Fourier Analysis

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This is an advanced undergraduate course. Offered in Spring 2014 at Columbia University. Recommended Texts: Folland, *Fourier Analysis and its Applications*; Kammler, *A First Course in Fourier Analysis*; Stein, *Fourier Analysis: an Introduction.* Office hours: Tu 2:00-3:00, Th 1:00-2:00.

Contents

1 Introduction

Lecture 1 (1/21/14)

This subject is interesting from both pure and applied math perspective. Fourier analysis studies infinite sum of sines and cosines, which are set of functions $f : \mathbb{R} \to \mathbb{C}$ forms a vector space, i.e.

$$(f+g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

From linear algebra, once we have a basis we can write any vector wrt to the basis and any linear operator becomes a matrix in that basis. If there are enough eigenvectors, then we can diagonalize the matrix so that computation becomes simpler.

We would like to do the same for some function space. The problem of function space is that it is ∞ dimensional, so that no canonical basis, and problem of convergence. Despite these issues, many things still work nicely.

Example 1. Write

$$f = \sum_{n=0}^{\infty} a_n x^n \tag{1.1}$$

basis $\{1, x, x^2, ...\}$, coordinate $(a_0, a_1, ...)$. Three problems:

1) convergence i.e. radius of convergence may be 0

2) many functions can not be represented in this way. e.g. $|x|$. That is because analyticity implies ∞ differentiability

3) locality. (1.1) says if we know that happens at 0, i.e. $f(0)$, $f'(0)$, $f''(0)$,.. then we know f everywhere in the radius of convergence.

1.1 Idea of Fourier

This is actually idea of Clairaut.

For $f : \mathbb{R} \to \mathbb{C}$ 2π periodic, i.e

$$f(x + 2\pi) = f(x)\ \forall\, x \in \mathbb{R}$$

try to write

$$f(x) = c_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Remark 2. 1) If $g : \mathbb{R} \to \mathbb{C}$ is T periodic, then

$$f(x) = g(\frac{2\pi x}{T})$$

is 2π periodic.

2) If we have a function on domain $[0, 2\pi]$. It can be made into 2π periodic.

3) If $f : S^1 \to \mathbb{C}$, S^1 unit circle, then we can make

$$g(\theta) = f(\cos\theta, \sin\theta)$$

is 2π periodic.

Why trigonometric basis?

1) less rigid than polynomials (1.1). More functions can be represented by Fourier

2) Useful for solving important physical problems, e.g. heat equation, wave equation, Schrodinger equation.

3) Very nice algebraic structure.

4) The coefficients have physical relevant: frequency, amplitude, regularity, etc

5) $\sin nx$, $\cos nx$ are eigenfunctions for d^2/dx^2

6) magic solution to many theoretical problems: e.g. we will do later using Fourier to derive Cauchy formula in complex variables.

1.2 1D Heat Equation

This is a classical problem stated and solved by Fourier.

x space, t time, $u(x, t)$ temperature of rod at location x and time t. Heat equation

$$\partial_t u(x, t) = k u_{xx}(x, t) \quad \forall\, x, t \tag{1.2}$$

$k > 0$ conductivity.

Lecture 2 (1/23/14)

Fix some initial state

$$u(x, 0) = u_0(x)$$

Fourier put the simplest Dirichlet boundary conditions

$$u(0, t) = u(2\pi, t) = 0 \;\forall\, t$$

this will enforce periodicity. Physically this means 2 ends in contact with reservoir $T = 0$. One can check that if 2 ends are kept at different temperature, say

$$u(0,t) = A \quad u(2\pi,t) = B$$

then

$$u(x,t) - A - \frac{x}{2\pi}(B - A)$$

solves (1.2) and satisfies the Dirichlet BC.

First observe that if $u_0 = \sin x$, then

$$u(x,t) = e^{-kt}\sin x$$

solves (1.2). If $u_0(x,t) = \sin nx$, n positive integer. Physically it means the initial temperature distribution has a lots of wiggles, so it should redistribute faster as $n \uparrow$. One can check that

$$u(x,t) = e^{-n^2kt}\sin nx$$

solves the problem.

Suppose

$$u_0(x) = c_1 \sin x + c_2 \sin 2x + ... + c_n \sin nx$$

then

$$u(x,t) = c_1 e^{-kt}\sin x + ... + c_n e^{-n^2kt}\sin nx$$

solves the heat equation. The question remains how to find c's.

