Introduction to General Relativity

Rachel Rosen

Transcribed by Ron Wu

This is an advanced undergraduate course. Offered *in* Fall 2013 at Columbia University. Required Course textbook: Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Recommended books: Schutz, *A First Course in General Relativity*. Hartle, *Gravity an Introduction to Einstein's General Relativity*. Zee, *Einstein Gravity in A Nutshell*. Weinberg, *Gravitation and Cosmology*. Office Hours: W 10:30-11:30.

Contents

1 Special Relativity

Lecture 1 (9/3/13)

The main object we study is gravity, which is one of the four fundamental forces. The other three are electromagnetic force, strong nuclear force (binding in nuclear) and weak nuclear force (responsible for nuclear decay).

We know since high school

$$F = -G\frac{m_1 m_2}{r^2} \tag{1.1}$$

when gravity is not too strong. This formula also contradicts special relativity which states that no signal propagates faster than speed of light. The goal of general relativity is to reconcile special relativity and Newtonian gravity (1.1). The study of general relativity is useful to study the end point of stellar evolution, black holes, big bang, etc. It's also applicable to our solar system. It gives small corrections, e.g. GPS system relies on these corrections.

In general relativity gravity is treated as curvature of spacetime. Let's first consider flat space.

1.1 Galilean Transformation

Newton 1st law: free particles move at constant v speed along straight line.

Define a reference frame (inertial frame), xyz, as one observer's choice of coordinates. The choices of coordinate is not unique, one can choose other coordinates $x'y'z'$, with respect to xyz, for example

Translated Coordinate

$$\begin{cases} x' = x - d \\ y' = y \\ z' = z \end{cases}$$

d is a constant.

Or rotated coordinate

$$\begin{cases} x' = \cos\theta x + \sin\theta y \\ y' = -\sin\theta x + \cos\theta y \\ z' = z \end{cases}$$

θ is a constant.

Or more interestingly uniform motion coordinate, called Galilean transformation

$$\begin{cases} x' = x - v_0 t \\ y' = y \\ z' = z \end{cases} \tag{1.2}$$

What about t? Newton: all observers agree on the same time and agree on the same displacement, i.e. in Cartesian coordinates infinitesimal line element

$$ds^2 = dx^2 + dy^2 + dz^2$$

later we will see this implies that the space is flat.

Galileo said

Theorem. *(principle of relativity) physical laws are unchanged in system under going uniform motion, i.e. his transformation* (1.2)*.*

This is manifested in Newton's 2nd law

$$\vec{F} = m\vec{a}$$

For Galilean transformation (1.2), we only have to check x axis, other directions no changes at all,

$$a'_x = \frac{d(v_x - v_0)}{dt} = \frac{dv_x}{dt} = a_x$$

so two observers see no difference.

1.2 Spacetime Interval

Special relativity (SR) has two postulates: Galilean principal of relativity and principal of invariant light speed. That is

Theorem 1. *Speed of light in vacuum is unchanged* ($3 \times 10^8 m/s$) *regardless of motion of source relative to observers.*

A point particle in Newton's mechanics is (x, y, z), while an point particle in SR is (t, x, y, z), an event. A trajectory in Newton's mechanics is a path, while a trajectory in SR is a worldline.

We adopt new units: so that $c = 1$, that is the new unit length is redefined

$$1 \text{ meter } = \text{ light travels in one second}$$

Suppose yz coordinates are constant. We plot x v.s t, (x axis horizontal) the worldline of an object at rest is a vertical line. Worldline of object moves at constant v is a straight line with slope $1/v > 1$. Worldline of photons is straight line with slope 1. Worldline of an accelerating particle is a curved line with decreasing slope but still > 1.

Consider two frames $O(x, t)$, $O'(\bar{x}, \bar{t})$. If we are in Galilean transformation, since

$$t = \bar{t} \qquad \bar{x} = x - vt$$

The worldline of the origin of O' in the O frame will be horizontal line.

In SR, suppose O' is moving wrt O in x direction with speed v, i.e. $\bar{x} = x - vt$ but we don't have $t = \bar{t}$. Suppose at $t = 0$, $t = \bar{t} = 0$ and $x = \bar{x} = 0$, the two origins coincide. The worldline of the origin O' in the frame O is a tilted line, denoted $l_{O'}$, passing the origin O and make an angle $\tan^{-1}(1/v)$. One can also interpret this as the $\bar{t}$ axis of the O' frame in the O frame, that is because in frame O', the motion of the origin O' is 0, i.e. always $\bar{x} = 0$, which is the $\bar{t}$ axis.

How to find $\bar{x}$ axis at $t = 0$ in the O frame? The trick is to mark two equal distances points E_1,E_2 on $l_{O'}$ in the O frame. E_1 on $-\bar{t}$ axis, and E_2 on $+\bar{t}$ axis, then make a line with slope 1 passing E_1 and a line with slope -1 passing E_2. Make the intersection of the two lines point p. Connecting O to p gives the $\bar{x}$ axis in the O frame.

The reason is that suppose a light is emitting from the $\bar{x} = 0$, denoted event E_1 and assuming there is mirror siting on the $\bar{x}$ axis, after light reaches p, it reflects back to $\bar{x} = 0$ denoted event E_2. During the time, t' advances from E_1 to E_2 along $l_{O'}$. Since p, the reflection point, is where the mirror is at, and it takes same amount of time for light traveling from E_1 to p and from p to E_2, so it is clear when light reaches p, O', the midpoint of E_1E_2 coincides O, hence we set the clock $t = \bar{t} = 0$, and p the mirror is on the $\bar{x}$ axis, so connecting Op gives the $\bar{x}$ axis in the O frame at $t = 0$.

More surprisingly the slope of $\bar{x}$ axis is v. Proof let

$$l_{O'} : \; y = \frac{1}{v}x$$

pick $E_2 = (v, 1)$ and $E_1 = (-v, -1)$, then the two light lines are

$$y = -x + (v + 1) \qquad y = x - 1 + v$$

then the intersection in the O coordinate

$$p = (1, v)$$

One sees that in the O' frame the distance between events E_1 and p is equal to the distance between events p and E_2, but in the O frame these two distances are clearly not the same, this suggests to define a new distance. Put

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2$$

then

$$\|E_1p\| = \|E_2p\|$$

is the same in both O and O' frame. This is equal to 0.

We define spacetime interval, for 3D

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Later we will show Δs is invariant for any two events for any inertial frame, using the fact that $(\Delta s)^2$ is a Lorentz scalar. Because of the $-$ sign in front of Δt, it is not 4d Euclidean space, it is called Minkowski space.

1.3 Lorentz Boost

Consider a curve in both O and O' frames. As before O' is moving constant speed v in x direction and two clocks are set to 0 when the two origins are coincided.

$$-t^2 + x^2 = -b^2 = -\bar{t}^2 + \bar{x}^2 \tag{1.3}$$

it should really be (Δt), but we take the difference between the origins $\Delta t = t - 0$.

Consider only the upper branch of the hyperbola (1.3) (i.e. $t > 0$) in the frame O, it intercepts t axis at event E_A,

$$E_A: \; x = 0 \; t = b$$

Can we identify the corresponding event with

$$E_B: \; \bar{x} = 0 \; \bar{t} = b$$

in O frame? Thanks to (1.3), event E_B in the O frame is the intersection point of the up branch of (1.3) with the line $l_{O'}$ defined before, because on $l_{O'}$ $\bar{x} = 0$. This is in fact how we can figure out the right scale (unit length) of $\bar{t}$ in O frame, i.e. define the scale of $\bar{t}$ as

$$|E_B O| = b$$

To figure out the right scale of $\bar{x}$ in O frame use $-t^2 + x^2 = a^2 = -\bar{t}^2 + \bar{x}^2$ and do intersection of $\bar{x}$ axis with the hyperbola.

Continue our thought experiment. Suppose there is a clock at origin of O' frame, it is moving at constant v. After $\Delta\bar{t} = b$, it arrives at E_B. Now in O frame what is the time elapses? i.e. what is the vertical coordinate of E_B?

Suppose the vertical coordinate of E_B, $(E_B)_t = \Delta t$, then the horizontal coordination of E_B, $(E_B)_x = v\Delta t$ because of the slope is $1/v$. Since E_B is on hyperbola (1.3), we have

$$-(\Delta t)^2 + (v\Delta t)^2 = (\Delta\bar{t})^2 \tag{1.4}$$

hence

$$\Delta t = \frac{\Delta \bar{t}}{\sqrt{1 - v^2}} > \Delta \bar{t}$$

showing time dilation. In that terminology, $\Delta \bar{t}$ is the proper time.

One can see indeed

$$\|OE_A\| = \|OE_B\|$$

with the Minkowski norm (1.4).

One can do Lorentz contraction as follows. Place a ruler of length l_0 on $\bar{x}$ axis with one end E_A at the origin and the other end E_B at $\bar{x} = l_0$. In O frame E_A moves along $l_{O'}$ and E_B moves along a line that is parallel to $l_{O'}$. The simultaneous events for O' are events that parallel to $\bar{x}$ axis, but for O, simultaneous events are events that are parallel to x axis. So the length measured by O is

$$l_{lab} = (E_B)_x - (E_B)_t v \quad \text{and} \quad l_0 = \|E_B O'\|$$

Because E_B is on $\bar{x}$ axis, $(E_B)_x v = (E_B)_t$, thus

$$l_0^2 = -(E_B)_t^2 + (E_B)_x^2 = (1 - v^2)(E_B)_x^2$$

and

$$l_{lab} = (1 - v^2)(E_B)_x = l_0 \sqrt{1 - v^2} < l_0$$

showing Lorentz contraction.

In summary time dilation is to impose $\Delta \bar{x} = 0$, we find

$$\Delta t = \gamma \Delta \bar{t}$$

length contraction is to impose $\Delta t = 0$, we find

$$\Delta x = \frac{\Delta \bar{x}}{\gamma}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}} > 1$$

In general if we have an event (t, x) in the O coordinate, what is it in O'

coordinate? Make a line passes (t, x) and is parallel to $\bar{t}$,

$$\tilde{y} = \frac{1}{v}(\tilde{x} - x) + t$$

then find the intersection, point A, of it with the $\bar{x}$ axis, $\tilde{y} = v\tilde{x}$, so

$$v\tilde{x} = \frac{1}{v}(\tilde{x} - x) + t \implies \tilde{x} = \gamma^2(x - vt)$$

Then use the same calculation in showing length contraction, the length of $OA = \bar{x}$ is

$$\bar{x} = \gamma(x - vt)$$

Similarly find $\bar{t}$, thus

$$\begin{cases} \bar{t} = \gamma(t - vx) \\ \bar{x} = \gamma(x - vt) \end{cases} \tag{1.5}$$

One can also find the inverse: given an event $(\bar{t}, \bar{x})$ in the O' coordinate, what is it in O?

$$\begin{cases} t = \gamma(\bar{t} + v\bar{x}) \\ x = \gamma(\bar{x} + v\bar{t}) \end{cases}$$

One can get addition of velocities. Say object is moving with speed w along $\bar{x}$ axis wrt to O', what is its wrt to O?

$$w' = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta\bar{x} + v\Delta\bar{t})}{\gamma(\Delta\bar{t} + v\Delta\bar{x})} = \frac{\frac{\Delta\bar{x}}{\Delta\bar{t}} + v}{1 + v\frac{\Delta\bar{x}}{\Delta\bar{t}}} = \frac{w + v}{1 + vw}$$

This ensures that w never > 1, since the both $w, v < 1$. If $w, v \ll 1$, $w' \approx w + v$.

1.4 Lorentz Transformation

Lecture 2 (9/5/13)

In 4D spacetime, we have seen 4 vectors. In the proof of general Lorentz transformation, we have used properties of vectors:

1) In any coordinate inertial system, pick basic 4 vectors with unit length and point along t, x, y, z

$$\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3$$

then any four-vector is sum of

$$\underset{\sim}{a} = a^0\hat{e}_0 + a^1\hat{e}_1 + a^2\hat{e}_2 + a^3\hat{e}_3$$

or write compactly

$$a^\mu = (a^0, a^1, a^2, a^3)$$

where a^μ sometimes means one of the component of $\underset{\sim}{a}$ and sometimes it means $\underset{\sim}{a}$. But in both cases the writing of a^μ implies a basis has been chosen, while $\underset{\sim}{a}$ is standing without a basis.

That is

$$\underset{\sim}{a} = \sum_{\mu=0}^{3} a^\mu \hat{e}_\mu$$

or Einstein summation convention

$$\underset{\sim}{a} = a^\mu \hat{e}_\mu$$

μ is clearly dummy indices. We use Greek indices, $\mu, \nu, \lambda, \alpha, ...$ etc, to denote 4-vector. use Latin indices, i, j, k for 3 space vector.

2) vector multiplied by a constant, changes its length.

3) Sum of two vectors is by the parallelogram.

That is why 4-vector is still called a vector.

The only unusual thing is the definition of norm, Minkowski norm. But it is still called a norm because it comes from an inner product space, i.e.

1) $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$

2) $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

3) $(\alpha \underset{\sim}{a}) \cdot \underset{\sim}{b} = \alpha(\underset{\sim}{a} \cdot \underset{\sim}{b})$

4) $\underset{\sim}{a} \cdot \underset{\sim}{b} = (a^\mu \hat{e}_\mu) \cdot (b^\nu \hat{e}_\nu) = (\hat{e}_\mu \cdot \hat{e}_\nu) a^\mu b^\nu$

where $\eta_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$ is called the metric.

Because we define the norm last time

$$\begin{aligned} (\Delta s)^2 &= \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \\ &= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \end{aligned} \tag{1.6}$$

So

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

η defines the inner product, and

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = -a^0 b^0 + \vec{a} \cdot \vec{b}$$

Why do we study Δs, infinitesimal line? It will gives us an idea of curved line in space which is the key objective to study in general relativity. From (1.6), we can integrate Δs to get a curved line

$$\int \Delta s = \int \sqrt{\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu}$$

In Newton mechanics t is the parameter of path, but now t is an independent variable, so we need other parameter λ, thus

$$\int \Delta s = \int d\lambda \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \tag{1.7}$$

What are the coordinate transformations that leave $(\Delta s)^2$ invariant?

There are 10, known as Poincare group.

4 Translations (static no relative motion)

$$x^\alpha \to x^\alpha + a^\alpha$$

3 Rotations (static no relative motion)

$$x^\alpha \to \Lambda^\alpha{}_\beta x^\beta$$

3 Boosts cf (1.5)

$$x^\alpha \to \Lambda^\alpha{}_\beta x^\beta$$

3 Boosts are the only interesting ones, but since 2 boosts can result a boost

followed by a rotation, we will study rotations as well. We call 3 Rotations + 3 Boosts = Lorentz transformations.

Mathematically $(\Delta s)^2$ invariant means

$$(\Delta s)^2 \quad = \quad (\Delta x)^T\eta(\Delta x) = (\Delta\bar{x})^T\eta(\Delta\bar{x}) = (\Delta x)^T\Lambda^T\eta\Lambda(\Delta x)$$

so Λ satisfies

$$\eta = \Lambda^T\eta\Lambda$$

or in components

$$\eta_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta_{\mu\nu}$$

because we are in component, we can commute scalar. But in matrix form, we can't.

For example, rotation in xz plane

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \tag{1.8}$$

Use (1.5), boost in x direction

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.9}$$

where $\tanh\phi = v$.

For any x

$$x = x^{\mu}\hat{e}_{\mu} = x^{\bar{\mu}}\hat{e}_{\bar{\mu}}$$

Since

$$x^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\mu}x^{\mu} \tag{1.10}$$

then

$$x^\mu \hat{e}_\mu = \Lambda^{\bar{\mu}}{}_\mu x^\mu \hat{e}_{\bar{\mu}}$$

or

$$\hat{e}_\mu = \Lambda^{\bar{\mu}}{}_\mu \hat{e}_{\bar{\mu}} \tag{1.11}$$

In comparison with (1.11) and (1.10), we will show that upper indices (contravariant vector) satisfying Lorentz transform and lower indices (covariant vector) satisfying inverse Lorentz transform. This is a general rule.

From (1.11), we can do

$$\hat{e}_{\bar{\mu}} = \Lambda^\mu{}_{\bar{\mu}} \hat{e}_\mu$$

We can call $\Lambda^\mu{}_{\bar{\mu}}$ inverse Lorentz transform. $\Lambda^\mu{}_{\bar{\mu}}$ and $\Lambda^{\bar{\mu}}{}_\mu$ satisfy

$$\Lambda^\nu{}_{\bar{\mu}} \Lambda^{\bar{\mu}}{}_\rho = \delta^\nu{}_\rho \tag{1.12}$$

$$\Lambda^{\bar{\mu}}{}_\nu \Lambda^\nu{}_{\bar{\rho}} = \delta^{\bar{\mu}}{}_{\bar{\rho}}$$

In matrix form, e.g. inverse of (1.8) is to put $\theta \to -\theta$. So is in (1.9).

1.5 Tensor Algebra

Utilize the idea of upper and lower indices, we define dual basis vector

$$\hat{\theta}^\mu$$

such that $\hat{\theta}^\mu$ acts on basis vector $\hat{e}_\nu$ is

$$\hat{\theta}^\mu(\hat{e}_\mu) = \delta^\mu{}_\nu$$

and dual vector

$$w = w_\mu \hat{\theta}^\mu$$

so dual vector acts on vector V as

$$w(V) = w_\mu V^\nu \hat{\theta}^\mu(\hat{e}_\nu) = w_\mu V^\mu \in \mathbb{R}$$

We sometimes take V acts on w, and define

$$V(w) = w(V) = w_\mu V^\mu$$

Apply the rule for upper and lower indices, we say the transformations of dual basic vector and dual vector are

$$w_{\bar{\mu}} = \Lambda^{\mu}{}_{\bar{\mu}} w_\mu \tag{1.13}$$

$$\hat{\theta}^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\mu} \hat{\theta}^\mu$$

Since both vectors and dual vectors satisfy properties of vectors, so commonly people call vectors that transform as (1.10) contravariant vectors and dual vectors that transform as (1.13) covariant vectors.

We will see a good example of V is tangent vector, and a good example of w is gradient. We define 4-tangent contravariant vector, $V(\lambda)$ of $x(\lambda)$, using the parametrization (1.7),

$$V^\mu = \frac{dx^\mu}{d\lambda}$$

or

$$V = \frac{dx^\mu}{d\lambda} \hat{e}_\mu$$

then V^μ transforms like a coordinate vector, indeed

$$V^\mu \to V^{\bar{\mu}} = \frac{d}{d\lambda} \Lambda^{\bar{\mu}}{}_{\nu} x^\nu = \Lambda^{\bar{\mu}}{}_{\nu} \frac{d}{d\lambda} x^\nu = \Lambda^{\bar{\mu}}{}_{\nu} V^\nu$$

Consider V is a function of spacetime, so $V(x)$ is a vector field. Suppose we want to get things like

$$w_\mu(x) V^\mu(x) = \phi(x)$$

a scalar function.

Let

$$w_\mu = \frac{\partial f}{\partial x^\mu}$$

or in short we write

$$w_\mu = \partial_\mu f$$

Claim w is indeed a 4-dual vector, i.e. check it transforms agreeing inverse Lorentz

$$\partial_{\bar{\mu}} f = \frac{\partial f}{\partial x^{\bar{\mu}}} = \frac{\partial x^\nu}{\partial x^{\bar{\mu}}} \frac{\partial f}{\partial x^\nu}$$

where

$$\frac{\partial x^\nu}{\partial x^{\bar{\mu}}} = \Lambda^\nu{}_{\bar{\mu}}$$

because $x^\nu = \frac{\partial x^\nu}{\partial x^{\bar{\mu}}} x^{\bar{\mu}} = \Lambda^\nu{}_{\bar{\mu}} x^{\bar{\mu}}$. Thus

$$\partial_{\bar{\mu}} f = \Lambda^\nu{}_{\bar{\mu}} \partial_\nu f$$

We can generalize vectors, dual vectors to tensor. It defines on a set of basis product:

$$\text{basis: } \hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \cdots \otimes \hat{\theta}^{\nu_l}$$

the ordering in the basis is important otherwise we have no idea what component corresponds to what. Then make

$$T = T^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_l} \hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \cdots \otimes \hat{\theta}^{\nu_l}$$

called a type (k, l) tensor. We use $T^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_l}$ to mean the component of a tensor, or sometimes it represents the whole tensor T.

