Complex Variables

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Transcribed by Ron Wu

This is an undergraduate course. Offered in Spring 2014 at Columbia University. Reference: Ahlfors, *Complex Analysis*. Office hours: TuWe 5:30-6:30.

Contents

From a truly dedicated teacher to his immature students:

A Warm Note to the Mermaid

by Patrick Gallagher

There's always danger in excess
Too much of anything can make a mess
Too many notes, with thoughts too subtle
Tossed out to sea, each in a bottle

Too late I wish I'd written less
A rain of asteroids would wreck no worse
Oh, what a mess, poor planet earth
All your blue seas boil out to space. Then

Higher forms come from some place
With waddling walks, and eyes on stalks
From every salty, bubbling puddle
They'll fish out a little bottle

They'll read the notes, and know the thoughts
That only you were meant to see
They'll feel so blue, not knowing you
They'll know how much you meant to me.

Course Overview

Lecture 1 (1/22/14)

For those interested in the historical development of the subject may look up brief biographies of

Gauss, Cauchy,

Abel, Jacobi, Liouville, Riemann, Weierstrass, Picard

Hadamard, Nevanlinna, Ahlfors

We will cover the following topics, tentative

1. complex numbers,
2. polynomials and their toots. fundamental theorem of algebra
3. open sets, paths, regions, convex set, analytic functions
4. derivatives of polynomials & rational functions, Gauss-Lucas theorem
5. path length and path integral
6. Cauchy integral theorem
7. Cauchy integral formula and Cauchy derivative formula
8. Morera theorem, Liouville theorem, Cauchy-Taylor formula
9. uniform convergence
10. Weierstrass test and Abel theorem
11. exponential, hyperbolic and trigonometric functions
12. zeros of analytic functions, identity theorem, reflection principle
13. isolated singularities of analytic functions
14. Cauchy residue theorem
15. Fourier transform and Fresnel integral
16. example of integrals using residue theorem
17. counting zeros, Rouche-Glicksberg theorem, local mapping
18. analytic functions without zeros, Jenson's formula
19. Euler formula, Weierstrass product, Stirling formula
20. elliptic functions, Liouville theorem and Weierstrass P-function

21. ODE for the P-function and Jacobi identities

option 1: advanced complex analysis

22. Riemann sphere and Nevanlinna first affinity theorem
23. Ramification and spherical derivative, Nevanlinna second affinity theorem
24. Nevanlinna main theorem, multiplicity theorem, and Picard theorem
25. Ahlfor's proof of Nevanlinna's main theorem

option 2: number theory

26. Elementary number theory
27. Theorem of Tcebychew, Mertens on prime numbers
28. Riemann zeta function
29. Modern proof of prime number theorem

1 Complex Numbers and Complex Functions

1.1 Complex Numbers

$\mathbb{P}$ is the set of all prime numbers, and $\mathbb{H}$ is the set of all quaternions, invented by Hamilton, but it doesn't have much usage.

$$\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$$

Around 1800, Gauss conjectured (when he was 15-16 years old) prime number theorem: an approximate formula that counts the number of prime numbers up to n. But he couldn't prove it. He was also the first person who used complex numbers professionally.

1800-1900 was the most active period in the development of mathematics. The two unrelated fields $\mathbb{P}$ and $\mathbb{C}$ finally married.

	$\mathbb{P}$	$\mathbb{C}$
1800	Gauss states PNT	Gauss used $\mathbb{C}$
1825		Cauchy studied curves, integral in space replaced line in $\mathbb{R}$ by curves in $\mathbb{R}^2$
1850	Riemann proved weaker verison PNT	He invented Riemann zeta function first time used complex varibles in number thoery
1875		Wiesstrass maked complex analysis rigiuous
1900	One hundred later Hadamard proved PNT using complex analysis	
1950	Selberg, Erdos found simpler proof. So far all proofs of PNT used complex analysis	

Definition 1. A complex number z is a point (x, y) in the xy plane, called complex plane and denoted by $\mathbb{C}$. The real coordinates x, y are called the real part and imaginary part of z,

$$x = \mathrm{Re}z \quad y = \mathrm{Im}z$$

Among the complex numbers are those with $\mathrm{Im}z = 0$, i.e. $z = (x, 0)$. We regard these points as real numbers and write simply $(x, 0) = x$, e.g. $(0, 0) = 0$. called 0 complex number. Thus the set $\mathbb{R}$ of all real numbers is just the x axis in $\mathbb{C}$, called the real axis.

Each point $z = (x, y)$ also has polar coordinates $r > 0$ and $\theta \in \mathbb{R}$, except for $z = (0, 0)$. r is called absolute value of z and θ is argument of z, written 0

$$r = |z| \quad \theta = \arg z$$

and θ is only determined up to an integer multiple of 2π.

Definition 2. The sum $z_1 + z_2$ of complex numbers z_1, z_2 is defined using Cartesian coordinates

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Geometrically addition of complex numbers is vector addition $z_1 + z_2$ is the point in $\mathbb{C}$ for which 0 z_1, $z_1 + z_2$, z_2 are the four vertices of a parallelogram.

Proposition 3. *For* z_1,z_2,z_3,$z \in \mathbb{C}$

1) $z_1 + z_2 = z_2 + z_1$

2) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

3) $z + 0 = z$

4) $z + (-z) = 0$ *with* $-z = (-x, -y)$ *for* $z = (x, y)$

Proof using Cartesian coordinates.

Definition 4. The product $z_1 z_2$ of nonzero complex numbers is defined using polar coordinates,

$$(r_1, \theta_1)(r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$$

If one of $z_1,\ z_2$ is 0, then the product is 0.

Proposition 5. *For* z_1,z_2,z_3,$z \in \mathbb{C}$

1) $z_1 z_2 = z_2 z_1$

2) $z_1(z_2 z_3) = (z_1 z_2) z_3$

3) $z1 = z$

4) $zz^{-1} = 1$ *if* $z \neq 0$ *with* $z^{-1} = (r^{-1}, -\theta)$ *for* $z = (r, \theta)$

Proof using polar coordinates

Proposition 6. *For* z_1,z_2,$z \in \mathbb{C}$

$$z(z_1 + z_2) = zz_1 + zz_2$$

$$(z_1 + z_2)z = z_1 z + z_2 z$$

Definition 7. The complex number i is the point $(0, 1)$. Equivalently in polar

$$|i| = 1 \qquad \arg i = \pi/2$$

Proposition 8.

$$i^2 = 1$$

Proof.

$$i^2 = i \cdot i = (1, \frac{\pi}{2})(1, \frac{\pi}{2}) = (1, \pi) = -1$$

□

Proposition 9. *Each* z *in* $\mathbb{C}$ *satisfies*

$$z = x + iy \text{ with } x = Rez,\ y = Imz$$

Proof. $z = (x, y) = (x, 0) + (0, y) = x + iy$. $\square$

Definition 10. The complex conjugate $\bar{z}$ of a complex number z is defined by

$$\bar{z} = x - iy \text{ for } z = x + iy$$

Equivalent in polar

$$|\bar{z}| = |z| \text{ and } \arg \bar{z} = -\arg z \text{ for } z \neq 0, \text{ and } z = 0 \text{ then } \bar{z} = 0$$

Proposition 11. *For* z_1,z_2,$z \in \mathbb{C}$

1) $Re z = \frac{1}{2}(z + \bar{z})$ $Im z = \frac{1}{2i}(z - \bar{z})$

2) z *is real iff* $\bar{z} = z$

3) $\bar{\bar{z}} = z$

4) $\overline{-z} = -\bar{z}$

5) $\overline{z^{-1}} = \bar{z}^{-1}$ *for* $z \neq 0$

6) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

7) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

8) $|z|^2 = z\bar{z}$

9) $|z_1 \pm z_2|^2 = |z_1|^2 \pm 2Re(z_1\overline{z_2}) + |z_2|^2$

10) For each parallelogram the sum of the square of the lengths of the two diagonals equals the sum of the square of the length of the four sides.

11) $Re(z_1\overline{z_2}) = \frac{1}{4}(|z_1 + z_2|^2 - |z_1 - z_2|^2)$

12) $z_1\overline{z_2} = \frac{1}{4}(|z_1 + z_2|^2 + i\,|z_1 + iz_2|^2 - |z_1 - z_2|^2 - i\,|z_1 - iz_2|^2)$

13) $|z_1 + z_2| \leq |z_1| + |z_2|$

1.2 Polynomials

Lecture 2 (1/27/14)

Definition 12. A polynomial is a complex valued function f of a complex variable z of the form

$$f(z) = c_0 z^n + c_1 z^{n-1} + ... + c_n \tag{1.1}$$

with complex coefficients $c_0, ..., c_n$. The terms are $c_0 z^n$, $c_1 z^{n-1}$,... The constant term is c_n. If $c_0 \neq 0$, then f has degree n and c_0 is the leading coefficient. The polynomials of degree 0 are the nonzero constant function with $c_0 \neq 0$. The $f = 0$ function has no degree and no leading coefficient.

Definition 13. For f a polynomial and $z_0 \in \mathbb{C}$, if

$$f(z_0) = 0 \text{ and } f \neq 0$$

then z_0 is a root of f. The $f = 0$ function has no root.

Proposition 14. *If f is a polynomial of degree n and z_0 is a root of f, then $n \geq 1$ and*

$$f(z) = (z - z_0)f_1(z)$$

for all z in $\mathbb{C}$, with f_1 is a polynomial of degree $n-1$ with the same leading coefficient as f.

Proof. Suppose f is a polynomial given by (1.1). Since z_0 is a root of f, it follows

$$c_0 z^n + c_1 z^{n-1} + ... + c_n = 0$$

Subtracting from (1.1)

$$c_0(z^n - z_0^n) + c_1(z^{n-1} - z_0^{n-1}) + ... + c_{n-1}(z - z_0) = f(z) \tag{1.2}$$

Use the identity

$$a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + ... + b^{k-1})$$

valid for all complex numbers a, b and all integer $k \geq 1$. Factor out $z - z_0$ from (1.2)

$$\begin{aligned} f_1(z) &= c_0(z^{n-1} + z^{n-2}z_0 + ... + z_0^{n-1}) + \text{terms of degee } < n-1 \\ &= c_0 z^{n-1} + \text{terms of degee } < n-1 \end{aligned}$$

showing f_1 is a polynomial of degree $n-1$ with leading coefficient c_0. $\square$

Theorem 15. *(Fundamental Theorem of Algebra, Gauss 1800) Each polynomial f of degree ≥ 1 with all coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Let f be a polynomial of degree $n \geq 1$. Assume that $|f|$ has a minimum (we will prove that later) for some $z_0 \in \mathbb{C}$, i.e.

$$|f(z)| \geq |f(z_0)| \;\; \forall \, z \in \mathbb{C}$$

Assume $f(z_0) \neq 0$ and put

$$g(z) = \frac{f(z + z_0)}{f(z_0)}$$

then g is also a polynomial of degree n satisfying

$$|g(z)| \geq g(0) = 1 \ \ \forall z \in \mathbb{C} \tag{1.3}$$

So the constant term of g is 1, and there is at least one other nonzero coefficient. Let cz^k be the term in g of least positive degree with nonzero coefficient, then

$$g(z) = 1 + cz^k + h(z)$$

so that $h(z)$ is a polynomial with no term of degree $\leq k$, h may be 0.

We can choose $z \neq 0$ with $|z|$ small s.t.

$$cz^k = -\left|cz^k\right|$$

and

$$\left|cz^k\right| < 1 \text{ and } |h(z)| < \left|cz^k\right|$$

They imply

$$|g(z)| \leq \underbrace{\left|1 + cz^k\right|}_{=1-\left|cz^k\right|} + |h(z)| < 1$$

Contradicting to (1.3). Hence $f(z_0) = 0$. □

Corollary 16. *Each polynomial f of degree $n \geq 1$ factors as follows*

$$f(z) = c_0(z - z_1)(z - z_2)...(z - z_n) \tag{1.4}$$

for some $z_1, z_2 ... \in \mathbb{C}$.

Proof by induction and use proposition 14 and theorem 15.

Definition 17. In corollary 16, the z_k may not be distinct. Suppose $z_1, ..., z_d$ are distinct and the remaining z_k are repetitions of these. For each $j = 1, ..., d$, denoted by m_j the number of $k = 1, ..., n$ with $z_k = z_j$ we call m_j the multiplicity of the root z_j. So (1.4) becomes

$$f(z) = c_0(z - z_1)^{m_1}...(z - z_d)^{m_d}, \text{ with } n = m_1 + ... + m_d \tag{1.5}$$

We still say that f has n roots, counted according to multiplicity.

Definition 18. For each positive integer n, the nth roots of unity are the solutions in $\mathbb{C}$ of the equation

$$z^n = 1$$

Proposition 19. *For each n in $\mathbb{N}$, there are exactly n distinct nth roots of unity.*

For $n \geq 3$. They are vertices of the polygon with n equal sides inscribed in the circle of radius 1 centered at 0 with one vertex at 1.

Proof. For ξ in $\mathbb{C}$, $\xi^n = 1$ means

$$|\xi^n| = 1 \text{ and } \arg \xi^n = 0$$

so

$$|\xi|^n = 1 \text{ and } n \arg \xi = 0$$

so

$$|\xi| = 1 \text{ and } \arg \xi = \frac{2\pi k}{n} \quad \text{for some integer } k$$

$\square$

Exercise 20. Prove for each integer $n > 1$

1) the sum of all the nth roots of unity is 0

2) the product of the distance from 1 to the other nth roots of unity is n.

Proposition 21. *For each n in $\mathbb{N}$ and each nonzero d in $\mathbb{C}$, there are exactly n distinct roots of $z^n - d$. If δ is any one of them, then they are the n complex numbers of the form $\rho = \xi\delta$, where ξ is an nth root of unity.*

Example 22. Solve polynomials of degree 3.

A special case (to which the general case easily reduces) is

$$z^3 - 3z - 2b = 0 \text{ for } b \in \mathbb{C} \tag{1.6}$$

with $z = w + w^{-1}$, thus w is root of

$$w^2 - wz + 1 = 0$$

the binomial theorem gives

$$z^3 = w^3 + 3w + 3w^{-1} + w^{-3}$$

or

$$z^3 - 3z = w^3 + w^{-3}$$

so (1.6) becomes

$$w^6 - 2bw^3 + 1 = 0$$

so

$$w^3 = b \pm \sqrt{b^2 - 1} \tag{1.7}$$

Since

$$(b + \sqrt{b^2 - 1})(b - \sqrt{b^2 - 1}) = 1$$

So the six solutions of w to (1.7) are really three and their reciprocals, giving at most three distinct values for z. If $b = \pm 1$, there are just two distinct roots, one of multiplicity 2.

1.3 Point Set Topology: Derivatives

Lecture 3 (1/29/14)

In complex variable one studies complex functions of a complex variable, defined in regions of $\mathbb{C}$ and taking values in $\mathbb{C}$, their derivatives, and their integrals over paths.

Definition 23. Let D be a subset $\mathbb{C}$. We say D is open if for each point p in D, there is an $r > 0$ so that the open disc

$$|z - p| < r$$

of radius r centered at p is entirely contained in D.

Example 24. (1) $\mathbb{C}$ itself is open, since every open disc is contained in $\mathbb{C}$; (2) For every choice of $z_1, z_2, ..., z_n$ in $\mathbb{C}$, the set $\mathbb{C} - \{z_1, z_2, ..., z_n\}$ is open. (3) Each open disc is open (4) the slit plane $\mathbb{C} - \{(-\infty, 0]\}$ is open.

