Mechanics

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This is an undergraduate course. Offered in Spring 2014 at Columbia University. Required Course textbook: Taylor, *Classical Mechanics*. Office Hours Fri 1:30–2:30.

Contents

1 Newtonian Mechanics

1.1 Newton Laws

Lecture 1 (1/22/14)

For those interested in the historical development of the subject may look up brief biographies of

Galileo
Newton
Einstein

Drake, *Galileo at Work: his scientific biography* explains some of the work which led Newton to formulate the equations of motion.

Article by the economist Keynes, *Newton, the Man.* One may find it surprising to learn some of Newton's extracurricular activities. The famous quote

> "I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

in which he deluded the greatest mysteries he was going to find is is alchemy. To avoid prosecution by the church, he had to keep it secret.

Pais, *Subtle is the Lord: the science and the life of Albert Einstein.* Ch6 describes how people thought about inertial frames before General Relativity was devised.

Equation of Motion

Pick a coordinate system, $\vec{r}$ be the position of the particle,

$$\begin{aligned} \text{velocity } \vec{v} &= \frac{d\vec{r}}{dt} \\ \text{acceleration } \vec{a} &= \frac{d^2\vec{r}}{dt^2} \end{aligned}$$

then Newton says

$$\vec{F} = m\vec{a}$$

where m is inertial mass, which is proportional to the gravitational mass

$$\vec{F}_{grav} = -m_{grav} g \hat{z}$$

Once we have the definition of $\vec{a}$, we can go back to see if coordinate system we chose is a good one, i.e. inertial frame. Meaning no acceleration if no obvious force

$$\vec{F} = 0 \iff \vec{a} = 0$$

Example. Free particle in an inertial frame

$$m \frac{d^2 \vec{r}}{dt^2} = 0$$

This is a 2nd order homogenous ODE, so it has to have 2 initial conditions.

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t \tag{1.1}$$

Example. Particle under constant gravitational field

$$m \frac{d^2 \vec{r}}{dt^2} = -mg\hat{z}$$

This is a 2nd order linear inhomogeneous ODE.

$$\begin{cases} x = x_0 + v_{x0} t \\ y = y_0 + v_{y0} t \\ z = z_0 + v_{z0} t - \frac{1}{2} g t^2 \end{cases} \tag{1.2}$$

so the solution $z = z_0 + v_{z0} t - \frac{1}{2} g t^2$ has homogenous part

$$z_h = z_0 + v_{z0} t$$

and particular solution

$$z_p = -\frac{1}{2}gt^2$$

Example. Particle under true gravitational field

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{GM_e m}{r^2}$$

This is a non-linear ODE, and choosing the origin of the coordinate to be the center of the earth. We will solve it later in terms of integrable system.

Ways to represent Solutions

1. formula
2. trajectory. Plot t v.s. x (usually 1D)
3. phase portrait. plot x v.s. v (usually 1D) This is useful for non-linear ODEs

E.g. (1.2) the formula representation is

$$z = z_0 + v_{z0}t - \frac{1}{2}gt^2$$

plot t v.s. z is family of parabolas with constant curvature.

Lecture 2 (1/27/14)

In phase space

substituting

$$v_z = v_{z0} - gt$$

and eliminate t, get

$$z = z_0 + \frac{v_0^2 + v^2}{2g}$$

family of parabolas but oriented so that the axis of symmetry is the z axis. Physically this is due to the fact that gravity is a conservative force.

1.2 Resistance and Drag

Example of drag: sliding friction, air resistance, liquid viscosity.

$$\vec{F}_{drag}(\vec{v}) = -\hat{v}F_d(|v|)$$

F_d is 0 if $v = 0$ and typically F_d is analytic around 0, i.e.

$$F_d = F_d(0) + F_d'(0)v + \frac{1}{2}F_d''(0)v^2 + ... = bv + cv^2 \tag{1.3}$$

However sliding friction is different, F_d is a step function, i.e. it is 0 if $v = 0$, and it becomes constant non 0 if $v \neq 0$. We won't deal with it.

Dimensional Analysis

Let us see what determine b, c in (1.3).

