Classical Fields & Waves

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This is a graduate course, offered in spring 2013 at Columbia University. Required Course textbooks: Landau, Lifshitz, *Physical Kinetic*. Landau, Lifshitz, *Fluid Mechanics.* 2 in class exams, weekly problem sets (no grade). Office Hours: MW 10-11.

Contents

Course Overview

Lecture 1 (1/23/13)

This course is the second semester of EM, and mainly focuses on applications

Things we will be studying.

(1) Scattering in Dielectric; Calculate index refraction (some other classical problems outlined in Jackson.)

(2) Waves in plasmas. We will do it in two stages. First in simple approach, then we will reproduce these results using Boltzmann equations for unequilibrium phenomenon, e.g. wave + dispersion.

(3) Ideal fluid

(4) Viscous fluids

(5) Turbulence. This is a complicated subject. In a sense that there are not many formulas to describe the problem. But it gives great applications e.g. heavy ion collision, field theory

(6) Diffusion in media

Reference for (1)-(2) is Landau "Physical Kinetic", reference for (3)-(6) is Landau "Fluid Mechanics".

1 Review of E+M

This will take about two lectures. For those just had EM last semester, will be nothing new but at least get the notations settle down.

1.1 Waves in Vacuum with Sources

Maxwell equations

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{1.1}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j} \tag{1.2}$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{1.3}$$

$$\nabla \cdot \vec{B} = 0 \tag{1.4}$$

$\rho(\vec{x},t)$ charge density, $\vec{j}(\vec{x},t)$ current density.

(1.1), (1.2) involve source, and (1.3), (1.4) do not.

Consistence condition

Continuity equation, it has not only practical usage but it is required from Maxwell equation to have solution, because it follows from (1.1), (1.2).

Taking $\nabla\cdot$ to (1.2), and use (1.1), we get

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

Lorentz Force Law

This is also important, and not follow from Maxwell.

Particle of charge q and moving with trajectory $\vec{x}(t)$, force acting on it is

$$\vec{F} = q[\vec{E}(\vec{x},t) + \frac{\vec{v}(t)}{c} \times \vec{B}(\vec{x},t)]_{\vec{x}(t)}$$

1.2 Potential + Gauge Choice

Maxwell equations are very hard to solve, one normally solves in terms of potentials and that requires gauge conditions.

As far as spatial part is concerned, we can define scalar potential $\phi(\vec{x},t)$, and vector potential $\vec{A}(\vec{x},t)$, then Maxwell 1.3, 1.4 are automatically equivalent to

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Substituting them into Maxwell (1.1), (1.2),

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \vec{A} = -4\pi\rho \tag{1.5}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \nabla(\nabla \cdot \vec{A}) + \frac{1}{c}\nabla\frac{\partial \phi}{\partial t} - \frac{4\pi}{c}\vec{j} \tag{1.6}$$

So these two equations contain four Maxwell equations, however, these equations are not well defined, in the sense that there is no systematic procedures to find the solutions, because there are infinitely many solutions.

Suppose one finds a solution for $\vec{A}'$, ϕ', then change to $\vec{A}$, ϕ

$$\vec{A} = \vec{A}' + \nabla\chi \tag{1.7}$$

$$\phi = \phi' - \frac{1}{c}\frac{\partial\chi}{\partial t} \tag{1.8}$$

give the exact same $\vec{E}$, $\vec{B}$, for any $\chi = \chi(\vec{x}, t)$.

So we will impose conditions on $\vec{A}$, ϕ to make (1.5), (1.6) well defined. It is called "fixing the gauge".

There are two gauges commonly used, other specific gauges may also used, but for very specific problems.

Coulomb Gauge

Require

$$\nabla \cdot \vec{A} = 0 \qquad \forall t \tag{1.9}$$

Suppose we are given $\vec{A}'$, ϕ' satisfying (1.5), (1.6), and we know from (1.7), (1.8), we can have different $\vec{A}$, ϕ, but now we impose (1.9), so from (1.7),

$$\nabla^2\chi = -\nabla \cdot \vec{A}'$$

This is Poisson's equation. It has a unique solution for χ, which will in term give $\vec{A}$ satisfying (1.9). If $\vec{A}'$ already satisfies (1.9), then we get

$$\nabla^2\chi = 0$$

This is Laplace's equation, and the unique solution is that $\chi \equiv 0$. So in any way, $\vec{A}$, ϕ are unique.

Coulomb gauge simplifies (1.5), (1.6),

$$\nabla^2\phi(\vec{x}, t) = -4\pi\rho(\vec{x}, t) \tag{1.10}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \frac{1}{c}\nabla\frac{\partial\phi}{\partial t} - \frac{4\pi}{c}\vec{j} \tag{1.11}$$

(1.10) has the same t on both sides, this shows that any change in the source will instantaneously charge ϕ, in other words, time causality is not manifested in this case. So generally people don't think ϕ represents degree of freedom, so ϕ is not a dynamical variable, and it is not quantized. It is used as a constraint.

(1.11) is a wave equation with source appeared on the right hand side. $\vec{A}$ is quantized and condition (1.9) reduces 3 degrees of freedom to 2. Indeed, in momentum space $\vec{A}$ has non-zero component perpendicular to $\vec{k}$.

Later we will show why Coulomb gauge is the best choice for atomic physics, while in radiation problem, Lorentz gauge is more attractive.

Lorentz Gauge

Require

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$

then (1.5), (1.6) become

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho \tag{1.12}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{j} \tag{1.13}$$

$\vec{A}$, ϕ are unique, because they are solutions of wave equations. More $\vec{A}$, ϕ represent 4 dynamical variables, and all manifest causality and Lorentz invariant.

Lorentz gauge adds more degree of freedom, so one would have to find the *hidden* conditions. This becomes very sophisticated tasks. It involves negative metric, adiabatic boundary condition, or fictitious fields. So for atomic physics, where relativity is not a big concerned, people prefer to use Coulomb gauge to limit degree of freedom, while for radiation from moving particles problems, one would have to use Lorentz gauge. This is at least what Jackson did.

1.3 Maxwell in Medium

$$\nabla \cdot \vec{D} = 4\pi\rho \tag{1.14}$$

ρ free charge, if assume linearity

$$\vec{D} = \epsilon\vec{E} \tag{1.15}$$

or

$$\vec{D} = \vec{E} + 4\pi\vec{P} \tag{1.16}$$

$\vec{P}$ polarization, and (1.16) is more general than (1.15). Of course (1.15) can appear in matrix form.

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j} \tag{1.17}$$

if assume linearity

$$\vec{B} = \mu\vec{H}$$

or more general

$$\vec{H} = \vec{B} - 4\pi\vec{M} \tag{1.18}$$

Notice in (1.16), (1.18) we write macroscopic quantity in terms of microscopic quantity.

So Maxwell in medium are equations (1.14), (1.17), and (1.3), (1.4). The two with no source are not effected by the medium.

We will use these equations, when we study plasma, because in plasma, we pretend it is dielectric.

1.4 Radiation

In Lorentz gauge, the solution to (1.12), (1.13)

$$\phi = \phi^0(\vec{x}, t) + \int d^3\vec{x}' \frac{1}{|\vec{x} - \vec{x}'|}\rho(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}) \tag{1.19}$$

$$\vec{A} = \vec{A}^0(\vec{x}, t) + \frac{1}{c}\int d^3\vec{x}' \frac{1}{|\vec{x} - \vec{x}'|}\vec{j}(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}) \tag{1.20}$$

Note (1) ϕ^0, $\vec{A}^0$ are related to boundary condition, they mean free radiation or background radiation. If no initial radiation or other things, we take them as 0.

(2) $1/|\vec{x}-\vec{x}'|$ here makes them look just like Coulomb potential.

(3) Notice the retarded time used here.

(4) We include (1.8) here just for completion, notice that scalar potential doesn't contribute to radiation.

(5) Although we include ϕ here just for completion, it doesn't contribute to the radiation because static charge has no radiation.

Because solving these two (1.19), (1.20) integrals for given ρ, $\vec{j}$ are very hard, we use dipole / wave approximations and do it in frequency space.

We Fourier in t, using Landau's notation

$$\vec{A}(\vec{x},t) = \frac{1}{\sqrt{2\pi}} \int dw e^{-iwt} \vec{A}_w(\vec{x})$$

$$\vec{A}_w(\vec{x}) = \frac{1}{\sqrt{2\pi}} \int dt e^{iwt} \vec{A}(\vec{x},t)$$

So (1.20) becomes

$$\vec{A}_w(\vec{x}) = \frac{1}{c} \int d^3\vec{x}' \frac{1}{|\vec{x}-\vec{x}'|} \vec{j}_w(\vec{x}') e^{ik|\vec{x}-\vec{x}'|} \tag{1.21}$$

where $k \equiv w/c$.

Wave Zone Approximation

If the source is limited, i.e. $|\vec{x}'|$ is not so big, we will expand $|\vec{x}-\vec{x}'|$ in term of $|\vec{x}'|/r$, $r=|\vec{x}|$

$$\begin{aligned}
|\vec{x}-\vec{x}'| &= \sqrt{(\vec{x}-\vec{x}')^2} \\
&= \sqrt{r^2-2\vec{x}\cdot\vec{x}'+\vec{x}'^2} \\
&= r\sqrt{1-\left(\frac{2\vec{x}\cdot\vec{x}'}{r^2}-\frac{\vec{x}'^2}{r^2}\right)} \because (\cdot) \ll 1 \\
&= r\left(1-\frac{\vec{x}\cdot\vec{x}'}{r^2}+\frac{\vec{x}'^2}{2r^2}+...\right) \\
&\approx r\left(1-\frac{\vec{x}\cdot\vec{x}'}{r^2}\right)
\end{aligned}$$

we still keep the $\vec{x}\cdot\vec{x}'/r^2$, because $|\vec{x}|=r$, $r\cdot(\vec{x}\cdot\vec{x}'/r^2)$ is still first order. Then (1.21) becomes

$$\vec{A}_w(\vec{x}) = \frac{e^{ikr}}{rc}\int d^3\vec{x}'\vec{j}_w(\vec{x}')e^{-i\vec{k}\cdot\vec{x}'}$$

where $\vec{k}=k\vec{x}/r$. The definition says you always look at the momentum coming towards you in the wave zone.

Dipole Radiation Formulation

Furthermore if $k\times$ (size of source) $=k|\vec{x}'|\ll 1$

$$\vec{A}_w(\vec{x}) = \frac{e^{ikr}}{rc}\int d^3\vec{x}'\vec{j}_w(\vec{x}')$$

or

$$\vec{A}(\vec{x},t) = \frac{1}{rc}\int d^3\vec{x}'\vec{j}(\vec{x}',t-r/c)$$

This approximation says we use dipole approximation when we don't have knowledge of the structure of the source, more precisely when $\lambda\sim 1/k\gg|\vec{x}'|$.

Lecture 2 (1/28/13)

For a general $\vec{j}(\vec{x},t)=\sum_n q_n\dot{\vec{x}}_n(t)\delta(\vec{x}-\vec{x}_n(t))$,

$$\vec{A}(\vec{x},t) = \frac{1}{rc}\sum_n q_n\dot{\vec{x}}_n(t-r/c) = \frac{1}{rc}\frac{d}{dt}\sum q_n\vec{x}_n(t-r/c)$$

set $\vec{p}(t-r/c)=\sum q_n\vec{x}_n(t-r/c)$, total dipole,

$$\vec{A}(\vec{x},t)=\frac{\dot{\vec{p}}(t-r/c)}{rc}$$

Recall

$$\vec{B}(\vec{x},t)=\frac{1}{c}\dot{\vec{A}}(\vec{x},t)\times\hat{n}=\frac{\ddot{\vec{p}}(t-r/c)\times\hat{n}}{rc^2}$$

Power detected

$$\frac{dP}{d\Omega}=\frac{cr^2}{4\pi}\left|\vec{B}\right|^2=\frac{1}{4\pi c^3}\left|\ddot{\vec{p}}(t-r/c)\times\hat{n}\right|^2$$

After integration

$$P=\frac{2}{3c^3}\left|\ddot{\vec{p}}(t-r/c)\right|^2$$

For single particle of charge e

$$P=\frac{2e^2}{3c^3}\left|\dot{\vec{v}}(t-r/c)\right|^2$$

In term of frequency

$$\frac{d\epsilon}{d\Omega}=\frac{cr^2}{2\pi}\vec{B}_w\cdot\vec{B}_w^*$$

where

$$\vec{B}_w=-\frac{w^2}{rc^2}\vec{p}_w\times\hat{n}e^{ikr}$$

e^{ikr} is same factor in $\vec{A}_w$ due to retarded time, but it does not effect power calculation.

$$\frac{d\epsilon}{d\Omega}=\frac{w^4\left|\vec{p}_w\times\hat{n}\right|^2}{2\pi c^3}$$

1.5 Scattering of E+M

Suppose we have incident plane wave coming from the $-z$ direction, we have some localized material sitting at the origin. We are measuring at P far from the origin. Assume the incident wave spread big so cover the whole material, but no too big, so at P, we only detect scattered wave.

