- = -ZI_-$$

At $x = 0$

$$Z_L(I_+ + I_-) = V_+ + V_- = Z(I_+ - I_-)$$

hence

$$\frac{I_{0-}}{I_{0+}} = \frac{Z - Z_L}{Z + Z_L}$$

So the ratio of power absorbed by the load to the power sent down the line is

$$T = 1 - \left(\frac{I_{0-}}{I_{0+}}\right)^2 = \frac{4ZZ_L}{(Z + Z_L)^2}$$

What happen if we short the circuit, $Z_L = 0$?

$$I_{0-} = I_{0+}$$

We will see V lagging I by 90^0, commonly seen if you work with oscilloscopes.

$$\tilde{V} = V_0 e^{i(wt-kx)} - V_0 e^{i(wt+kx)} = -2iV_0 e^{iwt} \sin kx$$

$$\tilde{I} = I_0 e^{i(wt-kx)} + I_0 e^{i(wt+kx)} = 2I_0 e^{iwt} \cos kx$$

$$\Re\tilde{V} = V(x,t) = 2V_0 \sin kx \sin wt$$

$$\Im\tilde{I} = I(x,t) = 2I_0 \cos kx \cos wt$$

What happen if we put $Z_L = Z$?

Then

$$I_- = 0$$

no reflection. All energy goes down the line. From power source perspective, it is generating energy loaded to a ∞ long cable.

The idea above is the starting point of microchannel plates (MCPs), which can capture sequential images of charged particle events at very high speeds. MCP is a slab with a regular array of tiny slots. Slots are parallel to each other. A particle or photon that enters one of the channels is guaranteed to reach the other end, because the load

$$Z_L = Z$$

Cascade of electrons that propagates through the channel amplifies the original signal by several orders of magnitude. MCPs were widely used in magnetic plasma (nuclear weapon) and laser plasma (e.g. take a snapshot of the plasma every 1 picosecond see how it cools down)

2.4 2D, 3D Waves

Plane Wave

Lecture 6 (9/23/13)

For example electric filed propagating in $\hat{x}$ direction and vibrating $\hat{y}$ direction

$$E_y = y_0 \cos(wt - kx)$$

How to describe a plane wave traveling along not just $\hat{x}$ but $\hat{n}$ with wave length λ?

Clearly $\hat{n}$ is orthogonal to the wave front. Let $\vec{r}$ be some arbitrary position point vector. Let $\vec{d}$ be the shortest distance from the origin to the wave front that $\vec{r}$ ends. Let

$$\vec{k} = \frac{2\pi}{\lambda}\hat{n}$$

then

$$\vec{k} \cdot \vec{r} = kd \equiv \Delta\phi$$

$\Delta\phi$ phase difference between the sheet where $\vec{r}$ ends and the phase at the origin, indeed because if $d = \lambda$, $kd = 2\pi$ no phase difference. Then assuming phase at the origin is wt, thus

$$\vec{\psi} = \vec{\psi}_0 \cos(wt - \vec{k} \cdot \vec{r}) \tag{2.6}$$

For those mathematical less picky, can see (2.6) is immediately true. Because plane wave is independent of coordinate, choose $\hat{x}$ along $\hat{k}$ and $\hat{y}$ along $\vec{\psi}_0$ then $\psi_0 \cos(wt - kx)$ has to become (2.6).

We can work backward to see what kind of 3D wave equation that will give solution (2.6). That is

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\psi(\vec{r}, t) - \frac{1}{v^2}\partial_t^2\psi(\vec{r}, t) = 0$$

or 2D wave equation

$$\left(\partial_x^2 + \partial_y^2\right)\psi(\vec{r}, t) - \frac{1}{v^2}\partial_t^2\psi(\vec{r}, t) = 0 \tag{2.7}$$

where $\vec{r}$ is a 2D vector.

Cylindrical Wave

Write Laplace in cylindrical

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

for most application ψ is independent of z and θ, so we get

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{2.8}$$

one approximation solution is

$$\psi(\rho, t) = \frac{\psi_0}{\sqrt{\rho}}\cos(wt - k\rho)$$

Ignoring order $\frac{1}{\rho^\alpha}$, $\alpha > 2$, because the order of pde (2.8) is 2

$$\begin{aligned}
\frac{\partial^2\psi}{\partial\rho^2} &\to -\frac{k^2\cos(wt-k\rho)}{\sqrt{\rho}} + \frac{3}{4}\frac{\cos(wt-k\rho)}{\rho^{5/2}} - \frac{k\sin(wt-k\rho)}{\rho^{3/2}} \\
\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} &\to -\frac{\cos(wt-k\rho)}{2\rho^{5/2}} + \frac{k\sin(wt-k\rho)}{\rho^{3/2}}
\end{aligned}$$

Therefore if we ignore $\frac{\cos(wt-k\rho)}{4\rho^{5/2}}$, $\frac{\psi_0}{\sqrt{\rho}}\cos(wt - k\rho)$ satisfies (2.8). The reason of $1/\sqrt{\rho}$ in the amplitude is that

$$\int \rho\left(\frac{\psi_0}{\sqrt{\rho}}\right)^2 d\theta$$

the energy carried by the wave crest should be conversed.

Spherical Wave

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

If assume ψ is spherical symmetric, $\psi = \psi(r, t)$, we only need to solve

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{2.9}$$

compare this to (2.8), there is a 2 in front of $\partial\psi/\partial r$. That is because (2.8) is circular symmetric, then in general spherical coordinates in N dimensions Laplace is

$$\Delta \;=\; \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + ... = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + ...$$

One particular solution to (2.9) is

$$\psi = \frac{\psi_0}{r}\cos(wt - kr)$$

Indeed

$$\begin{aligned}
\frac{2}{r}\frac{\partial\psi}{\partial r} &\to \frac{2k\sin(wt-kr)}{r^2} - \frac{2\cos(wt-kr)}{r^3} \\
\frac{\partial^2\psi}{\partial r^2} &\to -\frac{k^2\cos(wt-kr)}{r} + \frac{2\cos(wt-kr)}{r^3} - \frac{2k\sin(wt-kr)}{r^2}
\end{aligned}$$

The $1/r$ factor in the amplitude makes total energy flux on a sphere conserved

$$4\pi r^2\left(\frac{\psi_0}{r}\right)^2$$

2.5 Elastic Sheets

The technique of solving higher dimensional wave leads to QM.

Consider a rectangular 2D sheet a long b wide, clump all sides. We solve (2.7), use separation of variables

$$\psi = X(x)Y(y)T(t)$$

plugging into (2.7), and dividing ψ out

$$\frac{1}{X}\frac{d^2}{dx}X + \frac{1}{Y}\frac{d^2}{dy^2}Y = \frac{1}{v^2}\frac{1}{T}\frac{d^2}{dt^2}T$$

RHS is a function of t, while LHS is a function of x, y. Since x, y, t are independent variables, RHS and LHS must be constants, denoted $-k^2$ (the reason to put negative constant is to have periodic solutions). Put

$$\frac{1}{T}\frac{d^2}{dt^2}T = -k^2v^2 = -w^2$$

Or

$$\frac{d^2}{dt^2}T = -w^2T$$

which is a typical eigenvalue equation. It has eigen solution

$$T \sim e^{\pm iwt}$$

Similarly

$$\begin{cases} \frac{1}{X}\frac{d^2}{dx^2}X = -k_1^2 \\ \frac{1}{Y}\frac{d^2}{dy^2}Y = -k_2^2 \\ k_1^2 + k_2^2 = k^2 \end{cases} \implies \begin{array}{l} X \sim e^{\pm ik_1x} \\ Y \sim e^{\pm ik_2y} \end{array} \tag{2.10}$$

Boundary conditions

$$\psi(0, y, t) = \psi(a, y, t) = 0$$

$$\psi(x, 0, t) = \psi(x, b, t) = 0$$

Thus

$$\psi_{nm}(x, y, t) = \psi_0 \sin k_1x \sin k_2y \cos(wt - \phi)$$

$$k_1 = \frac{n\pi}{a} \quad k_2 = \frac{m\pi}{b} \quad w_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}v$$

(n, m) mode number. The energy of the sheet is just like before (1.10) $\propto w^2$, so the lowest energy state is w_{11}, w_{12}, w_{21}, or ... depending on how big a, b are. If $a = b$, then the lowest energy state is w_{11}, and for other energy levels there may be many degenerated states.

Lecture 7 (9/25/13)

We have seen that (2.10) gives continuous available k's and the boundary conditions impose discrete allowed k's.

Another example: circular sheet

Suppose a sheet of radius a is clumped along the edge. Clearly we have circular symmetry so use polar coordinate, we solve (2.8). Separation of variable

$$\psi(\rho, t) = \tilde{\psi}(\rho) \cos wt$$

Plugging into (2.8)

$$\frac{d^2\tilde{\psi}}{d\rho^2} + \frac{1}{\rho}\frac{d\tilde{\psi}}{d\rho} = -\frac{w^2}{v^2}\tilde{\psi}$$

Change variable $z = \frac{w}{v}\rho$

$$\frac{d^2\tilde{\psi}}{dz^2} + \frac{1}{z}\frac{d\tilde{\psi}}{dz} + \tilde{\psi} = 0$$

This is Bessel's equation of 0th order. If we have θ dependence, we will get higher order Bessel equations. The solutions are Bessel function

$$\tilde{\psi}(z) = J_0(z) = \int_0^\pi \cos(z \sin\theta) d\theta$$

Thus

$$\psi(\rho, t) = J_0(\frac{w}{v}\rho) \cos(wt - \phi)$$

The boundary condition implies w is discrete,

$$\frac{w}{v}a = z_j$$

where z_j's are the jth zeros of Bessel function. Therefore

$$\psi_j(\rho, t) = J_0(\frac{z_j}{a}\rho) \cos(wt - \phi)$$

2.6 Wave Guides

Consider a ∞ long strip of width b is clumped at two sides $y = 0$ and $y = b$. Suppose a wave is propagating in the sheet with

$$\tilde{\psi}_i \sim e^{i(wt - \vec{k}_i \cdot \vec{r})}$$

or

$$\psi_i(x,y,t) = \Re\tilde{\psi}_i \sim \cos(wt - k_1 x - k_2 y) \tag{2.11}$$

$$\vec{k}_i = (k_1, k_2)$$

once it reaches the other side, it gets reflected

$$\psi_r(x,y,t) \sim \cos(wt - k_1 x + k_2 y) \tag{2.12}$$

$$\vec{k}_r = (k_1, -k_2)$$

so the total wave propagating in the sheet

$$\tilde{\psi}(\vec{r},t) = A_1 e^{i(wt - \vec{k}_i \cdot \vec{r})} + A_2 e^{i(wt - \vec{k}_r \cdot \vec{r})}$$

One of the boundary conditions says

$$\tilde{\psi}(0,t) = 0 \implies A_1 = -A_2$$

So

$$\tilde{\psi}(x,y,t) = Ae^{iwt}\left(e^{-ik_1 x}e^{-ik_2 y} - e^{-ik_1 x}e^{ik_2 y}\right) = -A2ie^{i(wt-k_1 x)}\sin k_2 y$$

By the other boundary condition

$$k_2 = \frac{n\pi}{b}$$

so

$$\psi_n(x,y,t) = \Re\tilde{\psi} = 2A_n \sin(wt - k_1 x)\sin\frac{n\pi}{b}y \tag{2.13}$$

We get standing wave in y direction, traveling wave in x direction.

For given w, $k_1^2 + k_2^2 = w^2/v^2$

$$k_1 = \sqrt{\left(\frac{w}{v}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{2.14}$$

Compare (2.11) and (2.13). They have two different wave length. The incident wave has wave length $\lambda = \frac{2\pi}{k}$ and it is impinging to the side at angle θ; the wave guide is traveling along x has wave length $\lambda_1 = \frac{2\pi}{k_1}$. Clearly

$$k_1 = k \sin\theta$$

so

$$\lambda_1 = \frac{\lambda}{\sin\theta}$$

λ_1 is called the guide wavelength. We know the velocity of prorogation $v = w/k$ is determined by the vibrating material. And v is derived from either (2.11) or (2.12).

But there are two more concepts of velocity: phase velocity and group velocity.

