Particle Phenomenology

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This is a graduate course. Offered in Fall 2013 at Columbia University. Recommended Books: Halzen, Martin, *Quarks and Leptons.* Quigg, *Gauge Theories of the Strong Weak and Electromagnetic Interactions.* Langacker, *The Standard Model and Beyond.*

Visit particles data updates: http://pdg.lbl.gov/index.html

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Course Overview

Lecture 1 (9/4/13)

This course will cover Klein-Gordon equation, Dirac equation, practical aspects of relativistic quantum mechanics. QM chromodynamics, week interaction, Gauge theory, etc.

The principle difference between this course and the 8000 level course: particle physics I is that 8000 level course uses top down approaches, starting with Lagrangian of the universe and develop QFT from there. We do bottom up approaches. We follow the historic development of the subjects.

What in front of us is the standard model.

Fermion are tabulated below

	Flavor			Charges	Color	Interaction
quarks	up	$charm$	top	$2/3$	Y	S,EM,W
	$down$	$strange$	$bottom$	$-1/3$	Y	S,EM,W
leptons	ν_1 (ν_e)	ν_2 (ν_μ)	ν_3 (ν_τ)	0	N	W
	e	μ	τ	-1	N	EM,W

The 1st column 1st generation, the 2nd column 2nd generation, and 3rd column are 3rd generation. They becomes lest and less stable as going from 1st generation to 3rd generation.

Bosons

Strong interaction:	8 gluons
Week interaction:	3 W Bosons
EM interaction:	B

Higgs Boson (spin 0) We need Higgs mechanism. Before it, fermion has no mass.

Hypothetical Graviton (spin 2). Not yet detected.

1 Symmetry

Physics is independent of reference frame. Whether we a prior call positive charges positive and negative charges negative, it should not matter. In EM Lorentz transformation can interchange $\vec{E}$ and $\vec{B}$, but the resulting forces we obtain should be the same.

1.1 Groups

One example of a group is rotation in space. Then we see some characters of a group:

1) If R_1, R_2 are rotations, so is $R_1 R_2$.

2) There exists a unit element I s.t.

$$IR = RI = R$$

3) $\exists$ inverse

$$R^{-1}R = I = RR^{-1}$$

Note that commutativity is not required.

Rotation can be decomposed into infinitesimal rotations, i.e.

$$R_\theta = R_{\left(\frac{\theta}{n}\right)^n}$$

This gives a model of Lie groups.

There are discrete symmetries, we will take about them later.

1.2 Noether's Theorem

Let's look at what conditions the statement "Physics is independent of reference frame" imposes on physical states $|\psi\rangle$.

Since physics is invariant under rotation of frame, i.e.

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle$$

For any real measurable quantity such as $\langle\phi|\psi\rangle$ should be the same, for any $|\psi\rangle$,

$|\phi\rangle$,

$$\langle\phi|\psi\rangle = \langle\phi'|\psi'\rangle = \left\langle\phi\middle|U^\dagger U\middle|\psi\right\rangle$$

Thus

$$U^\dagger U = I$$

Such U is called unitary.

We can correlate U with R to say that U's form a group of unitary representation of the group of rotations.

More is true. Since physics is unchanged, Hamiltonian is preserved under a symmetry operation,

$$\langle\phi|H|\psi\rangle = \langle\phi'|H|\psi'\rangle = \left\langle\phi\middle|U^\dagger HU\middle|\psi\right\rangle$$

Thus

$$[H, U] = 0$$

So if we take

$$i\frac{d}{dt}\langle\psi(t)|U|\psi(t)\rangle = \langle\psi(t)|[H, U]|\psi(t)\rangle = 0$$

This is saying that the expectation value of U at the state $\psi(t)$ is independent of time, synonym: constant of motion or conserved. If ψ happens to be eigenstate of U, then the eigenvalue is independent of time.

Continue the infinitesimal rotation example, put

$$U = I - i\epsilon J_z$$

J_z called generator of rotations, then

$$1 = U^\dagger = (I + i\epsilon J_z^\dagger)(I - i\epsilon J_z) = 1 + i\epsilon(J_z^\dagger - J_z) + O(\epsilon^2)$$

Thus J_z is Hermitian, i.e. has real eigenvalues, so J_z is observable.

We want to get a form of J_z, suppose ψ is scalar function, and R is an infinitesimal rotation about z axis, i.e.

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\epsilon & \\ \epsilon & 1 & \\ & & 1 \end{pmatrix}$$

then

$$\begin{aligned} \psi'(\vec{r}) &= U\psi(x,y,z) = \psi(R^{-1}\vec{r}) \\ &= \psi(x+\epsilon y, y-\epsilon x, z) \\ &= \psi(x,y,z) + \epsilon(y\frac{\partial\psi}{\psi x} - x\frac{\partial\psi}{\partial y}) \\ &= \psi(x,y,z) + \epsilon(yip_x - xip_y)\psi \\ &= (I - i\epsilon J_z)\psi \end{aligned}$$

where $J_z = xp_y - yp_x$, hence we identify J_z as the z component of angular momentum operator.

Noether says symmetry implies conservation law. In particular

invariant under rotation→conservation of angular momentum

invariant under space translation→conservation of momentum

invariant under time translation→conservation of energy

The significance of Noether is that the left hand sides are symmetries. They are found in the mathematical formulation of the states, while the right hand sides are fundamental observable physical quantities.

1.3 Clebsch-Gordan

Given two fermions, we cannot add like two vectors classically, for we cannot access each component of the state vectors. Since

$$[J_i, J_j] = i\epsilon_{ijk}J_k \qquad [J^2, J_i] = 0$$

so QM only enables to observe one component of J at a time. Spin is much alike.

We can have definitive total angular momentum j, and projection on z axis, m, namely the state

$$|j, m\rangle$$

Recall for $|j, m\rangle$, measure J^2 yield

$$j(j+1)\hbar \qquad j = 0, 1, 2, ...$$

measure projection on z, yield

$$m_j\hbar \qquad m_j = -j, -j+1, ..., 0, 1, ..., j$$

Add $|j_1, m_1\rangle$, $|j_2, m_2\rangle$ to $|j, m\rangle$. The z component add $m = m_1 + m_2$. j takes all values from $|j_1 - j_2|$ to $j_1 + j_2$.

$$|j, m\rangle = (\ )\,|j_1, m_1\rangle + (\ )\,|j_2, m_2\rangle$$

the coefficients are tabulated.

The calculable way to obtain Clebsch-Gordan is taught in QM class.

1.4 Isospins

Lecture 2 (9/9/13)

We will talk about group theory in the next three lectures then we will do other staff, then come back to group theory in two months.

$$\begin{aligned} O(n) &= \text{n}\times\text{n real orthogonal matrices} = O^T O = I \\ SO(n) &= \text{n}\times\text{n real orthogonal matrices with } det = 1 \end{aligned}$$

e.g. rotation in space is $SO(3)$.

$$\begin{aligned} U(n) &= \text{n}\times\text{n unitary matrices} = U^T U = I \\ SO(n) &= \text{n}\times\text{n unitary matrices with } det = 1 \end{aligned}$$

One can represent groups (GL) by a set of matrices, e.g.

$$R_1 R_2 = R_3 \implies M_1 M_2 = M_3$$

Not all representations are useful, e.g. the trivial representation mapping everything to I.

Some representations are bizarre as in the following example using $SU(2)$ to represent $SO(3)$.

Consider fermion

$$\left|\frac{1}{2} + \frac{1}{2}\right\rangle$$

is represented by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\left|\frac{1}{2} - \frac{1}{2}\right\rangle$$

is represented by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then the state is

$$|\psi\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $|\alpha|^2 + |\beta|^2 = 1$. Suppose we want to rotate reference frame by $\vec{\theta} = \theta \hat{n}$. $\hat{n}$ is the rotation axis. Then $SO(3)$ is not that useful, since we need a 2D representation here. $SU(2)$ is good,

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U(\vec{\theta}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Our goal is to find $U(\vec{\theta})$. This is not an infinitesimal rotation, so we put

$$U = e^{-i\vec{\theta}\cdot\vec{J}}$$

$\vec{J}$ generator of rotation. One can show

$$J_i = \frac{1}{2}\sigma_i$$

σ_i Pauli matrix, forms a basis of $SU(2)$.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\frac{1}{2}\vec{\sigma}} = \sum_\alpha \cos\frac{\theta_\alpha}{2} - i\sigma_\alpha \sin\frac{\theta_\alpha}{2}$$

The detail calculation is taught in QM class.

Application: Strong Interaction

1st order, one cannot tell difference between proton and neutron. So the concept of "isospin" is invented, put

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Unlike spin up down for electrons is perfect symmetry, here p and n are not exactly the same particles, but in strong interaction, 1st order p, n are the same, $m_p \approx m_n$. Both have 2 degrees of freedom. Just like spin, internally we don't know what that is, externally it has magmatic effects, and they somehow agree the classical interpretation

$$H = \mu \cdot \vec{B}$$

For isospin, internally we don't know what that is; externally one is p and the another is n. That is why we say isospin is an internal symmetry.

All evidences show that

$$pp = |1\,1\rangle := \Pi^+$$

and

$$nn = |1\,-1\rangle := \Pi^-$$

are not stable. So we speculate that $L = 1$ don't exist, i.e. no potential well, no bound states

Also

$$\Pi^0 =: |1,0\rangle = \frac{1}{\sqrt{2}}(pn + np)$$

doesn't exists. Only

$$d =: |0,0\rangle = \frac{1}{\sqrt{2}}(pn - np)$$

stable, called deuteron, the nucleus of deuterium ^{2}H, called heavy water. They produced in nuclear reactions. To get a chain of reaction, one has to keep deuteron.

In isospin language, Π^-, Π^+, Π^0 are called isospin triplet, p, n are called isospin doublet, and d is isospin singlet. As an exercise, one can compute ratio of cross section

$$\frac{\sigma(pp \to \Pi^+ d)}{\sigma(pn \to \Pi^0 d)}$$

Because $pp \to \Pi^+ d$ is

$$|11\rangle \to |11\rangle + |00\rangle = |11\rangle$$

$pn \to \Pi^0 d$ is

$$\frac{|10\rangle + |00\rangle}{\sqrt{2}} \to |10\rangle + |00\rangle = |10\rangle$$

Hence

$$\frac{\sigma(pp \to \Pi^+ d)}{\sigma(pn \to \Pi^0 d)} = \frac{|\langle 11|11\rangle|^2}{|\left\langle \frac{|10\rangle+|00\rangle}{\sqrt{2}} \middle| 10 \right\rangle|^2} = 2$$

1.5 Multiplets

Rotate the reference frame affect Π^+, Π^0, Π^- triplet. Similar we put

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we need 3D representation, $SU(3)$, which has $n^2 - 1 = 8$ degrees of freedom, or 8 generator

$$g_i = \frac{\lambda_i}{2}$$

or 8 basis λ_i, in particle physics, the convention is to use Gell-Mann matrices. There are only two diagonal ones

$$\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

and

$$U = e^{-i\vec{\theta}\cdot\vec{g}}$$

After the invention of cyclotron, all kinds of particles came out. At the time people named them arbitrary, then Gell-Mann and independently Zweig discovered a way to symmetrize them. The idea is to combine group multiplets. One can formally prove what we are going to do using tensors, but we have an easy way.

Define $SU(n)$ multiplet using $n-1$ numbers, called number of steps.

For $SU(2)$, one number

$$\text{doublet} \quad \underset{p}{\cdot} \;\text{---}\; \underset{n}{\cdot}$$

$\alpha = 1$.

$$\text{triplet} \quad \underset{\Pi^-}{\cdot} \;\text{---}\; \underset{\Pi^0}{\cdot} \;\text{---}\; \underset{\Pi^+}{\cdot}$$

$\alpha = 2$.