Example. of tensors

1. metric $\eta_{\mu\nu}$ type $(0, 2)$ tensor for the inner product

   $$\eta_{\mu\nu} A^\mu B^\nu = A \cdot B$$

   It also gives the norm

   $$A \cdot A = \eta_{\mu\nu} A^\mu A^\nu = \begin{cases} < 0 & \text{timelike} \\ = 0 & \text{null (lightlike)} \\ > 0 & \text{spacelike} \end{cases}$$

   timelike: two events separated by a line with slope > 1, so there is a moving frame O' in which the two events are collinear on the $\bar{t}$ axis, so for O' observer the two events only happens in time. The idea of spacelike is similar.

2. delta $\delta^{\nu}{}_{\mu}$ type $(1,1)$ tensor. We have seen the usage of δ in defining inverse Lorentz. We can also define inverse metric $\eta^{\mu\nu}$ for the inner product of dual vectors

$$\eta^{\mu\nu}\eta_{\nu\lambda} = \delta^{\mu}{}_{\lambda} \tag{1.14}$$

3. Levi-Civita $\epsilon_{\mu\nu\rho\sigma}$ type $(0,4)$

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{even permutation e.g. } 0123 \\ -1 & \text{odd permutation e.g. } 1023 \\ 0 & \text{otherwise} \end{cases}$$

Transformation of tensor is just generalized from contravariant vectors and covariant vectors

$$T^{\bar{\mu}_1,\ldots,\bar{\mu}_k}{}_{\bar{\nu}_1,\ldots,\bar{\nu}_l} = \Lambda^{\bar{\mu}_1}{}_{\mu_1}\cdots\Lambda^{\bar{\mu}_k}{}_{\mu_k}\Lambda^{\nu_1}{}_{\bar{\nu}_1}\cdots\Lambda^{\nu_l}{}_{\bar{v}_l}T^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_l} \tag{1.15}$$

It turns out that the above three examples of tensors are the only tensors that have the same matrix form in all inertial frames.

Lecture 3 (9/10/13)

We would like to change a little of the notation. We don't want to put bar on the index, because this refers to the picture of lecture one, we had O' and O frames. The vectors wrt to O' is $V^{\bar{\mu}}$ and its basis vectors are unit vectors along $\bar{t}, \bar{x}$ axes, so writing V^{μ} and $V^{\bar{\mu}}$ somehow suggests that they are wrt to different basis vectors and in the picture of O' in O frame, the two sets of basis vectors are very different, e.g. basis vectors of O' frame in O are not orthogonal. The picture of lecture one was good for finding the Lorentz boost relation, i.e. the components of V in O and O', but now we should think two observers, one in O, and the other in O', both think their basis vectors are orthogonal and the two sets of basis vectors are the same $(\hat{e}_0,\ \hat{e}_i)$. Thus we should remove the bar from the indices, and put bar on the component of V, i.e.

$$V^{\mu} \to \bar{V}^{\mu}$$

Of course $\bar{V}^\mu$ is $V^{\bar{\mu}}$.

The only effect of this change of notation, Lorentz and inverse Lorentz are no longer

$$\Lambda^{\bar{\mu}}{}_{\mu} \text{ and } \Lambda^{\mu}{}_{\bar{\mu}}$$

but

$$\Lambda^{\nu}{}_{\mu} \text{ and } \Lambda_{\mu}{}^{\nu}$$

So (1.13) becomes

$$w_\mu \to \Lambda_\mu{}^\nu w_\nu$$

and (1.15) becomes

$$\bar{T}^{\mu_1,\ldots,\mu_k}{}_{\nu_1,\ldots,\nu_l} = \Lambda^{\mu_1}{}_{\alpha_1} \cdots \Lambda^{\mu_k}{}_{\alpha_k} \Lambda_{\nu_1}{}^{\beta_1} \cdots \Lambda_{\upsilon_l}{}^{\beta_l} T^{\alpha_1,\ldots,\alpha_k}{}_{\beta_1,\ldots,\beta_l}$$

We can rederive (1.12), (1.14) in the new notation,

$$\begin{aligned}
\Lambda_\mu{}^\nu \Lambda^\mu{}_\rho &= \Lambda^\alpha{}_\delta \eta_{\alpha\mu} \eta^{\delta\nu} \Lambda^\mu{}_\rho \\
&= \Lambda^\alpha{}_\delta \Lambda^\mu{}_\rho \eta_{\alpha\mu} \eta^{\delta\nu} \\
&= \Lambda^\alpha{}_\delta \eta_{\alpha\rho} \eta^{\delta\nu} \\
&= \eta_{\delta\rho} \eta^{\delta\nu} \\
&= \eta_{\rho\delta} \eta^{\delta\nu} \\
&= \delta^\nu{}_\rho
\end{aligned}$$

in 1st step, we use η to change between contravariant and covariant indices. In 2nd step switching is allowed because they are scalar. In 3rd and 4th steps we apply Lorentz Λ to tensor η. In 5th step we use the fact that η is symmetric.

In fact the matrix for η

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \eta^{\mu\nu}$$

are the same. This shows why we choose tensor notations but not matrix notations, because it doesn't apply to high rank tensors and because it doesn't distinguish between contravariant and covariant indices. (The distinguishing between contravariant and covariant indices will be very important in curved spaces.)

In summary we have the following tensor algebra:

1. Rising and downing operation. Given V^α, find the corresponding V_α? Clearly

$$V_\alpha(V^\alpha) = \|V\|^2 = V \cdot V = \eta_{\alpha\beta} V^\alpha V^\beta$$

So

$$V_\alpha = \eta_{\alpha\beta} V^\beta$$

Similarly given w_α,

$$w^\alpha = \eta^{\alpha\beta} w_\beta$$

Or more generally for any tensor, e.g.

$$T^\alpha_{\ \beta\gamma} \eta^{\gamma\delta} = T^{\alpha\ \gamma}_{\ \beta}$$

2. Contract two indices, one up one down, e.g.

$$T^\alpha_{\ \beta\gamma} \delta^\gamma_{\ \alpha} = T^\alpha_{\ \beta\alpha} = S_\beta$$

The resulting S_β is $(0,1)$ tensor. This is a generalization of taking trace from rank 2 to rank 0

$$T^\alpha_{\ \alpha} = T^0_{\ 0} + T^1_{\ 1} + T^2_{\ 2} + T^3_{\ 3}$$

showing scalar is too Lorentz invariant. Similarly one can show $(\Delta s)^2$ is Lorentz invariant.

If one want to contract two indices that are both up or both down, use η, e.g.

$$T^\alpha_{\ \beta\gamma} \eta^{\gamma\beta} = T^{\alpha\ \beta}_{\ \beta}$$

3. Direct sum, i.e. linear combination

$$aR^\alpha_{\ \beta} + bS^\alpha_{\ \beta} = T^\alpha_{\ \beta}$$

If R, S are tensors, then the resulting T^{α}_{β} is a tensor.

4. Direct product e.g. suppose A is a $(1,1)$ tensor and B is a contravariant vector,

$$A^{\alpha}{}_{\beta} \otimes B^{\gamma} = T^{\alpha}{}_{\beta}{}^{\gamma}$$

meaning that attaches the basis of B to A, so resulting another tensor.

5. Differentiation, e.g.

$$\frac{\partial}{\partial x^{\alpha}} T^{\beta\gamma} = \partial_{\alpha} T^{\beta\gamma} = S_{\alpha}{}^{\beta\gamma}$$

resulting a new tensor.
Gradient

$$\partial_{\alpha} f = \vec{\nabla}^{(4)} f$$

Divergence

$$\partial_{\alpha} V^{\alpha} = \vec{\nabla}^{(4)} \cdot V$$

Curl is only defined in 3D space

$$\epsilon^{ijk} \partial_j V_k = (\vec{\nabla} \times \vec{V})^i$$

1.6 Electromagnetism

Rewrite EM in tensor notations.

Recall charge density

$$\rho(\vec{x}, t) = \sum_n e_n \delta^3(\vec{x} - \vec{x}_n(t))$$

$\vec{x}_n(t)$ is the position of the nth charge at t.

Charge current

$$\vec{j}(\vec{x}, t) = \sum_n e_n \frac{d\vec{x}_n(t)}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$$

Combine the two

$$J^\alpha \;=\; (\rho, \vec{j}) = \sum_n e_n \frac{dx_n^\alpha}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$$

the delta function doesn't look like covariant. To make spacetime interchangeable, we increase dimension on δ

$$J^\alpha = \int dt' \sum_n e_n \frac{dx_n^\alpha}{dt'} \delta^4(x - x_n(t'))$$

where $x_n^0(t') = t'$. Since t' is a dummy variable, can be anything, choose t' to be the proper time of the observer in his/her observing frame,

$$J^\alpha = \int d\tau \sum_n e_n \frac{dx_n^\alpha}{d\tau} \delta^4(x - x_n(\tau))$$

By the continuity equation $\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$, or

$$\partial_\alpha J^\alpha = 0 \tag{1.16}$$

we get a covariant form, hence the above is true in any frame. This is clearly from Newton's perspective too, since continuity equation is a local relation.

One can use (1.16) to show that total charge of universe

$$Q = \int d^3x J^0$$

is constant

$$\frac{dQ}{dt} = \int d^3x \partial_0 J^0 = -\int d^3x \nabla \cdot \vec{j} = 0$$

by divergence theorem.

It takes a bit more work to show Q is Lorentz scalar, see Weinberg.

Recall Maxwell's equations

$$\begin{aligned} \nabla \times \vec{B} - \partial_t \vec{E} &= \vec{j} \qquad \nabla \cdot \vec{E} = \rho \\ \nabla \times \vec{E} + \partial_t \vec{B} &= 0 \qquad \nabla \cdot \vec{B} = 0 \end{aligned}$$

In component form

$$
\begin{aligned}
\epsilon^{ijk}\partial_j B_k - \partial_0 E^i &= J^i \qquad \partial_i E^i = J^0 \\
\epsilon^{ijk}\partial_j E_k + \partial_0 B^i &= 0 \qquad \partial_i B^i = 0
\end{aligned}
\tag{1.17}
$$

Introduce field tensor $F^{\mu\nu}$ s.t.

$$
\begin{cases}
F^{\mu\nu} = -F^{\nu\mu} \\
F^{0i} = E^i \\
F^{ij} = \epsilon^{ijk} B_k
\end{cases}
$$

In matrix form

$$
F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ & 0 & B^3 & -B^2 \\ & & 0 & B^1 \\ & & & 0 \end{pmatrix}
$$

Define dual tensor $(*F^{\mu\nu})$ of $F^{\mu\nu}$

$$
*F^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}
$$

Later we will show

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= J^\nu \\
\partial_\mu (*F^{\mu\nu}) &= 0
\end{aligned}
\tag{1.18}
$$

The 1st equation gives the 1st line in (1.17), and the 2nd equation gives the 2nd line in (1.17). More they show F is indeed a true Lorentz invariant tensor, hence Maxwell equations hold in any inertial frame.

1.7 Four-Momentum Vector

Define proper time: time elapses between two timelike events, measured by observer moving on straight path between the two events.

So the distance between the two events is the proper time. Suppose a massive particle is moving, i.e. slope > 1, we can parametrize its worldline x^μ by τ, which is measured as an observer moving with the particle.

We define

$$(d\tau)^2 = -(ds)^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$$

so

$$(d\tau)^2 = (dt)^2 - (dx^i)^2$$

ensuring $(d\tau)^2 \geq 0$ and for the an observer moving with the particle, he sees

$$(d\tau)^2 = (dt)^2$$

We take the tangent line of $x^\mu(\tau)$ to be

$$U^\mu = \frac{dx^\mu(\tau)}{d\tau}$$

clearly this is four-velocity. For particle at rest in frame O,

$$U = (1, 0, 0, 0)$$

Moreover

$$\begin{aligned} U^\mu U^\nu \eta_{\mu\nu} &= \frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau}\eta_{\mu\nu} \\ &= \frac{-(d\tau)^2}{(d\tau)^2} = -1 \end{aligned}$$

on the other hand

$$U^\mu U^\nu \eta_{\mu\nu} = U^\mu U_\nu$$

hence

$$U^\mu U_\nu = -1 \tag{1.19}$$

is independent of frame.

Define 4-memontum

$$P^\mu = mU^\mu$$

where m is always the rest mass. This is the only type of mass we will talk about.

So in the rest frame

$$P^\mu = (m, 0, 0, 0)$$

hence P^0 = rest energy. It becomes clearer, if we restore to c, we see that $P^0 = mc^2 = E$.

Lecture 4 (9/12/13)

From (1.19), we get

$$P^\mu P_\mu = -m^2 \tag{1.20}$$

is invariant in any frame, so define

$$E = \sqrt{\vec{p}^2 + m^2}$$

in any frame, where $\vec{p} = P^i$. (1.20) says that we should take

$$P^\mu = (E, \vec{p}^i)$$

For example in a moving frame, boost along x

$$P^0 = mU^0 = m\frac{dt}{d\tau} = m\Lambda^0{}_0 = m\gamma$$

Similarly we find

$$P^\mu = (\gamma m, v\gamma m, 0, 0)$$

In the Newton limit $v \ll 1$

$$E = P^0 = m(1 - v^2)^{-\frac{1}{2}} \approx m + \frac{1}{2}mv^2$$

$$p^x = v\gamma m = mv(1 - v^2)^{-\frac{1}{2}} \approx mv$$

For massless particles (1.20) becomes

$$P^\mu P_\mu = 0$$

so

$$E = |\vec{p}|$$

1.8 Collections of Many Particles

A single P^α is insufficient to describe the system of particles. We borrow the idea of field tensor $F^{\mu\nu}$ from EM, and define energy momentum tensor $T^{\mu\nu}$. Conceptually it is the flux of four-momentum P^μ across a surface of normal, and impose $T^{\mu\nu}$ to be symmetric.

Energy-momentum density

$$\begin{cases} T^{00} = \text{energy density} \\ T^{i0} = \text{momentum density} = T^{0i} \end{cases}$$

Current of fluid of particles

$$T^{ij} = \begin{cases} \text{pressure} & i = j \\ \text{shear stress} & i \neq j \end{cases}$$

For simplicity suppose we look at dust of n number density of particles that all have same mass m, and in rest with each other, so no viscosity, no shear stress. Define dust number, a 4-vector

$$N^\mu = nU^\mu$$

So in the rest frame of dust

$$N^\mu = (n, 0, 0, 0), \quad P^\mu = (m, 0, 0, 0)$$

We write

$$T^{\mu\nu} = N \otimes P = N^\mu P^\nu$$

So in the rest frame of dust

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

where $\rho = mn$ energy density and all other entries are zeros.

This is oversimplification of perfect fluid (ideal fluid, no viscosity). Usually for ideal fluid, in the rest frame

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}$$

P denotes pressure. Because the fluid is isotropy, P should be the same in any direction. So in tensorial notation we guess

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + P\eta^{\mu\nu} \tag{1.21}$$

For dust $P = 0$.

Photons gas $P = \frac{1}{3}\rho$.

Vacuum energy $P = -\rho$, this may explain why the universe is expanding, that is vacuum energy is anti gravity.

One important property of $T^{\mu\nu}$ is that it is conserved,

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.22}$$

because

$$\begin{aligned} \partial_\mu T^{\mu 0} &= 0 \text{ conservation of energy} \\ \partial_\mu T^{\mu i} &= 0 \text{ conservation of momentum} \end{aligned} \tag{1.23}$$

Apply this to prefect fluid,

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\rho + P)U^\mu U^\nu + (\rho + P)[(\partial_\mu U^\mu)U^\nu + U^\mu \partial_\mu U^\nu] + \partial_\mu P\eta^{\mu\nu}$$

Because of (1.19), then

$$U_\nu \partial_\mu T^{\mu\nu} = -U^\mu \partial_\mu(\rho + P) + (\rho + P)[-(\partial_\mu U^\mu) + U^\mu U_\nu \partial_\mu U^\nu] + U_\nu \partial_\mu P\eta^{\mu\nu} \tag{1.24}$$

use (1.19) again

$$\begin{aligned}0=\partial_\mu U^\nu U_\nu &= U^\nu\partial_\mu U_\nu + U_\nu\partial_u U^\nu\\ &= U^\nu\partial_\mu U^\alpha\eta_{\alpha\nu} + U_\nu\partial_u U^\nu\\ &= \eta_{\alpha\nu}U^\nu\partial_\mu U^\alpha + U_\nu\partial_u U^\nu\\ &= 2U_\nu\partial_u U^\nu\end{aligned}$$

Thus (1.24) becomes

$$\begin{aligned}0 &= -U^\mu\partial_\mu(\rho+P)-(\rho+P)(\partial_\mu U^\mu)+U^\mu\partial_\mu P\\ &= -U^\mu\partial_\mu\rho-(\rho+P)(\partial_\mu U^\mu)\\ &= -\partial_\mu(\rho U^\mu)-P\partial_\mu U^\mu \qquad (1.25)\end{aligned}$$

The above equation is the relativistic equation of energy conservation for a perfect fluid.

Take non-relativistic limit of above equation, not assume LHS equals 0. Since $|v^i|\ll 1$, $P\ll\rho$ (dust model), ignore $-P\partial_\mu U^\mu$ term, and $U^0\approx 1$, then (1.25) says

$$0=-\partial_0(\rho)-\partial_i(\rho v^i)$$

Thus

$$\partial_t\rho+\nabla\cdot(\rho\vec{v})=0$$

Continuity equation for energy density.

See Carroll, project along $(U_\nu)^\perp$ then take non-relativistic limit gives Euler's equation in fluid dynamics

$$\rho[\partial_t\vec{v}+(\vec{v}\cdot\nabla)\vec{v}]=-\nabla P$$

2 Non Euclidean Geometry

For flat space we can easily show that the motion of free particle is along a straight line in Minkowski space,

Theorem 2. *(principle of Maximize proper time) Worldline of free particles between two timelike separated points extremizes the proper time between them.*

In old language

$$d\tau^2 = dt^2 - dx^2$$

that is to minimize dx^2 in 3D space.

Proof. From two points A to B, the path is parametrized by λ

$$\begin{aligned}
\tau_{AB} &= \int_A^B d\tau = \int_A^B (dt^2 - dx^2 - dy^2 - dz^2)^{1/2} \\
&= \int_0^1 d\lambda[(\frac{dt}{d\lambda})^2 - (\frac{dx}{d\lambda})^2 - (\frac{dy}{d\lambda})^2 - (\frac{dz}{d\lambda})^2]^{1/2} \\
&= \int_0^1 d\lambda \left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2}
\end{aligned} \tag{2.1}$$

Put $\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} = L$, Lagrangian of the particle. Here is only a function of $\frac{dx^\mu}{d\lambda}$, "velocities", so the Euler-Lagrange equation

$$-\frac{d}{d\lambda}\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\lambda}\right)} + \frac{\partial L}{\partial x^\mu} = 0 \tag{2.2}$$

implies

$$-\frac{d}{d\lambda}\frac{-2\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}}{2L} = 0$$

i.e.

$$\frac{d^2x^\mu}{d\lambda^2} = 0$$

showing a straight line.

2.1 Curved Spaces

$\mathbb{R}^2$ Euclidean geometry have 5 axioms. The fifth is called parallel axiom: one can find a third line intersect with 2 lines with right angles, then the 2 lines are parallel. From 1 thousand year, people had tried to derive the 5th axiom from the first 4. No one succeeded, because it is not true if the space is not flat, while the first 4 axioms work in flat or curved space.

One notable difference of curved space is that the sum of internal angles of a triangle is not 180^0. It can be $> 180^0$ e.g. on a part of a sphere whose curvature is constantly positive. Or it can be $< 180^0$ e.g. on a part of a horse saddle whose curvature is constantly negative.

The essential character of a surface is the metric function which gives the distance between two points. Let's consider 2D space. Then the general metric, the distance square is

$$dl^2 = g_{ab}dx^a dx^b$$

For flat 2D

$$\begin{aligned} dl^2 &= dx^2 + dy^2 \\ &= dr^2 + r^2 d\theta^2 \end{aligned}$$

Cartesian metric

$$g_{ab} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

Polar metric

$$g_{ab} = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

A line on a sphere of radius a, related to curvature.

$$dl^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

so

$$g_{ab} = \begin{pmatrix} a^2 & \\ & a^2\sin^2\theta \end{pmatrix}$$

Define curvature K, for flat space $K = 0$, sphere $K = \frac{1}{a^2}$, saddle space $K = -\frac{1}{a^2}$.

Lecture 5 (9/17/13)

There are geometry that are curved and they cannot be embedded in flat space of higher dimension. E.g. space of constant negative curvature. If it were able to embedded in higher dimension, it would intersect itself, so there is no 1-1 mapping from the flat space to the curved space. For these reasons distance, straight line, angles are more accessible quantities to study. They are local and intrinsic. They are the main elements of differential manifold. The another branch of geometry: topology, which studies the globe shape.