Phase velocity in this case is the actual velocity of the moving crest of the total wave guide (2.13), so

$$v_p = \frac{w}{k_1} = \frac{w}{k}\frac{1}{\sin\theta} = \frac{v}{\sin\theta} > v$$

and group velocity is the actual velocity that carries energy. Since the energy is carried by either (2.11) or (2.12) the effective velocity of the moving energy is only the x component of speed of (2.11), i.e.

$$v_g = v \sin\theta < c$$

From this specific case, we get some general equations that are always true

$$\begin{aligned} (v_p)_i &= \frac{w}{k_i} \\ (v_g)_i &= \frac{dw}{dk_i} \end{aligned}$$

Indeed if we are interested in x direction

$$v_{gx} = \frac{dw}{dk_1} = \frac{d}{dk_1} vk = \frac{d}{dk_1} v k_1 \sin\theta = v \sin\theta$$

We will revisit group and phase velocities later.

Lecture 8 (10/7/13) Another interesting feature of (2.14) is that there is a cutoff frequency,

$$w \geq \frac{n\pi}{b} v$$

If $w < \frac{\pi v}{b}$, k_1 is imaginary, the wave will die out. We will see that in the case EM wave guide passing in metal sheet.

2.7 Geometric Optics: Snell's Law

We will show Snell's law on the base that lights are EM waves.

(Griffiths Problem 9.16) Suppose the coordinate are chosen so that x goes up z goes right and y out of paper. Let $z < 0$ be filled with dielectric medium n_1 and $z > 0$ filled with n_2. Send in plane light wave $\vec{E} \parallel \hat{y}$, $\vec{B}$ lie in the plane of xz. So $\vec{k}_i$ lie in the plane of xz. Suppose $\vec{k}_i$ makes angle θ_i with the normal axis, z axis. Consider $\vec{B}$ wave

$$\psi = \begin{cases} \psi_i e^{i(wt - \vec{k}_i \cdot \vec{r})} + \psi_r e^{(wt - \vec{k}_r \cdot \vec{r})} & z < 0 \\ \psi_t e^{i(wt - \vec{k}_t \cdot \vec{r})} & z > 0 \end{cases}$$

where ψ_i, ψ_r and ψ_t are complex vectors. We choose $\vec{B}$ wave to analyze because $\vec{B}$ is continuous across the boundary, while as continuity for $\vec{E}$ wave depends on whether charges built up at the interface. Then at $z = 0$

$$\psi_i e^{i(wt - \vec{k}_i \cdot \vec{r})} + \psi_r e^{i(wt - \vec{k}_r \cdot \vec{r})} = \psi_t e^{i(wt - \vec{k}_t \cdot \vec{r})} \tag{2.15}$$

for all t,x, and y. This can be true only if

$$\begin{cases} k_{ix} = k_{rx} = k_{tx} \\ k_{iy} = k_{ry} = k_{ty} \end{cases}$$

Since $k_{iy} = 0$, reflection and refraction happen in the same xz plane.

Because $k = \frac{w}{v} = \frac{w}{c} n$, $|\vec{k}_i| = |\vec{k}_r|$, then

$$k_{ix} = k_{rx} \implies \sin\theta_i = \sin\theta_r$$

and $|\vec{k}_i|/n_1 = |\vec{k}_t|/n_2$

$$k_{ix} = k_{tx} \implies n_1 \sin\theta_i = n_2 \sin\theta_t$$

we derived Snell's law.

We will continue to derive some relations between the amplitudes which will be useful later. We need to look at $\vec{E}$ field too

$$\begin{cases} E_i = \tilde{E}_{0i} e^{i(wt - \vec{k}_i \cdot \vec{r})} \hat{y} \\ B_i = \frac{1}{v_1} \tilde{E}_{0i} e^{i(wt - \vec{k}_i \cdot \vec{r})} (-\cos\theta_i \hat{x} + \sin\theta_i \hat{z}) \end{cases}$$
$$\begin{cases} E_r = \tilde{E}_{0r} e^{i(wt - \vec{k}_r \cdot \vec{r})} \hat{y} \\ B_r = \frac{1}{v_1} \tilde{E}_{0r} e^{i(wt - \vec{k}_r \cdot \vec{r})} (\cos\theta_i \hat{x} + \sin\theta_i \hat{z}) \end{cases}$$
$$\begin{cases} E_t = \tilde{E}_{0t} e^{i(wt - \vec{k}_t \cdot \vec{r})} \hat{y} \\ B_t = \frac{1}{v_2} \tilde{E}_{0t} e^{i(wt - \vec{k}_t \cdot \vec{r})} (-\cos\theta_t \hat{x} + \sin\theta_t \hat{z}) \end{cases}$$

Maxwell implies two boundary conditions

$$E^{\parallel}_{z\to 0^-} = E^{\parallel}_{z\to 0^+} \qquad B^{\parallel}_{z\to 0^-} = B^{\parallel}_{z\to 0^+}$$

then

$$\tilde{E}_{0i} + \tilde{E}_{0r} = \tilde{E}_{0t}$$
$$\frac{1}{v_1}\tilde{E}_{0i}(-\cos\theta_i) + \frac{1}{v_1}\tilde{E}_{0r}\cos\theta_i = \frac{1}{v_2}\tilde{E}_{0t}(-\cos\theta_t)$$

Therefore we find

$$\tilde{E}_{0r} = \frac{1-\alpha}{1+\alpha}\tilde{E}_{0i} \qquad \tilde{E}_{0t} = \frac{2}{1+\alpha}\tilde{E}_{0i} \tag{2.16}$$
$$\alpha = \frac{\cos\theta_t}{\cos\theta_i}\frac{n_2}{n_1} = \frac{\sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i}$$

Since α is always positive, $\tilde{E}_{0t}$ is always in phase with $\tilde{E}_{0i}$.

If $n_2 > n_1$,

$$(n_2/n_1)^2 > 1 = \sin^2\theta_i + \cos^2\theta_i \implies \alpha > 1$$

Hence incident light from n_1 reflected at n_2 with $n_1 < n_2$ will gain an extra π phase. If $n_2 < n_1$, no extra phase. This result is the generalization of the 1D case we studied before (1.11).

2.8 Group Velocity Revisit

Combine two 1D waves of same amplitude and similar frequencies

$$\psi_1 = a\cos(w_1 t - k_1 x) \qquad \psi_2 = a\cos(w_2 t - k_2 x)$$

Let

$$\bar{w} = \frac{w_1 + w_2}{2} \quad \Delta w = \frac{w_1 - w_2}{2} \quad \bar{k} = \frac{k_1 + k_2}{2} \quad \Delta k = \frac{k_1 - k_2}{2}$$

then

$$\psi_1 + \psi_2 = 2a\cos(\bar{w}t - \bar{k}x)\cos(\Delta w t - \Delta k x) \tag{2.17}$$

If $\frac{w_1 - w_2}{\bar{w}} = \frac{\Delta w}{2\bar{w}} \ll 1$, we get phenomena of beats. The graph y v.s. t for fixed x has envelope given by $\cos(\Delta w t - \Delta k x)$ and oscillation inside given by $\cos(\bar{w}t - \bar{k}x)$. The beat frequency we hear is twice of the envelope frequency Δw.

$$T_{beat} = \frac{1}{2|\nu_1 - \nu_2|}$$

We find the motion of the envelope is

$$\begin{aligned} v_g &= \frac{\Delta w}{\Delta k} \\ &= \frac{w_1 - w_2}{k_1 - k_2} = v \end{aligned} \tag{2.18}$$

because $w = kv$, linear dispersion relation. So is the phase velocity

$$v_p = \frac{\bar{w}}{\bar{k}} = v$$

This means that as the wave moves both the oscillation pattern and the envelope move at the same speed, thus the sharp of the wave stays the same. This will not be the case if $w = w(k) \neq kv$, non-linear, $v_g = v = v_p$ is not likely. Later we will see another example that as wave moves it attenuates (2.23).

We can generalize (2.18) to

$$v_g = \frac{dw}{dk}$$

Lecture 9 (10/9/13)

Suppose $n = n(w)$

$$k = \frac{n(w)w}{c}$$

then

$$\frac{dk}{dw} = \frac{n}{c} + \frac{w}{c}\frac{dn}{dw}$$

Since $v_p = c/n$,

$$v_g = \frac{dw}{dk} = \frac{\frac{c}{n}}{1 + \frac{w}{n}\frac{dn}{dw}} = \frac{v_p}{1 + \frac{w}{n}\frac{dn}{dw}} \tag{2.19}$$

In vacuum $n = 1$, $dn/dw = 0$, so for EM wave $v_g = v_p$.

Continue (2.19), for normal dispersion

$$\frac{dn}{dw} > 0 \implies v_g < v_p$$

For anomalous dispersion

$$\frac{dn}{dw} < 0 \implies v_g > v_p$$

Light through prism has spectrum: red, orange, yellow, green, blue. If prism is made of anomalous glass, the spectrum is reversed.

If $n = n(\lambda)$, from (2.19),

$$\frac{dn}{dw} = \frac{dn}{d\lambda}\frac{d\lambda}{dw}$$

$$\lambda = \frac{2\pi c}{wn}$$

$$\frac{d\lambda}{dw} = -\frac{2\pi c}{w^2 n} = -\frac{\lambda}{w}$$

so (2.19) becomes

$$v_g = \frac{v_p}{1 - \frac{\lambda}{n}\frac{dn}{d\lambda}} \approx v_p(1 + \frac{\lambda}{n}\frac{dn}{d\lambda})$$

2.9 Dispersive Waves: Plasma Wave

We now study the radiation of plasma due to excitation by the external field. Think electrons are bounded to the nuclei like a string k and under external $\vec{E}$ field, e^- are making oscillations. Let m be the mass of e^-, x be the position coordinate, then

$$P = qx$$

is the dipole momentum. The intensity of radiation

$$I \propto (\ddot{P})^2$$

First we want to show that the relative permittivity ϵ_r of the material that made of molecule with density $n = $ #moles/volume, and dipole momentum of a molecule P is

$$\epsilon_r = 1 + \frac{nP}{E\epsilon_0} \tag{2.20}$$

One way to derive the equation above is to fill such material into a parallel planes capacitor with separation d and area of two planes A. Then the electric field inside is

$$E_{net} = E_{free} - E_{bound} = \frac{\sigma_f}{\epsilon_r \epsilon_0} = \frac{E_f}{\epsilon_r}$$

where σ_f is charge density that disposed on the planes. Then

$$E_{bound} = (\epsilon_r - 1)E_f = \frac{\sigma_b}{\epsilon_0} \tag{2.21}$$

The bound charges are polarized by the free charges, and their effects are canceled in the internal of the capacitor, only those populated near the planes contribute to E_{bound}. They makes a thin layer of width d_b, which is half of the distance between $+$ and $-$ poles of the polarized molecule, because the another half of the charges are immersed in the internal and hence canceled. Thus

$$\sigma_b = \frac{q_b}{A} = \frac{qn2Ad_b}{A} = nq2d_b = nP \tag{2.22}$$

$q2d_b = P$, dipole momentum. Combining (2.21) and (2.22) gives (2.20).

Now back to the situation we discussed in the beginning of this section. External generating field $E = E_0 e^{iwt}$ acts on plasma, equation of motion for one e^-

$$m\ddot{x} = -\frac{m}{T}\dot{x} - kx + qE$$

we add a damping force term $-\frac{m}{T}\dot{x}$, because without it resonance goes to ∞. Suppose we look for wave solution

$$\tilde{X} = X_0 e^{iwt}$$

put $w_0 = \sqrt{k/m}$, natural frequency, in many cases it is the quantum transition frequency. Then

$$\left(-w^2 X_0 + \frac{iw}{T}X_0 + w_0^2 X_0\right) e^{iwt} = \frac{q}{m}E_0 e^{iwt}$$

$$X_0 = \frac{qE_0/m}{-w^2 + w_0^2 + \frac{iw}{T}}$$

X_0 is the average displacement, plugging into (2.20), with $P = qX_0$

$$\epsilon_r = 1 + \frac{\frac{nq^2}{m\epsilon_0}}{-w^2 + w_0^2 + \frac{iw}{T}}$$

define plasma frequency, w_p

$$w_p^2 = \frac{nq^2}{m\epsilon_0}$$

then

$$\epsilon_r = 1 + \underbrace{\frac{w_p^2(w_0^2 - w^2)}{(w_0^2 - w^2)^2 + \frac{w^2}{T^2}} - i\frac{w_p^2 \frac{w}{T}}{(w_0^2 - w^2)^2 + \frac{w^2}{T^2}}}_{\chi}$$

and

$$\tilde{n} = \sqrt{\epsilon_r} = \underbrace{1 + \frac{1}{2}\frac{w_p^2(w_0^2 - w^2)}{(w_0^2 - w^2)^2 + \frac{w^2}{T^2}}}_{n_R} - i\underbrace{\frac{1}{2}\frac{w_p^2 \frac{w}{T}}{(w_0^2 - w^2)^2 + \frac{w^2}{T^2}}}_{n_I}$$

Now we see that the external generating field $E = E_0 e^{iwt}$ becomes wave

$$\begin{aligned} E &= E_0 e^{i(wt - \frac{w}{c} n x)} \\ &= E_0 e^{i(wt - \frac{w}{c} n_R x)} e^{-\frac{w}{c} n_I x} \end{aligned} \tag{2.23}$$

The graph of $\Im \tilde{n}$ show that it has resonance near $w = w_0$, hence the wave die out fastest when $w = w_0$. One can graph $\Re(\tilde{n} - 1)$, at $w = 0$, $\Re(\tilde{n} - 1) = \frac{w_p^2}{2}$ then it goes up reaches its peak at $w = \sqrt{w_0^2 - \frac{w_0}{T}}$, then falls down $\Re(\tilde{n} - 1) = 0$ at $w = w_0$, then continues down reaches bottom at $w = \sqrt{w_0^2 + \frac{w_0}{T}}$, then up but stays negative and asymptotically approaches 0. Hence $w \in (\sqrt{w_0^2 - \frac{w_0}{T}}, \sqrt{w_0^2 + \frac{w_0}{T}})$

$$\frac{dn_R}{dw} < 0$$

anomalous dispersion.

