Elliptic PDE

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This is a graduate course, offered in fall 2013 at New York University. Course textbooks are *Elliptic Partial Differential Equations* by Han, Lin; *Elliptic Partial Differential Equations of Second Order* by Gilbarg, Trudinger; and *Fully Nonlinear Elliptic Equations* by Caffarelli, Cabre. There would be no midterm or final. Textbooks used in PDE I & II may be good for review. They are *Partial Differential Equations* by Fritz John, and *Partial Differential Equations* by Evans.

Contents

1 Course Overview

1.1 Basic Methods

Lecture 1 (9/4/13)

Elliptic PDE is when a_{ij} positive definite, it has two forms: Divergence Form v.s. non Divergence Form.

Divergence Form

$$\partial_{x_i}(a_{ij}(x)u_{x_j}) = f$$

one uses *Variational Method*, or *Energy Method.*

Assume (1) $a_{ij}(x) \in L^\infty$ (2) $a_{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2$ $\xi \in \mathbb{R}^n$, then in Hilbert space, solution exists always in integral form (often involve integration by parts). More recently result (1960s) we will study include *De Giorgi-Nash-Moser* gives Holder continuous.

More specifically one can use *Dirichlet Principle.* That is if

$$\Delta u = 0$$

then u is the energy minimizer,

$$\min \int |\nabla u|^2 \, dx$$

Non-Divergence Form

$$a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u = f$$

Assume (1) symmetric $a_{ij} = a_{ji}$ (2) uniformly elliptic $\Lambda I \geq (a_{ij}) \geq \lambda I$.

one uses *Maximum principle for Hopf-Tpye or Alexandroff-Tpye,* or uses *Viscosity Solutions.* All of them give C^α regularities. More recently (1980s) *Krylov-Safonov* show more general result, no smooth/continuity assumptions on a_{ij}, give $u \in C^\alpha(\bar{\Omega})$.

Affine Invariant

The theorems mentioned above are stated in Balls B_R in $\mathbb{R}^n$.

For example, pde

$$det(u_{ij}) = 1, \quad (u_{ij}) > 0$$

In 2-D, this equals to

$$u_{xx}u_{yy} - u_{xy}^2 = 1$$

then $u(x)$ is a solution, implies $u(Ax+b)$ is a solution $det(A) = 1$, more specifically, this means the solution is invariant under rotation, and scalaring, i.e. *Affine Invariant.* That is $R \in O(n)$, $u \to u(Rx+a)$ or $u \to u(\lambda x)$.

Instead of working with Balls, we can also use *convex* set.

Theorem 1. *(John's Lemma) Let Ω be a convex body in $\mathbb{R}^n$, then there is an affine transformation $T : x \to Ax + b$ $det(A) = 1$ such that*

$$B_R^n \subseteq T(\Omega) \subseteq B_{c(n)R}^n$$

Note 2. where $c(n) = \sqrt{n}$ (?)

1.2 Classical Theories (1900 - 1955)

Review Function Theory, Geometric Function Theory (conformal Mapping)

Cauchy Integral

$$f(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi \implies f^n(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f^n(\xi)}{\xi - z} d\xi$$

so

$$|f(z)| \leq \max|f(\xi)| \left(\frac{l(\partial\Omega)}{2\pi dist(z,\xi)}\right)^{1/n} \leq_{n\to\infty} \max|f(\xi)|$$

Maximum Principle

$$a_{ij}(x)u_{x_ix_j} = 0$$

in Ω, then $\forall\, x \in \Omega$

$$|u(x)| \leq \max_{y\in\partial\Omega}|u(y)|$$

Notice this is *no* correct for vector functions $u = (u^1, u^2, ..., u^n)$

$$Lu = 0 \iff a_{\alpha\beta}^{ij} u_{x_ix_j}^{\beta} = 0 \text{ in}\Omega,\ \alpha = 1, 2, ..., n$$

then

$$\sup_{x\in\Omega}|u(x)| \not\leq \sup_{y\in\partial\Omega}|u(y)|$$

Gradient Estimate

$$f^{(n)}(z) = \frac{c(n)}{2\pi i} \oint_{\partial\Omega} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

Looking for an integral representation for the solution. Things involved are *Power Series* (Analyticity), *Unique Continuation, Liouville Theorem* (local estimate). 1950s Power Series for non-linear pde developed.

Theorem 3. *(Hopf Theorem 2-D case)*

$$a_{ij}(x)u_{x_i x_j} = 0$$

in $\mathbb{R}^2$, $(a_{ij}) > 0$. *If* $u = o(|x|)$ *as* $|x| \to \infty$, *then* $u(x) =$ *constant.*

Singularities

Is it *removable* or not? Other involves *Fundamental Solutions*, *eigenvalues & eigenfunctions.*

For example

$$\begin{cases} \Delta u^\lambda + \lambda u = 0 & \text{in } \Omega \\ u^\lambda = 0 & \text{on } \partial\Omega \end{cases}$$

we require the solution to be normalized

$$\int_\Omega \left|u^\lambda\right|^2 dx = 1$$

choose a probability measure $\mu_\lambda = \left|u^\lambda\right|^2 dx$, let $\mu_\lambda \rightharpoonup \mu$.

Questions (1) What's the support of μ? (2) Do we have to take subsequence to make the weak convergence to work?

Potential Theory

More of these are in Harmonic Analysis.

1.3 More Recent Results (1955 -)

DeGiogi-Moser-Nash

Late 1950's DeGiogi-Moser-Nash developed C^α regularity for $(a_{ij}(x)u_{x_j})_{x_i} = 0$. This solves *Quasilinear* Elliptic PDEs. The problem is motivated by Hilbert's 19th Problem: "whether the solutions of regular problems in the calculus of variations are always analytic." More improved results see Serrin, Trudinger.

Krylov-Safonov

1980's Krylov-Safonov showed

$$a_{ij}(x)u_{x_ix_j} = 0 \implies u \in C^\alpha$$

with no smooth assumption on a_{ij}. This problem is motivated by solving $F(D^2u) = 0$.

Example

$$\begin{cases} F(D^2u) = 0 \\ u|_{\partial\Omega} = g \end{cases}$$

is uniformly elliptic at u iff

$$\lambda\,|N| \leq F(M+N) \leq \Lambda\,|N|$$

or

$$\lambda I \leq \left(\frac{\partial F}{\partial M_{ij}}\right) \leq \Lambda I$$

where $M \in S_{n\times n}$, $N \in S^+_{n\times n}$.

Now

$$0 = \partial_l F(D^2u) = F_{ij}(D^2u)D^2_{ij}u_l = 0$$

Let $a_{ij}(x) = F_{ij}(D^2u)$ uniformly elliptic, let $V = u_l$ so

$$a_{ij}(x)v = 0$$

By Krylov-Safonov

$$\begin{aligned} &\implies \quad D^2u_l \in C^\alpha(\Omega) \\ &\implies \quad u \in C^{1,\alpha} \end{aligned}$$

In addition, if F is concave, then $u \in C^{2,\beta}(\Omega)$, $\beta > 0$ by Evans, Krylov, Caffarelli.

Regularity Theory (Linear)

$$\begin{cases} a_{ij}(x)u_{x_ix_j} = f(x) \qquad \textbf{in } \Omega \subseteq \mathbb{R}^n \\ u|_{\partial\Omega} = 0 \\ \Lambda I \geq (a_{ij}(x)) \geq \lambda I \end{cases}$$

(i) $a_{ij}(x) \in C^\alpha(\bar{\Omega})$ Holder continuous $(1 > \alpha > 0) \implies \|u\|_{C^{\alpha+2}(\bar{\Omega})} \leq c(n, \alpha, \lambda, \Lambda, \|a_{ij}\|_{C^\alpha}) \|f\|_{C^\alpha(\bar{\Omega})}$.

This is called *Schauder Estimate*. See Trudinger Chap 6.

(ii) $a_{ij}(x) \in C^0(\bar{\Omega}) \implies a_{ij} \in VMO$.

This was proved by C. B. Morrey, L. Nirenberg. Furthermore Calderon-Zygmund, showed

$$a_{ij}(x) \in C^0(\bar{\Omega}) \implies \|u\|_{W^{2,p}(\Omega)} \leq c(n, \alpha, \lambda, \Lambda, \|a_{ij}\|_{C^\alpha}) \|f\|_{L^p(\Omega)}$$

$\forall\, 1 < p < \infty$, then by Sobolev invariant, we got $u \in C^{1-\alpha}$ $\forall\, 0 < \alpha < 1$.

