Electricity-Magnetism II

Brian Metzger

Transcribed by Ron Wu

This is an undergraduate course. Offered in Spring 2014 at Columbia University. Required Course textbook: Griffiths, *Introduction to Electrodynamics*; Hecht, *Optics*. Suggested Readings: Rybicki, *Radiative Processes in Astrophysics*; Purcell, *Electricity and Magnetism.* Office Hours Mon 12:30–1:30.

Contents

1 Electromagnetic Waves

Lecture 1 (1/22/14)

One may have heard about unification: combine gravity to quantum field theory.

$$\begin{tikzcd}
 & & & \text{nothing interesting} \\
 & & \hbar \arrow[ur, "=0"] \arrow[dr, "{>0}"'] & \\
 & & & \text{non-rel qm} \\
 & G \arrow[uur, "=0"] \arrow[ddr, "{>0}"'] & & \\
 & & & \text{Newton gravity} \\
 & & \hbar \arrow[ur, "=0"] \arrow[dr, "{>0}"'] & \\
v/c \arrow[uuur, "=0"] \arrow[ddddr, "{>0}"'] & & & \text{nothing interesting} \\
 & & & \text{special rel} \\
 & & \hbar \arrow[ur, "=0"] \arrow[dr, "{>0}"'] & \\
 & & & \text{qft/qed} \\
 & G \arrow[uur, "=0"] \arrow[ddr, "{>0}"'] & & \\
 & & & \text{general rel} \\
 & & \hbar \arrow[ur, "=0"] \arrow[dr, "{>0}"'] & \\
 & & & \text{TOE}
\end{tikzcd}$$

1.1 Maxwell Equations

Gauss Law

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

integral form

$$\iint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

No Magnetic Monopole Law

$$\nabla \cdot \vec{B} = 0$$

integral form

$$\iint_S \vec{B} \cdot d\vec{A} = 0$$

Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

integral form

$$\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$$

Ampere's Law

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

integral form

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} + \mu_0 I_{enc}$$

Consider some concentration of charge particles in volume ΔV, the Lorentz Force on particle i

$$\vec{F}_i = q_i(\vec{E} + \vec{v}_i \times \vec{B})$$

so the average force density

$$\vec{f} = \lim_{\Delta V \to 0} \sum_i \frac{q_i}{\Delta V}(\vec{E} + \vec{v}_i \times \vec{B}) = \rho \vec{E} + \vec{J} \times \vec{B} \tag{1.1}$$

because

$$\rho = \lim_{\Delta V \to 0} \sum_i \frac{q_i}{\Delta V} \quad \vec{J} = \lim_{\Delta V \to 0} \sum_i \frac{q_i \vec{v}_i}{\Delta V}$$

From the definition of ρ and $\vec{J}$ above, we get continuity equation

$$\frac{\partial}{\partial t} \iiint_V \rho dV = -\iint_S \vec{J} \cdot d\vec{A} = -\iiint_V \nabla \cdot \vec{j} dV \tag{1.2}$$

so

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$$

One can check that continuity equation is consistent with Maxwell equations

$$\frac{\partial \rho}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} = \nabla \cdot \left(\epsilon_0 \frac{\partial}{\partial t} \vec{E} - \frac{1}{\mu_0} \nabla \times \vec{B} \right) = -\nabla \cdot \vec{J}$$

1.2 Poynting Flux and Momentum Tensor

(1.2) shows a general scheme.

$$\frac{\partial}{\partial t}(\text{density}) + \nabla \cdot (\text{flux}) = 0 \tag{1.3}$$

	density	flux
charge	ρ	$\vec{J}$
energy	$u_{KE} + u_{EM}$	$\vec{S}$
momentum	$\vec{P}_{mech} + \vec{P}_{EM}$	$\overleftrightarrow{T}$

Lecture 2 (1/27/14)

Now we show the flux of energy is $\vec{S}$, poynting flux,

$$[\vec{S}] = \frac{\text{energy}}{\sec \cdot m^2}$$

For single particle i

$$\frac{d}{dt} U_{KE_i} = \frac{d}{dt} \frac{1}{2} m v_i^2 = \vec{v}_i \cdot \vec{F}_i = q_i \vec{v}_i (\vec{E} + \vec{v}_i \times \vec{B}) = q_i \vec{v}_i \cdot \vec{E}$$