1.3 Complex Notation

We try to represent functions

$$f : \mathbb{R} \to \mathbb{C}$$

that are 2π periodic as series

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

where $c_n \in \mathbb{C}$ and $\sum_{-\infty}^{\infty}$ is understood as $\lim_{N\to\infty}\sum_{-N}^{N}$. By switching to complex notation, many formulas become simpler, but not necessary becomes computationally easier. The sin, cos have computationally advantages, because many physical systems have even or odd parities.

One can check

$$u(x,t) = \sum_{-\infty}^{\infty} c_n e^{-n^2 kt} e^{inx}$$

solves (1.1) with

$$u_0 = \sum_{-\infty}^{\infty} c_n e^{inx}$$

so the problem is still to find c's.

Lemma 3. *For $k \in \mathbb{Z}$*

$$\int_0^{2\pi} e^{ikx} dx = \begin{cases} \frac{e^{ikx}}{ik}\Big|_0^{2\pi} = 0 & k \neq 0 \\ 2\pi & k = 0 \end{cases}$$

So if $f(x) = \sum_{-N}^{N} c_n e^{inx}$

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) dx \\ c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \end{aligned}$$

2 Fourier Main Theory

Summarize the idea of Fourier: for a 2π periodic $f : \mathbb{R} \to \mathbb{C}$ the Fourier coefficients $c_k(f)$ is given by

$$c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \tag{2.1}$$

assume f is Riemann integrable because we need to perform the integration (2.1), then

$$f(x) = \sum c_k(f) e^{ikx}$$

We want to know when the sum does converge and $f(x) = \sum$ the equality in what sense.

Rewrite this in language of linear algebra, vector space

basis $e_k(x) = e^{ikx}$. That $k \in \mathbb{Z}$ is important because of the assumption in lemma 3.

In analogy of inner product in $\mathbb{C}^N$

$$\langle v, w\rangle = \sum \overline{v_k} w_k$$

we define

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)} g(x) dx$$

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |f|^2\, dx\right)^{1/2}$$

It is easy to check that

$$\begin{aligned}
\langle f_1 + f_2, g\rangle &= \langle f_1, g\rangle + \langle f_2, g\rangle \\
\langle f, g_1 + g_2\rangle &= \langle f, g_1\rangle + \langle f, g_2\rangle \\
\langle f, g\rangle &= \overline{\langle g, f\rangle} \\
\langle \alpha f, g\rangle &= \bar{\alpha}\langle f, g\rangle \\
\langle f, \alpha g\rangle &= \alpha \langle f, g\rangle
\end{aligned}$$

Hence in this notion

$$\langle e_k, e_l\rangle = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

and the Fourier coefficients become

$$c_k(f) = \langle e_k, f\rangle$$

Lecture 3 (1/28/14)

Define partial Fourier sum

$$S_N = \sum_{n=-N}^{N} c_n(f) e_n$$

Does S_N converge? converge to what? Preferable to f, in what sense. If it does converge to f, for an operator on f, e.g. taking derivative, can we perform operations on the $c_n(f)$ instead of f? and the resulting sum converges? in what sense?

Example 4. Fourier series of $f(x) = |\sin x|$ is

$$\frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

The $\sum \frac{1}{n^2}$ looks like converge nicely.

In general, the convergence is a difficult problem. Many fundamental results were just found recently in 60-70s.

We will prove the conditions on f for $S_N f$ to converge to f in three different kinds of convergence and each of them has different implications.

- Point wise

  $$\forall\, x \in \mathbb{R}\; \epsilon > 0\, \exists\, N_0 \text{ s.t. } |S_N f - f| < \epsilon \quad \forall\, N \geq N_0$$

  symbolically

  $$\text{for all } x \quad S_N f(x) \to f(x)$$

- Uniform

  $$\forall\, \epsilon > 0\, \exists\, N_0\, \forall\, N \geq N_0\, \forall\, x \in \mathbb{R} \quad |S_N f(x) - f(x)| < \epsilon$$

  symbolically

  $$\|S_N f - f\|_\infty \to 0$$

  where $\|\cdot\|_\infty$ is called the super norm

  $$\|g\|_\infty = \sup_{x \in [0,2\pi]} |g(x)|$$

- L^2 convergence

  $$\|S_N f - f\|_2 \to 0$$

  where $\|\cdot\|_\infty$ is called the Lˆ2 norm

  $$\|g\|_2 = \sqrt{\langle g, g \rangle}$$

Lecture 4 (1/30/14)
Lecture 5 (2/4/14)
Lecture 6 (2/6/14)
Lecture 7 (2/11/14)
Lecture 8 (2/13/14)
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