We define scattering cross section by

$$\frac{d\sigma}{d\Omega} = \frac{1}{|\vec{S}|}\frac{dP}{d\Omega}$$

dimension P is energy/time, dimension of $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}$ ($1/4\pi$ for real formalism; $1/2\pi$ for complex) is energy/(area.time). In QED, scatter wave are measured by number of photons, but in classical EM, no photons, so the definition given here is quite natural, it looks at energy in both denominator and numerator.

Example. Wave acts on free particle of charge e of mass m in dipole approximation when $t = 0$, particle is at origin.

$$m\ddot{\vec{x}}(t) = e\vec{E}(\vec{x}(t), t)$$

here t is time on the particle. By dipole approximation

$$m\ddot{\vec{x}}(t) = e\vec{E}_w e^{-iwt}$$

($\because$plane wave $\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - wt)$ but $\vec{k} \cdot \vec{x}$ is small, can be ignored, we use $\vec{E}_w$ to indicate that the wave is a combination of plane waves with various frequencies, so $\vec{E}_w$ is the Fourier transform of $\vec{E}$ at frequency w.)

Use the real formalism, equation of motion

$$\ddot{\vec{x}}(t) = \frac{e}{m}\vec{E}_w \cos wt$$

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{\left|\ddot{\vec{p}}(t - r/c) \times \hat{n}\right|^2}{4\pi c^3} \\ &= \frac{1}{4\pi c^3}\left(\frac{e^2}{m}\right)^2 \left|\vec{E}_w \times \hat{n}\right|^2 \cos^2 w(t - r/c) \end{aligned}$$

$$\overline{\frac{dP}{d\Omega}} = \frac{c}{4\pi}\left(\frac{e^2}{mc^2}\right)^2 \left|\vec{E}_w \times \hat{n}\right|^2 \frac{1}{2}$$

Similarly

$$\vec{S}(t') = \frac{c}{4\pi}\hat{n}|\vec{E}_w|^2 \cos^2 wt'$$

$$\overline{\left|\vec{S}\right|} = \frac{c}{8\pi}|\vec{E}_w|^2$$

For the two time averaging above, we want T is big compare to 1/freq, but T should not be too big compare to the incident wave on/off.

Therefore

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{\left|\vec{E}_w \times \hat{n}\right|^2}{|\vec{E}_w|^2} = \left(\frac{e^2}{mc^2}\right)^2 \sin^2\Theta \tag{1.22}$$

where Θ is the angle between $\vec{E}_w$ and $\hat{n}$.

Now we can integrate once more

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$$

This is *Thomson cross section* formula, which is correct in QED.

Notice $\left(\frac{e^2}{mc^2}\right)$ has unit of length. For electron, $e^2/mc^2 \approx 3 \times 10^{-13} cm$.

Example. Scattering on harmonically bound electron with radiation damping.

(the value we will get is small, so one will hardly consider this effect. Nevertheless, the calculation gives an correct universal form that will be used later in wave damping for example.)

Recall we have instantaneous power radiated by electron is

$$p = \frac{2e^2}{3c^3}\dot{\vec{v}}^2$$

Now we want to consider radiation reaction, we'll do it in a non-relativistic way, it breaks down in relativistic limit. In fact there is no consistent way treat both relativistic and non-relativistic limits.

Equation of motion of harmonically bound electron

$$m\ddot{\vec{x}} = \vec{F}_{internal} + \vec{F}_{ext\,field} + \vec{F}_{rad\,reaction}$$

By energy conservation

$$\vec{F}_{rad} \cdot \vec{v} = -\frac{2e^2}{3c^3}\dot{\vec{v}}^2 = -\frac{2e^2}{3c^3}\left[\frac{d}{dt}(\vec{v}\cdot\dot{\vec{v}}) - \vec{v}\cdot\ddot{\vec{v}}\right]$$

Because the electron stays in one region on average, we claim $\frac{d}{dt}(\vec{v}\cdot\dot{\vec{v}}) = 0$. Hence

$$\vec{F}_{rad} \cdot \vec{v} = \frac{2e^2}{3c^3}\vec{v}\cdot\ddot{\vec{v}}$$

or

$$\vec{F}_{rad} = \frac{2e^2}{3c^3}\dddot{\vec{x}}$$

We have a very dangerous equation, involving third derivative, but we will use it anyway,

$$m\ddot{\vec{x}} = -mw_0^2\vec{x} - e\vec{E}_w e^{-iwt} + \frac{2e^2}{3c^3}\dddot{\vec{x}}$$

Here we used dipole approximation again. In fact in Landau "classical field" states dipole approximation is non relativistic approximation. (cf problem 1 in PS 1)

We look for solution (steady state)

$$\vec{x} = \vec{x}_w e^{-iwt}$$

$$[-mw^2 + mw_0^2 - iw^3\frac{2e^2}{3c^3}]\vec{x}_w = -e\vec{E}_w$$

So

$$\vec{x}_w = \frac{-e}{m}\frac{\vec{E}_w}{w_0^2 - w^2 - i\gamma w}$$

This is the universal form.

Here

$$\gamma = w^2\frac{2e^2}{3mc^3} = \left(\frac{e^2}{mc^2}\right)\frac{2w^2}{3c}$$

γ is not important until $w \sim w_0$, then the damping factor will play a role so $\vec{x}$ doesn't go to infinity. So we can just replace w by w_0 in the expression of γ. Hence we pretend

$$\gamma = \left(\frac{e^2}{mc^2}\right)\frac{2w_0^2}{3c} \tag{1.23}$$

or

$$\frac{\gamma}{w_0} \simeq \frac{1}{c}\frac{e^2}{mc^2} w_0$$

If $\frac{\gamma}{w_0} \simeq 1$, i.e. that is when the photon energy $\hbar w_0$ becomes

$$\hbar w_0 \simeq \frac{\hbar c}{(e^2/mc^2)}$$

which is about 70MeV. (recall $mc^2 \approx 1/2$MeV)

$$\frac{dP}{d\Omega} = \frac{w^4 c}{4\pi}\left(\frac{e^2}{mc^2}\right)^2 \frac{\left|\vec{E}_w \times \hat{n}\right|^2}{(w^2 - w_0^2)^2 + \gamma^2 w^2} \tag{1.24}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{w^4 \sin^2\Theta}{(w^2 - w_0^2)^2 + \gamma^2 w^2} \tag{1.25}$$

If $w \gg w_0$, we are back to the previous example free electron (1.22).

2 Scattering in a Dielectric Medium

Lecture 3 (1/30/13)

The material can be ionized plasmas, gas, or solid. They are characterized by either dielectric constant or index of refraction.

2.1 Scattering Amplitude

We always use plane wave $\vec{E}_0 e^{i\vec{k}_0\cdot\vec{x}-iwt}$ incident from $-z$ direction onto the plane xy, always assume frequency incoming is the same as frequency coming out, also assume no change in polarization. These assumptions are in fact close to real applications.

At observe point $\vec{x}$ with solar angle $d\Omega$ the scattering wave detected denoted by $\vec{E}_{sc}$ with

$$\vec{k} = \hat{n}\frac{w}{c} \quad \vec{k} \parallel \vec{x}$$

and total wave detected is

$$\vec{E} = \vec{E}_0 e^{i\vec{k}_0\cdot\vec{x}-iwt} + \vec{E}_{sc}$$

Recall

$$\frac{dP}{d\Omega} = \frac{cr^2}{4\pi}|\vec{E}_{sc}|^2 \qquad \frac{d\sigma}{d\Omega} = \frac{1}{|\vec{S}|}\frac{dP}{d\Omega}$$

we want to develop some general results (in general wave zone) that beyond dipole approximation.

$$\vec{A}_w^{se} = \frac{e^{ikr}}{rc}\int d^3x'\vec{j}_w(\vec{x}')e^{-i\vec{k}\cdot\vec{x}'}$$

$$\vec{E}_{sc} = \vec{B}_{sc}\times\hat{n} = \frac{1}{c}(\dot{\vec{A}}_{sc}\times\hat{n})\times\hat{n}$$

$$\vec{E}_{sc} = \frac{e^{ikr-iwt}}{r}\frac{-iw}{c^2}\int d^3x'(\vec{j}_w(\vec{x}')\times\hat{n})\times\hat{n}e^{-i\vec{k}\cdot\vec{x}'}$$

compare this to QM scattering, in QM there is no e^{-iwt} term, because QM is about stationary solution, no time dependence.

We set

$$\vec{F}(\vec{k},\vec{k}_0) = \frac{-iw}{c^2}\int d^3x'(\vec{j}_w(\vec{x}')\times\hat{n})\times\hat{n}e^{-i\vec{k}\cdot\vec{x}'}$$

so clearly $\vec{F}(\vec{k},\vec{k}_0)$ has meaning of amplitude of a spherical wave because in front of it, $\frac{e^{ikr-iwt}}{r}$ is a spherical wave.

But people normal define *scattering amplitude* as

$$\vec{f}(\vec{k},\vec{k}_0) = \frac{\vec{F}(\vec{k},\vec{k}_0)}{|\vec{E}_0|}$$

This makes sense to make scattering amplitude as an intrinsic property of scattering as long as the effect on $\vec{j}$ is linear in $\vec{E}_0$, which indeed is normally the case.

Now we arrived

$$\frac{d\sigma}{d\Omega} = |\vec{f}(\vec{k},\vec{k}_0)|^2$$

2.2 Index of Refraction

Same incident wave setup, now suppose there is a very big (infinite extend in xy direction) slab of material sitting in the xy plane with thickness d, observing point is on z axis with z above the material, z is big too.

Use cylindrical coordinate, for an arbitrary point $\vec{x}'$ in the material, ρ is distance in xy plane of $\vec{x}'$ from origin, i.e. $\rho = \sqrt{x'^2 + y'^2}$. So

$$\vec{E}_{sc} = \int d^3x' N \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \vec{f} \left|\vec{E}_0\right| e^{-i\vec{k}\cdot\vec{x}'} e^{-iwt}$$

where $e^{i\vec{k}\cdot\vec{x}'}$ is the extra term, which was not there in last section because there material was at the origin, (and this is the reason we don't do dipole approximation here), N is density of scattering with unit 1/length^3 due to integration, number of molecules that can cause scattering in d^3x.

We now use wave zone approximation

$$\begin{aligned} |\vec{x}-\vec{x}'| &= \sqrt{(z-z')^2 + \rho^2} \\ &= (z-z')\left(1 + \frac{\rho^2}{2(z-z')^2} + ...\right) \end{aligned} \quad (2.1)$$

So

$$\vec{E}_{sc} = N \int_{-d/2}^{d/2} dz' \int_0^\infty 2\pi\rho d\rho \frac{e^{ik(z-z')+i\frac{k\rho^2}{2(z-z')}+ikz'-iwt}}{z-z'} \vec{f}(\vec{k},\vec{k}_0) \left|\vec{E}_0\right|$$

To replace the exponential

$$e^{ik\sqrt{(z-z')^2+\rho^2}}$$

by

$$e^{ik(z-z')+i\frac{k\rho^2}{2(z-z')}}$$

require $\frac{k\rho^2}{2(z-z')}$, or

$$\frac{k\rho^2}{z} \approx 1$$

i.e. this term dominates the exponential, because we have exponential here, we cannot simply drop lower order terms in 2.1.