For $SU(3)$, two numbers α,β, numbers of steps across top and bottom

$$\nabla \quad (1,0) \text{ triplet}$$

$$\triangle \quad (0,1) \text{ antitriplet}$$

$$(1,1) \text{ octet}$$

looking like a hexagon with two nodes at the center.

$$(3,0) \text{ decuplet}$$

Number of nodes N is

$$
\begin{aligned}
SU(2) &: N = \frac{\alpha+1}{1} \\
SU(3) &: N = \frac{\alpha+1}{1}\frac{\beta+1}{1}\frac{\alpha+\beta+2}{2} \\
SU(4) &: N = \frac{\alpha+1}{1}\frac{\beta+1}{1}\frac{\gamma+1}{1}\frac{\alpha+\beta+2}{2}\frac{\beta+\gamma+2}{2}\frac{\alpha+\gamma+2}{2}\frac{\alpha+\beta+\gamma+3}{3}
\end{aligned}
$$

To visualize $SU(4)$, we need 3D drawing.

What can we do with them?

$$
\begin{array}{cccccccccccccc}
 & & & & & & & & \alpha b & \text{---} & \alpha c & & & \\
b & & c & & \alpha & & & & & & & & & \\
 & \nabla & & \otimes & \Delta & & = & r\beta & & & & \beta c & \oplus & \cdot \\
 & a & & \gamma & & \beta & & & & & & & & \\
 & (1,0) & & & (0,1) & & & & \gamma a & \text{---} & \beta a & & & (0,0) \\
 & & & & & & & & & & & & & \\
 & N=3 & & & N=\bar{3} & & & & & N=8 & & & & \text{singlet}
\end{array}
$$

There must be 2 at the center of the hexagon. Since the singlet is

$$\frac{1}{\sqrt{3}}(\alpha a + \beta b + \gamma c)$$

the other 2 nodes must be some other linear combination of $\alpha a, \beta b, \gamma c$. We find consistently

$$3 \times 3 = 8 + 1$$

1.6 Young Diagrams

This method above becomes hard to illustrate when number is large.

Young has some rules:

1) Leave no gap of the paper, e.g. first one is good. Second is bad.

$$\begin{array}{l} \square \\ \square \end{array} \qquad \begin{array}{ll} \square & \\ & \square \end{array}$$

2) Each row has at least as many boxes as row below
3) At most n rows for $SU(n)$
4) $\alpha = \#$ of boxes in row $1 - \#$ of boxes in row 2,
$\beta = \#$ of boxes in row $2 - \#$ of boxes in row 3, etc.
For example

$$\begin{array}{l} \square \end{array} \quad (1,0) \qquad \begin{array}{l} \square \\ \square \end{array} \quad (0,1) \qquad \begin{array}{l} \square \\ \square \\ \square \end{array} \quad (0,0) \quad \text{trivial}$$

$$\begin{array}{ll} \square & \square \\ \square & \end{array} \quad (2,1) \qquad \square\square\square \quad (3,0)$$

We want to do

$$(1,1) \otimes (1,1)$$

Label same letter for each row

$$\begin{array}{ll} \square & \square \\ \square & \end{array} \otimes \begin{array}{ll} \boxed{a} & \boxed{a} \\ \boxed{b} & \end{array}$$

Add a's to the leg, maximum 1 a per column

$$\begin{array}{llll} \square & \square & \boxed{a} & \boxed{a} \\ \square & & & \end{array} \oplus \begin{array}{lll} \square & \square & \boxed{a} \\ \square & \boxed{a} & \end{array} \oplus \begin{array}{lll} \square & \square & \boxed{a} \\ \square & & \\ \boxed{a} & & \end{array} \oplus \begin{array}{ll} \square & \square \\ \square & \boxed{a} \\ \boxed{a} & \end{array}$$

Add b's the same way, and reading from right to left and reading from top to bottom there are at least many a's as b's.

$$
\begin{array}{cccc}
\square & \square & \boxed{a} & \boxed{a} \\
\square & \boxed{b} & &
\end{array}
\oplus
\begin{array}{ccc}
\square & \square & \boxed{a} \\
\square & \boxed{a} & \boxed{b}
\end{array}
\oplus
\begin{array}{ccc}
\square & \square & \boxed{a} \\
\square & \boxed{b} & \\
\boxed{a} & &
\end{array}
\oplus
\begin{array}{cc}
\square & \square \\
\square & \boxed{a} \\
\boxed{a} & \boxed{b}
\end{array}
$$

$$
\begin{array}{cccc}
(2,2) & (0,3) & (1,1) & (0,0) \\
N = 27 & \bar{10} & 8 & 1
\end{array}
$$

$$
\oplus \qquad \oplus
$$

$$
\begin{array}{cccc}
\square & \square & \boxed{a} & \boxed{a} \\
\square & & & \\
\boxed{b} & & &
\end{array}
\oplus
\begin{array}{ccc}
\square & \square & \boxed{a} \\
\square & \boxed{a} & \\
\boxed{b} & &
\end{array}
$$

$$
\begin{array}{cc}
(3,0) & (1,1) \\
N = 10 & 8
\end{array}
$$

Total N

$$N = 8 \times 8 = 27 + 10 + 10 + 8 + 8 + 1$$

1.7 Meson

Lecture 3 (9/11/13)

Last time we showed we derive Pions Π^-, Π^+, Π^0 from isospin of p and n, constituents of u and d. Later Kaons K^+, K^-, K^0, $\bar{K}^0$ were discovered, the solution was to go high dimension. Introduces strange quarks. Put them together as

$$
\begin{array}{ccc}
d & & u \\
& \nabla & \\
& s &
\end{array}
\otimes
\begin{array}{ccc}
& \bar{s} & \\
& \Delta & \\
\bar{u} & & \bar{d}
\end{array}
=
\begin{array}{ccccccc}
& & d\bar{s} & \text{---} & u\bar{s} & & \\
& / & & & & \backslash & \\
d\bar{u} & & & & & & u\bar{d} \\
& \backslash & & & & / & \\
& & \bar{u}s & \text{---} & \bar{d}s & &
\end{array}
\oplus \cdot
$$

Hypercharge

$$Y = B + S$$

B Baryon number, S strangeness.

$$B = \begin{cases} \frac{1}{3} & \text{quarks} \\ 0 & \text{leptons} \end{cases} \quad S = \begin{cases} -1 & \text{strange} \\ 0 & \text{other quarks} \end{cases}$$

And the charge is given by

$$Q = I_3 + \frac{Y}{2}$$

I_3 is the z component of the isospin. E.g.

$$\begin{aligned} Q_u &= \frac{1}{2} + \frac{\frac{1}{3}+0}{2} = \frac{2}{3} \\ Q_d &= -\frac{1}{2} + \frac{\frac{1}{3}+0}{2} = -\frac{1}{3} \\ Q_s &= 0 + \frac{\frac{1}{3}-1}{2} = -\frac{1}{3} \end{aligned}$$

What particle is $d\bar{u}$?

$$Q = -\frac{1}{3} - \frac{2}{3} = -1, \quad I_3 = -\frac{1}{2} - \frac{1}{2} = -1 \implies \Pi^-$$

What particle is $\bar{u}s$?

$$Q = -\frac{2}{3} - \frac{1}{3} = -1, \quad I_3 = -\frac{1}{2}, \quad S = -1 \implies K^-$$

In sum, we identify

$$\begin{array}{ccccccccc} & & K^0 & \text{---} & K^+ & & & & \\ & / & & & & \backslash & & & \\ \Pi^- & & & \Pi^0, \eta & & & \Pi^+ & \oplus & \cdot \\ & \backslash & & & & / & & & \eta' \\ & & K^- & \text{---} & \bar{K}^0 & & & & \end{array}$$

1.8 Baryons

The real test of the Gell-Mann model was the attempt to figure out Baryons.

$$\begin{array}{ccccccccccc} d & & u & & d & & u & & d & & u \\ & \nabla & & \otimes & & \nabla & & \otimes & & \nabla & \\ & s & & & & s & & & & s & \end{array}$$

Use Young boxes

$$\begin{array}{ccccccc} \square & \otimes & \square & \otimes & \square & & \\ & & \Downarrow & & & & \\ (\square\boxed{a} & \oplus & \begin{array}{c}\square) \\ \boxed{a}\end{array} & \otimes & \square & & \\ & & \Downarrow & & & & \\ \square\square\boxed{a} & \oplus & \begin{array}{cc}\square & \square \\ \boxed{a} & \end{array} & \oplus & \begin{array}{cc}\square & \boxed{a} \\ \square & \end{array} & \oplus & \begin{array}{c}\square \\ \square \\ \boxed{a}\end{array} \\ (3,0) & & (1,1) & & (1,1) & & (0,0) \end{array}$$

that is

$$N = 3 \times 3 \times 3 = 10 + 8 + 8 + 1$$

10 is decuplets

$$\begin{array}{ccccccccccccc} \Delta & \text{---} & \text{---} & \text{---} & \Delta^0 & \text{---} & \text{---} & \text{---} & \Delta^+ & \text{---} & \text{---} & \text{---} & \Delta^{++} & 1240\text{MeV} & S=0 \\ & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & & \\ & & \Sigma^{*-} & \text{---} & \text{---} & \text{---} & \Sigma^{*0} & \text{---} & \text{---} & \text{---} & \Sigma^{*+} & & & 1385\text{MeV} & S=-1 \\ & & & \diagdown & & \diagup & & \diagdown & & \diagup & & & & & \\ & & & & \Xi^{*-} & \text{---} & \text{---} & \text{---} & \Xi^{*0} & & & & & 1520\text{MeV} & S=-2 \\ & & & & & \diagdown & & \diagup & & & & & & & \\ & & & & & & \Omega^- & & & & & & & & S=-3 \end{array}$$

8 is octet (there is another 8, but people have disputes about what they correspond to)

$$\begin{array}{ccccccc}
 & n & ___ & p & & & 940\text{MeV} \\
 & / & & & \backslash & & \\
\Sigma^- & & \Sigma^0, \Lambda & & & \Sigma^+ & 1190\text{MeV} \\
 & \backslash & & & / & & \\
 & \Xi^- & ___ & \bar{\Xi}^0 & & & 1315\text{MeV}
\end{array}$$

We see that particles on the same line have same mass, and they prefer decay give up a unit of strangeness. One success of the model was the prediction of Ω^- and later found it. Ω^- decays

$$\Omega^- \rightarrow \Xi^- + \Pi^0$$

or

$$\Omega^- \rightarrow \Xi^0 + \Pi^-$$

or

$$\Omega^- \rightarrow \Sigma^0 + K^-$$

or

$$\Omega^- \rightarrow \Lambda + K^-$$

with a long life time $10^{-10}s$, which the characteristic time of weak decay.

The strong decay time is much shorter, e.g.

$$\Delta^{++} \rightarrow p + \Pi^+$$

with a life time $10^{-24}s$, which convert to length, it is within the nuclear distance, not observable.

Note: Baryons cannot be the center of multiplets, so Mesons must be the centers.

1.9 Colors

So far we have successfully explain Baryons and Mesons using $SU(3)_{flavor} = \{u, p, s\}$. Question: why no free quarks? Gell-Mann introduced 3 colors: red, blue, and yellow. They combine to give white. There are also 3 anti colors.

We think 3 colors are triplet of $SU(3)_L$. The generators of $SU(3)_L$ are the 8 gluons. Each gluon carries a color and an anti color.

Gell-Mann says only white particles can be free on macroscopic scales, i.e. only see bound states of quarks.

So $(3 \otimes \bar{3})$ $q\bar{q}$ $\rightarrow$mesons, color with its anti color.

Or $(3 \otimes 3 \otimes 3)$ qqq $\rightarrow$baryons, with three different color.

This is fundamentally different from EM. In EM photon serves the interaction, but itself has no charge. Gluon serves strong interaction, and it carries charge. This is related to the fact that $SU(3)$ is not abelian.

Because the charge causes screening effect, as it gets closer. This says to get accurate measurements and get closer, one need high energy. In other words the coupling constants are energy dependent.

$$\alpha(E = 0) = \frac{1}{137}$$

$$\alpha_{EM}(E = 90\text{GeV}) = \frac{1}{127}$$

What is screening effect in the sense of particle experiments? EM we know the medium will get polarized and shell the charges, so the actual value of the charge is greater. In QFT vacuum is full of e^- and e^+, they will too be created and shell the charges.

At high very energy $\sim$ 10GeV, we get weak coupling, α_s approaches a constant, called asymptotic freedom. At low energy $\Lambda_{QCD} \approx$ 150MeV, $\alpha_s \rightarrow 0$ vertically asymptote, called confinement, where perturbation theory fail, because too much screening, too complicated.