3 Semi-Classical Waves

3.1 Ultraviolet Catastrophe

Lecture 10 (10/14/13)

1900 there were 2 clouds, ultraviolet catastrophe from black body radiation and undetectable aether from Michelson-Morley experiment.

Consider a conductor square box with side l. Since $E = 0$ on the faces of the box, the $\vec{E}$ field inside

$$E_{n_1,n_2,n_3}(x, y, z, y) = E_{0_{n_1,n_2,n_3}} \sin \frac{n_1 \pi x}{l} \sin \frac{n_2 \pi y}{l} \sin \frac{n_3 \pi z}{l} \sin wt \tag{3.1}$$

$$w = kc = c\sqrt{\left(\frac{n_1\pi}{l}\right)^2 + \left(\frac{n_2\pi}{l}\right)^2 + \left(\frac{n_3\pi}{l}\right)^2}$$

$$\nu = \frac{w}{2\pi} = \sqrt{\left(\frac{cn_1}{2l}\right)^2 + \left(\frac{cn_2}{2l}\right)^2 + \left(\frac{cn_3}{2l}\right)^2}$$

Typical value $l \sim 1m$, $c = 3 \times 10^8 m/s$, $n \sim 10^7$, so

$$\nu \sim 10^7$$

adjacent frequency are closed spaced

$$\frac{\Delta \nu}{\nu} = \frac{\frac{c}{l}}{\frac{c}{l} n} = 10^{-7}$$

This allows to calculate things in continuum limit. In the modal space, $\hat{x}$ axis $\frac{cn_1}{2l}$, $\hat{y}$ axis $\frac{cn_2}{2l}$, and $\hat{z}$ axis $\frac{cn_3}{2l}$. A small cube encloses one discrete dot in the modal space, the volume of the cube

$$\left(\frac{c}{2l}\right)^3 = \frac{c^3}{8V}$$

and the volume in the modal space from frequency ν to $\nu + d\nu$ is

$$\frac{1}{8} 4\pi \nu^2 d\nu$$

So the number of states from ν to $\nu + d\nu$ is

$$\frac{\frac{1}{8} 4\pi \nu^2 d\nu}{\frac{c^3}{8V}} = V \frac{4\pi \nu^2 d\nu}{c^3} = dN \tag{3.2}$$

If we divide $dN/V = dn$, we get number of states per volume from ν to $\nu + d\nu$.

Equipartition theorem from statistical mechanics says each degree of freedom of the system at equilibrium temperature T gets $kT/2$ of the total energy of the system. k is Boltzmann constant.

E.g. SHO

$$\frac{p^2}{2m} + \frac{1}{2} k x^2$$

has two degree of freedom

$$KE = UE = \frac{kT}{2}$$

E.g. monatomic atom, non interacting gas, no potential

$$E = N \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) = \frac{3}{2} kTN$$

In our case the box is at temperature T. The atoms and molecules of the conductor vibrating, and radiation are emitted, The populated radiation are standing waves. Each mode has energy

$$Energy_{n_1,n_2,n_3} \propto E^2_{0_{n_1,n_2,n_3}} + B^2_{0_{n_1,n_2,n_3}}$$

Treat each mode (n_1, n_2, n_3) as SHO, so each has

$$2 \times 2 \times \frac{kT}{2}$$

one 2 is from E and B fields, and the other 2 is from polarization, because the amplitude in (3.1) has two polarization directions.

So the total energy per volume from frequency 0 to ∞ is

$$U = \int_0^\infty u(\nu)d\nu = \int_0^\infty (2kT)\frac{4\pi\nu^2 d\nu}{c^3} = \infty \tag{3.3}$$

This is UV catastrophe. It says even before the microwave oven is on, it contains ∞ radiation.

Planck had the experimental curve $u(\nu)$ v.s. ν. For small ν,

$$u(\nu) \sim \nu^2$$

agrees (3.3). For large ν

$$u(\nu) \sim \nu^3 e^{-h\nu/kT}$$

suggested by Wien. So it is called Wien tail.

$$h = 6.67 \times 10^{-27}\text{erg}\cdot\text{s}$$

is Planck constant. Planck said "I am going to fit the 2 ends"

$$u(\nu) \sim \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{3.4}$$

and his explanation was that equipartition assumes $E \sim kT$, since T changes continuously, E changes continuously. Planck said No. Only

$$nh\nu \qquad n = 1, 2, 3, ...$$

goes into each mode.

Lecture 11 (10/16/13)

Now by (3.4)

$$U = \int_0^\infty u(\nu)d\nu < \infty$$

Planck said that counting in (3.2) showed that higher ν, higher E, and most states are available, but states with high v.s low energy are not equally likely being occupied. Planck said the probability of occupying states with energy E is

$$P(E) = Ne^{-E/kT} \tag{3.5}$$

Planck distribution. N is normalization

$$\int_0^\infty P(E)dE = NkT = 1 \implies N = \frac{1}{kT}$$

The reason Planck made such guess (3.5) because

$$\langle E\rangle = \int_0^\infty EP(E)dE = kT \tag{3.6}$$

agrees classical result for SHO. The important feature about (3.5) is that the lowest energy has the highest probability being occupied.

For the blackbody radiation, in addition, Planck said energy was quantized

$$E_m = mh\nu$$

for each mth mode. So Planck distribution is

$$P(E_m) = Ne^{-\frac{mh\nu}{kT}}$$

Find normalization

$$1 = \sum_m P(E_m) = \sum_{m=0}^\infty Ne^{-\frac{mh\nu}{kT}} = N\frac{1}{1-e^{-\frac{h\nu}{kT}}}$$

That is

$$P(E_m) = (1-e^{-\frac{h\nu}{kT}})e^{-\frac{mh\nu}{kT}}$$

One can show that similar thing happens to (3.6) when T is large, so we return to classical result.

$$\begin{aligned}
\langle E\rangle &= -kT\,\frac{\partial}{\partial\alpha}\bigg|_{\alpha=1}\sum e^{-\frac{mh\nu}{kT}\alpha}(1-e^{-\frac{h\nu}{kT}}) \\
&= \lim_{\alpha\to 1} -kT\frac{\partial}{\partial\alpha}\left(\frac{1}{1-e^{-\frac{h\nu}{kT}\alpha}}\right)(1-e^{-\frac{h\nu}{kT}}) \\
&= \frac{h\nu e^{-\frac{h\nu}{kT}\alpha}}{1-e^{-\frac{h\nu}{kT}}} \\
&= \frac{h\nu}{e^{\frac{h\nu}{kT}}-1} \qquad (3.7) \\
&\to \frac{h\nu}{1+\frac{h\nu}{kT}-1} = kT
\end{aligned}$$

We can now correctly count number of states with proper weight factor (Planck distribution) and the correct energy levels (not kT, but E_m), back to (3.3)

$$\begin{aligned}
u(\nu)d\nu &= \frac{8\pi\nu^2 d\nu}{c^3}\sum_m E_m P(E_m) \\
&= \frac{8\pi\nu^2 d\nu}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}}-1} = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1}d\nu
\end{aligned}$$

This is still semi-classical argument, stating from string and imposing quantization.

Modern interpretation of (3.7) is that $h\nu$ is photon energy and $\frac{1}{e^{\frac{h\nu}{kT}}-1}$ is the number of photons. This is first time for us to think waves as particles. Later we will treat particles as waves. It was easily accepted by 19 century hero's that EM waves as particles, but many of them (including Einstein) didn't fully embrace that e.g. e^- are waves.

3.2 Fourier Analysis

The idea of particles are waves builds on forming wave packets. Suppose we add N waves each has frequency δw difference from each other

$$R = \sum_{n=0}^{N-1} a\cos[(w+n\delta w)t]$$

and

$$N\delta w = \Delta w = \text{"Bandwidth"}$$

difference between the upper and lower frequencies in a continuous set of frequencies, is so narrow that

$$\frac{\Delta w}{w} \ll 1 \tag{3.8}$$

One can show that

$$\tilde{R} = a\sum e^{iwt}e^{in\delta wt} = ae^{iwt}\frac{1-e^{iN\delta wt}}{1-e^{i\delta wt}} = ae^{iwt}\frac{e^{i\frac{N\delta wt}{2}}\sin\frac{N\delta wt}{2}}{e^{i\frac{\delta wt}{2}}\sin\frac{\delta wt}{2}}$$

so

$$R \approx a\frac{\sin\frac{\Delta wt}{2}}{\frac{\delta wt}{2}}\cos\bar{w} = aN\frac{\sin\frac{\Delta wt}{2}}{\frac{\Delta wt}{2}}\cos\bar{w}t$$

$$\bar{w} = w + \frac{1}{2}(N-1)\delta w$$

which is about the middle of the upper and lower frequencies. Because of (3.8), $\left|\frac{\sin\frac{\Delta wt}{2}}{\frac{\Delta wt}{2}}\right|$ is the envelope similar to (2.17) and if we put R into a traveling wave

$$R = \sum_{n=0}^{N-1} a\cos[(w+n\delta w)(t-\frac{x}{v})] = aN\frac{\sin\frac{\Delta w(t-\frac{x}{v})}{2}}{\frac{\Delta w(t-\frac{x}{v})}{2}}\cos\bar{w}(t-\frac{x}{v}) \tag{3.9}$$

we have a moving envelope.

Above example motivates Fourier.

Recall we can write any function $f(x)$ with periodicity L

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{2\pi nx}{L} + b_n\sin\frac{2\pi nx}{L} \tag{3.10}$$

Similar we can write any function $f(t)$ with periodicity T

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T}$$

Let

$$w_n = \frac{2\pi n}{T} = nw$$

so

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos w_n t + b_n \sin w_n t$$

As showed before, the energy stored in the nth mode is $\propto a_n^2 + b_n^2$.

We can use complex notation

$$f(t) = \sum_{n=-\infty}^{\infty} d_n e^{-iw_n t} \tag{3.11}$$

with the identifications

$$\begin{aligned} a_n &= d_n + d_{-n} \\ b_n &= i(d_n - d_{-n}) \end{aligned}$$

or

$$\begin{cases} d_n = \frac{a_n - ib_n}{2} & n \geq 0 \\ d_n = \frac{a_{|n|} + ib_{|n|}}{2} & n < 0 \end{cases}$$

and

$$E_n \propto a_n^2 + b_n^2 = 4|d_n|^2 \tag{3.12}$$

and

$$\tilde{f}(w) = d_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{iw_n t} dt$$

hence the Fourier transform of $f(t)$, because

$$\int_{-T/2}^{T/2} e^{-iw_n t} e^{iw_m t} = \begin{cases} 0 & n \neq m \\ T & n = m \end{cases}$$

Similarly we can write (3.10)

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos k_n x + b_n \sin k_n x \\ &= \sum_{n=-\infty}^{\infty} d_n e^{ik_n x} \end{aligned} \tag{3.13}$$

where

$$\tilde{f}(k) = d_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-ik_n x} dx$$

hence the Fourier transform of $f(x)$.

Lecture 12 (10/21/13)

What if k_n or w_n in (3.13) or (3.11) change continuously, so the summation has to become integral. For this to converge, there are some mathematical requirements for f but we won't care about them.