Then drop z' for $z' \ll z$

$$\begin{aligned} \vec{E}_{sc} &= N \int_{-d/2}^{d/2} dz' \int_0^\infty 2\pi\rho d\rho \frac{e^{ikz+i\frac{k\rho^2}{2z}-iwt}}{z} \vec{f}(\vec{k},\vec{k}_0) \left|\vec{E}_0\right| \\ &= e^{ikz-iwt} \frac{2i\pi N d}{k} \vec{f}(\vec{k},\vec{k}_0) \left|\vec{E}_0\right| \end{aligned}$$

we used

$$\lim_{a\to\infty}\int_0^a d\rho^2 e^{i\frac{k\rho^2}{2z}} = i\frac{2z}{k}(1-\lim_{a\to\infty}e^{i\frac{ka^2}{2z}})$$

and

$$\lim_{x\to\infty}e^{ix}=0$$

by Riemann Lebesgue lemma. We also cheat when add all material points, but we didn't integrate over $\vec{f}$ or more specifically over $\vec{k}$, waves coming from different direction. Here we essentially say

$$\vec{k}\approx\vec{k}_0$$

or

$$\vec{f}(\vec{k},\vec{k}_0)=\vec{f}(\vec{k}_0,\vec{k}_0)$$

this is called *forward scattering amplitude*, only the wave coming right in the direction of $\vec{k}_0$ will be considered. But for this to work, we need z big, at point z, we only see the wave coming from right under us. More preciously ρ/z is small so $\theta=\arctan\rho/z$ is small, which is already used for wave zone approximation, so the two are consistent.

Now we can assume $\vec{f}\parallel\vec{E}_0$, (although this is not universal, but okay for now. When we get to plasma, we will see dielectric constant is a tensor), so

$$\vec{E}_{total}=\vec{E}_0e^{ikz-iwt}\left(1+\frac{2\pi iNd}{k}f(k)\right)$$

where $f=|\vec{f}|$.

Now stacking M slabs one on the top of another, from $z=0$ to z observing point, so $d=z/M$, and the total $\vec{E}$ coming out of one slab comes the incident wave for the next slab, so we should have the recursive

$$\begin{aligned}
\vec{E}_{total} &= \vec{E}_0 e^{ikz-iwt}\left(1+\frac{2\pi iNd}{k}f(k)\right)^M \\
&= \vec{E}_0 e^{ikz-iwt}\left(1+\frac{2\pi iNz}{kM}f(k)\right)^M \\
&= \vec{E}_0 e^{ikz-iwt+\frac{2\pi iNz}{k}f} \\
&= \vec{E}_0 e^{ik\left(1+\frac{2\pi Nf}{k^2}\right)z-iwt}
\end{aligned}$$

The assumptions for the above to work are 1) Nf/k^2 is small, so we can exponentiate; the condition implies weak scattering, this has to do with absorption (cf problem 3 in PS1). 2) From one slab derivation, we assumed z is big. Here we make the product of all possible scattering at each slab, but that z is big requires that actual scatter should not happen at every slab, there should a big spatial separation between scatterer at one point to next scatterer, the separation is at the order of z. (cf problem 5 in PS2)

Now we see the total $\vec{E}$ is given by a new k, we define

$$k_{med} = \frac{w}{c}n \text{ or } k_{med} = \frac{w}{c}\sqrt{\epsilon}$$

where n, index refraction $n = 1 + \frac{2\pi Nf}{k^2}$, or ϵ, dielectric constant,

$$\epsilon = 1 + \frac{4\pi Nf}{k^2} \tag{2.2}$$

But one can see they are the same thing.

Lecture 4 (2/4/13)

Last lecture's discussion did not dealt with two issues in detail. (cf problems 4, 5 in PS2)

(i) Because we only consider forward scattering amplitude, what about polarization of scattered wave? (ϵ would be a tensor). (ii) How big N can be for our discussion to hold? If N is too big, the assumption that Nf/k^2 is small violated. More preciously the distance between two scatterers should be big.

2.3 Simple Picture of Dielectric Medium

(reference in Jackson)

The general picture is to imagine molecule move along, as we put in external field (which is weak wave) that interact with electrons in the medium, we see how electrons respond. We pretend everything is on QM stable states (electrons in ground state). We have seen electrons in harmonic oscillator example in lecture 2, later we will use that to calculate specific heat.

The simple picture is to pretend that electrons in the medium making up dielectric, and they respond to weak external forces as coupled oscillation. That makes sense because if one uses perturbation for potential, then harmonic oscillator is the dominated term.

Later we will take the limit to make dielectric to be conductors, we will calculate conductivity constant by first considering the damping, or dissipation. We have showed the universal form last time, although the problem is very complicated to deal with in fundamental level, but we will get there when we discuss Landau damping in plasma, where γ is big, strong damping.

For now let's suppose the electrons in a molecule have small oscillation frequency w_j (normal mode) and corresponding "widths" γ_j (the widths here is the same concept in (1.24), (1.25) plots.) Let f_j be the degeneracy of each normal mode. (In classical 1D coupled harmonic problem, we know w_j are eigenfrequencies, γ_j coupled frequency, and f_j number of coupled electrons in one group. For 3D, we get spherical harmonic but the notion of normal modes are the same).

The sum $\sum f_i$ gives the total number of electrons in material, or sum of degree of freedom. We only look at translational freedom, (i.e. linear model), we are not looking at motion of nuclear, so no rotational modes.

Recall for one particle oscillation with external field

$$\vec{E} = \vec{E}_w e^{-iwt}$$

with

$$\vec{x}(t) = \vec{x}_w e^{-iwt}$$

with

$$\vec{x}_w = \frac{-e}{m}\frac{\vec{E}_w}{w_0^2 - w^2 - i\gamma w}$$

we get dipole moment $\vec{p} = -e\vec{x}$

$$\vec{p}_w = \frac{e^2}{m}\frac{\vec{E}_w}{w_0^2 - w^2 - i\gamma w}$$

the form always same, because the mechanism depends on dissipation. Note in $\vec{p}_w$ the coordinate disappeared.

Putting in all modes of the molecule, we get dipole moment of molecule

$$\vec{p}_w = -\frac{e^2}{m}\vec{E}_w \sum_j \frac{f_j}{w_j^2 - w^2 - i\gamma_j w}$$

The polarization = dipole/volume, $\vec{P}$, of molecule is

$$\vec{P}_w = N\vec{p}_w$$

multiplied by density of molecules. Since

$$\vec{D} = \epsilon\vec{E} = \vec{E} + 4\pi\vec{P},$$

we have

$$\epsilon\vec{E}_w = \vec{E}_w + 4\pi N\frac{e^2}{m}\vec{E}_w \sum_j \frac{f_j}{w_j^2 - w^2 - i\gamma_j w}$$

This gives

$$\epsilon(w) = 1 + \frac{4\pi N e^2}{m}\sum_j \frac{f_j}{w_j^2 - w^2 - i\gamma_j w} \tag{2.3}$$

compare to (2.2), we find

$$f = \left(\frac{e^2}{mc^2}\right)\sum_{j=0} \frac{w^2 f_j}{w_j^2 - w^2 - i\gamma_j w}$$

Suppose $w_0 = 0$, and $f_0 \neq 0$, physically this means these electrons have no internal restore force, so they are free electrons, then

$$\epsilon(w) = \underbrace{1 + \frac{4\pi Ne^2}{m}\sum_{j\neq 0}\frac{f_j}{w_j^2 - w^2 - i\gamma_j w}}_{\epsilon_0} - \frac{4\pi Ne^2}{mw}\frac{f_0}{i\gamma_0 + w}$$

so above we declare all electrons are either bound electrons (characterized by ϵ_0), or free electrons, which give currents.

From Maxwell equation

$$\nabla \times \vec{B} = \frac{4\pi}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\frac{\partial \vec{D}_0}{\partial t} + \frac{4\pi}{c}\vec{j}$$

If $\vec{D} = \vec{D}_w e^{-iwt}$, then

$$-iw\epsilon\vec{E}_w = -iw\epsilon_0\vec{E}_w + \vec{j}_w$$

so

$$\vec{j}_w = -iw(\epsilon - \epsilon_0)\vec{E}_w$$

This gives conductivity, σ, since $\vec{j} = \sigma\vec{E}$

$$\sigma(w) = -iw(\epsilon - \epsilon_0) = i\frac{4\pi Ne^2}{m}\frac{f_0}{i\gamma_0 + w}$$

take limit $w \to 0$, physically this means we apply external field, then turn down w slowly to zero, to get a steady state current, and we get σ for one electron ($f_0 = 1$)

$$\sigma = \frac{4\pi Ne^2}{m\gamma}$$

This equation makes prefect sense. m mass of electron at down stair, because it is the inertial. We don't have mass of nuclei in the equation, because we consider they are heavy and don't move. γ here should not be interpreted as in (1.23), since for free electrons $w_0 = 0$. Because (1.24) gives the universal form, that is the reason we call γ the width as discussed early, or $1/\gamma$ the mean free time between ionic collisions.

What we have done above is called *Drude model of conductivity.*

2.4 Problem Set 1 (due 2/12/13)

1)

For the problem of a free charged particle, say an electron, moving in a plane electromagnetic field

$$m\ddot{\vec{x}}(t) = -e\vec{E}_0 \cos(kz - wt),$$

show that the dipole approximation is the same as non relativistic motion. Give a heuristic argument that is the case generally.

2)

Consider a charged article moving under the forces due to a simple harmonic oscillator of frequency w_0, radiation damping and a plane em wave

$$\vec{E} = \vec{E}_0 e^{i\vec{k}_0 \cdot \vec{x} - iwt}$$

$\vec{k}_0 = \hat{e}_z w/c$. Evaluate the scattering amplitude $\vec{f}(\vec{k}, \vec{k}_0)$. Show that the total cross section for scattering of the incident wave is given by

$$\sigma = \frac{4\pi}{k} \mathrm{Im} \vec{f}(\vec{k}, \vec{k}_0) \cdot \frac{\vec{E}_0}{|\vec{E}_0|}$$

that is the scattering obeys the *optical theorem.*

3)

An em wave going through a dielectric suffers some attenuation. For a wave impinging on a dielectric one can write

$$|\vec{E}|^2(z,t) = |\vec{E}_0|^2 e^{-z/\lambda}$$

where z is the length that the wave has penetrated and λ is the mean free path. Show that $1/\lambda = N\sigma$ where N is the density of scatterers in the dielectric and σ

is the cross section.

2.5 Problem Set 2 (due 2/12/13)

4)

For a charged particle bound in a harmonic oscillator show that $\vec{f}(\vec{k}, \vec{k}_0) = f(k)\vec{E}_0 / \left|\vec{E}_0\right|$, that is the polarization of the radiation is the same as that of the incoming plane wave.

5)

We found

$$\sqrt{\epsilon} = 1 + \frac{2\pi N}{k^2} f(k)$$

for a single scatterer. For a given $k = w/c$ how big can N be before the above formula break down? Hint the formula will break down if a scattered wave has good chance of rescattering before it goes free enough to be in the wave zone.

6)

Consider the charge distribution

$$\rho(\vec{x}) = \frac{q}{a}\delta(x)\delta(y)$$

where q has dimension of charge e, a is a fixed length. Suppose ρ is put into a plasma having Debye wave number K_D. Solve for the scalar field ϕ in the plasma in the presence of ρ.

3 Plasmas-Simple Discussion

3.1 Debye Screening

Suppose a plasma is at temperature T, $\rho_e(\vec{x})$ charge density of electrons; $\rho_B(\vec{x})$ the background charge density, assume to be low. If things are in prefect equilibrium

$$\rho_e(x) = \rho_e^{(0)} = -\rho_B$$

so no $\vec{x}$ dependence, all uniform throughout, and the two densities cancel each other.

Here we imagine electrons in a box, we heat the box to temperature 10^6K$\approx$ 80eV, next lecture we'll show to have a some plasma wave that is the minimal temperature, 10^5K would not be enough. Even at 10^6K, those very inner shell electrons are still not free.

We will use Coulomb gauge, because it is a static problem Coulomb gauge is the natural choice. We are going to insert a charge q at $\vec{x} = 0$ to see how electrons respond, let ϕ be scalar electric potential,

$$\begin{aligned} \nabla^2 \phi &= -4\pi\rho \\ &= -4\pi q \delta(\vec{x}) - 4\pi\rho_e(\vec{x}) - 4\pi\rho_B(\vec{x}) \end{aligned}$$

(here if we put a negative charge at $\vec{x} = 0$, $q < 0$, electrons will repel, and attract ions. But we will assume ions are fixed)

We first assume

$$\rho_e(\vec{x}) = \rho_e^{(0)} + \delta\rho_e(\vec{x}) \text{ and } \rho_B(\vec{x}) = -\rho_e^{(0)} \tag{3.1}$$

This is saying the charge density of all electrons changes a little bit due to the new coming q, but background charge is still the negative of $\rho_e^{(0)}$.