Recall $f \in VMO$ (*vanishing mean oscillation*) if

(a) $f \in BMO$ (*bounded mean oscillation*) i.e.

$$\fint_{B_r} \left|f - \bar{f}_{B_r}\right| dx \leq C,\ \forall\, B_r \in \Omega$$

(b)

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r} \left| f(x) - \fint_{B_r} f \right| \to 0$$

uniformly for $x \in \Omega$.

Theorem 4. *If $f \in W^{1,n}(\Omega)$, $dim\Omega = n$, then $f \in VMO$.*

Proof. By Poincare inequality

$$\fint_{B_r} \left|f - \bar{f}_{B_r}\right|^n \leq c(n) \int_{B_r} |\nabla f|^n\, dx \to 0$$

as $r \to 0$ $\square$

The reverse version of the theorem goes like

$\forall\, p \in (1, \infty)$, $\exists\, \delta_p = \delta_p(n, \lambda, \Lambda)$ st if

$$\limsup_{r \to 0^+} \fint_{B_r(x)} |a_{ij}(y) - (a_{ij})_{B_r}| \leq \delta_p$$

uniformly $x \in \Omega$, then

$$u \in W^{2,p}(\Omega)$$

Divergent Type

$$\begin{cases} \partial_{x_i}(a^{ij}(x)u_{x_i}) = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

(1) Assume $a^{ij} \in C^\alpha$, $u \in C^{1,\alpha}$ if $f \in L^p$, $\alpha = 1-n/p(\alpha)$ by G.Stampacchia, M. Giaquinta, E. Giusti in later 1970's.

(2) $a^{ij} \in C^0 \implies u \in W^{1,p}\ \forall\, 1 < p < \infty,\ f \in L^n$

(3) $a^{ij} \in VMO \implies u \in W^{1,p}$ or a^{ij} $\in$small BMO norm $\implies u \in W^{1,p_0}$.

(4) $\partial_{x_i}(a^{ij}(x)u_{x_i}) = f \implies u \in C^\alpha$ for some optimal $\alpha = \alpha(n,\lambda,\Lambda) > 0$

2-D Cases

Non-Divergent case

Worked by Morrey, Nirenberg

$$a_{ij}(x)u_{x_ix_j} = 0$$

in $C^{1,\alpha}$, then $(\nabla u) = (u_x, u_y)$ the map $\Omega \to \mathbb{R}^2$ is quasi-conformal.

Divergent case

Worked by Ahlfors, Courant, Bernstein $Du \in C^\alpha \implies u \in C^\alpha$.

2 Laplace Operators

Definition 5. A function $u \in C^2$ is harmonic if $\Delta u = 0$ in Ω.

But most of the discussion below will only assume $\Delta u = 0$ in Ω, not $u \in C^2$.

2.1 Mean-Value Properties

Lecture 2 (9/18/13)

Definition 6. Let $u \in L^1(\Omega)$, we say u satisfies mean-value property if

$$u(x) = \frac{1}{|B_r|}\int_{B_r(x)} u(y)dy$$

for a.e. $x \in \Omega$, for any $r > 0$, s.t. $B_r(x) \subset \Omega$.

This definition is equivalent to

$$u(x) = \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)dS_y$$

Theorem 7. *If u satisfies the mean-value property, then u is smooth.*

(For proof, see Lin Theorem 1.8) Basic idea, let $\phi(r) \in C_0^1(B_r)$ with $\int_{B_1}\phi(r)dr = 1$, let $p_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$, one can show

$$p_\varepsilon * u(x) = u_\varepsilon(x) \in C^\infty$$

and

$$u_\varepsilon(x) \equiv u(x), \forall\, \varepsilon > 0$$

Theorem 8. *If $u \in C^2(\Omega)$ and $\Delta u = 0$, then u satisfies the mean value property.*

That is because

$$0 = \int_{B_r(x)} \Delta u(y)dy = \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu} dS_y = r^{n-1}\frac{\partial}{\partial \nu}\fint_{\partial B_r(x)} u(y).$$

Theorem 9. *If u satisfies mean-value property, then u satisfies the maximum principle,*

$$\max_\Omega u \leq \max_{\partial\Omega} u$$

If $u(x_0) = \max_\Omega u$, $x_0 \in \Omega \implies u \equiv$constant in a connected component of Ω that contains x_0.

Theorem 10. *Let u be a distribution, then Δu is also a distribution.*

Proof. By Wyel Theorem, $\Delta u = 0$ in $\mathcal{D}' \iff u$ is smooth and $\Delta u = 0$ point wise. That$\Delta u = 0$ in $\mathcal{D}'$ implies

$$\Delta \rho_\varepsilon * u = \rho_\varepsilon * \Delta u = 0$$

So $u_\varepsilon = \rho_\varepsilon * u$ is harmonic and $u_\varepsilon \to u$ as $\varepsilon \to 0$, we have

$$u_\varepsilon \xrightarrow{C^{2,\alpha}_{loc}(\Omega)} u$$

Thus $\Delta u = 0$ in C^2 sense. ∎

Now we study Gradient Estimate for harmonic functions $\Delta u = 0$ in B_1, $\Delta u_{x_i} = 0$, by mean-value property

$$\begin{aligned} u_{x_i}(x_0) &= \frac{1}{|B_r|} \int_{B_r(x_0)} u_{x_i} dx_i \\ &= \frac{1}{|B_r|} \int_{\partial B_r(x_0)} u \nu_i dS_y \end{aligned}$$

Theorem 11. *(Gradient Estimate) Given u above,*

$$|u_{x_i}(x_0)| \leq \frac{|\partial B_r(x_0)|}{|B_r|} \fint_{\partial B_r(x_0)} |u| = \frac{c_n}{r} \fint_{\partial B_r(x_0)} |u| \, , \, \forall \, r$$

Moreover if $u \geq 0$ in Ω

$$|\nabla u| \, (x_0) \leq \frac{c_n}{r} \fint_{\partial B_r(x_0)} u \leq \frac{c_n}{r} u(x_0)$$

The last line is equal to

$$|\nabla \log u| \, (x_0) \leq \frac{c_n}{r}$$

Theorem 12.

$$\begin{cases} \Delta u = 0 & \text{in } B_R(0) \\ u|_{\partial B_R} = \phi \end{cases}$$

is solvable if $\phi \in C^0(\partial B_R)$, and Poisson Formula

$$u(x) = c_n \int_{\partial B_R} \frac{R^2 - |x|^2}{|x-y|^n} \phi(y) dS_y$$

2.2 Harnack Inequality

Use Poisson Formula, Gradient Estimate, and Mean-Value property, one can get Harnack Inequality, which states

Theorem 13. *Let $u \geq 0$ be s.t. $\Delta u = 0$ in Ω then*

$$u(x) \leq c(K, \Omega) u(y)$$

for all $x, y \in K \Subset \Omega$.

c depends on the ratio of the size of Ω and $\operatorname{dist}(K, \Omega)$. We can check this by a special case

If $\Delta u = 0$ in B_R and $u \geq 0$ in B_R,

$$\begin{cases} u(x) \leq c_n u(y) \\ x, y \in B_{R/2} \end{cases}$$

let

$$P(x,y) = c_n \frac{R^2 - |x|^2}{|x-y|^2}$$

x in B_R and $y \in \partial B_R$, choose r s.t. $|x| < r < R$, then

$$c_n \frac{R-r}{(R+r)^{n-1}} \leq P(x,y) \leq c_n \frac{R^2 - r^2}{(R-r)^n} = c_n \frac{R+r}{(R-r)^{n-1}}$$

$c_n \frac{R-r}{(R+r)^{n-1}}$ gives the optimal constant. That is because for fixed $y \in \partial B_R$

$$\begin{cases} \Delta_x P(x,y) = 0 & \text{in } B_R \\ P(x,y)|_{x=\partial B_R} \equiv \delta_y \end{cases}$$

One can also see that for fixed $y \in \partial B_R$, $\{P(x,y)\} = H_+$, the set of positive harmonic function on B_R, forms a *positive cone* in $C(B_R)$. Here $K =$ extreme points of H_+.