Empirically

$$\begin{aligned} b &\sim \text{linear dimension of moving body } D \\ &\sim \text{dynamic viscosity } \eta \end{aligned}$$

$$[\eta] = \frac{\text{mass}}{\text{length} \cdot \text{times}}$$

The dimensional analysis will be important for studying Reynolds number.

$$\eta = \frac{m}{lt} = \frac{\rho l^2}{t} = \rho_o \nu$$

ν is kinematic viscosity, ρ_o is the density of the object. So

$$b \sim D\rho_o\nu$$

For (1.3)

$$c = \frac{\text{force}}{\text{velocticy}^2}$$

The momentum gained by the medium due to the moving object of cross section D^2 with velocity v, sweeping a volume of the medium with density ρ_m in time interval Δt is

$$\rho_m (v D^2 \Delta t) v$$

so

$$c \sim \rho_m D^2$$

At small v

$$bv \gg cv^2$$

At big v

$$bv \ll cv^2$$

the critical v^*

$$v^* = \frac{b}{c} \sim \frac{\rho_o \nu}{\rho_m D}$$

For baseball $D = 10^{-2}$m, and $\rho_{water} \sim \rho_{baseball} \sim 10^3 \rho_{air}$

$$\begin{aligned}
\nu_{air} &= 10^{-5}\text{m}^2/\text{s} \implies v^* = 1\text{m/s} \\
\nu_{water} &= 10^{-6}\text{m}^2/\text{s} \implies v^* = 10^{-4}\text{m/s}
\end{aligned}$$

Linear Drag

So if v is smaller than v^*

$$m\frac{dv}{dt} = -bv$$

solve

$$v(t) = v_0 e^{-\frac{b}{m}t} = v_0 e^{-t/\tau} \tag{1.4}$$

put $\tau = m/b$

$$x(t) = x_0 + v_0 \tau (1 - e^{-t/\tau})$$

In phase space

$$x = x_0 + \tau(v_0 - v)$$

hence a straight line, and stop at $x = x_0 + \tau v_0$

One may be interested in how far particle can move relative to its size

$$\frac{\tau v_0}{D}$$

Let us compute this for bacteria in water,

$$\tau \sim \frac{\rho_o D^3}{D\rho_m \nu} = \frac{D^2}{\nu} \tag{1.5}$$

so

$$\frac{\tau v_0}{D} = \frac{v_0 D}{\nu} = \frac{10^{-6} \cdot 10^{-6}}{10^{-6}} = 10^{-6}$$

stop immediately

For more about Reynolds number and dimension analysis read article by Purcell, *Life at Low Reynolds Number*.

Lecture 3 (1/29/14)

Suppose we add gravity

$$m\frac{dv}{dt} + bv = -mg$$

Homogenous solution is (1.4)

Particular solution

$$v_p = -\tau g$$

so

$$v = -\tau g + ce^{-t/\tau} = (v_0 + g\tau)e^{-t/\tau} - g\tau$$

As $t \to \infty$,

$$v \to -g\tau$$

terminal velocity, regardless whether the initial v_0 is $<$ or $>$ $-g\tau$. This is a general rule: when damping involves, initial condition become irrelevant, after some characteristic time τ.

Example. A cat is falling from the roof of an apartment.

$$\tau \sim \frac{m}{D\rho\nu} \propto \frac{1}{D}$$

This looks different from (1.5), because in (1.5) we want to use $\rho_{bacteria} = \rho_{water}$. We don't have that now.

Hence the larger the cross section in the direction of motion, the smaller the terminal speed is. Study have shown that cat can survive falling from the roof building, but die falling from the 1st or 2nd floor windows, because it takes time to adjust its body to increase D.

For skydiving, terminal velocity is the same, so no need to open parachute too early. But don't want to open it too later, because it takes time to reach terminal velocity.

Quadratic Drag

$$m\frac{dv}{dt} = -cv^2 \text{sign} v \tag{1.6}$$

Physical no external force, the motion will slow down. We can choose the direction of v_0 to be the $+x$ direction, then (1.6) becomes

$$m\frac{dv}{dt} = -cv^2$$

so

$$v(t) = \frac{1}{\frac{c}{m}t + \frac{1}{v_0}}$$

1.3 SHO

Cyclotron Motion

Particle of charge q and mass m moving in a constant magnetic field $\vec{B} = (0, 0, B)$

$$\vec{F} = (\vec{v} \times \vec{B})$$

$F_z = 0$

$$\begin{cases} \frac{dv_x}{dt} = wv_y \\ \frac{dv_y}{dt} = -wv_x \end{cases} \tag{1.7}$$

$$w = \frac{qB}{m} \text{ cyclotron freq}$$

Guess solution

$$\begin{cases} v_x = A\sin(wt + \phi) \\ v_y = A\cos(wt + \phi) \\ v_z = v_{z0} \end{cases}$$

check they are indeed solutions with 3 free parameters A, ϕ and v_{z0}. Suppose at $t = 0$, $v_x = v_0$, $v_y = v_z = 0$.