If we try to get free quarks, e.g. Π^+ made of $u\bar{d}$, if we pull apart the two quark and anti quark. It will initial become a color flux tube, continue pulling, A q and $\bar{q}$ will pop out of the vacuum. Each goes with u and $\bar{d}$, so that color neutral, so no free quarks, like breaking a magnet.

1.10 Homework 1 (due 9/23/2013)

When beam of p collide head on with very large energy $\gg \Lambda_{QCD}$. If it happens that 2 quarks interact, they will come out in opposite directions, and the same time quarks, anti quarks pop out of vacuum. This results 2 jets of particles coming out.

Write a report with no more than 2 pages. Discuss what is jet algorithm. Research on 3 jet events. What is that?

1.11 Discrete Symmetries

Lecture 4 (9/16/13)

We discuss three types of discrete symmetries: Charge conjugation, Parity, Time reversal. We won't talk a lot about time reversal, which is not much relevant to us.

Charge conjugation

C changes particles to antiparticles. E.g.

$$\begin{aligned} C\left|p\right\rangle &= \left|p^-\right\rangle \\ C\left|e^-\right\rangle &= \left|e^+\right\rangle \end{aligned}$$

It has two group elements and $C^2 = I$.

Note: some neutral particles can be their own antiparticles, e.g. γ (photon), Π^0 (pion), η, J/ψ (two letters for one particle.)

Some of them have $-$ eigenvalue of C

$$C\left|r\right\rangle = -\left|r\right\rangle$$

Such particles are called "magorana" particles.

For $q\bar{q}$ bound states

$$C\left|q\bar{q}\right\rangle = (-1)^{l+s}\left|q\bar{q}\right\rangle$$

where l is orbital angular momentum, s spin angular momentum. E.g.

$$\begin{aligned} C\left|\Pi^0\right\rangle &= \left|\Pi^0\right\rangle \\ C\left|\rho\right\rangle &= -\left|\rho\right\rangle \end{aligned}$$

ρ has spin 1.

Eigenvalues of C is conserved in strong and EM interactions. E.g.

$$\Pi^0 \to \gamma\gamma$$

but three photon decay is not allowed,

$$\Pi^0 \not\to \gamma\gamma\gamma$$

Parity

It means to reverse reference frame

$$\begin{cases} x \to -x \\ y \to -y \\ z \to -z \end{cases} \tag{1.1}$$

It's two group elements P, I

One can differentiate physical quantities by their eigenvalues with P

$$\begin{aligned} P(a) &= a \text{ scalar} \\ P(\vec{a}) &= \vec{a} \text{ vector} \\ P(\vec{a}\times\vec{b}) &= \vec{a}\times\vec{b} \text{ psudo vector} \\ P(\vec{a}\cdot(\vec{b}\times\vec{c})) &= -\vec{a}\cdot(\vec{b}\times\vec{c}) \text{ psudo scalar} \end{aligned}$$

Fermions have + parity; anti fermions have − parity.

$$P\left|q\bar{q}\right\rangle_{groundstate} = -\left|q\bar{q}\right\rangle$$

If not in ground state, multiply by $(-1)^l$.

Notation

$$J^{PC}$$

where $J =$ spin $P =$ partity$C =$ charges.

For examples Π's, K's, family have spin 0, $P = -1$, so we denote them

$$0^-$$

hence pseudoscalar.

ρ spin 1, $P = -1$, denoted

$$1^-$$

hence vector mesons.

In particular apply parity to reference frame,

$$P(\text{right handed frame}) = \text{left handed frame}$$

For example Helicity: projection of spin on direction of motion, with respect to the right hand rule.

$$m = \pm\frac{1}{2}$$

Say particle is moving in $+z$ direction and the spin is in the xy plane, counterclockwise, hence helicity is $m = +\frac{1}{2}$. Now apply P, by (1.1), the particle is moving in $-z$ direction and spin is clockwise, hence $m = -\frac{1}{2}$. So under parity helicity changes sign.

The physical realization of such frame changes is that one observer with move in the same direction as a massive particle. If the particle starts with a lower speed than the observer, then later it has higher speed than the observer. In the observer's frame, he will see a parity change.

From above discussion, helicity is not Lorentz invariant, so it is bad physical quantity.

In the 50's people thought about θ, τ particles

$$\theta^+ \to \bar{\nu}^+, \pi^0$$

$$\tau^+ \to \Pi^+\Pi^0\Pi^0$$

or

$$\tau^+ \to \Pi^+\Pi^+\Pi^-$$

θ^+, τ^+ have same mass, spin and same charge, except parity. θ^+ has $+$ under parity while τ^+ has $P = -1$. Nowadays we know they are same particle called K^+. At that time, everybody assumed parity was conserved.

Lee & Young proposed parity may not be 100% conserved in week decays.

Madam Wu experimented with ^{60}Co. In the beta decay, ^{60}Co nuclei emit electrons. Under very strong $\vec{B}$ field and low temperature, the spin (i.e. magnetic moment) of ^{60}Co nuclei align along $\vec{B}$. Wu found that there was higher probability for electrons emitted toward the opposite direction of the aligned nuclei spin, which marked the first experimental demonstration of the parity violation in the weak interactions.

In fact

$$\Pi^+ \to \mu^+ + \nu_\mu \tag{1.2}$$

has maximal parity violation. μ^+ to be 100% right handed.

Weak interaction produces left-handed fermions and right-handed anti fermions. E.g. right handed ν could exist, but it doesn't interact in weak interaction, only left-handed does.

What if we apply C to (1.2)?

$$\Pi^- \to \mu^-_{\text{r.h.}} + \bar{\nu}_\mu$$

not good, unless it is followed by P

$$\Pi^- \to \mu^-_{\text{l.h.}} + \bar{\nu}_\mu$$

So maybe weak interaction respect CP?

Another example–Kaon

$$K^0 = d\bar{s} \quad \bar{K}^0 = \bar{d}s$$

K^0 and $\bar{K}^0$ oscillate into one another. K^0 is pseudoscalar

$$P\left|K^0\right\rangle = -\left|K^0\right\rangle \qquad C\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle$$

$$P\left|\bar{K}^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad C\left|\bar{K}^0\right\rangle = \left|K^0\right\rangle$$

Then

$$CP\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad CP\left|\bar{K}^0\right\rangle = -\left|K^0\right\rangle$$

If we put

$$\left|K_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) \quad \left|K_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right)$$

$$CP\left|K_1\right\rangle = \left|K_1\right\rangle \quad CP\left|K_2\right\rangle = -\left|K_2\right\rangle$$

we call $\left|K_1\right\rangle$ CP-even, $\left|K_2\right\rangle$ CP-odd.

If CP is conserved,

$$K_1 \to \Pi\Pi \tag{1.3}$$

$$K_2 \to \Pi\Pi\Pi \tag{1.4}$$

Q (of reaction) $= E_{after} - E_{before}$ is much large for (1.3) than (1.4), so K_1's decays much faster than K_2. ($E\Delta t \sim \hbar$)

Although $K_{1,2}$ are the CP eigenstates, they are not eigenstates of weak interaction. In fact the eigenstates are

$$\left|K_L^0\right\rangle = \frac{1}{\sqrt{1+\epsilon_K^2}}\left(\left|K_2\right\rangle + \epsilon_K\left|K_1\right\rangle\right)$$

$$\left|K_R^0\right\rangle = \frac{1}{\sqrt{1+\epsilon_K^2}}\left(\left|K_1\right\rangle + \epsilon_K\left|K_2\right\rangle\right)$$

For the Kaon system, weak interaction violate CP by $\epsilon_K = 2.2 \times 10^{-3}$.

In summary, for the neutral Kaons

Strong Eigenstates: K^0, $\bar{K}^0$

CP Eigenstates: K_1, K_2

Weak Eigenstates: K_L^0, K_R^0

For mesons containing $b-$quarks (B-mesons), CP violation is much larger.

Time Reversal

T is anti-unitary in group theory. We won't talk about much.

Theorem. *(CPT) Any reasonable QFT conserves CPT.*

Hence T is violated in weak interaction too in order to compensate CP violation.

T reversal experiment is hard to do. First experiment by Bell entangle $\beta-$decays.

In conclusion: weak interaction is characterized by CPT.

2 Relativistic Quantum Mechanics

2.1 Resonances

Lecture 5 (9/18/13)

Consider

$$\rho^0 \to \Pi^+\Pi^-$$

with $\tau \sim 10^{-23}s$. τ is so small that uncertainty in mass is measurable, which means the particles have measurable width $\Gamma = 1/\tau$

$$|\psi(t)|^2 = |\psi(0)|^2 e^{-Tt} \implies \psi(t) = e^{-iMt}e^{-\Gamma t/2}$$

We want to measure in energy so do Fourier

$$\chi(E) = \int \psi(t)e^{iEt}dt \sim \frac{1}{(E-M)+i\Gamma/2}$$

$$|\chi(E)|^2 \sim \frac{1}{(E-M)^2-(\frac{\Gamma}{2})^2}$$

so Γ is the width and the graph of $|\chi(E)|^2$ vs E is peaked at M

2.2 Schrodinger

In non-relativistic quantum mechanics, we relate

$$E \to i\hbar\frac{\partial}{\partial t}$$

$$\vec{p} \to -i\hbar\nabla$$

For free particle $V = 0$

$$(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2)\psi = 0 \tag{2.1}$$

we learn that $\psi(x,t)$ is a complex wave function

$$\rho = |\psi|^2 \text{ prob density}$$

$$-\frac{\partial}{\partial t}\int \rho dV = \int_S \vec{j}\cdot d\vec{S} = \int \nabla\cdot\vec{j}dV$$

then we arrive continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

Apply $i\psi^*$ to (2.1)

$$-\psi^*\frac{\partial\psi}{\partial t} + \frac{i}{2m}\psi^*\nabla^2\psi = 0$$

Apply $i\psi$ to $(2.1)^*$

$$\psi\frac{\partial\psi^*}{\partial t} + \frac{i}{2m}\psi\nabla^2\psi^* = 0$$

Subtracting the two above

$$\frac{\partial|\psi^*\psi|}{\partial t} + \frac{i}{2m}(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi) = 0$$

or

$$\frac{\partial\rho}{\partial t} + \frac{i}{2m}\nabla(\psi\nabla\psi^* - \psi^*\nabla\psi) = 0$$

i.e.

$$\vec{j} = \frac{i}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi)$$

Look for plane wave solution

$$\psi = Ne^{i\vec{p}\cdot\vec{x}-iEt}$$

One can check it gives

$$\rho = |N|^2$$

$$\vec{j} = \frac{\vec{p}}{m}|N|^2$$

they agree the our classical intuitions.

2.3 Klein-Gordon

In particle physics, we need special relativity (still in flat space).

Recall 4-vector

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{x}) = (t, \vec{x})$$

4-momentum vector

$$p^\mu = (\frac{E}{c}, \vec{p})$$

$$AB = A^\mu B_\mu = A_\mu B^\mu = g^{\mu\nu} A_\mu B_\nu = A^0 B^0 - \vec{A}\cdot\vec{B}$$

Scalar product is invariant under Lorentz transformation, in particular

$$E^2 - \vec{p}^2 = m^2 \tag{2.2}$$

Theorem. *An equation is manifestly Lorentz covariant if all unrepeated indices (upper and lower separately) balance on either side and all repeated indices appear once as upper and once as lower indices.*

Forget about Schrodinger. It cannot even be written in 4 vector equation.

We use (2.2), and quantum mechanics says

$$p^\mu \to i\partial^\mu$$

Since

$$\partial^\mu = (\frac{\partial}{\partial t}, -\nabla) \qquad \partial_\mu = (\frac{\partial}{\partial t}, \nabla)$$

$$\partial^\mu \partial_\mu = \Box^2 \text{ d'Alembert}$$

(2.2) becomes

$$(\Box^2 + m^2)\phi = 0 \tag{2.3}$$

Klein-Gordon equation, later we will show it is only for spin 0.

We can write explicitly

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi \tag{2.4}$$

Can we find ρ, $\vec{j}$?