Recall in real function of real variable, the concepts of limit, continuous, derivative are defined on a closed interval $[a, b]$ of $\mathbb{R}$, or on a puncture interval $[a, b] - \{t_0\}$. Here we will do the same:

Definition 25. A complex valued function $\gamma = \gamma(t)$ with real variable, where t is defined on a closed interval $[a, b]$ of $\mathbb{R}$, or on a puncture interval $[a, b] - \{t_0\}$. For such γ defined on $[a, b] - \{t_0\}$, and for L in $\mathbb{C}$, we write the *limit* L as

$$\gamma(t) \to L \text{ for } t \to t_0$$

if for each real $\epsilon > 0$, $\exists$real $\delta > 0$ so that

$$|\gamma(t) - L| < \epsilon \text{ for all } t \text{ in } [a, b] \text{ with } 0 < |t - t_0| < \delta$$

For γ defined on $[a, b]$ we say γ is *continuous* if for each t_0 in $[a, b]$,

$$\gamma(t) \to \gamma(t_0) \text{ for } t \to t_0$$

and γ has *derivative* γ' if for each t_0 in $[a, b]$

$$\frac{\gamma(t) - \gamma(t_0)}{t - t_0} \to \gamma'(t_0) \text{ for } t \to t_0, t \neq t_0$$

Definition 26. Let p, q be points in $\mathbb{C}$. A smooth path from p to q is any complex valued function $\gamma = \gamma(t)$ defined on $[a, b]$ with a continuous derivative γ' and satisfying $\gamma(a) = p$ and $\gamma(b) = q$..

Notice that a smooth path is the function γ, not the set of points $\gamma(t)$, though we sometimes use the same notation for both.

Example 27. (1) For any point p and q in $\mathbb{C}$ consider the line segment $[p, q]$ in $\mathbb{C}$. To get a smooth path from p to q we usually take

$$\gamma(t) = (1 - t)p + tq \text{ for } t \in [0, 1]$$

and denote this path by $[p, q]$. Here $\gamma' = q - p$ is constant, so is continuous.

(2) Let C be the circle centered at z_0 with radius r. The usual smooth path is given by

$$\gamma(\theta) = z_0 + r\cos\theta + ir\sin\theta \text{ for } \theta \in [0, 2\pi]$$

with

$$\gamma'(\theta) = -r\sin\theta + ir\cos\theta = i(\gamma - z_0)$$

here $p = q = z_0 + r$.

(3) For each smooth path γ from p to q, defined on $[a, b]$ the opposite smooth path $-\gamma$ from q to p is defined on $[-b, -a]$ by $-\gamma(t) = \gamma(-t)$.

Definition 28. A piecewise smooth path γ from p to q is a finite sequence of $n \geq 1$ smooth paths $\gamma_1, ..., \gamma_n$ from z_0 to z_1, ..., from z_{n-1} to z_n with $z_0 = p$ and $z_n = q$. We write

$$\gamma = \gamma_1 + ... + \gamma_n$$

From now on path means piecewise smooth path.

Definition 29. A region is an open subset D of $\mathbb{C}$ which is connected, i.e. for each p and q in D there is a path γ in D from p to q.

Definition 30. A subset D of $\mathbb{C}$ is convex if for each p and q in D, the line segment $[p, q]$ is in D.

Exercise 31. Prove that each convex open set is a region.

Exercise 32. Prove $\mathbb{C} - \{0\}$ is a region but not convex.

Exercise 33. Prove $\mathbb{C} - \mathbb{R}$ is not a region.

Definition 34. A complex valued function $f = f(z)$ with complex variable, defined on an open set D, or a puncture open set $D - \{z_0\}$ with $z_0 \in D$. For such f defined on such $D - \{z_0\}$ and L in $\mathbb{C}$, we say L is the limit

$$f(z) \to L \text{ as } z \to z_0$$

if for each $\epsilon > 0$ there is $\delta > 0$ so that

$$|f(z) - L| < \epsilon \text{ for all } z \text{ in } D \text{ with } 0 < |z - z_0| < \delta$$

For f defined on such D, we say f is continuous if for each z_0 in D,

$$f(z) \to f(z_0) \text{ for } z \to z_0$$

and f has derivative f' in D if for each $z_0 \in D$

$$\frac{f(z) - f(z_0)}{z - z_0} \to f'(z_0) \text{ for } z \to z_0, z \neq z_0$$

Proposition 35. *Let D be an open set in $\mathbb{C}$, let z_0 be a point in D, and let f, g be complex-valued functions defined on the punctured set $D - \{z_0\}$. Assume that for $z \to z_0$*

$$f(z) \to L \text{ and } g(z) \to M \tag{1.8}$$

the for $z \to z_0$

$$\begin{aligned} f(z) + g(z) &\to L + M \\ f(z)g(z) &\to LM \end{aligned} \tag{1.9}$$

If in addition $g(z) \neq z$ for z in $D - \{z_0\}$ and $M \neq 0$ then

$$\frac{f(z)}{g(z)} \to \frac{L}{M} \tag{1.10}$$

Proof. let's prove (1.9). Using an identity

$$f(z)g(z) - LM = (f(z) - L)M + L(g(z) - M) + (f(z) - L)(g(z) - M)$$

and using triangle inequality

$$|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$$

we have

$$|f(z)g(z) - LM| \leq |f(z) - L|\,|M| + |L|\,|g(z) - M| + |f(z) - L|\,|g(z) - M|$$

To make the right side of this $< \epsilon$ we make each of the three terms $< \epsilon/3$, i.e. we choose

$$|f(z) - L| < \frac{\epsilon/3}{|M| + 1}, \;\; |g(z) - M| < \frac{\epsilon/3}{|L| + 1}$$

and

$$|f(z) - L| < \sqrt{\epsilon/3}, \;\; |g(z) - M| < \sqrt{\epsilon/3}$$

they can be met because of (1.8).

Now prove (1.10). It suffices to show

$$\frac{1}{g(z)} \to \frac{1}{M}$$

start with

$$\frac{1}{g(z)} - \frac{1}{M} = \frac{M - g(z)}{g(z)M} \text{ for } z \in D - z_0$$

for small $|z - z_0| \neq 0$ we have $|g(z)| \geq |M|/2$ since $g(z) \to M$ and $M \neq 0$. It follows that for small $|z - z_0| \neq 0$

$$\left| \frac{1}{g(z)} - \frac{1}{M} \right| \leq \frac{2\,|g(z) - M|}{M^2}$$

The right side will be $< \epsilon$ provided $|g(z) - M| < |M|^2 \epsilon/2$ which can be met because $g(z) \to M$ for $z \to z_0$. □

Theorem 36. *If f,g are complex-valued functions with derivatives f', g' on an open set D then $f + g$ and fg have derivatives on D given by*

$$(f + g)' = f' + g' \tag{1.11}$$

$$(fg)' = f'g + fg' \tag{1.12}$$

If $g(z) \neq 0$ for $z \in D$ then f/g has a derivative

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \tag{1.13}$$

Proof. Prove (1.12), use (1.9). For $z_0 \in D$ and $z \in D - \{z_0\}$

$$(fg)(z)-(fg)(z_0) = (f(z)-f(z_0))g(z_0)+f(z_0)(g(z)-g(z_0))+(f(z)-f(z_0))(g(z)-g(z_0))$$

or

$$\frac{(fg)(z)-(fg)(z_0)}{z-z_0} = \frac{f(z)-f(z_0)}{z-z_0}g(z_0)+f(z_0)\frac{g(z)-g(z_0)}{z-z_0}+\frac{f(z)-f(z_0)}{z-z_0}\frac{g(z)-g(z_0)}{z-z_0}(z-z_0)$$

then $z \to z_0$, we have (1.12).

Similarly to show (1.13), suffice

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

and use

$$\frac{\frac{1}{g(z)} - \frac{1}{g(z_0)}}{z-z_0} = -\frac{g(z)-g(z_0)}{z-z_0}\frac{1}{g(z)}\frac{1}{g(z_0)}$$

□

Exercise 37. Let

$$\gamma(t) = t^2 \sin\frac{1}{t} \text{ for } 0¡\,|t| \leq 1 \text{ and } \gamma(0) = 0$$

Graph γ show that γ has derivative on $[-1,1]$ with $\gamma'(0) = 0$ but $\gamma'(0)$ is not continuous on $[-1,1]$: i.e. $\gamma'(t)$ has no limit for $t \to 0$.

1.4 Analytic Functions

Lecture 4 (2/3/14)

Definition 38. A complex valued function of a complex variable defined on a region D and having a derivative there is said to be analytic in D.

Proposition 39. *(1) Each polynomial* (1.1) *is analytic in* $\mathbb{C}$, *its derivative is*

$$f'(z) = nc_n z^{n-1} + ... + c_1$$

(2) Each rational function $f = g/h$, g, h are polynomial, $h \neq 0$ is analytic in the region $D = \mathbb{C} - \{z_1, ..., z_n\}$ where $z_1, ..z_n$ are the roots of h. The derivative of f is (1.13) *is also an analytic rational function in the same D.*

Proof. Let's prove part (1). It suffices in view of (1.11) to observe that $c' = 0$ for each constant c and to show that $(cz^k)' = kcz^{k-1}$ for constant c and $k \in \mathbb{N}$. In fact for z_0 and $z \in \mathbb{C}$

$$cz^k - cz_0^k = c(z - z_0)(z^{k-1} + z_0 z^{k-2} + ... + z_0^{k-1})$$

for $z \to z_0$ with $z \neq z_0$,

$$\frac{cz^k - cz_0^k}{z - z_0} \to c(z^{k-1} + z_0 z^{k-2} + ... + z_0^{k-1}) = kcz_0^{k-1}$$

□

Proposition 40. *Let $k > 1$. If $f_1, ..., f_k$ are analytic in D, then so is $f_1 \cdot ... \cdot f_k$ and*

$$(f_1 \cdot ... \cdot f_k)' = f_1' f_2 ... f_k + ... + f_1 f_2 ... f_k' \tag{1.14}$$

in particular if f is analytic in D, so is f^k and

$$(f^k)' = kf^{k-1}f'$$

Proof. The case $k = 2$ is the product rule, then use induction. □

Proposition 41. *Let z_0 be a root of multiplicity m of a polynomial f, (a) if $m = 1$, then $f'(z_0) \neq 0$. (b) if $m > 1$, then z_0 is a root of f' of multiplicity $m - 1$.*

Proof. By (1.5), $f(z) = (z - z_0)^m g(z)$, where g is a polynomial with $g(z_0) \neq 0$, i follows that

$$f'(z) = m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z) = (z - z_0)^{m-1} h(z)$$

with $h(z) = mg(z) + (z - z_0)g'(z)$. Thus h is a polynomial with $h(z_0) \neq 0$. Therefore if $m = 1$ then $f'(z_0) \neq 0$, while if $m > 1$ then z_0 is a root of f' of multiplicity $m - 1$. □

Proposition 42. *If f is a polynomial of degree n, then there are fewer than n distinct complex numbers w for which $f - w$ has a multiple root i.e. a root of multiplicity > 1.*

Proof. If $w_1, ..., w_n$ are distinct, and $f(z_1) - w_1 = 0, ..., f(z_n) - w_n = 0$, then $z_1, ..., z_n$ are distinct. If in addition z_k is a multiple root of $f - w_k$ for each k, then each z_k is a root of f', too many roots for a polynomial of degree $n - 1$. □

Exercise 43. Let f be a polynomial of degree $n \geq 1$ which is real on $\mathbb{R}$ Prove that

(1) f has at most one more real root than f'

(2) f' has no more non-real roots than f

(3) if all the roots of f are real, then all the roots of f' are real.

Lemma 44. *For each polynomial f of degree $n \geq 1$, with $z_1, ..., z_n$*

$$\frac{f'}{f}(z) = \sum_{k=1}^{n} \frac{1}{z - z_k} \text{ for } z \neq z_1, ..., z_n \tag{1.15}$$

Proof. From $f(z) = c_0(z - z_1)...(z - z_n)$ and (1.14)

$$f'(z) = c_0 \sum_{k=1}^{n} (z - z_1)...(\widehat{z - z_k})...(z - z_n)$$

$\hat{w}$ means that the factor w is to be omitted. Dividing the formula for $f'(z)$ by the formula for $f(z)$ gives, after cancellation, the formula for $(f'/f)(z)$. □

Definition 45. The convex hull H of complex numbers $z_1, ..., z_n$ (not necessarily distinct) is the set of all points z of the form

$$z = \lambda_1 z_z + ... + \lambda_n z_n \text{ with all } \lambda_k \geq 0 \text{ and } \lambda_1 + ... + \lambda_n = 1 \tag{1.16}$$

Exercise 46. Prove that the convex hull of $z_1, ..., z_n$ is contained in every convex set which contains $z_1, ..., z_n$.

Theorem 47. *(Gauss-Lucas) Let f be a polynomial of degree $n \geq 1$. Then each root of f' is contained in the convex hull H of the roots of f.*

Proof. By lemma 44 let $z_1, ..., z_n$ be the roots of f. Let z^*be the roots of f'. If $z^*$is one of $z_1, ..., z_n$, then z^* is in H, set one $\lambda_k = 0$ the rest 0. If z^* is not one of $z_1, ..., z_n$, we put $z = z^*$ in (1.15) gives

$$\sum_{k=1}^{n} \frac{1}{z^* - z_k} = 0$$

conjugating this, and using $1/\bar{w} = w/|w|^2$ for $w \neq 0$ gives

$$\sum_{k=1}^{n} \frac{z^* - z_k}{|z^* - z_k|^2} = 0$$

writing this sum as a difference of two sums, and factoring z^* from the numerators of the first sum gives

$$z^* = \sum_{k=1}^{n} \lambda_k z_k$$

with

$$\lambda_k = \frac{\frac{1}{|z^*-z_k|^2}}{\sum_{k=1}^{n} \frac{1}{|z^*-z_k|^2}}$$

for $k = 1, ..., n$. Clearly λ_k satisfies (1.16). $\square$

Exercise 48. Prove if f is a polynomial of degree n with n distinct real roots $x_1 < ... < x_n$ then $\exists$ a root x_j^* of f' with $x_{j-1} < x_j^* < x_j$ for $j = 2, ..., n$.

Theorem 49. *(Marcel Riesz 1926) Let f be a polynomial of degree ≥ 3 with n distinct real roots separated as in above exercise by the $n-1$ distinct real roots of f'. Then the smallest difference between consecutive roots of f' is greater than the smallest difference between consecutive roots of f.*

Proof. By (1.16) with the $z_k = x_k$ and with $z = x_j^*$ or x_{j-1}^*,

$$\sum_{k=1}^{n} \frac{1}{x_j^* - x_k} - \sum_{k=1}^{n} \frac{1}{x_{j-1}^* - x_k} = 0$$

for all $j = 3, ..., n$. Combining the $k = 2, ..., n$ terms in the first sum with the $k = 1, ..., n-1$ terms in the second sum gives for all $j = 3, ..., n$

$$\frac{1}{x_j^* - x_1} + \sum_{k=1}^{n} \left(\frac{1}{x_j^* - x_k} - \frac{1}{x_{j-1}^* - x_{k-1}} \right) + \frac{1}{x_n - x_{j-1}^*} = 0 \qquad (1.17)$$

Note that the two terms at the ends are positive. If the assertion of the theorem is false, then

$$x_j^* - x_{j-1}^* \leq x_k - x_{k-1} \qquad (1.18)$$

for some $j = 3, ..., n$ and all $k = 2, ..., n$, from this we will show that for some $j = 3, ..., n$ all the terms in the middle sum in (1.17) are ≥ 0 giving a contradiction.