For this type of problem, we don't have to know all the dynamics, thermodynamics takes care of all.

We have a result from Boltzmann distribution in equilibrium

$$\rho_e(\vec{x}) = \rho_e^{(0)} e^{-\left[\frac{-e\phi(\vec{x})}{kT}\right]}$$

where $\rho_e^{(0)} = -en_e^{(0)}$, $n_e^{(0)}$ is number of electrons density. The equation above appears in many places, for example, for particles distribution in a gravitational field.

Therefore we have

$$\nabla^2\phi(\vec{x}) = -4\pi q\delta(\vec{x}) + 4\pi en_e^{(0)}\left(e^{\frac{e\phi(\vec{x})}{kT}} - 1\right)$$

This is hard to do anything analytically but since kT really big, we get

$$\left(\nabla^2 - \frac{4\pi e^2 n_e^{(0)}}{kT}\right)\phi(\vec{x}) = -4\pi q\delta(\vec{x}) \tag{3.2}$$

we define $\frac{4\pi e^2 n_e^{(0)}}{kT} = K_D^2$, K_D is called Debye wave number and $K_D = 1/\lambda_D$, this is dimensional correct, since dimension of ∇^2 is 1/length^2.

So

$$\left(\nabla^2 - K_D^2\right)\phi(\vec{x}) = -4\pi q\delta(\vec{x}) \tag{3.3}$$

(if away from 0, (3.3) become D'alembert with 0 source) Solution to (3.3) is

$$\phi(\vec{x}) = \frac{q}{|\vec{x}|}e^{-K_D|\vec{x}|}$$

that is almost Coulomb field with screen factor $e^{-K_D|\vec{x}|}$.

The interesting thing about T here is if $T \gg 1$, that gives little screen, i.e. K_D is small, but when temperature is big, electrons tend to screen themselves and temperature brings in chaotic elements.

Lecture 5 (2/6/13)

Another interesting dichotomy. In the discussion above, we fix ions, because they are not important, but sometimes ions are important. What we are doing here is static problems, we wait long time till system comes to equilibrium. If one looks right way (e.g after new charge puts in) electron respond much fast, and ions become important, and we will then define Debye

$$K_D^2 = \frac{4\pi}{kT}\left(e^2 n_e^{(0)} + q_I^2 n_I^{(0)}\right).$$

3.2 Plasma Waves

For this discussion, wave in plasma, ion is irrelevant.

Before we turn to plasma waves, let's see some example of numbers.

$$10^6 K$$

is hot. It basically melt any material. For $n_e^{(0)} \approx 10^{15}/\text{cm}^3$, $\lambda_D \approx 2.4 \times 10^{-4}$cm,

$$T = 10^6 K = \frac{10^6}{(300)(40)}\text{eV} = 80\text{eV}$$

(Recall $1/40\text{eV} \approx 300^0 K$ where $^0K = KT = 10^6 K$)

Recall -80eV is still higher than some lowest ion blinding energy, so we still don't get free ion, but this is typical temperature we need, we don't want T to be too high, then things get complicated.

Compare this to relativistic rest energy

$$m_e c^2 = \frac{1}{2}\text{MeV} = \frac{1}{2}10^6\text{eV} \gg 80\text{eV} = T$$

so we are still in very non-relativistic region.

In this section, we will see there is a definite frequency (plasma mode). Above that we get longitudinal wave , below that we get oscillation, not wave prorogation. In this simple picture wave has no dissipation. We then use Boltzmann equation, we will see that plasma frequency doesn't depend on wave frequency and we'll get damping.

Consider a system of dielectric, turn on temperature to $10^6 K$. Since we don't take ions into account, we take all ions frequency to 0. (Later in using Boltzmann equation, we will see all the leading terms are correct in this simple method given here.)

We start with molecular gas. From (2.3), letting $w_j \to 0$, $\gamma_j \to 0$ means all electrons have no internal restoring forces (similar to free electrons, but no so because that $\gamma \to 0$ means no collision either, so no current) Hence we are actually treating all electrons are bound electrons, then (2.3) gives

$$\epsilon(w) = 1 - \frac{4\pi n_e e^2}{mw^2}$$

This is the simple way getting plasma from dielectric.

We now send waves

$$(\vec{E}, \vec{B}) = (\vec{E}_0, \vec{B}_0) e^{i\vec{k}\cdot\vec{x} - iwt}$$

through plasma, and look for what $\vec{k}$ and w are allowed from Maxwell equations. Everything that is allowed by Maxwell exists in plasma.

$$\nabla \cdot \epsilon \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\epsilon \vec{E} = 0$$

give

$$\epsilon(w)\vec{k} \cdot \vec{E}_0 = 0 \tag{3.4}$$

$$\vec{k} \cdot \vec{B}_0 = 0 \tag{3.5}$$

$$\vec{k} \times \vec{E}_0 - \frac{iw}{c}\vec{B}_0 = 0 \tag{3.6}$$

$$\vec{k} \times \vec{B}_0 + \frac{iw}{c}\epsilon \vec{E}_0 = 0 \tag{3.7}$$

Consider two cases:

1)

$\epsilon(w) \neq 0$, $\epsilon(w) > 0$.

Then (3.4), (3.5) imply $\vec{k}$ is perpendicular to both polarization, so the wave is transverse, just like in vacuum.

From (3.6), (3.7), we get

$$\vec{k} \times (\frac{c}{w}\vec{k} \times \vec{E}_0) = -\frac{w}{c}\epsilon\vec{E}_0$$

or

$$\vec{k}(\vec{k}\cdot\vec{E}) - \vec{E}_0 k^2 = -(\frac{w}{c})^2\epsilon\vec{E}_0$$

It follows

$$\begin{aligned} k^2 &= \epsilon(\frac{w}{c})^2 \\ &= (\frac{w}{c})^2\left(1 - \frac{4\pi n_e e^2}{mw^2}\right) \\ &= \frac{w^2}{c^2} - \frac{w_p^2}{c^2} \end{aligned}$$

where $w_p^2 \equiv 4\pi n_e e^2/m$, is called plasma frequency. Compare this to Deybe wave number, there is a corresponding

$$kT \leftrightarrow m$$

We can write above in a different form: Deybe equation

$$w^2 = k^2c^2 + w_p^2$$

The calculation above is non-relativistic, and there is no temperature dependence, which will come later.

2)

$\epsilon = 0$, so

$$1 = \frac{4\pi n_e e^2}{mw^2} \implies w = w_p$$

Then (3.5) still works, and it implies $\vec{k}$ is $\perp$ to $\vec{B}_0$, from (3.7), $\vec{k}$ is parallel to $\vec{B}_0$, so $\vec{B}_0 = 0$. (we don't say $\vec{k} = 0$, because we want to find any possible allowed $\vec{k}$.) By (3.6), $\vec{k}$ is parallel to $\vec{E}_0$.

From those two cases, we see that if we start at high frequency w, then turn it down, before it gets w_p, we have transverse wave and k is real no dissipation. When $w = w_p$, we have longitudinal vibration

$$\vec{E} = \frac{\vec{k}}{|k|} E_0 e^{i\vec{k}\cdot\vec{x} - iw_p t}$$

which has phase velocity (i.e. $\vec{x}$, t change in a way that phase stays constant $v_p = w_p/k$.) Notice it has no group velocity, because $v_g = dw/dk$, here frequency has no wave number dependence. So it is not a wave, since it doesn't transport energy nor charge, etc.

If $w < w_p$, k becomes pure imaginary, that corresponds to decay, with no wave propagation.

Later we will use more sophisticated method, that will show longitudinal vibration gives wave and we will show it has group velocity depended on temperature, kT/mc^2.

3.3 Group Velocity Review

Let's review the concept of group velocity. Suppose we have a 1d wave package

$$f(z,t) = \int dk \tilde{f}(k) e^{ikz - iw(k)t}$$

(we didn't put in a normalization factor, for it is not important.) We assume (1) w has k dependence; (2) $\tilde{f}(k)$ is concentrated around some k_0.

As an example, let's take

$$\tilde{f}(k) = f_0 e^{-\frac{(k-k_0)^2}{2\mu^2}} \tag{3.8}$$

where μ is at our disposal. This choice of $\tilde{f}$ is useful, so our discussion is pretty analytic.

$$f = f_0 \int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{2\mu^2} + ikz - it\left[w(k_0) + (k-k_0)w'(k_0) + \frac{(k-k_0)^2}{2} w''(k_0) \ldots\right]}$$

We will choose μ small w.r.t. the scale of variation of w around k_0, more preciously

$$\mu^2 w''(k_0) \ll 1 \tag{3.9}$$

and usually

$$(k - k_0) < \mu$$

Therefore

$$\begin{aligned} f &= f_0 e^{ik_0 z - itw(k_0)} \int_{-\infty}^{\infty} dk e^{-\frac{1}{2\mu^2}\left[(k-k_0)-\mu^2 i(z-tw'(k_0))\right]^2 - \frac{\mu^2}{2}[z-tw'(k_0)]^2} \\ &= f_0 e^{ik_0 z - itw(k_0)} \mu\sqrt{2\pi} e^{-\frac{\mu^2}{2}[z-tw'(k_0)]^2} \end{aligned} \tag{3.10}$$

the $e^{-\frac{\mu^2}{2}[z-tw'(k_0)]^2}$ factor gives the wideness of the amplitude of the wave package, i.e. the envelope. Then clear the envelope is moving at speed of $v_g = w'(k_0)$. And clearly v_g has same dimension as v_p.

Now we understand what (3.9) means physically. It means that the Gaussian we chose dies quickly for k away from k_0, so choosing a small μ but not too small so that the wave package is not too broad, so we won't get too much interference from the different waves in the packages. From QM, we know if μ is too small then it is highly concentrated in (3.8), but then by the uncertainty principle, the wave package will spread very fast, so it no longer looks a wave, consistent with (3.10).

This also shows that it is the group velocity that has physical meaning, such as transporting energy or particles, since particles can be considered as wave packets. One can even have v_p larger than speed of light, but only v_g matters.

3.4 Problem Set 2 (continued)

6)

Consider the charge distribution

$$\rho(\vec{x}) = \frac{q}{a}\delta(x)\delta(y)$$

where q has dimension of charge e, a is a fixed length. Suppose ρ is put into a plasma having Debye wave number K_D. Solve for the scalar field ϕ in the plasma in the presence of ρ.

4 Boltzmann Equation and Plasma Waves

Lecture 6 (2/11/13)

(Reference not in Jackson, from Landau "Physical Kinetics")

We will slowly introduce Boltzmann equation, which is key for solving equilibrium or non equilibrium systems, or systems transition from non equilibrium toward equilibrium, then we will deduce two new things from last section: 1) longitudinal oscillation gives propagation, 2) Landau damping. The methods are widely used in heavy ion collision, astrophysics, or early universe. The steps are not complicated, but bit subtle.

4.1 The Louisville Theorem

This is somewhat discrete version of Boltzmann equation. Suppose N identical particles of mass m in a box.

We shall assume N is very big, we also assume we can label them as $1, 2, ..., N$, we will not make distinguish whether particles are Fermions and Bosons. The interactions among particles is generated by a Hamiltonian

$$H(\vec{x}_1, \vec{p}_1, ..., \vec{x}_N, \vec{p}_N)$$

The potential in the Hamiltonian should have some symmetry feature among interchange labeling, since they are identical particles. Notice although Boltzmann equation came from thermodynamics, and thermos knows nothing about interaction, Boltzmann equation can handle interaction but not too much, only those conservative interaction, due to Hamiltonian description works for conservative forces.

Boltzmann equation cannot generally be reduced to single particle description, because 1) we don't know all the microscopic dynamics of the system, not all initial conditions are known either, 2) we mostly talk about averaging.

For statical system, there are always two views.