Apply $i\phi^*$ to (2.2)

$$-i\phi^* \frac{\partial^2 \phi}{\partial t^2} + i\phi^* \nabla^2 \phi = im^2 \phi^* \phi$$

Apply $i\phi$ to $(2.2)^*$

$$-i\phi \frac{\partial^2 \phi^*}{\partial t^2} + i\phi \nabla^2 \phi^* = im^2 \phi \phi^*$$

Subtract two above

$$-i(\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2}) + i(\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^*) = 0$$

$$i\frac{\partial}{\partial t}(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}) - i\nabla(\phi^* \nabla \phi - \phi \nabla \phi^*) = 0$$

Hence

$$\begin{aligned} \rho &= \phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \\ \vec{j} &= \phi^* \nabla \phi - \phi \nabla \phi^* \end{aligned}$$

or in 4 density

$$j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

so continuity says

$$\partial_\mu j^\mu = 0$$

Use free particle solution we can find

$$\rho = i(-iE - iE)|N|^2 = 2E|N|^2 \tag{2.5}$$

proportional to E. From ρ if one want to integrate

$$\int \rho d^3x$$

this has to be Lorentz invariant, so d^3x has to be $d^3x\sqrt{1-v^2}$ contracted, so

$$\rho \to \frac{\rho}{\sqrt{1-v^2}}$$

which is built in E.

One will also find

$$\vec{j} = 2\vec{p}|N|^2$$

or

$$j^\mu = 2p^\mu|N|^2 \tag{2.6}$$

2.4 Antiparticles

Inject free particle solution $\phi = Ne^{i\vec{p}\cdot\vec{x}-iEt}$ in to KG, yield

$$E = \pm\sqrt{\vec{p}^2 + m^2}$$

States with negative energy or by (2.5) negative ρ?!

Dirac concerns about this a lot. He spent a lot time trying to get ride of it, and he invented somethings more useful for semiconductors industry. He proposed that there are sea of negative energy states fully occupied initially, due to quantum fluctuation, some e^- pop out leaving a hole state e^+. His idea works only for fermions.

Feynman-Stueckelberg interpretation says there should really be a $-e$ in (2.6),

so

$$j^\mu(e^-) = -2ep^\mu |N|^2$$

e is the absolute value of the electron charge.

Then

$$\begin{aligned} j^\mu(e^+) &= 2e|N|^2(E, \vec{p}) \\ &= -2e|N|^2(-E, -\vec{p}) \end{aligned}$$

i.e. e^+ with positive E is equivalent to e^- with negative E moving in opposite time.

For a system emitting an e^+ is equivalent for a system to absorb an e^-.

In other words, negative energy particles going backwards in time is same as a positive energy particle going forward in time.

So we end up with two choices:

always works time forwards, then accept $E < 0$

or always works with particles, then accept time goes forward and backward. This is the approach we follow.

2.5 Fermi Golden Rule

Lecture 6 (9/23/13)

Recall at high energy we have weak coupling. At low energy $E < \Lambda_{QCD}$, perturbation doesn't work. Unperturbed solution, suppose H_0 independent of time.

$$H_0 \phi_n = E_n \phi_n$$

and

$$\int \phi_n^* \phi_m d^3x = \delta_{mn}$$

Now add potential may or may not dependent on t, V is concentrated locally

$$\left(H_0 + V(\vec{x}, t)\right)\psi = i\frac{\partial \psi}{\partial t}$$

suppose solution

$$\psi = \sum a_n \phi_n(\vec{x}) e^{-iE_n t}$$

Suppose before hit the target, $t = -\frac{T}{2}$, initial wave function has only one stationary state $a_i \neq 0$ and all other $a_{j\neq i} = 0$. The transition T_{fi} from state i to final after scattering state f

$$T_{fi} = a_f(\frac{T}{2}) = -i\int \phi_f^*(\vec{x})V(\vec{x})\phi_i(\vec{x})d^4x$$

$|T_{fi}|^2$ is the probability for particle to go $i \to f$. But it is not useful.

Assume V is time-independent,

$$T_{fi} = -i\int [\phi_f(\vec{x})e^{-iE_f t}]^* V(\vec{x})[\phi_i(\vec{x})e^{-iE_i t}]d^3x dt$$

Let

$$V_{fi} = \int [\phi_f(\vec{x})e^{-iE_f t}]^* V(\vec{x})[\phi_i(\vec{x})e^{-iE_i t}]d^3x \tag{2.7}$$

so

$$T_{fi} = -iV_{fi}\int e^{-i(E_f - E_i)t}dt = -iV_{fi}2\pi\delta(E_f - E_i)$$

hence $E_i = E_f$ then by uncertainty $\tau = 0$, not so useful, so we should define

$$W_{fi} = \lim_{T\to\infty}\frac{|T_{fi}|^2}{T}$$

transition probability per unit time.

$$|T_{fi}|^2 = |V_{fi}|^2 2\pi\delta(E_f - E_i)\underbrace{\int_{-T/2}^{T/2} e^{i(E_f - E_i)t}dt}_{T}$$

Hence

$$W_{fi} = 2\pi|V_{fi}|^2\delta(E_f - E_i)$$

To give this physical meaning, we integrate over all final states, let

$$\rho(E_f) = \text{density of final states}$$

so

$$
\begin{aligned}
W_{fi} &= 2\pi |V_{fi}|^2 \int \delta(E_f - E_i)\rho(E_f) dE_f \\
&= 2\pi |V_{fi}|^2 \rho(E_i)
\end{aligned}
$$

which is Fermi golden rule.

Cf (2.7), this is 1st order perturbation (Born approximation)

For higher order

$$
T_{fi} = -2\pi i V_{fi}\delta(E_f - E_i) - 2\pi i \sum_{n\neq i} \frac{V_{fn}V_{ni}}{E_i - E_n + i\epsilon}\delta(E_f - E_i) + ... \tag{2.8}
$$

the $i\epsilon$ is for integrals to converge. And we put the second order (including the 1st order)

$$
V_{fi}^{(2)} = V_{fi} + \sum_{n\neq i} \frac{V_{fn}V_{ni}}{E_i - E_n + i\epsilon}
$$

and the

$$
\frac{1}{E_i - E_n}
$$

is called propagator between vertices. As in (2.8)

$$
E_i = E_f
$$

but we don't need $E_i = E_n$, so intermediate states can be virtual, meaning exists in a time of fluctuation allowed by uncertainty principle, and during at time, particles can be created and energy is not conserved.

2.6 Homework 2 (due 10/7/2013)

Write a report, discuss Sakharov Conditions.

2.7 Minimal Substitution

Lecture 7 (9/30/13)

In classical EM particles in field A^μ, we can get the Humiliation and Schrodinger equation by simply

$$p^\mu \to p^\mu + eA^\mu$$

Similarly in turns of relativistic QM operator

$$i\partial^\mu \to i\partial^\mu + eA^\mu$$

then

$$\begin{aligned}\partial^\mu\partial_\mu \to (\partial^\mu + \frac{e}{i}A^\mu)(\partial_\mu + \frac{e}{i}A_\mu) &= \partial^\mu\partial_\mu + \frac{e}{i}\partial^\mu A_\mu + \frac{e}{i}A^\mu\partial_\mu - e^2A^2 \\ &= \partial^\mu\partial_\mu - ie(\partial^\mu A_\mu + A^\mu\partial_\mu) - e^2A^2\end{aligned}$$

so KG (2.3) becomes

$$(\partial^\mu\partial_\mu + m^2)\phi = \underbrace{[ie(\partial^\mu A_\mu + A^\mu\partial_\mu) + e^2A]}_{-V}\phi$$

by setting

$$V = -ie(\partial^\mu A_\mu + A^\mu\partial_\mu)$$

and ignoring e^2A^2 because it is small.

What is T_{fi} now?

$$\begin{aligned}T_{fi} &= -i\int \phi_f^* V_{fi}\phi_i d^4x \\ &= i\int \phi_f^*[-ie(\partial^\mu A_\mu + A^\mu\partial_\mu)]\phi_i d^4x\end{aligned}$$

The $\partial^\mu A_\mu$ term integration give

$$\phi_f^* A_\mu\ \phi_i|_{-\infty}^{\infty} - \int A_\mu\phi_i\partial^\mu\phi_f^* d^4x$$

So

$$\begin{aligned}T_{fi} &= i\int (\phi_f^*(ieA^\mu\partial_\mu\phi_i) - ieA_\mu\phi_i\partial^\mu\phi_f^*)d^4x \\ &= -i\int j_{fi}^\mu A_\mu d^4x\end{aligned}$$

setting

$$j^{\mu}_{fi} = -ie(\phi^*_f \partial_\mu \phi_i - \phi_i \partial^\mu \phi^*_f)$$

Assume initial state and final state are plane waves

$$\phi_{i,f} = N_{i,f} e^{-ip_{i,f}\cdot x}$$

then

$$j^{\mu}_{fi} = -eN_i N_f (p_i + p_f) e^{i(p_f - p_i)x} \tag{2.9}$$

compare to the non interacting picture $\rho = 2E|N|^2$. Here we have a particle coming in state i interact with the field and coming out in state j.

3 Quantum Electrodynamics

The next more interesting interaction we study is two particles collision and the interactions are purely electrodynamincal.

3.1 e^- μ^- Scattering

e^- before collision p_A and after collision p_C, and μ^- before collision p_B and after collision p_D. There are two vertices $(p_A + p_C)^\mu$ and $(p_B + p_D)^\nu$

Denote $j^{\mu(1)}$ to be the probability current connecting p_A to p_c and denote $j^{\mu(2)}$ to be the current connecting p_B to $p_{D.}$ What is connecting the vertices are A^μ field with propagator

$$\frac{-ig_{\mu\nu}}{q^2} \tag{3.1}$$

Why? We need to find A^μ. By (2.9)

$$j^{\mu(2)} = -eN_B N_D (p_B + p_D)^\mu e^{i(p_D - p_B)x}$$

and since

$$\Box^2 A^\mu = j^{\mu(2)}$$

and

$$\Box^2 e^{iqx} = -q^2 e^{iqx}$$

hence

$$A^\mu = -\frac{1}{q^2} j^{\mu(2)}$$

where

$$q = p_D - p_B \tag{3.2}$$

This slows (3.1) is right.

Compute T for e^-

$$\begin{aligned}
T_{fi} &= -i \int j^{\mu(1)} A^\mu d^4x \\
&= -i \int j^{\mu(1)} \left(-\frac{1}{q^2}\right) j^{\mu(2)} A^\mu d^4x \\
&= -ieN_A N_B N_C N_D \int (p_A + p_C)_\mu \frac{-1}{(p_D - p_B)^2} (p_B + p_D)^\mu e^{i(p_D + p_C - p_B - p_A)x} d^4x \\
&= -iN_A N_B N_C N_D (2\pi)^4 \delta^4 (p_D + p_C - p_B - p_A) \mathcal{M}
\end{aligned}$$

with

$$-i\mathcal{M} = ie(p_A + p_C)^\mu \frac{-ig_{\mu\nu}}{q^2} ie(p_B + p_D)^\nu$$

$\mathcal{M}$ is called matrix element or invariant amplitude.

Cf (3.2) shows as μ^- changes from p_B to p_D, since $q \neq 0$, virtual photons (not massless photons) are created. It is called off shell mass. If $q = 0$, means

$$p_B = p_D$$

so no interaction, the two particles don't collide.

3.2 Cross Section

As before T_{fi} is not so useful, we use

$$W_{fi} = \frac{|T_{fi}|^2}{T}$$

transition prob unit time

What are $N_{A,..,D}$? From the normalization

$$\phi = Ne^{-ipx}$$

$$\rho = 2E|N|^2 \qquad \int \rho dV = E \implies N \sim V^{-\frac{1}{2}}$$

so

$$W_{fi} = \frac{(2\pi)^4 \delta^4 (p_D + p_C - p_B - p_A)|\mathcal{M}|^2}{V^4}$$

Experimentally we measure cross section

$$\sigma = \frac{W_{fi}}{\text{(initial flux)}} \times \text{(number of final states)}$$

μ^- mass 200 times of e^-, the CM is at μ^-, initial flux is product of incoming A's per unit time and area

$$\frac{|\vec{v}_A| 2E_A}{V}$$

and number of target particles per unit volume

$$\frac{2E_B}{V}$$

They have $2E$ because $p = 2E$.