In fact (1.18) is

$$x_j^* - x_k \leq x_{j-1}^* - x_{k-1}$$

for some $j = 3, ..., n$ and all $k = 2, ..., n$. For $k < j$ we have $x_j^* - x_k > 0$ and for $k \geq j$ we have $k - 1 \geq j - 1$ so $x_j^* - x_k < 0$. Thus in both cases the two differences $x_j^* - x_k$ and $x_{j-1}^* - x_{k-1}$ have the same sign. Now we use the fact that if u, v have same sign, and $u \leq v$ then $1/u \geq 1/v$, so

$$\frac{1}{x_j^* - x_k} - \frac{1}{x_{j-1}^* - x_{k-1}} \geq 0$$

for some $j = 3, ..., n$ and all $k = 2, ..., n$. □

2 Complex Integration

2.1 Integral and Path Integral

Lecture 5 (2/5/14)

We define the integral

$$\int_a^b \psi = \int_a^b \psi(t)dt$$

of a complex-valued continuous function ψ on a closed interval $[a, b] \subset \mathbb{R}$.

Definition 50. Given a closed interval $[a, b] \subset \mathbb{R}$, a partition of $[a, b]$ is any finite increasing sequence in $\mathbb{R}$ from a to b

$$\mathcal{P} : a = t_0 \leq t_1 \leq ... \leq t_n = b$$

the norm $\|\mathcal{P}\|$ is defined by

$$\|\mathcal{P}\| := \max\{t_1 - t_0, ..., t_n - t_{n-1}\}$$

Definition 51. Given a partition $\mathcal{P}$ of a closed interval $[a, b]$ of $\mathbb{R}$, we call intermediate points any finite sequence

$$\mathcal{P}^* := t_1^*, ..., t_n^* \text{ with } t_{k-1} \leq t_k^* \leq t_k \text{ for } k = 1, ..., n$$

Definition 52. Given a complex-valued function ψ defined on a closed interval $[a, b]$, a partition $\mathcal{P}$ of $[a, b]$ and a sequence $\mathcal{P}^*$ of intermediate points, the corresponding Riemann sum $S(\psi)$ is defined by

$$S(\psi) = \sum_{k=1}^{n} \psi(t_k^*)(t_k - t_{k-1})$$

observe that $S(\psi)$ depends on all three of $\psi, \mathcal{P}, \mathcal{P}^*$.

Proposition 53. *(also served as a definition for integral) For each continuous ψ on $[a, b]$ as above there is a unique complex number called the integral of ψ and denoted as above which has the following property: For each $\epsilon > 0$ $\exists\, \delta > 0$ so that*

$$\left| S(\psi) - \int_a^b \psi \right| < \epsilon \text{ for all those } \mathcal{P}, \mathcal{P}^* \text{ as above for which } \|\mathcal{P}\| < \delta$$

In this sense, the integral is the limit of Riemann sums as $\|\mathcal{P}\| \to 0$. Proof see introduction to modern analysis I notes.

Proposition 54. *For continuous ψ and ϕ on $[a, b]$ and constant c*

$$\begin{aligned} \int_a^b (\psi + \phi) &= \int_a^b \psi + \int_a^b \phi \\ \int_a^b c\psi &= c \int_a^b \psi \\ \int_a^b \psi &= \int_a^c \psi + \int_c^b \psi \text{ for } a \leq c \leq b \\ \left| \int_a^b \psi \right| &\leq \int_a^b |\psi| \qquad (2.1) \\ \int_a^b 1 &= b - a \end{aligned}$$

One can prove those using Riemann sums.

Corollary 55. *For value-valued continuous ψ and ϕ on $[a, b]$*

$$\int_a^b \psi \geq 0 \text{ if } \psi \geq 0 \text{on } [a, b] \qquad (2.2)$$

$$\int_a^b \psi \geq \int_a^b \phi \text{ if } \psi \geq \phi \text{on } [a, b] \qquad (2.3)$$

Proof. (2.2) follows from (2.1), since $\psi \geq 0$ implies $|\psi| = \psi$. (2.3) follows from (2.2) on replacing ψ by $\psi - \phi$. □

Definition 56. For each smooth path γ, (here smooth doesn't mean piecewise smooth) the length $L = L(\gamma)$ is defined by

$$L := \int_a^b \left| \gamma'(t) \right| dt$$

Example 57. Prove the opposite path $-\gamma$ has the same length as γ. Recall if γ is defined on $[a,b]$, then $-\gamma$ is defined on $[-b,-a]$, and $(-\gamma)(t)=\gamma(-t)$, thus the length of $-\gamma$ is

$$\int_{-b}^{-a}\left|-\gamma'(-t)\right|dt=-\int_{b}^{a}\left|\gamma'(q)\right|dq=\int_{a}^{b}\left|\gamma'(q)\right|dq$$

Definition 58. For each piecewise smooth path $\gamma=\gamma_1+\gamma_2+...+\gamma_n$ with length $L_1,...,L_n$, the length of γ is

$$L_1+L_2+...+L_n$$

Definition 59. For each region D in $\mathbb{C}$ each complex-valued f defined and continuous on D and each piecewise smooth path γ in D the path integral

$$\int_\gamma f=\int_\gamma f(z)dz$$

of f over γ is defined by

$$\int_\gamma f=\int_\gamma f(\gamma(t))\gamma'(t)dt \text{ for smooth } \gamma \text{ defined on } [a,b]$$

and if γ is piecewise smooth

$$\int_\gamma f:=\int_{\gamma_1} f+...+\int_{\gamma_n} f,\ \text{ for } \gamma_1,...,\gamma_n \text{ each smooth}$$

Proposition 60. *For all D, f, g, α, β, γ, L as above and c in $\mathbb{C}$*

$$\begin{aligned}
\int_\gamma(f+g) &= \int_\gamma f+\int_\gamma g\\
\int_\gamma cf &= c\int_\gamma f\\
\int_\gamma f &= \int_\alpha f+\int_\beta f\ \textit{ if } \gamma{=}\alpha{+}\beta\\
\left|\int_\gamma f\right| &\leq ML \textit{ where } M \textit{is the maximum of } |f(z)| \textit{ for } z \textit{ on } \gamma
\end{aligned} \tag{2.4}$$

Proof. We prove (2.4). If γ is a smooth path, then

$$\begin{aligned}
\left|\int_\gamma f\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right|\\
&\leq \int_a^b |f(\gamma(t))|\left|\gamma'(t)\right|dt\leq\int_a^b M\left|\gamma'(t)\right|dt=ML
\end{aligned}$$

If γ is piecewise smooth, then

$$\begin{aligned}\left|\int_\gamma f\right| &= \left|\int_{\gamma_1} f + ... + \int_{\gamma_n} f\right| \\ &\leq \left|\int_{\gamma_1} f\right| + ... + \left|\int_{\gamma_n} f\right| \leq M_1L_1 + ... + M_nL_n \leq M(L_1 + ... + L_n) = ML\end{aligned}$$

□

Proposition 61. *For each counterclockwise circle C centered at z_0*

$$\int_C \frac{dz}{z - z_0} = 2\pi i$$

Proof. By example 27(2) $\gamma' = i(\gamma - z_0)$ for C, so

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{\gamma'(\theta)d\theta}{\gamma(\theta) - z_0} = \int_0^{2\pi} id\theta = 2\pi i$$

□

Below is our first version of fundamental theorem of calculus for path integrals:

Proposition 62. *Let D be a region in $\mathbb{C}$, f a complex-valued continuous function on D, and γ a path in D from p to q. Assume that $\exists$ a function F defined on D with $F' = f$ i.e. f has an anti derivative F, then*

$$\int_\gamma f = F(q) - F(p)$$

If besides f having an anti derivative, γ is a closed path i.e. with $p = q$ then

$$\int_\gamma f = 0$$

Proof. For smooth γ

$$\begin{aligned}\int_\gamma f &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b (F(\gamma(t)))'dt \text{ (by calculus chain rule)} \\ &= F(\gamma(b)) - F(\gamma(a)) \text{ (by calculus fundamental thm)} \\ &= F(q) - F(p)\end{aligned}$$

Then complete the proof for piecewise smooth γ. □

Exercise 63. Prove that there is no F define on $D = \mathbb{C} - \{z_0\}$ satisfying

$$F'(z) = \frac{1}{z - z_0} \text{ on } D \tag{2.5}$$

Exercise 64. Prove that $\int_\gamma f = 0$ for each polynomial f and each closed path γ.

2.2 Cauchy Integral Theorem

Lecture 6 (2/10/14)

This section is the most important section, from which all the rest of the course follows. The Cauchy's integral theorem (1820) is related to Green's theorem (also found in 1820) in vector integral in $\mathbb{R}^2$. Here we derive it by the method of Goursat, which requires weaker hypotheses than those for Green's theorem stated in Calculus.

Theorem 65. *(Cauchy's integral theorem) Let D be a convex region in C, and f is analytic in D, and γ is a closed path in D then*

$$\int_\gamma f(z)dz = 0$$

The proof depends on two lemmas.

Lemma 66. *(Goursat, 1900) If D is a convex region in $\mathbb{C}$ and f is analytic in D and Δ is a triangle in D then*

$$\int_\Delta f(z)dz = 0$$

Proof. Starting with Δ we get four triangles $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$, $\Delta^{(4)}$ each quarter the size of Δ by connecting the midpoints of the sides of Δ. After adding canceling integrals in both directions over the new line segments, we find that the path integral is equal to

$$\int_\Delta f = \int_{\Delta^{(1)}} f + \int_{\Delta^{(2)}} f + \int_{\Delta^{(3)}} f + \int_{\Delta^{(4)}} f$$

Let $\left|\int_{\Delta^{(1)}} f\right|$ be the largest of the four i.e.

$$\left|\int_{\Delta^{(1)}} f\right| \geq \left|\int_{\Delta^{(2,3,4)}} f\right|$$

then by triangle inequality

$$\int_{\Delta} f \leq 4 \left| \int_{\Delta^{(1)}} f \right|$$

then repeat this process for $\Delta^{(1)}$, cut it into four triangles (with relabeling) $\Delta^{(1)}, \Delta^{(2)}, \Delta^{(3)}, \Delta^{(4)}$ with $\left| \int_{\Delta^{(2)}} f \right|$ be the largest of the four, etc, thus

$$\int_{\Delta} f \leq 4^n \left| \int_{\Delta^{(n)}} f \right| \quad n \in \mathbb{N} \tag{2.6}$$

we have the length of $\Delta^{(n)}$

$$L_n = \frac{L}{2^n} \tag{2.7}$$

Now we use a fact to be proven later: As $n \to \infty, \exists\, z^*$ is contained in all Δ_n (possible on some edges Δ_n). Since $z^* \in \Delta_n \subset \Delta \subset D$, and f is analytic in D

$$\frac{f(z) - f(z^*)}{z - z^*} \to f'(z^*) \text{ for } z \to z^*$$

define

$$r(z) = \begin{cases} 0 & z = z^* \\ \frac{f(z)-f(z^*)}{z-z^*} - f(z^*) & z \neq z^* \end{cases}$$

then

$$f(z) = f(z^*) + f'(z^*)(z - z^*) + r(z)(z - z^*)$$

and

$$r(z) \to 0 \text{ as } z \to z^* \tag{2.8}$$

Thus

$$\int_{\Delta} f(z) = \int_{\Delta} \big(f(z^*) + f'(z^*)(z - z^*) \big)\, dz + \int_{\Delta} r(z)(z - z^*) dz$$

the first integral on the right is 0, for the integrand is a polynomial of degree 1 in z. We estimate the second integral on the right

$$\left| \int_{\Delta^{(n)}} f(z) dz \right| \leq M_n L_n^2$$

where M_n is the maximum of $|r(z)|$ for z on Δ_n. Here we have used that $|z - z^*| \leq L_n$ for z on Δ_n. By (2.8) and $L_n \to 0$ for $n \to \infty$, we have

$$M_n \to 0 \text{ as } n \to \infty \tag{2.9}$$

By (2.6), (2.7)

$$\int_{\Delta} f \leq M_n L^2$$

then by (2.9)

$$\int_{\Delta} f(z)dz = 0$$

□

Lemma 67. *Let D be a convex region in $\mathbb{C}$, let f be continuous on D, and assume that*

$$\int_{\Delta} f = 0$$

for every triangle Δ in D. Choose any point p in D, and define F in D by

$$F(z) := \int_{[p,z]} f$$

with $[p, z]$ the line segment from p to z. Then $F" = f$ in D. In particular f has an analytic anti derivative in D.

Proof. Because D is convex the line segment $[p, z]$ is contained in D for every $z \in D$. Therefore F is well-defined. It remains to show that $F' = f$. For z_0 and z in D consider the triangular path

$$\Delta = [p, z] + [z, z_0] + [z_0, p]$$

Since D is convex, Δ is contained in D, so by assumption $\int_{\Delta} f = 0$ i.e.

$$\int_{[p,z]} f + \int_{[z,z_0]} f + \int_{[z_0,p]} f = 0$$

which we may write as

$$F(z) - F(z_0) = \int_{[z,z_0]} f = \int_{[z,z_0]} f(z_0) + \int_{[z,z_0]} (f - f(z_0))$$

From which

$$|F(z) - F(z_0) - f(z_0)(z - z_0)| \leq M(z, z_0)\,|z - z_0|$$

i.e.

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \leq M(z_0, z) \text{ for } z \neq z_0$$

where $M(z_0, z)$ is the maximum of $|f - f(z_0)|$ on the line segment $[z_0, z]$. Since f is continuous in D, we have $M(z_0, z) \to 0$ for $z \to z_0$, giving

$$F'(z_0) = f(z_0) \text{ for all } z_0 \text{ in } D$$

$\square$

Now we prove Cauchy's integral theorem.

Proof. By above two lemmas, each function f analytic in convex regions D has an anti derivative F in D. It follows from proposition 62. $\square$

Use anti derivative lemma one can define log. Recall natural logarithm of calculus is defined by

$$\log x = \int_1^x \frac{d\xi}{\xi} \text{ for real } x > 0$$

The following 4 exercise introduce complex log, which is defined and analytic in the slit plane $S = \mathbb{C} - \{(-\infty, 0]\}$ and agrees with the log of calculus on $(0, \infty)$.

Exercise 68. Define *log* by

$$\log(z) = \int_{[1,z]} \frac{d\zeta}{\zeta} \text{ for } z \text{ in } S$$

$[1, z]$ is a straight path on the complex plant from $(1, 0)$ to complex z. Since the integrand $1/\zeta$ is continuous in $\mathbb{C} - \{0\}$ and therefore in S, and since $[1, z]$ is in S the integral is defined. Prove that

$$(\log z)' = \frac{1}{z} \text{ for } z \text{ in } S$$

Hint: although S is not convex, we can take a smaller convex region.

Exercise 69. With *log* defined above, prove that

$$\log z = \int_\gamma \frac{d\zeta}{\zeta} \text{ for } z \text{ in } S$$

where γ is the piecewise smooth path consisting of the line segment $[1, |z|]$ following by an arc from $|z|$ to z of the circle of radius $|z|$ centered at 0.

Exercise 70. Use exercise 69 to prove that

$$\log z = \log|z| + i \arg z \text{ for } z \text{ in } S$$

where log in $\log|z|$ is the log of calculus and we take $-\pi < \arg z < \pi$.

Exercise 71. Use exercises 70 to show that

$$\log i = i\pi/2 \text{ and } \log -i = -i\pi/2$$

and

$$\log z \to \begin{cases} i\pi & \text{Im} z > 0 \\ -i\pi & \text{Im} z < 0 \end{cases} \text{ for } z \to -1$$

this shows log cannot be extended continuously from S to $\mathbb{C} - \{0\}$.