2.3 Harmonic Measure

Let $h \geq 0$, $\Delta h = 0$ in B_R,

$$h = \int_K \phi_y d\mu(y)$$

We say u satisfies the *Martin boundary*, if $u > 0$, and

$$\Delta u - c^2 u = 0$$

or

$$\Delta u + \vec{b} \cdot \overrightarrow{\nabla u} + c(x) u = 0$$

in $\mathbb{R}^n$. Let

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

$\phi \in C(\partial\Omega)$, Ω open and bounded. let $x_0 \in \Omega$, then for all $\phi \in C(\partial\Omega)$, the map l

$$\phi \xrightarrow{l} u_\phi(x_0)$$

is linear and positive representation,

$$l(\phi) = u_\phi(x_0) = \int_{\partial\Omega} dw_{x_0}(y)$$

$\{w_{x_0}(y)\}$ probability measure on $\partial\Omega$.

The support of $w_{x_0}(\cdot)$, K, is a subset of $\partial\Omega$. For x_1, $x_2 \in K \Subset \Omega$, (K compact) one has

$$w_{x_1} \leq c w_{x_2} \leq c^2 w_{x_1}$$

where $c = c(K)$.

When $n = 2$, the *Hausdorff dimension* of K is 1. This is due to T. Wolff.

$$\eta(r) = r \log(\log(\log r))$$

When $n \geq 3$, the *Hausdorff dimension* of $K \leq n - \varepsilon(n)$. This is due to J. Bourgain.

More facts: $\exists\, \Omega$ s.t. *Hausdorff dimension* of $K > n - 1$. And the following

Theorem 14. *(Dahlberg's theorem) If Ω is Lipschitz, then w_{x_0} is an A_p weight.*

C. Kenig studied harmonic measure on Lipschitz domains, his main result

Theorem 15. *For $1 < p \leq 2 + \delta$*

$$\left(\fint_E w^p \right)^{1/p} \leq c_p \fint_E w$$

Let us now consider distribution function. let

$$Lu = \frac{\partial}{\partial x_i}(a_{ij}(x) u_{x_j}) = 0$$

$(a_{ij}(x))$ positive definite, and $\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda\,|\xi|^2$. Since

$$\Delta \frac{c_n}{r^{n-2}} = -\delta_0$$

consider $G(x,y)$

$$LG = -\delta_x$$

and

$$E_r(x) = \{y \in \mathbb{R}^n : G(y,x) \geq r^{2-n}\}$$

Let $Lu = 0$, then

$$\begin{aligned}
u(x) &= \int_{E_r} (GLu - uLG) \\
&= \int_{\partial E_r} (GA\nabla u \cdot \nu - u(A\nabla G) \cdot \nu) ds \\
&\geq r^{2-n} \int_{\partial E_r} A\nabla u \cdot \nu + \int_{\partial E_r} u(A\,|\nu|)\,|\nabla G| \\
&= r^{2-n} \int_{E_r} Lu + \int_{\partial E_r} u(A\,|\nu|)\,|\nabla G| \\
&= \int_{\partial E_r} Au(y)\,|\nabla G|\,dS_y
\end{aligned}$$

Exercise 16. Find mean-value formula for

$$\Delta u + c^2 u = 0$$

$$\Delta u - c^2 u = 0$$

Exercise 17. Consider the region Ω in $\mathbb{R}^2$ is bounded by x, y axes and a line $y = -ax + b,\ a, b > 0$. Solve $\Delta u = 0$ in Ω with the boundary conditions $u = 0$ on y axis and $\frac{\partial u}{\partial y} = 0$ on the other two edges. Show that

$$u = cx, \quad c \in \mathbb{R}$$

are all solutions. Generalize this problem to n-dim.

2.4 Perron Methods

We solve

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi \in C(\partial\Omega) \end{cases}$$

Let $S_\phi = \{v | v \text{ is subharmonic in } \Omega \text{ and } v \leq \phi \text{ on } \partial\Omega\}$.

Definition 18. $v \in C(\Omega)$ is called subharmonic if $v \leq u$ in $B_r(x_0) \subset \Omega$.

Here $\Delta u = 0$ in $B_r(x_0)$, $u|_{\partial B_r(x_0)} = v|_{\partial B_r(x_0)}$. These are equivalent to $v(x_0) \leq \fint_{B_r(x_0)} v(y)dy$.

Theorem 19. *(i)* $S_\phi \neq \varnothing$, $v = \min_{\partial\Omega} \phi \equiv m \in S_\phi$.

(ii) $u_\phi(x) = \sup_{v \in S_\phi} v(x)$ *is well-defined, and it is* $\leq M \equiv \max_{\partial\Omega} \phi$.

Theorem 20. u_ϕ *is harmonic in* Ω, $u_\phi \leq \phi(x)\ \forall\, x \in \partial\Omega$.

Proof. Take $x_0 \in \Omega$, $B_r(x_0) \subset \Omega$, $\exists\, v_i \in S_\phi$ s.t. $v_i(x_0) \to u_\phi(x_0)$.

Let $\tilde{v}_i = \begin{cases} w_i & \text{in } \overline{B_r(x_0)} \\ v_i & \text{outside } B_r(x_0) \end{cases}$, then $\begin{cases} \Delta w_i = 0 & \text{in } B_r(x_0) \\ w_i = v_i & \text{on } \partial B_r(x_0) \end{cases}$. More

$$v_i \leq \tilde{v}_i \implies \tilde{v}_i(x_0) \to u_\phi(x_0)$$

and

$$m \leq w_i \leq M$$

implies $v \in S_\phi$, $m \in S_\phi$ and $\max(v, m) \in S_\phi$. Thus

$$w_i \xrightarrow{C^{2,\alpha}_{loc}} w \qquad \text{in } B_r(x_0)$$

with $\Delta w = 0$. Next $u_\phi \equiv w$ on $B_r(x_0)$. Let $x_1 \in B_r(x_0)$ $\exists\, \bar{v}_i \in S_\phi$ s.t. $\bar{v}_i(x_1) \to u_\phi(x_1)$. And $\Delta \bar{w}_i = 0$ in $B_r(x_0)$

$$\bar{w}_i = \begin{cases} \max(\tilde{v}_i, \bar{v}_i) \in S_\phi \\ \max(\tilde{v}_i, \bar{v}_i)|_{\partial B_r} \end{cases}$$

So $\bar{w}_i$ is harmonic and $\bar{w}_i \to \bar{w}$ is harmonic in $B_r(x_0)$, $\bar{w} \geq w$ in $B_r(x_0)$. Combining $\bar{w}(x_0) \geq w(x_0) = u_\phi(x_0)$ and $\bar{w}(x_0) \leq u_\phi(x_0)$, $\bar{w} \equiv w \implies w(x_1) = u_\phi(x_1)\ \forall\, x_1 \in B_r(x_0)$.

Now we show $u_\phi(x_0) = \phi(x_0)$ for any $x_0 \in \partial\Omega$.

Definition $x_0 \in \partial\Omega$, $B_{x_0}(x)$ is called a barrier function for Δ at x_0 if $B_{x_0}(x) \in C(\Omega)$, $\Delta B_{x_0}(x) = 0$ and $B_{x_0}(x_0) = 0$ and $B_{x_0}(x) < 0\, \forall\, x \in \partial\Omega$.

If at $x_0 \in \Omega$ s.t. such $B_{x_0}(x)$ exists, then $u_\phi(x_0) = \phi(x_0)$. (see theorem 22 below) $\square$

Summary of Perron's Method

Lecture 3 (9/25/13)

For $\Delta u = 0$ in $\Omega \subseteq \mathbb{R}^n$ (bounded, open), $u|_{\partial\Omega} = \phi \in C(\partial\Omega)$, Perron method is based on

(a) Maximum Principle (comparison principle)

(b) Solvability of the problem on a ball B.

$S_\phi = \{v \in C(\bar{\Omega}) : v \text{ subharmonic in } \Omega \ \&\ v \leq \phi \text{ on } \partial\Omega\}$ then (i) $S_\phi \neq \varnothing$ (ii) $u_\phi(x) = \sup_{v \in S_\phi} v(x)$, $x \in \Omega$ is the solution.

2.5 Regular Point on $\partial\Omega$ for Δ

Definition 21. x_0 is called regular point of Δ if $\exists\, B_{x_0}(x)$ continuous in $B_R(x_0)\cap\bar{\Omega}$ s.t. $B_{x_0}(x_0)=0$, $\Delta B_{x_0}(x)\geq 0$ and $B_{x_0}<0\ \forall\, x\in\partial\Omega$.