Let

$$P_N(\vec{x}_1, \vec{p}_1, ..., \vec{x}_N, \vec{p}_N, t)$$

with normalization condition

$$\int P_N d^3x_1 d^3p_1 ... d^3x_N d^3p_N = 1$$

This is of course classical discussion, since in QM, $\vec{x}$ and $\vec{p}$ are not known at the same time.

(1) We can view P_N as the probability density of $6N$ phase space for finding the particles at a particular point in phase space. So there is one system, we follow it over time, get probability distribution.

(2) View P_N as density of ensembles having particles at that point in phase space. There are many identical system begin with, having random initial conditions, after time, averaging different parts over these ensembles.

If the problem is ergodic, the two views are equivalent.

We define a single particle distribution by

$$\begin{aligned}
f(\vec{x}, \vec{p}, t) &= \int P_N(\vec{x}_1 = \vec{x}, \vec{p}_1 = \vec{p}, ..., \vec{x}_N, \vec{p}_N, t) d^3x_2 d^3p_2 ... d^3x_N d^3p_N \\
&\quad + \int P_N(\vec{x}_1, \vec{p}_1, \vec{x}_2 = \vec{x}, \vec{p}_2 = \vec{p}, ..., \vec{x}_N, \vec{p}_N, t) d^3x_1 d^3p_1 d^3x_3 d^3p_3 ... d^3x_N d^3p_N \\
&\quad + ... \\
&\quad + \int P_N(\vec{x}_1, \vec{p}_1, ..., \vec{x}_N = \vec{x}, \vec{p}_N = \vec{p}, t) d^3x_1 d^3p_1 d^3x_3 d^3p_3 ... d^3x_{N-1} d^3p_{N-1} \\
&= N \int P_N(\vec{x}_1 = \vec{x}, \vec{p}_1 = \vec{p}, ..., \vec{x}_N, \vec{p}_N, t) d^3x_2 d^3p_2 ... d^3x_N d^3p_N \qquad (4.1)
\end{aligned}$$

Because the Hamiltonian is symmetric among interchange labeling, we believe so is P_N. (4.1) is called Boltzmann distribution function.

Note

$$\int d^3p f(\vec{x}, \vec{p}, t) = n(\vec{x}, t)$$

is number density of particle at $\vec{x}$, t. Of course

$$\int d^3xn(\vec{x},t) = N$$

If we want to use $\vec{v}$ instead of $\vec{p}$, we define

$$f(\vec{x},\vec{v},t) = m^3 f(\vec{x},\vec{p},t) \tag{4.2}$$

so

$$\begin{aligned}
\int d^3vf(\vec{x},\vec{v},t) &= \int d^3vm^3 f(\vec{x},\vec{p},t) \\
&= \int d^3pf(\vec{x},\vec{p},t) \\
&= n(\vec{x},t)
\end{aligned}$$

Recall for a charge particle in electrodynamics

$$\frac{\partial}{\partial t}\rho(\vec{x},t) + \nabla \cdot \vec{j}(\vec{x},t) = 0$$

where $\rho(\vec{x},t) = q\delta(\vec{x} - \vec{x}(t))$, $\vec{j}(\vec{x},t) = q\dot{\vec{x}}(t)\delta(\vec{x} - \vec{x}(t)) = \rho\vec{v}$. Here we do analogously, we have particle conservation: total number particles changes in a region of the phase space is equal to the flow in to the region in the phase space,

$$\frac{\partial}{\partial t}P_N + \sum_i^N \left[\nabla_{\vec{x}_i}(P_N\dot{\vec{x}}_i) + \nabla_{\vec{p}_i}(P_N\dot{\vec{p}}_i)\right] = 0 \tag{4.3}$$

the quality in the square bracket is equal to

$$(\dot{\vec{x}}_i \cdot \nabla_{\vec{x}_i} + \dot{\vec{p}}_i \cdot \nabla_{\vec{p}_i})P_N + P_N(\nabla_{\vec{x}_i}\dot{\vec{x}}_i + \nabla_{\vec{p}_i}\dot{\vec{p}}_i)$$

We know $\dot{\vec{x}}_i = \nabla_{\vec{p}_i}H$, $\dot{\vec{p}}_i = -\nabla_{\vec{x}_i}H$, so

$$\nabla_{\vec{x}_i}\dot{\vec{x}}_i + \nabla_{\vec{p}_i}\dot{\vec{p}}_i = 0$$

(recall if $H = \vec{p}^2/2m + V(\vec{x})$, then $\nabla_{\vec{p}_i}H = \frac{\vec{p}}{m} = \vec{v} = \dot{\vec{x}}$, $\nabla_{\vec{x}}H = \nabla_{\vec{x}}V = -\vec{F} = -\dot{\vec{p}}$.

Recall there are two formulations: Lagrangian, 2nd order coordinate; Hamilton, 1st order in coordinate and momentum. They are the same in classical mechanics. But QM always prefer the latter, because in which one gets Poisson's bracket, which is classical version of the commutator.)

So (4.3) becomes

$$\frac{\partial}{\partial t}P_N + \sum_i^N (\dot{\vec{x}}_i \cdot \nabla_{\vec{x}_i} + \dot{\vec{p}}_i \cdot \nabla_{\vec{p}_i}) P_N = 0 \tag{4.4}$$

This is call the Liouville theorem.

In summary, we start from a Hamiltonian that has no time dependence, so energy is conserved, also entropy is conserved, but Boltzmann allows entropy to change.

4.2 The Boltzmann Equation

From (4.3), integrate over $d^3x_2 d^3p_2 ... d^3x_N d^3p_N$, then use Gauss turn into surface integral. Because particles are in a box, the boundary of surface at ∞ gives 0 for $i \geq 2$, namely for example $i = 2$

$$\int d^3x_2 d^3p_2 ... d^3x_N d^3p_N \nabla_{\vec{x}_2}(P_N \dot{\vec{x}}_2)$$

$$\begin{aligned} &= \int d^3p_2 d^3x_3 d^3p_3 ... d^3x_N d^3p_N \left(\int d^3x_2 \nabla_{\vec{x}_2}(P_N \dot{\vec{x}}_2) \right) \\ &= \int d^3p_2 d^3x_3 d^3p_3 ... d^3x_N d^3p_N \int_S P_N \dot{\vec{x}}_2 dS = 0 \end{aligned}$$

after multiplying N, we get by (4.4) for $i = 1$

$$\frac{\partial f(\vec{x}, \vec{p}, t)}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f(\vec{x}, \vec{p}, t) + N \int d^3x_2 d^3p_2 ... d^3x_N d^3p_N \dot{\vec{p}}_1 \cdot \nabla_{\vec{p}_1} P_N \tag{4.5}$$

we take $\dot{\vec{x}}_1 \cdot \nabla_{\vec{x}_1}$ out of the integral, but we cannot do that for the second term, that would mean that the force $\dot{\vec{p}}_1$ acting on the particle is independent of other particles, which is clearly no true.

But to process, we need to use approximation to trickle the last term in (4.5). First recognize that

$$\dot{\vec{p}}_1 \cdot \nabla_{\vec{p}_1} = \dot{\vec{v}} \cdot \nabla_{\vec{v}}$$

Then use mean field, saying that the expectation value of product of two operators is about the product of two expectations if the correlation is small

$$\langle AB \rangle \simeq \langle A \rangle \langle B \rangle$$

In our case, that means all particles are far away, so the last term in (4.5) becomes

$$N \left\langle \dot{\vec{v}} \right\rangle \langle \nabla_{\vec{v}} \rangle + \delta \dot{f}$$

where

$$\left\langle \dot{\vec{v}} \right\rangle = \int d^3x_2 d^3p_2 ... d^3x_N d^3p_N \dot{\vec{p}}_1 P_N$$

$$\begin{aligned} N \langle \nabla_{\vec{v}} \rangle &= N \int d^3x_2 d^3p_2 ... d^3x_N d^3p_N \nabla_{\vec{v}} P_N \\ &\approx \nabla_{\vec{v}} f(\vec{x}, \vec{p}, t) \end{aligned}$$

here we used another approximation.

Then write $\vec{p}$ in $\vec{v}$ using (4.2), and drop the average symbol (but remember Boltzmann is about average, not single particle), we get Boltzmann equation

$$\frac{\partial f(\vec{x}, \vec{p}, t)}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f(\vec{x}, \vec{p}, t) + \dot{\vec{v}} \cdot \nabla_{\vec{v}} f(\vec{x}, \vec{p}, t) = -\left(\frac{\partial f}{\partial t}\right)$$

the term on the right is called the collision term. This term cannot be written into the conservative Hamiltonian. This is the term randomize the system (e.g. Brownian motion), this term makes the system not deterministic. Please read Landau “Statistical Mechanics”, “Physical Kinetics” to learn more about the effect of collision in connection to the cross section in QM.

The general method to treat collision term is to consider two-body collision, three collide at the same time is very unlikely, because we only talk about collision due to short range forces. For long range forces, like Coulomb force, are dealt by

EM screening.

If we drop the collision term, we get the Vlasov equation

$$\frac{\partial f(\vec{x},\vec{p},t)}{\partial t} + \vec{v}\cdot\nabla_{\vec{x}} f(\vec{x},\vec{p},t) + \dot{\vec{v}}\cdot\nabla_{\vec{v}} f(\vec{x},\vec{p},t) = 0$$

This equation will preserve entropy (see PS3). Therefore it will not create any heat. Because macroscopic heat is due to collision, some high momentum particle collides along the way, transferring energy toward mean momentum of the system. Under this equation framework one will not give a method of describing system coming to equilibrium, because that will increase entropy.

Lecture 7 (2/13/13)

Although it has so many limitations, it is also very useful. It does many surprising things: describing small amplitude oscillations, equilibrium phenomena, plasma wave, Landau damping, exchange between plasma and em waves.

4.3 Permittivity from the Vlasov Equation

The discussion here is more general than before. Put a EM wave in plasma, as before we view charges as free electrons (recall $w_0 \to 0$) and as bound charges so as part of dielectric medium (recall $\gamma \to 0$). We don't assume $\vec{D} = \epsilon\vec{E}$, $\vec{D}$ linear to $\vec{E}$, instead we use $\vec{D} = \vec{E} + 4\pi\vec{P}$ or more generally

$$D_i(\vec{x},t) = E_i(\vec{x},t) + \int_{-\infty}^{\infty} dt' \int d^3x' K_{ij}(\vec{x}-\vec{x}';t-t')E_j(\vec{x}',t') \tag{4.6}$$

so there are spatial effects (one point affects another point) and time delay. Here K is translational invariant, which is due to the fact that we assume the medium is uniform. And causality should imply $K_{ij} = 0$ if $t' > t$. One can also assume $K_{ij} = 0$ if $\vec{x} - \vec{x}'$ is too big, i.e. no long range effect, but we don't need this assumption.

Whenever we have convolution, we should try Fourier. We use same symbol D, but they are two completely different functions

$$D_i(\vec{x},t) = \int \frac{d^3k dw}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x} - iwt} D_i(\vec{k},w)$$

Note: Just as before we will no longer take $k = w/c$, because we are in medium that may have dissipation.

Now Fourier (4.6) or stick in $\int$,

$$\int \frac{d^3kdw}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}-iwt} D_i(\vec{k}, w) = \int \frac{d^3kdw}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}-iwt} E_i(\vec{k}, w) + \int dt' \int d^3x' K_{ij}(\vec{x}-\vec{x}'; t-t') \int \frac{d^3kdw}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}'-iwt'} E_j(\vec{k}, w) \tag{4.7}$$

The last term is equal to

$$\int d(t-t') \int d^3(x-x') K_{ij}(\vec{x}-\vec{x}'; t-t') \int \frac{d^3kdw}{(2\pi)^2} e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')+iw(t-t')} E_j(\vec{k}, w) e^{i\vec{k}\cdot\vec{x}-iwt}$$

So by inverse Fourier, (4.7) gives

$$D_i(\vec{k}, w) = E_i(\vec{k}, w) + \int dtd^3x K_{ij}(\vec{x}; t) e^{i\vec{k}\cdot\vec{x}-iwt} E_j(\vec{k}, w) \tag{4.8}$$

After defining

$$\epsilon_{ij}(\vec{k}, w) = \int dtd^3x K_{ij}(\vec{x}; t) e^{i\vec{k}\cdot\vec{x}-iwt} + \delta_{ij} \tag{4.9}$$

we get

$$D_{ij}(\vec{k}, w) = \epsilon_{ij}(\vec{k}, w) E_j(\vec{k}, w)$$

This is the matrix form of non-relativistic plasma, where $\epsilon_{ij}(\vec{k}, w)$ is a 2nd rank tensor. ϵ has to satisfy the rotational symmetry, for the medium is uniform. This suggests to write ϵ in sum of transverse part ϵ_t and longitudinal part ϵ_l,

$$\epsilon_{ij}(\vec{k}, t) = \epsilon_t(|\vec{k}|, w)(\delta_{ij} - \frac{k_ik_j}{k^2}) + \epsilon_l(|\vec{k}|, w)(\frac{k_ik_j}{k^2})$$

This method is also used in relativity for Lorentz tensor.