Below is a simplest case we study, where S^2 can be embedded in $\mathbb{R}^3$.

Example 3. Suppose we live on a S^1 space of curvature $K = \frac{1}{a^2}$. What will the circumference of a circle of radius r be?

Choose spherical coordinate so that the circle is placed at constant θ, i.e. $r = a\theta$

$$\text{circumference} = \int dl = \int_0^{2\pi} a \sin\theta d\phi = 2\pi a \sin\frac{r}{a}$$

In the limit $r \ll a$.

$$\text{circumference} \to 2\pi r$$

notice we used $\sin\theta$, trigonometric function. Does the curved space change trigonometric function? No, $1 + 1 = 2$ math operations don't change.

We will come back with more math later.

2.2 Equivalence Principle

Newtonian gravity:

$$\vec{F} = -\frac{GMm}{r^2}\hat{e}_r = -m\nabla\Phi(\vec{x}_m) \tag{2.3}$$

$\vec{x}_m$ is the position of m wrt M.

$$\Phi(\vec{x}) = -\frac{GM}{r} = -\frac{GM}{|\vec{x} - \vec{x}_M|}$$

If there are more than one sources

$$\Phi(\vec{x}) = -\sum_A \frac{GM_A}{|\vec{x} - \vec{x}_A|}$$

If the sources are continuous distributed

$$\Phi(\vec{x}) = -\int d^3x' \frac{G\mu(\vec{x}')}{|\vec{x} - \vec{x}'|} \tag{2.4}$$

Combining (2.3) and Newton 2nd law $\vec{F} = m\vec{a}$. m drop out

$$\vec{a} = -\frac{GM}{r^2}$$

this lad to GR. Because this line assumes

$$\text{inertial mass} = \text{gravitational mass} \tag{2.5}$$

This has been well tested. Starting from Newton: he used pendulum verified above with very high accuracy, see Newton's Principal

> "In simple pendulums whose centers of oscillation are equally distant from the center of suspension, the quantities of matter are in a ratio compounded of the ratio of the weights and the squared ratio of the times of oscillation in a vacuum."

1889 Eotvos experiment was a famous one. His experiment even counts the effect of earth spin. He gave (2.5) with 1 part of 10^9 uncertainty.

Einstein got very impressed. He proposed

Theorem 4. *(Equivalence principle) No static homogeneous gravitational field can be detect by a freely falling frame.*

Such frame has an infamous name: Einstein elevator.

Proof. Suppose there are N particles in an homogeneous gravitational field g. The equation of motion for the ith particle with mass m is

$$m_I \frac{d^2\vec{x}_i}{dt^2} = m_g\vec{g} + \sum_{j\neq i} F(\vec{x}_j - \vec{x}_i) \tag{2.6}$$

where F is interaction term. Now we do a coordinate transformation, move to the falling frame

$$\vec{\tilde{x}} = \vec{x} - \frac{1}{2}\vec{g}t^2$$

Because g is independent of t and x, (2.6) becomes

$$m_I \frac{d^2 \vec{\tilde{x}}_i}{dt^2} + m_I \vec{g} = m_g \vec{g} + \sum_{j \neq i} F(\vec{\tilde{x}}_j - \vec{\tilde{x}}_i)$$

By (2.5), cancel $m\vec{g}$, we get

$$m_I \frac{d^2 \vec{\tilde{x}}_i}{dt^2} = \sum_{j \neq i} F(\vec{\tilde{x}}_j - \vec{\tilde{x}}_i) \tag{2.7}$$

looks like no gravity. □

In sum an observer freely falling in a static homogenous gravitational field $\Longleftrightarrow$ an observer at rest (or uniform motion) in absence of gravitational field.

In other words the equivalence principle implies that at every spacetime point, in an arbitrary gravitational field it's possible to choose a locally inertial coordinate system, such that in a sufficiently small region that laws of nature are the same as in an unaccelerated frame in the absence of gravity, i.e. (2.7).

Lecture 6 (9/19/13)

Equivalence principle also implies that an observer who's stationary in a static homogenous gravitational field $\vec{g}$ $\Longleftrightarrow$ an observer is uniformly accelerating with $\vec{a} = -\vec{g}$ in absence of gravitational field.

Suppose a rocket is accelerating up with $\vec{a} = -\vec{g}$ in the empty universe, i.e. no gravity. The top point A of the rocket emits pulse and the bottom point B of the rocket receives pulse. The distance between A, B is h. Since A, B are moving at same speed, no relative speed, it looks like we don't get the effects of special relativity, but in fact we will get time dilation and length contraction. Suppose there are ∞ many stationary inertial observers lining along x axis. They record the time t and position x. From inertial observers' view

$$x_A = \frac{1}{2} g t^2 + h \quad x_B = \frac{1}{2} g t^2 \tag{2.8}$$

1st pulse emitted at $t = 0$, 1st pulse received at $t = t_B$.

2nd pulse emitted at $t = \Delta t_A$, 2nd pulse received at $t = t_B + \Delta t_B$

The t is the time recorded by the inertial observers, and $\Delta t_{A,} \Delta t_B$ are the time elapsed should be noticed down by points A, B in the view of the inertial observers.

Since c is constant in the inertial frame, distance traveled by 1st pulse

$$ct_B = x_A(0) - x_B(t_B) \tag{2.9}$$

Distance traveled by 2nd pulse

$$c(t_B + \Delta t_B - \Delta t_A) = x_A(\Delta t_A) - x_B(t_B + \Delta t_B)$$

Using (2.8),

$$c(\Delta t_B - \Delta t_A) = \frac{1}{2}g\Delta t_A^2 + \frac{1}{2}g[(t_B + \Delta t_B)^2 - t_B^2]$$

Assume $\Delta t_{A,}\Delta t_B \ll 1$, we get

$$c(\Delta t_B - \Delta t_A) = \frac{1}{2}g(2t_B\Delta t_B) \tag{2.10}$$

Solve t_B from (2.9)

$$t_B = \frac{-c + \sqrt{c^2 + 2gh}}{g} \approx \frac{-c + c(1 + \frac{gh}{c^2})}{g} = \frac{h}{c}$$

so (2.10) becomes

$$\Delta t_A = \Delta t_B(1 + \frac{gh}{c^2})$$

or

$$\Delta t_B = \Delta t_A(1 - \frac{gh}{c^2})$$

Hence

$$\Delta t_B < \Delta t_A$$

the clock runs faster at point A, the top of rocket. By equivalence principle, this rocket of accelerating $\vec{a} = -\vec{g}$ is equivalent to a stationary rocket in the gravitational field $\vec{g}$, therefore we conclude that clock runs faster at higher gravitational potential, so

$$\Delta t_B = \Delta t_A(1 + \frac{\Phi_B - \Phi_A}{c^2}) \tag{2.11}$$

where Φ is given by (2.4).

By equivalence principle, there exist local Minkowski spaces at points A, B. Thus time dilation will inevitably lead to length contraction. So length contraction in what direction? only x? No, unlike Lorentz boost we did before, here time dilation is connected with a scalar function Φ, we will get length contraction in all directions. And the contraction is given by the same factor as the time dilation.

This leads to try the metric, now forget about c, $(c = 1)$

$$ds^2 = -(1 + 2\Phi(\vec{x}))dt^2 + (1 - 2\Phi(\vec{x}))(dx^2 + dy^2 + dz^2) \tag{2.12}$$

This is Schwarzschild metric for weak field.

Does this agree the time dilation factor? Yes. Indeed the proper time

$$\Delta\tau = \sqrt{1 + 2\Phi(\vec{x})}\Delta t = (1 + \Phi(\vec{x}))\Delta t$$

(why is Φ small? because don't forget the true Φ is Φ/c^2) so we roughly reproduce (2.11)

$$\Delta\tau_B - \Delta\tau_A = [\Phi(\vec{x}_B) - \Phi(\vec{x}_A)]\Delta t \tag{2.13}$$

where Δt is time in the inertial frame. Because the two local inertial frames are not comparable, strictly speaking (2.13) doesn't make a lot sense.

One can also show that the lowest order of (2.12) is consistent with Newtonian gravity. Suppose a particle in gravitational field

$$\begin{aligned}
\tau_{AB} &= \int_A^B d\tau = \int_A^B (-ds^2)^{1/2} \\
&= \int_A^B \frac{d(-ds^2)^{1/2}}{dt} dt
\end{aligned}$$

Take t derivative of (2.12), recall t, $\vec{x}$ are independent coordinates

$$\begin{aligned}
ds\frac{ds}{dt} &= -(1 + 2\Phi(\vec{x}))dt + (1 - 2\Phi(\vec{x}))(v_x dx + v_y dy + v_z dz) \\
&= dt\left[-(1 + 2\Phi) + (1 - 2\Phi)(v_x^2 + v_y^2 + v_z^2)\right]
\end{aligned}$$

ignoring the product of $\Phi\vec{v}^2$

$$\frac{ds}{st} = \sqrt{-1+\vec{v}^2-2\Phi} \approx -1+\frac{1}{2}\vec{v}^2-\Phi$$

so

$$\tau_{AB} = \int dtL = \int -\frac{ds}{dt}dt = \int dt\left(1-(\frac{1}{2}v^2-\Phi)\right)$$

By Euler-Lagrange equation (2.2)

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\Phi$$

recovering Newtonian gravity.

Now we study GR in formal mathematical language. (2.3).

2.3 Metric & Vectors In Curved Space

Let $g_{\mu\nu}(\vec{x})$ be the metric at every spacetime points. There are quantities can be defined globally and there are quantities can only be defined locally. E.g.

$$\tau_{AB} = \int_A^B d\tau = \int_A^B [-g_{\mu\nu}dx^\mu dx^\nu]^{1/2} \tag{2.14}$$

is true globally. But the infinitesimal $d\tau$ in the integrand above from the vector inner product is only defined locally. That is because in the pre-GR time, we allow to parallel move vectors. Say we want to compute $\vec{V}\cdot\vec{W}$, we move the tails of the two vectors together, find norm and find angle between them. Now we cannot parallel transport vectors. E.g. suppose there are two vectors, lying on the equator of the earth, (so if the earth is a unit sphere one vector is at $(1,0,0)$ pointing in $+y$ direction; the other is at $(0,1,0)$ pointing in $-x$ direction) If we move one of them parallel along the equator (i.e. great circle) to the other vector, so that the direction of the moving vector kept fixed, we find the two vectors are actually parallel. But if we bring one of them along the latitude keeping the direction of the vector fixed, i.e. perpendicular to the latitude all the way to the north pole, then move it down along another latitude keeping the direction of vector fixed, i.e. parallel to the latitude, we find that the two vectors are perpendicular to each other.

Hence inner products are defined only locally. The good news is by equivalence principle, there exists local inertial frame with coordinate x_p^μ and

$$\bar{g}_{\mu\nu}(\bar{x}^\mu|_{\bar{x}=x_p}) = \eta_{\mu\nu}$$

and

$$\left.\frac{\partial \bar{g}_{\mu\nu}}{\partial \bar{x}^\sigma}\right|_{\bar{x}=x_p} = 0 \tag{2.15}$$

This means the g is constant wrt to the tangent space vectors, although the actually $g(\vec{x})$ does change if we talk about its derivative wrt an arbitrary curve.

So we can define the physical lengths, areas, volumes as usual Minkowski way

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$dV = \sqrt{-\det g_{\mu\nu}} d^4x \tag{2.16}$$

and vector scalar product

$$V \cdot W = (V^\mu \hat{e}_\mu)(W^\nu \hat{e}_\nu) = V^\mu W^\mu g_{\mu\nu}$$

$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$ is the angle between basis vectors.

We can too do Lorentz transformation in the local inertial frame.

Lecture 7 (9/24/13)

2.4 Geodesics

How does free test particle move in curved spacetime? "Free" means no to count gravitation as a force. "Test particle" means small mass so itself does not change the spacetime geometry. The similar extremal principle applies

Theorem 5. *Worldline of a free test particle extremize the proper time between two points.*

The extremized worldline is knows as geodesics.

The equation of motion derived from variation is the geodesic equation.

Start from (2.14),

$$\tau_{AB} = \int_A^B d\tau = \int_A^B d\lambda[-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}]^{1/2}$$

and the difference between now and (2.1) is that g depends on spacetime. We introduce Christoffel Symbol (it is not a tensor).

$$\Gamma^\delta_{\beta\gamma} = \frac{1}{2}g^{\delta\alpha}(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) \tag{2.17}$$

which is symmetric wrt $\beta\gamma$. It means torsion free. We will show why it is defined this way later, end of section 2.5.

By Euler-Lagrange equation (2.2), one can get geodesics equation (derivation Carroll pages 106-108) we will give an alternative derivation later (cf (2.33))

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{2.18}$$

or

$$\frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma = 0$$

We now show that it agrees the Newtonian limit: assume

1) slow moving test particle. Why slow? Because we will show later that it takes time for gravity to propagate.

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$$

so (2.18) is dominated by $\frac{dt}{d\tau}\frac{dt}{d\tau}$ term

2) weak static gravitational field, i.e.

$$\partial_0 g_{\mu\nu} = 0$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

So

$$\Gamma^\alpha_{00} = -\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{00}}{\partial x^\beta} = -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^\beta} \tag{2.19}$$

so

$$\frac{d^2x^\alpha}{d\tau^2} - \frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^\beta}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{2.20}$$

Thus

$$\begin{cases} \frac{d^2\vec{x}}{d\tau^2} = \frac{1}{2}\nabla h_{00}\left(\frac{dt}{d\tau}\right)^2 \implies \frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\nabla h_{00} & (1) \\ \frac{d^2t}{d\tau^2} = 0 \implies \frac{dt}{d\tau} = \text{const} & (2) \end{cases} \tag{2.21}$$

If we choose

$$h_{00} = -2\Phi, \tag{2.22}$$

or $g_{00} = -(1+2\Phi)$, cf (2.12), (2.21)(1) says

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\Phi$$

and (2.21)(2) says proper time and inertial coordinate time run at the same speed.

2.5 Covariant Derivative

As we discussed the globe defined quantities like (2.14) are easier to deal with. We now study local properties in the tangent space. Let $\bar{x}^\mu$ denote the vector x^μ in the tangent space of origin p.

First show V^μ, w_μ are still tensors in the local frame

$$V^\mu = \frac{dx^\mu}{d\lambda} \quad \rightarrow \quad \bar{V}^\mu = \frac{d\bar{x}^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda}\frac{\partial\bar{x}^\mu}{\partial x^\nu} = \frac{\partial\bar{x}^\mu}{\partial x^\nu}V^\nu$$

$$w_\mu = \frac{d\phi}{dx^\mu} \rightarrow \bar{w}_\mu = \frac{d\phi}{d\bar{x}^\mu} = \frac{d\phi}{dx^\nu}\frac{\partial x^\nu}{\partial\bar{x}^\mu} = \frac{\partial x^\nu}{\partial\bar{x}^\mu}w_\nu$$

$g_{\mu\nu}$ is also a tensor

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial\bar{x}^\mu}\frac{\partial x^\beta}{\partial\bar{x}^\nu}g_{\alpha\beta} \tag{2.23}$$

But if we take

$$\frac{\partial}{\partial\bar{x}^\lambda}\bar{V}^\mu = \frac{\partial\bar{x}^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial\bar{x}^\lambda}\frac{\partial V^\nu}{\partial x^\rho} + \frac{\partial^2\bar{x}^\mu}{\partial x^\nu\partial x^\rho}\frac{\partial x^\rho}{\partial\bar{x}^\lambda}V^\nu$$

which is not a tensor. It would be a tensor if it didn't have the second term on the right.

It turns out if we define covariant derivative by combining with the Christoffel symbol

$$\nabla_\lambda V^\mu = \partial_\lambda V^\mu + \Gamma^\mu_{\lambda\nu} V^\nu \tag{2.24}$$

we get a tensor, i.e.

$$\bar{\nabla}_\lambda \bar{V}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial \bar{x}^\lambda} \nabla_\rho V^\nu$$

For now we should think of ∇_λ just a symbol, it doesn't have a same physical meaning as ∂_λ does, the physical derivative.

Lecture 8 (9/26/13)

Of course in flat space $\Gamma = 0$, so covariant derivative is the usual derivative.

What is the covariant derivative of a covariant vector? Try

$$\nabla_\mu w_\lambda = \partial_\mu w_\lambda + \tilde{\Gamma}^\kappa_{\mu\lambda} w_\kappa$$

Consider

$$\nabla_\mu (w_\lambda V^\lambda) = (\partial_\mu w_\lambda + \tilde{\Gamma}^\kappa_{\mu\lambda} w_\kappa) V^\lambda + w_\lambda (\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\kappa} V^\kappa) \tag{2.25}$$

We impose that

$$\nabla_\mu (w_\lambda V^\lambda) = \partial_\mu (w_\lambda V^\lambda)$$

i.e. ∇_λ acts on scalar is the usual gradient. Thus exchange κ and λ in the 2nd term on the right of (2.25)

$$\tilde{\Gamma}^\kappa_{\mu\lambda} = -\Gamma^\kappa_{\mu\lambda}$$

or

$$\nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma^\kappa_{\mu\lambda} w_\kappa$$

This leads to for a general tensor

$$\begin{aligned} \nabla_\lambda T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} &= \partial_\lambda T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} + \Gamma^{\mu_1}_{\lambda\sigma} T^{\sigma\mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} + \ldots + \Gamma^{\mu_k}_{\lambda\sigma} T^{\mu_1 \ldots \mu_{k-1}\sigma}{}_{\nu_1 \ldots \nu_l} \\ &\quad - \Gamma^\sigma_{\lambda\nu_1} T^{\mu_1 \ldots \mu_k}{}_{\sigma\nu_2 \ldots \nu_l} - \ldots - \Gamma^\sigma_{\lambda\nu_l} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_{l-1}\sigma} \end{aligned} \tag{2.26}$$

Some book uses the following notations

$$\begin{aligned} \partial_\mu T^\alpha{}_{\beta\gamma} &\equiv T^\alpha{}_{\beta\gamma,\mu} \\ \nabla_\mu T^\alpha{}_{\beta\gamma} &\equiv T^\alpha{}_{\beta\gamma;\mu} \end{aligned}$$

Although Γ is not a tensor, the difference of two different Γ is a tensor. Pf. Suppose one defines two covariant derivatives from the two Γ

$$\begin{aligned} \nabla_\lambda V^\mu &= \partial_\lambda V^\mu + \Gamma^\mu_{\lambda\sigma} V^\sigma \\ \tilde{\nabla}_\lambda V^\mu &= \partial_\lambda V^\mu + \tilde{\Gamma}^\mu_{\lambda\sigma} V^\sigma \end{aligned}$$

Then

$$\nabla_\lambda V^\mu - \tilde{\nabla}_\lambda V^\mu = (\Gamma^\mu_{\lambda\sigma} - \tilde{\Gamma}^\mu_{\lambda\sigma}) V^\sigma$$

Since LHS is a tensor and V^σ is a tensor,

$$\Gamma^\mu_{\lambda\sigma} - \tilde{\Gamma}^\mu_{\lambda\sigma} \text{ is a tensor} \tag{2.27}$$

Finally we show what led to define Γ as such (2.17). We want Γ to have two properties

1) Torsion free

$$\Gamma^\mu_{\lambda\sigma} = \Gamma^\mu_{\sigma\lambda}$$

2) metric compatibility

$$\nabla_\lambda g_{\mu\nu} = 0 \tag{2.28}$$

By (2.26), and cycle permute indices

$$\begin{cases} \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma} = 0 & (1) \\ \nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\sigma_{\nu\lambda} g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} g_{\lambda\sigma} = 0 & (2) \\ \nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^\sigma_{\mu\nu} g_{\sigma\lambda} - \Gamma^\sigma_{\mu\lambda} g_{\nu\sigma} = 0 & (3) \end{cases}$$

then $(1) - (2) - (3) = 0$, we get (2.17).

2.6 Tensor Density

Let us contemplate (2.16). It is the physical volume, so it has to be invariant. Indeed (2.23) gives

$$\bar{g} = \det \bar{g}_{\alpha\beta} = \left(\det \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right)^2 \det g_{\alpha\beta} = \left(\det \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\right)^{-2} g$$

so by Jacobian

$$d^4\bar{x} = \det \frac{\partial \bar{x}^\mu}{\partial x^\alpha} d^4x$$

therefore

$$\sqrt{-\bar{g}}d^4\bar{x} = \sqrt{-\left(\det \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\right)^{-2} g} \det \frac{\partial \bar{x}^\mu}{\partial x^\alpha} d^4x = \sqrt{-g}d^4x \tag{2.29}$$

In general if we do

$$M_{\mu_1 \dots \mu_n} \to \bar{M}_{\mu_1 \dots \mu_n}$$

and want to the norm unchanged, should add the weight factor

$$\bar{M}_{\mu_1 \dots \mu_n} = \left(\det \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\right)^w \frac{\partial x^{\alpha_1}}{\partial \bar{x}^{\mu_1}} \cdots \frac{\partial x^{\alpha_n}}{\partial \bar{x}^{\mu_n}} M_{\mu_1 \dots \mu_n}$$

This is a nice formula for the infinitesimal variation of g we will use later when we study variation of path with weight factor $\sqrt{-g}$.