Take (3.11) for example,

$$w_n = \frac{2\pi n}{T} = w_0 n$$

now label $w_0 n \to w$ a continuous variable.

$$dw = w_{n+1} - w_n = w_0 = \frac{2\pi}{T}$$

$$f(t) = \lim_{T\to\infty} \frac{1}{2\pi} \underbrace{\sum_{n=-\infty}^{\infty}}_{\int} \underbrace{T d_n}_{\tilde{f}(w)} e^{-iwt} \underbrace{\frac{2\pi}{T}}_{dw}$$

We guess

$$\begin{cases} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(w) e^{-iwt} dw \\ \tilde{f}(w) = \int_{-\infty}^{\infty} f(t) e^{iwt} dt \end{cases} \tag{3.14}$$

Let's verify

$$
\begin{aligned}
\tilde{f}(w) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-iwt}\tilde{f}(w')e^{iw't}dw'dt \\
&= \int_{-\infty}^{\infty} dw' \frac{1}{2\pi}\lim_{T\to\infty}\underbrace{\int_{-T}^{T} dt e^{i(w'-w)t}}_{\frac{2\sin[(w'-w)T]}{w'-w}}\tilde{f}(w')
\end{aligned}
$$

Change of variable $y = (w' - w)T$

$$
\begin{aligned}
\tilde{f}(w) &= \frac{2}{2\pi}\int_{-\infty}^{\infty} dy \left(\frac{\sin y}{y}\right)\lim_{T\to\infty}\tilde{f}(\frac{y}{T}+w) \\
&= \frac{2}{2\pi}\underbrace{\int_{-\infty}^{\infty} dy \left(\frac{\sin y}{y}\right)}_{\pi}\tilde{f}(w) \\
&= \tilde{f}(w)
\end{aligned}
$$

Indeed (3.14) is right. One can simplify the derivation above by associating

$$
\lim_{T\to\infty}\frac{1}{\pi}\frac{\sin[(w'-w)T]}{w'-w} = \delta(w'-w)
$$

One application of Fourier in electronics is the amplifier.

$$
A(w)V_{in}(t) = V_{out}(t)
$$

Since $A(w)$ is multiplied to the frequency components due to the mechanism of the amplifier, we first Fourier V_{in} in frequency space, then multiply $A(w)$, then inverse Fourier back to $V_{out}(t)$. Suppose V_{in} is a square wave, we know from Fourier analysis, the Fourier of square is made of a few sine and cosines and a cheap sound system that only picks up the first few leading terms doesn't resemble or duplicate V_{in} very well, if V_{out} happens to have a lot of sharp edges.

3.3 Bandwidth Theorem

If one studies atom spontaneous or stimulated transitions radiations, the theory predicts that transition occurs at a specific frequency w_0, hence $I(w)$, the detected intensity of radiations due to that transition should a spark or $\delta(w_0)$, but actually one gets a Gaussian like curve around w_0 with some width. Why?

That is because physically no radiation goes on forever. If radiation $E(t) = E_0 \cos w_0 t$, $t \in (-\infty, \infty)$, then Fourier of $E(t)$ would indeed be $\delta(w_0)$. But in reality each atom only radiates in a finite interval τ, called characteristic time. Typical $\tau \sim 10^{-9}$s, $\nu_0 \sim 10^{17}\text{s}^{-1}$.

$$\begin{aligned} E(w) &= \int_{-\tau/2}^{\tau/2} E_0 \cos w_0 t e^{iwt} dt \\ &= \int_{-\tau/2}^{\tau/2} E_0 \frac{e^{iw_0 t} + e^{-iw_0 t}}{2} e^{iwt} dt \\ &= E_0 \tau \left\{ \frac{\sin[(w_0 + w)\frac{\tau}{2}]}{(w_0 + w)\frac{\tau}{2}} + \frac{\sin[(w_0 - w)\frac{\tau}{2}]}{(w_0 - w)\frac{\tau}{2}} \right\} \end{aligned} \tag{3.15}$$

which is made of two broad curves around $\pm w_0$. Normally one is only interested in positive frequency. But the two curves are identical, meaning that if we are interested in energy flux, energy per unit volume per time pass unit area

$$F = \frac{1}{2}\epsilon_0 E^2$$

we only have to calculate it using the positive w_0

$$F(w) \propto (E_0 \tau)^2 \left\{ \frac{\sin[(w_0 - w)\frac{\tau}{2}]}{(w_0 - w)\frac{\tau}{2}} \right\}^2$$

or

$$\frac{F(w)}{F(w_0)} = \left\{ \frac{\sin[(w_0 - w)\frac{\tau}{2}]}{(w_0 - w)\frac{\tau}{2}} \right\}^2$$

because recall $\lim_{x\to 0} \sin x/x = 1$.

Observe (3.15), the first zero next to the peak w_0 is at

$$(w_0 - w)\frac{\tau}{2} = \pi$$

so the width of the curve is

$$\Delta w = \frac{2\pi}{\tau}$$

this is called Bandwidth theorem.

Application: we know Laser has very narrow frequency. So it has a metastable state, whose transition time is very long. In our words, it oscillates for long time.

Another application: student cheapo oscilloscope. Bandwidth $\Delta\nu = 100\text{MHz}$, then

$$\tau = \frac{1}{\Delta\nu} = 10^{-8}\text{s} = 10\text{ns}$$

so it cannot detect oscillations that last less than 10ns. Good oscilloscope, Bandwidth $\Delta\nu = 1\text{GHz}$, $\tau = 1\text{nsec}$.

Similarly one can get Bandwidth theorem in spatial space. From

$$\Delta w = \Delta k v$$

multiplied by τ

$$\underbrace{\tau\Delta w}_{2\pi} = \Delta k \underbrace{v\tau}_{\Delta x}$$

Get uncertainty principle in QM

$$\Delta x \Delta k = 2\pi$$

Since

$$\Delta k = \frac{2\pi}{\lambda^2}\Delta\lambda$$

we get

$$\Delta x = \frac{\lambda^2}{\Delta\lambda}$$

called coherence length, which is the length of the coherent radiation spreads. We will use it in optics.

Lecture 13 (10/23/13)

We have seen that we can freely do Fourier in time space or spatial space. Both give Bandwidth theorem and the connection between the two spaces is dispersion relation.

3.4 Incoherence v.s Coherence

Consider a focusing lens that takes N light rays (think this as dividing a light beam into N parallel rays) into a point. Suppose the light rays are from light bulbs, i.e. incoherent sources

$$E_{total} = \sum_{r=1}^{N} E_0 e^{i(wt-kx+\phi_r)} = E_0 e^{i(wt-kx)} \sum_r e^{i\phi_r}$$

E_0 is the same by equipartition theorem. ϕ_r is roughly equally distributed.

$$\begin{aligned} I &\propto |E_{total}|^2 \\ &= E_0^2 \sum_r \sum_s (\cos\phi_r + i\sin\phi_r)(\cos\phi_r - i\sin\phi_r) \\ &= E_0^2 [\sum_{r=s} (\sin^2\phi_r + \cos^2\phi_r) + \underbrace{\sum_{r\neq s} (\cos\phi_r \sin\phi_s + \sin\phi_r \cos\phi_s)}_{0}] \qquad (3.16) \\ &= N E_0^2 \end{aligned}$$

The zero is because first fix r, then sum s, there are equal number positive and negative s except r so

$$\sum_{r\neq s} \cos\phi_r \sin\phi_s = \sum_r \cos\phi_r \sin\phi_r$$

similarly

$$\sum_{r\neq s} \sin\phi_r \cos\phi_s = \sum_s \sin\phi_s \cos\phi_s$$

so

$$\sum_{r\neq s} (\cos\phi_r \sin\phi_s + \sin\phi_r \cos\phi_s) = \sum_r \sin 2\phi_r = 0$$

The result (3.16) i.e.

$$I = N I_0$$

shows no interference, if the sources is incoherence, because I_0 is $1/N$ of intensity of the total light before entering the lens.

If the source is coherence, above calculation will give

$$I_{focused} \propto N^2 E_0^2$$

or

$$I_{focused} = N^2 I_0 \tag{3.17}$$

(3.17) doesn't imply that if our initial choice of N was ∞, then $I_{focused}$ would be ∞. N is limited by the fact that if N is too big, then our construction would allow even parallel rays to interfere each other, so the initial light intensity would be intensified by our method of calculation. Below example shows the properness of choosing N.

Consider coherence source sending parallel rays to a screen and focusing lens focuses rays into a point. We put a glass index of refraction n, thinness l, onto the path of half of the rays in front of the lens. So we intentionally create optical path difference (note: like example before the lenses themselves are designed so that rays get focused without creating any optical path difference.)

$$\Delta\phi = k_1 l - k_2 l = \frac{2\pi}{\lambda_0}(n-1)l$$

$$k_1 = \frac{wn}{c} \quad k_2 = k_0 = \text{wave number in vacuum}$$

or

$$\lambda_1 = \frac{\lambda_0}{n} \quad \lambda = \lambda_0 = \text{wavelength in vacuum}$$

Thus

$$\begin{aligned} E_{total} &= E_0 \left(e^{i(wt-k_1 l)} + e^{i(wt-k_2 l)}\right) e^{ik_0 x} \\ &= E_0 e^{i(wt-k_1 l)} \left(1 + e^{i\Delta\phi}\right) e^{ik_0 x} \end{aligned}$$

$$I \propto \left(1 + e^{i\Delta\phi}\right)\left(1 + e^{-i\Delta\phi}\right) = 2(1+\cos\Delta\phi) = 4\cos^2\frac{\Delta\phi}{2}$$

so

$$I_{focused} = 4 I_0 \cos^2\frac{\Delta\phi}{2} \tag{3.18}$$

cf (3.17); while if the source is incoherent. Putting a glass changes nothing

$$I_{focused} = 2I_0$$

where I_0 is half of the intensity of the total light.

3.5 Interference

One can do the whole business of optics using Huygen's geometric constructions. It will match qualitatively to results obtained from Maxwells equations. We will show the equivalence of the two when we study single slit diffraction.

2-Slit Interference

Lecture 14 (10/28/13)

In an introductory course, we learned that parallel rays that are parallel to the axis of lens will be focused to the focal point. In fact lens has property that focus parallel rays that are not parallel to the axis of lens. They will focused to a point on the focal plane. One can prove this fact geometrically by drawing two lines among the parallel rays: one goes to the center of the lens; the other goes to the focal point before entering the lens.

Consider two slits separated by d, plane wave light enter the slits. A focusing lens placed behind the slits. Observe interfere on the screen. Ignoring any background signals produced by lights going through the slit without interference.

Mathematically it is the same as (3.18), commonly write

$$I(\theta) = I_0 \cos^2 \frac{\phi(\theta)}{2}$$

I_0 here is the maximum intensity on the screen, of course happens at $\theta = 0$.

$$\phi(\theta) = kd \sin\theta = \frac{2\pi d \sin\theta}{\lambda}$$

where θ is the angle of elevation from the point of observation to the midpoint of the 2-slit. One can find the maxima at

$$\frac{\phi(\theta)}{2} = m\pi$$

In fact

$$I(\theta) = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

is not completely correct. Because experimentally one will see that the central peak has the maximum intensity. The correct expression takes into account the width of the slit a

$$I(\theta) = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)\left(\frac{\sin\frac{\pi a \sin\theta}{\lambda}}{\frac{\pi a \sin\theta}{\lambda}}\right)^2 \tag{3.19}$$

the extra term is due to single slit diffraction, which we will discuss later.

N-Slit Interference

Same setup, N slits. Similar math

$$\phi(\theta) = kd\sin\theta$$

where θ is the angle of elevation from the point of observation to the midpoint of the $N-$slit. Nd should be small so that from the lens prospective, the N rays are still near axis parallel rays.

$$\begin{aligned} E &= E_0 + E_0 e^{i\phi} + Ee^{2i\phi} + ... \\ &= E_0 \sum_{n=0}^{N-1} e^{in\phi} = E_0 \frac{1-e^{iN\phi}}{1-e^{i\phi}} \end{aligned}$$

$$I = I_0 \left(\frac{\sin\frac{N\phi}{2}}{N\sin\frac{\phi}{2}}\right)^2$$

because $\lim_{x\to 0}\frac{\sin Nx}{\sin x} = N$, where I_0 maximum intensity on the screen. Of course just like (3.19) we may want to multiply the single slit effect. But if we deal with small angle i.e.$\frac{\pi a \sin\theta}{\lambda} \ll 1$ then the single slit factor ≈ 1.