$$\begin{cases} v_x = v_0 \cos(wt) \\ v_y = -v_0 \sin(wt) \\ v_z = 0 \end{cases} \tag{1.8}$$

with $A = v_0$ $\phi = -\pi/2$.

Integrate

$$\begin{cases} x = x_0 + \frac{v_0}{w} \sin wt \\ y = y_0 + \frac{v_0}{w} \cos wt \\ z = z_0 \end{cases}$$

Hence particle is moving in a circle of radius v_0/w.

Application: Cyclotron

Put charged particle in a circular motion with constant B. Leave one small slit open no B. put in an alternating E. Since the periodic of the circle motion is independent of velocity and radius, ignoring the time spent in the slit. One can set the alternating E with the right periodic so that the particle will be gaining energy each time passing through the slit.

Application: Mass spectrometer.

Let known v_0, q charged particles entering a constant B. From the radius one can separate particles with different m. Such motions are observed in natural such as astrophysical jets due to magnetic field of large plane.

In phase space $(x - x_0)$ v.s. v_x is an ellipse, hence a closed orbit with periodic $2\pi/w$, because

$$\begin{cases} v_x = v_0 \cos wt \\ x - x_0 = \frac{v_0}{w} \sin wt \end{cases}$$

however the angular frequency is not constant w, because

$$\frac{v_x}{x - x_0} = w \cot wt = w \tan(\frac{\pi}{2} - wt)$$

$$\text{instantaneous angular frequency} = \left| \frac{d}{dt} \tan^{-1} \Big(w \tan(\frac{\pi}{2} - wt) \Big) \right| \neq w$$

One can use complex notation, to write (1.8) in a compact form

$$\eta(t) = v_x(t) + i v_y(t)$$

so

$$\frac{d\eta}{dt} = w v_y - i w v_x = -i w \eta$$

thus

$$\eta = v_0 e^{-iwt}$$

to be consistent with the initial condition in (1.8).

Spring Motion

Particle driven by 1D force $F = -kx$, by choosing the origin of the coordinate to be the equilibrium point.

$$\begin{cases} \frac{dx}{dt} = v \\ \frac{dv}{dt} = -\frac{k}{m} x \end{cases}$$

Lecture 4 (2/3/14)

same pattern as (1.7), if we set

$$w = \sqrt{\frac{k}{m}} \qquad u = wx$$

then

$$\begin{cases} \frac{du}{dt} = wv \\ \frac{dv}{dt} = -wu \end{cases} \implies \begin{cases} v = A\cos(wt + \phi) \\ u = A\sin(wt + \phi) \end{cases}$$

Pendulum

Mass m is suspended by a massless rigid rod l, swing in a vertical xz plane, let T be tension

$$\begin{cases} m\frac{d^2x}{dt^2} = T_x \\ m\frac{d^2z}{dt^2} = -g + T_z \end{cases}$$

with constraint

$$x^2(t) + z^2(t) = l^2$$

This problem will show how to get a linear ODE from nonlinear ODE. This problem also exemplifies how to deal with constraints: using generalized coordinates so that the constraints are automatically taken into account. Later when we study Lagrange mechanics, we will learn a systematic way of choosing good generalized coordinate.

Choose polar coordinates $(r(t),\ \theta(t))$. The origin is the point of suspension. $+$ sign for r is radial outward, $+$ sign for θ is swing in from $+x$ to $+z$ in the smallest angle.