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu} \tag{2.30}$$

Indeed

$$\delta(g^{\lambda\mu} g_{\mu\nu}) = \delta(\delta^\lambda{}_\nu) = 0$$

Notice the two δ's are completely different things.

$$\text{LHS} = \delta(g^{\lambda\mu}) g_{\mu\nu} + g^{\lambda\mu} \delta(g_{\mu\nu})$$

This proves(2.30).

The matrix version of (2.30) is

$$M = \det M_{\mu\nu}$$

$$\delta M = M (M^{-1})^{\mu\nu} \delta M_{\mu\nu}$$

2.7 Parallel Transport

We discussed before, parallel transporting a vector from the equator to another point. The end result is path dependent. Suppose parallel transport along a path

$x^\mu(\lambda)$

In flat space

$$\frac{d}{d\lambda}T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}=0 \tag{2.31}$$

the Cartesian components don't change

or

$$\frac{dx^\sigma}{d\lambda}\frac{\partial}{\partial x^\sigma}T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}=0$$

In curved space we use covariant derivative

$$\frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}=0$$

So by (2.28), metric is always parallel transported.

Lecture 9 (10/1/13)

So if two vectors V, W are parallel transported

$$\frac{dx^\sigma}{d\lambda}\nabla_\sigma V^\mu=0$$

then the norm is parallel transported, because

$$\frac{dx^\sigma}{d\lambda}\nabla_\sigma(g_{\mu\nu}V^\mu W^\nu)=0$$

We can now define what a straight line is in a curved space.

Definition 6. A straight line is a path which the tangent vector to the path is parallel transported.

Indeed the tangent vector of $x^\mu(\lambda)$ is

$$\frac{dx^\mu}{d\lambda}$$

then

$$\frac{dx^\sigma}{d\lambda}\nabla_\sigma\frac{dx^\mu}{d\lambda}=0 \tag{2.32}$$

i.e.

$$\frac{dx^\sigma}{d\lambda}\frac{\partial}{\partial x^\sigma}\frac{dx^\mu}{d\lambda}+\frac{dx^\sigma}{d\lambda}\Gamma^\mu_{\sigma\nu}\frac{dx^\nu}{d\lambda}=0$$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{2.33}$$

which is exactly the geodesic equation. However in (2.18) the parametrization λ is τ. For (2.33) to be strictly equivalent to the geodesic equation, we need to choose affine parameter

$$\lambda = a\tau + b$$

But usually people prefer (2.33). Because e.g. for massless particle $d\tau^2 = 0$, so τ is not a good parameter.

So the definition of a straight line is what we want a straight to be. This definition of a straight line has another good feature. Recall 4 velocity

$$U^\mu = \frac{dx^\mu}{d\tau}$$

so (2.32) says

$$U^\sigma \nabla_\sigma U^\mu = 0$$

or

$$p^\sigma \nabla_\sigma p^\mu = 0 \tag{2.34}$$

i.e. p^σ is perpendicular to $\nabla_\sigma p^\mu$. In words freely falling (straight line) motion moves in the same direction as the momentum, which we think that is what a momentum is. This also leads to put a normalization condition on the parameter λ.

$$p^\mu = \frac{dx^\mu(\lambda)}{d\lambda} \tag{2.35}$$

2.8 Expanding Universe & Cosmological Redshift

We have encountered Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and Schwarzschild metric

$$ds^2 = -(1 + 2\Phi(\vec{x}))dt^2 + (1 - 2\Phi(\vec{x}))dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.36}$$

Both are static metric. For the expanding universe we want something like

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{2.37}$$

this is homogenous (translational invariant) and isotropic (rotational invariant) metric in the local frame. $a(t)$ is the scale factor. Since (2.37) has a universal form, it is the same in all space, not just in one local frame, where t is measured by one specific frame, i.e. people on the earth.

Such expansion is observed. Although our galaxy cluster are still bounded together, intergalaxy do move away from ether other.

In matrix form

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^{-2} & & \\ & & a^{-2} & \\ & & & a^{-2} \end{pmatrix} \tag{2.38}$$

We can compute the Γ symbol

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^i_{00} = \Gamma^i_{jk} = 0 \tag{2.39}$$

$$\Gamma^0_{ij} = a(t)\dot{a}(t)\delta_{ij} \tag{2.40}$$

$$\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i{}_j \tag{2.41}$$

Take a massless photon, see what the geodesic path it takes. From (2.40)

$$\frac{d^2t}{d\lambda^2} + a(t)\dot{a}(t)\left(\frac{d\vec{x}}{d\lambda}\right)^2 = 0 \tag{2.42}$$

since $ds^2 = -dt^2 + a^2(t)dx^2 = 0$, we get

$$dt = a(t)dx$$

or

$$\frac{1}{a}\frac{dt}{d\lambda} = \frac{dx}{d\lambda}$$

so (2.42) says

$$\frac{d^2t}{d\lambda^2} + \frac{\dot{a}(t)}{a(t)}\left(\frac{dt}{d\lambda}\right)^2 = 0$$

Solve 1st order ode for $dt/d\lambda$

$$\frac{dt}{d\lambda} = \frac{w_0}{a(t)}$$

where w_0 is a constant.

Since energy in the rest frame is p^0, and by (2.35)

$$E = p^0 = \frac{dt}{d\lambda} = \frac{w_0}{a(t)}$$

we are talking about photons, so $E = \hbar w$, for us $\hbar = 1$. Hence w_0 is defined to be the frequency of the photon when $a(t) = 1$. Therefore if the light signal is traveling in the space, as the universe expands, the photon emitted at t_1 and received at t_2 have different frequencies

$$\frac{E_1}{E_2} = \frac{a(t_2)}{a(t_1)} \tag{2.43}$$

This is known as cosmological redshift. This is additional shift, add to the usual relativistic redshift due to the relative motion of the emitter and receiver.

As $a \uparrow$, energy goes down. So the total energy of the universe is decreasing.

Lecture 10 (10/3/13)

Let us reconsider perfect fluids (1.21) in the metric (2.38). Because the metric is universal, the form of the energy momentum tensor in the fluid rest frame $U = (1, 0, 0, 0)$,

$$\begin{aligned} T^{\mu\nu} &= (\rho + P)U^\mu U^\nu + P g^{\mu\nu} \qquad (2.44) \\ &= \begin{pmatrix} \rho & & & \\ & P/a^2 & & \\ & & P/a^2 & \\ & & & P/a^2 \end{pmatrix} \end{aligned}$$

is the same in any local frame. The conservation of energy momentum (1.22) becomes

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.45}$$

For $\nu = 0$,($\nu = i$ not so interesting) (2.45) says

$$\partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} = 0 \tag{2.46}$$

Using (2.39), (2.40), (2.41),

$$\Gamma^\mu_{\mu\lambda} T^{\lambda 0} = \Gamma^\mu_{\mu 0} T^{00} = 3\frac{\dot{a}}{a}\rho$$

$$\Gamma^0_{\mu\lambda} T^{\mu\lambda} = \Gamma^0_{ij} T^{ij} = \dot{a}a\delta_{ij}\frac{P}{a^2}\delta^{ij} = 3\frac{\dot{a}}{a}P$$

So (2.46) gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{2.47}$$

This is in direct contract to (1.23), hence energy is not conserved in the local frame.

(2.47) says

$$\frac{d}{dt}(\rho a^3) = -P\frac{d}{dt}(a^3) \tag{2.48}$$

multiplying r^3 to both sides and recognizing $a^3 r^3$ is the physical volume, we get

$$dE = -PdV$$

This identifies the cause of energy decrease, because from thermal dynamics there are two causes of energy to decrease

$$dE = -PdV + Tds$$

Now back to (2.47), one can solve $P(\rho)$, equation of state.

Assume

$$P = w\rho$$

then (2.47) becomes

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + w)$$

so

$$\rho \propto a^{-3(1+w)}$$

Then if

1. $w = 0$,
$$\rho \propto a^{-3} \tag{2.49}$$
this is non relativistic matter dominated universe, because density is inverse of distance cube

2. $w = 1/3$,
$$\rho \propto a^{-4} \tag{2.50}$$
this is radiation (or relativistic particle) dominated universe, because of Stefan–Boltzmann law

3. $w = -1$
$$\rho = \text{const} \tag{2.51}$$
this is negative vacuum energy (or dark energy) dominated universe, because $w = -1 \implies P = -\rho$, which agrees the hypotheses that dark energy is anti gravity, which is used to explain the fact that our universe is not only expanding but accelerating expansion.

3 Einstein Equation

3.1 Riemann Curvature Tensor

We now want to quantify the curveness of the curved space. We have seen several characters of flat space that are not true in curved space.

1) parallel geodesics remain parallel

2) parallel transport around a loop leaves vector unchanged

3) covariant derivative commute

In flat space a 2D surface square loop element in the 4D spacetime is characterized by a vector
$$\hat{n} = V^\mu$$
only need one index.

In curved space a 2D square loop element in the 4D spacetime is characterized by a tensor

$$\Delta f^{\mu\nu} = dx^{\mu} dx'^{\nu} - dx^{\nu} dx'^{\mu}$$

where dx is the infinitesimal distance at one corner of the loop, and $dx'^{\mu}$is the infinitesimal distance at the other diagonal corner of the loop. Why?

Recall 4D Stokes

$$\oint_{\partial\Sigma} F dr = \int_{\Sigma} \partial F d\Sigma$$

Parallel transport of a vector V^{μ} by dx^{α} results the same vector, but in curved space because

$$\nabla_{\alpha} V^{\mu} = \partial_{\alpha} V^{\mu} + \Gamma^{\mu}_{\alpha\nu} V^{\nu}$$

the difference between the resulting vector is

$$\delta V^{\mu} = -\Gamma^{\mu}_{\alpha\nu} V^{\nu} dx^{\alpha}$$

the total effect of parallel transport V^{μ} around a loop

$$\Delta V^{\mu} = -\oint \Gamma^{\mu}_{\alpha\nu} V^{\nu} dx^{\alpha} = -\frac{1}{2} \int \left(\partial_{\beta}(\Gamma^{\mu}_{\alpha\nu} V^{\nu}) - \partial_{\alpha}(\Gamma^{\mu}_{\nu\beta} V^{\nu}) \right) \Delta f^{\alpha\beta} \tag{3.1}$$

and define Riemann tensor by equating $\left(\partial_{\beta}(\Gamma^{\mu}_{\alpha\nu} V^{\nu}) - \partial_{\alpha}(\Gamma^{\mu}_{\beta\nu} V^{\nu}) \right) = -R^{\mu}{}_{\nu\alpha\beta} V^{\nu}$, thus

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_{\alpha} \Gamma^{\mu}_{\beta\nu} - \partial_{\beta} \Gamma^{\mu}_{\alpha\nu} + \Gamma^{\mu}_{\alpha\lambda} \Gamma^{\lambda}_{\beta\nu} - \Gamma^{\mu}_{\beta\lambda} \Gamma^{\lambda}_{\alpha\nu} \tag{3.2}$$

It is possible to write it in more compact form by using covariant derivative

$$\nabla_{\mu} \nabla_{\nu} V^{\alpha} - \nabla_{\nu} \nabla_{\mu} V^{\alpha} = R^{\alpha}{}_{\beta\mu\nu} V^{\beta} - 2\Gamma^{\lambda}_{[\mu,\nu]} \nabla_{\lambda} V^{\alpha} \tag{3.3}$$

where

$$\Gamma^{\lambda}_{[\mu,\nu]} = \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}) \tag{3.4}$$

is the torsion. If torsion free, (3.3) acting on any tensor is simply

$$\begin{aligned} \left[\nabla_{\alpha}, \nabla_{\beta} \right] T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \quad &= \quad R^{\mu_1}_{\ \lambda\alpha\beta} T^{\lambda\mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} + R^{\mu_2}_{\ \lambda\alpha\beta} T^{\mu_1 \lambda \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} + \ldots \\ & \quad - R^{\lambda}{}_{\nu_1\alpha\beta} T^{\mu_1 \ldots \mu_k}{}_{\lambda\nu_2 \ldots \nu_l} - \ldots \end{aligned}$$

Lecture 11 (10/8/13)

Symmetry of Riemann indices in matrix form

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu}$$

1) $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$
2) $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$
3) $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$
4) $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$ or denoted

$$R_{\alpha[\beta\mu\nu]} = 0$$

5) Bianchi identity

$$\nabla_{[\lambda} R_{\alpha\beta]\mu\nu} = 0$$

These symmetric requirement gives 20 free parameters.

Define Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{3.5}$$

so it is symmetric.

Define Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \tag{3.6}$$

aka curvature scalar.

Define Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{3.7}$$

then the Bianchi identity implies

$$\nabla^{\mu} G_{\mu\nu} = 0$$

and

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R \tag{3.8}$$

and hence

$$\nabla_{\mu} g^{\lambda\rho} = 0$$

regularity of the space.

Example. S^2

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Ricci scalar

$$R = \frac{2}{a^2}$$

Before we discuss Einstein equation, we would like to discuss symmetry of the metric, isometrics. Consider transformation

$$x^\mu \to \bar{x}^\mu = x^\mu + \epsilon K^\mu(x^\mu) \qquad \epsilon \ll 1$$

such that K satisfies Killing equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{3.9}$$

Then

$$p^\mu p^\nu \nabla_\mu K_\nu = 0$$

because (3.9) shows the anti commutator is 0, i.e.

$$\nabla_{(\mu} K_{\nu)} = 0$$

then if we are on a geodesics (2.34)

$$p^\mu \nabla_\mu p_\nu = 0$$

we can do

$$p^\mu p^\nu \nabla_\mu K_\nu + K_\nu p^\mu \nabla_\mu p_\nu = 0$$

which is equivalent to

$$p^\mu \nabla_\mu (K_\nu p^\mu) = 0$$

hence

$$K_\nu p^\mu$$

is conserved along geodesics. This agrees perfectly to the Noether: symmetry gives conserved quantity.

Carroll problem 3.12 shows

$$\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$$

or in term of Ricci tensor, Ricci scale

$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu$$

$$K^\lambda \nabla_\lambda R = 0$$

meaning that in the direction of Killing vector curvature stays the same.

Let's see an example of conserved quantity. Let

$$J^\mu = K_\nu T^{\mu\nu} \tag{3.10}$$

be current. $T^{\mu\nu}$ be EM tensor, and then

$$\begin{aligned} \nabla_\mu J^\mu &= (\nabla_\mu K_\nu) T^{\mu\nu} + K_\nu (\nabla_\mu T^{\mu\nu}) \\ &= 0 \end{aligned}$$

First term on the right is 0 by (3.9), second term on the right is 0 by (2.45). Then by the continuity equation

$$E = \int_\Sigma J^\mu n_\mu \sqrt{\gamma} d^3 y$$

is conversed. where Σ is a 3D in space surface, (if K_ν in (3.10) is a timelike vector). n is normal vector and $\gamma = \det \gamma_{ij}$, γ_{ij} is induced metric, defined by the 3D in space surface $x^\mu(y^i)$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \underbrace{g_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}}_{\gamma_{ij}} dy^i dy^j$$

Some examples of Killing vectors. E.g. in $\mathbb{R}^3$,

$$ds^2 = dx^2 + dy^2 + dz^2$$

we clearly have 3 Killing vectors, because it stays the same if we translate the coordinates

$$\begin{aligned} X &= (1, 0, 0) \\ Y &= (0, 1, 0) \\ Z &= (0, 0, 1) \end{aligned}$$

of course the conserved quantities Kp^μ are the momentum in three directions. However we can express $\mathbb{R}^3$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

then the metric is independent of ϕ, i.e. one can rotate the coordinates about ϕ and $g_{\mu\nu}$ stays the same. This implies one Killing vector

$$R = (-y, x, 0)$$

the other two are similarly found

$$\begin{aligned} S &= (z, 0, -x) \\ T &= (0, -z, y) \end{aligned}$$

Of course the conserved quantities Kp^μ are the angular momentum in three directions.

In general if the number of totally independent Killing vectors gives us a clue of how symmetric the space is. If

$$\text{number of totally independent Killing vectors} = \frac{1}{2}d(d+1)$$

d to the dimension of the space, we say that the space is maximally symmetric, i.e. curvature is the same everywhere. In such space

$$R_{\alpha\beta\mu\nu} = \frac{R}{d(d-1)}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$$

So it turns out that in Euclidean space, there are only three kinds of maximally symmetric spaces.

$$\begin{aligned} R = 0 \quad &\Longleftrightarrow \quad \text{planes}\rightarrow\text{Minkowski} \\ R > 0 \quad &\Longleftrightarrow \quad \text{sphere}\rightarrow\text{deSitter} \\ R < 0 \quad &\Longleftrightarrow \quad \text{hyperbole}\rightarrow\text{antideSitter} \end{aligned}$$

Lecture 12 (10/10/13)

In summary: why Riemann curvature tensor is useful.

1. It measures failure of vector to return to itself. Parallel transport a vector around an arbitrary small loop, cf (3.1)

$$\Delta V^\mu = -\oint \Gamma^\mu_{\alpha\nu} V^\nu dx^\alpha = \frac{1}{2}\int R^\mu{}_{\nu\alpha\beta} V^\nu \Delta f^{\alpha\beta}$$

2. It measures failure of covariant derivatives to commute, cf (3.3)

$$\left[\nabla_\alpha, \nabla_\beta\right] V^\sigma = R^\sigma{}_{\beta\mu\nu} V^\beta$$

assuming torsion free.

3. Lastly we'll show it measures failure of parallel lines to remain parallel: Geodesic derivation. Define

$$\text{tanget vector } T^\mu = \frac{\partial x^\mu}{\partial \lambda}$$

vector tangent to a geodesics, and define

$$\text{deviation vector } S^\mu = \frac{\partial x^\mu}{\partial \kappa}$$

vector describe the variation between two nearby geodesic. Then the relative velocity of geodesics

$$V^\mu = (\nabla_T S)^\mu = T^\nu \nabla_\nu S^\mu$$

and the relative acceleration of geodesic

$$A^\mu = (\nabla_T V)^\mu = T^\nu \nabla_\nu (T^\alpha \nabla_\alpha S^\mu)$$

and it turns out that

$$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

this is manifestation of gravitational tidal force.

3.2 Einstein Equation

The theory of general relativity has two parts: find EOM, and trajectories for known gravity. In Newtonian gravity theory

$$\vec{a} = -\nabla\Phi$$

Since we are no longer interpreting gravity as a force, we say that the particle moves in a gravitational field following geodesics. So if we know $g_{\mu\nu}(x^\alpha)$ we know how to find geodesics.

The second part is to determine the gravitational field. In Newtonian gravity theory

$$\nabla^2\Phi = 4\pi G\rho \tag{3.11}$$

where G is the Newton constant, and ρ is matter density. We will replace (3.11) by Einstein equation. First change ρ to $T_{\mu\nu}$ because relativistic speaking energy mass are interchangeable, and change Φ to $g_{\mu\nu}$. But

$$\nabla_\lambda g_{\mu\nu} = 0 \text{ so } \nabla_\lambda \nabla^\lambda g_{\mu\nu} = 0$$

bad. We want some thing that is a (0,2) rank tensor because the RHS of (3.11) after changing ρ to $T_{\mu\nu}$ is a a (0,2) rank tensor, and we want that to be a second order derivatives on $g_{\mu\nu}$. So the good candidate is Ricci tensor $R_{\mu\nu}$.

But if we rewrite (3.11) as

$$R_{\mu\nu} = kT_{\mu\nu} \tag{3.12}$$

for some constant k, we will have a problem. By conservation of energy $\nabla^\mu T_{\mu\nu} = 0$ then (3.8)

$$\nabla_\nu R = 0$$

then by (3.12)

$$R = g^{\mu\nu}R_{\mu\nu} = kg^{\mu\nu}T_{\mu\nu} = kT$$

implies

$$\nabla_\nu T = 0$$

but this is not true. Because $T = 0$ in vacuum and non-zero in matter.