Total Reflection

This is destructive interference and it is simpler to be found by Snell's law. Consider a piece of fiber optic, which is made of 2 kinds of glass. Clad wrap the core. no air

$$n_{core} > n_{clad}$$

Send light into the cable at incident angle θ,

$$n_{air} \sin\theta = n_{core} \sin\theta_1 \quad n_{air} = 1$$

then the light will hit the clad from inside at incident angle $\theta_2 = \pi/2 - \theta_1$, we want to have no light refracted into clad glass,

$$n_{core} \sin\theta_2 = n_{clad} \sin\frac{\pi}{2}$$

this gives the smallest θ_2. Hence the largest θ is $\theta_{critical}$

$$\sin\theta_{critical} = n_{core}\cos\theta_2 = n_{core}\sqrt{1 - \left(\frac{n_{clad}}{n_{core}}\right)^2} = \sqrt{n_{core}^2 - n_{clad}^2}$$

Thin Film

Before we do thin film, we need to understand why it only works for thin films not thick films.

Michelson interferometer consists two highly polished mirrors, one fixed, one is movable. Plane wave light beam (may be produced by a point source putting at the focal point of a lens) hits a half-silvered mirror, splits into two beams. Both then hit the mirrors and get reflected and recombined to give interference, forming fringes. One may put addition glass in one of paths to cancel any discrepancy in the optical path difference due to one beam may travel longer in the half-silvered mirror.

The experimentalists may move the movable mirror, and find the fringes pattern moves too. It gives a precise way to measure distance. Interestingly fringes disappear if the movable mirror, is moved beyond certain distance. This is because interference patter is produced by coherent lights. If the optic path difference is beyond the coherence length

$$\frac{\lambda^2}{\Delta\lambda}$$

cannot produce interference. Therefore we require the thinness of the film to be within the coherence length.

Lecture 15 (10/30/13)

Consider a thin film of thinness t, refraction index n. The medium above and below the film is air. So we have 2 interfaces to consider. Send plane wave light to the film from the air with incident angle θ. Consider two light rays that are coincide after they get reflected by the film. One is reflected right at the first interface; the other had penetrated the film and gets reflected at the bottom interface. (note: it is common mistake to draw one incident ray and two parallel reflected rays. It should be two parallel incident rays and one reflected ray, although math is the same.) Recall (2.16) reflected light from $n_1 < n_2$ medium gain extra phase. Refraction doesn't gain phase. So the optical path difference of the two rays

$$\sin\theta = n\sin\theta_t$$

$$\begin{aligned} \phi &= \frac{2\pi}{\lambda}\left(\frac{2t}{\cos\theta_t}n - 2t\tan\theta_t\sin\theta\right) - \pi \qquad (3.20) \\ &= \frac{2\pi}{\lambda}2tn\cos\theta_t - \pi = \frac{4\pi tn}{\lambda}\sqrt{1-(\sin\theta/n)^2} - \pi \end{aligned}$$

It is the same as (3.18), so

$$I(\theta) = I_0\cos^2\frac{\phi}{2}$$

maxima at $\frac{\phi}{2} = m\pi$.

Many other variations: inclined plane, Newton's rings, ...

3.6 Diffraction

Single Slit Diffraction

Slit width a, focusing lens behind the slit. In the introductory course, we learn to divide the bean into 2. The top half balances the bottom half. If it happens to be that

$$\frac{a}{2}\sin\theta = m\frac{\lambda}{2} \quad m = 1, 3, 5, ... \tag{3.21}$$

We get destructive interference, hence minima. This formula also reveals that if

$$a \gg \lambda$$

we will not distinguish minima and maxima, hence no diffraction, only see a slit image on the screen.

We now want to do a better job. First applying Huygen's principle to a general situation, consider a plane wave moving in z direction. Point A to the one of the points on the constant wave phase plane, ψ_I, by Huygens it products a spherical wave

$$d\psi_A = g\psi_I \frac{e^{-i\vec{k}\cdot\vec{r}}}{r} dS$$

g is the fraction of ψ_I that spherical wave element shares, which will give the right normalization. dS is the area element that A occupies on the phase plane. Consider a point A' on another constant phase plane in distance l ahead of ψ_I, so

$$\psi_{A'} = \psi_I e^{-ikl} \tag{3.22}$$

Another way to calculate $\psi_{A'}$ is to think $\psi_{A'}$ is generated by the spherical waves from ψ_I, so

$$\psi_{A'} = g\psi_I \iint_{-\infty}^{\infty} \frac{e^{-i\vec{k}\cdot\vec{r}}}{r} dS \tag{3.23}$$

We can reasonably argue that only these spherical waves that are near behind A' will contribute to A', so we say that $x^2 + y^2$ is small, so approximately

$$r = \sqrt{l^2 + x^2 + y^2} \approx l[1 + \frac{1}{2l}(x^2 + y^2)]$$

and

$$\frac{1}{r} \approx \frac{1}{l}$$

thus

$$\psi_{A'} = g\psi_I \frac{e^{-ikl}}{l} \underbrace{\iint_{-\infty}^{\infty} dxdy e^{-\frac{ik}{2l}(x^2+y^2)}}_{\pi\frac{2l}{ik}}$$

equating to (3.22)

$$g = \frac{ik}{2\pi} = \frac{i}{\lambda}$$

We now do single slit diffraction. Let (ξ, η) be slit coordinate, $-\frac{a}{2} < \xi < \frac{a}{2}$, $-\infty < \eta < \infty$. Let (x, y) be screen coordinate. And $(0, 0)$ in both coordinates systems are aligned in z direction. To use (3.23), we need

$$\vec{r} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + l^2}\hat{n}$$

where $\hat{n}$ is pointing from (ξ, η) in the slit to (x, y) on the screen. Let $R^2 = x^2+y^2+l^2$

$$r = R\sqrt{1 - \frac{2x\xi}{R^2} - \frac{2y\eta}{R^2} + \frac{\xi^2+\eta^2}{R^2}}$$

If we ignore $\frac{\xi^2+\eta^2}{R^2}$, i.e. treat $\xi \ll x$, $\eta \ll y$, we get Fraunhofer diffraction. That is far field approximation. While Fresnel diffraction, near-field approximation, will keep this term. We do far field

$$r \approx R - \frac{x\xi}{R} - \frac{y\eta}{R}$$

So by (3.23), we get

$$\begin{aligned} \psi(x, y) &= \frac{i\psi_I}{\lambda R} e^{-ikR} \underbrace{\int_{-a/2}^{a/2} d\xi e^{ik\frac{x\xi}{R}}}_{\frac{\sin\frac{kax}{2R}}{\frac{kx}{2R}}} \underbrace{\int_{-\infty}^{\infty} d\eta e^{ik\frac{y\eta}{R}}}_{\sim\delta(y)} \\ &\propto \frac{\sin\frac{kax}{2R}}{\frac{kx}{2R}} \end{aligned}$$

$$I = I_0 \left(\frac{\sin \frac{kax}{2R}}{\frac{kx}{2R}} \right)^2 = I_0 \left(\frac{\sin \frac{\pi a \sin\theta}{\lambda}}{\frac{\pi \sin\theta}{\lambda}} \right)^2$$

with $\sin\theta = \frac{x}{\sqrt{l^2+x^2}}$, this justifies (3.19).

Lecture 16 (11/6/13)

One can draw above I, for three cases: $\lambda \approx a$, $\lambda \ll a$, and $\lambda \gg a$. One will see that only $\lambda \approx a$ gives nice diffraction pattern. $\lambda \ll a$ gives a very narrow peak, and $\lambda \gg a$ gives a very broad peak. That is because if $\lambda \gg a$, the first zero

$$\frac{\pi a \sin\theta}{\lambda} = \frac{\pi}{2}$$

can never be met.

Circular Hole Diffraction

Hole of radius a, do the similar integral (3.23) in polar, get

$$I = I_0 \left(\frac{J_1(k\theta a)}{k\theta a} \right)^2$$

where J_n, Bessel's equation

$$J_n(u) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{iu\cos\phi} e^{in\phi} d\phi$$

The zeros of J_1 at $u = 3.83, 7.02, ...$

The first minimum
$$\theta_1 = \frac{3.83}{2\pi}\frac{\lambda}{a} = 0.61\frac{\lambda}{a}$$

or in term of the opening angle

$$\delta\theta = 2\theta_1 = 1.22\frac{\lambda}{a} \tag{3.24}$$

This gives Rayleigh Criteria, saying that two point sources are resolved when the principal diffraction maximum of one image coincides or separated from the first minimum of the other. So better resolution, bigger a, that means larger lens.

Another application of (3.24), at night, infrared camera is used. So λ is longer, so the resolution gets worse.

An old sage says spy satellite sees license plate. Can it see? No, A satellite is D altitude from the earth and it can resolve the object of size on earth

$$\delta S = D\delta\theta = \frac{400\text{km} \times 1.22 \times 4000\dot{\text{A}}}{2\text{m}} = 8\text{cm}$$

Babinet's Principle

Not only hole gives diffraction, but also an opaque disk gives diffraction. And if we fill the hole with an opaque disk, we get nothing.

$$\psi_{hole} + \psi_{disk} = 0 \implies \psi_{disk} = -\psi_{hole}$$

$$I_{disk} = I_{hole}$$

4 Quantum Waves

4.1 Photons

Photoelectric Effects

1905 good year for Einstein. He figured out special relativity, photoelectric effects and Brownian motion implied existence of atoms.

Two observations in front of Einstein

Observation 1: There was a minimum ν to eject e^- from a metal. E.g. Na, 500$\dot{\text{A}}$ ejects, 600$\dot{\text{A}}$ noting.

Observation 2: There is no time leg from shinning lights on metal to ejection of e^-.

Both contradict wave theory

$$\text{energy} = \text{intensity} \times \text{time}$$

independent on ν, and depending on t.

Einstein postulated photons:

1) light composed photons γ

2) γ interacts with 1 e^-

3) γ transfers energy $h\nu$

The maximum energy of the ejected e^-

$$E_{max\, e^-} = h\nu - \phi$$

this is maximum because the ejected e^- may not be right on the surface. γ may have to penetrate in some depth and the ejected e^- is scattered before coming out of the metal. ϕ, work function, energy used to break binding.

The experimental setup is to put metal on one of the electrodes and space between another electrode. Shin light on the metal. Put adjustable voltage V to the electrodes. Use V_{max} to stop current, from there to determine E_{max}

$$eV_{max} = E_{max}$$

The graph V v.s. ν.

$$V_{max} = \frac{h}{e}\nu - \frac{\phi}{e}$$

The measurement is kind of tricky. Initially data gave $V(\nu) \sim V^{1/2}$. It was Millikan, being the first doctoral student graduated from our department, who obtained clean surface metals, and got a straight line. Millikan later got his Nobel prize for oil drop experiment.

X rays Production

This is the reverse of photoelectric effect. Use a secondary voltage to heat up the wire. When wire gets hot, e^- detaches. Then use another high voltage V source to accelerate those e^-. The place e^- takes off is called cathode, and then e^- hit the metal target, called anode, at very high energy. Not only current is produced, but also photons come out with high frequency from the metal.

Energy of the e^- before hit the metal

$$E_{e^-} = eV$$

the maximum energy transfer to photons

$$eV = E_{e^-} = E_{max} = h\nu = \frac{hc}{\lambda_{min}}$$

or

$$\lambda_{min} = \frac{hc}{eV} \tag{4.1}$$

This is in fact what one will get if he measures the intensity of X rays v.s. λ of the X rays. See that no X rays with λ less than λ_{min}. This contradicts classical wave theory that predicts all λ can be produced.

Compton Scattering

Lecture 17 (11/11/13)

We have seen two phenomena, photoelectric effect and X rays, that give definite support that lights are particles. Some historical remarks. X rays were discovered before photoelectric effects, but X rays were not understood until photoelectric effects were understood 1905. 1924 Compton explained Compton effect. By that time people had well accepted lights are particles, but Compton got Nobel prize anyway.