2.3 Cauchy's Integral Formula

Theorem 72. *(Cauchy's Integral Formula) Let D be a convex region, let f be a function analytic in D, let C be a counterclockwise circle in D and let z_0 be any point inside C then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

Corollary 73. *Each such $f(z_0)$ is determined by the values of f for z on C.*

Proof. Of the theorem. We first show that all sufficiently small $r > 0$

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_r} \frac{f(z)}{z - z_0} dz \tag{2.10}$$

where C_r is the counterclockwise circle of radius centered at z_0 and inside C. In fact we can connect C and C_r with some paths so the annulus looks like consists of some fan-shaped regions. The internal path integrals cancel.

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_r} \frac{f(z)}{z - z_0} dz + \int_{Fan_1} \frac{f(z)}{z - z_0} dz + \int_{Fan_2} \frac{f(z)}{z - z_0} dz + ...$$

One then applies Cauchy's integral theorem to each fan. because $f(z)/(z - z_0)$ is analytic in a convex region containing the fan. The convex region could be the intersection of D with the half plane whose edge across z_0. Thus each $\int_{Fan} \frac{f(z)}{z - z_0} dz = 0$.

Next show

$$\int_{C_r} \frac{f(z)}{z - z_0} dz \to 2\pi i f(z_0) \text{ for } r \to 0 \tag{2.11}$$

In fact, using (2.5) for C_r gives

$$\int_{C_r} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \tag{2.12}$$

We have

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq M_r L_r$$

with $L_r = 2\pi r$ and

$$M_r = \max \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \quad \text{for } z \text{ on } C_r$$

Since f is analytic in D and z_0 is in D, we have $M_r \to |f'(z_0)|$ for $r \to 0$, with $L_r \to 0$ for $r \to 0$, we conclude that the integral on the right in (2.12) has limit 0. Thus combining (2.10), (2.11), prove the theorem. □

Exercise 74. In the proof we assumed (1), so prove the following

(1) intersection of two convex sets is a convex set;

(2) intersection of two open sets is an open set;

(3) intersection of two convex regions is a convex region.

Exercise 75. Use Cauchy's Integral theorem and Cauchy's Integral Formula to show for each counterclockwise circle C and each z_0 not on C

$$\int_C \frac{dz}{z - z_0} = \begin{cases} 2\pi i & z_0 \text{ inside } C \\ 0 & z_0 \text{ outside } C \end{cases}$$

Lecture 7 (2/12/14)

Definition 76. An entire function is a function analytic in all of $\mathbb{C}$.

Thus each polynomial is entire, but log is not.

Definition 77. A function f on D is bounded if for some positive B

$$|f(z)| \leq B$$

for all $z \in D$.

Lemma 78. *No non constant polynomial f is bounded. In fact for each such f*

$$|f(z)| \to \infty \tag{2.13}$$

for $|z| \to \infty$.

Proof. For each such $f = c_0 z^n + ... + c_n$,

$$\frac{f(z)}{z^n} = c_0 + \frac{c_1}{z} + ... + \frac{c_n}{z^n}$$

which $\to c_0$ for $|z| \to \infty$, from which (2.13) follows. $\square$

Theorem 79. *(Liouville 1840) Each bounded entire function is constant.*

Proof. Let f be entire and suppose $|f(z)| \leq B$ for all z in $\mathbb{C}$. Let p, q be any two points in $\mathbb{C}$. For each $r > \max\{|p|, |q|\}$, Cauchy integral formula gives

$$f(q) - f(p) = \frac{1}{2\pi i} \int_{C_r} \left(\frac{1}{z-q} - \frac{1}{z-p} \right) f(z) dz$$

where C_r is the counterclockwise circle of radius r centered at 0.

For each z on C_r we have $|z| = r$ so $|z-p| \geq r - |p|$ and $|z-q| \geq r - |q|$, so

$$\left| \frac{1}{z-q} - \frac{1}{z-p} \right| = \left| \frac{q-p}{(z-q)(z-p)} \right| \leq \frac{|q-p|}{(r-|q|)(r-|p|)}$$

It follows by the ML inequality, cf (2.4),

$$|f(q) - f(p)| \leq \frac{B\,|q-p|}{(r-|q|)(r-|p|)} 2\pi r \to 0 \text{ for } r \to \infty$$

Thus $f(q) - f(p)$ doesn't depend on r, so f is constant. $\square$

One of the application of Liouville's theorem is a short proof of the fundamental theorem of algebra (already proved before theorem 15.

Proof. (new proof) suppose f is a non constant polynomial with no root. Put $g = 1/f$. then g is an entire function. By the lemma above, it follows that

$$g(z) \to 0 \tag{2.14}$$

for $|z| \to \infty$. Therefore g is bounded. (In more detail, (2.14) implies $|g(z)| \leq 1$ for $|z| > R$ for some large constant R. However $|g(z)|$ being continuous, by a general theorem in analysis, is also bounded for $|z| \leq R$)

By Liouville's theorem g as a bonded entire function must be constant. By (2.14) the constant must be 0, an impossible value for a reciprocal. Contradiction. $\square$

Finally we show another proof of fundamental theorem of algebra, due to N. Ankeny (1950's). His proof uses the Cauchy integral theorem, but not the Cauchy integral formula:

Proof. For each non constant polynomial $f(z) = c_0 z^n + ... + c_n$ put

$$f^*(z) = \bar{c}_0 z^n + ... + \bar{c}_n$$

then

$$\overline{f(z)} = f^*(\bar{z}) \tag{2.15}$$

so the roots of f^* are the conjugates of the roots of f. Thus if f has no roots, neither does f^* so if we put $h = ff^*$then h is a polynomially with no roots, and again by (2.15)

$$h(x) = f(x)f^*(x) = f(x)\overline{f(x)} = |f(x)|^2 > 0, \text{ for all real } x \tag{2.16}$$

also since h has degree ≥ 2, $\exists$ constant $c > 0$ s.t.

$$|h(z)| \geq c\,|z|^2 \text{ for all sufficiently large } |z| \tag{2.17}$$

Since h is entire and has no roots, $1/h$ is also entire. By Cauchy's integral theorem, for each $r > 0$

$$\int_{[-r,r]} \frac{1}{h(x)} dx = \int_{\sigma_r} \frac{1}{h(z)} dz$$

where σ_r is the clockwise semicircle connecting $-r, r$. By (2.16) the integral on the left is a positive increasing function of r for $r > 0$. By (2.17) and the ML inequality the integral on the right for sufficiently large r has absolute value

$$\left| \int_{\sigma_r} \frac{1}{h(z)} dz \right| \leq \frac{\pi r}{cr^2} \to 0 \text{ for } r \to \infty$$

not possible for a positive increasing function. The contradiction shows that a non constant polynomial with no roots is impossible. $\square$

2.4 Cauchy's Derivative Formulas

The Cauchy's integral formula can be used to reveal a bunch of pleasant properties of analytic functions. For example we can take derivative of it (for now not worrying about strict rigor)

$$f'(z_0) = \frac{d}{dz_0}\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i}\int_C \frac{d}{dz_0}\frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2} dz \tag{2.18}$$

The pass of derivative and integral will be make rigor later.

Let's look at this in a bit more detail: for z_0,$z_1$both inside C,

$$f(z_1)-f(z_0) = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-z_1} - \frac{1}{z-z_0}\right) f(z)dz = \frac{1}{2\pi i}\int_C \frac{z_1-z_0}{(z-z_1)(z-z_0)} f(z)dz$$

so for z_0,$z_1$both inside C, and $z_0 \neq z_1$

$$\frac{f(z_1)-f(z_0)}{z_1-z_0} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_1)(z-z_0)}dz$$

All the remains is to let $z_1 \to z_0$ and say that

$$\int_C \frac{f(z)}{(z-z_1)(z-z_0)}dz \to \int_C \frac{f(z)}{(z-z_0)^2}dz$$

this requires uniform convergence, which we will study later.

Theorem 80. *(Cauchy's Derivative Formulas) Let f be analytic in any region D. Then for $n = 1, 2, ...,$*

1. *$f^{(n)}$ is analytic in D. In particular f' is analytic in D.*
2. *For D is convex, C a counterclockwise circle in D and z_0 inside C*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Proof. With the usual convention $f^{(0)} = f$, prove by induction $n = 0$ is Cauchy's integral formula. To go from $n-1$ to n, we use similar thing in (2.18)

$$f^{(n)}(z_0) = \frac{d}{dz_0}f^{(n-1)} = \frac{(n-1)!}{2\pi i}\int_C \frac{d}{dz_0}\frac{f(z)}{(z-z_0)^n}dz = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

the passing of derivative will be justified later. Because we have been able to calculate $f^{(n)}$ at each point inside C, it follows that $f^{(n-1)}$ is analytic in this open disc. However each point in D is inside some circle C in D (because D is open), so $f^{(n-1)}$ is analytic in D. □

There are three well-known consequences of Cauchy's Derivative Formulas: theorem 81, theorem 84 & Taylor's formula theorem 85.

Theorem 81. *(Morera) Let f be continuous in a convex region D. If*

$$\int_\Delta f = 0$$

for each triangle Δ in D, then f is analytic in D.

Proof. The hypothesis implies that $f = F'$ for some F, by anti derivative lemma 67. F is analytic, so f is analytic by theorem 80. $\square$

Corollary 82. *For f continuous in a convex region D. f is analytic in D iff*

$$\int_{\Delta} f = 0$$

for each triangle Δ in D.

Proof. In one direction this follows from Cauchy's integral theorem, and in the other direction it is Morera's theorem. $\square$

Corollary 83. *For f continuous in a convex region D. f is analytic in D iff*

$$\int_{\gamma} f = 0$$

for each closed path γ in D.

Theorem 84. *(Cauchy's Derivative Estimates) Let f be analytic in a convex region D, let C be a circle in D of radius r centered at z_0. Then*

$$\frac{\left|f^{(n)}(z_0)\right|}{n!} \leq \frac{B}{r^n}$$

for $n = 0, 1, 2, ...$ where B is the maximum of $|f(z)|$ for z on C.

Proof. By the Cauchy derivative formula

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Since $|z - z_0| = r$ for z on C, the ML inequality gives

$$\frac{\left|f^{(n)}(z_0)\right|}{n!} \leq \frac{1}{2\pi i} \frac{B}{r^{n+1}} 2\pi r = \frac{B}{r^n}$$

$\square$

2.5 Cauchy-Taylor Formulas

Lecture 8 (2/17/14)

Theorem 85. *(Cauchy-Taylor Formulas) Let f be analytic in a convex region D and let C be a circle in D centered at z_0. Then for all z inside C*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

It follows that the Taylor series for f about z_0 converges to f in each open disc in D centered at z_0.

Proof. We deal with the case $z_0 = 0$. Let C be a counterclockwise circle in D centered at 0. For z in C the Cauchy integral formula (with z_0 and z replaced by z and ζ) reads

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \tag{2.19}$$

Using the identity $1 - w^N = (1 - w)(1 + w + ... + w^{N-1})$ in the form

$$\frac{1}{1-w} = \sum_{n=0}^{N-1} w^n + \frac{w^N}{1-w} \text{ for } w \neq 1$$

with $w = z/\zeta$ and dividing by ζ gives

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\frac{1}{1 - z/\zeta} = \sum_{n=0}^{N} \frac{z^n}{\zeta^{n+1}} + (\frac{z}{\zeta})^N \frac{1}{\zeta - z} \tag{2.20}$$

for ζ on C and z inside C.

Substituting (2.20) into (2.19) gives

$$f(z) = \sum_{n=0}^{N-1} \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta z^n + r_N = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + r_N$$

by the Cauchy derivative formulas for $n = 0, 1, ..., N - 1$ and with

$$r_N = \frac{1}{2\pi i}\int_C (\frac{z}{\zeta})^N \frac{f(\zeta)}{\zeta - z} d\zeta$$

Let B be the maximum of $|f(\zeta)|$ for ζ on C. Using also $|\zeta| = r$ (the radius of C) and $|z| < r$ for z inside C, the ML inequality gives

$$|r_N| \leq \frac{Br}{r - |z|}(\frac{|z|}{r})^N \to 0 \text{ for } N \to \infty$$

thus

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \text{ for } z \text{ inside } C$$

For z in any open disc in D centered at z_0 there is a circle C in D centered at z_0 with z inside C to which the series formula above applies.

Finally the general case, with z_0 not necessarily equal to 0: Put $g(z) = f(z + z_0)$, so that g is analytic in the convex region $D - z_0$ gotten by translating D by $-z_0$, we have $g^{(n)}(0) = f^{(n)}(z_0)$, so for z inside any open disc in D centered at z_0, i.e. $z - z_0$ inside the translated disc in $D - z_0$ centered at 0,

$$f(z) = g(z - z_0) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} (z - z)_0^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

□

Considering the Cauchy-Taylor Theorem for $\mathbb{R}$, we may wonder if

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n \text{ for some } x \neq 0$$

could be proved for each real valued function h of a real variable, assumed to be defined and with derivatives of all orders on all of $\mathbb{R}$.The answer is surprisingly NO. This is the goal of the next four exercises.

Exercise 86. Using the fact that

$$e^x = \sum_{=0}^{\infty} \frac{x^n}{n!}$$

prove that

$$x^n \leq n! e^x$$

for $x \geq 0$ and each non negative integer n.

Exercise 87. Replacing n by $n + 1$ in the inequality above, prove that

$$x^n e^{-x} \to 0$$

for $x \to \infty$, for each non negative integer n.

Exercise 88. Replacing x by $1/x^2$ in the limit above to prove that

$$\frac{e^{-1/x^2}}{x^{2n}} \to 0$$

for $x \to 0$, $x \neq 0$ for each non negative integer n.

Exercise 89. Put

$$h(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(1) prove that h is continuous on $\mathbb{R}$, using the case $n = 0$ in exercise 88 to deal with $x = 0$.

(2) prove by induction that

$$h^{(n)}(x) = p_n(\frac{1}{x})e^{-1/x^2}$$

for $x \neq 0$ for each non negative integer n, for some polynomials p_n and calculate $p_1(u)$, $p_2(u)$, $p_3(u)$, $p_4(u)$.

(3) Conclude from (2) and exercise 88 that h has derivative of all orders at each point of $\mathbb{R}$ and

$$h^{(n)}(0) = 0$$

for each non negative integer n, so

$$\sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n = 0$$

hence it is not equal to $h(x)$.