Theorem 22. *x is regular $\implies u_\phi(x_0)=\phi(x_0)$.*

Proof. $m\equiv\inf_{\partial\Omega}\phi\leq u_\phi\leq M\equiv\sup_{\partial\Omega}\phi$, $\forall\, x\in B_R(x_0)\cap\Omega$, $\forall\,\varepsilon>0$, $\exists\, K(R,m,M,\varepsilon)$, s.t.

$$\phi(x_0)-\varepsilon+KB_{x_0}\leq u_\phi(x)\leq\phi(x_0)+\varepsilon-KB_{x_0}(x)$$

$\lim_{x\to x_0}u_\phi(x)=\phi(x_0)$

$$B_{x_0}(x)\leq -c_\delta<0,\ \text{on}\ B^c_\delta(x_0)\cap\partial\Omega$$

then Maximum principle implies the conclusion. $\square$

Example 23. If Ω has an exterior cone at $x_0\in\partial\Omega$, with angle $\theta_0\in(0,\pi)$, use conformal mapping, one can show

$$-B_{x_0}(x)=r^{1+\alpha}\sin(\frac{\theta}{\theta_0}\pi)$$

$\alpha=\alpha(\theta_0)>0$, and $\Delta B_{x_0}(x)=0$ in Ω, $B_{x_0}(x_0)=0$, and $B_{x_0}\leq 0$.

Theorem 24. *(Wiener Criterion) If $x_0\in\partial\Omega$, then x_0 is regular for $\Delta\iff$*

$$\int_0^{r_0}\frac{c_{x_0}(\rho)}{\rho^{n-1}}d\rho=\infty.$$

where $c_{x_0}(\rho)=$*capacity of* $\Omega^c\cap B_\rho(x_0)$.

Definition 25. Let E be compact in $\mathbb{R}^n$, then

$$\begin{aligned}\mathrm{cap}(E) &= \int_{\mathbb{R}^n}|\nabla u|^2\,dx\\ &= \inf\{\int_{\mathbb{R}^n}|\nabla v|^2\,dx,\ v\in H^1_0(\mathbb{R}^n), v\geq 1\ \text{on}\ E\}\end{aligned}$$

Example 26. Capacity on B_{r_0}

$$u(r)=\begin{cases}1 & r<r_0\\ \left(\frac{r_0}{r}\right)^{n-2} & r>r_0\end{cases}$$

This is close related to Bernstein Method (1930's).

Theorem 27. *(Bernstein) $\Delta u = 0$ in B_1, then $|\nabla u(0)| \leq c(n) \max_{B_1(0)} |u|$.*

Proof.
$$\Delta(|\nabla u|^2(x)) = 2\,|Hess\, u|^2 = 2\sum_{ij} u_{x_i x_j}^2 \geq 0$$

implies
$$|\nabla u|^2(0) \leq \sup_{\partial B_1(0)} |\nabla u(x)|^2$$

$\forall\, \phi \in C_0^\infty(B_1)$ s.t. $\phi = 1$ for $0 < x < 1/2$, $\phi = 0$ for $x > 1$, then by $\Delta u^2 = 2\,|\nabla u|^2$,
$$\Delta(\phi^2 |\nabla u|^2 + M_\phi u^2)^{1/2} \geq 0$$

this shows that $(\phi^2 |\nabla u|^2 + M_\phi u^2)^{1/2}$ is subharmonic, then

$$|\nabla u|^2(0) \leq (\phi^2 |\nabla u|^2 + M_\phi u^2)(0) \leq \sup_{\partial B_1}(\phi^2 |\nabla u|^2 + M_\phi u^2) = M_\phi \sup_{\partial B_1}(u^2)$$

since $\phi|_{\partial B_1} = 0$. $\square$

H.F. Weinberger, Shing-Tung Yau showed $\Delta_M u = 0$.

2.6 Fundamental Solutions

We use $\Delta u = 0$ in B_1, $\Delta \frac{1}{|x|^{n-2}} = -c(n)\delta_0$.

Proposition 28. *Let $u(x)$ be a harmonic function in $B_1 \backslash \{0\}$. Suppose $u(x) = o(|x|^{2-n})$, as $|x| \to 0$, then $u(x)$ can be defined at $x = 0$ in a way $\Delta u = 0$ in B_1.*

For proof see Lin Theorem 1.28.

The following two theorems also deal with *Riemann removable singularity.*

Theorem 29. *$f(z)$ is holomorphic on $B_1 \backslash \{0\}$ and if $|f(z)| \leq M$ or $|f(z)| \leq o(\frac{1}{|z|})$, as $z \to 0$, then $f^*(z)$ is holomorphic in B_1.*

Theorem 30. *$u(z)$ harmonic in $B^2 \backslash \{0\}$ and if $u(z) = o(|\log |z||)$, then u is harmonic in $B(0)$.*

Proof. Solve

$$\begin{cases} \Delta h = 0 & \text{in } B_1 \\ h|_{\partial B_1} = u \end{cases}$$

then $u - h = v$ is harmonic in $B_1 \backslash \{0\}$, and $v(x) = o(|x|^{2-n})$, as $|x| \to 0^+$. $v|_{\partial B_1} = 0$.

That is

$$-\varepsilon |x|^{2-n} \leq v(x) \leq \varepsilon |x|^{2-n}$$

$\forall \varepsilon \in (0, 1)$ and $x \in B_1 \backslash \{0\}$, then let $\varepsilon \to 0^+$, we get the desire results. □

2.7 Maximum Principle on Unbounded Boundary

Suppose we want to do

$$\Delta u = 0$$

in $B_1^c = \{x \in \mathbb{R}^n : |x| > 1\}$ with $u|_{\partial B_1} = 0$.

$u = 0$ is a solution and $u = 1 - |x|^{2-n}$ is also a solution, and we require solution to be $u(x) \to 0$, as $|x| \to \infty$, by *involution*.

3 Classical Results

We now begin to study general elliptic pde operators.

3.1 Dini Continuous

Example 31. $Lu = a_{ij}(x) u_{x_i x_j} = 0$, $\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2$, $\forall\, x, \xi \in \mathbb{R}^n$.

Consider

$$a_{ij}(x) = \delta_{ij} + g(r) \frac{x_i x_j}{r^2}$$

then

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_\theta u$$

and

$$\frac{u_{rr}}{u_r} = \frac{1-n}{(g(r)+1)r}$$

So

$$u(r) = \int_0^r \exp\left[\int^\gamma \frac{1-n}{g(\rho)}\frac{d\rho}{\rho}\right] d\gamma$$

and

$$\lambda = \min(1, 1+g)\ \Lambda = \max(1, 1+g).$$

If $n = 2$, $g(r) = -\frac{2}{2+\log r}$, $0 < r < e^{-2}$, then this $g(r)$ is continuous and $g(0) = 0$, but not Holder i.e.

$$|g(x) - g(y)| \leq C\,|x-y|^\alpha$$

for some C, α. g is also not *Dini continuous.* i.e.

$$\int_0^1 \frac{w_g(r)}{r} dr = \infty$$

where the *modulus of continuity* of g, $w_g(r) = \sup_{d(x,y)\leq r} d(g(x), g(y)) \sim 1/\log r$.

Recall

$$f(r) = a + \frac{b}{\log r}$$

is not Dini continuous.

Things go wrong when u is not Dini continuous.

(a) Extended Maximum Principle is not true.

(b) No positive fundamental solution. (No positive solution $v(x)$ in neighborhood of 0 s.t. $v(x) \to +\infty$, as $x \to 0$.)

(c) Removable singularity theorem is also not true.

(d) If $a_{ij} \in C^\alpha(Dini)$, then exists fundamental solution.

Example 32. $n = 2$, let $g(r) = \frac{2}{\log r - 2} \implies u = \left(\log \frac{1}{r}\right)^3$. So existence of positive solution $v(x)$ near $r = 0$, s.t. $v(x) \to +\infty$ as $|x| \to 0$. It is not like $\log 1/r$.