Initial flux is

$$\frac{4|\vec{v}_A|E_A E_B}{V^2}$$

Number of final states accessible to a particles in V is

$$\frac{Vd^3p}{(2\pi)^3}$$

Since we have $2E_C$ particles, the number of final states

$$\frac{Vd^3p_C}{(2\pi)^3 2E_C} \frac{Vd^3p_D}{(2\pi)^3 2E_D}$$

so

$$d\sigma = \frac{V^2}{4|\vec{v}_A|E_A E_B}\frac{1}{V^4}|\mathcal{M}|^2\frac{(2\pi)^4}{(2\pi)^6}\delta^4(p_D + p_C - p_B - p_A)\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}V^2$$

V is canceled, good for it is arbitrary to start with.

We write

$$d\sigma = \frac{|\mathcal{M}|^2}{F}dQ \tag{3.3}$$

where

$$dQ = \frac{1}{(2\pi)^2}\delta^4(p_D + p_C - p_B - p_A)\delta(E_f - E_i)\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D} \tag{3.4}$$

dQ is Lorentz invariant phase space factor.

$$F = 4|\vec{v}_A|E_A E_B$$

represents flux.

In sum what is $d\sigma$? It is the number of scatters per unit time per target particles per incoming particle into a given set of final states.

For simplicity in calculation let's assume

1) neglect mass i.e. $E = |p|$

2) choose center of mass frame (moving almost with μ^-) and set up the initial energies for e^- and μ^- so that

$$\vec{p}_B = -\vec{p}_A \tag{3.5}$$

Therefore $\delta^4(p_D + p_C - p_B - p_A)d^3p_C$ in (3.4) implies

$$\vec{p}_C = -\vec{p}_D \tag{3.6}$$

so

$$E_C = \sqrt{m_C^2 + |\vec{p}_D|}$$

so

$$dQ = \frac{1}{(2\pi)^2}\delta(E_f - E_i)\frac{d^3p_D}{4E_C E_D}$$

$$d^3p_D \to |\vec{p}|^2 \sin\theta dp d\theta d\phi = |\vec{p}_D|^2\frac{dp_D}{dE_f}dE_f d\Omega$$

$$E_f = E_C + E_D = \sqrt{m_C^2 + \vec{p}_D} + \sqrt{m_D^2 + \vec{p}_D^2}$$

$$\frac{dE_f}{dp_D} = \frac{|p_D|}{E_C} + \frac{|p_D|}{E_D} = \frac{|p_D|(E_C + E_D)}{E_C E_D}$$

Hence

$$\begin{aligned} dQ &= \frac{1}{(2\pi)^2}\delta(E_f - E_i)\frac{|p_D|^2}{4E_C E_D}\frac{E_C E_D}{|p_D|(E_C + E_D)}dE_f d\Omega \\ &= \frac{1}{16\pi^2}\frac{|p_f|}{(E_C + E_D)}d\Omega \end{aligned}$$

of course $E_C + E_D = E_A + E_B$ and we define it to be

$$E_C + E_D = \sqrt{s}$$

center of mass energy.

Lecture 8 (10/2/13)

We continue in computing (3.3). We have seen dQ only involved measurable quantities like $p_{A,\ldots,D}$ we do not need to know V_{fi}. This is also true in $\mathcal{M}$ and F.

From the assumption last time, mass small and $p_A = -p_B$

$$p_A = (|p_A|, 0, 0, |p_A|)$$

$$p_B = (|p_A|, 0, 0, -|p_A|)$$

$$p_C = (|p_A|, 0, |p_A|\sin\theta, |p_A|\cos\theta)$$

$$p_D = (|p_A|, 0, -|p_A|\sin\theta, -|p_A|\cos\theta)$$

where θ is the scattering angle in CM frame.

$$\mathcal{M} = -e^2(p_A + p_C)^\mu \frac{1}{q^2}(p_B + p_D)_\mu$$

$$\begin{aligned} q^2 &= (p_A - p_C)^2 = \underbrace{p_A^2}_{0} - 2p_A p_C + \underbrace{p_C^2}_{0} \\ &= |p_A|^2 - |p_A|^2 - 2|p_A|^2 + 2|p_A|^2\cos\theta + |p_A|^2 - |p_A|^2\sin^2\theta - |p_A|^2\cos^2\theta \\ &= 2|p_A|^2(\cos\theta - 1) \end{aligned}$$

$$
\begin{aligned}
(p_A+p_C)^\mu(p_B+p_D)_\mu &= p_Ap_B+p_Ap_D+p_Cp_B+p_Cp_D \\
&= 2|p_A|^2+|p_A|^2+|p_A|^2\cos\theta+|p_A|^2+|p_A|^2\cos\theta+2|p_A|^2 \\
&= 6|p_A|^2+2|p_A|^2\cos\theta \\
&= 2|p_A|^2(3+\cos\theta)
\end{aligned}
$$

$$
\mathcal{M} = -e^2\frac{3+\cos\theta}{\cos\theta-1}
$$

Now F, in the CM frame, velocityof A is $\vec{v}_A-\vec{v}_B$

$$
\begin{aligned}
F &= |\vec{v}_A-\vec{v}_B|2E_A2E_B \\
&= 4((p_Ap_B)^2-m_A^2m_B^2)^{\frac{1}{2}} \\
&= 4((E_AE_B+\vec{p}_A^2)^2-m_A^2m_B^2)^{\frac{1}{2}} \\
&= 4(E_A^2E_B^2+2\vec{p}_AE_AE_B+\vec{p}_A^4-m_A^2m_B^2)^{\frac{1}{2}} \\
&= 4(|p_A|^4+(m_A^2+m_B^2)|p_B|^2+2|\vec{p}_A|^2E_AE_B+|\vec{p}_A|^4)^{\frac{1}{2}} \\
&= 4(|p_A|^2(2|p_A|^2+2E_AE_B+m_A^2+m_B^2))^{\frac{1}{2}} \\
&= 4(|p_A|^2(E_A+E_B))^{\frac{1}{2}} \\
&= 4|p_A|\sqrt{s}
\end{aligned}
$$

Therefore $|p_A|=|p_i|$, $|p_C|=|p_f|$

$$
\begin{aligned}
d\sigma &= \frac{|\mathcal{M}|^2}{F}dQ \\
&= \frac{e^4}{4|p_i|\sqrt{s}}\frac{(3+\cos\theta)^2}{(\cos\theta-1)^2}\frac{1}{4\pi^2}\frac{|p_f|}{4\sqrt{s}}d\Omega \\
&= \frac{\alpha^2}{4s}\frac{(3+\cos\theta)^2}{(\cos\theta-1)^2}d\Omega
\end{aligned}
$$

$\alpha = e^2/4\pi$. Convert that to lab frame, since μ^- moves at relativistic speed, length contraction gives pancake.

For $e^-\mu^- \to e^-\mu^-$, we can think the out coming p_B is $-p_B$ associated with e^+. In other words, the collision becomes two events: $\mu^+\mu^-$ creation and a little bit later e^-e^+ annihilation. In between intermediate state γ was created,

$$
E_n = E_i + E_\gamma + E_f
$$

$$E_n = E_\gamma$$

The $T_{fi}^{(2)}$ only 2nd order excluding 1st order is

$$\begin{aligned}
T_{fi} &= -i\sum_{n\neq i} V_{fn}\frac{1}{E_i - E_n}V_{ni}2\pi\delta(E_f - E_i) \sim V_{fn}(\frac{1}{E_i - E_\gamma} + \frac{1}{-E_i - E_\gamma})V_{ni} \\
&\sim \frac{2E_\gamma}{E_i^2 - E_\gamma^2} = \frac{2E_\gamma}{(p_A + p_B)^2 - m_\gamma^2}
\end{aligned}$$

where m_γ is called on shell mass. $E_i^2 = (p_A+p_B)^2+(\vec{p}_A+\vec{p}_B)$, $E_\gamma^2 = (\vec{p}_A+\vec{p}_B)^2+m_\gamma^2$.

This completes our general study of 2 particles $\rightarrow$ 2 particles. Later we will revisit them and introduce spins.

3.3 Decays

What about 1 particle $A\rightarrow$ n particles? E.g.

$$K \rightarrow \Pi\Pi$$

$$\bar{\mu} \rightarrow \bar{e}\nu_\mu\bar{\nu}_e$$

Lorentz invariant phase space

$$dQ = \frac{d^3p_1}{(2\pi)^3E_1}\frac{d^3p_2}{(2\pi)^3E_2}\cdots\frac{d^3p_n}{(2\pi)^3E_n}$$

Let

$$\Gamma = \frac{1}{N_A}\frac{dN_A}{dt} \qquad N_A(t) = N_A(0)e^{-\Gamma t}$$

then

$$d\Gamma = \frac{1}{2E_A}|\mathcal{M}|^2 dQ$$

3.4 Scattering with Identical Particles

For example $e^-e^- \rightarrow e^-e^-$.

Because we don't know the one coming out on is the one coming in from the right or left, we have combine the two possibilities. They are called $t-$channel and

$u-$channel.

$$\mathcal{M}_{e^-e^-\to e^-e^-} = -e^2 \left(\frac{(p_A+p_C)^\mu (p_B+p_D)_\mu}{(p_A-p_C)^\nu} + \frac{(p_A+p_D)^\mu (p_B+p_C)_\mu}{(p_A-p_D)^\nu} \right)$$

After working it out, we will see it gets not just doubled from what we have before, but we will see interference.

Similarly for $e^+e^- \to e^+e^-$, we should combine both $t-$channel and $s-$channel. One takes no intermediate time, and the other takes some time.

$$\mathcal{M}_{e^+e^-\to e^+e^-} = -e^2 \left(\frac{(p_A+p_C)^\mu (-p_B-p_D)_\mu}{(p_C-p_A)^\nu} + \frac{(p_A-p_D)^\mu (p_C-p_D)_\mu}{(p_A+p_B)^\nu} \right)$$

Hence

$$\mathcal{M}_{e^+e^-\to e^+e^-}(p_A,p_B,p_C,p_D) = \mathcal{M}_{e^-e^-\to e^-e^-}(p_A,-p_D,p_C,-p_B)$$

Lecture 9 (10/7/13)

It is possible to simplify above expression by introducing Mandelstam variables, which are three Lorentz invariant variables

$$\begin{aligned} s &= (p_A+p_B)^2 \\ t &= (p_A-p_C)^2 \\ u &= (p_A-p_D)^2 \end{aligned}$$

then

$$\mathcal{M}_{e^-e^-\to e^-e^-} = e^2 \left(\frac{u-s}{t} + \frac{t-s}{u} \right)$$

3.5 Dirac Equation

KG has negative energy. Dirac didn't like it, he said we should do 1st order in $\frac{\partial}{\partial t}$ and first order in ∇. So something like

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi \tag{3.7}$$

What are $\vec{\alpha}$ and β?

$$H^2\psi = E^2\psi = (\vec{p}^2 + m^2)\psi$$

and the other hand

$$H^2\psi = [\alpha_i^2 p_i^2 + (\alpha_i\alpha_j + \alpha_j\alpha_i)p_i p_j + \alpha_i\beta p_i m + \beta^2 m^2]\psi$$

Hence

$$\begin{cases} \alpha_i^2, \beta^2 = 1 \\ \{\alpha_i, \alpha_j\} = \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \\ \{\alpha_i, \beta\} = 0 \end{cases}$$

so α_i, β_i has to be matrices. The lowest dimension possible are 4×4.

The conventional choose is Dirac-Pauli representation

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

ψ is a 4 component column vector.

Define

$$\gamma^\mu = (\beta, \beta\vec{a})$$

γ algebra or γ gymnastics. So

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

Some useful results

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma^{0\dagger} = \gamma^0$$

For $k = 1, 2, 3$

$$\gamma^{k\dagger} = (\beta\alpha^k)^\dagger = \alpha^k\beta = -\gamma^{k\dagger}$$

$$(\gamma^k)^2 = -I$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

Write (3.12) explicitly

$$i\frac{\partial\psi}{\partial t} = -i\vec{\sigma}\cdot\nabla\psi + \beta m\psi$$

Apply β to above

$$i\beta\frac{\partial\psi}{\partial t} + i\beta\vec{\sigma}\cdot\nabla\psi - m\psi = 0$$

Thus we obtain Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3.8}$$

This is 4 coupled differential equations.