Next a generalization of Liouville's theorem, proved by combining the Cauchy-Taylor formula with the Cauchy derivative estimates

Theorem 90. *(generalized Liouville) Let f be an entire function, and let $N \geq 0$ be an integer, then*

$$f \text{ is a polynomilal of degree } \leq N \tag{2.21}$$

iff there is a constant $A > 0$ s.t.

$$|f(z)| \leq A\,|z|^N \text{ for } |z| \geq 1 \tag{2.22}$$

Proof. Suppose first that f is entire and satisfies (2.21). Then

$$f(z) = c_0 z^N + ... + c_N$$

By the triangle inequality

$$|f(z)| \le |c_0|\,|z|^N + ... + |c_N|$$

For $|z| \ge 1$ we have $|z|^n \le |z|^N$ for $n = 0, ..., N$ so f satisfies (2.22) with

$$A = |c_0| + ... + |c_N|$$

Suppose f is entire and satisfies (2.22). Applying the Cauchy derivative estimates for $r \ge 1$ with $z_0 = 0$ gives for each integer $n \ge 0$

$$\frac{\left|f^{(n)}(0)\right|}{n!} \le \frac{Ar^N}{r^n}$$

for each $n > N$, the upper bound $\to 0$ for $r \to \infty$ so

$$f^{(n)}(0) = 0 \text{ for } n > N$$

Thus the Cauchy-Taylor formula with $z_0 = 0$ reduces to

$$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n$$

a polynomial of degree $\le N$. □

Exercise 91. Apply Cauchy-Taylor to prove that for $|z| < 1$,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ... \tag{2.23}$$

Actually above is true for $|z| \le 1$ except for $z = -1$. To show that, we start with

$$\log(1+z) = \int_{[1,1+z]} \frac{dw}{w} = \int_{[0,z]} \frac{d\zeta}{1+\zeta} \tag{2.24}$$

for $1 + z$ in the slit plane. For $\zeta \ne 1$ and each integer $N \ge 1$,

$$\frac{1}{1+\zeta} = 1 - \zeta + \zeta^2 - \zeta^3 + ... \mp \zeta^{N-1} \pm \frac{\zeta^N}{1+\zeta} \tag{2.25}$$

substituting (2.25) into (2.24) gives

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - ... \mp \frac{z^N}{N} \pm \int_{[0,z]} \frac{\zeta^N}{1+\zeta} d\zeta \tag{2.26}$$

For $|z| \leq 1$ with $z \neq 1$ the ML inequality does not show that the integral in (2.26) $\to 0$ since for $|z| = 1$ the bound M doesn't $\to 0$. However the integral does $\to 0$ because putting $\zeta = tz$ with $t \in [0, 1]$ we get

$$\left| \int_{[0,z]} \frac{\zeta^N}{1+\zeta} d\zeta \right| = \left| \int_0^1 \frac{t^N z^N}{1+tz} z dt \right| \leq \frac{1}{m} \int_0^1 t^N dt \tag{2.27}$$

with m the minimum of $|1 + tz|$ for t in $[0, 1]$ and the integral on the right in (2.27) is $1/(N + 1) \to 0$ proving (2.23).

One application of above is that for $-\pi < \theta < \pi$

$$\log(2\cos\frac{\theta}{2}) = \cos\theta - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \frac{\cos 4\theta}{4} + ... \tag{2.28}$$

and

$$\frac{\theta}{2} = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + ... \tag{2.29}$$

To see these, for $|z| = 1$ with $z \neq -1$, put $\arg z = \theta$ with $-\pi < \theta < \pi$. For the isosceles triangle of vertics 0, 1 and $1 + z$

$$\arg(1 + z) = \phi = \frac{\theta}{2}$$

and

$$|1 + z| = 2\cos\phi = 2\cos\frac{\theta}{2}$$

so

$$\log(1 + z) = \log(2\cos\frac{\theta}{2}) + i\frac{\theta}{2}$$

Since $|z^n| = 1$ and $\arg(z^n) = n\theta$, equation (2.28), (2.29) express the equality of the real, imaginary parts of the two sides of (2.23).

Observe that for $\theta = \pm\pi$ the series in (2.28) diverges to $-\infty$ so (2.28) still holds provided we interpret $\log 0$ as $-\infty$. For $\theta = \pm\pi$ the terms of series (2.29) are all 0, the series converges to 0 so (2.29) is not true under any interpretation. However there is a rationale for what happens with (2.29) at $\pm\pi$ about which later we study Fourier series.

3 Little Analysis

3.1 Uniform Convergence

Lecture 9 (2/24/14)

First we adapt the definition of convergence to complex sequences.

Definition 92. Let s_n and s be complex numbers, $n \in \mathbb{N}$ then s_n converges to s written $s_n \to s$ if for each real $\epsilon > 0$ there is a positive integer N s.t. $|s_n - s| < \epsilon$ for all $n > N$. s is the limit of the sequence.

Definition 93. Let f_n and f be complex functions defined on a set S, $n \in \mathbb{N}$

(1) $f_n \to f$ pointwise on S means $f_n(z) \to f(z)$ for each $z \in S$ i.e.

$$\forall\, z \in S, \forall\, \epsilon > 0, \exists\, N_{\epsilon,z} \in \mathbb{N} : |f_n(z) - f(z)| < \epsilon, \forall\, n > N$$

(2) $f_n \to f$ uniformly on S means

$$\forall\, \epsilon > 0, \exists\, N_\epsilon \in \mathbb{N} : |f_n(z) - f(z)| < \epsilon, \forall\, n > N, \forall\, z \in S$$

Example 94. Let $f_n = x^n$ for $0 \leq x \leq 1$. Then $f_n \to f$ pointwise on $[0, 1]$ with

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

Recall the definition of continuity

Definition 95. Let S be a subset of $\mathbb{C}$, f a complex valued function defined on S. Let z_0 be a point in S. We say f is continuous at z_0 if

$$\forall\, \epsilon > 0, \exists\, \delta > 0 : |f(z) - f(z_0)| < \epsilon, \forall\, z \in S \text{ with } |z - z_0| < \delta$$

we say f is continuous on S if f is continuous at each point of S.

Example 96. Continue from example 94, each of the f_n is continuous on $[0, 1]$ but the pointwise limit f is not. Thus

$$\text{if } f_n \to f \text{ pointwise on } S \text{and each } f_n \text{ is continuous on } S$$

$$\text{then } f \text{ is continuous on } S$$

is in general false. Uniform convergence is better than pointwise convergence.

Proposition 97. *If f_n and f are defined on $S \subset \mathbb{C}$ if $f_n \to f$ uniformly on S, and if each f_n is continuous on S, then f is continuous on S. That is*

A uniform limit of continuous functions is continuous.

Proof. Let $z_0 \in S$. To show f is continuous at z_0. For each $n \in \mathbb{N}$ and $z \in S$

$$f(z) - f(z_0) = f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)$$

by triangle inequality

$$|f(z) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)|$$

Since $f_n \to f$ uniformly on S, we have the first and third absolute differences $< \epsilon$ and f_n is continuous so the second absolute differences is too $< \epsilon$. □

Proposition 98. *If f_n and f are complex functions defined on $[a, b]$ in $\mathbb{R}$ with each f_n continuous and $f_n \to f$ uniformly on $[a, b]$ then f is continuous on $[a, b]$ and*

$$\int_a^b f_n \to \int_a^b f$$

That is

The integral of a uniform limit is the limit of the integrals

Proof. The continuity of f is a special cases $S = [a, b]$ of the previous proposition. For each $n \in \mathbb{N}$

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq M_n(b - a)$$

with M_n equal to the maximum of $|f_n - f|$ on $[a, b]$. Since $f_n \to f$ uniformly on $[a, b]$ it follows that $M_n \to 0$. □

Corollary 99. *If f_n are complex functions defined on a region D in $\mathbb{C}$ with each f_n continuous and $f_n \to f$ uniformly on D then f is continuous on D and*

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz \textit{ for each path } \gamma \textit{ in } D$$

Proof. The continuity of f on D follows from previous proposition. If γ is smooth, i.e. γ complex function on $[a, b]$ with values in D and continuous derivative, then

$$\left|\int_\gamma f_n - \int_\gamma f\right| = \left|\int_a^b (f_n(\gamma(t)) - f(\gamma(t)))\, \gamma'(t)dt\right| \leq M_n L$$

where L is the length of γ and M_n is the maximum of $|f_n - f|$ on γ, since $f_n \to f$ uniformly on D, it follows that $M_n \to 0$ giving that we want.

Then the case that γ is piecewise smooth is easily followed $\square$

Proposition 100. *Let f_n and f be complex functions on a region D in $\mathbb{C}$ with each f_n analytic in D and $f_n \to f$ uniformly on E for each closed disc E in D. Then f is also analytic in D. That is*

A uniform limit of analytic funtions is analytic

Proof. Since each z in D is inside some closed disc E in D, it suffices to show that f is analytic inside each closed disc E in D. For each triangle Δ inside E Cauchy integral theorem gives

$$\int_\Delta f_n = 0 \text{ for each } n \in \mathbb{N} \tag{3.1}$$

Since $f_n \to f$ uniformly on E previous proposition shows that f is continuous on E and

$$\int_\Delta f_n \to \int_\Delta f \tag{3.2}$$

Combining (3.1) & (3.2) gives

$$\int_\Delta f = 0 \text{ for each triangle } \Delta \text{ inside } E \tag{3.3}$$

By Morera's theorem it follows (3.3) that f is analytic inside E. $\square$

Proposition 101. *Same hypotheses as in proposition 100.*

$$f_n' \to f' \textit{ uniformly on each closed disc } E \textit{ in } D$$

That is

The derivative of a uniform limit is the uniform limit of the derivatives.

Proof. For each closed disc E in D there is (by analysis) a slightly larger concentric closed disc F in D, let C be the counterclockwise boundary circle of F.

By Cauchy's derivative formula for f_n' and f',

$$f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} d\zeta$$

for z inside C. Let r and s be the radii of E and F. Then $|\zeta - z| \geq s - r$ for ζ on C and z on E, so

$$\left|f_n'(z) - f'(z)\right| \leq \frac{M_n s}{(s-r)^2} \text{ for } z \text{ on } E$$

with M_n the maximum of $|f_n - f|$ on C. Since $M_n \to 0$ this gives $f_n' \to f'$ uniformly on E. □

3.2 Cauchy Criterion

Lecture 10 (2/26/14)

In this section we discuss a subtle "completeness" property of $\mathbb{R}$ and $\mathbb{C}$, which was recognized explicitly only in the late 19th century. It is expressed by the Axiom below. In treatments where $\mathbb{R}$ is "constructed" (see Intro to Modern Analysis I notes), this Axiom is a theorem.

Definition 102. Let S be a subset of the set of real numbers, i.e. $S \subset \mathbb{R}$. A real number B is an upper bound (u.b.) for S if $x \leq B$ for each $x \in S$. Lower bound (l.b.) is defined similarly.

Remark 103. (1) It is not required of an u.b. for S that it belongs to S.

(2) some subsets S of $\mathbb{R}$ have no u.b. e.g. $\mathbb{N}$ and $\mathbb{R}$ itself have none.

(3) The empty set $\varnothing$ has o elements has every $B \in \mathbb{R}$ as an u.b. (Does every cat which plays the violin bark rather than meow?)

Definition 104. Let $S \subset \mathbb{R}$, a least upper bound (l.u.b.) for S is an u.b. B^* satisfying $B^* \leq B$ for all u.b. B. Greatest lower bound (g.l.b.) is defined similarly.

Exercise 105. Prove if $S \subset \mathbb{R}$ has a least upper bound, then it is unique.

Axiom 106. *Each nonempty $S \subset \mathbb{R}$ which has an u.b. has a unique l.u.b.*

Proposition 107. *If s_n and B are real numbers with*

$$s_1 \leq s_2 \leq \ldots \leq B$$

then $s_n \to s$ for some real s (with $s \leq B$). That is

Each bounded increasing sequence of real numbers converges.

Proof. The set S of all s_n has B as u.b. let s be the l.u.b for S, guaranteed by the Axiom. Since s is an u.b for S, we have $s_n \leq s$ for all n. For each $\epsilon > 0$, the number $s - \epsilon$ which is $< s$, is not an u.b. for S, because s was least so there is an N in $\mathbb{N}$ s.t. $s_N > s - \epsilon$. Since the sequence is increasing, it follows that $s_n > s - \epsilon$ for all $n > N$. Thus for each $\epsilon > 0$ there is an N in $\mathbb{N}$ so that $|s_n - s| < \epsilon$ for all $n > N$, showing $s_n \to s$. □

Definition 108. Let $s_1, s_2, ...$ be any sequence. For each strictly increasing sequence of positive integers $n_1 < n_2 < ...$, we get a subsequence $s_{n_1}, s_{n_2}, ...$ of the sequence $s_1, s_2, ...$

Proposition 109. *Let $s_n \in [A, B]$ then for some $n_1 < n_2 < n_3 < ...$ and some $s \in [A, B]$ we have $s_{n_k} \to s$. That is*

Each bounded sequence of real numbers has a convergent subsequence.

Proof. $[A_1, B_1] = [A, B]$ and choose $n_1 \in \mathbb{N}$ arbitrarily. Then $s_{n_1} \in [A_1, B_1]$. Let M_1 be the midpoint of segment $[A_1, B_1]$. Either the left half $[A_1, M_1]$ or the right half $[M_1, B_1]$ of $[A_1, B_1]$ contains s_n for infinitely many n or perhaps both halves do. Choose such a half call it $[A_2, B_2]$ and choose any $n_2 > n_1$with $s_{n_2} \in [A_2, B_2]$.

Repeat this process again and again, getting a sequence of intervals

$$[A_1, B_1], [A_2, B_2], ...$$

each after the first either the left half or the right half of the proceeding one, and a subsequence

$$s_{n_1}, s_{n_2}, ... \text{ of } s_1, s_2, s_3, ... \text{ with } s_{k_k} \in [A_k, B_k] \text{ for all } k \in \mathbb{N}$$

Since $A_1 \leq A_2 \leq ... \leq B$, 85 gives $A_k \to s$ for some $s \in [A, B]$. Since $B_k - A_k = (B - A)/2^{k-1}$ we have $B_k - A_k \to 0$ so $B_k \to s$ also. Since $s_{n_k} \in [A_k, B_k]$ it follows that $s_{n_k} \to s$ too. □

Proposition 110. *Let $s_n \in \mathbb{C}$ satisfy $|s_n| \leq B$ for some $B \geq 0$, then some subsequence $s_{n_k} \to s$ for some $s \in \mathbb{C}$ with $|s| \in B$. That is*

Each bounded sequence of complex numbers has a convergent subsequence.

Proof. Let $s_n = x_n + iy_n$. Since each $|x_n| \leq B$, some subsequence of the x_n converges to some $x \in \mathbb{R}$. After discarding the unchosen indices and reindexing the remaining s_n by $1, 2, 3, ...$ we may assume that $x_n \to x$.

Working with this new sequence of s_n, we may choose by a similar argument a subsequence for which also $y_{n_k} \to y$ for some $y \in \mathbb{R}$ giving $s_{n_k} \to s = x + iy$. Clearly $|s| \leq B$. $\square$

Definition 111. A subset F of $\mathbb{C}$ is closed if F contains the limit of each convergent sequence in F.

Exercise 112. Let F be a subset of C and let D be the complement of F in $\mathbb{C}$, i.e. D is the set of all points in $\mathbb{C}$ which are not in F. Prove that F is closed iff D is open.

Exercise 113. Let f be a complex function defined on a subset D of $\mathbb{C}$ and let p be any point in D. Prove that f is continuous at p iff $f(z_n) \to f(p)$ for each sequence of points z_n in D with $z_n \to p$.

Proposition 114. *Let F be a nonempty closed and bounded subset of $\mathbb{C}$. Then each real continuous function ψ on F has a maximum, i.e. there is a point p in F with $\psi(z) \leq \psi(p)$ for all z in F.*

Proof. Let S be the set of all values $\psi(z)$ of ψ for all z in F. We show first that S has an u.b. If not then for each n in $\mathbb{N}$ there is a point z_n in F with $\psi(z_n) > n$. Since F is bounded, z_n is bounded , so some subsequence $z_{n_k} \to p$ for some p in $\mathbb{C}$ by the previous proposition. Since F is closed, p is in F. Since ψ is continuous at p. it follows by exercise 113 that $\psi(z_{n_k}) \to \psi(p)$, contrary to $\psi(z_{n_k}) > n_k$ and $n_k \to \infty$.