Example 33. $n > 2$, let $g(r) = -\frac{1}{1+(n-1)\ln r}$, then $u(r) = \frac{r^{2-n}}{\log r}(1+\varepsilon(r))$, where $\varepsilon(r) \to 0$ as $r \to 0^+$. $u(r) = o(r^{2-n})$. $u(x) \neq O(|x|^{2-n+\delta})$ for any $\delta > 0$.

Example 34. $n > 2$, let $g(r) = \frac{(n-2)\ln r - 2}{\ln r + 2}$, $r > 0$, $g(0) = n-2$, so $\frac{1-n}{1+g(r)} = -1 - \frac{2}{\ln r} \implies u(r) = a + b/\ln r$.

The following theorem is very important. See paper "On isolated singularities of solutions of second order elliptic differential equations" (1953-Acta Math) by D. Gilbarg, James Serrin.

Theorem 35. *If (a_{ij}) is Dini at $r = 0$, and if u is a non-constant solution of $Lu = a_{ij}(x)u_{x_ix_j} \geq 0$ in $\{0 < r \leq r_0\}$. Let $M = \max_{|x|=r_0} u$ and if $u(x) = \begin{cases} o(lnr) & n = 2 \\ o(r^{2-n}) & n \geq 3 \end{cases}$, then $u(x) < M$ in $\{r : 0 < r < r_0\}$. More $\overline{\lim}_{|x|\to 0^+} u(x) < M$.*

Corollary 36. *(of theorem 35) If (a_{ij}) are continuous at $r = 0$ and u is as above. If $u(x) = O(|x|^{2-n+\delta})$, $\delta > 0$, then conclusion of theorem 35 is true.*

Proof. (Corollary 36) Let $g = r^{2-n}$, $h = r^{2-n+\delta'} = g^{1-r}$. $0 < \delta' < \delta$, $r = \delta'/(n-2)$. Calculate, we find

$$Lh \leq \varepsilon_1(r)r^{-n}$$

and

$$a_{ij}g_{x_i}g_{x_j} \geq (n-2)^2(1-\varepsilon_2(r))r^{2-2n}$$

we need

$$\varepsilon_1 - r(n-2)^2(1-\varepsilon_2) \leq 0$$

$0 < r < r_1$. Consider $v(x) = u - \varepsilon h - M$, then apply Maximum principle, we got $v < 0$ in $\{0 < r < r_1\}$

$$\therefore u \leq M \text{ in } \{0 < r < r_1\}.$$

$\square$

Now we give a sketch of proof of theorem 35.

Proof. Let

$$h(x) = h(r) = \int_0^g \exp K(r + \int_0^r \phi(s)\frac{ds}{s})dg$$

where $\phi(s)$ =modulo of continuity of a_{ij}. and let $g = r^{2-n}$.

Remark 37. if $a_{ij}(x)$ is Holder, then $h = g(1 + Kr^\alpha)$

Therefore

$$Lh \leq 0$$

$\square$

Lecture 4 (10/2/13)

Summary

Assume $Lu = a_{ij}(x)u_{x_ix_j} = 0$, $|a_{ij}(x) - \delta_{ij}| \ll c(\delta_1)$

(i) $a_{ij} \in C^0(B_{r_0})$, $L(r^{2-n+\delta_1}) \leq 0$ on B_{r_1}, then $h > 0$, $\Delta h = 0 \implies \Delta h^\alpha \leq 0$ for $0 \leq \alpha < 1$.

$Lu = 0$ in $\{0 < |x| < r_0\}$ and $u(x) = o(|x|^{2-n+\delta}) \implies \lim_{x\to 0}|u(x)| \leq M = \sup_{\partial B_{r_0}}|u(x)|$. This is because $x = 0$ is removable singularity.

(ii) a_{ij} $\in$Holder (Dini), $h(r) = r^{2-n}(1 + Kr^\alpha)$, $Lh \leq 0$ in $B_{r_0} \implies u(x) = o(|x|^{n-2})$, $x = 0$ is removable singularity.

$$\overline{\lim_{x\to 0}}\,|u(x)| \leq M = \sup_{\partial B_{r_0}}|u(x)|$$

and

$$Lu = 0 \quad \text{in } \{0 < |x| < r_0\}.$$

3.2 Unbounded Boundary

We first take $u(x)$ which is bounded on one-side.

Theorem 38. *Assume $u(x) \geq -M$, then $\lim_{x\to 0} u(x) = +\infty$, or finite, both exists.*

Proof. $\underline{\lim}_{x\to 0}u(x) = u_0 \geq -M$, consider $v_\varepsilon(x) = u(x) - v_0 + \varepsilon$, $\forall\, \varepsilon > 0$. $\underline{\lim}_{x\to 0}v(x) = \varepsilon > 0$. By Harnack inequality,

$$\sup_{|x|=r} v(x) \leq \inf_{|r|=r} v(x) \cdot c_*$$

(since the value at each ball is comparable.)

$$\sup_{|x|=r} v_\varepsilon \leq \inf_{|r|=r} v(x) \cdot c_* \to c_*\varepsilon,$$

as $r \to 0$. If $u_0 < \infty$,

$$\lim_{|x|\to 0} \sup v_\varepsilon \leq c_*\varepsilon$$

$\square$

$$\therefore \overline{\lim_{x\to 0}} v(x) = u_0$$

Theorem 39. *If $u \in H^1(B_1)$, $E \subset \bar{B}_1$ closed, $\Delta u = 0$ in $B_1 \backslash E$, then $\Delta u = 0$ in B_1 if $Cap(E) = 0$.*

Proof. $Cap(E) = \min\{\int_{\mathbb{R}^n} |\nabla u|^2 : v \in H_0^1(\mathbb{R}^n), v \geq 1 \text{ on } E\}$, and

$$\begin{cases} -\Delta v_E = \mu_E \geq 0 \\ v_E = 1 & \text{on } E \end{cases}$$

$v_E(\infty) = 0$, $Cap(E) = \int_{\mathbb{R}^n} |\nabla v_E|^2 \, dx = \int_{\mathbb{R}^n} (-\Delta v_E \cdot v_E) dx = \int_{\mathbb{R}^n} \mu_E v_E$

$$v_E(x) = c(n) \int_{\mathbb{R}^n} \frac{d\mu_E(y)}{|x-y|^{n-2}} = c(n) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\mu_E(x) d\mu_E(y)}{|x-y|^{n-2}}$$

That $cap(E) = 0$ gives $\exists \{v_i\} \in C_0^\infty(\mathbb{R}^n)$ s.t. $v_i \geq 1$ on E, $0 \leq v_i \leq 1$ in B_1

$$\int_{\mathbb{R}^n} |\nabla v_i|^2 \, dx \to 0$$

$\Delta u = 0$ in B_1. $\forall \phi \in C_0^\infty(B_1)$,

$$0 = \int_{B_1} \phi(1 - v_i) \Delta u$$

$\phi v_i \xrightarrow{H^1} 0$, so

$$\int_{B_1} \phi v_i \Delta u \to 0$$

as $i \to \infty$. $\Delta u \in H^{-1}$, hence

$$0 = \int_{B_1} \phi \Delta u$$

□

3.3 Probability Measure

Definition 40. Given a *Radon measure* μ on $\mathbb{R}^n$ (with point separated), we define

$t-$potential of μ,

$$\phi_t = \int \frac{d\mu(y)}{|x-y|^t}.$$

$t-$energy of μ,

$$I_t(\mu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\mu_t(x)d\mu_t(y)}{|x-y|^t}.$$

Given compact set E, we define

$$C_t(E) = \sup_{\mu} \{ \frac{1}{I_t(\mu)} : \mu \text{ is a probabilty measure on } E \}$$

One can check that the notion in the proof of theorem 39,

$$v_E(x) = c(n) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\mu_E(x)d\mu_E(y)}{|x-y|^{n-2}}$$

$$\begin{aligned} \mu_E(E) &= \int_E d\mu_E(x) \\ &= \int_E v_E(x) d\mu_E(x) \\ &= c(n) I_{n-2}(\mu_E) \end{aligned}$$

$$\tilde{\mu} = \frac{\mu_E}{c(n) I_{n-2}(\mu_E)}$$

is probability measure.

$$I_{n-2}(\tilde{\mu}) = \frac{1}{c^*(\mu) I_{n-2}(\mu_E)}.$$

Theorem 41. *Let μ be a probability measure, let*

$$E_0 = \{ x : \overline{\lim_{r\to 0}} \frac{\mu(B(x,r))}{r^t} > 0 \}$$

then $I_t(\mu) < \infty \implies \mu(E_0) = 0$.