We are going to interplay the two views of electrons:

(1) As bound charges, from 2 Maxwell equations with sources

$$\nabla \cdot \vec{E} + 4\pi \nabla \cdot \vec{P} = 0$$

$$\frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\frac{\partial \vec{P}}{\partial t}$$

(2) As free charge, $\vec{P} = \vec{d}/V = -q\vec{x}/V$,

$$\nabla \cdot \vec{P} = -\rho \tag{4.10}$$

$$\frac{\partial \vec{P}}{\partial t} = \vec{j}$$

We will also interplay the views of density of the electrons and probability distribution of charges in phase space, i.e. we will change ρ in (3.1) to

$$f(\vec{x}, \vec{p}, t) = f_0(\vec{p}) + \delta f(\vec{x}, \vec{p}, t) + f_{Background}$$

where δf is fluctuation since we perturbed the system.

So

$$\rho(\vec{x}, t) = -e \int d^3p f(\vec{x}, \vec{p}, t) = -e \int d^3p \delta f \tag{4.11}$$

$$\vec{j}(\vec{x}, t) = -e \int \frac{\vec{p}}{m} d^3p \delta f$$

Because plasma itself is neutral, f_B always cancels f_0, when we integrate all space. One can think f_B is due to protons.

Now use Vlasov equation

$$\frac{\partial f(\vec{x}, \vec{p}, t)}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f(\vec{x}, \vec{p}, t) + \dot{\vec{p}} \cdot \nabla_{\vec{p}} f(\vec{x}, \vec{p}, t) = 0 \tag{4.12}$$

where

$$\dot{\vec{p}} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

the force due to EM wave, we don't put collision force here, because we ignore collision, and in this way, we'll get Maxwell distribution.

Suppose $\vec{E}$ and $\vec{B}$ are weak, so $\vec{v}$ is small. Since $\nabla_{\vec{p}} f \sim \vec{v}$, then the product is even smaller, so we drop $\vec{B}$ term. Hence when field is weak, magnetic field plays no role, so (4.12) becomes

$$\frac{\partial}{\partial t}\delta f(\vec{x}, \vec{p}, t) + \vec{v} \cdot \nabla_{\vec{x}} \delta f(\vec{x}, \vec{p}, t) = e\vec{E} \cdot \nabla_{\vec{p}} f_0(\vec{p}) \tag{4.13}$$

We drop δf on the right, because product of E and δf is very small.

Here $\vec{E}$ can have t dependence. Now Fourier

$$\delta f(\vec{x},\vec{p},t) = \int \frac{d^3kdw}{(2\pi)^2}\delta f(\vec{k},\vec{p},t)e^{i\vec{k}\cdot\vec{x}-iwt}$$

So (4.13) gives

$$(-iw + i\vec{k}\cdot\vec{v})\delta f(\vec{k},\vec{p},t) = e\vec{E}(\vec{k},w)\cdot\nabla_{\vec{p}}f_0(\vec{p})$$

or

$$\delta f = \frac{e\vec{E}\cdot\nabla_{\vec{p}}f_0}{-iw + i\vec{k}\cdot\vec{v}} \tag{4.14}$$

The method given above is called linear response, because δf is now an linear function of f_0.

Now from (4.10), (4.11)

$$i4\pi\vec{k}\cdot\vec{P} = 4\pi e\int d^3p\delta f$$

substitute in δf, and use (4.8), (4.9), one gets

$$4\pi P_i = (\epsilon_{ij} - \delta_{ij})E_j \tag{4.15}$$

so

$$ik_i\left[\epsilon_t(\delta_{ij} - \frac{k_ik_j}{k^2}) + \epsilon_l\frac{k_ik_j}{k^2} - \delta_{ij}\right]E_j = \frac{4\pi}{i}e^2E_i\int d^3p\frac{\nabla_{p_i}f_0(\vec{p})}{\vec{k}\cdot\vec{v} - w} \tag{4.16}$$

Let's first find ϵ_l, so choose $\vec{k}\parallel\vec{E}$, i.e.

$$E_j = \frac{k_j}{|\vec{k}|}E$$

So

$$k_i\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)k_j = k_i^2 - \frac{k_i^2k_j^2}{k^2} = 0$$

(4.16) becomes

$$i(\epsilon_l|\vec{k}| - |\vec{k}|)E = \frac{4\pi}{i}e^2\frac{k_i}{|\vec{k}|}E\int d^3p\frac{\nabla_{p_i}f_0(\vec{p})}{\vec{k}\cdot\vec{v} - w}$$

That gives

$$\epsilon_l(\vec{k}, w) = 1 - \frac{4\pi e^2}{k^2} \int d^3p \frac{1}{\vec{k}\cdot\vec{v} - w} \vec{k}\cdot\nabla_{\vec{p}} f_0(\vec{p}) \tag{4.17}$$

So we get rid of E. The integral above will diverge, so we can put in a convergent factor

$$\int d^3p \frac{1}{\vec{k}\cdot\vec{v} - (w + i\eta)}$$

This convergent factor will actually come in for free if we had at the beginning considered causality in momentum space

$$K_{ij}(\vec{x} - \vec{x}', \vec{p} - \vec{p}', t)$$

Lecture 8 (2/18/13)

Let's recap what we have done. We take some plasma in equilibrium and put in a EM wave, then we suppose that plasma responds linearly by the Boltzmann dynamics (i.e. Vlasov equation). We look for self consistent solutions, namely the EM wave has both E and B parts, but B is much weak then E so we drop B. E produces δf then induces $\vec{j}$, then it in turns produces E. The self consistent solution will allow us to cancel E as we did in equation (4.17).

Later we'll show what the condition for wave to propagate is, and that leads to many applications, such as Boltzmann gas, see "Kinetic Theory" for more of these.

Now take the unperturbed distribution

$$f_0 = \frac{n_e e^{-\frac{\vec{p}^2}{2mT}}}{(2\pi m T)^{3/2}}$$

We will always assume T as Boltzmann temperature, i.e. $T = kT$. And it is a constant throughout out the plasma.

f_0 is normalized to be $\int d^3p f_0 = n_e$. Now plug in f_0 into (4.17). Assume

$\vec{k} = k\hat{z}$, we get

$$\begin{aligned}
\epsilon_l &= 1 - \frac{4\pi e^2 n_e}{k^2} \underbrace{\int dp_x dp_y \frac{e^{-\frac{p_x^2+p_y^2}{2mT}}}{(2\pi mT)}}_{=1} \int dp_z \frac{-\frac{1}{T} k v_z e^{-\frac{p_z^2}{2mT}}}{k(v_z - w/k)\sqrt{2\pi mT}} \\
&= 1 + \frac{K_D^2}{k^2} \int dp_z \frac{(v_z - w/k + w/k) e^{-\frac{p_z^2}{2mT}}}{(v_z - w/k)\sqrt{2\pi mT}} \\
&= 1 + \frac{K_D^2}{k^2} (1 + \frac{w}{k} \sqrt{\frac{m}{2\pi T}} \int dv_z \frac{e^{-\frac{mv_z^2}{2T}}}{v_z - \frac{w}{k} - i\eta})
\end{aligned}$$

Consider the special function

$$F(x) = \frac{x}{\sqrt{\pi}} \int_{-\infty}^{\infty} dz \frac{e^{-z^2}}{z - x - i\eta}$$

with the substitution $z^2 = \frac{mv_z^2}{2T}$, i.e. $dv_z = \sqrt{\frac{2T}{m}} dz = \sqrt{2} v_T dz$, where we define

$$v_T = \sqrt{\frac{T}{m}}$$

because from equal partition theorem in 1D $\frac{mv_T^2}{2} = \frac{T}{2}$.

If we further take $x = \frac{w}{k}\sqrt{\frac{m}{2T}} = \frac{w}{\sqrt{2} k v_T}$, so finally

$$\epsilon_l = 1 + (\frac{K_D}{k})^2 (1 + F(\frac{w}{\sqrt{2} k v_T}))$$

We now find ϵ_t. If one still uses (4.16) would find $\epsilon_t = 0$, that is because (4.16) was derived from Coulomb field which is intrinsically longitudinal. So now we have to do things differently. We look at radiation field produced by oscillating currents, which is intrinsically transverse.

$$\begin{aligned}
\vec{j} &= -e \int d^3p \vec{v} f(\vec{p}, t) \\
&= -e \int d^3p \vec{v} \delta f
\end{aligned}$$

since $f = f_0 + \delta f$ and f_0 is symmetric.

By (4.14),

$$RHS = -\frac{e^2}{i} v_i \int d^3p \frac{\vec{E} \cdot \nabla_{\vec{p}} f_0(\vec{p})}{\vec{k} \cdot \vec{v} - w}$$

and using (4.15)

$$LHS = \frac{d\vec{P}}{dt} = -iw\vec{P} = \frac{-iw}{4\pi} (\epsilon_{ij} - \delta_{ij}) E_j$$

Now take $\vec{k} = k\hat{z}$, and $\vec{E} = E\hat{x}$ (the exact choice doesn't matter), and write $(\epsilon_{ij} - \delta_{ij})E_j$ similar to (4.16) we get

$$\begin{aligned}
(\epsilon_t - 1)E &= -\frac{4\pi}{w} e^2 v_i \int d^3p \frac{\vec{E} \cdot \nabla_{\vec{p}} f_0(\vec{p})}{\vec{k} \cdot \vec{v} - w} \\
&= \frac{4\pi e^2}{wT} \int d^3p \frac{v_x^2 f_0}{kv_z - w - i\eta} E
\end{aligned}$$

So similar as before

$$\begin{aligned}
\epsilon_t &= 1 + \frac{4\pi e^2 n_e}{wT} \int dp_x \frac{v_x^2 e^{-\frac{p_x^2}{2mT}}}{\sqrt{2\pi mT}} \int dp_y \frac{e^{-\frac{p_y^2}{2mT}}}{\sqrt{2\pi mT}} \int dp_z \frac{\frac{e^{-\frac{p_z^2}{2mT}}}{\sqrt{2\pi mT}}}{kv_z - w - i\eta} \\
&= 1 + \frac{w_p^2 m}{wT} (\frac{T}{m})(1)(\frac{1}{w} F(\frac{w}{\sqrt{2}kv_T})) \\
&= 1 + (\frac{w_p}{w})^2 F(\frac{w}{\sqrt{2}kv_T})
\end{aligned}$$

here the evaluation of the x integration is using equal partition theorem $\langle mv_x^2/2 \rangle = T/2$, so $\langle v_x^2 \rangle = T/m$.