Because F is not empty, S is not empty, so since S has an u.b. S has a l.u.b B^*. Since B^* is the l.u.b for S, it follows that for each $n \in \mathbb{N}$ there is a point z_n in F with

$$\psi(z_n) > B^* - \frac{1}{n}$$

Since F is bounded and closed, some subsequence z_{n_k} $\to$some p in F. Since ψ is continuous at p, exercise 113 gives $\psi(z_{n_k}) \to \psi(p)$, so $\psi(p) \geq B^*$. Since B^* is an u.b. for the values of ψ, we conclude that $\psi(p) = B^*$, so $\psi(p)$ is the maximum value of ψ. $\square$

Definition 115. A complex sequence $s_1, s_2, ...$ satisfies Cauchy's criterion if for each $\epsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t.

$$|s_m - s_n| < \epsilon, \ \forall\, m, n > N$$

Cauchy's criterion is easier to verify than convergence since Cauchy's criterion doesn't involve s. Nevertheless

Proposition 116. *A complex sequence satisfies Cauchy's criterion iff it converges to some $s \in \mathbb{C}$.*

Proof. First assume $s_n \to s$. To show Cauchy's criterion is satisfied, since

$$|s_m - s_n| \leq |s_m - s| + |s_n - s|$$

and each term on the right is $< \epsilon/2$ for $m, n > N_{\epsilon/2}$ we get

$$|s_m - s_n| < \epsilon \text{ for } m, n > N_{\epsilon/2}$$

showing that Cauchy's criterion is satisfied.

Second assume only that Cauchy's criterion is satisfied. To show $s_n \to$some s in $\mathbb{C}$. We show first that Cauchy's criterion implies the sequence is bounded: Cauchy's criterion with $\epsilon = 1$ gives

$$|s_m - s_n| < 1 \text{ for all } m, n > N = N_1$$

it follows that $|s_n| \leq B$ for all n, with

$$B = \max\{|s_1|, ..., |s_N|, |s_{N+1}| + 1\}$$

By theorem 110 some subsequence $s_{n_k} \to s$ for some $s \in \mathbb{C}$. Using Cauchy's criterion again we show that $s_n \to s$. In fact for $n, k \in \mathbb{N}$

$$|s_n - s| \leq |s_n - s_{n_k}| + |s_{n_k} - s|$$

For n, n_k sufficiently large, the first term on the right is $< \epsilon/2$, and for k sufficiently large the second term on the right is also $< \epsilon/2$. Since we are free to choose k as large as we want, it follows that $|s_n - s| < \epsilon$ for sufficiently large n. Thus $s_n \to s$. $\square$

Definition 117. An infinite series $\sum a_n(z)$ whose terms are complex functions $a_n = a_n(z)$, all defined on a subset E of $\mathbb{C}$, converges uniformly on E if the sequence of partial sums $s_n = a_1 + ... + a_n$ converges uniformly on E.

Proposition 118. *(Weierstrass Test) If $|a_n(z)| \leq b_n$ for z in E and n in $\mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n(z)$ converges uniformly on E.*

Proof. Let $s_n = a_1 + ... + a_n$ and $t_n = b_1 + ... + b_n$. For $m > n$

$$s_m - s_n = a_{n+1} + ... + a_m$$

so for all z in E

$$|s_m(z) - s_n(z)| \leq |a_{n+1}(z)| + ... + |a_m(z)| \leq b_{n+1} + ... + b_m = t_m - t_n$$

Since $\sum b_n$ converges, proposition 116 implies that $t_m - t_n < \epsilon$ for $m, n > N_\epsilon$, with the same bound for $|s_m(z) - s_n(z)|$. Thus for each z in E the sequence of $s_n(z)$ satisfies Cauchy's criterion and therefore converges, by proposition 116. From $s_m(z) \to s(z)$ we get $|s_n(z) - s(z)| \leq \epsilon$ for $n > N_\epsilon$ and all z in E, giving uniform convergence. □

3.3 Exponential, Hyperbolic, Trigonometric Functions

Lecture 11 (3/3/14)

Now we define by power series and get the main properties of five closely related entire functions: exp, cosh, sinh, cos & sin.

Definition 119. For each $z \in \mathbb{C}$

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \ \& \ \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \ \& \ \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

The five series reduce on the real line, i.e. for $z \in \mathbb{R}$, to the series derived in Calculus for the real functions on $\mathbb{R}$.

Proposition 120. *Each of the above five series converges for all $z \in \mathbb{C}$, uniformly in each closed disc $|z| \leq r$ to an entire function.*

Proof. Here we prove the exponential series. For each $r > 0$, the series for e^r converges by the ratio test, since

$$\frac{\frac{r^{n+1}}{(n+1)!}}{\frac{r^n}{n!}} = \frac{r}{n+1} \to 0 \text{ for } n \to \infty$$

since

$$\left|\frac{z^n}{n!}\right| \leq \frac{r^n}{n!} \text{ for } |z| \leq r$$

Weierstrass test with $b_n = r^n/n!$ shows tat the series for e^z converges uniformly for $|z| \le r$.

By proposition 100 the function e^z is analytic in each open disc $|z| < r$. Therefore e^z is an entire function. □

Proposition 121. *For all $z \in \mathbb{C}$*

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \& \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \& \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Proof. Replace z by $-z$ in the exponential series. Now both

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \exp -z = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!}$$

adding or subtracting gives the series for $2\cosh z$ or $2\sinh z$.

Similarly replacing z by iz in the exponential series and adding or subtracting gives $2\cos z$ or $2i\sin z$. □

Corollary 122. *(Euler) For all real θ*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

i.e.

$$\left| e^{i\theta} \right| = 1 \text{ and } \arg e^{i\theta} = \theta$$

Corollary 123.

$$e^{i\pi} = 1$$

What are some applications of these things? Corollary 122 may be applied to give an improved version of the story about Euler and Diderot in the Court of Catherine the Great–The proof of existence of God.

Lemma 124. *(Binomial) For all integers $n \ge 0$ and all $a, b \in \mathbb{C}$. l, m are nonnegative integers.*

$$\frac{(a+b)^n}{n!} = \sum_{l+m=n} \frac{a^l}{l!} \frac{b^m}{m!}$$

Proposition 125. *For all $a, b \in \mathbb{C}$,*

$$e^{a+b} = e^a e^b$$

Proof. We cannot say "rule of exponents", since that rule was proved in Calculus only works for real a, b. We use Binomial lemma instead

$$\sum_{l \le N} \frac{a^l}{l!} \sum_{m \le N} \frac{b^m}{m!} = \sum_{l,m \le N} \frac{a^l}{l!} \frac{b^m}{m!} = \sum_{n=0}^{2N} \sum_{\substack{l+m=n \\ l,m \le N}} \frac{a^l}{l!} \frac{b^m}{m!}$$

Bounding the inner sums with $n > N$ crudely and using the binomial lemma for $0 \le n \le 2N$, we get

$$\left| \sum_{l \le N} \frac{a^l}{l!} \sum_{m \le N} \frac{b^m}{m!} - \sum_{n \le N} \frac{(a+b)^n}{n!} \right| \le \sum_{n=N+1}^{\infty} \frac{(|a|+|b|)^n}{n!}$$

For $N \to \infty$, the sums on the left $\to e^a$, e^b, e^{a+b} while the right series $\to 0$. $\square$

Corollary 126. *For all $z = x + iy$ with $x, y \in \mathbb{R}$*

$$e^z = e^x \cos y + i e^x \sin y$$

and

$$|e^z| = e^x \text{ and } \arg e^z = y$$

Proof. These follow from $e^z = e^x e^{iy}$ and corollary 122 with $\theta = y$. $\square$

Corollary 127. *For all $a, b \in \mathbb{C}$*

$$\begin{aligned} \cosh(a+b) &= \cosh a \cosh b + \sinh a \sinh b \\ \sinh(a+b) &= \sinh a \cosh b + \cosh a \sinh b \end{aligned}$$

Corollary 128. *For all $z \in \mathbb{C}$*

$$\cosh^2 z - \sinh^2 z = 1 \ \& \ \cos^2 z + \sin^2 z = 1$$

Corollary 129. *For all $z = x + iy$, $x, y \in \mathbb{R}$*

$$\begin{aligned} \cosh z &= \cosh x \cos y + i \sinh x \sin y \\ \sinh z &= \sinh x \cos y + i \cosh x \sin y \\ |\cosh z|^2 &= \sinh^2 x + \cos^2 y \\ |\sinh z|^2 &= \sinh^2 x + \sin^2 y \end{aligned} \tag{3.4}$$

Exercise 130. For each nonconstant analytic function f, the lines or curve on which $f(z)$ is real, i.e. $\text{Im} f(z) = 0$ separate the regions where $\text{Im} f(z) > 0$ from those where $\text{Im} f(z) < 0$ and the lines or curves on which $|f(z)| = 1$, separate the regions where $|f(z)| > 1$ from those where $|f(z)| < 1$. Do that for

$$f(z) = z^2,\ f(z) = e^z,\ f(z) = \cosh z$$

4 Residue Theorem

4.1 Zeros

Recall in exercise 89 we had a real function h defined on $\mathbb{R}$, positive except at 0 with derivatives of all orders at every pint of $\mathbb{R}$ all of which are 0 at the point 0: $h^{(n)}(0) = 0$ for $n = 1, 2, ...$ Nothing like this can happen with analytic functions:

Proposition 131. *Let f be analytic in a region D in $\mathbb{C}$. Suppose there is a point p in D for which*

$$f^{(n)}(p) = 0 \textit{ for } n = 0, 1, 2, ... \tag{4.1}$$

Then $f = 0$ i.e. $f(z) = 0$ for all $z \in D$.

The proof uses a lemma:

Lemma 132. *(Disc-Chain) Let D be a region in $\mathbb{C}$, and p, q pints in D. Then there is a finite sequence of points $z_0, ..., z_k$ in D, with $z_0 = p$, $z_k = q$ and open discs $D_0, ..., D_{k-1}$ in D so that D_j is centered at z_j and contains z_{j+1} for $j = 0, ..., k-1$.*

The proof of the lemma uses the following two lemmas:

Lemma 133. *(Bowstring) For each path γ of length L going from p to q we have $|q - p| \leq L$. That is*

The bowstring is not longer than the bow.

Proof. If γ is a smooth path from p to q, then

$$q - p = \gamma(b) - \gamma(a) = \int_a^b \gamma'(t)dt$$

so

$$|q-p| = \left| \int_a^b \gamma'(t)dt \right| \leq \int_a^b |\gamma'(t)|\, dt = L$$

In the general case of piecewise smooth γ, we have $\gamma = \gamma_1 + ... + \gamma_s$ with each γ_r smooth of length L_r from z_{r-1} to z_r for $r = 1, ..., s-1$ and $z_0 = p$, $z_s = q$. it follows that

$$q - p = z_r - z_0 = (z_1 - z_0) + ... + (z_r - z_{r-1})$$

so

$$|q-p| \leq |z_1 - z_0| + ... + |z_r - z_{r-1}| \leq L_1 + ... + L_r = L$$

$\square$

Lecture 12 (3/5/14)

Lemma 134. *(Worm) For each path γ in a region D of $\mathbb{C}$, there is an $r > 0$ so that each open disc of radius r centered at a point on γ is contained in D.*

Proof. Since each smooth piece of γ is continuous, and the pieces attach together at the ends, we may suppose that γ is a continuous function on a closed interval $[a, b]$ of $\mathbb{R}$ with all values in D. If there is no $r > 0$ so that each open disc of radius r centered on γ is contained in D, then choosing $r = 1, 1/2, 1/3, ...$ we can find points $z_1, z_2, ...$ on γ and points $z_1', z_2', ...$ not in D s.t. $|z_n' - z_n| < 1/n$ for $n = 1, 2, 3, ...$

We have $z_n = \gamma(t_n)$ for a sequence $t_1, t_2, ...$ in $[a, b]$. Because it is a bounded sequence, there is a subsequence $t_{n_1}, t_{n_2}, ...$ which converges to a point $t^* \in [a, b]$. Since γ is continuous, $z_{n_k} \to z^* := \gamma(t^*)$. it follows that $z_{n_k}' \to z^*$ too.

Since z^* is on γ, which is in D, and D is open, the open disc $|z - z^*| < \epsilon$ is contained in D for some positive ϵ. Since $\left|z_{n_k}' - z^*\right| < \epsilon$ for sufficiently large k, it follows that $z_{n_k}' \in D$ for these k. Contradiction. $\square$

Now we prove Disc-Chain Lemma

Proof. Let p, q be points in a region D. Since D is connected, there is path γ in D from p to q. By Worm, there is an $r > 0$ s.t. each open disc of radius r centered at a pint of γ is contained in D. Choose points $z_0, ..., z_k$ on γ with $z_0 = p$, $z_k = q$, and s.t. that part of γ between z_j and z_{j+1} has length $< r$ for $j = 0, ..., k-1$, the disc D_j of radius r centered at z_j contains z_{j+1} and is contained in D. $\square$

We now prove proposition 131.

Proof. Let D_0 be any open disc in D centered at p, by the Cauchy-Taylor theorem

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n \text{ for all } z \in D_0 \tag{4.2}$$

It follows from (4.1), (4.2) that $f(z) = 0$ for all $z \in D_0$. If $D_0 = D$, we are done. If not, use Disc-chain lemma. (4.1) implies $f = 0$ in D_0, so $f^{(n)} = 0$ in D_0 for all integers $n \geq 0$. In particular, $f^{(n)}(z_1) = 0$ for all $n \geq 0$ so $f = 0$ in D_1 by the previous argument. After k such steps we reach $f = 0$ in D_{k-1}. In particular $f(q) = 0$. Thus (4.1) implies $f(q) = 0$ for all $q \in D$. □

Definition 135. Let $f \neq 0$ be analytic in a region D, and let z_0 be a point in D. If $f(z_0) = 0$ then z_0 is a zero of f.

Remark 136. Thus $f = 0$ has no zeros, and the zeros of a polynomial are its roots.

Proposition 137. *Let f be analytic in a region D and let z_0 be a zero of f in D. Then there is a positive integer m called the order of z_0 so that*

$$f(z_0) = 0, ..., f^{(m-1)}(z_0) = 0, \text{ but } f^{(m)}(z_0) \neq 0 \tag{4.3}$$

Proof. Since f has a zero, $f \neq 0$ so by proposition 131 $f^{(n)}(z_0) \neq 0$ for some positive integer n. Then (4.3) holds, with m the least such integer. □

Lemma 138. *Assume the convergence, for some $z_1 \neq z_0$, of the power series*

$$g = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{4.4}$$

Let D_1 be the open disc centered at z_0 with radius $r_1 = |z_1 - z_0|$, then

(1) the series (4.4) *also converges for all z in D.*

(2) it converges uniformly in each closed disc E centered at z_0 with radius $r < r_1$

(3) its sum g is analytic in D_1

(4) it is the Cauchy-Taylor series of g in D_1. That is

Each convergent power series is the Cauchy-Taylor series of its sum.

Proof. (1) follows from (2) since each z in D_1 is in some E.

Prove (2). For z in E

$$|c_n(z-z_0)^n| \leq b_n := |a_n|\,(r/r_1)^n, \text{ with } a_n = c_n(z_1-z_0)^n$$

since the series $\sum a_n$ converges, its sequence of terms $a_0, a_1, ...$ is bounded. Since the geometric series $\sum (r/r_1)^n$ converges, it follows by Weierstrass test that (4.4) converges uniformly in E.

(3) follows from (2) by proposition 100.

Prove (4). By proposition 101 it follows that for each integer $k \geq 0$

$$g^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)...(n-k+1)c_n(z-z_0)^{n-k} \text{ for all } z \in D_1$$

For $z = z_0$ this gives $g^{(k)}(z_0) = k!c_k$ i.e. (4) is the Cauchy-Taylor series of g. □

Proposition 139. *Let f be analytic in a region D, and let z_0 be a zero of f, of order m, then for each open disc D_0 in D centered at z_0*

$$f(z) = (z-z_0)^m f_0(z), \forall\, z \in D_0 \tag{4.5}$$

for some f_0 analytic in D_0 with $f_0(z_0) \neq 0$.