What are the energy density and energy current?

Apply $(3.8)^\dagger$

$$[i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^k\frac{\partial\psi}{\partial x^k} - m\psi]^\dagger = 0$$

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial\psi^\dagger}{\partial x^k}(-\gamma^k) - m\psi^\dagger = 0 \tag{3.9}$$

The first 2 terms give

$$-\partial^\mu\gamma^\mu$$

which is not covariant. Bad!

Multiply γ^0 to (3.9)

$$-i\frac{\partial\psi^\dagger}{\partial t}\gamma^0\gamma^0 + i\frac{\partial\psi^\dagger}{\partial x^k}\underbrace{\gamma^k\gamma^0}_{-\gamma^0\gamma^k} - m\psi^\dagger\gamma^0 = 0$$

Define the adjoint spinor

$$\bar{\psi} = \psi^\dagger\gamma^0$$

then above becomes

$$-i\frac{\partial\bar{\psi}}{\partial t}\gamma^0 - i\frac{\partial\bar{\psi}}{\partial x^k}\gamma^k - m\bar{\psi} = 0$$

That is

$$(i\partial_\mu\gamma^\mu + m)\bar{\psi} = 0 \tag{3.10}$$

We can now play with (3.8) and (3.10). Consider

$$\begin{aligned}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi &= 0\\ \psi(i\partial_\mu\gamma^\mu + m)\bar{\psi} &= 0\end{aligned}$$

We get

$$i(\bar{\psi}\gamma^\mu\partial_\mu\psi + \psi\partial_\mu\gamma^\mu\bar{\psi}) = 0$$

That is

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

So we should have

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{3.11}$$

and

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \sum_{\mu=0}^{3}|\psi_\mu|^2$$

which is positive definite.

Apply $\gamma^\nu\partial_\nu$ to (3.8), one gets

$$i(\Box^2 + m^2)\psi = 0$$

Hence each component of ψ has to satisfy KG.

Try "plane wave" solution

$$\psi = U(\vec{p})e^{-ip\cdot x}$$

U is 4 component spinor independent of x.

Plug it into Dirac equation

$$(\gamma^\mu p_\mu - m)U(\vec{p}) = 0$$

Commonly people denote

$$\not{A} = \gamma^\mu A_\mu$$

so

$$(\not{p} - m)U(\vec{p}) = 0 \tag{3.12}$$

we get Dirac equation in momentum space.

Suppose particles at rest

$$\vec{p} = 0$$

then by (3.7)

$$EU = HU = \beta m U = \begin{pmatrix} m & & & \\ & m & & \\ & & -m & \\ & & & -m \end{pmatrix}$$

eigenvectors are

$$\begin{pmatrix} 1 \\ \\ \\ \\ \end{pmatrix}, \begin{pmatrix} \\ 1 \\ \\ \\ \end{pmatrix}, \begin{pmatrix} \\ \\ 1 \\ \\ \end{pmatrix}, \begin{pmatrix} \\ \\ \\ 1 \end{pmatrix}$$

the first 2 are e^- with $E > 0$ and last two are e^- with $E < 0$. So we get antiparticles.

If $\vec{p} \neq 0$

$$HU = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & m \end{pmatrix} \begin{pmatrix} U_A \\ U_B \end{pmatrix} = E \begin{pmatrix} U_A \\ U_B \end{pmatrix}$$

$U_{A,B}$ 2 component spinors.

Setting $U_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get one solution

$$U^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} \tag{3.13}$$

Other independent solutions

$$U^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad U^{(3)} = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad U^{(4)} = N \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

$U^{(1,2)}$ have $E > 0$, and $U^{(3,4)}$ have E with $-$sign.

Look for a way to distinguish the 2 degeneracies. Look for an operator that commutes with H and $\vec{p}$: spin projection on momentum vector

$$\vec{\Sigma} \cdot \hat{p} = \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & \\ & \vec{\sigma} \cdot \vec{p} \end{pmatrix}$$

The eigenvalues are

$$\frac{1}{2} \vec{\sigma} \cdot \hat{p} = \pm \frac{1}{2}$$

which is called Helicity, it is a good quantum number, refer to spin up and down.

If $\hat{p} \parallel \hat{z}$,

$$\vec{\Sigma} \cdot \hat{p} = \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

In the presence of angular momentum, the operator that commutes with H and $\vec{p}$ is

$$\vec{J} = \vec{L} + \vec{\Sigma} \cdot \hat{p}$$

Lecture 10 (10/9/13)

Continuing from (3.12). For an e^- of energy $-E$, momentum $-p$, (3.12) becomes

$$(-\not{p} - m) U(-\vec{p}) = 0$$

For its antiparticle, positron of $(E, \vec{p})$,

$$(\not{p} + m) V(\vec{p}) = 0$$

We too have completeness relations

$$\sum_{s=1,2} U^{(s)}(p)\bar{U}^{(s)}(p) = \not{p} + m$$

$$\sum_{s=1,2} V^{(s)}(p)\bar{V}^{(s)}(p) = \not{p} - m$$

Let

$$\Lambda_+ = \frac{\not{p} + m}{2m}$$

be project out positive E states, and

$$\Lambda_+ = \frac{\not{p} - m}{2m}$$

be project out negative E states.

3.6 Homework 3 (due 10/21/2013)

Write a short report on "Muon g-2". What is it? What is interesting about it? etc.

3.7 Bilinear Covariant

Cf (3.11) where we sort of define a new dot product. We can generalize this to

$$\bar{\psi} X \psi$$

where X is a 4×4 matrix.

One example of X is product of γ matrices

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

notice that

$$\gamma^{5\dagger} = \gamma^5, \qquad (\gamma^5)^2 = I, \qquad \{\gamma^5, \gamma^\mu\} = 0$$

We list some common X's

	number of element components	parity
$\bar{\psi}\psi$	1	$+ \to$ scalar
$\bar{\psi}\gamma^\mu\psi$	4	$- \to$ vector
$\bar{\psi}\sigma^{\mu\nu}\psi$	6	$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ tenor
$\bar{\psi}\gamma^5\gamma^\mu\psi$	4	$+ \to$ psudovector
$\bar{\psi}\gamma^5\psi$	1	$- \to$ psudoscalar

3.8 Minimal Substitution

Continuing from (3.12),

$$p_\mu \to p_\mu + eA_\mu$$

we get

$$(\gamma^\mu p_\mu + e\gamma^\mu A_\mu - m)\psi = 0$$

or

$$(\gamma^\mu p_\mu - m)\psi = \underbrace{(-e\gamma^\mu A_\mu)}_{=\gamma^0 V}\psi$$

setting

$$\gamma^0 V = -e\gamma^\mu A_\mu$$

or

$$V(x) = -e\gamma^0\gamma^\mu A_\mu$$

Now compute T_{fi}

$$\begin{aligned} T_{fi} &= -i\int \psi_f^\dagger(x)V(x)\psi_i(x)d^4x \\ &= ie\int \bar{\psi}_f(x)\gamma^\mu A_\mu\psi_i(x)d^4x \\ &= i\int j^\mu(x)A_\mu d^4x \end{aligned}$$

with

$$\begin{aligned} j^\mu(x) &= e\bar{\psi}_f(x)\gamma^\mu\psi_i(x) \\ &= -e\bar{U}_f\gamma^\mu U_i e^{i(p_f - p_i)x} \end{aligned}$$

Compare this to $j^\mu(x)$ obtained from KG, (2.9).

It can be shown

$$\bar{U}_f\gamma^\mu U_i = \frac{1}{2m}\bar{U}_f[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu]U_i$$

where $(p_f+p_i)^\mu$ is electric interaction and $(p_f-p_i)_\nu$ gives magnetic interaction.

3.9 Cross Section

$$\begin{aligned} T_{fi} &= -i\int j^{(k)}_\mu(x)\left(-\frac{1}{q^2}\right)j^{\mu(k)}d^4x \\ &= \underbrace{-i[-e\bar{U}(k')\gamma_\mu U(k)]\left(-\frac{1}{q^2}\right)[-e\bar{U}(k')\gamma^\mu U(k)]}_{\mathcal{M}}\underbrace{\int e^{-(k'-k+p'-p)x}d^4x}_{(2\pi)^4\delta(k'+p'-k-p)} \end{aligned} \tag{3.14}$$

The products are unpolarized. We sum over outgoing spins and average over incoming spins.

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{1}{2s_e+1}\frac{1}{2s_\mu+1}\sum_{\text{spins}}|\mathcal{M}|^2 \\ &= \frac{e^2}{q^4}L^{(e)}_{\mu\nu}L^{(\mu)\mu\nu} \end{aligned} \tag{3.15}$$

$s_e = s_\mu = \frac{1}{2}$.

$$L^{(e)}_{\mu\nu} = \frac{1}{2}\sum_{\text{spins}}[\bar{U}(k')\gamma_\mu U(k)][\bar{U}(k')\gamma_\nu U(k)]^*$$

Since $[\bar{U}(k')\gamma_\nu U(k)]^*$ is a number, we do

$$\begin{aligned} [\bar{U}(k')\gamma_\nu U(k)]^* &= [\bar{U}(k')\gamma_\nu U(k)]^\dagger \\ &= [U^\dagger(k')\gamma^0\gamma_\nu U(k)]^\dagger \\ &= [U^\dagger(k)\gamma_\nu\gamma^0 U(k')] \\ &= [\bar{U}(k)\gamma_\nu U(k')] \end{aligned}$$

$$L^{(e)}_{\mu\nu} = \frac{1}{2}\sum_{\text{spins}} [\bar{U}(k')\gamma_\mu U(k)][\bar{U}(k)\gamma_\nu U(k')]$$

Say k has spin s and k' has spin s'

$$L^{(e)}_{\mu\nu} = \frac{1}{2}\sum_{s'} \bar{U}^{(s')}_\alpha(k')\gamma_{\mu_{\alpha\beta}} \sum_{s} U^{(s)}_\beta(k)\bar{U}^{(s)}_\gamma(k)\gamma_{\nu_{\gamma\delta}} U^{(s')}_\delta(k')$$

Above implies summing over $\alpha, \beta, \gamma, \delta$. After writing out components, we can commute them

$$L^{(e)}_{\mu\nu} = \frac{1}{2}\sum_{s'} U^{(s')}_\delta(k')\bar{U}^{(s')}_\alpha(k')\gamma_{\mu_{\alpha\beta}} \sum_{s} U^{(s)}_\beta(k)\bar{U}^{(s)}_\gamma(k)\gamma_{\nu_{\gamma\delta}}$$

We see they are matrix multiplications

$$L^{(e)}_{\mu\nu} = \frac{1}{2}(\not{k}' + m)_{\delta\alpha}\gamma_{\mu_{\alpha\beta}}(\not{k} + m)_{\delta\alpha}\gamma_{\nu_{\gamma\delta}}$$

By indices contraction, we get

$$L^{(e)}_{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(\not{k}' + m)\gamma_\mu(\not{k} + m])\gamma_\nu]$$

The following properties will help us simplify the expression.