We now want to study the asymptotic limit of $F(x)$ function. Write $F(x)$ using principal value, recall for $\eta \to 0$

$$\frac{1}{x - i\eta} = P\frac{1}{x} + i\pi\delta(x)$$

that is because

$$\frac{1}{x - i\eta} = \frac{x}{x^2 + \eta^2} + i\frac{\eta}{x^2 + \eta^2}$$

where the second term is a representation of $\delta(x)$, since $\lim_{\eta\to 0}\int f(x)\frac{\eta}{x^2+\eta^2} = 0$ if $x \neq 0$.

$$F(x) = \frac{x}{\sqrt{\pi}} P \int dz \frac{e^{-z^2}}{z-x} + i\sqrt{\pi} x e^{-x^2}$$

Moreover for x large

$$\frac{x}{\sqrt{\pi}} P \int dz \frac{e^{-z^2}}{z-x} \quad = \quad -\frac{1}{\sqrt{\pi}} P \int_{-\infty}^{\infty} dz e^{-z^2} (1 + z/x + z^2/x^2 + z^3/x^3 + z^4/x^4 ...)$$

We neglect the odd terms, because they give 0. And use a standard trick

$$\begin{aligned} \int_{-\infty}^{\infty} dz e^{-z^2} z^{2n} &= (-1)^n \left.\frac{\partial^n}{\partial \lambda^n}\right|_{\lambda=1} \int_{-\infty}^{\infty} dz e^{-\lambda z^2} \\ &= (-1)^n \frac{\partial^n}{\partial \lambda^n} \frac{\sqrt{\pi}}{\sqrt{\lambda}} \\ &= \frac{(2n-1)!!}{2^n} \end{aligned}$$

Hence for x large

$$F(x) \sim -1 - \frac{1}{2x^2} - \frac{3}{4x^4} + ... + i\sqrt{\pi} x e^{-x^2}$$

4.4 Plasma Waves

Lecture 9 (2/20/13)

We are able to study plasma waves in its full generality. We are going to compare the results to our simple picture before, and we will see that for the transverse wave, there is essentially no change; while longitudinal waves are now significantly different. We will use the large x limit for $F(x)$, because $x = w/\sqrt{2} k v_T = c/\sqrt{\epsilon} v_T \gg 1$, i.e. we are in the non-relativistic situation: the speed of propagation is much larger than the speed of thermal vibration.

Transverse Waves

$$\begin{aligned} k^2c^2 &= \epsilon_t w^2 \\ &= w^2[1+\frac{w_p^2}{w^2}(-1-\frac{1}{2}\frac{2k^2v_T^2}{w^2})] \end{aligned}$$

We only keep the first term expansion in $F(x)$,

$$w^2 = w_p^2 + k^2c^2 + \frac{w_p^2k^2v_T^2}{w^2} \approx w_p^2 + k^2c^2 + \frac{w_p^2k^2v_T^2}{w_p^2+k^2c^2}$$

(i) $kc \ll w_p$,

$$w^2 = w_p^2 + k^2c^2(1+\frac{v_T^2}{c^2})$$

(ii) $kc \gg w_p$,

$$w^2 = k^2c^2 + w_p^2(1+\frac{v_T^2}{c^2})$$

Both extreme cases (i), (ii) tell us there is just a small correction to the simple picture

$$w^2 = w_p^2 + k^2c^2$$

so we can reasonably believe in the intermediate region it works too, so the simple picture is quite right, that is because transverse EM waves much alike in vacuum, where w_p doesn't know much detail of the medium (density, charge, mass). If we turn down the temperature to 0, $v_T \to 0$, then we back to simple picture.

Longitudinal Wave

In the simple picture there is no longitudinal EM waves, $\epsilon_l = 0$. Let us start small k, so $w/\sqrt{2}kv_T \gg 1$, then move k up. As k becomes very big, we will see that dissipation takes over the propagation, but at the boarder before the dissipation takes over, our discussion should be still valid.

Now in the expansion of $F(x)$ we need to keep the first 3 terms and the imaginary term. So

$$\begin{aligned} 0 &= 1+\frac{K_D^2}{k^2}(1-1-\frac{1}{2}\frac{2k^2v_T^2}{w^2}-\frac{3}{4}\frac{4k^4v_T^4}{w^4}+i\sqrt{\pi}\frac{w}{\sqrt{2}kv_T}e^{-\frac{w^2}{2k^2v_T^2}}) \\ &= 1-\frac{K_D^2}{w^2}v_T^2[1+3\frac{k^2v_T^2}{w^2}-i\sqrt{\pi}(\frac{w}{kv_T})^3e^{-\frac{w^2}{2k^2v_T^2}}] \end{aligned}$$

since $K_Dv_T = w_p$, approximately

$$w = w_p(1+\frac{3}{2}\frac{k^2v_T^2}{w^2}-i\sqrt{\frac{\pi}{8}}(\frac{w}{kv_T})^3e^{-\frac{w^2}{2k^2v_T^2}})$$

The minus is good, which is going to give damping.

So the real part of

$$w \approx w_p(1+\frac{3}{2}\frac{k^2v_T^2}{w^2})$$

that is a dispersion relation, so group velocity

$$v_g = \frac{\partial w}{\partial k} \approx \frac{3kv_T}{w_p} = 3\frac{k}{k_D}v_T$$

Compare to the dispersion relation for transverse wave, we see $v_g \approx c$ even when $w \approx w_p$.

Let $\gamma = -\mathrm{Im}w$, called Landau damping.

$$\gamma = w_p\sqrt{\frac{\pi}{8}}(\frac{w}{kv_T})^3e^{-\frac{w^2}{2k^2v_T^2}}$$

or after substituting $w^2 \approx w_p^2(1+3\frac{k^2v_T^2}{w_p^2})$

$$\gamma = w_p\sqrt{\frac{\pi}{8}}(\frac{k_D}{k})^3e^{-\frac{w_p^2}{2k^2v_T^2}-\frac{3}{2}}$$

We know for wave to propagate we require under damping, $\gamma \ll w$, hence $k_D \ll k$. note: in our calculation we assume k is small, but it turns out the condition for wave to propagate is the same $k_D \ll k$ even k is not so small. Landau damping is somehow unconventional. Normally damping go into heat and increase entropy, but here we started from Vlasov equation. That Vlasov equation gives damping is little bit surprising. The mechanism is that the creation of wave here is due to exchange of waves with the original motion of the particles in the medium, so the random thermal motion of particles gives formation of the wave. Hence part of particle motion is in the direction of the wave so they are not thermal energy, the motion is deterministic. If $k_D \sim k$, EM force acts on particle over extended period of time, wave loss energy to particle and eventually attenuates.

4.5 Problem Set 3 (due 2/26/13)

7)

Use the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f + q(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}) \cdot \nabla_{\vec{p}} f = 0$$

to show that the entropy

$$S = \int d^3x d^3p f(\vec{x}, \vec{p}, t) \ln \frac{e}{f}$$

is time independent in the absence of collisions.

In problems 8), 9) consider only the electrons and imagine the ions are fixed in space.

8)

Consider a slab of plasma $0 < z < L$, $-\infty < x, y < \infty$ in equilibrium. Put on an external electric field $\vec{E}(\vec{x}, t) = \hat{e}_z E_0$ and wait for the plasma to reach a new equilibrium. Suppose the new equilibrium distribution f is given as

$$f(\vec{x}, \vec{p}) = g(z) f_0(p)$$

where f_0 is the usual Maxwell-Boltzmann distribution. Evaluate g.

9)

Consider a plasma covering all of space $-\infty < x, y, z < \infty$ in equilibrium. Again turn on an external field $\vec{E}(\vec{x}, t) = \hat{e}_z E_0$. Now one expects a flow of electrons in a steady state limit. Use the Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f - e\vec{E} \cdot \nabla_{\vec{p}} f = -\frac{f - f_0}{\tau_c}$$

where τ_c can be taken as a constant $1/\tau_c = n_e^{(0)} \sigma \langle v_e \rangle$ to evaluate the induced current $\vec{j}$. What is the conductivity?

5 Ideal Fluids

Reference Landau Lifshitz "fluid" chapter 1, which is very old book. It has a lots words.

Fluid is made of particles all of same kind. Ideal means no dissipation, or viscosity is 0.

The dynamics are defectively simple. only a few equations: conversation of mass, conversation of energy,... We will look at a few examples: waves under gravity, fluid pass a solid. We define

$$\rho(\vec{x}, t) = \text{mass density of fluid} = m \cdot n(\vec{x}, t)$$

to study mass conservation in analogous to conservation of charges for electric currents. $n(\vec{x}, t)$ is number density and m mass of single molecule.

Let $\vec{v}(\vec{x}, t)$ is velocity field of fluid. There is one restriction to $\vec{v}$ is that the neighbor fluid element should have not too different velocities. If the neighbor elements moves in different directions, we are talking about high temperature gas. That is not what we want to study.

So conservation of mass gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

or

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = 0$$

Define convective derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$, then

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{v}$$

This gives the change of the density function along the motion of the fluid, because

$$\frac{d\rho}{dt} = \lim_{\Delta t \to 0} \frac{\rho(x(t+\Delta t), t+\Delta t) - \rho(x,t)}{\Delta t} = \vec{v} \cdot \nabla + \rho \frac{\partial \rho}{\partial t}$$

whiles partial derivative $\frac{\partial \rho}{\partial t}$ gives the change of the density function of some fixed point $\vec{x}$.

If the fluid is incompressible (for almost all applications we study), we should have $d\rho/dt = 0$ hence

$$\nabla \cdot \vec{v} = 0$$

And for most applications, we will consider the moving frame, namely moving with one fluid element. Note: even in the moving frame, the magnitude and the direction of the velocity vector are still respect to the rest frame, the only change is the position of the origin. However there are cases that we may want to use the usual rest frame, such as we do fluid pass solid problems.

5.1 Euler Equations

There three different Euler equations in fluid dynamics.

Consider a small sub volume V of the fluid, F total force exert on V by the fluid, the minus is due to $\vec{S}$ outward normal direction.

$$F = -\int d\vec{S} p = -\int_V d^3 x \nabla p$$

where pressure $p(\vec{x}, t)$ is not a vector, because it acts the same to all directions at particular $\vec{x}$ and t.

Let V be very small and apply Newton's to element V, in the moving frame.

(we don't have to worry about non-inertial forces, because all velocities are respect to the rest frame)

$$V\rho\frac{d\vec{v}}{dt} = -V\nabla p$$

or we get Euler equation of 1st form

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p \tag{5.1}$$

The right hand side is force per density. If there is gravity in the problem, we can add g to RHS.

(5.1) is a very complicated non-linear partial differential equations. For typical problem we shall find a region of estimation that allows us to drop the non-linear term $(\vec{v}\cdot\nabla)\vec{v}$.

Lecture 10 (2/25/13)

In the derivation last lecture Euler equation, we didn't assume that fluid was incompressible.

Use thermodynamics, we assume particles are in equilibrium (although in different region T may not be the same), entropy S is constant, or more close to what we are doing

$$\frac{S}{V\rho} = s(\vec{x}, t) = \text{entropy per unit mass or entropy per particle}$$

is constant, i.e.

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \vec{v}\cdot\nabla s = 0$$

Recall internal energy $U(S, V)$, and

$$dU = TdS - pdV$$

one can define enthalpy (work function) $W(S,p) = U + pV$, by Legendre transformation

$$dW = TdS + Vdp$$

Of course $dS = 0$ and all differentials are understood as convective derivatives.

Let

$$w = \frac{W}{\rho V} = \text{enthalpy/mass}$$

And

$$dw = \frac{dW}{\rho V}$$

here ρV comes out of differentiation because ρV along the motion of the particles is of course constant, that is why in fluid dynamics, we always talk about quantities /mass or /particle.

So

$$dw = \frac{dp}{\rho} \tag{5.2}$$

we now arrive the second form of Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla w$$

Use vector identity

$$\frac{1}{2}\nabla(\vec{v} \cdot \vec{v}) = \vec{v} \times (\nabla \times \vec{v}) + (\vec{v} \cdot \nabla)\vec{v}$$

one can check above by

$$(LHS)_i = \frac{1}{2}\nabla_i(v_j v_j) \; (RHS)_i = v_j \nabla_i v_j - v_j \nabla_j v_i + v_j \nabla_j v_i$$

Therefore we obtain the third form of Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\nabla v^2 - \vec{v} \times (\nabla \times \vec{v}) = -\nabla w \tag{5.3}$$

The three forms are equivalent, and each is good for some particular problems.

For example, incompressible fluid in a fixed box under the force of gravity. What is the equilibrium pressure?

clearly Euler form 1 is good. Since no fluid current

$$\vec{v} = 0$$

so Euler 1 gives

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

If we further assume ρ doesn't change (this is very good approximation. In ocean, pressure changes tremendously as go down deep into the ocean, but water density doesn't change much.)

