# Intro Differentiable Manifolds

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This is an advanced undergraduate course. Offered in Spring 2015 at Columbia University. Course textbooks: Lee, *Introduction to Smooth Manifolds*; Bröcker, *Introduction to Differential Topology.* Office hours: MW 9:00-10:00pm.

## Contents

# 1 Smooth Manifolds & Smooth Maps

## 1.1 Manifolds

Lecture 1 (1/21/15)

**Definition 1.** $M$, a topological space, is a manifold of dimension $n$ if

- $M$ Hausdorff space, i.e. Given any 2 points $p_1 \neq p_2 \in M$, $\exists\, U_1, U_2$ open, $p_1 \in U_1$,$p_2 \in U_2$, $U_1 \cap U_2 = \varnothing$.
- second countable, i.e. countable basis for its topology. (A basis of topology of $X$ is a collection of open subsets $U_\alpha \subset X$ s.t. every open subset of $X$ is a union of some $U'_\alpha s$.)
- $M$ is locally homeomorphic to $\mathbb{R}^n$, i.e. $\forall\, p \in M\ \exists$ open neighborhood $p \in U \subset M$ and a homeomorphism (i.e. continuous map that has continuous inverse)

$$\psi : U \to \psi(U) \subset \mathbb{R}^n$$

  $\psi(U)$ is an open subset in $\mathbb{R}^n$ .

**Example 2.** (of manifold) $\mathbb{R}^n$ is second countable, because it has basis $B(p, r)$ open ball centered at $p \in \mathbb{Q}$ and radius $r \in \mathbb{Q}_{>0}$.

There are only two connected 1-manifold up to homeomorphism:

- $\mathbb{R} \cong (0, 1)$ not compact
- $S^1 \cong$ circle, that is compact

One can classify connected 2-manifolds

- not compact ones, e.g. $\mathbb{R}^2$, infinite cylinder, cylinder with points removal, etc, are very complicated. Hard to classify.
- compact ones upto homeomorphism are grouped to two kinds:
  - can be embedded in $\mathbb{R}^3$, and they are orientable
    $S^1$, 1-hole donut, 2-hole donut, 3-hole donut,... the hole is called genus.
  - cannot be embedded in $\mathbb{R}^3$, and they are non orientable
    $\mathbb{RP}^2$, Klein bottle, ... later we will learn how to construct them.

For connected 3-manifolds , there is a classification theorem, conjugated in 1970's and proved ten years ago.

In dimension 4, it is proven that there is no algorithms to decide that 2 arbitrary 4-manifold are homeomorphic or not, so no classification can be made. It relates to some undecidable problems in group theory.

**Example 3.** (of non manifolds due to non second countable) "the long line"

Consider

$$1, 2, 3, ..., w_1, w_1 + 1, ...., w_2, w_2 + 1, ...$$

where $w_\alpha$ are not ordinary numbers. But from set theory, they can be ordered too. Then

$$\underbrace{(0,1) \cup [1,2) \cup ...}_{(0,\infty)} \bigcup_{w_\alpha}^{w} [w_\alpha, w_{\alpha+1})$$

is not second countable.

The non second countable ones do not exist in nature. But non-Hausdorff ones are quite natural, so some people define manifold without Hausdorff.

**Example 4.** (of non manifolds due to non-Hausdorff) Consider

$$X = \mathbb{R}^n \cup 0'$$

$0'$ is another copy of 0. Put basis: usual open balls $B^n(p,r)$, which don't contain $0'$ and combining the sets

$$B^n(p,r) \cup \{0'\} \text{ for } r > |p|$$

therefore 0, $0'$ are not separatable.

### 1.2 Lie Groups

### 1.3 Coordinate Maps

## 2 Tangent & Cotangent Bundles

Lecture 2 (1/26/15)

Lecture 3 (1/28/15)

Lecture 4 (2/2/15)

Lecture 5 (2/4/15)

Lecture 6 (2/9/15)