Suppose e^- at rest, γ hits e^- then γ and e^- scatter with angles θ, ϕ with respect to the incident axis.

$$\begin{aligned} \text{energy conservation} \qquad & h\nu + mc^2 = h\nu' + E_e' \\ \text{momentum conservation} \qquad & p_\gamma = p_\gamma' \cos\theta + p_e' \cos\phi \\ & 0 = p_\gamma' \sin\theta - p_e' \sin\phi \end{aligned}$$

By $E^2 = p^2c^2 + m^2c^4$

$$\text{photon } p_\gamma = \frac{h\nu}{c}$$

Thus

$$\begin{aligned} (h\nu/c - h\nu' \cos\theta/c)^2 + (h\nu' \sin\theta/c)^2 &= (p_e')^2 = E_e'^2/c^2 - m^2c^2 \\ &= (h\nu + mc^2 - h\nu')^2/c^2 - m^2c^2 \end{aligned}$$

therefore

$$\nu' = \frac{\nu}{1 + \frac{h\nu}{mc^2}(1 - \cos\theta)}$$

This is an example of elastic scattering. This happens when photon energy $h\nu \ll$ the rest energy of $e^- = mc^2 = 511\text{keV}$.

Experiment to verify above is to send photon of known energy $h\nu$ to a metal target, measure scatter photons θ and ν'. How to prepare $h\nu$? Use X ray setup (4.1). How to measure frequency of scattering light and precisely the scattering angle? Use crystal of Bloch spectrum, which we will discuss later.

We now fully establish particle nature of light. We need to reconcile particle wave duality. Because wave interferes but particles don't. This requires quantum mechanics.

4.2 Bohr Model

Rutherford 1910-1912 used α particle, He nucleus, to hit metal sheet. α particles were obtained from α decay, which we will discuss later in detail.

$$\text{U} \to \text{Th} + \alpha$$

Rutherford found that while most α passing through without any deflection, there were some α got almost reflected back, so he believed that those reflected back α must hit the nuclei of the metal. This contracts the early model that nuclei and e^-s were uniformly distributed inside of the atom. If it were the case, α would jiggle through like Brownian and not reflected back.

Thus Rutherford proposed that nuclei is highly concentrated. If the size of atom is $1\dot{\text{A}}$, then size of nuclei is $10^{-5}\dot{\text{A}}$. This model immediately created a problem. If e^- orbits around nuclei, dipole momentum

$$I \propto \left|\ddot{P}\right|^2 \neq 0$$

hence it radiates energy, hence atoms are unstable.

Motivated by planetary model, Bohr said

1) All classical law of physics are valid except those pertaining to radiation

2) radiation is only emitted when e^- jump from one planetary orbit to another

3) angular momentum is quantized

$$L = n\hbar = n\frac{h}{2\pi}$$

Bohr's rule solved the mystery put forward by a high school teacher, Balmer: the Rydberg formula for Hydrogen

$$\frac{1}{\lambda_{if}} = R\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) \tag{4.2}$$

The radiated photon wavelength λ_{if} is related to the orbital labels, where R is Rydberg constant.

Because

$$\frac{mv^2}{r} = \frac{ke^2}{r^2} \qquad k = \frac{1}{4\pi\epsilon_0}$$
$$L = mvr = n\hbar$$

That is

$$\begin{cases} mv^2 = \frac{ke^2}{r} \\ m^2v^2 = \frac{n^2\hbar^2}{r^2} \end{cases} \implies r_n = \frac{n^2\hbar^2}{e^2mk} = n^2a_0$$

where

$$a_0 = \frac{\hbar^2}{e^2mk} = \text{Bohr radius}$$

one can replace m by reduced mass μ to make more precise. $a_0 = 0.5\dot{\text{A}}$.

Therefore

$$E_n = \frac{1}{2}mv^2 - \frac{ke^2}{r} = -\frac{ke^2}{2r_n} = -\frac{ke^2}{2a_0n^2} = -\frac{1}{2}\frac{k^2me^4}{n^2\hbar^2} = -\frac{1}{2}\frac{mc^2}{n^2}\alpha^2$$

where

$$\alpha = \frac{ke^2}{\hbar c} = \text{fine structure constant} = \frac{1}{137}$$

This agrees exactly to quantum mechanics calculation.

Since

$$h\nu = E_{n_f} - E_{n_i}$$

we derived (4.2)

$$\frac{1}{\lambda_{if}} = \underbrace{\frac{1}{2\pi\hbar c}\frac{ke^2}{2a_0}}_{\frac{\alpha}{4\pi a_0}}\left(\frac{1}{n_i} - \frac{1}{n_f}\right)$$

This derivation works for any hydrogen like atom with replacement

$$e^2 \to Ze^2$$

But fails miserably for others, e.g. He

4.3 Matter Waves

de Broglie was the first person to evoke symmetry argument: if waves $\to$ particles, then particles $\to$ waves.

To complete his program, he needed

1) a formula for converting particles to wave: Broglie wavelength

2) a propagation equation of the matter waves: Schrodinger equation.

Broglie interpreted Bohr third rule

$$L = n\hbar$$

as wave wraps around a circle

$$n\lambda = 2\pi r$$

so

$$\lambda = \frac{2\pi r}{n} = \frac{2\pi\hbar}{n\hbar/r} = \frac{h}{L/r} = \frac{h}{p}$$

Broglie generalized this to that all parties have wavelength

$$\lambda = \frac{h}{p}$$

or

$$p = \frac{h}{\lambda} = \frac{\hbar 2\pi}{\lambda} = \hbar k$$

Then for free particle

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{4.3}$$

and the associated photon with that energy

$$E = h\nu = \hbar w \tag{4.4}$$

thus

$$w(k) = \frac{\hbar k^2}{2m}$$

dispersion relation, we can get group velocity

$$v_g = \frac{dw}{dk} = \frac{\hbar k}{m} = \frac{p}{m}$$

agrees particle definition.

Davisson–Germer Experiment verified Broglie. Shin e^- beam to crystal obtained some image as shinning X ray. Crystal has latter layer, so similar to thin film (3.20), but no refraction index hence $\theta_t = \theta$ and no extra π phase, so we get Bragg's law

$$n\lambda = 2t\cos\theta \tag{4.5}$$

where t is the latter spacing. But unlike thin film, the direction of latter plane may not be obvious (see Eisberg, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles.* figure 3.33). From the angle of deflection, i.e. the angle between incident beam and reflected beam ϕ, then $\theta = \phi/2$.

Davisson–Germer used energy e^- 54eV, $t = 0.91\dot{\text{A}}$ spacing and detect $\phi = 50^{\text{O}}$. By (4.5)

$$\lambda = 2 \cdot 0.91\dot{\text{A}} \cdot \cos 25^{\text{O}} = 1.65\dot{\text{A}}$$

Using de Broglie

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = 1.65\dot{\text{A}}$$

In perfect agreement.

4.4 Uncertainty Principle

Lecture 18 (11/13/13)

It says one can measure conjugate variable simultaneously with an uncertainty given by

$$\begin{aligned}\Delta x \Delta p_x &\geq \frac{\hbar}{2} \\ \Delta y \Delta p_y &\geq \frac{\hbar}{2} \\ \Delta z \Delta p_z &\geq \frac{\hbar}{2}\end{aligned}$$

Let's see what they mean.

Example 1:

Suppose we shin e^- beam through a slit with width a, then we know that at the moment it passes the slit its vertical position has uncertainty less than a. Since e^- is wave, it will form an diffraction pattern on the screen. Hence at the moment it passes the slit it had some momentum in vertical direction too, which can be calculated. Since e^- will likely fall within the first minimum of the diffraction pattern, cf (3.21)

$$\frac{\Delta p_y}{p} = \delta\theta \sim \frac{\lambda}{a} = \frac{h/p}{\Delta x_y} \implies \Delta x_y \Delta p_y \sim h$$

Example 2: Heisenberg Microscope

Suppose we use ordinary microscope to see e^-. There is a "microscope" L distance above an e^-. It sees the e^- by photons, with a resolution

$$\delta\theta = 1.22\frac{\lambda}{D} \sim \frac{\lambda}{D}$$

where D is the size of the lens, and λ is the wavelength of the photons. So the uncertainty in the horizontal direction of the $e^-$is

$$\Delta x_e = L\delta\theta \sim L\frac{\lambda}{D}$$

The reason it sees the e^- because many photons beam are sent near and parallel to the lens axis to illuminate the object, in this case is e^-. So some photons are scattered by the e^- and enters the lens. For those photons to enter the lens their horizontal momentum must not greater than

$$\Delta p_{x_\gamma} \sim \frac{E}{c}\frac{D}{L} = \frac{h\nu}{c}\frac{D}{L} = \frac{h}{\lambda}\frac{D}{L}$$

By conservation of momentum, the e^- must recoil and gain some horizontal momentum in opposite direction

$$\Delta p_{x_e} \sim h\lambda\frac{D}{L}$$

Thus

$$\Delta x_e \Delta p_{x_e} \sim L\frac{\lambda}{D}\frac{h}{\lambda}\frac{D}{L} = h$$

Example 3: Simple harmonic oscillation

Suppose a mass is hang and oscillating by a spring k, assume that the average x is 0 and average p is 0, so

$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2}k(\Delta x)^2$$

By uncertainty principle $\Delta p \sim \hbar/\Delta x$

$$E = \frac{\hbar^2}{2m(\Delta x)^2} + \frac{1}{2}k(\Delta x)^2$$

One can find that the minimum E is at

$$\Delta x = \left(\frac{\hbar^2}{mk}\right)^{\frac{1}{4}}$$

and

$$E_{min} = \hbar w = \hbar\sqrt{\frac{k}{m}}$$

This is exactly equal to what we will get from quantum mechanics.

Example 4:

Hydrogen

$$E = \frac{(\Delta p)^2}{2m} - \frac{ke^2}{\Delta x} = \frac{\hbar^2}{2m(\Delta x)^2} - \frac{ke^2}{\Delta x}$$

minimum at

$$\Delta x = \frac{\hbar^2}{kme^2} = a_0$$

and

$$E_{min} = -\frac{ke^2}{2a_0}$$

Both in perfect agreement to quantum mechanics.

Example 5:

By uncertainty principle, e^- cannot get in the nuclei, then $\Delta x \to 0$, $\Delta p \to \infty$. This explanation is more fundamental than Bohr rules, but still doesn't answer why orbiting causes no radiation.

One can get E v.s. t version of the uncertainty principle, by $E = \frac{p^2}{2m}$

$$\Delta E \Delta t = \frac{p\Delta p}{m}\Delta t = \frac{mv\Delta p\Delta t}{m} = \Delta p \Delta x \geq h$$

Example 6: Lamb Shift

Due to vacuum fluctuations, e^- e^+ are created within Δt allowed by uncertainty principle, which causes screening

$$k\frac{Ze^2}{r}e^{-\frac{hr}{mc}}$$

4.5 Schrodinger Equation

Quantum Postulates

1) To each classical observable assign a quantum operator which will operator the wave function, e.g. $\hat{x}$, $\hat{p}$, $\hat{E}$

$$\hat{A}\psi$$

2) The result of operating on ψ with operator $\hat{A}$ must be a physically possible result if ψ is an eigenstate. The results are eigenvalues.

3) The definition relative to classical physics not involving derivative are unchanged

$$x \to \hat{x} \qquad p \to \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\hat{E} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \qquad \hat{L}_z = \hat{x}\hat{p} - \hat{p}\hat{x}$$

Lecture 19 (11/18/13)

Example: Free Particle

$$\hat{E} = \frac{\hat{p}^2}{2m}$$

find eigenfunction try

$$\hat{E}e^{i(kx-wt)} = -\frac{\hbar^2\partial^2}{2m\partial x^2}e^{i(kx-wt)} = \frac{\hbar^2k^2}{2m}e^{i(kx-wt)}$$

Hence

$$E = \frac{\hbar^2k^2}{2m}$$

We have other quantity for $E = \hbar w$, cf (4.3), (4.4), so put Humiliation

$$\hat{H} = i\hbar\frac{\partial}{\partial t}$$

so for $\psi = e^{i(kx-wt)}$

$$(\hat{H} - \hat{E})\psi = 0$$

In general if potential presents

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$$

Or 3D

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\vec{x})\Psi$$

We solve 1D Schrodinger equation by separation of variables. This works only if V has no t dependence.

$$\Psi(x,t) = \psi(x)f(t)$$

get

$$\frac{i\hbar}{f}f' = -\frac{\hbar^2}{2m}\frac{\psi''}{\psi} + V$$

Since left depends on t, right depends on x, both should be constant $= E$

$$f = e^{-iEt/\hbar}$$

Or

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

where $\psi(x)$ solves

$$-\frac{\hbar^2}{2m}\frac{\psi''}{\psi}(x) + V(x) = E$$

Time independent Schrodinger equation. The solution is normally chosen to be real. With one exception in hydrogen wave function, complex ψ is more useful. Because the eigenfunctions of the rotation operator are complex.