Theorem. *(trace theorem)*

$$\begin{aligned} Tr(\not{a}\not{b}) &= \frac{1}{2}Tr(\not{a}\not{b} + \not{b}\not{a}) \\ &= \frac{1}{2}2g^{\mu\nu}a_\mu b_\nu TrI = 4a\cdot b \end{aligned}$$

Because $\mathrm{Tr}I = 4$, and $\not{a} = \gamma^\mu a_\mu$, and $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

More generally

Theorem.

$$Tr(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)] \tag{3.16}$$

Most useful of all: trace of product of odd number of γ matrices is 0

$$
\begin{aligned}
\mathrm{Tr}(\gamma^5) &= 0 \\
\mathrm{Tr}(\gamma^5 \not{a}\not{b}) &= 0 \\
\mathrm{Tr}(\gamma^5 \not{a}\not{b}\not{c}\not{d}) &= 4ik_{\alpha\beta\gamma\delta}a^\alpha b^\beta c^\gamma d^\delta \\
\gamma^\mu\gamma_\mu &= 4 \\
\gamma^\mu \not{a}\gamma_\mu &= -2\not{a} \\
\gamma^\mu \not{a}\not{b}\gamma_\mu &= 4\not{a}\not{b} \\
\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu &= -2\not{c}\not{b}\not{a}
\end{aligned}
\tag{3.17}
$$

Now back to

$$
\begin{aligned}
L^{(e)}_{\mu\nu} &= \frac{1}{2}\mathrm{Tr}[(\not{k}' + m)\gamma_\mu(\not{k} + m])\gamma_\nu] \\
&= \frac{1}{2}\mathrm{Tr}[\not{k}'\gamma_\mu\not{k}\gamma_\nu] + \frac{1}{2}\mathrm{Tr}[\not{k}'\gamma_\mu m\gamma_\nu] \\
&\quad + \frac{1}{2}\mathrm{Tr}[m\gamma_\mu\not{k}\gamma_\nu] + \frac{1}{2}\mathrm{Tr}[m^2\gamma_\mu\gamma_\nu]
\end{aligned}
$$

The 2nd and 3rd terms are 0 because they are product of 3 γ matrices.

By (3.16)

$$\mathrm{Tr}[\not{k}'\gamma_\mu\not{k}\gamma_\nu] = 4[k'_\mu k_\nu - (k'\cdot k)g_{\mu\nu} + k'_\nu k_\mu]$$

$$\mathrm{Tr}[m^2\gamma_\mu\gamma_\nu] = m^4 4g_{\mu\nu}$$

Finally

$$L^{(e)}_{\mu\nu} = 2[k'_\mu k_\nu + k'_\nu k_\mu - ((k'\cdot k) - m^2)g_{\mu\nu}]$$

Similarly

$$L^{(\mu)\mu\nu} = 2[p'^\mu p^\nu + p'^\mu p^\nu - ((p'\cdot p) - M^2)g_{\mu\nu}]$$

Therefore

$$L^{(e)}_{\mu\nu}L^{(\mu)\mu\nu} = 8[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - m^2(p'\cdot p) - M^2(k'\cdot k) + 2m^2M^2]$$

Lecture 11 (10/14/13)

Using Mandelstam variables

$$\begin{aligned}
s &= (k+p)^2 = k^2 + 2kp + p^2 = 2k'p' = 2kp \\
u &= (k-p')^2 = -2kp' = -2k'p
\end{aligned}$$

$$q^2 = (k-k')^2 = t^2$$

for $E \gg m, M$ (3.15) becomes

$$|\bar{\mathcal{M}}|^2 = \frac{8e^4}{q^2}[\frac{s^2}{4} + \frac{s^2}{4} + O(m^2, M^2)] = 2e^4 \frac{s^2+u^2}{t^2}$$

One can easily exchange

$$k' \leftrightarrow -p$$

of the above derivation to change $e^-\mu^- \to e^-\mu^-$ scattering to

$$e^+e^- \to \mu^+\mu^-$$

scattering. Many of the results carry over. E.g. s channel

$$s_{e^-\mu^-\to e^-\mu^-} \to t_{e^+e^-\to\mu^+\mu^-}$$

and

$$t_{e^-\mu^-\to e^-\mu^-} \to s_{e^+e^-\to\mu^+\mu^-}$$

and

$$|\bar{\mathcal{M}}|^2_{e^+e^-\to\mu^+\mu^-} = 2e^4 \frac{t^2+u^2}{s^2}$$

and cross section in center of mass

$$\begin{aligned}
\left.\frac{d\sigma}{d\Omega}\right|_{e^+e^- \to \mu^+\mu^-} &= \frac{1}{64\pi^2 s}\frac{|p_f|}{|p_i|}|\bar{\mathcal{M}}|^2 \\
&= \frac{1}{64\pi^2 s} 2e^4[\frac{1}{2}(1+\cos^2\theta)]
\end{aligned}$$

Integration

$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s}$$

where $\alpha = \frac{e^2}{4\pi}$.

3.10 Interference

In section 3.4, we mentioned that scattering with identical particles one cannot distinguish t or s channel. The similar thing happens for

$$e^+e^- \to e^+e^-$$

scattering. As of $\mathcal{M}$ for $e^-\mu^- \to e^-\mu^-$ cf (3.14), we now have to include both channels, and notice it is $'-'$ in combining the two, because e^- is antisymmetric,

$$\begin{aligned}\mathcal{M} &= -e^2\Bigg[\bar{U}(k')\gamma^\mu U(k)\frac{1}{(k-k')^2}\bar{U}(-p)\gamma_\mu U(-p') \\ &\quad - \bar{U}(-p)\gamma^\nu U(k)\frac{1}{(k-(-p))^2}\bar{U}(k')\gamma_\nu U(-p')\Bigg]]\end{aligned}$$

Similar to (3.15), average over spins

$$|\bar{\mathcal{M}}| \propto \frac{1}{4}\sum_{\text{spins}}[\bar{U}(k')\gamma^\mu U(k)\bar{U}(-p)\gamma_\mu U(-p')][\bar{U}(-p)\gamma^\nu U(k)\bar{U}(k')\gamma_\nu U(-p')]^*$$

After cross multiply we should obtain some interference terms, one of them looks like

$$\begin{aligned}I &= \frac{1}{4}\sum_{\text{spins}}\bar{U}(k')\gamma^\mu U(k)\bar{U}(k)\gamma^\nu U(-p)\bar{U}(-p)\gamma_\mu U(-p')\bar{U}(-p')\gamma_\nu U(k') \\ &= \frac{1}{4}\sum_{\text{spins}}U(k')\bar{U}(k')\gamma^\mu U(k)\bar{U}(k)\gamma^\nu U(-p)\bar{U}(-p)\gamma_\mu U(-p')\bar{U}(-p')\gamma_\nu \\ &= \frac{1}{4}\mathrm{Tr}[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu(-\not{p} + m)\gamma_\mu(-\not{p}' + m)\gamma_\nu]\end{aligned}$$

Neglecting terms with m^2 or higher, and first order in m is 0 because it is product of 7 γ matrices, so

$$I = \frac{1}{4}\mathrm{Tr}[\not{k}'\gamma^\mu\not{k}\gamma^\nu\not{p}\gamma_\mu\not{p}\gamma_\nu]$$

Using (3.17)

$$\begin{aligned} I &= -\frac{2}{4}\mathrm{Tr}[\not{k}'\underbrace{\gamma^\mu \not{k}\not{p}'\gamma_\mu}_{4k\cdot p'}\not{p}] \\ &= -2(k\cdot p')\mathrm{Tr}\not{k}'\not{p} \\ &= -8(k\cdot p')(k'\cdot p) \end{aligned}$$

One can show the full interference term is

$$|\bar{\mathcal{M}}|^2_{interf} = e^4\frac{4u^2}{st}$$

The other two non-interference terms are proportional to

$$\frac{1}{s^2} \text{ and } \frac{1}{t^2}$$

This completes our study of 1st order Born scattering.

3.11 High Order Scattering

Higher order corrections have more than 2 vertices, and there two kinds: virtual corrections and real corrections. Virtual corrections are additional loops in forms of virtual photons (off shell mass, massive photons). Real corrections are detectable real photons (on shell mass, massless) coming out of the reactions. The real difference between the two is somehow arbitrary, depending on the cutoff scale, and the scattering angle. If the radiated photon comes out colinearly with the particles it is not detectable regardless of the cutoff scale.

Another problem of higher order scattering is that some perturbation becomes ∞ if we just calculate one term at a time. Some smart people have proved that in standard model there are always some another perturbations of ∞ to compensate and cancel that ∞, which is saying the standard model is renormalizable, although it is not always apparent how to find the another ∞.

3.12 Helicity and Chirality

Lecture 12 (10/21/13)

First we show photons have 2 polarizations.

Recall Gauge invariant

$$\begin{cases} \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V \\ \vec{B} = \nabla \times \vec{A} \end{cases}$$

and

$$\Box^2 A_\mu - \partial_\mu(\partial^\nu A_\nu) = j_\mu$$

Adding a curless time-independent function to $\vec{A}$ and a constant to V, $\vec{E}$ and $\vec{B}$ stay the same, consequently physical quantities calculated in one gauge choice should be the same all gauge choices.

For radiation problem it is convenience to choose Lorentz gauge

$$\partial^\mu A_\mu = 0 \tag{3.18}$$

$$\Box^2 A^\mu = j^\mu$$

For a photon,

$$\Box^2 A^\mu = 0 \tag{3.19}$$

Assume plane wave

$$A^\mu = \epsilon^\mu(\vec{q}) e^{-iqx} \tag{3.20}$$

$\epsilon^\mu(\vec{q})$ polarizations. Plug (3.20) into (3.19)

$$q^2 = 0$$

Because $q^2 = m^2$ Lorentz invariant.

$$m = 0$$

showing gauge invariance implies photon is massless.

Plug (3.20) into (3.18)

$$q^\mu \epsilon_\mu = 0 \tag{3.21}$$

This eliminates that ϵ_μ can have only 3 independent components, which is good, because photon spin is 1, so it has 3 degrees of freedom: ± 1, 0.

But Lorentz condition is not total fixed, we can still have

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

if

$$\Box^2 \Lambda = 0$$

or for any Λ such that $\Lambda = iae^{-iqx}$ for any constant a. Thus we are free to choose

$$\epsilon'_\mu = \epsilon_\mu + aq_\mu$$

We now pick a a such that

$$\epsilon'_0 = 0$$

called Coulomb Gauge. Then (3.21) becomes

$$\vec{\epsilon} \cdot \vec{q} = 0$$

so only two independent polarizations!

In sum we have shown that for on shell photons, there are only 2 possible pole, if we choose moving in $\hat{z}$ direction, i.e. $\vec{q} \parallel \hat{z}$

$$\epsilon_1 = (1, 0, 0)$$

$$\epsilon_2 = (0, 1, 0)$$

called transverse polarization.

We now study general cases of helicity and chirality

γ^5 is the chirality operator. What does that mean?

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Apply γ^5 to $U^{(1)}$ (3.13) first free solution to Dirac equation

$$\gamma^5 U^{(1)} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} = \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

which is $U^{(3)}$ with $-m$.

Take $E \gg m$, i.e. $|\vec{p}| \approx E$. (if E is comparable with m we don't need Dirac, Schrodinger works fine.)

Recall the short hand notation,

$$U^{(s)} = \begin{pmatrix} \chi^{(s)} \\ \vec{\sigma} \cdot \vec{p} \chi^{(s)} \end{pmatrix}$$

$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\frac{\vec{\sigma} \cdot \vec{p}}{E+m} \approx \vec{\sigma} \cdot \hat{p}$$

so

$$\gamma^5 U^{(s)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi^{(s)} \end{pmatrix} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

Claim

$$\gamma^5 U^{(s)} = \vec{\sigma} \cdot \hat{p} \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi^{(s)} \end{pmatrix}$$

Indeed because

$$(\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^2$$

Thus

$$\gamma^5 U^{(s)} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & \\ & \vec{\sigma} \cdot \hat{p} \end{pmatrix} \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi^{(s)} \end{pmatrix}$$

Recall $\vec{\sigma}\cdot\hat{p}$ is the helicity operator for spin 1, i.e.

$$\vec{\sigma}\cdot\hat{p}\chi^{(s)} = \begin{cases} \chi^{(s)} & s=1 \\ -\chi^{(s)} & s=2 \end{cases}$$

Consider the follow operation

$$\begin{aligned}
\frac{1}{2}(1-\gamma^5)U^{(s)} &= \frac{1}{2}\left[\begin{pmatrix} \chi^{(s)} \\ \vec{\sigma}\cdot\hat{p}\chi^{(s)} \end{pmatrix} - \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & \\ & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\begin{pmatrix} \chi^{(s)} \\ \vec{\sigma}\cdot\hat{p}\chi^{(s)} \end{pmatrix}\right] \\
&= \begin{cases} \frac{1}{2}\begin{pmatrix} \chi^{(1)}-\chi^{(1)} \\ \vec{\sigma}\cdot\hat{p}\chi^{(1)}-\vec{\sigma}\cdot\hat{p}\chi^{(1)} \end{pmatrix} = 0 & s=1 \\ \frac{1}{2}\begin{pmatrix} \chi^{(2)}+\chi^{(2)} \\ -\chi^{(2)}-\chi^{(2)} \end{pmatrix} = \begin{pmatrix} \chi^{(2)} \\ -\chi^{(2)} \end{pmatrix} & s=2 \end{cases}
\end{aligned}$$

Hence

$$\frac{1}{2}(1-\gamma^5) \equiv U_L$$

is a projection operator selecting the left-chiral component, and

$$\frac{1}{2}(1+\gamma^5) \equiv U_R$$

For massless particles, chiral is same as helicity. For massive particles, helicity depends on frame of reference while chiral is fixed due to Higgs mechanism.