We now transfer dynamics variables from xz Cartesian to polar

$$\begin{aligned}\vec{r}(x,y,z) &= (r\sin\theta, 0, -r\cos\theta)\\ \vec{v}(x,y,z) &= (\dot{r}\sin\theta + r\cos\theta\dot{\theta}, 0, -\dot{r}\cos\theta + r\sin\theta\dot{\theta})\end{aligned} \tag{1.9}$$

If we make unit vectors

$$\hat{r} = (\sin\theta, 0, -\cos\theta) \quad \hat{\theta} = (\cos\theta, 0, \sin\theta)$$

We can now find components of $\vec{v}$ in polar

$$v_r = \vec{v}\cdot\hat{r} = \sin\theta(\dot{r}\sin\theta + r\cos\theta\dot{\theta}) - \cos\theta(-\dot{r}\cos\theta + r\sin\theta\dot{\theta}) = \dot{r} \tag{1.10}$$

$$v_\theta = \vec{v}\cdot\hat{\theta} = \cos\theta(\dot{r}\sin\theta + r\cos\theta\dot{\theta}) + \sin\theta(-\dot{r}\cos\theta + r\sin\theta\dot{\theta}) = r\dot{\theta} \tag{1.11}$$

Now do $\vec{a}$, notice always use Cartesian $\vec{v}$

$$\vec{a} = (\ddot{r}\sin\theta + 2\dot{r}\cos\theta\dot{\theta} - r\sin\theta\dot{\theta}^2 + r\cos\theta\ddot{\theta}, 0, -\ddot{r}\cos\theta + 2\dot{r}\sin\theta\dot{\theta} + r\cos\theta\dot{\theta}^2 + r\sin\theta\ddot{\theta})$$

Similarly find components

$$\begin{aligned}a_r &= \ddot{r} - r\dot{\theta}^2\\ a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta}\end{aligned}$$

If one doesn't use (1.9), but instead uses (1.10), (1.11) will miss

$$\begin{aligned} -r\dot{\theta}^2 &\rightarrow \text{centripetal} \\ \dot{r}\dot{\theta} &\rightarrow \text{Coriolis} \end{aligned}$$

We can now back to pendulum problem

$$r = \text{const}$$

$$\begin{aligned} F_r &= ma_r \Longrightarrow mg\cos\theta - T = -mr\dot{\theta}^2 \\ F_\theta &= ma_\theta \Longrightarrow -mg\sin\theta = mr\ddot{\theta} \qquad (1.12) \end{aligned}$$

Since we don't T, we only need (1.12).

Suppose θ is small

$$\sin\theta \approx \theta$$

so

$$\ddot{\theta} = -\frac{g}{r}\theta$$

so

$$\theta(t) = A\cos(wt + \phi) \quad w = \sqrt{\frac{g}{r}}$$

provided A is not too big.

One may study (1.12) without assuming θ small. Use phase space and energy, which we will study more detailed later

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

so the graph of θ v.s. $\dot{\theta}$ is given by

$$\dot{\theta} = \pm\sqrt{2(E + mgl\cos\theta)/ml^2}$$

If $E > mgl$, i.e. the pendulum has energy to roll over the cliff and continue rotating. Confirmed by the graph. There are two branches, where $\dot{\theta} > 0$ is counterclockwise rotating and $\dot{\theta} < 0$ is clockwise, and keep on forever $\theta \to \pm\infty$. If $E < mgl$, then θ is bounded. The graph is closed around the center 0, 2π,..., and the pendulum is swing back and forth. [One should not confuse this with Bertrand theorem: only closed orbits are motion under inverse square force and Hooks force. In Bertrand theorem, orbits are spatial orbits and only central forces are involved. Here the pendulum is fixed at the suspension, so the spatial orbit is closed no matter what.]

So we encounter 2 types of equilibrium. stable 0, $\pm 2\pi$,...,; unstable $\pm\pi$, $\pm 3\pi$. How to see this mathematically?

Suppose we perturb π

$$\theta = \pi + \psi$$

ψ is small. Then (1.12) says

$$\ddot{\psi} = \frac{g}{r}\sin\psi = \frac{g}{r}\psi \implies \psi = e^{\pm wt}$$

ψ grows so unstable.

1.4 Momentum

1.5 Angular Momentum

1.6 Energy

1.7 Motion Near Equilibrium Point

1.8 Kepler's Problem

Lecture 5 (2/5/14)
Lecture 6 (2/10/14)
Lecture 7 (2/12/14)
Lecture 8 (2/17/14)
Lecture 9 (2/22/14)