Particle Astrophysics and Cosmology

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This is an advanced undergraduate course. Offered in Spring 2014 at Columbia University. Required Course textbook: Ryden, *Introduction to Cosmology*.

Contents

1 Expending Universe

We will study the standard model of cosmology: big bang model, cosmic microwave background radiation, nucleosynthesis, baryon asymmetry, neutrinos, dark matter, dark energy, cosmic Rays... We will also study the standard model of particle physics: standard model, quarks and leptons. Feynman diagram, compute cross section, supersymmetry, gravitational waves... The underlining theories are general relativity and quantum field theory. They work quite well for most cases. The key problem of the present days is to find a description of quantum gravity, which may enable us to recap what universe looked like within 10^{-43}sec, Planck time, after the big bang.

1.1 Special Relativity

Lecture 1 (1/22/14)

When Einstein developed special relativity he didn't know 4d (Minkowski) space, etc. He used ruler and clock. Later math facile people reformulated it. That is the version we study today.

4d spacetime diagram: as we will show later that coordinates transformation (called Lorentz transformation) between inertial reference frames, i.e. constant velocity between coordinates, are boosts, translation and rotation. Since pure translation and rotation in space do not involve relative velocity, so they are not interesting. However it is possible to use translation and rotation to make every Lorentz transformation look like a boost along x^1 direction, meaning two frames: lab frame and moving frame, moving frame moves at speed v along x^1 with synchronization that initial time $t = t' = 0$, $x = x' = 0$. Therefore when we draw 4d space time diagram we only have to draw ct and x^1 axis.

Principle of relativity

speed of light is same in all frames of reference

World line for a light beam is a straight line with slope 1, or symbolically

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

is equal to 0. Important to notice that $(\Delta s)^2$ is the norm of a 4 vector

$$x^\mu = (ct, x^1, x^2, x^3)$$

in the Minkowski space, hence it is invariant, which leads to define proper time τ

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 = (c\Delta t)^2(1 - v^2/c^2)$$

so

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1 \tag{1.1}$$

known as moving clock kicks slow.

Let $\beta = \frac{v}{c}$, we have Lorentz boost along x^1

$$\begin{cases} t' = \gamma(t - \beta x) \\ x' = \gamma(x - \beta t) \\ y' = y \\ z' = z \end{cases} \tag{1.2}$$

sanity check: 1) suppose a clock is sitting at the origin of the lab frame, hence $x = 0$. From the observer in the moving frame prospective

$$t' = \gamma t$$

$$\begin{aligned} t &= \text{proper time} \\ t' &= \text{measured time} \end{aligned}$$

consistent with (1.1). This also shows that the for any τ proper time measureed on the t' axis drawing in the lab frame become $\gamma\tau$, i.e. the time componenet of τ in the lab frame is $\gamma\tau$, so the t' axis in the lab frame is tilted with angle

$$\cos\theta = \frac{\tau}{\gamma\tau} \implies \tan\theta = \frac{v}{c} \tag{1.3}$$

hence the slope of t' axis in the lab frame is v/c wrt t axis.

2) suppose a ruler is place in a moving frame, with one end at the origin of the moving frame, the other end is at x'. From the observer in the lab frame prospective, we need to measure two ends simultaneously, i.e. $t = 0$, so

$$x' = \gamma x$$

or

$$x = \frac{x'}{\gamma}$$

$$\begin{aligned} x' &= \text{proper length} \\ x &= \text{measured length} \end{aligned}$$

showing length contraction. This also shows that the for any L proper length measureed on the x' axis drawing in the lab frame become L/γ, so the x' axis in the lab frame is tilted with angle

$$\cos\theta = \frac{L/\gamma}{L} \implies \tan\theta = \frac{v}{c} \tag{1.4}$$

hence the slope is of x' axis in the lab frame is v/c wrt x axis.

(1.2) in matrix form, define θ such that

$$\sinh\theta = \gamma\beta \;\; \cosh\theta = \gamma$$

This θ somehow looks like the angle defined in (1.3), (1.4).

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & & \\ -\sinh\theta & \cosh\theta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

so boost along x^1 has one free parameter.

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{1.5}$$

Above contains 4 equation. E.g. $x'^0 = \Lambda^0{}_\nu x^\nu$ means

$$x'^0 = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1 + \Lambda^0{}_2 x^2 + \Lambda^0{}_3 x^3$$

In the same spirit of Einstein notation, we can formulate norm

$$ds^2 = dt^2 - (dx)^2 - (dy)^2 - (dz)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$\eta_{\mu\nu}$ metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

And inner product

$$a^\mu \cdot b^\nu = \eta_{\mu\nu} a^\mu b^\nu = a_\nu b^\nu$$

We mentioned before Lorentz transformations are boosts, translation, and rotation. Why?

E.g. translation

$$x^\mu \to x'^\mu = (x^0 + a, x^1 + b, x^2 + c, x^3 + d)$$

rotation in space e.g. around x^3 axis

$$\Lambda^{rot} = \begin{pmatrix} 1 & \\ & \boxed{\begin{matrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{matrix}} \end{pmatrix}$$

so rotation about x^3 has one free parameter.

So we can extend the short hand formula (1.5) to work for boost, rotation and translation.

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

The condition for Lorentz transformation (forget about translation) is

$$\Lambda^T \eta \Lambda = \eta \tag{1.6}$$

i.e. inner product is preserved

$$\begin{aligned} \langle a|b\rangle &= a\eta b \\ \langle a'|b'\rangle &= \langle \Lambda a|\Lambda b\rangle = a\Lambda^T \eta \Lambda b \end{aligned}$$

In index notation

$$a_\nu b^\nu = \eta_{\mu\nu} a^\mu b^\nu = \Lambda_\mu{}^\rho \eta_{\rho\sigma} \Lambda^\sigma{}_\nu a^\mu b^\nu = \Lambda_\mu{}^\rho a^\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu b^\nu = \eta_{\rho\sigma} a'^\rho b'^\sigma = a'_\sigma b'^\sigma$$

By (1.6)

$$\det(\Lambda) = \pm 1$$

Proper Lorentz is to choose $\det(\Lambda) = 1$. In addition considering infinitesimal transformation, hence small deviation from the identity

$$\Lambda = \delta + \lambda$$

where $\delta^\mu_\nu = I$, we will show that λ is antisymmetric.

$$a \to a' = \Lambda a$$

and

$$a^2 = a'^2 = \left(a_\mu + \lambda_\mu{}^\nu a_\nu\right)\left(a^\mu + \lambda^\mu{}_\nu a^\nu\right) = a^2 + \lambda^\mu{}_\nu a^\nu a_\mu + \lambda_\mu{}^\nu a_\nu a^\mu + O(\lambda^2)$$

$$0 = \lambda^\mu{}_\nu a^\nu a_\mu + \lambda_\mu{}^\nu a_\nu a^\mu$$

exchange ν and μ for the second term

$$\lambda^\mu{}_\nu a^\nu a_\mu + \lambda_\nu{}^\mu a_\mu a^\nu$$

inserting pair of raising and lower operators

$$\left(\lambda^{\mu\nu} + \lambda^{\nu\mu}\right) a_{\nu\mu} = 0$$

Thus

$$\lambda^{\mu\nu} = -\lambda^{\nu\mu}$$

so the matrix of λ is antisymmetric, so Λ has 6 independent components: 3 rotation angles, 3 boost velocities. They form Lorentz group. The Poincare group includes translations, which adds 4 more parameters.

Example. Relativistic Doppler Effect

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