U_L and U_R are orthogonal, so

$$\begin{aligned}
\bar{U}\gamma^\mu U &= (\bar{U}_L+\bar{U}_R)\gamma^\mu(U_L+U_R) \\
&= \bar{U}_L\gamma^\mu U_L + \bar{U}_R\gamma^\mu U_R
\end{aligned}$$

Indeed

$$\begin{aligned}
\bar{U}_L &= U_L^\dagger\gamma^0 = \left(\frac{1}{2}(1-\gamma^5)U\right)^\dagger\gamma^0 \\
&= \frac{1}{2}U^\dagger(1-\gamma^5)\gamma^0 = U^\dagger\gamma^0\frac{1}{2}(1+\gamma^5) = \bar{U}\frac{1}{2}(1+\gamma^5)
\end{aligned}$$

so

$$\begin{aligned}
\bar{U}_L\gamma^\mu U_R &= \bar{U}\frac{1}{2}(1+\gamma^5)\gamma^\mu\bar{U}\frac{1}{2}(1+\gamma^5)U \\
&= \frac{1}{4}\bar{U}\gamma^\mu(1-\gamma^5)\frac{1}{2}(1+\gamma^5)U \\
&= \frac{1}{4}\bar{U}\gamma^\mu(1-(\gamma^5)^2)U = 0
\end{aligned}$$

In other words, at high energy QED interaction conserves chiral (i.e. conserve helicity because at high energy they are the same.)

4 Weak Interactions

4.1 Muon Decay

Recall we discussed

$$\mu \to e + \nu \tag{4.1}$$

has maximally violation parity. The Feynman diagram has two vertices. One has μ^- in with 4 momentum p and ν_μ out with k and the other vertex has e^- out with p' and $\bar{\nu}_e$ in with k' (ν_e back in time). The two vertices are connected by W boson.

There are number yet not understood conservation laws in weak interactions:

Total Lepton number is conserved. In the example above we have

$$1 = 2 - 1$$

the -1 is ν_e back in time.

Lepton flavor number, the total number of lepton per generation, is conserved. In the example above

1st generation e^- out ν_e in

2nd generation μ in ν_μ out

4.2 Homework 4 (due 11/4/13)

This conservation does not hold in cases where only ν is evolved in the reaction. That is ν will change itself to different generation.

Write a brief report about ν oscillation. What is it? How was discovered? How accurately is the experimental data?

4.3 Muon Decay (continued)

One can show from Green function the matrix element for three body (4.1) problem

$$i\mathcal{M} = i\frac{g}{\sqrt{2}}\bar{U}(k)\gamma^{\mu}\frac{1}{2}(1-\gamma^5)U(p)\frac{-g_{\mu\nu}+q^{\mu}q^{\nu}/M^2}{q^2-M^2}\frac{g}{\sqrt{2}}\bar{U}(p')\gamma_{\mu}\frac{1}{2}(1-\gamma^5)U(-k')$$

The $\frac{1}{2}(1-\gamma^5)$ extra term is due to the fact that ν are purely left handed. M is the mass of W boson, hence $M \gg q$. So the propagator

$$\frac{-g_{\mu\nu}+q^{\mu}q^{\nu}/M^2}{q^2-M^2} \approx \frac{g_{\mu\nu}}{M^2}$$

We can simplify the current vertex

$$\bar{U}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)U = \bar{U}\gamma^{\mu}\frac{1}{2}U - \bar{U}\gamma^{\mu}\frac{1}{2}\gamma^5 U$$

where the 1st term on the right is a vector and the 2nd on the right is a psudovector.

Or writing

$$\bar{U}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)U = \bar{U}\gamma^{\mu}(g_V - g_A\gamma^5)U$$

called $V-A$ structure of weak interactions. In our case

$$g_V = g_A = \frac{1}{2} \tag{4.2}$$

If it were right handed, we would use $g_V + g_A\gamma^5$, called $V+A$ structure.

Experiment can measure g_V, g_A very well. For μ decay they agree (4.2), showing indeed it is purely 100% left handed, with 0 deviation from (4.2).

Lecture 13 (10/23/13)

We're now ready to compute $\mathcal{M}$, let

$$\frac{g^2}{M^2} = \frac{G}{\sqrt{2}}$$

then

$$\mathcal{M} = \frac{G}{\sqrt{2}}[\bar{U}(k)\gamma^\mu(1-\gamma^5)U(p)\bar{U}(p')\gamma_\mu(1-\gamma^5)U(-k')]$$

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{2}\sum_{\text{spins}}\frac{G^2}{2}[\bar{U}(k)\gamma^\mu(1-\gamma^5)U(p)\bar{U}(p')\gamma_\mu(1-\gamma^5)U(-k')] \\
&\qquad [\bar{U}(-k')\gamma_\nu(1-\gamma^5)U(p')\bar{U}(p)\gamma^\nu(1-\gamma^5)U(k)] \\
&= \frac{G^2}{4}\sum_{\text{spins}}[\bar{U}(k)\gamma^\mu(1-\gamma^5)U(p)\bar{U}(p')\gamma^\nu(1-\gamma^5)U(k)] \\
&\qquad [\bar{U}(-k')\gamma_\nu(1-\gamma^5)U(p')\bar{U}(p)\gamma_\mu(1-\gamma^5)U(-k')] \\
&= \frac{G^2}{4}\mathrm{Tr}[\not{k}\gamma^\mu(1-\gamma^5)\not{p}\gamma^\nu(1-\gamma^5)]\mathrm{Tr}[-\not{k}'\gamma_\nu(1-\gamma^5)\not{p}'\gamma_\mu(1-\gamma^5)] \\
&= -\frac{256G^2}{4}(k\cdot p')(k'\cdot p)
\end{aligned}$$

Then compute cross section

$$d\Gamma = \frac{1}{2E_\mu}|\mathcal{M}|^2 dQ$$

$$dQ = \frac{d^3k}{(2\pi)^3 2w}\frac{d^3k'}{(2\pi)^3 2w'}\frac{d^3p'}{(2\pi)^3 2E'}(2\pi)^4\delta^4(p-p'-k-k')$$

After long phase space integral, we get

$$\Gamma = \frac{G^2 m_\mu^5}{192\pi^3}$$

4.4 Pion Decay

When protons from outer space entering atmosphere, they collide and produce Π because it has largest mass of leptons, it is easier to product then it decays to K, μ then decays to e^-.

Consider Π^- bound state of $\bar{u}d$,

$$\Pi^- \to \mu^- + \bar{\nu}_\mu$$

out μ^- with p and in ν_μ with $-k$, then 4 momentum of Π^- is $(p+k)$

$$\mathcal{M} = \frac{G}{\sqrt{2}}(\qquad)_\mu[\bar{U}(p)\gamma^\mu(1-\gamma^5)\bar{U}(-k)]$$

We need a correct prefactor goes into the parentheses. The prefactor gives the gluon binding of the Π^- and it must be Lorentz invariant. In fact it is

$$q_\mu f_\Pi$$

where f_Π is called Pion form factor, and q is the propagator. So

$$M = \frac{G}{\sqrt{2}} f_\Pi (p+k)_\mu [\bar{U}(p)\gamma^\mu(1-\gamma^5)\bar{U}(-k)]$$

From Dirac equation

$$(\not{p} - m_\mu)U(p) = 0$$

or

$$\bar{U}(p)(\not{p} - m_\mu) = 0$$

so

$$\bar{U}(p)\not{p} = m_\mu \bar{U}(p)$$

and

$$kU(-k) = 0$$

because $m_\nu \approx 0$.

Therefore

$$\mathcal{M} = \frac{G}{\sqrt{2}} f_\Pi m_\mu [\bar{U}(p)(1-\gamma^5)U(-k)]$$

so

$$\begin{aligned}
|\bar{\mathcal{M}}| &= \frac{G^2}{2} f_\Pi^2 m_\mu^2 \mathrm{Tr}[\not{p}(1-\gamma^5)\not{k}(1+\gamma^5)] \\
&= 4G^2 f_\Pi^2 m_\mu^2 (p-k)
\end{aligned}$$

Take Pion at rest $\vec{p} = -\vec{k}$, let

$$p = (E, \vec{p}) \qquad k = (w, \vec{k})$$

$k^2 = w^2$ because $m_\nu \approx 0$, then

$$p \cdot k = Ew + w^2$$

Then cross section

$$\begin{aligned}
\Gamma &= \frac{4G^2 m_\mu^2 f_\Pi^2}{2m_\Pi} \int \frac{d^3p}{(2\pi)^3 2E} \frac{d^3k}{(2\pi)^3 2w} (2\pi)^4 (p \cdot k) \delta(m_\Pi - E - w) \delta^3(\vec{p} + \vec{k}) \\
&= \frac{G^2 m_\mu^2 f_\Pi^2}{(2\pi)^2 2m_\Pi} \int \frac{d^3k}{E} \delta(m_\Pi - E - w)(E + w)
\end{aligned}$$

Since $E = \sqrt{m_\mu^2 + w^2}$, let

$$\tilde{f}(w) = m_\Pi - \sqrt{m_\mu^2 + w^2} - w$$

so

$$\frac{\partial \tilde{f}}{\partial w} = -\frac{1}{2}\frac{2w}{E} - 1 = -\frac{w}{E} - 1$$

Since

$$\delta(\tilde{f}(w)) = \frac{\delta(w - w_0)}{|\partial \tilde{f}/\partial w|_{w=w_0}}$$

we get

$$\Gamma = \frac{G^2 m_\mu^2 f_\Pi^2}{(2\pi)^2 2m_\Pi} 4\pi \int w^2 dw \delta(w - w_0) \frac{E + w}{w_0 + E}$$

4π and w^2 factors are from spherical integration. w_0 satisfies

$$m_\Pi - E - w_0 = 0$$

or

$$m_\Pi - w_0 = \sqrt{m_\mu^2 + w_0^2}$$

solving for

$$w_0 = \frac{m_\Pi^2 - m_\mu^2}{2m_\Pi}$$

Finally

$$\Gamma = \frac{G^2 m_\mu^2 f_\Pi^2 (m_\Pi^2 - m_\mu^2)^2}{8\pi m_\Pi^3}$$

showing Π is much more likely to decay to μ than to e^-,

$$\frac{\Gamma(\Pi^- \to \mu^- \bar{\nu}_\mu)}{\Gamma(\Pi^- \to e^- \bar{\nu}_e)} = \frac{m_\mu^2 (m_\Pi^2 - m_\mu^2)^2}{m_e^2 (m_\Pi^2 - m_e^2)^2} = 0.8 \times 10^4$$

Is there a physical reason why Π^- is likely to decay to μ^-?

The charged current of the weak interaction couples to left-handed particles (i.e., the coupling is proportional to $(1-\gamma^5)$) and right-handed antiparticles. In the rest frame of Π^-, one has positive helicity and the other has negative helicity, but since they are flying in opposite directions, that implies in the rest frame of Π^- the total angular momentum is not 0, so it must be the case that μ flips spin right away. We will show later the flipping happens in some matrix elements in the Language. Since mass of μ is much larger than e^-, so it is easier for μ than for e.

4.5 Interactions with Hadrons

Heavier the better

Lecture 14 (10/28/13)
Lecture 15 (10/30/13)
Lecture 16 (11/11/13)
Lecture 17 (11/13/13)
Lecture 18 (11/18/13)
Lecture 19 (11/20/13)
Lecture 20 (11/25/13)
Lecture 21 (12/2/13)

Lecture 22
(12/4/13)
Lecture 23
-Last Lec-
(12/9/13)