Quantum Interpretation Postulates

1) The probability of a particle being at position x to $x + dx$ is

$$|\psi(x)|^2 \, dx$$

The implies ψ that ψ has to be proper normalized,

$$\int |\psi(x)|^2 \, dx = 1$$

or 3D

$$\int |\psi(x,y,z)|^2 \, dxdydz = 1$$

And also means that

$$|\psi|^2 \sim \frac{1}{[\text{length}]}$$

In the spirit of (3.12), we put

2) Given an eigenfunction u_a with eigenvalue a of an observable $\hat{A}$ and a general state vector ψ, then the probability of measure ψ and obtain a is

$$P(a) = \left| \int u_a^*(x)\psi(x)dx \right|^2$$

call overlapping integral.

E.g. If $a = \hat{x}$, then $u_a(x) = \delta(x - a)$, so

$$P(a) = |\psi(a)|^2$$

agreeing 1). Another example $u_p(x) = e^{ipx/\hbar}$momentum eigenfunction, i.e.

$$\frac{\hbar}{i}\frac{\partial}{\partial x}e^{ipx/\hbar} = pe^{ipx/\hbar}$$

Measure

$$\psi(x) = \frac{1}{\sqrt{\sqrt{2\pi}\sigma_x}}e^{-x^2/4\sigma_x^2}$$

whose $|\psi(x)|^2$ has deviation σ_x

$$\tilde{\psi}(p) = \int u_p^*(x)\psi(x)dx = \int_{-\infty}^{\infty} e^{-ipx/\hbar}\frac{1}{\sqrt{\sqrt{2\pi}\sigma_x}}e^{-x^2/4\sigma^2}dx \sim e^{-p^2\sigma_x/\hbar}$$

whose $\left|\tilde{\psi}(p)\right|^2$ = Probability of measuring $\psi(x)$ and obtain momentum p, has deviation $\sigma_p = \hbar/2\sigma_x$. Hence we get the minimum of uncertainty

$$\sigma_x\sigma_p = \frac{\hbar}{2}$$

3) The expectation value of observable on a general state ψ is

$$\left\langle \hat{A} \right\rangle = \int aP(a) = \int \psi(x)^*\hat{A}\psi(x)dx$$

E.g.

$$\begin{aligned}
\langle \hat{x} \rangle &= \int \psi(x)^* x\psi(x)dx \\
\langle \hat{p} \rangle &= \int \psi(x)^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)dx
\end{aligned}$$

4.6 Infinite Square Well

$$V = \begin{cases} 0 & -\frac{L}{2} < x < \frac{L}{2} \\ \infty & \text{otherwise} \end{cases} \tag{4.6}$$

Physical example like p n in nuclear like in a box.

First consider parity $x \to -x$, since $V(-x) = V(x)$ symmetric potential

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \tag{4.7}$$

change $x \to -x$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{\partial(-x)^2} + V(-x)\psi(-x) = E\psi(-x)$$

then

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{\partial x^2} + V(x)\psi(-x) = E\psi(-x)$$

hence $\psi(-x)$ is a solution of (4.7) too. Since (4.7) is linear, we can also construct solutions of (4.7) have even or odd parities.

$$\psi(x) + \psi(-x) \text{ even sol}$$

$$\psi(x) - \psi(-x) \text{ odd sol}$$

Now solve (4.6) for $-L/2 < x < L/2$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$$

or

$$\psi'' = -k^2\psi$$

Hence

$$\psi_n = \begin{cases} \psi_{even} = \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} & n = 1, 3, 5, .. \\ \psi_{odd} = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} & n = 2, 4, 6 \end{cases} \tag{4.8}$$

and

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

Lecture 20 (11/25/13)

This example shows some very important general rules

1) Ground state is always even, if the potential is symmetric.

2) Eigenstates are stable, because full wave function

$$\Psi = \psi e^{-iEt/\hbar}$$

$$|\Psi(x,t)|^2 = |\psi(x)|^2$$

independent of time. Hence dipole momentum

$$\langle P \rangle = e \int x |\Psi_{n,m}|^2 dx$$

is independent of time. This shows why e^- orbits without radiation.

But if we put e^- in the box in state

$$\Psi_{n,m}(x,t) = a\psi_n e^{-iE_n t/\hbar} + b\psi_m e^{-iE_m t/\hbar}$$

then

$$|\Psi_{n,m}|^2 = a^2|\psi_n| + b^2|\psi_m| + 2ab|\psi_n \psi_m| \cos wt$$

where $w = \frac{E_n - E_m}{\hbar}$.

Compute dipole momentum

$$\langle P \rangle = e \int x |\Psi_{n,m}|^2 dx \sim \cos wt + \text{terms independent of } t$$

Hence

$$I \sim \frac{d^2}{dt^2} \langle P \rangle \neq 0$$

such e^- will radiate.

If eigenstates are truly stationary, then how come spontaneous radiative decay happen in nature? That is because 0 point vibration of vacuum, the true Hamiltonian is not truly time independent.

3) Selection rule: prohibit electric dipole transition unless change parity. Refer to (4.8) clearly

$$\int x \psi_{even} \psi_{even} = 0 \quad \int x \psi_{odd} \psi_{odd} = 0$$

This doesn't mean no transition can happen between these states. Higher poles can happen, but they are weak.

4.7 Bound States

We study two more bound states: finite square well and simple harmonics oscillator.

Finite Square Well

e^- energy $-\epsilon > -V_0$ look for bound solutions

$$V = \begin{cases} -V_0 & |x| < a \\ 0 & |x| > a \end{cases} \qquad V_0 > 0$$

Need to solve

$$\begin{cases} -\frac{\hbar^2}{2m}\psi'' - V_0\psi = -\epsilon\psi & |x| < a \\ -\frac{\hbar^2}{2m}\psi'' = -\epsilon\psi & |x| > a \end{cases}$$

Let

$$k^2 = \frac{2m}{\hbar^2}(V_0 - \epsilon) \qquad \alpha^2 = \frac{2m}{\hbar^2}\epsilon$$

Look for even or odd solutions

	even	odd
$\lvert x\rvert < a$	$b_1 \cos kx$	$b_2 \sin kx$
$x > a$	$c_1 e^{-\alpha x}$	$c_2 e^{-\alpha x}$
$x < a$	$c_1 e^{\alpha x}$	$-c_2 e^{\alpha x}$

At $x = \pm a$, ψ is continuous, ψ' is continuous.
For the even solution

$$\begin{aligned} b_1 \cos ka &= c_1 e^{-\alpha a} \\ -b_1 k \sin ka &= -c_1 \alpha e^{-\alpha a} \end{aligned}$$

So

$$\tan ka = \frac{\alpha}{k} = \frac{\sqrt{\lambda - (ka)^2}}{ka}$$

where $\lambda = \frac{2ma^2}{\hbar^2}V_0$. Put $y = ka$

$$\tan y = \frac{\sqrt{\lambda - y^2}}{y}$$

One can draw the LHS and the RHS. And see that if

$$(n-1)\pi \leq \lambda < n\pi$$

there are n different solutions of y, so n different energies

$$-\epsilon_n = \frac{\hbar^2 k^2}{2m} - V_0 = \frac{\hbar^2 y_n^2}{2ma^2} - V_0$$

Lecture 21 (12/2/13)

In fact if the well is shallower or narrower, i.e. smaller λ, less number of y, but no matter how small λ is, there is at least one bound state.

Question: the solution has a piece in $|x| > a$ region. That is possible to find e^- outside of well, hence e^- will have negative kinetic energy. What does negative kinetic energy mean? Outside of well $\psi = e^{\pm\alpha x}$, so

$$\Delta x \sim \frac{1}{\alpha}$$

By uncertainty

$$\Delta p \sim \hbar\alpha$$

so the uncertainty in energy when the e^- is outside

$$\Delta E = \frac{(\Delta p)^2}{2m} \sim \frac{\hbar^2\alpha^2}{2m} = \epsilon$$

hence the uncertainty in energy as large as the energy itself, then negative kinetic energy has no definite confirmation.

Question: Why are bound states discrete? Outside of well, ψ satisfies

$$\psi'' = \alpha^2\psi$$

and ψ has to be normalizable, we will see only special α, i.e. special ϵ, will make this happen. Suppose at the boundary of the well, $\psi(a) > 0$ then $\psi''(a) > 0$ so $\psi'(a)$ has to be negative otherwise ψ will keep growing. So $\psi'(a) < 0$, even then ψ' will become less negative because $\psi''(a) > 0$. If α is too big, or ψ' reaches 0 too fast, i.e. ψ' becomes positive before $x \to \infty$, then $\psi \to \infty$, not normalizable. If α is too small, or ψ' reaches 0 too slow, i.e. ψ becomes negative before ψ' reaches 0, then ψ'' becomes negative, then ψ' will keep negative, then $\psi \to -\infty$, not normalizable.

Simple Harmonics Oscillation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + \frac{1}{2}mw^2x^2\psi = E\psi \tag{4.9}$$

First choose dimensionless variables

$$u = \gamma x = \sqrt{\frac{mw}{\hbar}}x \qquad \alpha = \frac{2E}{\hbar w}$$

(4.9) becomes

$$\psi''(u) = (u^2 - \alpha)\psi(u)$$

The solutions are

$$\psi_n(u) = c_n e^{-\frac{1}{2}u^2} H_n(u)$$

$$c_n = \left(\frac{\gamma}{\sqrt{\pi}2^n n!}\right)^{\frac{1}{2}}$$

H_n Hermite polynomials

$$\begin{aligned} H_0(u) &= 1 \\ H_1(u) &= 2u \\ H_2(u) &= 4u^2 - 2 \\ H_3(u) &= 8u^3 - 12u \end{aligned}$$

with

$$\alpha_n = 2n + 1$$

or

$$E_n = (n + \frac{1}{2})\hbar w \quad n = 0, 1, 2, ...$$

We see that ground state energy is $\frac{1}{2}\hbar w$, so there's always energy in the system.

4.8 Scattering, Tunneling

Step function Scattering

(Griffiths problem 2.34)

$$V = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases} \quad V_0 > 0$$

Particle coming from the left, with energy $E > V_0$, we get

$$\begin{cases} \psi'' + k_1^2\psi = 0 & x < 0 \\ \psi'' + k_2^2\psi = 0 & x > 0 \end{cases}$$

$$k_1^2 = \frac{2m}{\hbar^2}E \qquad k_2^2 = \frac{2m}{\hbar^2}(E - V_0)$$

This is exactly equivalent to section 1.6, impedance mismatch.

$$\psi = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < 0 \\ Ce^{ik_2x} & x > 0 \end{cases}$$

Compare to section 1.6 we don't have to put e^{iwt} because in QM, $e^{iwt} \leftrightarrow e^{iEt/\hbar}$.

And the same boundary conditions ψ, ψ' continuous, therefore

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2} \text{ and } \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

and reflected ratio, transmitted ratio are the same

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \qquad T = \frac{4k_1k_2}{(k_1 + k_2)}$$

$$R + T = 1$$

However the interpretations of these ratio are different. In section 1.6, R, T are energy ratio. Here they are ratio of finding particles. Suppose initial N particles coming from the far left. Let

$$F_\pm = \text{flux of particles to right or left}$$

then

$$\begin{cases} F_+ = N|A|^2 v_1 \\ F_- = N|B|^2 v_1 \end{cases} x < 0 \qquad F_+ = N|C|^2 v_2 \, x > 0$$

$v_1 = \sqrt{2E/m}$, $v_2 = \sqrt{2(E - V_0)/m}$, thus

$$R = \left|\frac{B}{A}\right|^2 \qquad T = \sqrt{\frac{E - V_0}{E}} \left|\frac{C}{A}\right|^2$$

The identification of the particle nature and wave nature is of course by de Broglie

$$mv = p = \hbar k$$

Step FUnction Tunneling

Lecture 22 (12/4/13)

We continue Griffiths problem 2.34, what happens if $E < V_0$? Tunneling, which is not only impossible for classical particles, but impossible for classical wave.