B field do no work. So the density

$$\frac{\partial}{\partial t} u_{KE} = \frac{1}{\Delta V} \sum_i q_i \vec{v}_i \cdot \vec{E}$$

Since $\vec{E}$ changes continuous within the small volume ΔV, treat it as constant. We change total time derivative to partial derivative, because we now take time derivative for each particle and not concern with the space variation of the charges.

$$\frac{\partial}{\partial t}u_{KE} = \vec{J}\cdot\vec{E} = (\frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial}{\partial t}\vec{E})\cdot\vec{E} = \frac{1}{\mu_0}\vec{E}\cdot(\nabla\times\vec{B}) - \frac{\partial}{\partial t}\underbrace{\epsilon_0\frac{\vec{E}\cdot\vec{E}}{2}}_{u_E}$$

Using product rule

$$\nabla\cdot(\vec{A}\times\vec{C}) = -\vec{A}\cdot(\nabla\times\vec{C}) + \vec{C}\cdot(\nabla\times\vec{A})$$

So

$$\frac{\partial}{\partial t}(u_{KE}+u_E) = \frac{1}{\mu_0}\nabla\cdot(\vec{B}\times\vec{E}) + \underbrace{\frac{1}{\mu_0}\vec{B}\cdot\underbrace{\nabla\times\vec{E}}_{-\frac{\partial\vec{B}}{\partial t}}]}_{-\frac{\partial}{\partial t}u_B}$$

Thus

$$\frac{\partial}{\partial t}(u_{KE}+u_E+u_B) = \nabla\cdot\underbrace{\frac{1}{\mu_0}(\vec{E}\times\vec{B})}_{\vec{S}}$$

Example. A cylindrical resistor of radius a and length l is connected to a battery kept voltage V and current flow through it I. Then inside of the resistor

$$\vec{E} = \frac{V}{L}\hat{I}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

$\hat{\phi}$ is the azimuthal angle as if $\hat{I} = \hat{z}$, then the poynting flux on the surface of the resistor is

$$\vec{S} = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi r}\hat{r}$$

$\hat{r}$ is radial outward. Compute the charge of energy density

$$\oint\vec{S}\cdot d\vec{A} = \frac{V}{L}\frac{I}{2\pi a}2\pi aL = VI$$

which agrees common sense.

We now do Maxwell stress tensor.

Starting from the average force density (1.1)

$$\vec{f} = \epsilon_0(\nabla \cdot \vec{E})\vec{E} + \left(\frac{1}{\mu_0}\nabla \times \vec{B} - \epsilon_0\frac{\partial}{\partial t}\vec{E}\right) \times \vec{B} \tag{1.4}$$

$$\frac{\partial \vec{E}}{\partial \vec{B}} \times \vec{B} = \frac{\partial}{\partial t}(\vec{E} \times \vec{B}) - \vec{E} \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{-\nabla \times \vec{E}}$$

so (1.4) becomes

$$\vec{f} = \epsilon_0\left((\nabla \cdot \vec{E})\vec{E} - \vec{E} \times (\nabla \times \vec{E})\right) + \frac{1}{\mu_0}\left((\nabla \cdot \vec{B})\vec{B} - \vec{B} \times (\nabla \times \vec{B})\right) + \epsilon_0\frac{\partial}{\partial t}(\vec{E} \times \vec{B}) \tag{1.5}$$

where we added $(\nabla \cdot \vec{B})\vec{B} = 0$ to make the expression symmetric.

Using product rule

$$\nabla(\vec{A} \cdot \vec{C}) = \vec{A} \times (\nabla \times \vec{C}) + \vec{C} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{C} + (\vec{C} \cdot \nabla)\vec{A}$$

we get

$$\vec{E} \times (\nabla \times \vec{E}) = \frac{1}{2}\nabla\vec{E}^2 - (\vec{E} \cdot \nabla)\vec{E}$$

So (1.5) becomes

$$\vec{f} = \underbrace{\epsilon_0\left((\nabla \cdot \vec{E})\vec{E} + (\vec{E} \cdot \nabla)\vec{E}\right) + \frac{1}{\mu_0}\left((\nabla \cdot \vec{B})\vec{B} + (\vec{B} \cdot \nabla)\vec{B}\right) - \frac{1}{2}\nabla\left(\epsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)}_{\nabla \cdot \overleftrightarrow{T}} - \epsilon_0\mu_0\frac{\partial \vec{S}}{\partial t}$$