Proof. $I_t(\mu) < \infty \implies \phi_t(x) < \infty$ for μ a.e. x. If at x_0 $\phi_t < \infty$, i.e. $\int \frac{d\mu(y)}{|x_0 - y|^t} < \infty$

$$\therefore \frac{\mu(B(x,r))}{r^t} \to 0$$

as $r \to 0^+$. □

Proposition 42. *(i) if $H^t(E) < \infty$, then $C_t(E) = 0$. (ii) if $C_t(E) = 0$, then $H^s(E) = 0 \ \forall\, s > t$.*

Proof. Let μ be a probability measure. $H^t(E) \geq H^t(E\backslash E_0)$, note $\forall\,\varepsilon > 0$, $E\backslash E_0$ can be finitely covered by balls $B_1(x)$, such that

$$\frac{\mu(B_r(x))}{r^t} < \varepsilon \implies r^t > \frac{\mu(B_r(x))}{\varepsilon}$$

Conversely $H^t(E\backslash E_0) \geq \frac{1}{\varepsilon}\mu(E\backslash E_0) = \frac{\mu(E)}{\varepsilon} = \frac{1}{\varepsilon} \implies H^t(E) = \infty$. If $C_t(E) > 0$, then $\exists\,\mu$ s.t. $I_t(\mu) < \infty$. $\square$

Corollary 43. *If $E \subset B_1$ closed and $H^{n-2}(E) < \infty$, then E is removable for any H^1 solution of $\Delta u = 0$ in $B_1\backslash E$.*

3.4 Quasi-Open, Quasi-Continuous

Definition 44. A set $E \subseteq \mathbb{R}^n$ (Borel) is called Quasi-open if $\forall\,\varepsilon > 0$, $\exists\,O_\varepsilon$ open set s.t.

$$cap(O_\varepsilon \Delta E) < \varepsilon.$$

Definition 45. $f : \mathbb{R}^n \to \mathbb{R}$ is called Quasi-continuous if $\forall\,\varepsilon > 0$, $\exists\,f_\varepsilon$ continuous $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$cap\{x : f_\varepsilon(x) \neq f(x)\} < \varepsilon.$$

Theorem 46. *(Federer-Ziemer) If $f \in H^1(\mathbb{R}^n)$, then f is Quasi-continuous.*

Definition 47. We define $E_n \Rightarrow E$ (bounded sets) $\subseteq \Omega$ to mean that if we solve $\begin{cases} -\Delta u_{E_n} = f, & f \in H^{-1} \\ u_{E_n} = 0 & \text{on } \Omega\backslash E_n \end{cases}$, then

$$u_{E_n} \xrightarrow{H^1} u_E \text{ s.t. } \begin{cases} -\Delta u_E = f \\ u_E = 0 & \text{on } \Omega\backslash E \end{cases}$$

Definition 48. We define $E_i \overset{*}{\rightharpoonup} E$ to mean that if we solve $\begin{cases} -\Delta u_i = 1 \\ u_i = 0 & \text{on } \Omega\backslash E_i \end{cases}$, then

$$u_i(x) \overset{H^1}{\rightharpoonup} w(x) \text{ where } E = \{x | w(x) > 0\}$$

Clearly if $w(x) = u_E(x)$, then $E_i \overset{*}{\rightharpoonup} E$. If $E_n \Rightarrow E$, $\lambda_i(E_i) \to \lambda(E)$.

Remark 49. $\exists\,Lu = a_{ij}(x)u_{x_ix_j} = 0$ in $B_{r_0}\backslash\{0\}$ uniformly elliptic, $a_{ij} \in C^w(B_{r_0}\backslash\{0\})$. $\exists\,u^*(x)$ solution of $LU = 0$ in $B_{r_0}\backslash\{0\}$ s.t. $u^*|_{\partial B} = 1$, $u^*(0) = 0$, $u^*(x)$ Holder.

3.5 Serrin-Weinberger Method

The following theorem is applied to divergence form

$$Lu = \partial_{x_i}(a_{ij}(x)u_{x_j}) = 0$$

in $B_{r_0}\backslash\{0\}$, $\lambda\,|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda\,|\xi|^2$.

Theorem 50. *(Littman-Stampacchia-Weinberger)* $\exists\, G(x) \geq 0$, $LG(x) = -\delta(x)$, $G(x) \cong |x|^{2-n}$, $\nabla G \in L^{\frac{n}{n-1}}$, $G \in L^{\frac{n}{n-1}-\varepsilon}$, $\forall\,\varepsilon > 0$, $G \in$*Holder in* $\mathbb{R}^n\backslash\{0\}$, *and* $G \in H^1_{loc}(\mathbb{R}^n\backslash\{0\})$ *and*

$$c\,|x|^{2-n} \leq G(x) \leq C\,|x|^{2-n}$$

Lemma 51. *If* $Lu = 0$ *in* $B_{r_0}\backslash\{0\}$ *&* $u(x) = o(|x|^{2-n})$, *then* $u(x) = aG(x) + w(x)$, *where* a *is a constant,* $Lw(x) = 0$ *in* B_{r_0} *and* $w \in$*Holder in* B_{r_0}.

Proof. (See Serrin) $\begin{cases} Lv = 0 & \text{in } B_{r_0} \\ v = G & \text{on } \partial B_{r_0} \end{cases}$. let $g = G - v$ in B_{r_0}. Since $u(x) = o(|x|^{2-n})$, let $u^*(x) = u(x) - w(x)$, $u^*|_{\partial B_{r_0}} = 0$, one need to show $u^*(x) = ag$. We have $u^*(x) = O(|x|^{2-n}) = O(g)$, $u^*|_{\partial B_{r_0}} = 0 \implies u^*(x) \leq Mg(x)$ for large M.

Let $a = \inf\{M \in \mathbb{R}, u^*(x) \leq Mg(x) \text{ for } x \in B_{r_0}\backslash\{0\}\}$, one wants to show $0 \leq ag(x) - u^*(x) = 0$, that is to show

$$\underline{\lim}_{|x|\to 0}\frac{ag(x) - u^*(x)}{|x|^{n-2}} = 0$$

suppose that $\underline{\lim}_{|x|\to 0}\frac{ag(x)-u^*(x)}{|x|^{n-2}} \geq \varepsilon_0 > 0$, then by Maximum principle

$$ag(x) - u^*(x) \geq \frac{\varepsilon_0}{c}g(x) \quad \text{for } 0 < |x| < r_0$$

then

$$ag(x) - u^*(x) \leq \varepsilon g(x) \quad \forall\,\varepsilon > 0,\ 0 < |x| < r_0$$

let $\varepsilon \to 0$,

$$ag(x) - u^*(x) \leq 0$$

□

Theorem 52. *(Serrin-Weinberger) Consider $Lu = 0$ in $B_R^c = \{|x| > R\}$,*

$$m(\sigma) = \inf_{\partial B_\sigma} u \quad M(\sigma) = \sup_{\partial B_\sigma} u$$

Either $u(x) \to u_\infty$ as $x \to \infty$; or else $M(\sigma) \geq A\sigma^\alpha$, $m(\delta) \leq -A\sigma^\alpha$, for some $A > 0$ and $\alpha = \alpha(n, \lambda, \Lambda)$. More u_∞ is finite if $n \geq 3$.

Note 53. $Lu = 0$ in $\mathbb{R}^n$. $M(\sigma) \uparrow$, $m(\sigma) \downarrow$, Moser showed $M(\sigma) - m(\sigma) \geq A\sigma^\alpha$.

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Proof. If u is bounded on one-side, then $\lim_{|x| \to \infty} u(x) = u_0$ exists (finite if $n \geq 3$). $\exists\, G(x)$ fundamental solution s.t.

$$\begin{cases} L(C - G) = 0 & \text{in } B_R^c \\ C - G|_{\partial B_R} = 0 \end{cases}$$

Since $Lv = 0$ in B_R^c, $v \geq 0$ in B_R^c, and $v(x) \to \infty$ as $|x| \to \infty$. $C - G \leq \varepsilon v$, $\forall\, \varepsilon > 0$. So the fundamental solution goes to C, hence u is not bounded on neither side.