Try again

$$G_{\mu\nu} = kT_{\mu\nu} \tag{3.13}$$

recall Einstein tensor satisfies $\nabla^\mu G_{\mu\nu} = 0$. We find k by arguing that (3.13) should be consistent with Newton gravity if the gravitational field is weak, static and test particle moves slow and we check for the spacial case that the universe is made of perfect fluid cf (2.44), with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

In Newton limit matter dominates $\rho \gg P$, (2.44) becomes

$$T^{\mu\nu} = \rho U^\mu U^\nu$$

Consulting with (2.20), only the g_{00} component of $g_{\mu\nu}$ matters, so from (3.13)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}$$

or

$$\underbrace{g^{\mu\nu}R_{\mu\nu}}_{R} - \frac{1}{2}\underbrace{g^{\mu\nu}g_{\mu\nu}}_{4}R = k\underbrace{g^{\mu\nu}T_{\mu\nu}}_{T}$$

so

$$R = -kT \tag{3.14}$$

thus

$$R_{\mu\nu} = k(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

and

$$T \approx g^{00}T_{00} = -\rho$$

thus

$$R_{00} = k(T_{00} - \frac{1}{2}g_{00}T) = k(\rho - \frac{1}{2}\rho) = \frac{1}{2}k\rho \tag{3.15}$$

On the other hand,

$$\begin{aligned} R_{00} &= R^{\mu}{}_{0\mu 0} = R^{i}{}_{0i0} \\ &= \partial_i \Gamma^i_{00} - \underbrace{\partial_0 \Gamma^i_{i0}}_{0\ \because \text{static}} + O(h^2) \end{aligned}$$

then by (2.19), (2.22)

$$R_{00} = \partial_i - \frac{1}{2}\eta^{ii}\frac{\partial h_{00}}{\partial x^i} = -\frac{1}{2}\nabla^2 h_{00} = \nabla^2 \Phi = 4\pi G\rho \tag{3.16}$$

Compare (3.15) and (3.16),

$$k = 8\pi G$$

So we get Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{3.17}$$

or by (3.14) sometimes written as

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

showing that the vacuum Einstein equation is

$$R_{\mu\nu} = 0 \tag{3.18}$$

Notice (3.17) is nonlinear because of the product $(g_{\mu\nu}R)$, so

1) superposition of two solutions is not a solution

2) effect of gravitational field of two sources is not the sum of each separately.

3.3 Schwarzschild Metric

Lecture 13 (10/15/13)

Consider a simplest universe that has mass M at the origin, outside is all vacuum. Later we will show the solution of Einsteins equation of such universe is

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\theta + r^2 \sin^2\theta d\phi \tag{3.19}$$

Since the metric is static, we find one Killing vector

$$\xi^\mu = (1, 0, 0, 0) \tag{3.20}$$

associated with time translation. The metric is also independent of ϕ, so another Killing vector is

$$\eta^\mu = (0, 0, 0, 1) \tag{3.21}$$

If r is so large that $GM \ll r$, we get the Newtonian limit (2.36). If r is so small that $r < 2GM$, we will get a metric that has positive coefficient in front of dt, which, as we will show, implies even lights cannot escape, i.e. it is a black hole. E.g.

$$r_{sum} \sim 7 \times 10^5 \text{km}$$

if sum shrinks into a point, then the space around the sum within radius

$$r = 2GM_{sum} = 3\text{km}$$

will become a black hole.

Before we study black holes, we look at two implications of (3.19).

Gravitational Redshift

As we showed before expanding universe causes redshift, static metric can produce redshift too. Consider two stationary observers, one at r_∞, the other at r_0. They measure the same photons traveling from r_0 to r_∞. By (3.20),

$$\xi^\mu p_\mu$$

is a conserved quantity, i.e. energy of photon is conserved along the geodesic: e.g. radial outward from r_0 to r_∞. We showed before that in the two local observers frames clocks run at different speeds. This will make them to get different frequencies for the photons. Formally for observer at r_0, cf (1.19)

$$U^\mu = \frac{dx^\tau}{d\tau}$$

$$g_{\mu\nu} U^\mu U^\nu = -1 \tag{3.22}$$

stationary observe in the local frame

$$U^\mu = (U^0_{obs}, 0, 0, 0)$$

so

$$U^0_{obs} = \sqrt{1 - \frac{2GM}{r_0}}$$

or

$$U^\mu = \sqrt{1 - \frac{2GM}{r_0}} \xi^\mu$$

and the photon energy observed

$$E_{obs} = w_0 = -p_\mu U^\mu_{obs} = -\sqrt{1 - \frac{2GM}{r_0}} p_\mu \xi^\mu$$

hence

$$\frac{w_0}{w_\infty} = \sqrt{1 - \frac{2GM}{r_0}}$$

hence frequency at ∞ is less than frequency at r_0. Since the dt component of the metric (3.19) is -1, the clock at ∞ is the reference standard clock. while as the clock at $r = 0$ stops.

Geodesics

By (3.21),

$$\xi^\mu p_\mu$$

is a conserved quantity, i.e. angular momentum is conserved. From there, we can define two conserved quantities along geodesic

$$\begin{aligned} e &= -\xi^\mu U_\mu = (1 - \frac{2GM}{r})U^t &(3.23)\\ l &= \eta^\mu U_\mu = r^2 \sin^2\theta U^\phi &(3.24)\end{aligned}$$

Hence we can look for orbits that lie on xy plane, $\theta = \pi/2$, $U^\theta = 0$. By (3.22)

$$-(1-\frac{2GM}{r})(U^t)^2 + (1-\frac{2GM}{r})^{-1}(U^r)^2 + r^2(U^\phi)^2 = -1$$

that is

$$-(1-\frac{2GM}{r})^{-1}e^2 + (1-\frac{2GM}{r})^{-1}(U^r)^2 + \frac{l^2}{r^2} = -1$$

which is simplified to

$$\underbrace{\frac{1}{2}(e^2-1)}_{\epsilon} = \frac{1}{2}(U^r)^2 + \underbrace{\frac{1}{2}\left((1-\frac{2GM}{r})(1+\frac{l^2}{r^2})-1\right)}_{V_{eff}(r)} \tag{3.25}$$

The term on the left is a conserved quantity, we call it ϵ, energy per mass. The first term on the right is the "kinetic energy" of the 1D radial motion, the only r dependence is in the second term on the right, so we say it is the effective "potential" of the radial motion.

$$V_{eff}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} \tag{3.26}$$

compare this to the classical Kepler's problem, we see that the last term is relativistic correction. How big is the correction? Let us restore c and m

$$\text{RHS (3.25)} = \frac{1}{2}mU^{r2} + \frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{GML^2}{c^2mr^3}$$

$L = ml$. One can plot (3.26). $r \to \infty$, $V_{eff} \to 0^-$; $r \to 0$, $V_{eff} \to -\infty$ this is the most striking feature of the relativistic correction. Hence if the mass M is condensed at the origin, black hole solution is possible.

One can also find that it has one maximum at r_- and one local minimum at r_+

$$\frac{dV_{eff}(r)}{dr} = \frac{GM}{r^2} - \frac{l^2}{r^3} + 3\frac{GMl^2}{r^4} = 0 \tag{3.27}$$

$$r_\pm = \frac{l^2}{2GM}[1 \pm \sqrt{1 - 12(\frac{GM}{l})^2}] \tag{3.28}$$

If l is very small, no solutions for $r_\pm$. So no bound state.

So if $\epsilon > \max V_{eff}$, the particle will be spirally sucked into the black hole. If $0 < \epsilon < \max V_{eff}$, unbounded meteor solution. If local $\min V_{eff} < \epsilon < 0$, we get bounded orbits (not necessary closed, because of the relativistic correction, however circular orbit is always closed).

Lecture 14 (10/22/13)

The smallest l to have a bound orbit is

$$l/GM = \sqrt{12}$$

then

$$r_+ = r_- = 6GM \tag{3.29}$$

called inner most stable circular orbit, aka "isco".

Let's look at a situation where a particle is at almost rest at ∞ with E slight above 0, and it should radially free fall into the black hole.

$$l = 0 \ e = 1$$

we have

$$0 = \frac{1}{2}mU^{r2} - \frac{GMm}{r} \implies U^r = \sqrt{\frac{2GM}{r}}$$

and so the 4 velocity of the particle in the local frame is

$$U^\mu = \left((1 - \frac{2GM}{r})^{-1}, -\sqrt{\frac{2GM}{r}}, 0, 0\right)$$

the - sign in the U^r component is because + is defined to be outward. In the particle rest frame

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}} \text{ or } r^{1/2}dr = -(2GM)^{1/2}d\tau$$

Integrate

$$r(\tau) = (\frac{3}{2})^{2/3}(2GM)^{1/3}(\tau_* - \tau)^{2/3}$$

where τ_* is proper time where particle reaches $r = 0$. Now find $t(r)$,

$$\frac{dt}{dr} = \frac{dt/d\tau}{dr/d\tau} = -\frac{1}{(1-\frac{2GM}{r})\sqrt{\frac{2GM}{r}}} \tag{3.30}$$

integrate, putting

$$\frac{2GM}{r} = y^2 \text{ or } -\frac{2GM}{r^2}dr = 2ydy \implies dr = -\frac{4GM}{y^3}dy$$

$$\begin{aligned}
t - t_* &= \int \frac{4GMdy}{(1-y^2)y^4} = 4GM\left(\int \frac{1/2}{1-y} + \frac{1/2}{1+y} + \frac{1}{y^2} + \frac{1}{y^4}\right)dy \\
&= 4GM\left(\frac{1}{2}\log\left|\frac{1+y}{1-y}\right| - \frac{1}{y} - \frac{1}{3y^3}\right) \\
t(r) &= t_* + 2GM\left(-\frac{2}{3}(\frac{r}{2GM})^{3/2} - 2(\frac{r}{2GM})^{1/2} + \log\left|\frac{(\frac{r}{2GM})^{1/2}+1}{(\frac{r}{2GM})^{1/2}-1}\right|\right)
\end{aligned}$$

As discussed before, this t is wrt to the standard clock, hence wrt to the observer at ∞.

We can do the similar thing like (3.30) to figure out the angular speed of the particle on circular orbits $r = r_\pm$, by (3.23), (3.24) the 4 velocity of the particle in the local rest frame is

$$U^\mu = \left(e(1-\frac{2GM}{r})^{-1}, 0, 0, \frac{l}{r^2}\right)$$

$$\Omega = \frac{d\phi/d\tau}{dt/d\tau} = (1-\frac{2GM}{r})\frac{l}{er^2}$$

thus

$$U^\mu = U^t(1,0,0,\Omega)$$

one can solve Ω by using (3.25) and (3.27).

$$\frac{l}{e} = (GMr)^{1/2}(1-\frac{2GM}{r})^{-1}$$

so

$$\Omega^2 = \frac{GM}{r^3} \tag{3.31}$$

saying that the square of the period is proportional to the distance cube, coincided with non-relativistic Kepler's law. Then

$$-1 = g_{\mu\nu}U^\nu U^\mu = -(1-\frac{2GM}{r})U^{t2} + r^2\Omega^2 U^{t2}$$

so

$$U^t = \sqrt{1-\frac{2GM}{r} - r^2\frac{GM}{r^3}} = \sqrt{1-\frac{3GM}{r}} \tag{3.32}$$

3.4 Precession of Perihelia

We now do two most important calculations, precession of Mercury orbit around Sun and defection of light, that gave support to general relativity. We can do the similar thing like (3.30) to figure out $\phi(r)$. If the orbit is a perfect ellipse, then as r goes from the perihelia r_1 to aphelia r_2 and back to perihelia, ϕ changes 2π.

$$\frac{d\phi}{dr} = \frac{d\phi/d\tau}{dr/d\tau} = \frac{l}{r^2 U^r} = \frac{l}{r^2}\left(e^2 - (1-\frac{2GM}{r})(1+\frac{l^2}{r^2})\right)^{-1/2}$$

$$\text{precession} \;=\; \underbrace{2l\int_{r_1}^{r_2}\frac{dr}{r^2}\left(e^2-(1-\frac{2GM}{r})(1+\frac{l^2}{r^2})\right)^{-1/2}}_{\equiv I} - 2\pi$$

We know that if we don't have the relativistic term $2GMl^2/r^3$, after restoring c,

$$\begin{aligned} I &= 2l\int_{r_1}^{r_2}\frac{dr}{r^2}\left(c^2(e^2-1)+\frac{2GM}{r}-\frac{l^2}{r^2}+\frac{2GMl^2}{c^2r^3}\right)^{-1/2} \\ &\to 2l\int_{r_1}^{r_2}\frac{dr}{r^2}\left(c^2(e^2-1)+\frac{2GM}{r}-\frac{l^2}{r^2}\right)^{-1/2} \end{aligned}$$

let $y = 1/r$, and complete the square, there are two roots $y_1 = 1/r_1$, $y_2 = 1/r_2$ because at which $U^r = 0$

$$2l\int_{r_1}^{r_2}\frac{dr}{r^2}\left(c^2(e^2-1)+\frac{2GM}{r}-\frac{l^2}{r^2}\right)^{-1/2} = 2\int_{y_1}^{y_2}\frac{dy}{\sqrt{(y_2-y)(y-y_1)}} = 2\pi$$

To really do the integral I, we Taylor expand. The next order correction in $1/c^2$ is

$$\text{precession} = 6\pi\left(\frac{GM}{cl}\right)^2$$

and

$$l^2 = \left(r^2\frac{d\phi}{d\tau}\right)^2 \approx \left(r^2\frac{d\phi}{dt}\right)^2 = GMa(1-\epsilon)$$

a is semi major axis, ϵ is eccentricity. So

$$\text{precession} \approx \frac{6\pi GM}{c^2a(1-\epsilon)}$$

so the effect is the largest for smallest a, that is why we measure Mercury, the closest planet to the sum. It has period of 88 days.

$$\begin{aligned} \frac{GM_{sum}}{c^2} &= 1.48\times10^5\text{cm} \\ a &= 5.79\times10^{12}\text{cm} \\ \epsilon &= 0.2056 \end{aligned}$$

so

$$\text{precession} = 0.013''/\text{orbit} = 43.0''/\text{century}$$

$$\begin{aligned} 1' &= \text{arcmin}=1/60 \text{ of day} \\ 1'' &= \text{arcsec} = 1/60 \text{ of } 1/60 \text{ of day} \end{aligned}$$

Observed value 575″/century, among it effect due to other planets 532″/century. Subtract the two, get exact 43.0″/century.

3.5 Defection of Light in Schwarzschild Metric

As we mentioned before the good parametrization for the trajectories of light is

$$\tau \to \text{affine } \lambda$$

So the conserved quantities are

$$\begin{aligned} e &= -\xi^\mu U_\mu = (1-\frac{2GM}{r})\frac{dt}{d\lambda} \\ l &= \eta^\mu U_\mu = r^2\sin^2\theta\frac{d\phi}{d\lambda} && (3.33) \\ U^\mu U_\mu &= g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 && (3.34) \end{aligned}$$

Again by (3.33), we can choose the trajectory in xy plane. $\theta = \pi/2$, $d\theta = 0$. By (3.34)

$$-(1-\frac{2GM}{r})(\frac{dt}{d\lambda})^2 + (1-\frac{2GM}{r})^{-1}(\frac{dr}{d\lambda})^2 + r^2(\frac{d\phi}{d\lambda})^2 = 0$$

or

$$-(1-\frac{2GM}{r})^{-1}e^2 + (1-\frac{2GM}{r})^{-1}(\frac{dr}{d\lambda})^2 + \frac{l^2}{r^2} = 0$$

Let

$$\frac{1}{b^2} = \frac{1}{l^2}(\frac{dr}{d\lambda})^2 + w_{eff}(r)$$

combining all r dependence into w_{eff}

$$w_{eff}(r) = \frac{1}{r^2}(1-\frac{2GM}{r}) \tag{3.35}$$

then

$$b^2 = \frac{l^2}{e^2}$$

i.e. b is a constant. What is the physical interpretation for b?

$$b = \left|\frac{l}{e}\right|$$

Suppose the photon comes in from ∞ parallel to the x axis with impact parameter d,

$$e^2 - 1 = c^2 \implies e \sim c$$

$$l = dc$$

or

$$b = d$$

hence b is the impact parameter. The graph of (3.35) shows that it has a maximal value of $1/27(GM)^2$ at $r = 3GM$, and $w_{eff} \to -\infty$ as $r \to 0$, So if the initial impact parameter

$$\begin{aligned}
\frac{1}{b^2} < \frac{1}{27(GM)^2} \quad &\implies \quad \text{the light will deflect and scatter out} \\
\frac{1}{b^2} > \frac{1}{27(GM)^2} \quad &\implies \quad \text{the light will continue move into } r = 3GM,
\end{aligned}$$

when it goes to $r < 2GM$, the metric is positive definite. Things become very interesting, which is subject of next lecture.

Consider the case that light scatters, denote r_{min} the closest distance to the origin as light approaches the origin, and r_{min} satisfies

$$w_{eff}(r_{min}) = \frac{1}{b^2}$$

$$\frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \frac{l}{r^2}\frac{1}{l\sqrt{\frac{1}{b^2} - w_{eff}}}$$

$$\text{deflection angle} = 2\int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}(1 - \frac{2GM}{r})}} - \pi$$

let $y = b/r$

$$\text{deflection angle} = 2\int_0^{y_{max}} \frac{dy}{\sqrt{1 - y^2(1 - \frac{2GM}{b}y)}} - \pi$$

expanding in GM/b

$$\text{deflection angle} = \frac{4GM}{b}$$

put b to the radius of sum, one can detect light coming from stars behind the sum

$$\text{deflection angle} \approx 1.7''$$

3.6 Black Holes in Schwarzschild Metric

Lecture 15 (10/29/13)

So far we have assumed the mass M is concentrated at the origin, but it may not be the case. Because Pauli exclusive principle, the gravity pulls M to the origin is balanced by the Fermi pressure. In other words, if the star is not massive enough, it cannot shrink into $r < 2GM$. There are mass hierarchy of what they can eventually become to

White dwarf (sun will become to) supported by degeneracy pressure of the electrons

Neutron star (denser) supported by degeneracy pressure of the neutron, electrons are combined with protons via beta decays.

Black hole (densest) singularities

Consider a light ray radially travel to/away the origin.

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 = 0$$

The slope of the world line of the light beam wrt observer at ∞

$$\frac{dt}{dr} = \pm(1 - \frac{2GM}{r})^{-1} \tag{3.36}$$

the $\pm$ give both directions traveling to or away from the origin. As $r \to 2GM$, $dt \to \infty$, i.e. the clock stops.

When $r = 2GM$, $g_{rr} = \infty$, it is a coordinate singularity, not a physical singularity. Meaning that there are interesting thing going on at $r = 2GM$, but certainly lights can go into that region. It is more convenient to change coordinates that will removed the singularity, e.g. "Tortoise Coordinate"

$$r \to r^* = r + 2GM \ln \left| \frac{r}{2GM} - 1 \right|$$

$$dr^* = dr + (\frac{r}{2GM} - 1)^{-1} dr = \frac{r}{r - 2GM} dr \implies dr = (1 - \frac{2GM}{r}) dr^*$$

thus

$$0 = ds^2 = \left(1 - \frac{2GM}{r(r^*)}\right)(-dt^2 + dr^{*2}) + \underbrace{r(r^*)^2 d\theta^2 + r(r^*)^2 \sin^2\theta d\phi^2}_{=0}$$

Now the slope

$$\frac{dt}{dr^*} = \pm 1 \tag{3.37}$$

and $g_{\mu\nu}$ is finite.

Because wrt clock at ∞, time stops at $r = 2GM$, thus t is not a good reference variable. So we change to Eddington-Finkelstein coordinate. Don't change r , change t

$$t \to t^* = t + r^* = t + r + 2GM \ln \left| \frac{r}{2GM} - 1 \right| \tag{3.38}$$

with the understand that when $r = 2GM$, $t = \infty$, and we will show that t^* is actually a finite constant for light infalling, which is exact what we want t^* to be, i.e. affine to the proper time of light.

By (3.38)

$$dt^* = dt + (1 - \frac{2GM}{r})^{-1} dr \tag{3.39}$$

then

$$0 = ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^{*2} + \underbrace{dt^* dr + dr dt^*}_{2dt^* dr} + \underbrace{r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}_{=0} \tag{3.40}$$

The nicest thing about this metric is that

$$g = \det g_{\mu\nu} = -1$$

so the transformation is valid for all $r > 0$. This allows us to study light inside and outside $r = 2GM$ with one continuous transformation. Since t^* is constant for infalling light, we can use the same standard clock at ∞ to calibrate the clock inside $2GM$.

One Solution to (3.40) is just

$$dt^* = 0 \implies t^* = \text{const} \tag{3.41}$$

or

$$\frac{dt}{dr} = -(1 - \frac{2GM}{r})^{-1} > 0 \quad \text{for } r > 2GM \tag{3.42}$$

so this describes infalling, and although (3.42) does not work for $r < 2GM$, but (3.41) is true continuously inside $r < 2GM$.