Particle coming from the left, with energy $E < V_0$, we get

$$\begin{cases} \psi'' + k_1^2 \psi = 0 & x < 0 \\ \psi'' - k_2^2 \psi = 0 & x > 0 \end{cases}$$

$$k_1^2 = \frac{2m}{\hbar^2} E \qquad k_2^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

So (simply recognize that solutions can be quickly obtain by replacing k_2 in the previous case by ik_2)

$$\psi = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < 0 \\ Ce^{-k_2x} & x > 0 \end{cases}$$

using ψ,ψ' continuous, find

$$\frac{B}{A} = \frac{k_1 - ik_2}{k_1 + ik_2} \text{ and } \frac{C}{A} = \frac{2k_1}{k_1 + ik_2}$$

and

$$R = \left| \frac{Be^{-ik_1x}}{Ae^{ik_1x}} \right|^2 = \frac{k_1^2 + k_2^2}{k_1^2 + k_2^2} = 1 \qquad T = \frac{k_2}{k_1} \left| \frac{Ce^{-k_2x}}{Ae^{ik_1x}} \right|^2 e^{-2k_2x} = \frac{4k_1k_2}{(k_1 + k_2)} e^{-2k_2x}$$

$x > 0$ $|\psi|^2 \sim e^{-2k_2x}$, so the uncertainty of finding particles outside well

$$\Delta x \sim \frac{1}{2k_2}$$

Hence the uncertainty in T is as large as T itself, so as far as uncertainty is concern

$$R + T \approx R = 1$$

Finite Square Well Scattering

$$V = \begin{cases} -V_0 & |x| < a \\ 0 & |x| > a \end{cases} \qquad V_0 > 0$$

Particle coming in from the left with energy $E > 0$. We will see this is exactly equivalent to the technique used in section 2.2, impedance matching.

We solve

$$\begin{cases} \psi'' + k^2\psi = 0 & |x| > a \\ \psi'' + q^2\psi = 0 & |x| < a \end{cases}$$

$$k^2 = \frac{2mE}{\hbar^2} \quad q^2 = \frac{2m}{\hbar^2}(E + V_0) \tag{4.10}$$

then

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < a \\ Ce^{iqx} + De^{-iqx} & |x| < a \\ Fe^{ikx} & x > a \end{cases}$$

After some exhausting exercise (Griffiths equations 2.167, 2.168)

$$\begin{aligned} B &= \frac{ie^{-2ika}(q^2 - k^2)\sin(2qa)}{2qk\cos(2qa) - i(k^2 + q^2)\sin(2qa)} A \\ F &= \frac{e^{-2ika}(2qk)}{2qk\cos(2qa) - i(k^2 + q^2)\sin(2qa)} A \end{aligned}$$

If $E \gg V_0$, then

$$q^2 - k^2 \ll 2qk$$

then

$$\left|\frac{B}{F}\right| = \frac{(q^2 - k^2)\sin(2qa)}{2qk} \ll 1$$

complete transmission. There is another way to get complete transmission: Ramsauer-Townsend effect. That is if $2qa = n\pi$, $B = 0$. And by (4.10)

$$E_n = -V_0 + \frac{n^2\hbar^2\pi^2}{8ma^2}$$

this is exactly the allowed energies in a ∞ square well, except it is now shifted by V_0 and n starts from some integer n_0 such that $E_{n_0} > 0$. If $n = 1$ is allowed, then choose

$$a = \frac{\pi}{2q} = \frac{\lambda}{4}$$

This is quarter wave transmission we discussed in section 2.2.

Finite Square Barrier Tunneling

$$V = \begin{cases} V_0 & |x| < a \\ 0 & |x| > a \end{cases} \qquad V_0 > 0$$

Particle coming in from the left with energy $V_0 > E > 0$.

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < a \\ Ce^{qx} + De^{-qx} & |x| < a \\ Fe^{ikx} & x > a \end{cases}$$

$$k^2 = \frac{2mE}{\hbar^2} \quad q^2 = \frac{2m}{\hbar^2}(V_0 - E) \tag{4.11}$$

So (simply recognize that solutions can be quickly obtain by replacing q in the previous case by iq),

$$F = \frac{e^{-2ika}(2qk)}{2qk\cosh(2qa) - i(k^2 - q^2)\sinh(2qa)}A \tag{4.12}$$

4.9 WKB, alpha Decay

From (4.12), if $qa \gg 1$

$$\cosh 2qa,\ \sinh 2qa \to \frac{1}{2}e^{2qa}$$

then

$$T = \left|\frac{F}{A}\right|^2 \to \left(\frac{4qk}{q^2 + k^2}e^{-2qa}\right)^2$$

then

$$\ln T = -2q\underbrace{2a}_{\Delta x} + \text{small terms}$$

Suppose instead of having one finite barrier, we have $V(x)$, e.g. a Gaussian. Particle coming from the left with energy $< \max V(x)$ tries to tunnel through. What is the probability of tunneling, P? We will approximate the tunnel region, i.e. where $V(x) > E$, by many finite square barrier of width Δx, so the probability of tunneling is the product of the probability, T_i, of tunneling through each finite square barrier, i.e.

$$\begin{aligned} P &= \prod T_i \\ \ln P &= \sum \ln T_i = -2\sum q_i(\Delta x)_i = -2\int q(x)dx \end{aligned}$$

Thus

$$P = e^{-2\int \sqrt{\frac{2m}{\hbar^2}(V(x)-E)}dx} = e^{-G}$$

the integral is taken over classically forbidden region.

α decay

Lecture 23 -Last Lec- (12/9/13)

The 1D model we develop can be used to estimate the lifetimes T of α decay

$$A = \alpha + D$$

A parent, D daughter, $\alpha =^4$ He ion. The potential is given by Griffiths figure 8.5.

$$r_1 = 1.2\text{fm}A^{\frac{1}{3}}$$

A =number of n and ps. 1fm $= 10^{-15}$m, r_2 is given by

$$E_\alpha = \frac{kZZ_\alpha e^2}{r_2}$$

Let Z be number of protons after α has passed r_1.

Assume within r_1, α is moving at speed v, then it takes

$$\frac{2r_1}{v} \text{ sec}$$

to for one α particle to begin tunneling again if it didn't succeed last time. The probability of escape is e^{-G}. So in 1sec the number of escaped α is at the order (because there may be multiple α bouncing in $r < r_1$)

$$\sim \frac{1}{\frac{2r_1}{v}}e^{-G}$$

which is the rate of tunneling, so lifetime

$$T \sim \frac{v}{2r_1}e^{G}$$

We estimate G.

$$\begin{aligned} G &\approx \frac{1}{\hbar}\int_{r_1}^{r_2} dr\sqrt{2m(V(r)-E_\alpha)} \\ &\sim r_2|p| \\ &\propto \frac{Z}{E_\alpha}\sqrt{E_\alpha} \\ &= \frac{Z}{\sqrt{E_\alpha}} \end{aligned}$$

Experimentalists prefer $T_{\frac{1}{2}}$ half time. The relation between lifetimes and half time is

$$T_{\frac{1}{2}} = T\ln 2$$

$$\log_{10}\frac{1}{T_{\frac{1}{2}}} = \frac{\ln\frac{1}{T\ln 2}}{\ln 10} = \frac{1}{\ln 10}\left(\ln\frac{1}{T} - \ln\ln 2\right)$$

Hence $\log_{10}\frac{1}{T_{\frac{1}{2}}}$ is linear wrt $\ln\frac{1}{T}$, so we can write

$$\log_{10}\frac{1}{T_{\frac{1}{2}}} = C_1 - \frac{C_2 Z}{\sqrt{E}}$$

Theoretical estimates for $C_1 = 27-28$, $C_2 = 1.73\sqrt{\text{MeV}}$.

The experiment data plot

$$\log_{10}\frac{1}{T_{\frac{1}{2}}} \text{ v.s. } \frac{1}{\sqrt{E}}$$

find $C_1 = 28.9$, $C_2 = 1.61$.

Nuclear Fusion

Fusion is the reverse of α decay. It happens inside of sun

$$^2\text{H} + {}^2\text{H} \to ^3\text{He} + n$$

$$^2\text{H} + {}^2\text{H} \to ^3\text{H} + p$$

$$^3\text{H} + {}^2\text{H} \to {}^4\text{He} + n$$

Why aren't there high Z fusion?

$$\mathrm{Ca} + \alpha \to ?$$

Because for high Z, the Coulomb potential

$$\frac{Z_\alpha Z e^2 k}{r}$$

is very high, for normal energy α the probability of tunneling is too low. Even at the center of the star,

$$kT = 1\mathrm{MeV} \implies T \sim 10^{10}\mathrm{K}$$

still too low for high Z fusion.

4.10 Hydrogen Wave function & Spin

Solve Schrodinger in spherical coordinate

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

becomes

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V\psi = E\psi$$

solution

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

$$Y_{lm}(\theta,\phi) \sim P_{lm}(\theta)e^{im\phi}$$

Y_{lm} spherical harmonics, P_{lm} associated Legendre polynomials.

$$E_n = -\frac{kme^4}{2\hbar^2 n^2} \qquad k = \frac{1}{4\pi\epsilon_0}$$

$$\begin{aligned} n &= 1, 2, 3, ... \\ l &= 0, 1, ..., n-1 \\ m &= -l, -l+1, ..., l \end{aligned}$$

Groundstate $n = 1$, $l = m = 0$

$$R_{10} \sim e^{-r/2a_0}$$

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}} \\ Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y_{1-1} &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \\ Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \end{aligned}$$

Probability of finding particles at volume element (r, θ, ϕ) is

$$|\psi|^2 r^2 dr d\Omega$$

Probability of finding particles at radius r is

$$|R_{nl}(r)|^2 r^2 dr$$

Probability of finding particles at radial direction θ, ϕ is

$$|Y_{lm}|^2 d\Omega$$

Y_{lm} spherical harmonics are eigenstate of L^2, L_z

$$\begin{aligned} L_z &= xp_y - yp_x \\ L^2 &= L_x^2 + L_y^2 + L_z^2 \end{aligned}$$

$$\begin{aligned}\hat{L}^2 Y_{lm} &= \hbar^2 l(l+1) Y_{lm}\\ \hat{L}_z Y_{lm} &= \hbar m Y_{lm}\end{aligned}$$

Spin

Spin is completely quantum mechanical phenomena that has no classical analogy. Pauli said no two e^{-}s can have same quantum numbers. Since the groundstate of hydrogen ψ_{100} is occupied by 2 e^{-}, there must be a missing quantum number. Pauli said spin is an internal degree of freedom

$$\chi_{s,m_s}$$

called spinor. It has the same properties as Y_{lm}

$$\begin{aligned}\hat{S}^2 \chi_{s,m_s} &= \hbar^2 s(s+1)\chi_{s,m_s}\\ \hat{S}_z \chi_{s,m_s} &= \hbar m_s \chi_{s,m_s}\end{aligned}$$

s is fixed for the particle, and

$$m_s = -s, ..., s$$

so the groundstates

$$\psi_{nlmm_s} = \psi_{100\frac{1}{2}} \text{ or } \psi_{100-\frac{1}{2}}$$

2 groups of particles and supersymmetry particles

		Superpartner	
fermion	half integer spin	Sfermion	integer spin
Boson	integer spin	Sboson	half integer spin

4.11 Afterword: Counterfactual Measurement

Thing to think about

A Mach-Zehnder interferometer is used in such a way that when γ reaches detector 1 always constructive interference and when γ reaches detector 2 always deconstructive interference, hence no γ detected by detector 2.

$$
\begin{tikzcd}
E_0 \arrow[r] & \boxed{\begin{array}{c}\text{bean spliter}\\ \text{half silvered mirror}\end{array}} \arrow[rr] \arrow[d] & & \boxed{\text{mirror}} \arrow[d] & \\
& \boxed{\text{mirror}} \arrow[rr] & & \boxed{\begin{array}{c}\text{bean spliter}\\ \text{half silvered mirror}\end{array}} \arrow[r] \arrow[d] & \boxed{\text{detector 1}} \\
& & & \boxed{\text{detector 2}} &
\end{tikzcd}
$$

Now block one of the two paths

$$
\begin{tikzcd}
E_0 \arrow[r] & \boxed{\begin{array}{c}\text{bean spliter}\\ \text{half silvered mirror}\end{array}} \arrow[rr] \arrow[d] & & \boxed{\text{mirror}} \arrow[d] & \\
& \boxed{\text{mirror}} \arrow[r] & \boxed{\text{stop}} & \boxed{\begin{array}{c}\text{bean spliter}\\ \text{half silvered mirror}\end{array}} \arrow[r] \arrow[d] & \boxed{\text{detector 1}} \\
& & & \boxed{\text{detector 2}} &
\end{tikzcd}
$$

The interference is destroyed, so both detectors detect γ. The spooky thing is that the information of the present of the stopper is conveyed via the photons that hit detector 2. Paradoxically these photons don't even pass the path where the stopper are. Hence we can know things about an object without interacting with it.