Claim $\overleftrightarrow{T}$ is a tensor

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \tag{1.6}$$

Check that

$$(\nabla \cdot \overleftrightarrow{T})_x := \nabla \cdot \vec{T}_x = \frac{\partial}{\partial x}T_{xx} + \frac{\partial}{\partial y}T_{xy} + \frac{\partial}{\partial z}T_{xz} \tag{1.7}$$

where $\vec{T}_x$ is a vector obtained after contacting one index of $\overleftrightarrow{T}$.

$$\text{LHS (1.7)} = \epsilon_0 \left((\nabla \cdot \vec{E})E_x + (\vec{E} \cdot \nabla)E_x \right) + \frac{1}{\mu_0} \left((\nabla \cdot \vec{B})B_x + (\vec{B} \cdot \nabla)B_x \right) - \frac{1}{2}\frac{\partial}{\partial x} \left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2 \right) \tag{1.8}$$

$$\begin{aligned} \text{RHS (1.7)} &= \frac{\partial}{\partial x}\epsilon_0 \left(E_x^2 - \frac{E^2}{2} \right) + \frac{\partial}{\partial x}\frac{1}{\mu_0} \left(B_x^2 - \frac{B^2}{2} \right) + \frac{\partial}{\partial y} \left(\epsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y \right) \\ &\quad + \frac{\partial}{\partial z} \left(\epsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z \right) \end{aligned} \tag{1.9}$$

Let's check the terms that have ϵ_0

$$\begin{aligned} (1.8) &\rightarrow \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) E_x + E_x \frac{\partial E_x}{\partial x} + E_y \frac{\partial E_x}{\partial y} + E_z \frac{\partial E_x}{\partial z} - \frac{1}{2}\frac{\partial}{\partial x} E^2 \\ &= \frac{\partial}{\partial x} \left(E_x^2 - \frac{1}{2}E^2 \right) + \frac{\partial}{\partial y} (E_x E_y) + \frac{\partial}{\partial z} (E_x E_z) \\ &\rightarrow (1.9) \end{aligned}$$

So (1.9) is correct.

What is the meaning of $\overleftrightarrow{T}$?

Find force

$$\underbrace{\vec{F}}_{\frac{\partial}{\partial t}\vec{P}_{mech}} = \int \vec{f} dV = \underbrace{\int \nabla \cdot \overleftrightarrow{T} dV}_{\oint \overleftrightarrow{T} \cdot dA} - \frac{\partial}{\partial t} \underbrace{\epsilon_0 \mu_0 \int \vec{S} dV}_{:= \vec{P}_{EM}}$$

or

$$\frac{\partial}{\partial t} \left(\vec{P}_{mech} + \vec{P}_{EM} \right) - \oint \overleftrightarrow{T} \cdot dA = 0 \tag{1.10}$$

The $-$ sign in front of the integral is different from the general expression (1.3) in which there is a $+$ sign in front of the integral is because here the $+$ direction for $\overleftrightarrow{T}$ is pressure acting to the volume, i.e. $+\ \overleftrightarrow{T}$ is radial inward, while before the flux is $+$ radial outward.

(1.10) contains three equations, e.g.

$$\frac{\partial}{\partial t} P_x = \oint \vec{T}_x \cdot d\vec{A} = \oint T_{xx} dA_x + \oint T_{xy} dA_y + \oint T_{xz} dA_z$$

We now understand what $\overleftrightarrow{T}$ means. E.g.

$$T_{xy} \text{ means the contribution to the } x\text{component of the force}$$
$$\text{due to momentum flux flowing acting on surface in } y\text{direction}$$

The mechanical analogy of this is the shear tensor, which says when pressure act on a block in opposite x directions cause a shear that twist the block so that y direction of the block gets squeezed which corresponds to produce a force in y direction.

According to (1.6), T_{ij} is symmetric, we can equally say that

$$T_{xy} \text{ means the contribution to the } y\text{component of the force}$$
$$\text{due to momentum flux flowing acting on surface in } x\text{direction}$$

1.3 EM Waves in Vacuum

1.4 Radiation Power Spectrum

1.5 EM Waves in Linear Dielectric

1.6 EM Waves in Dispersive Media

Lecture 3 (1/29/14)
Lecture 4 (2/3/14)
Lecture 5 (2/5/14)
Lecture 6 (2/10/14)
Lecture 7 (2/12/14)
Lecture 8 (2/17/14)
Lecture 9 (2/19/14)