The other solution to (3.40) is

$$\left(1 - \frac{2GM}{r}\right) dt^* = 2dr \tag{3.43}$$

so this describes outgoing. Notice as light travels from inside and approaches $r = 2GM$, the slope is negative, so "outgoing" is not really outgoing and the slope becomes $-\infty$, so from outside and inside observes views $t^* \to \infty$, it takes ∞ amount of time to cross $2GM$, hence no thing can cross $2GM$, which is the defining feature of a black hole and we say $r = 2GM$ sphere is an event horizon.

The interesting thing about Eddington-Finkelstein metric is that infalling worldline slope

$$\frac{dt^*}{dr} = 0$$

so it is nice to plot draw many little light cones $t^* - r$ v.s. r then the left side of a light cone

$$\frac{dt^* - r}{dr} = -1$$

the right side of a light cone is

$$\frac{dt^* - r}{dr} = \frac{2}{1 - \frac{2GM}{r}} - 1 = \frac{1 + \frac{2GM}{r}}{1 - \frac{2GM}{r}}$$

3.7 Derivation Schwarzschild Metric

We now derive Schwarzschild metric. Consider a generic static, spherical symmetric metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + \underbrace{r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}_{r^2 d\Omega^2}$$

put $e^{\alpha,\beta(r)}$ to make calculation simple. Why is that $g_{\Omega\Omega} = r^2$? Because we can rescale r so that $g_{\Omega\Omega} = r^2$.

By (3.18) we need to compute Ricci tensor.

$$\begin{aligned} R_{tt} &= e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha] = 0 \\ R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta = 0 \end{aligned}$$

so

$$e^{-2(\alpha-\beta)}R_{tt} + R_{rr} = \frac{1}{r}\partial_r(\alpha+\beta) = 0$$

or

$$\alpha = -\beta + \text{const}$$

and rescale t so that

$$\alpha = -\beta$$

then by

$$R_{\theta\theta} = e^{-2\beta}[r\partial_r(\beta-\alpha) - 1] + 1 = 0$$

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \implies e^{2\alpha} = 1 - \frac{R_0}{r}$$

so constant R_0. By the weak field approximation (2.12)

$$g_{tt} = -(1 - \frac{2GM}{r})$$

so

$$e^{2\alpha} = (1 - \frac{2GM}{r})$$

QED.

In fact we have uniqueness: but we are not going to prove this.

Theorem 7. *(Birkhoff)If one looks for a static, spherical symmetric, vacuum solution to Einstein equation, then Schwarzschild metric is the only solution.*

Finally we consider that a massive spherical star mass M, radius R sitting at the origin. Assume the star is made of perfect fluid. Clear outside R, we have Schwarzschild metric. What abut inside?

Consider a generic static, spherical symmetric metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + \underbrace{r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}_{r^2 d\Omega^2}$$

By (2.44)

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + Pg^{\mu\nu} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\beta}P & & \\ & & r^2 P & \\ & & & r^2 P \sin^2\theta \end{pmatrix}$$

then compute the relevant Einstein tensors, cf (2.17), (3.2), (3.5), (3.6)

$$G_{tt} = \frac{1}{r^2} e^{2(\alpha-\beta)}(2r\partial_r\beta - 1 + e^{2\beta}) = 8\pi G e^{2\alpha}\rho$$

so

$$-r\frac{de^{-2\beta}}{dr} - e^{-2\beta} + 1 = 8\pi G\rho r^2$$

then the solution is

$$e^{-2\beta} = 1 - \frac{2Gm(r)}{r}$$

where

$$m(r) = 4\pi \int_0^r dr' r'^2 \rho(r')$$

Then compute

$$G_{rr} = \frac{1}{r^2}(2r\partial_r\alpha + 1 - e^{2\beta}) = 8\pi G e^{2\beta} P \tag{3.44}$$

so

$$2r\partial_r\alpha = 8\pi G r^2 e^{2\beta} P - 1 + e^{2\beta}$$

one can show that

$$e^{2\alpha} = \frac{3}{2}(1 - \frac{2Gm(R)}{R})^{1/2} - \frac{1}{2}(1 - \frac{2Gmr^2}{R^3})^{1/2} \quad r < R$$

Lecture 16 (10/31/13)

One can see immediately that if

$$m(R) > \frac{4R}{9G}$$

there exists some r_ϵ region around the origin where $-g_{tt} = e^{2\alpha}$ will be negative. We will get unstable solution, so $R \downarrow$ then it becomes more unstable therefore the star will keep shrinking and become a singularity, a black hole.

More precise statement is to look at pressure $P(r)$ and see if it gets ∞ large. From (3.44)

$$\frac{d\alpha}{dr} = \frac{Gm(r) + 4\pi G r^3 P}{r(r - 2Gm(r))}$$

then by $\nabla_\mu T^{\mu\nu} = 0$,

$$(\rho + P)\frac{d\alpha}{dr} = -\frac{\partial P}{\partial r}$$

so we get Tolman–Oppenheimer–Volkoff equation

$$\frac{dP}{dr} = -(\rho + P)\frac{Gm(r) + 4\pi G r^3 P}{r(r - 2Gm(r))}$$

4 Black Holes

4.1 Kruskal–Szekeres Coordinates

As we mentioned before in Eddington-Finkelstein coordinate the slopes of a light cone have different value on the two sides of the light cone. We introduce Kruskal–Szekeres coordinates (T, R, θ, ϕ), whose light cone slope is always

$$\frac{dT}{dR} = \pm 1$$

The textbook defines

$$\begin{aligned} T &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh \frac{t}{4GM} &\quad (4.1) \\ R &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh \frac{t}{4GM} &\quad (4.2) \end{aligned}$$

which makes a little bit awkward for $r < 2GM$, but actually to derive the metric

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dR^2) + r^2 d\Omega^2 \tag{4.3}$$

from Schwarzschild metric. We only need the following implicit definitions of T, R

$$\begin{cases} T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} & (1) \\ \frac{T}{R} = \tanh \frac{t}{4GM} & (2) \end{cases} \tag{4.4}$$

This will work for $r < 2GM$ as well.

First let's derive (4.3),

$$2TdT - 2RdR = e^{r/2GM} \frac{r}{4G^2M^2} dr$$

$$\frac{dT}{R} - \frac{T}{R^2} dR = \frac{1}{4GM}(1 - \tanh^2 \frac{t}{4GM})dt = \frac{1}{4GM}(1 - \frac{T^2}{R^2})dt$$

then

$$\begin{aligned} dr^2 &= \frac{64G^4M^4}{r^2} e^{-r/GM}(TdT - RdR)^2 \\ dt^2 &= \frac{16G^2M^2}{(R^2 - T^2)^2}(RdT - TdR)^2 = \frac{64G^2M^2}{r^2} \frac{1}{(1 - \frac{2GM}{r})^2} e^{-r/GM}(RdT - TdR)^2 \end{aligned}$$

so

$$\begin{aligned}
-(1-\frac{2GM}{r})dt^2+(1-\frac{2GM}{r})^{-1}dr^2 &= \frac{64G^4M^4}{r^2}\frac{1}{1-\frac{2GM}{r}}e^{-r/GM}(T^2-R^2)(dT^2-dR^2)\\
&= \frac{32G^3M^3}{r}e^{-r/GM}(-dT^2+dR^2)
\end{aligned}$$

This proves (4.3).

Although the definitions of T, R, look the same, from the metric (4.3) we see that T is time like and R is space like and the $2GM$ positivity issue of the g_{TT} is completely resolved.

From (4.4) (1)

$$T=\pm R \iff r=2GM$$

being a null surface, event horizon.

One can draw Kruskal diagram, by (4.4). Constant r gives

$$T^2-R^2=\text{const}\to\begin{cases} >0 & r<2GM\\ <0 & r>2GM\end{cases}$$

hence we get hyperbola oriented in R axis if $r<2GM$, and we get hyperbola oriented in T axis if $r>2GM$. Constant

$$\frac{T}{R}=\text{const}\to\begin{cases} >0 & t>0\\ <0 & t<0\end{cases}$$

So the worldline for a infalling photon will look like the following, starting at $t=0$, $r=r_0>2GM$, so the initial position is

$$(R,T)=\Big(R_0=(\frac{r_0}{2GM}-1)^{1/2}e^{r_0/GM}),0\Big) \tag{4.5}$$

then it goes on a 45^0 line with slope -1, as time goes on it will pass $R=T$ line which is the surface $r=2GM$, then it will end up on the upper branch of the hyperbola $r=0$

$$T=\sqrt{R^2+1}$$

By in the definition (4.4), changing $R \to -R$ and $T \to -T$ is completely symmetric, people still would like to impose that R is spacelike coordinate so the initial position R_0 in (4.5) must be positive then we see that the RT plane is separated by $R = -T$. Upper half describes a black hole, and lower half plane contains

$$T = -\sqrt{R^2 + 1}$$

that cannot be reached from outside, describing a white hole.

We now study more general black holes, such as rotating back holes, non-spherical collapse. There are much more topics than we have time for. Like singularity theorem, cosmic censorship conjecture, area theorem, and some detection methods for searching black holes, such as X ray binary, Doppler shifts in spectra. We will focus on rotating black holes and a recent experiment testing general relativity: gravity probe B.

4.2 Gyroscope Equation

Lecture 17 (11/7/13)

Consider a spin object, in its rest frame

$$U^\mu = (1, 0, 0, 0)$$

and likewise we define a spin vector in its rest frame

$$S^\mu = (0, \vec{S})$$

and called

$$S = \sqrt{S^\mu S_\mu}$$

total spin. Assume that it is a constant of motion (independent of τ), which is somehow to say no torque acting on the object (some people call this effect as "frame dragging", but it should not be interpreted literally, because this is no dragging force and spin is conserved.) In local internal frame it gives

$$\frac{dS^\alpha}{d\tau} = 0$$

Following the same logic we derived transport equations (cf (2.31) to (2.33)), and use

$$U^\alpha S_\alpha = 0 \tag{4.6}$$

so in general spacetime

$$\frac{dS^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} S^\beta U^\gamma = 0$$

this is known as the gyroscope equation. One check it indeed gives

$$S^a S_\alpha = \text{const}$$

$$S^\alpha U_\alpha = \text{const}$$

along geodesic.

Consider a spinning test object moving in the Schwarzschild metric in a circular orbit $r = R$, $\theta = \pi/2$. So the 4 velocity points in ϕ direction

$$U^\phi = \frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \Omega U^t$$

$$U^\mu = U^t(1, 0, 0, \Omega) \tag{4.7}$$

Ω is orbital angular velocity in the local inertial frame, which we have computed before (3.31).

Suppose we have some spin S_*(which is an invariant constant number) initially points in r direction. By the symmetry of the z direction, S^θ should remain 0 for all time. First find a relation between S^t and S^ϕ. By (4.6), (4.7), (3.31) and (3.32)

$$-(1 - \frac{2GM}{R})S^t U^t + R^2 S^\phi U^\phi = 0$$

so

$$S^t = R^2\Omega(1 - \frac{2GM}{R})^{-1} S^\phi$$

Then find relation between S^r and S^ϕ. From gyroscope equation, and compute

Christoffel symbols for Schwarzschild metric, the relevant non-zero ones are

$$\begin{aligned}
\Gamma^r_{tt} &= \frac{GM}{R^2}(1-\frac{2GM}{R}) \\
\Gamma^r_{rr} &= -\frac{GM}{R^2}(1-\frac{2GM}{R})^{-1} \\
\Gamma^r_{\theta\theta} &= -(R-2GM) \\
\Gamma^r_{\phi\phi} &= -(R-2GM)\sin^2\theta = -(R-2GM) \\
\Gamma^\phi_{r\phi} &= \frac{1}{R} \\
\Gamma^\phi_{\theta\phi} &= \frac{1}{\tan\theta} = 0
\end{aligned}$$

then

$$\begin{aligned}
\frac{dS^r}{d\tau} + \Gamma^r_{\phi\phi} S^\phi U^\phi + \Gamma^r_{tt} S^t U^t &= 0 \\
\frac{dS^\phi}{d\tau} + \Gamma^\phi_{r\phi} S^r U^\phi &= 0 \\
U^t d\tau = \frac{dt}{d\tau} d\tau &= dt
\end{aligned}$$

or

$$\begin{cases} \frac{dS^r}{dt} - (R-3GM)\Omega S^\phi = 0 \\ \frac{dS^\phi}{dt} + \frac{\Omega}{R} S^r = 0 \end{cases}$$

we get coupled SHO

$$\frac{d^2 S^\phi}{dt^2} + (1-\frac{3GM}{R})\Omega^2 S^\phi = 0$$

associated with frequency

$$w = \sqrt{1-\frac{3GM}{R}}\,\Omega$$

with the correct normalization S_* and initial condition $t=0$, $S^\phi = 0$

$$\begin{cases} S^r(t) = S_*(1-\frac{2GM}{R})^{1/2}\cos wt \\ S^\phi(t) = -S_*(1-\frac{2GM}{R})^{1/2}\frac{\Omega}{wR}\sin wt \end{cases}$$

After one cycle, the angle of S^μ changes by

$$\left(\frac{S^\mu(2\pi/\Omega)}{S_*}\right)\cdot\hat{e}^\mu_r = \cos 2\pi\sqrt{1-\frac{3GM}{R}}$$

Hence the precession is

$$\Delta\phi_{geodetic} = 2\pi - \left(2\pi\sqrt{1-\frac{3GM}{R}}\right) \tag{4.8}$$

knows as geodetic precession.

If $GM/R \ll 1$, (it is definitely true for the case below)

$$\Delta\phi_{geodetic} \approx \frac{3\pi GM}{R} \tag{4.9}$$

Gravity-Probe B is a satellite that carries some gyroscopes in circular orbit around the earth

$$\Delta\phi_{geodetic} \approx \frac{3\pi GM_{earth}}{c^2 R_{earth}}\frac{R_{earth}}{R} = 6.5\times 10^{-9}\left(\frac{R_{earth}}{R}\right)\text{rad}$$

$R_{earth} = 6378$km. The experiment was done in 2008, confirming with an accuracy 0.5%.

Reversely one can also use (4.9) to find M by putting some gyroscope to some planet and measuring $\Delta\phi_{geodetic}$.

4.3 Metric Due to Rotating Body

After studying the motion of a test spin object in Schwarzschild metric, we now study the effect of distortion to spacetime due to a massive rotating body. Assume it is spherical and slowly rotating. We claim the metric has form

$$ds^2 = (ds^2)_{Schwartz} - \frac{4GJ}{c^3R^2}\sin^2\theta(rd\phi)(cdt) + O(J^2) \tag{4.10}$$

where $\vec{J} \parallel \hat{z}$. The correction has order v/c because

$$J \sim I\Omega \sim MR^2\Omega \sim MvR$$

or

$$\frac{GJ}{c^3R^2} \sim \frac{GM}{c^2R}\frac{v}{c}$$

and it is called gravitomagnetic effect, because J will act like a magnetic dipole moment.

Now we put in a test spin object (gyroscope) in such spacetime (4.10). We will get a different kind of precession than (4.8), known as Lense–Thirring precession. We choose gyroscope to freely fill along $\hat{z}$ and it's initial spin is perpendicular to $\hat{z}$, i.e.

$$\begin{aligned} U^\alpha &= (U^t, 0, 0, U^z) \\ S^\alpha &= (0, S^x, S^y, 0) \end{aligned}$$

in such a set up we will get maximum Lense–Thirring precession and get 0 geodetic precession (4.8). Because we are moving vertical not circular as before, it is easier to work in Cartesian coordinate, write (4.10)

$$ds^2 = (ds^2)_{Schwartz} - \frac{4GJ}{c^3R^2}(cdt)\frac{xdy - ydx}{R}$$

Following the same steps before, to solve gyroscope equation, first compute non-zero Christoffel

$$\begin{aligned} \left(\Gamma^x_{ty}\right)_{x,y=0} &= \frac{2GJ}{c^2z} \\ \left(\Gamma^y_{tx}\right)_{x,y=0} &= -\frac{2GJ}{c^2z} \end{aligned}$$

then we get coupled SHO for S^x, S^y and the precession rate is

$$\Omega_{LT} = \frac{2GJ}{c^2z^3} \tag{4.11}$$

and it precess in the same direction as J.

For the earth

$$\Omega_{LT} = .22'' \left(\frac{6378\text{km}}{z}\right)^2 /\text{year}$$

and Gravity-Probe B confirmed with an accuracy 1%.

Reversely one can also use (4.11) to find J by putting some gyroscope to some planet and measuring Ω_{LT}.

4.4 Kerr Metric

In (4.10) we ignore J^2 term, for slow rotation, small mass like the earth, it is okay. But for Black Holes we need the complete expression. It is called Kerr metric,

$$\begin{aligned} ds^2 &= -(1-\frac{2GMr}{\rho^2})dt^2 - \frac{4GMar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ &\quad +(r^2+a^2+\frac{2GMra^2\sin^2\theta}{\rho^2})\sin^2\theta d\phi^2 \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} a &= \frac{J}{M} \\ \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2GMr + a^2 \end{aligned}$$

and (t, r, θ, ϕ) is Boyer–Lindquist coordinates.

Lecture 18 (11/12/13)

One can check when $r \gg GM$, $r \gg a$,

$$(4.12) \to (4.10) \tag{4.13}$$

when $a = 0$, not rotating

$$(4.12) \to \text{Schwarzschild metric}$$

(4.12) is independent of t and ϕ, so two killing vectors

$$\xi^\mu = (1,0,0,0) \quad \eta^\mu = (0,0,0,1) \tag{4.14}$$

To see it produces BH, we need to look at singularities.

1)

$$\rho = 0 \iff r = 0, \theta = \pi/2$$

this is a real spacetime singularity, whose curvature is ∞.

2)

$$\Delta = 0 \iff r_{\pm} = GM \pm \sqrt{G^2M^2 - a^2} \text{ for } a \leq GM \tag{4.15}$$

We call

$$r_+ = \text{BH horizon}$$

Why not r_-? because we're only interested in life outside of BH. The horizon $r = r_+$ is a 3D surface at which light can neither escape nor fall in, i.e. a null surface. Suppose we have a light ray trying come out of the horizon

$$V^\mu = (V^t, V^r, V^\theta, V^\phi)$$

Because it's light, it's norm is 0, by (4.12), when $r = r_+$

$$V^\mu V_\mu = 0 \neq g_{tt}(V^t)^2 + \underbrace{g_{r_+r_+}}_{\infty}(V^r)^2 + ...$$

hence we cannot satisfy $V^\mu V_\mu = 0$ unless $V^r = 0$ at $r = r_+$.

Consider a light ray moving on the surface, so its trajectory is given by

$$V^\mu = (V^t, 0, V^\theta, V^\phi) \tag{4.16}$$

Because it's light, it's norm is 0, by (4.12)

$$V^\mu V_\mu = 0 = g_{tt}(V^t)^2 + 2g_{t\phi}(V^t)(V^\phi) + g_\phi g_\phi(V^\phi)^2 + g_{\theta\theta}(V^\theta)^2$$

we simplify

$$(\frac{2GMr_+ \sin\theta}{\rho_+})^2(V^\phi - \frac{a}{2GMr_+}V^t)^2 + \rho_+^2(V^\theta)^2 = 0$$

$$\rho_+ = r_+^2 + a^2\cos^2\theta$$

we solve

$$V^\theta = 0 \quad V^\phi = \frac{a}{2GMr_+}V^t$$

so if we normalize $c = 1$,

$$V^\mu = (1, 0, 0, \Omega_H) \tag{4.17}$$

then

$$\Omega_H = \frac{d\phi}{dt} = \frac{d\phi/d\lambda}{dt/d\lambda} = \frac{V^\phi}{V^t} = \frac{a}{2GMr_+}$$

so the light is moving on the horizon surface and making circles, or we can integrate this (since for BH, the whole space is empty only singularity point at the origin) as the whole space is rotating with angular speed Ω_H. If the BH is a singular point, what does J mean? J in fact means if we put a gyroscope at ∞, then we have (4.13), then J corresponds to the precession rate, cf (4.11).