Proof. By the Cauchy-Taylor formula, for $z \in D_0$

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^m \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

giving (4.5) with f_0 the power series on the right, convergent in D_0 and therefore by lemma 138 it's analytic in D_0 and satisfying $f_0(z_0) \neq 0$. □

Theorem 140. *(isolation of zeros) Let f be analytic in a region D. Then each zero z_0 of f is isolated, i.e. there is a $\delta > 0$ so that f has no zeros in the punctured disc of radius δ centered at z_0.*

Proof. The function f_0 in proposition 139 is analytic in D_0. In particular f_0 is continuous at z_0. Since $f_0(z_0) \neq 0$ there is a $\delta > 0$ s.t.

$$|f_0(z) - f_0(z_0)| < |f_0(z_0)| \text{ for } |z-z_0| < \delta$$

In particular f_0 has no zero in $|z-z_0| < \delta$, so f has no zero in $0 < |z-z_0| < \delta$. □

Theorem 141. *(Identity theorem) Let f_1, f_2 be analytic in a region D. If $f_1 = f_2$ on some line segment $[p, q]$ in D with $p \neq q$ then $f_1 = f_2$ in D.*

Proof. Put $f = f_1 - f_2$. then f is analytic in D and $f = 0$ on $[p, q]$. In particular $f(p) = 0$. If $f \neq 0$, then p is a zero of f. Since $f = 0$ on $[p, q]$, the zero at p is not isolated, contrary to theorem 140. $\square$

Theorem 142. *(reflection principle) Let f be an entire function which is real on $\mathbb{R}$, i.e. $f(x)$ is real for all real x, then*

$$f(\bar{z}) = \overline{f(z)} \quad \forall\, z \in \mathbb{C}$$

Proof. Since f is entire, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \forall\, z \in \mathbb{C}$$

since f is real on $\mathbb{R}$, so are $f^{(n)}$ for all n (see exercise below). In particular all $f^{(n)}(0)$ are real, so

$$\overline{f(z)} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \overline{z^n} = f(\bar{z})$$

$\square$

Exercise 143. Show that if f is an entire function real on $\mathbb{R}$, then f' is also real on $\mathbb{R}$. Hint: show first that the derivative of an analytic function is its partial derivative with respect to x.

Exercise 144. Show for each root of a polynomial

multiplicity=order

Exercise 145. For e^z show it has no zeros, and it has period $2\pi i$, i.e.

$$e^{z+2\pi i} = e^z, \forall\, z \in \mathbb{C}$$

Exercise 146. Find all zeros in $\mathbb{C}$ of cosh,cos, sinh, sin and show that all these zeros have order 1. Also show that cos, sin are unbounded on the imaginary axis.

Exercise 147. Use the series definitions to find the derivatives of exp,cosh,cos, sinh, and sin.

4.2 Poles & Singularities

Lecture 13 (3/10/14)

Definition 148. If f is analytic in a punctured region $D - \{z_0\}$, then z_0 is an isolated singularity of f.

Isolated singularities come in three flavors:

Definition 149. Let f be defined and analytic in a punctured region $D - \{z_0\}$.

(i) If some choice of $f(z_0)$ makes f analytic in D, then z_0 is a removable singularity of f.

(ii) If $|f(z)| \to \infty$ for $z \to z_0$ $z \neq z_0$ then z_0 is a pole of f.

(iii) If z_0 is neither removable nor a pole, then z_0 is an essential singularity of f.

Example 150. The function f defined by

$$f(z) = \frac{\sin z}{z}$$

is defined and analytic in $\mathbb{C} - \{0\}$, for $z \neq 0$

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + ...$$

The power series on the right converges for all $z \in \mathbb{C}$ to an entire function with value 1 at 0. Therefore the isolated singularity $z_0 = 0$ of f is removed by choosing $f(0) = 1$.

Example 151. The function

$$f(z) = \frac{1}{z}$$

is defined and analytic in $\mathbb{C} - \{0\}$ with a pole at $z_0 = 0$.

Example 152. The function

$$f(z) = \exp\frac{1}{z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + ...$$

is defined and analytic in $\mathbb{C} - \{0\}$, since each term of the series is analytic there and the series converges uniformly for $|z| > s$ for each $s > 0$. In this example, f is real on $\mathbb{R} - \{0\}$. To see $z = 0$ is not a pole

$$f(x) \to \infty \text{ for } x \to 0^+$$

$$f(x) \to 0 \text{ for } x \to 0^-$$

One can also find the behavior of $f(z)$ along another punctured line and punctured circles.

From above examples we understand an essential singularity is a tiger, a removable singularity is a lamb. To get some poetic feeling of these, read William Blake "The Tyger", "The Lamb".

Exercise 153. Prove if f has a removable singularity at z_0, there is only one way for $f(z_0)$ which removes the singularity.

Proposition 154. *(on removable singularity) If f is analytic and bounded in $0 < |z - z_0| < r_1$ then z_0 is a removable singularity of f.*

Proof. Let γ be any counterclockwise circle centered at z_0 with radius $< r_1$. We will show first that for each $z \neq z_0$ inside γ and for sufficiently small counterclockwise circles C and C_0 centered at z and z_0,

$$\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} = \int_C \frac{f(\zeta)d\zeta}{\zeta - z} + \int_{C_0} \frac{f(\zeta)d\zeta}{\zeta - z}$$

this can be done by the same kind of deformation made in the proof of corollary 73. By Cauchy's integral formula

$$\int_C \frac{f(\zeta)d\zeta}{\zeta - z} = 2\pi i f(z)$$

and since $f(\zeta)/(\zeta - z)$ is bounded for ζ near to z_0, so

$$\int_{C_0} \frac{f(\zeta)d\zeta}{\zeta - z} \to 0$$

thus

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} \text{ for } z \text{ inside } \gamma \text{ with } z \neq z_0 \tag{4.6}$$

Of course (4.6) doesn't hold at $z = z_0$ since f is not defined at z_0. However if we now choose $f(z_0)$ to be the right side of (4.6) for $z = z_0$, then we have extended the definition of f to the unpunctured disc inside γ. Now the same differentiation argument which proves Cauchy's derivative formula shows that this extended f which satisfies (4.6) for all z in this disc inside γ is analytic in this disc. Thus we have removed the singularity of f at z_0. □

Proposition 155. *(on poles) If f is analytic in $D - \{z_0\}$ with a pole at z_0 then*

$$f(z) = \frac{f_0(z)}{(z - z_0)^m} \text{ for } z \in D - \{z_0\} \tag{4.7}$$

with f_0 analytic in D with $f_0(z_0) \neq 0$ and $m \in \mathbb{N}$ the order of the pole. Also

$$f(z) = P\left(\frac{1}{z - z_0}\right) + g(z) \text{ for } z \in D - \{z_0\} \tag{4.8}$$

with P a polynomial of degree m with constant term 0, and g analytic in D.

The first term on the right is the principal part of f at the pole. The order m and the three functions f_0, P, g are uniquely determined by f and z_0.

Proof. Since $|f(z)| \to \infty$ for $z \to z_0$, $z \neq z_0$ there is an $r > 0$ so that the punctured disc $|z - z_0| < r$ belongs to $D - \{z_0\}$ and contains no zeros of f.

Put $\psi = 1/f$ in this punctured disc. Thus ψ is also analytic and without zeros in the punctured disc, and $\psi(z) \to 0$ for $z \to z_0$, $z \neq z_0$. By proposition 154 ψ has removable singularity at z_0. The only choice of $\psi(z_0)$ is 0, so after removing the singularity, z_0 is zero of ψ. By proposition 139

$$\psi(z) = (z - z_0)^m \psi_0(z) \text{ in the disc } |z - z_0| < r$$

with ψ_0 analytic and without zeros in this disc.

Put $f_0 = 1/\psi_0$ in this disc. Then f_0 is analytic and without zeros in the disc, and (4.7) holds in the unpunctured disc. To get (4.7) throughout $D - z_0$ supplement the definition of f_0 by

$$f_0(z) = (z - z_0)^m f(z) \text{ for } z \in D - \{z_0\}$$

In the disc, f_0 is the sum of a Cauchy-Taylor series:

$$f_0(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \text{ with } c_0 \neq 0$$

Combined with this gives in the punctured disc

$$f(z) = \sum_{n=0}^{m-1} \frac{c_n}{(z - z_0)^{m-n}} + \sum_{n=m}^{\infty} c_n (z - z_0)^{n-m} = \mathrm{P}\left(\frac{1}{z - z_0}\right) + g(z)$$

with P a polynomial of degree m constant term 0 and g analytic in the disc. To get (4.8) throughout $D - \{z_0\}$ supplement the definition of g by

$$g(z) = f(z) - \mathrm{P}\left(\frac{1}{z - z_0}\right) \text{ for } z \in D - \{z_0\}$$

The uniqueness part of the proof is left as an exercise: show (1) there is only one pair m, f_0 with $m \in \mathbb{N}$ and f_0 analytic in $D - \{z_0\}$ with $f(z_0) \neq 0$ which satisfies (4.7); (2) there is only one pair P, g with P a polynomial with constant term 0, and g analytic in D which satisfies (4.8). □

The next proposition gives an "additive" representation of an arbitrary rational function analogous to the "multiplicative" factorization of an arbitrary nonzero polynomial. Analogous to the powers of linear factors corresponding to the distinct roots and the leading coefficient for a polynomial are the principal parts at the distinct poles, and a polynomial P_∞ for a rational function.

Proposition 156. *(partial fractions) For each rational function f with distinct poles $z_1, ..., z_r$*

$$f(z) = \sum_{j=1}^{r} P_j \frac{1}{z - z_j} + P_\infty(z) \text{ for } z \in \mathbb{C} - \{z_1, ..., z_r\}$$

where for $j = 1, ..., r$ the jth term in the sum is the principal part of f at the pole z_j and P_∞ is a polynomial.

Proof. The principal part at z_j is analytic in $\mathbb{C} - \{z_j\}$. By proposition 155

$$\mathrm{P}_\infty(z) = f(z) - \sum_{j=1}^{r} \mathrm{P}_j \frac{1}{z - z_j}$$

has removable singularities at $z_1, ..., z_r$. After removing them, P_∞ is an entire rational function, and is therefore a polynomial. □

Proposition 155 shows that the blowup of an analytic function at a pole no worse than the behavior of a polynomial for large z. The next theorem shows that the behavior of an analytic function at an essential singularity is essentially more chaotic:

Theorem 157. *(on essential singularities -Casorati & Weierstrass) If f is analytic in $D - \{z_0\}$ with an essential singularity at z_0, then*

$$\forall\, \epsilon > 0, \forall\, \delta > 0, \forall\, w \in \mathbb{C}, \exists\, z \in D - \{z_0\} : |z - z_0| < \delta \text{ \& } |f(z) - w| < \epsilon$$

that is

$f(z)$ is arbitrarily close to each $w \in \mathbb{C}$ at points arbitrarily close to z_0.

Proof. Suppose f doesn't satisfy the conclusion,

$$\exists\, \epsilon > 0, \exists\, \delta > 0, \exists\, w \in \mathbb{C} : \forall\, z \in D - \{z_0\}, \text{ if } |z - z_0| < \delta, \text{ then } |f(z) - w| \geq \epsilon$$

we may choose $\delta > 0$ s.t. the open disc of radius δ is contained in D, then

$$g = \frac{1}{f - w} \text{ satisfies } |g(z)| \leq \frac{1}{\epsilon} \text{ for } 0 < |z - z_0| < \delta$$

By proposition 154 g has a removable singularity at z_0. After removing the singularity,

(i) if $g(z_0) \neq 0$, then $f(z) - w$ is bounded for z near z_0, so z_0 is a removable singularity of f

(ii) if $g(z_0) = 0$, then $|f(z) - w| \to \infty$ for $z \to z_0$ so z_0 is a pole of f. Either way f doesn't satisfy the hypothesis. □

4.3 Cauchy's Residue Theorem

Lecture 14 (3/12/14)

Cauchy's Residue Theorem is an important generalization of Cauchy's integral theorem which ma be applied to evaluate certain definite integrals over $\mathbb{R}$ and to study the distribution of zeros and poles of meromorphic functions. This results was also used by Riemann to uncover the connection between set of all prime numbers and the set of all zeros of the Riemann zeta functions.

Definition 158. Let f be analytic in a punctured disc centered at z_0, with a pole of order m at z_0. Writing the principal part of f at the pole z_0 as

$$\mathrm{P}\frac{1}{z - z_0} = \frac{a_1}{z - z_0} + \ldots + \frac{a_m}{(z - z_0)^m} \tag{4.9}$$

the residue of f at z_0 is the coefficient a_1 :

$$Res(f, z_0) = a_1$$

Definition 159. Let D be a region and $z_1, ..., z_n$ be distinct points in D. A function f, analytic in $D - \{z_1, ..., z_n\}$, with a pole at each z_k is said to be meromorphic in D.

For example, each rational function is meromorphic in $\mathbb{C}$.

Definition 160. A closed path γ from p to q in $\mathbb{C}$ defined on $[a, b]$ is simple if

$$\gamma(t_1) = \gamma(t_2) \text{ with } a \leq t_1 < t_2 \leq b \text{ only for } t_1 = a,\ t_2 = b$$

Remark 161. As with circles and rectangles, each simple closed path γ is either clockwise or counterclockwise, and each such γ separates $\mathbb{C}$ into two regions, made up of the points inside γ and the points outside γ (Jordan Curve Theorem). Although precise definitions of these terms and a proof of Jordan Curve Theorem are difficult. We will use only circles, rectangles, halfmoons, and other easy γ, made up of a few line segments and arcs of circles, for which the meanings of these terms and the truth of Jordan Curve Theorem are clear.

Theorem 162. *If f is meromorphic in a convex region D, and γ is a counterclockwise simple closed curve in D not passing through any pole of f, then*

$$\int_\gamma f(z)dz = 2\pi i \sum Res(f, z_j)$$

The sum is over those poles z_j of f which are inside γ.

Proof. We will suppose γ is a rectangle, and leave the proof for a circle or halfmoon as exercises. Let $z_1, ..., z_n$ be the distinct poles of f in D, put

$$g(z) = f(z) - \sum_{j=1}^{n} \mathrm{P}\frac{1}{z - z_j} \text{ for } z \in D - \{z_1, ..., z_n\}$$

where the jth term in the sum is the principal part of f at z_j. Then g has a removable singularity at each z_j. We denote by the same symbol g the function gotten by removing all these removable singularities. By Cauchy's integral theorem, the integral of g around γ is 0, so

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_\gamma \mathrm{P}\frac{1}{z - z_j} dz$$

To finish, it suffices to show that for each z_0 in $D - \gamma$ and each P as in (4.9),

$$\int_\gamma \mathrm{P}\frac{1}{z - z_j}dz = 0 \text{ for } z_0 \text{ outside } \gamma, \text{ and } 2\pi i a_1 \text{ for } z_0 \text{ inside } \gamma$$

If z_0 is outside γ, then we draw a convex region containing γ but not z_0. In this region the integrand is analytic, so the integral is 0 . If z_0 is inside γ, we draw a counterclockwise circle C inside γ and centered at z_0, then

$$\int_\gamma \mathrm{P}\frac{1}{z - z_j}dz = \int_C \mathrm{P}\frac{1}{z - z_j}dz$$

It remains to show that for positive integers l,

$$\int_C \frac{dz}{(z - z_0)^l} = \begin{cases} 2\pi i & l = 1 \\ 0 & l > 1 \end{cases}$$

For $l = 1$, this is proposition 61. For $l > 1$, the integrand has an antiderivative in $\mathbb{C} - \{z_0\}$

$$\frac{1}{(z - z_0)^l} = \frac{1}{1 - l}\left(\frac{1}{(z - z_0)^{l-1}}\right)'$$

so the integral is 0 by proposition 62. □

Proposition 163. *(i) If f has a simple pole at z_0, i.e. a pole of order 1, then*

$$Res(f, z_0) = \lim(z - z_0)f(z) \textit{ for } z \to z_0, z \neq z_0$$

(ii) If $f = g/h$ with g, h analytic in a region containing z_0, $g(z_0) \neq 0$ and z_0 is a simple zero, i.e. a zero of order 1, of h, then

$$Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

(iii) If f has a pole of order m at z_0 then

$$Res(f, z_0) = \frac{1}{(m - 1)!}\left(\frac{d}{dz}\right)^{m-1}\{(z - z_0)^m f(z)\}$$

at z_0. Where $\{...\}$ means with removable singularity at z_0 removed.