By maximum principle $M(\sigma) = \max_{|x|=\sigma} u(x)$ can have at most one relative maximum; $m(\sigma) = \min_{|x|=\sigma} u(x)$ can have at most one relative minimum. And $M(\sigma) \to \infty$, $m(\sigma) \to -\infty$, as $\sigma \to \infty$. So $\exists\, \sigma_0$ s.t. for $\sigma \in [\sigma_0, \infty)$, $M(\sigma) \uparrow \infty$, $M(\sigma) \geq 0$ and $m(\sigma) \downarrow -\infty$, $m(\sigma) \leq 0$.

Let $A_\sigma = \{\frac{\sigma}{2} < |x| < 2\sigma\}$, then

$$0 \leq M(2\sigma) - u(x),\ 0 \leq u(x) - m(2\sigma), \quad \text{on} A_\sigma$$

$$\therefore M(\sigma) - m(2\sigma) \leq c\,(m(\sigma) - m(2\sigma))$$

Apply Harnack inequality of Moser on $|x| = \sigma \implies$

$$M(2\sigma) - m(\sigma) \leq c\,(M(2\sigma) - M(\sigma))$$

Combining the two,

$$M(\sigma) - m(2\sigma) + M(2\sigma) - m(\sigma) \leq c\,(M(2\sigma) - m(2\sigma)) - c\,(M(\sigma) - m(\sigma))$$

Or

$$(M(\sigma) - m(\sigma)) \leq \frac{c-1}{c+1}\,(M(2\sigma) - m(2\sigma))$$

That gives

$$M(\sigma) - m(\sigma) \geq A\delta^\alpha,\ \sigma \geq 2\sigma_0$$

and

$$\begin{aligned} M(2\sigma) - M(\sigma) &\geq \frac{1}{c}\left(M(2\sigma) - m(\sigma)\right) \\ &\geq \frac{1}{c}\left(M(\sigma) - m(\sigma)\right) \\ &\geq \frac{A}{c}\sigma^{\alpha} \end{aligned}$$

Thus

$$\sup_{|x|=r} u(x) \geq Ar^{\alpha}$$

$\square$

Corollary 54. *If $Lu = 0$ in $\{|x| > r_0\}$ and if either $\overline{\lim}\, |x|^{-\alpha} u(x) \leq 0$ or $\underline{\lim}\, |x|^{-\alpha} u(x) \geq 0$, then $\lim_{|x|\to\infty} u(x) = u_0$ is finite.*

Theorem 55. *If $Lu = 0$ in $\{0 < |x| \leq r_0\}$, $\exists\, \delta = \delta(m, \lambda, \Lambda) > 0$ s.t. suppose either $\overline{\lim}\, |x|^{2-n-\delta} u(x) \leq 0$ or $\underline{\lim}\, |x|^{2-n-\delta} u(x) \geq 0$, then $u(x) = aG(x) + w(x)$, where $Lw(x) = 0$ and $w(x)$ is regular solution.*

If $\Delta u(x) = 0$ in $|x| > 1$, then

$$v(y) = \frac{u(\frac{y}{|y|^2})}{\frac{1}{|y|^{n-2}}} = |y|^{n-2}\, u(\frac{y}{|y|^2})$$

solves $\Delta v(y) = 0$ in $|y| < 1$.

For more general elliptic operator, if $a^{ij}(x) u_{x_i x_j} = 0$ in $|x| < 1$, define

$$v(y) = \frac{u(\frac{y}{|y|^2})}{G(y)}$$

to be solution of $\tilde{L}v = 0$ in $|y| > 1$, where

$$\tilde{L} = \partial_{y_i}(\tilde{a}^{ij}(y)\partial_{y_j}),\ \tilde{a}^{ij}(y) = \frac{G^2(y)}{J(y)} a^{kj}(x) \frac{\partial y^i}{\partial x_i}\frac{\partial y^j}{\partial x_i}$$

$$c\lambda \leq (\tilde{a}^{ij}) \leq c\Lambda,\ \frac{\partial y^i}{\partial x_i} \sim \frac{1}{|x|^2},\ J(y) \sim \frac{1}{|x|^{2n}},\ G(y) \sim |x|^{2-n}$$

so

$$\tilde{a}^{ij}(y) \sim \frac{|x|^{2n}}{|x|^4}\left(|x|^{2-n}\right)^2$$

3.6 Maximum Principle

We now discuss two aspects of Maximum principle.

Theorem 56. *(i) If $a_{ij}(x)u_{x_i x_j} \geq 0$ in B_1 and $u \in C^2(B_1)$, then*

$$\sup_{x \in B_1} u(x) \leq \sup_{\partial B_1} u(x)$$

(ii) (Hopf boundary point lemma or strong Maximum principle) If

$$u(x_0) = \max_{x \in \bar{B}_1} u(x)$$

$u \in C^2$, and $u \neq$ constant, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Another version: $u(x) < u(x_0)\ \forall\, x \in \bar{B}_1 \backslash \{x_0\}$, $u \in C^0(\bar{B}_1) \cap C^2(B_1)$, then

$$\frac{\partial u}{\partial \nu}(x_0) \geq c(n, \lambda, \Lambda)\,(u(x_0) - u(0))$$

Proof. Consider $h(x) = e^{-\alpha |x|^2} - e^{-\alpha}$, h vanishes on boundary and ≥ 0 inside.

$$Lh = (4\alpha^2 a_{ij}(x) x_i x_j + o(\alpha)) e^{-\alpha |x|^2}$$

$Lh > 0$ if $|x| \geq 1/2$, $0 \geq \alpha(n, \lambda, \Lambda)$

$$v = u(x) - u(x_0) + \varepsilon h(x) \leq 0$$

in $\{1/2 < |x| < 1\}$, implies

$$Lv \geq 0$$

in $\{1/2 < |x| < 1\}$, implies

$$v(x) \leq 0 \text{ in } \{\frac{1}{2} < |x| < 1\},\ v(x_0) = 0$$

So

$$\frac{\partial v}{\partial \nu}(x_0) \geq 0$$

which gives

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial h}{\partial \nu}(x_0) \geq \varepsilon \alpha > 0$$

How big is $u(x_0) - u(x)$ on $|x| = \frac{1}{2}$? Since

$$c\,(u(x) - u(x_0)) \leq u(x_0) - u(x) \leq c\,(u(x_0) - u(0))$$

we have

$$(u(x) - u(x_0)) \sim (u(x_0) - u(0)) \sim \varepsilon$$
$$\therefore \frac{\partial u}{\partial \nu}(x_0) \geq c(n, \lambda, \Lambda)\,(u(x_0) - u(0))$$

□

Exercise 57. $Lu = 0$ in B_1. Show Hopf boundary point lemma $\Longrightarrow$ Holder of solution.

$M(R) = \max_{|x|\leq R}(u(x) - u(0))$, $RM'(R) \geq c(n, \lambda, \Lambda)M(R)$ ($R = 1$ by scaling), $M(R) \geq 2^c M(R/2)$, for $0 \leq r < 1$, $M(r) \leq M(1)r^c$.

Theorem 58. *(Alexandroff Estimate or ABP-Estimate [Alexandroff–Bakelman–Pucci]) $a_{ij}(x)u_{x_ix_j}(x) \geq f(x) \in L^n(\Omega)$, $u \in W^{2,n}(\Omega)$, $u|_{\partial\Omega} \leq 0$, then*

$$\sup_{\Omega} u^+ \leq c(n)(dim\Omega)\left(\int_{\Omega}\left|\frac{f^-}{A}\right|^n dx\right)^{1/2}$$

where $A = det^{1/n}(a_{ij}(x))$. $f = f^+ - f^-$.

Remark 59. If $(a_{ij}) \geq \lambda I$, $A \geq \lambda$.