It is easy to see the horizon surface is not a physical sphere, and don't get tricked by the fact that from (4.15) r_+ looks like a constant, because we are in Boyer–Lindquist coordinates, not physical coordinates. Put

$$r = r_+, \ t = \text{const}$$

in (4.12). We get

$$ds^2 = \rho_+^2 d\theta^2 + (r_+^2 + a^2 + \frac{2GMr_+a^2\sin^2\theta}{\rho_+^2})\sin^2\theta d\phi^2$$

using (4.15), substituting $a^2 = 2GMr_+ - r_+^2$, we get

$$ds^2 = \rho_+^2 d\theta^2 + (\frac{2GMr_+}{\rho_+})^2 \sin^2\theta d\phi^2 \tag{4.18}$$

which will be a sphere iff

$$2GMr_+ = \rho_+^2 \iff a = 0$$

and integrating (4.18), we find the horizon surface area is

$$A = 8\pi GMr_+$$

In (4.16), we say that if a light ray that moves on the horizon surface, it will move in a horizontal plane cf (4.17). That is a very situation. In general the geodesic path will not remain horizontal which is in direct contract to Schwarzschild metric. However by rotational antisymmetry (4.14), geodesic path will remain in the equatorial plane $\theta = \pi/2$ if it starts off on the equatorial plane, so (4.12) becomes

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 - \frac{4GMa}{r}d\phi dt + \frac{r^2}{\Delta}dr^2 + (r^2 + a^2 + \frac{2GMa^2}{r})d\phi^2$$

$$U = (U^t, U^r, 0, U^\phi)$$

with the normalization

$$U^\mu U_\mu = -1 \tag{4.19}$$

In terms of killing vectors (4.14)

$$\begin{aligned} e &= -\xi U \\ l &= \eta U \end{aligned}$$

are conversed quantities: energy per mass and momentum per mass respectively. Because the metric has off diagonal elements, we get

$$\begin{aligned} e &= -g_{tt}U^t - g_{t\phi}U^\phi \\ l &= g_{\phi t}U^t + g_{\phi\phi}U^\phi \end{aligned}$$

solve for U^t, U^ϕ

$$\begin{aligned} U^t &= \frac{1}{\Delta}\left((r^2 + a^2 + \frac{2GMa^2}{r})e - \frac{2GMa}{r}l\right) \\ U^\phi &= \frac{1}{\Delta}\left((1 - \frac{2GM}{r})l + \frac{2GMa}{r}e\right) \end{aligned}$$

Combining with (4.19), we get an expression explicitly involving U^r

$$\frac{1}{2}(e^2-1)=\frac{1}{2}(U^r)^2+\underbrace{-\frac{GM}{r}+\frac{l^2-a^2(e^2-1)}{2r^2}-\frac{M(l-ae)^2}{r^3}}_{V_{eff}(r)} \tag{4.20}$$

i.e. the LHS is total energy, $\frac{1}{2}(U^r)^2$ is radial kinetic energy. When $a=0$, we are back to (3.26).

If we are interested in light ray trajectory, we will replace (4.19) by

$$U^\mu U_\mu = 0$$

then we find

$$\frac{1}{b^2}=\frac{1}{l^2}(U^r)^2+\underbrace{\frac{1}{r^2}\left(1-(\frac{a}{b})^2-\frac{2GM}{r}(1-\sigma\frac{a}{b})^2\right)}_{W_{eff}(r)}$$

where

$$\begin{aligned} b &= \text{impact parameter} = \left|\frac{l}{e}\right| \\ \sigma &= \text{sign} l \end{aligned}$$

when $a=0$, we are back to (3.35).

Lecture 19 (11/14/13)

Unlike for Schwarzschild metric at the horizon $g_{rr}\to\infty$ and g_{tt} flips signs, for Kerr metric g_{tt} flips signs before it gets to the horizon. We call $r=r_e$ s.t.

$$g_{tt}=1-\frac{2GMr}{\rho^2}=0 \implies r_e=GM+\sqrt{G^2M^2-a^2\cos\theta}$$

the ergosphere (not a sphere either) see Carroll figure 6.7 page 264. Horizon is where light cannot escape, and ergosphere is where mass object cannot escape except some cases. Suppose we have a test object trying come out of the ergosphere radially

$$U^\mu=(U^t,U^r,0,0)$$

with the normalization,

$$U^\mu U_\mu = -1 \neq \underbrace{g_{tt}}_{>0}(U^t)^2 + \underbrace{g_{rr}}_{>0}(U^r)^2 + ...$$

hence no solution, no matter how large U^r is. However since the $g_{t\phi}$ term is negative, it is possible to some rotational energy out known as Penrose process.

5 Gravitational Waves

5.1 Introduction and Detection

Gravitational waves are ripples in spacetime, more precisely changes in spacetime curvatures.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.1}$$

$h_{\mu\nu}$ is the amplitude of the gravitational wave. Because gravity is weak, $F_{gravity}/F_{em} = 10^{-36}$, usually

$$|h| \sim 10^{-22}$$

it is not detectable unless it is from some significant sources: Binary pulsar (e.g. Hulse-Taylor 1974 experiment, Nobel prize 1993), supernova explosion, black hole collapse, big bang.

For example: a plane wave propagating in z direction, vibrating in x, y directions (as we will see this is + polarized), is given by

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z) \tag{5.2}$$

where f is any function. Later we will show that such metric

$$ds^2 = -dt^2 + [1 + f(t-z)]dx^2 + [1 - f(t-z)]dy^2 + dz^2 \tag{5.3}$$

solves the linearized Einstein equation.

How to detect gravitational waves for above $h_{\mu\nu}$? Putting two test particles at rest, one at the origin, the other on the $\hat{x}$ axis at x_0 initially

$$U_A = U_B = (1, 0, 0, 0)$$

By the geodesics equation (2.18)

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{tt} U^t U^t = 0$$

Now let a gravitational wave passing through $\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$

$$\Gamma^\alpha_{tt} \to \Gamma^\alpha_{tt} + \delta\Gamma^\alpha_{tt}$$

By (5.3),

$$\delta\Gamma^\alpha_{tt} = 0$$

so

$$\frac{dU^\alpha}{d\tau} = 0$$

i.e. the two particles will remain stationary. What changes? The separation by (5.2)

$$\int_0^{x_0} dx \sqrt{1 + h_{xx}(t, x)} \approx x_0 \left(1 + \frac{f(t)}{2}\right) \tag{5.4}$$

if f is an oscillatory function, then the separation will oscillate too.

Next time we will see there is another solution to linearized Einstein equation independent to (5.2)

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f_\times(t - z)$$

is $\times$ polarized. So the general solution is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.5}$$

5.2 Linearized Einstein Equation

Lecture 20 (11/19/13)

As before we assume $h_{\mu\nu}$ is small, so we will keep upto the 1st order. First linearized Einstein equation.

Christoffel (2.17) in 1st order

$$\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\eta^{\delta\alpha}(\partial_\gamma h_{\alpha\beta} + \partial_\beta h_{\alpha\gamma} - \partial_\alpha h_{\beta\gamma})$$

Riemann tensor (3.2) in 1st order, forget about $\Gamma\Gamma$ term

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha}$$

or

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= \eta_{\mu\lambda}\partial_\alpha\Gamma^\lambda_{\nu\beta} - \eta_{\mu\lambda}\partial_\beta\Gamma^\lambda_{\nu\alpha} \\ &= \frac{1}{2}(\partial_\alpha\partial_\nu h_{\mu\beta} + \partial_\mu\partial_\beta h_{\nu\alpha} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\beta h_{\mu\alpha}) \end{aligned}$$

Ricci tensor (3.5) in 1st order

$$\begin{aligned} R_{\nu\beta} &= \eta^{\alpha\mu}R_{\mu\nu\alpha\beta} \\ &= \frac{1}{2}(\partial_\alpha\partial_\nu h^\alpha{}_\beta + \partial_\mu\partial_\beta h^\mu{}_\nu - \underbrace{\partial^\alpha\partial_\alpha}_{\Box} h_{\nu\beta} - \partial_\nu\partial_\beta \underbrace{\eta^{\alpha\mu}h_{\mu\alpha}}_{\equiv h}) \end{aligned}$$

Ricci scalar (3.6) in 1st order

$$\begin{aligned} R &= \eta^{\nu\beta}R_{\nu\beta} = \frac{1}{2}(\partial_\alpha\partial_\nu h^{\alpha\nu} + \partial_\mu\partial_\beta h^{\mu\beta} - \Box\underbrace{\eta^{\nu\beta}h_{\nu\beta}}_{h} - \underbrace{\eta^{\nu\beta}\partial_\nu\partial_\beta}_{\Box} h) \\ &= \partial_\alpha\partial_\nu h^{\alpha\nu} - \Box h \end{aligned}$$

Einstein tensor (3.7) in 1st order

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\
&= \frac{1}{2}(\partial_\alpha\partial_\nu h^\alpha{}_\mu + \partial_\mu\partial_\beta h^\beta{}_\nu - \Box h_{\nu\mu} - \partial_\nu\partial_\mu h - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \eta_{\mu\nu}\Box h)
\end{aligned}$$

which is indeed linear.

5.3 Vacuum Gravitational Waves

$$G_{\mu\nu} = 0$$

It turns out we can choose a good coordinate so that above equation becomes

$$\Box h_{\alpha\beta} = 0$$

Why are we free to do so? because in the end of day we only measure physical distance cf (5.4).

$h_{\mu\nu}$ is symmetry as it is true for any metric. For computational convenience, we put

$$\begin{aligned}
h_{00} &= -2\Phi \\
h_{0i} &= w_i \\
h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}
\end{aligned}$$

where

$$\begin{aligned}
s_{ij} &= \frac{1}{2}(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}) = \text{traceless} \\
\Psi &= -\frac{1}{6}\delta^{kl}h_{kl}
\end{aligned}$$

Then the metric

$$ds^2 = -(1+2\Phi)dt^2 + w_i(dtdx^i + dx^i dt) + [(1-2\Psi)\delta_{ij} + 2s_{ij}]dx^i dx^j$$

From there one can show Φ,Ψ and w_i are constants of motions. See Carroll page 281-282, notice the notation Carroll uses

$$\begin{aligned}\partial_{[j,w_i]} &= \frac{1}{2}(\partial_j\partial_{w_i} - \partial_{w_i}\partial_j)\\ \partial_{(j,w_i)} &= \frac{1}{2}(\partial_j\partial_{w_i} + \partial_{w_i}\partial_j)\end{aligned}$$

same notation in (3.4). So only s_{ij} describes propagating. It can also be shown that choosing transverse traceless (TT) gauge, for gravitational wave in vacuum Φ,Ψ and w_i are zero and we have a wave equation for

$$\Box s_{ij} = 0 \quad \text{so is for } \Box h^{TT}_{\mu\nu} = 0 \tag{5.6}$$

with gauge condition

$$\partial_\mu h^\mu_{\ \nu} - \frac{1}{2}\partial_\nu h = 0 \tag{5.7}$$

Try plane wave

$$h^{TT}_{\mu\nu} = c_{\mu\nu}e^{ik_\lambda x^\lambda}$$

By (5.6),

$$\eta^{\sigma\rho}\partial_\sigma\partial_\rho h^{TT}_{\mu\nu} = 0 \implies -\eta^{\sigma\rho}k_\sigma k_\rho h^{TT}_{\mu\nu} = 0$$

or

$$k^\rho k_\rho = 0$$

showing k is a null vector. In other words, gravitational wave propagates at the speed light.

Applying $\eta^{\alpha\nu}$ to (5.7),

$$\partial_\mu h^{\mu\alpha} = 0 \tag{5.8}$$

implies

$$c_{\mu\alpha}k^\mu = 0$$

So if we choose the wave to propagate in z direction, i.e.

$$k^\mu = (w, 0, 0, k_z)$$

then

$$c_{3\alpha} = 0$$

because $c_{0\alpha}$ is 0. Hence $h_{\mu\nu}$ has only 4 non-zero entries, cf (5.5). The + polarization gives vibration in x, y directions; the $\times$ polarization gives vibration in 45^0 lines off x, y directions. One also choose different basis

$$\begin{aligned} h_R &= \frac{1}{\sqrt{2}}(h_+ + ih_\times) = \text{right hand circular polarization} \\ h_L &= \frac{1}{\sqrt{2}}(h_+ - ih_\times) = \text{left hand circular polarization} \end{aligned}$$

and thumb points in k direction.

5.4 Gravitational Waves with a Source

Lecture 21 (11/21/13)

Last time, the gauge transformation gives a simple wave equation without source. It turns out that when the RHS of Einstein equation is not 0, we don't have such simple expression. The trick is to define another wave amplitude

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$$

called 'trace-reversed' perturbation because

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$$

Then Einstein equation becomes

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{5.9}$$

with gauge condition

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \tag{5.10}$$

Use Green function method, solve

$$\bar{h}_{\mu\nu}(x^\lambda) = -16\pi G \int d^4y G(x^\lambda - y^\lambda) T_{\mu\nu}(y^\lambda)$$

where

$$\Box_x G(x^\lambda - y^\lambda) = \delta^{(4)}(x^\lambda - y^\lambda)$$

The retarded green function:

$$G(x^\lambda - y^\lambda) = -\frac{1}{4\pi\,|\vec{x} - \vec{y}|}\delta(|\vec{x} - \vec{y}| - (x^0 - y^0))\theta(x^0 - y^0)$$

therefore

$$\bar{h}_{\mu\nu}(x^\lambda) = 4G\int d^3y \frac{T_{\mu\nu}(t_{tr}, \vec{y})}{|\vec{x} - \vec{y}|}$$

where

$$t_{tr} = t - |\vec{x} - \vec{y}|$$

TO do further, we use far field approximation, replacing

$$|\vec{x} - \vec{y}| \to r$$

First Fourier in time

$$\begin{aligned}\tilde{\bar{h}}_{\mu\nu}(w, \vec{x}) &= \frac{1}{\sqrt{2\pi}}\int dt e^{iwt}\bar{h}_{\mu\nu}(t, \vec{x}) \\ &= \frac{4G}{\sqrt{2\pi}}\int dt \int d^3y e^{iwt_{tr}} e^{iwr}\frac{T_{\mu\nu}(t_{tr}, \vec{y})}{r}\end{aligned}$$

then change integrating variable

$$dt \to dt_{tr}$$

and Fourier again

$$\begin{aligned}\tilde{\bar{h}}_{\mu\nu}(w, \vec{x}) &= \frac{4G}{\sqrt{2\pi}}\int dt_{tr} \int d^3y e^{iwt_{tr}} e^{iwr}\frac{T_{\mu\nu}(t_{tr}, \vec{y})}{r} \\ &= \frac{4G}{\sqrt{2\pi}}\frac{e^{iwr}}{r}\int d^3y \tilde{T}_{\mu\nu}(w, \vec{y})\end{aligned}$$

Fourier (5.10) in time,

$$iw\tilde{\bar{h}}^{0\nu} + \partial_i \tilde{\bar{h}}^{i\nu} = 0 \qquad (5.11)$$

Hence we only need to known $\tilde{\bar{h}}^{ij}$, 9 components, then we know $\tilde{\bar{h}}^{0i}$, another 6 components, then we know $\tilde{\bar{h}}^{00}$, all 16 components. Lucky to compute $\int d^3y\tilde{T}_{ij}(w,\vec{y})$, we only need to know $\tilde{T}_{00}$, because by conservation of energy momentum (1.22), we get in time Fourier

$$iw\tilde{T}^{0\nu} + \partial_i\tilde{T}^{i\nu} = 0$$

so

$$\begin{aligned}
\int d^3y\tilde{T}^{ij}(w,\vec{y}) &= \underbrace{\int d^3y\partial_k(y^i\tilde{T}^{kj})}_{0} - \int d^3yy^i\partial_k\tilde{T}^{kj} \\
&= iw\int d^3yy^i\tilde{T}^{0j} \\
&= \frac{iw}{2}\int d^3y(y^i\tilde{T}^{0j} + y^j\tilde{T}^{0i}) \\
&= \frac{iw}{2}\int d^3y[\partial_k(y^iy^j\tilde{T}^{0k}) - y^iy^j\partial_k\tilde{T}^{0k}] \\
&= -\frac{w^2}{2}\underbrace{\int d^3yy^iy^j\tilde{T}^{00}(w,\vec{y})}_{\tilde{I}_{ij}(w)}
\end{aligned}$$

where

$$I_{ij}(t) = \int d^3yy^iy^jT^{00}(t,\vec{y}) \tag{5.12}$$

is called quadrupole moment. Thus

$$\tilde{\bar{h}}_{ij}(w,\vec{x}) = -2Gw^2\frac{e^{iwr}}{r}\tilde{I}_{ij}(w)$$

so

$$\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{r}\frac{d^2I_{ij}(t_{tr})}{dt^2} \tag{5.13}$$

Example. Gravitational radiation from binary stars. Suppose two equal mass plane M orbiting around their center of mass. They are separated by $2R$, the gravitational force acts on of them by the other is equal to the centripetal force

$$\frac{GM^2}{(2R)^2} = M\frac{v^2}{R}$$

so

$$v = \sqrt{\frac{GM}{4R}}$$

angular frequency of the rotation is

$$w = \frac{v}{R} = \sqrt{\frac{GM}{4R^3}}$$

Choose coordinate xyz such that motion is in the xy plane and center of mass is the origin, then

$$\begin{aligned} x_A &= R\cos wt \qquad y_A = R\sin wt \\ x_B &= -R\cos wt \quad y_B = -R\sin wt \end{aligned}$$

The energy density is

$$T^{00}(t,\vec{x}) = M\delta(z)[\delta(x - R\cos wt)\delta(y - R\sin wt) + \delta(x + R\cos wt)\delta(y + R\sin wt)]$$

then compute quadrupole moments (5.12),

$$\begin{aligned} I_{11} &= 2MR^2\cos^2 wt \\ I_{22} &= 2MR^2\sin^2 wt \\ I_{12} &= I_{21} = 2MR^2\cos wt\sin wt \\ I_{i3} &= 0 \end{aligned}$$

thus by (5.13), for $r \gg R$

$$\bar{h}_{ij}(t,\vec{x}) = \frac{8GMw^2R^2}{r}\begin{pmatrix} -\cos 2wt_{tr} & -\sin 2wt_{tr} & 0 \\ -\sin 2wt_{tr} & \cos 2wt_{tr} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

one can figure the rest of $\bar{h}_{\mu\nu}$ by (5.11). The final $\bar{h}_{\mu\nu}$ should be traceless, we also see the two modes $+$, $\times$ appear.

One can also compute the energy delivered or power emitted by the gravitational waves. The power is given by

$$P = -\frac{G}{5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3} \right\rangle$$

where

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl} I_{kl}$$

reduced quadrupole moment.

For binary system

$$P = -\frac{2}{5}\frac{G^4 M^5}{R^5}$$

confirmed by Hulse-Taylor 1974, marking the first indirect proof of gravitational wave.

6 Cosmological Models

6.1 Robertson Walker Metric

Lecture 22 (11/26/13)

When Einstein thought of general relativity and the universe, he thought universe was static. But many observation pointed to different directions. Later he regarded it as the biggest blunder in his life.

Copernican principle, which has two statements, serves as the basis for cosmological models as fundamental as equivalent principle to general relativity and relativistic postulate to special relativity. It says at large

1) universe is homogenous, i.e. invariant under translation

2) universe is isotropic, i.e. invariant under rotation

The two are related. In a sense that if isotropic in two points then it has to be homogenous. The simplest model will be Robertson Walker metric, which we've already discussed in section 2.8, expanding universe & cosmological redshift

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) && (6.1)\\ &= -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] && (6.2) \end{aligned}$$

which is a flat space and time dependent metric.

If we go back to Schwarzschild metric on page 57, we see there is a reference clock and reference ruler, i.e. observer at $r = \infty$. For Roberson Walker metric the reference is typically chosen to be today,

$$a(t_0) = 1 \tag{6.3}$$

and the t in $a(t)$ is referred to the past and future age of the universe according to the clock on the earth, but since at every instance the metric is flat, it doesn't matter whether the clock and rule are on the earth or other planets. Therefore at a particular instance the whole universe will agree on the same t and r values. This is the basis for the cosmological redshift discussion below.

What is the r in (6.2) referred to? It is the distance to the origin. Since it will be multiplied by $a(t)$ to give the true physical distance to the origin (by Copernican principle the origin is chosen totally arbitrary), we say r is a co-moving coordinate. In the sense that if today the earth is r distance from the origin, then the value of r will be the same for the past and future as earth moving away.