Proof. (i) By hypothesis in some punctured disc centered at z_0

$$f(z) = \frac{a_1}{z - z_0} + h(z)$$

with h analytic in the unpunctured disc. Since h is bounded near z_0,

$$(z - z_0)f(z) = a_1 + (z - z_0)h(z) \to a_1 \text{ for } z \to z_0, z \neq z_0.$$

(ii) Apply (i) the hypothesis imply f has a simple pole at z_0, so for $z \neq z_0$

$$(z - z_0)f(z) = \frac{g(z)}{\frac{h(z)-h(z_0)}{z-z_0}} \to \frac{g(z_0)}{h'(z_0)} \text{ for } z \to z_0, z \neq z_0.$$

(iii) By (4.8) and (4.9)

$$f(z) = \frac{a_m}{(z - z_0)^m} + ... + \frac{a_1}{z - z_0} + c_0 + c_1(z - z_0) + ...$$

in some punctured disc centered at z_0, so

$$(z - z_0)^m f(z) = a_m + ... + a_1(z - z_0)^{m-1} + c_0(z - z_0)^m + ...$$

with the same formula for $\{(z - z_0)^m f(z)\}$ in the unpunctured disc. Thus a_1 is the $(m - 1)$st Cauchy-Taylor coefficient for this function at z_0. □

Example 164. Let us calculable some residue

$$f(z) = \frac{1}{(z^2 + 1)^m}, \ m \in \mathbb{N}$$

with poles of order m at $\pm i$. Let us do residue at $z_0 = i$

$$(z - i)^m f(z) = \frac{1}{(z + i)^m}$$

for $z \neq \pm i$. Using (iii) above

$$\begin{aligned} Res(f, i) &= \frac{1}{(m-1)!}\frac{(-m)(-m-1)...(-m-(m-2))}{(z+i)^{2m-1}} \\ &= \frac{(-1)^{m-1}}{(m-1)!}\frac{m(m+1)...(2m-2)}{(2i)^{2m-1}} = \frac{1}{2i}\frac{1}{4^{m-1}}\begin{pmatrix} 2m-2 \\ m-1 \end{pmatrix} \end{aligned}$$

Exercise 165. Prove if f has a zero of order m at z_0, then $1/f$ has a pole of order m at z_0.

Example 166. Let us use residue theorem to compute some integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi \tag{4.10}$$

This was done in Calculus,

$$\int_{-R}^{R} \frac{dx}{x^2+1} = \arctan R - \arctan -R = 2\arctan R \to 2\frac{\pi}{2} = \pi$$

Now we use residue and upper halfmoon path $\gamma_R = [-R, R] + \sigma_R$ with $\sigma_R(\theta) = Re^{\theta}$, $0 \leq \theta \leq \pi$ to

$$f(z) = \frac{1}{z^2+1}$$

thus

$$\int_{\gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\sigma_R} f(z)dz = 2\pi i Res(f, i) = \frac{2\pi i}{2i} = \pi$$

The ML inequality gives

$$\left| \int_{\sigma_R} f(z)dz \right| \leq \frac{1}{R^2-1}\pi R \to 0 \text{ as } R \to \infty$$

Remark 167. The pole at i "explains" the evaluation (4.10) without using an antiderivative. The pole at i is also behind the fact that for real x

$$1 - x^2 + x^4 - ... = \frac{1}{x^2+1}$$

only works for $-1 < x < 1$ even though the function on the right has derivatives of all orders for all real x.

Proposition. *Let f be a rational function with no real poles, i.e. no poles on $\mathbb{R}$. If $f = g/h$ with g, h polynomials of degree k and l with $l \geq k+2$, then*

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum Res(f, z_j)$$

the sum is over the poles of f in the upper half plane.

Proof. As in the proceeding example, we need to show

$$\left|\int_{\sigma_R} f(z)dz\right| \to 0$$

Let bz^k and cz^l be the highest degree terms in g, h. Then $g(z) \sim bz^m$ $|z| \to \infty$, meaning that $\frac{g(z)}{bz^m} \to 1$ as $|z| \to \infty$. Similarly $h(z) \sim cz^l$. It follows that $f(z) \sim az^d$ with $a = b/c$, $d = k - l$. The maximum M_R of $|f(z)|$ for $z \in \sigma_R$ satisfies $M_R \sim |a| R^d$ for $R \to \infty$. For sufficiently large R

$$\left|\int_{\sigma_R} f(z)dz\right| \leq M_R \pi R$$

since $d \leq -2$, so $\left|\int_{\sigma_R} f(z)dz\right| \to 0$. $\square$

Example 168. Thus example 164 gives

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^m} = \frac{\pi}{4^{m-1}} \binom{2m-2}{m-1}$$

a generalization of (4.10).

5 Applications of Residue Theorem

5.1 Fresnel's Integrals

Lecture 15 (3/24/14)

We begin by related evaluations of two integrals

$$\begin{aligned} \mathcal{G} &= \int_{-\infty}^{\infty} e^{-\pi x^2} dx \\ \mathcal{F} &= \int_{-\infty}^{\infty} e^{i\pi x^2} dx \end{aligned}$$

$\mathcal{G}$ relates to the Gaussian distribution and the Laplace central limit theorem in probability theory; $\mathcal{F}$ was introduced by Fresnel to study diffraction of light and used also by Riemann and Debye in the method of stationary phase.

Proposition 169.

$$\mathcal{G} = 1$$

Proof. $\mathcal{G}^2$ is

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\pi x^2}dx \int_{-\infty}^{\infty} e^{-\pi y^2}dy &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy \\ &= \int_0^\infty \int_0^{2\pi} e^{-\pi r^2} d\theta r dr = 1\end{aligned}$$

The conversion between $\mathbb{R}^2$ Cartesian to polar can be more precise. For $R > 0$

$$\left(\int_{-R}^{R} e^{-\pi x^2}dx\right)^2 = \int_{Square} e^{-\pi r^2}dA$$

$Square = [-R, R] \times [-R, R]$. It is bounded be two discs of radii R and $\sqrt{2}R$ thus

$$\underbrace{\int_{discR} e^{-\pi r^2}dA}_{1-e^{-\pi R^2}} < \int_{Square} e^{-\pi r^2}dA < \underbrace{\int_{disc\sqrt{2}R} e^{-\pi r^2}dA}_{1-e^{-2\pi R^2}}$$

and both $\to 1$, so $\int_{Square} e^{-\pi r^2}dA \to 1$.

Since $\mathcal{G} > 0$, $\mathcal{G} = 1$. ☐

Proposition 170.

$$\mathcal{F} = \zeta_8 := e^{i\pi/4}$$

Proof. By Cauchy's integral theorem,

$$\int_\gamma f(z)dz = 0$$

for every entire function f and every closed path γ in $\mathbb{C}$, for $f(z) = e^{-\pi z^2}$ and γ the counterclockwise eighthmoon path

$$\gamma = [0, R] + \delta_R + [\zeta_8 R, 0] \text{ with } \delta_R(\theta) = Re^{i\theta}, 0 \le \theta \le 2\pi/8.$$

We get

$$\int_0^R e^{-\pi x^2}dx + \int_{\delta_R} e^{-\pi z^2}dz - \int_0^R e^{-\pi(\zeta_8 r)^2}\zeta_8 dr = 0$$

The first integral $\to 1/2$ by proposition 169.

Since $(\zeta_8)^2 = i$, the last integral $\to \zeta_8\bar{\mathcal{F}}/2$, assuming it has a limit, i.e. assuming the integral defining $\mathcal{F}$ converges.

For $z = Re^{i\theta}$, we have $\mathrm{Re} z^2 = R^2 \cos 2\theta$, so

$$\left| \int_{\delta_R} e^{-\pi z^2} dz \right| \leq \int_0^{2\pi/8} e^{-\pi R^2 \cos 2\theta} R d\theta = \frac{R}{2} \int_0^{\pi/2} e^{-\pi R^2 \cos\beta} d\beta$$

Put $\beta = \pi/2 - \alpha$, use $\sin\alpha \geq 2\alpha/\pi$ for $0 \leq \alpha \leq \pi/2$

$$\left| \int_{\delta_R} e^{-\pi z^2} dz \right| \leq \frac{R}{2} \int_0^{\pi/2} e^{-\pi R^2 \sin\alpha} d\alpha \leq \frac{R}{2} \int_0^{\infty} e^{-2R^2 \alpha} d\alpha \leq \frac{1}{4R}$$

showing that the integral over δ_R goes to 0 for $R \to \infty$.

It follows that the integral defining $\mathcal{F}$ does converge, and

$$\mathcal{G} = \zeta_8 \bar{\mathcal{F}} \implies \bar{\mathcal{F}} = 1/\zeta_8 \implies \mathcal{F} = \zeta_8$$

□

5.2 Fourier Transforms

Definition 171. For each complex valued F continuous on $\mathbb{R}$,and satisfying

$$\int_{-\infty}^{\infty} |F(x)|\, dx < \infty$$

the Fourier transform $\hat{F}$ is defined on $\mathbb{R}$ by

$$\hat{F}(t) = \int_{-\infty}^{\infty} F(x) e^{2\pi i x t} dx \text{ for } t \in \mathbb{R}$$

Proposition 172. *If*

$$F(x) = \frac{1}{1+x^2},\ x \in \mathbb{R}$$

then

$$\hat{F}(t) = \pi e^{-2\pi|t|},\ t \in \mathbb{R}$$

Proof. For each $t \in \mathbb{R}$ the residue theorem applied to $e^{2\pi i z t}/(1+z^2)$ and the halfmoon path, with $R > 1$ gives

$$\int_{-R}^{R} \frac{e^{2\pi i x t}}{1+x^2} dx + \int_{\sigma_R} \frac{e^{2\pi i z t}}{1+z^2} dz = 2\pi i \frac{e^{-2\pi t}}{2i} = \pi e^{-2\pi t}$$

For $t \geq 0$ and $y \geq 0$ we have $\left|e^{2\pi izt}\right| = e^{-2\pi yt} \leq 1$, so for $t \geq 0$ and $R > 1$

$$\left|\int_{\sigma_R} \frac{e^{2\pi izt}}{1+z^2} dz\right| \leq \frac{\pi R}{R^2 - 1} \to 0$$

giving what we want for $t \geq 0$, for $t \leq 0$ either use the lower halfmoon path or use the following general fact: $\square$

Lemma 173.

$$\bar{\hat{F}}(-t) = \hat{\bar{F}}(t) \textit{ for all } t \in \mathbb{R}$$

Theorem 174. *If*

$$F(x) = e^{-\pi x^2} \quad x \in \mathbb{R}$$

then

$$\hat{F}(t) = e^{-\pi t^2}, \ t \in \mathbb{R}$$

Proof. To show:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi ixt} dx = e^{-\pi t^2}, \text{ for } t \in \mathbb{R} \tag{5.1}$$

Since $-x^2 + 2ixt = -(x - it)^2 - t^2$ it suffices by proposition 169 to show that

$$\int_{-\infty}^{\infty} e^{-\pi(x-it)^2} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} dx, \text{ for } t \in \mathbb{R} \tag{5.2}$$

to prove it, apply Cauchy's integral theorem to the function $\exp -\pi z^2$ and the rectangular path with corners $-R - it$, $R - it$, R, $-R$ to get

$$\int_{-\infty}^{\infty} \left(e^{-\pi(x-it)^2} - e^{-\pi x^2}\right) dx = i \int_0^t \left(e^{-\pi(R-iy)^2} - e^{-\pi(-R-iy)^2}\right) dy \tag{5.3}$$

For y between 0 and t,

$$\left|e^{-\pi(\pm R-iy)^2}\right| = e^{-\pi(R^2-y^2)} \leq e^{\pi(t^2-R^2)}$$

so for $R \to \infty$, the integral on the right in 5.3$\to 0$, giving 5.2. $\square$

Theorem 175. *If*

$$F(x) = \frac{1}{\cosh \pi x}, \ x \in \mathbb{R}$$

then

$$\hat{F}(t) = \frac{1}{\cosh \pi t}, \ t \in \mathbb{R}$$

Proof. For each $t \in \mathbb{R}$, the residue theorem applied to the function $e^{2\pi izt}/\cosh \pi z$ and the counterclockwise rectangular path γ_R with corners $\pm R$, $\pm R + i$ gives

$$\int_{\gamma_R} F(z)e^{2\pi izt}dt = 2\pi i Res\left(\frac{e^{2\pi izt}}{\cosh \pi x}, \frac{i}{2}\right) = 2\pi i \frac{e^{-\pi t}}{\pi \sinh(\pi i/2)} = 2e^{-\pi t}$$

Since the simple poles of the integrand are at $\pm i/2$, $\pm 3i/2$, $\pm 5i/2$,... with $i/2$ inside the rectangle, and all the others outside.

The contribution of the bottom side of the rectangle is

$$\int_{-R}^{R} \frac{e^{2\pi ixt}}{\cosh \pi x} dx$$

the integral over the top side of the rectangle is

$$-\int_{-R}^{R} \frac{e^{2\pi i(x+i)t}}{\cosh \pi(x+i)} dx = e^{-2\pi t} \int_{-R}^{R} \frac{e^{2\pi ixt}}{\cosh \pi x} dx$$

since $\cosh \pi(x+i) = -\cosh \pi x$.

The two vertical sides of the rectangle contribute

$$\pm i \int_{0}^{1} \frac{e^{2\pi i(\pm R+iy)t}}{\cosh \pi(\pm R + iy)} dy$$

The identity (3.4) implies

$$|\cosh \pi(\pm R + iy)| \geq \sinh \pi R$$

It follows that for each $t \in \mathbb{R}$, the absolute value of each vertical contribution is

$$\leq \frac{e^{2\pi|t|}}{\sinh \pi R} \to 0 \text{ for } R \to \infty$$

Combining all the contributions and letting $R \to \infty$ gives

$$(1 + e^{-2\pi t})\hat{F}(t) = 2e^{-\pi t} \implies \hat{F}(t) = \frac{1}{\cosh \pi t}$$

□

5.3 Cauchy Principal Values

5.4 Rouche's Theorem

6 More Analysis

6.1 Jensen's Theorem

6.2 Euler's Formulas

6.3 Stirling Approximation

7 Intro to Analytic Number Theory

7.1 Elliptic Functions

7.2 Riemann Zeta Functions

7.3 Gamma Functions

Lecture 16 (3/26/14)

Lecture 17 (4/7/14)

Lecture 18 (4/9/14)

Lecture 19 (4/14/14)

Lecture 20 (4/16/14)

Lecture 21 (4/21/14)

Lecture 22 (4/23/14)

Lecture 23 (4/28/14)

Lecture 24 (4/30/14)

Lecture 25 -Last Lec- (5/5/14)