Proof. Let $\Gamma_u(x)$ be the minimum concave function which vanishes on $\partial\Omega$ and lies above u. So

$$M = \max \Gamma_u = \max u^+$$

Let $d = dim\Omega$.

$$Vol\left(\frac{B^n_M(0)}{d}\right) = c(n)\left(\frac{M}{d}\right)^n$$

$D\Gamma_u(\Omega) = \{D\Gamma_u(x) : x \in \Omega\} \supseteq B^n_M(0)/d$, any plane from $u = \infty$ with slope $\leq M/d$ will have to touch the graph Γ_u before it hits $\partial\Omega$, that implies

$$\begin{aligned}
c(n)\left(\frac{M}{d}\right)^n &\leq |D\Gamma_u(\Omega)| \\
&\leq \int_{\Omega} |det\,(\text{Hessian of } \Gamma_u)|\,dx \\
&\leq \int_{\Omega} \left|det(D^2u(x))\right| dx \\
&\leq \int_{\{(D^2u)\leq 0\}} \left|det(D^2u(x))\right| dx
\end{aligned}$$

here we used det (Hessian of Γ_u) $= J_{ac}(D\Gamma_u(\Omega))$ for Lipschitz map Γ_u. Moreover

$$-a_{ij}(x)u_{ij}(x) \leq -f(x) \leq f^-(x)$$

$$\begin{aligned} det(A)det(-D^2u) &= det(A(-D^2u)) \\ &\leq c(n)\left[tr(A(-D^2u))\right]^n \\ &\leq c(n)f^n(x) \end{aligned}$$

on $(D^2u \leq 0)$, use $\lambda_1\lambda_2 ... \lambda_n \leq \left(\frac{\lambda_1 + ... + \lambda_n}{n}\right)^n$, we get

$$c(n)\left(\frac{M}{d}\right)^n \leq \int_{\{(D^2u)\leq 0\}} \left|det(D^2u(x))\right| dx \leq c(n)\int_\Omega \frac{(f^-)^n}{detA}dx$$

$$\therefore \sup u^+ = M \leq c(n)dim\Omega\left(\int_\Omega \frac{(f^-)^n}{detA}\right)^{1/n}$$

□

We will apply Alexandroff estimate to get the following result

Proposition 60. *(small-volume maximum principle) Let* $d = dim\Omega$, $\delta = Vol\Omega$, *and* $|c(x)| \leq M$. *If* $\Delta u + c(x)u = 0$ *in* Ω, $u \leq 0$ *on* $\partial\Omega$, *then* $\exists\, \delta_0 = \delta_0(n, M, d)$ *s.t. if* $\delta \leq \delta_0$ *then* $u(x) \leq 0$ *in* Ω.

Proof. $\sup u^+ \leq c(n)d\left\|(c(x)u)^+\right\|_{L^n(\Omega)} \leq c(n)dM\sup u^+\delta^{1/n} \implies \sup u^+ = 0$, if $\delta^{1/n}c(n)Md < 1$. □

3.7 Moving Plane Method

We borrow an idea from Geometry: Constant-mean-curvature surface in $\mathbb{R}^3$ must be sphere by Alexandrov.

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Theorem 61. *If*

$$\begin{cases} \Delta u + f(x) = 0 & in\ B_1 \\ u = 0 & on\ \partial B_1 \\ u > 0 & in\ B_1 \end{cases}$$

with $f \in Lip$, *then* $u(x) = u(|x|)$.

See Lin Lemma 2.35 gives a different version of the theorem.

Lemma 62. *(Similar to Lin 2.35) Consider the plane $T_\lambda = \{x_1 = \lambda\}$, $0 \le \lambda \le 1$, let $\Sigma_\lambda = B_1 \cap \{x_1 > \lambda\}$ and Σ^*_λ = reflection of Σ_λ with respect to T_λ. Then*

$$u(x^\lambda) \le u(x)$$

$x \in \Sigma^*_\lambda$ for all $0 \le \lambda \le 1$, where $x^\lambda = (2\lambda - x_1, x')$.

Notice when $\lambda = 0$, the lemma says $u(x_1, x') \le u(-x_1, x')$.

Idea of proof Lemma: Let $w^\lambda = u(x^\lambda) - u(x)$, $x \in \Sigma^*_\lambda$, then $w^\lambda = 0$ on T_λ, $w^\lambda < 0$ on $\partial\Sigma^*_\lambda \backslash T_\lambda$ and

$$\Delta w^\lambda(x) + c^\lambda(x) w^\lambda(x) = 0$$

in Σ^*_λ, where $\left|c^\lambda(x)\right| \le L = \max|f'|$, and $c^\lambda(x)w^\lambda(x) = f(u^\lambda) - f(u)$, where $u^\lambda(x) = u(x^\lambda)$.

Proof. (of Lemma 62) If $1 - \delta_0 \le \lambda \le 1$ ($\delta_0 > 0$ small), $|\Sigma^*_\lambda| \le c\delta_0$ and $dim\Sigma_\lambda \le 2$. By ABP-estimate, $w^\lambda(x) \le 0$, $x \in \Sigma^*_\lambda$ and $w^\lambda(x) < 0$ inside Σ^*_λ.

Since $\sup_{\Sigma^*_\lambda}(w^\lambda_+) \le c\mathrm{dim}\,|\Sigma^*_\lambda| \cdot \left\|c^\lambda w^\lambda\right\|_{L^n(\Sigma^*_\lambda)}$,

$$\sup w^\lambda_+ \le c\mathrm{dim}\,|\Sigma^*_\lambda|\, L \sup w^\lambda_+ \,|\Sigma^*_\lambda|^{1/n}$$

$$\therefore \sup w^\lambda_+ = 0$$

for $\sup w^\lambda_+$ appears on both sides but $c\mathrm{dim}\,|\Sigma^*_\lambda|\,|\Sigma^*_\lambda|^{1/n}$ is arbitrarily small.

Let $\lambda_0 = \inf\{\lambda_* \in (0,1) : w^\lambda(x) \le 0 \,\forall\, \lambda \ge \lambda_*\}$, if $\lambda_0 = 0$ we are done, i.e. $u(x_1, x') = u(-x_1, x')$, $x_1 \ge 0$. (choose all planes)

If $\lambda_0 > 0$, then $u^{\lambda_0}(x) \le u(x)$ on $\Sigma^*_{\lambda_0}$, $u(x) = u^{\lambda_0}(x)$ on T_{λ_0} and $u^{\lambda_0}(x) \le u(x)$ on $\partial\Sigma^*_{\lambda_0} \backslash T_{\lambda_0}$. By Hopf,

$$u^{\lambda_0}(x) < u(x)$$

inside $\Sigma^*_{\lambda_0}$, it implies $w^{\lambda_0}(x) < 0$, so $w^{\lambda_0}(x)$ is continuous on $\Sigma^*_{\lambda_0}$ $\implies$

$$w^{\lambda_0}(x) \le -\eta_0(K) < 0$$

on $K \Subset \Sigma^*_{\lambda_0}$. Choose K large s.t. $\left|\Sigma^*_{\lambda_0} \backslash K\right| < \delta_0/2$ for $\lambda_0 - \delta \le \lambda < \lambda_0$ ($\delta \ll \delta_0$)

Consider $w^\lambda(x)$ continuous in λ on $\Sigma^*_{\lambda_0} \implies w^\lambda(x) \le -\frac{\eta_0}{2}\ \forall x \in K$ and $|\Sigma^*_\lambda \backslash K| < \delta_0$. Consider w^λ on $\Sigma^*_\lambda \backslash K$, $w^\lambda \lneq 0$ on $\partial(\Sigma^*_\lambda \backslash K)$,

$$\Delta w^\lambda = -c^\lambda(x) w^\lambda$$

in Σ^*_λ. Then by ABP-estimate,

$$\begin{aligned}
\sup_{\Sigma^*_\lambda \backslash K} \left| w^\lambda_+ \right| &\le c_0 2L \left| w^\lambda_+ \right|_{L^n(\Sigma^*_\lambda \backslash K)} \\
&\le 2c_0 L \left| \Sigma^*_\lambda \backslash K \right| \sup_{\Sigma^*_\lambda \backslash K} \left| w^\lambda_+ \right| \\
&\le 2c_0 L \delta_0^{1/n} \sup_{\Sigma^*_\lambda \backslash K} (w^\lambda_+)
\end{aligned}$$

This gives

$$\sup_{\Sigma^*_\lambda \backslash K} (w^\lambda_+) = 0$$

□

One can use the same idea in the proof to show: M^2 embedded in $\mathbb{R}^3$ consider constant mean curvature $H_M = const > 0 \implies M^2 \cong S^2$ standard sphere.

4 Modern Results

4.1 Krylov-Safonov Theorem

4.2 Calderon-Zygmund Lemma

4.3 Continuity Method

4.4 Lax-Milgram

4.5 DeGiorgi Theory

4.6 John-Nirenberg Estimate

4.7 Morrey-Stampacchia Theory

5 More Recent Developments