How to find $a(t)$, the scale factor? We look at cosmological redshift, which relates to Hubble's law. Suppose a light signal emits at the origin with period δt_e. The reason that we can choose signal emits at the origin is that $a(t)$ will be the same for any chosen origin because (6.1) works for any origin. Suppose the receiver is at $r_0 = R$. The first signal is sent at $t = t_e$, and received at $t = t_0$ today, and the second signal is sent at $t = t_e + \delta t_e$ and received at $t = t_0 + \delta t_0$. But since R doesn't change,

$$R = \int dr = \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)} \tag{6.4}$$

adding $\int_{t_0}^{t_e+\delta t_e} \frac{dt}{a(t)}$ to both sides, we get

$$\int_{t_e}^{t_e+\delta t_e} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)}$$

If δt_e, δt_0 are small,

$$\frac{\delta t_e}{a(t_e)} = \frac{\delta t_0}{a(t_0)}$$

or

$$1 + z \equiv \frac{\lambda_0}{\lambda_e} = \frac{w_e}{w_0} = \frac{a(t_0)}{a(t_e)} \tag{6.5}$$

z is the redshift. This is the same as (2.43), but derived from a different way.

$$z = \frac{\frac{a(t_e)-a(t_0)}{t_e-t_0}}{a(t_e)}(t_e - t_0) = \frac{\dot{a}}{a}d$$

if t_e is close to t_0, and $d = (t_e - t_0)c$, $c = 1$. Because Hubble showed by observation

$$z \propto d$$

that is light from outer galaxy is red shifted (the orbital motion is negligible), from Doppler formula, this is due to the fact that receiver and emitter are moving away from each other, Hubble found that relative separating speed (which corresponds to z) of the two planets is proportional to the separation d.

Therefore

$$\frac{\dot{a}(t_0)}{a(t_0)} = H(t_0) \tag{6.6}$$

is a constant. It is a constant of the universe everywhere at a particular instance but it is not a constant of all time. The inverse of $H_0 = H(t_0)$ is the Hubble time

$$t_H = \frac{1}{H_0} = 13.97 \times 10^9 \text{yr} \tag{6.7}$$

This turns out to be a good estimate of the age of the universe, because assume H is constant of all time, then clearly $a(t) = H_0 t$ because $t = 0$, big bang $a = 0$. (6.6) becomes

$$t_0 = \frac{1}{H_0}$$

which also agrees (6.3).

6.2 Three Stages Evolution

Continue our discussion in section 2.8, expanding universe & cosmological redshift. We showed for the perfect fluid universe, i.e. $T_{\mu\nu}$ is (2.44), we have further assumed that if the universe is made of 3 kinds of fluid: matter, radiation, vacuum. Each occupies part of the universe with fraction

$$\Omega_M, \Omega_R, \Omega_V \text{ with } \Omega_M + \Omega_R + \Omega_V = 1 \tag{6.8}$$

and they don't interact each other. Then from 1st law thermodynamic (2.48), we derived cf (2.49), (2.50), (2.51)

$$\begin{aligned}
\rho_M(t) &= \rho_M(t_0)a^{-3}(t) \\
\rho_R(t) &= \rho_R(t_0)a^{-4}(t) \\
\rho_V(t) &= \text{constant, chosen to be} = \frac{c^4\Lambda}{8\pi G}
\end{aligned}$$

where cosmological constant

$$\Lambda = 3H_0^2 \tag{6.9}$$

In section 2.8, we also derived Christoffel symbols for RW metric, cf (2.39), (2.40), (2.41), so we can solve for Einstein equation for $T_{\mu\nu}$ (2.44),

$$\begin{aligned}
G_{tt} &= \frac{3}{a^2}\dot{a}^2 = 8\pi\rho_c \\
G_{rr} &= G_{\theta\theta} = G_{\phi\phi} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 8\pi P_c
\end{aligned}$$

where ρ_c is some kind of average energy density, combining matter, radiation and vacuum.

The two equations above are not independent. We can get the second equation from the first by using 1st law thermodynamic (2.48). So let's focus on the first equation

$$3\frac{\dot{a}^2}{a^2} = 8\pi G\rho_c \tag{6.10}$$

by the way, it is called Friedmann equation. and

$$H_0^2 = \frac{8\pi G}{3}\rho_c \tag{6.11}$$

This justifies why we put (6.9).

$\rho_c(t)$ is clearly related to

$$\Omega_M = \frac{\rho_M}{\rho_c}, \Omega_R = \frac{\rho_R}{\rho_c}, \Omega_V = \frac{\rho_V}{\rho_c} \tag{6.12}$$

so

$$\rho_c(t) = \frac{\rho(t_0)}{\Omega_V(t) + \frac{\Omega_M(t)}{a^3(t)} + \frac{\Omega_V(t)}{a^4(t)}} \tag{6.13}$$

For simplicity, let's assume the universe consists only one of the three kinds. In fact it is indeed what the universe has gone through after it was born. Initially after big bang, the universe was too hot, only radiation, no matter formed. Then it expanded and cooled down

1) radiation dominated universe $\Omega_R = 1$

$$\rho_c = \rho_R$$

so by (6.10)

$$ada \sim dt$$

so

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

2) matter dominated universe $\Omega_M = 1$

$$\rho_c = \rho_M$$

so by (6.10)

$$\sqrt{a}da \sim dt$$

so

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{6.14}$$

3) vacuum dominated universe $\Omega_V = 1$

$$\rho_c = \rho_V$$

so by (6.10)

$$a^{-1} da \sim dt$$

so

$$a(t) = e^{H_0(t-t_0)} \tag{6.15}$$

When we say we are in the stage of vacuum dominated universe, we don't mean that the universe expand to such a degree that it is almost empty. If in that case, we would still say matter dominated. vacuum dominated universe means $\rho =$ const as the universe expand, and it has negative pressure (dark energy) see (2.51). There has been intense research on this topic, but at the moment we still have no idea what dark energy is. Dark matter is not dark energy; dark matter belongs to Ω_M, but it has very special property that doesn't interact with lights, so we have no way to see it directly.

One reason we propose vacuum dominated universe is the following calculation.

If we were in the matter dominated universe, then

$$H_0(t_0) = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{2}{3t_0} \tag{6.16}$$

so by (6.7), the age of the universe would be

$$t_0 \sim 9 \times 10^9 \text{yr}$$

however we know the oldest stars in our galaxy is 12×10^9yr old.

If we propose vacuum dominated universe, then in (6.15) H_0 is indeed constant over time (ignoring the other 2 stages), and if we trace back in time, we will get the age of the universe given by (6.7).

From there we can compute the observable size of the universe: the domain of the universe where signal can reach to us as far as from the beginning of the time

$$d_H = ct_H$$

6.3 Friedmann Robertson Walker Metric

Lecture 23 (12/3/13)

We mentioned in section 2.1 that a sphere has constantly positive curvature. A horse saddle has constantly negative curvature. If we make them to be the universe, they will too satisfy Copernican principle.

We need a 3-sphere or 3-saddle to represent our 3D universe.

1) A 3-sphere embedded in 4D space is

$$x^2 + y^2 + z^2 + w^2 = 1$$

or in spherical coordinate $0 \leq \chi, \theta \leq \pi$, $0 \leq \phi < 2\pi$

$$\begin{aligned} x &= \sin\chi \sin\theta \cos\phi \\ y &= \sin\chi \sin\theta \sin\phi \\ z &= \sin\chi \cos\theta \\ w &= \cos\chi \end{aligned}$$

then one can show

$$\begin{aligned} dx^2 + dy^2 + dz^2 + dw^2 &= d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \\ &= \frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{6.17}$$

by defining

$$r = \sin\chi$$

then

$$d\chi^2 = \frac{dr^2}{1-r^2}$$

so that (6.17) alike more or less (6.2). This is called closed universe.

2) A 3-hyperboliod embedded (after cutting and paste) in 4D space is

$$x^2 + y^2 + z^2 - w^2 = -1$$

or in hyperbolic coordinate $0 \le \chi < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$

$$\begin{aligned}
x &= \sinh\chi \sin\theta \cos\phi \\
y &= \sinh\chi \sin\theta \sin\phi \\
z &= \sinh\chi \cos\theta \\
w &= \cosh\chi
\end{aligned}$$

then one can show

$$\begin{aligned}
dx^2 + dy^2 + dz^2 - dw^2 &= d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \\
&= \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (6.18)
\end{aligned}$$

by defining

$$r = \sinh\chi$$

so that (6.18) alike more or less (6.2). This is called open universe.

Therefore we can combine them in one expression for the three types of expanding universe

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

$$k = \begin{cases} 1 & \text{closed} \\ 0 & \text{flat} \\ -1 & \text{open} \end{cases}$$

The (r, θ, ϕ) are completely different things for the three different types, but at the end of day, we're only interested in physical affect, so we follow the same steps in deriving Friedmann equation, we get the general Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \qquad (6.19)$$

for the Friedmann Robertson Walker metric. This provides us a clue which universe we live in by looking at

$$H_0^2 = \frac{8\pi G\rho(t_0)}{3} - \frac{k}{a^2(t_0)} = \frac{8\pi G}{3}\rho_c \tag{6.20}$$

ρ_c, critical, is defined in (6.11), for flat universe, so if the current $\rho(t_0)$

$$\begin{aligned} \rho(t_0) > \rho_c &\implies \text{closed universe} \\ \rho(t_0) < \rho_c &\implies \text{open universe} \end{aligned}$$

Before the 1990's people didn't know vacuum dominated universe, they thought universe was matter dominated, i.e. $\rho \sim a^{-3}$ so Friedmann equation says

$$\dot{a}^2 =\sim \frac{1}{a} - k \tag{6.21}$$

then if $k = 0$, as the universe expands eventually $\dot{a} = 0$; if $k = -1$, the universe expands eventually at constant rate. The shocking thing is if $k = 1$, at some point $\dot{a} = 0$, after that since the RHS of (6.21) has to be positive, a has to decrease. In other words after some point the universe will turn around and accelerating shrinks.

6.4 Solution of Friedmann Equation

It turns out that we can keep the same definitions in (6.12) for $\Omega_M, \Omega_R, \Omega_V$, but when $k \neq 0$, (6.8) doesn't hold. We should replace (6.8) by setting $t = t_0$ and using (6.20)

$$\Omega_M + \Omega_R + \Omega_V = \frac{\rho_0}{\rho_c} = \frac{\rho_c + \frac{k}{\frac{8\pi G}{3}}}{\rho_c} = 1 + \frac{k}{H_0^2}$$

We define $\Omega_C = -\frac{k}{H_0^2}$, then

$$\Omega_M + \Omega_R + \Omega_V + \Omega_C = 1$$

and Friedmann equation (6.19) becomes

$$\dot{a}^2 = \frac{H_0^2}{\rho_c}\rho a^2 + \Omega_C H_0^2$$

Putting

$$\tilde{t} = H_0 t$$

then by (6.13) (hand waving a little, in (6.13), $\rho(t_0)$ is evaluated at t_0, here we pretend $\rho(t) \approx \rho(t_0)$.)

$$\frac{1}{2}\left(\frac{da}{d\tilde{t}}\right)^2 + \underbrace{-\frac{1}{2}\left(\Omega_V a^2 + \frac{\Omega_M}{a} + \frac{\Omega_R}{a^2}\right)}_{U_{eff}(a)} = \frac{\Omega_C}{2}$$

we get analogy of energy relation for Newtonian particle. So the function $a(t)$ is determined by the four parameters H_0,$\Omega_M, \Omega_R, \Omega_V$ of FRW model.

How to find them?

1) H_0 by Hubble's law we need to measure cosmological redshift and the distance to the emitter: another galaxies. It is found to be

$$H_0 \sim 70\text{km/s/Mpc}$$

How to measure distance to another galaxy? Suppose we know the luminosity (L) of a distance (d)known star, called "standard candle". we can measure its flux f reach to us

$$f = \frac{L}{4\pi d^2} \tag{6.22}$$

then if we can find another similar star (similar age, size, e.g. so they have similar L) in another galaxy, we can compare f to find d.

Actually (6.22) is incorrect for as the universe expands, total energy is not conversed. The correct expression is

$$f = \frac{L}{4\pi d_{eff}^2}\frac{1}{(1+z)^2} \tag{6.23}$$

z is defined in (6.5). d_{eff} is some kind of physical distance + retarded distance. If we assume the universe is flat and matter dominated, and put the emitter at the origin, so we can forget about retarded behavior, we compute just as in (6.4)

$$d_{eff} = d_{phys} = a(t_0) \int dr = a(t_0) \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

using (6.14), (6.5), (6.16)

$$d_{eff} = \int_{t_e}^{t_0} \frac{dt}{a(t)} = 3t_0 \left(1 - (\frac{t_e}{t_0})^{1/3}\right) = 3t_0 \left(1 - (\frac{a(t_e)}{a(t_0)})^{1/2}\right) = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

plugging in (6.23) and assume $z \ll 1$, we recover (6.22) with Hubble's law

$$f = \frac{LH_0^2}{4\pi z^2}$$

2) Ω_V, in 1998, 1999 by two terms uses Type Ia Supernova found dark energy

$$\Omega_V \sim 0.7$$

and got Nobel prize 2011.

3) Ω_R, is cosmic background radiation + neutrinos

$$\Omega_R \sim 8 \times 10^{-5}$$

so we usually ignore it in the calculation. Among them

$$\Omega_{CBR} \sim 5 \times 10^{-5}$$

obtained from black body radiation at temp $T = 2.73K$.

4) Ω_M, it leaves

$$\Omega_M \sim 0.3$$

but only

$$\Omega_B \sim 0.04$$

are Baryonic (leptons are much lighter, so no effect on Ω_B), obtained from big bang nucleosynthesis and standard model. The rest are dark matter; they are evidenced by gravitational lensing, galactic rotation curve, etc

7 Lagrangian Formulation

7.1 Lagrangian for Fields

Lecture 24 -Last Lec- (12/5/13)

Recall single particle 1D coordinate $q(t)$, we have

$$S = \int dt L(q, \dot{q}) \qquad L = K - V$$

vary the extreme, get EL

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

If $L = \frac{1}{2}\dot{q}^2 - V(q)$

$$EL \implies \ddot{q} + \frac{\partial V}{\partial q} = 0$$

In general relativity both t and $\vec{x}$ are coordinate. So we would like to develop Lagrangian for scalar fields $\Phi(x)$, $\mathcal{L} = \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))$, $\mathcal{L}$ is Lagrangian density.

$$S = \int dt L = \int d^4x \mathcal{L}$$

First EL

$$\begin{aligned} \Phi(x) &\rightarrow \Phi(x) + \delta\Phi(x) \\ \partial_\mu \Phi(x) &\rightarrow \partial_\mu \Phi(x) + \delta(\partial_\mu \Phi(x)) \\ &= \partial_\mu \Phi(x) + \partial_\mu(\delta\Phi(x)) \end{aligned}$$

Taylor expand

$$\mathcal{L} \rightarrow \mathcal{L}(\Phi + \delta\Phi, \partial_\mu \Phi + \partial_\mu \delta\Phi) = \mathcal{L}(\Phi, \partial_\mu \Phi) + \frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\mu \delta\Phi$$

so $S \to S + \delta S$

$$\begin{aligned}
\delta S &= \int d^4x \frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\mu \delta\Phi \\
&= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \delta\Phi
\end{aligned}$$

So EL

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0$$

for Φ that extremes S. This is also true for vector field,

$$\frac{\partial \mathcal{L}}{\partial \Phi_\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_\alpha)} = 0 \tag{7.1}$$

Example. EM vector field A_μ with source J_μ. Maxwell tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.2}$$

is antisymmetric. The scalar Lagrangian density is

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \tag{7.3}$$

which is also gauge invariant under

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

We will show (7.3) gives 2 of the Maxwell, the other 2 are already in (7.2).

Compute

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{\partial A_\nu J^\nu}{\partial A_\mu} = \delta^\mu{}_\nu J^\nu = J^\mu$$

$$F_{\alpha\beta}F^{\alpha\beta} = \eta^{\alpha\sigma}\eta^{\beta\rho} F_{\alpha\beta}F_{\sigma\rho}$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} &= \frac{\partial F_{\alpha\beta}F^{\alpha\beta}}{\partial(\partial_\nu A_\mu)} \\
&= \eta^{\alpha\sigma}\eta^{\beta\rho}\left(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\nu A_\mu)}F_{\sigma\rho} + F_{\alpha\beta}\frac{\partial F_{\sigma\rho}}{\partial(\partial_\nu A_\mu)}\right) \\
&= \eta^{\alpha\sigma}\eta^{\beta\rho}\left((\delta^\nu{}_\alpha\delta^\mu{}_\beta - \delta^\mu{}_\alpha\delta^\nu{}_\beta)F_{\sigma\rho} + F_{\alpha\beta}(\delta^\nu{}_\sigma\delta^\mu{}_\rho - \delta^\mu{}_\sigma\delta^\nu{}_\rho)\right) \\
&= (\eta^{\nu\sigma}\eta^{\rho\mu} - \eta^{\mu\sigma}\eta^{\rho\nu})F_{\sigma\rho} + F_{\alpha\beta}(\eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\nu}\eta^{\beta\mu}) \\
&= 4F^{\mu\nu}
\end{aligned}
$$

so

$$\text{EL (7.1)} \implies J^\alpha - \partial_\mu F^{\mu\alpha} = 0$$

proving (1.18).

7.2 Einstein-Hilbert Action

As in the gravitational wave, we want field

$$\Phi(x) \leftrightarrow g_{\mu\nu}(x)$$

We want a scalar $\mathcal{L}$ so that it gives Einstein equation as EL. The answer is

$$\mathcal{L} = R = g^{\mu\nu}R_{\mu\nu}$$

Ricci scalar. This is unusual. Usually $\mathcal{L}$ contains Φ and $\partial\Phi$, but here R is only function of $\partial^2 g_{\mu\nu}$. Unlike in the EM example, we are not going to use EL directly, instead we will do a variance on the action. The action is special too, it has extra weight

$$S = \int d^4x\sqrt{-g}R(x)$$

where $d^4x\sqrt{-g}$ is an invariant measure, shown in (2.16), proven in (2.29).

Let

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

then

$$\delta S = \int d^4x\underbrace{\delta\sqrt{-g}R}_{I} + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \underbrace{\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}}_{II}$$

First deal with I, by (2.30),

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

Next compute, using (2.30), (2.26)

$$\begin{aligned}
\delta\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\delta g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) + \frac{1}{2}g^{\lambda\rho}(\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\nu}) \\
&= -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\nu}) \\
&= \frac{1}{2}g^{\lambda\rho}(\nabla_\mu \delta g_{\nu\rho} + \nabla_\nu \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\nu})
\end{aligned}$$

hence indeed $\delta\Gamma$ is a tensor as we prove it in the finite difference case in (2.27).

Thus by (3.5), (3.2), and (2.26)

$$\begin{aligned}
\delta R_{\mu\nu} &= \partial_\lambda \delta\Gamma^\lambda_{\nu\mu} - \partial_\nu \delta\Gamma^\lambda_{\lambda\mu} + \delta\Gamma^\kappa_{\kappa\lambda}\Gamma^\lambda_{\nu\mu} - \delta\Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\mu\kappa} + \Gamma^\kappa_{\kappa\lambda}\delta\Gamma^\lambda_{\nu\mu} - \Gamma^\kappa_{\nu\lambda}\delta\Gamma^\lambda_{\mu\kappa} \\
&= \nabla_\lambda \delta\Gamma^\lambda_{\nu\mu} - \nabla_\nu \delta\Gamma^\lambda_{\lambda\mu}
\end{aligned}$$

Use a trick

$$\sqrt{-g}\nabla_\mu V^\mu = \partial_\mu(\sqrt{-g}V^\mu)$$

So

$$\begin{aligned}
II &= \sqrt{-g}\nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}) \\
&= \partial_\lambda(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}) - \partial_\nu(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu})
\end{aligned}$$

so by stokes, II gives 0.

Therefore

$$\begin{aligned}
\delta S &= \int d^4x - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}R + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} \\
&= \int d^4x\sqrt{-g}\delta g^{\mu\nu}\left(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right)
\end{aligned}$$

that is Einstein equation in vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

To incorporate the source term, we add matter term to the action

$$S = S_{Vacuum} + S_{matter}$$

$$S_{matter} = 16\pi G \int d^4x \sqrt{-g} \mathcal{L}_{matter}(\Phi, \partial_\mu \Phi, g_{\mu\nu})$$

where Φ is the matter field. In the argument, we assume gravity commutes with matter field.

Then

$$\delta S_{matter} = 16\pi G \int d^4x \frac{\delta \sqrt{-g} \mathcal{L}_{matter}(g_{\mu\nu})}{\delta g^{\mu\nu}} \delta g^{\mu\nu}$$

setting

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \mathcal{L}_{matter}(g_{\mu\nu}) \right)}{\delta g^{\mu\nu}} = -\frac{1}{\sqrt{-g}} \frac{1}{8\pi G} \frac{\delta S_{matter}}{\delta g^{\mu\nu} \delta^4 x}$$

$$\delta S_{matter} = 8\pi G \int d^4x \sqrt{-g} \delta g^{\mu\nu} T_{\mu\nu}$$

we get

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$