Therefore we have

$$\begin{aligned} p &= -g\rho z + \text{const} \\ &= P_{\text{atmosphere}} - g\rho(z - z_0) \end{aligned}$$

5.2 Bernoulli's Equation

Suppose there is a steady flow. Suppose we pick a particle and paint it red, the path of the particle indicates a streamline. At each point in fluid define a tangent vector to a streamline as

$$\vec{e}(\vec{x}, t) = \frac{\vec{v}(\vec{x}, t)}{\|\vec{v}(\vec{x}, t)\|}$$

Take Euler 3 (5.3), if no gravity involved, $\partial\vec{v}/\partial t = 0$ because of steady flow, as soon as some particles leaving a position, some other particles will take the place, so streamline looks the same all time. Take dot product between Euler 3 and $\vec{e}(\vec{x}, t)$, and call $\vec{e} \cdot \nabla = \frac{\partial}{\partial l}$ the change along a streamline, we obtain

$$\frac{\partial}{\partial l}(\frac{1}{2}v^2 + w) = 0 \tag{5.4}$$

From (5.2), we notice that if particle moves from high pressure to low pressure, there is force acting on it, so kinetic energy goes up and w goes down. In short v goes slow in high pressure and goes fast in low pressure. We also notice that (5.4) says $\frac{1}{2}v^2 + w$ is constant along streamline, but $\frac{1}{2}v^2 + w$ may not be the same between different streamlines.

5.3 Potential Flow

Because Euler equation is non-linear, it takes outrageous amount of efforts to solve, but if it is potential flow, it becomes much easier.

Suppose $\nabla \times \vec{v}(\vec{x}, t) = 0$, called irrotational or potential flow, by stokes

$$\int d\vec{S} \cdot (\nabla \times \vec{v}) = \oint_C \vec{v} \cdot d\vec{x} = 0$$

showing there is no vortices.

Moreover we can then put ϕ such that

$$\vec{v}(\vec{x}, t) = \nabla\phi(\vec{x}, t)$$

of course ϕ is not unique, we can add any $u(t)$ to it.

Use Euler 3, we get

$$\nabla(\frac{\partial\phi}{\partial t} + \frac{1}{2}v^2 + w) = 0$$

Partly by choice, we have

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}v^2 + w = 0 \tag{5.5}$$

If steady current, we get Bernoulli, because $\frac{\partial\phi}{\partial t}$, v^2, w all have no t dependence, so

$$\frac{1}{2}v^2 + w = \text{const} \tag{5.6}$$

but now we have a stronger result than Bernoulli, namely (5.6) is true throughout the fluid, not just on a streamline, but here we have to assume irrotational, which is not required by Bernoulli.

In addition, if the fluid is incompressible

$$\nabla \cdot \vec{v} = 0$$

then we have

$$\nabla^2\phi = 0$$

5.4 Incompressible Fluid

Lecture 11 (2/27/13)

Incompressibility is property of fluid (of course it has its domain of validity e.g. pressure is not too big). Irrotational motion is not property of fluid, but more of a property of the type of motion we are looking at. Ideal fluid can have vortices but we will show it mainly comes from the initial setup. For viscous fluid, whether it has vortices depends on the initial setup as we will study later and the velocity of the sphere going through it for the particular problem we will solve.

People usually assume incompressibility is equivalent to that ρ constant in $\vec{x}$ and t. So from (5.2)

$$w = \frac{p}{\rho} + \text{const}$$

so (5.5) gives

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} = 0 \tag{5.7}$$

Example: Solid sphere moving at constant velocity $\vec{u} = u\hat{z}$ in ideal incompressible fluid. We suppose irrotational motion. Later we will redo it in viscous fluid.

Say we start the sphere at rest, then push it to get velocity v, then keep constant, during the process never produce vortices.

We are at the rest frame of fluid so all velocity vectors are with respect to the rest frame, but our origin of coordinate is moving with the ball. In doing so we will get all convective derivatives right.

And

$$\vec{v} = \nabla \phi(\vec{x})$$

where ϕ has no t dependence, because from the person sitting on the ball, the motion of the fluid is the same over time to time. Now we only have to solve Laplace equation

$$\nabla^2 \phi = 0$$

for $|\vec{x}| > R$, radius of the ball. There is no azimuthal dependence, so $\phi = \phi(r, \theta)$.

The solution should be built from

$$\frac{1}{r^{l+1}} P_l(\cos\theta) \qquad r^l P_l(\cos\theta)$$

Matching conditions:

At $r = R$,

$$\hat{n} \cdot \vec{u} = \hat{n} \cdot \vec{v} \tag{5.8}$$

showing neither can fluid penetrate nor running away from the sphere causing empty space. There is no condition on the tangential velocity at $r = R$ of the fluid. Later when we do viscous fluid, we will add to it.

Hence

$$\vec{n} \cdot \nabla\phi = \frac{\partial \phi}{\partial r}\bigg|_{r=R} = u\cos\theta$$

so only $l = 1$. And $v = 0$ for large r, so

$$\phi = \frac{c\cos\theta}{r^2}$$

Landau Lipschitz gave a more quick argument. They argued that ϕ must be linear in $\vec{u}$, then to make ϕ be a scalar, ϕ has to be

$$\phi = \frac{c\vec{u} \cdot \vec{n}}{r^2}$$

the r^2 is needed to make conservation of fluid around the region containing the ball.

Now find c,

$$\vec{v} = \nabla\phi = c\nabla\frac{\vec{u} \cdot \vec{x}}{r^3} = c(\frac{\vec{u}}{r^3} - \frac{3\vec{x}(\vec{u} \cdot \vec{x})}{r^5}) = c\frac{\vec{u} - 3\hat{n}(\hat{n} \cdot \vec{u})}{r^3}$$

By (5.8),

$$u\cos\theta = \hat{n} \cdot \vec{v} = c(\frac{u\cos\theta - 3u\cos\theta}{R^3}) \implies c = -\frac{R^3}{2}$$

Therefore

$$\vec{v} = \frac{R^3}{2r^3}[3\hat{n}(\hat{n} \cdot \vec{u}) - \vec{u}]$$

If we make a top down view of the fluid, we get velocity on the xz plane, $\varphi = 0$,

$$v_z = \hat{z} \cdot \vec{v} = \frac{R^3 u}{2r^3}(3\cos^2\theta - 1)$$

$$v_x = \frac{R^3 u}{2r^3}(3\cos\theta\sin\theta)$$

One can then draw a velocity field around the ball.

One may ask that we solved the whole problem without referring to the dynamics at all, nowhere in the solution we consider the force of the ball acting on the fluid and causes what kind of motion. Everything gets done by setting

$$\nabla \cdot \vec{v} = 0 \qquad \nabla \times \vec{v} = 0$$

and assuming some analyticity (smoothness of the solutions). This happens exactly in electrostatic problem $\nabla \cdot \vec{E} = 0$, $\nabla \times \vec{E} = 0$ where no charges.

5.5 Surface Gravity Wave

This gives applications to study tsunamis.

Suppose fluid fills the region $-\infty < x, y < \infty$, $-\infty < z < 0$. Look for small amplitude waves at the surface. We will see how it die out as z goes down. We'll look for waves with motion in x (propagation) and z (oscillation) directions. We suppose fluid is incompressible and ideal.

Claim: Small amplitude means no vortices can be created if no vortices to start with. From Euler 1

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g}$$

We can denote a as the amplitude of motion, τ typical time scale of motion, λ wavelength, assume $a \ll \lambda$, then

$$v \sim a/\tau \quad \nabla \sim 1/\lambda$$

so

$$(\vec{v} \cdot \nabla)\vec{v} \sim \frac{a}{\tau^2}\frac{a}{\lambda} \sim 0$$

so we drop the non-linear term, now

$$\frac{\partial}{\partial t}(\nabla \times \vec{v}) = -\frac{1}{\rho}\nabla \times \nabla p + \nabla \times \vec{g} = 0$$

Hence no vortices can be created. Later we will see this discussion cannot be applied to viscous fluid. In fact in viscous fluid vortices are commonly produced.

Lecture 12 (3/4/13)

From (5.7),

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} + gz = 0$$

We then drop v^2 term because it is small, now ϕ must have t dependence, otherwise we got $p = -\rho g z$ not interesting solution.

Let's look near boundary ($z \sim 0$). The boundary here is very subtle, and it's a moving boundary, not quite like anything we do in EM. We can safely assume wave has no y dependence, i.e. we look for a plane wave. At the boundary $p = p_0$ atmospheric pressure, change ϕ to $\phi - p_0/\rho t$, we get

$$\rho(\frac{\partial \phi}{\partial t} + gz) = 0$$

we don't cancel ρ, because ρ is constant in the water and 0 above the water, but ϕ doesn't change much cross the boundary.

Take t derivative again, we get

$$\rho(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial z}{\partial t}) = 0$$

we can approximate $\frac{\partial z}{\partial t} \approx \frac{dz}{dt} = v_z$ near the boundary, because v_z is small. Putting things together

$$\rho(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}) = 0 \tag{5.9}$$

We have no reason to believe this will also be true in the bulk, but it turns out to be a big surprise, this allows us to connect the solution with the solution in the bulk.

In the bulk, incompressible+irrotational gives

$$\nabla^2 \phi = 0$$

Look for solution

$$\phi = f(z)\cos(kx - wt)$$

(if one wants to form wave packages, then $e^{i(kx-wt)}$ should be used instead, but here cosine is enough to get the group velocity.) That $\nabla^2\phi = 0$ implies

$$(\frac{d^2}{dz^2} - k^2)f(z) = 0$$

Thus

$$\phi = Ae^{kz}\cos(kx - wt)$$

this gives $\phi \to 0$ and $z \to -\infty$, and we don't have to worry about ϕ above the water, where ρ is automatically 0.

Now matching boundary (5.9),

$$-w^2 + gk = 0 \implies w = \sqrt{kg}$$

we have phase velocity

$$v_p = w/k = \sqrt{g/k}$$

group velocity

$$v_g = dw/dk = \frac{1}{2}\sqrt{g/k}$$

Some tsunami stimulated from earthquake, forming a coherence disturbance. say its group velocity is 100 mile/hr, then

$$\frac{100\text{miles}}{60 \cdot 60} = \sqrt{\frac{g\lambda}{8\pi}} \implies \lambda = 4\text{miles}$$

To estimate the destructive landfall power from the tsunami, one can multiply the size of the wave front by v_g.

We now study what kind of motion the water at the boundary is.

$$\frac{dz}{dt} = v_z = \frac{\partial\phi}{\partial z} = Ake^{kz}\cos(kx - wt)$$
$$\frac{dx}{dt} = v_x = \frac{\partial\phi}{\partial x} = -Ake^{kz}\sin(kx - wt)$$

hence the point at the boundary oscillates as

$$\begin{cases} x = x_0 - \frac{Ak}{w}\cos(kx - wt) \\ z = -\frac{Ak}{w}\sin(kx - wt) \end{cases} \tag{5.10}$$

Hence for our discussion to be valid we need at least two things:

(1) the depth of the ocean is $\gg 1/k \sim \lambda$, so that $\phi \to 0$ as $z \to -\infty$. If the wave is approaching ashore, where the depth is not too big, we need to consider second boundary.

(2) for no vortices, we assume small amplitude wave, i.e.

$$\frac{Ak}{w} \ll \lambda \quad \text{or} \quad \frac{A}{w\lambda^2} \ll 1$$

(not $A \ll \lambda$ because (5.10) are truly the motion of the fluid)

5.6 Energy Flux and Momentum Tensor

Let's consider ideal fluid in its full generality, no assumptions on absence of vortices nor incompressibility.

Energy flux

2 kinds of energy: internal energy (e.g. molecule binding) including thermal energy (random motion, macroscopically are kinetic too, but macroscopically are not); kinetic energy of fluid (flows).

Define

$$\frac{U}{\rho V} = \epsilon = \text{internal energy/mass}$$

then

$$\frac{\text{total energy}}{\text{volume}} = \frac{1}{2}\rho v^2 + \rho\epsilon = e$$

Let's do the two separately

(1) Kinetic, use continuity of mass and Euler 1,

$$\begin{aligned}
\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2) &= \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\vec{v}\cdot\frac{\partial \vec{v}}{\partial t} \\
&= -\frac{1}{2}v^2\nabla\cdot(\rho\vec{v}) + \rho\vec{v}\cdot[-(\vec{v}\cdot\nabla)\vec{v} - \frac{1}{\rho}\nabla p]
\end{aligned}$$

Recall

$$\nabla w = T\nabla s + \frac{1}{\rho}\nabla p$$

so $\vec{v}\cdot\nabla p = \rho\vec{v}\cdot\nabla w - \rho T\vec{v}\cdot\nabla s$,

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2) = -\frac{1}{2}v^2\nabla\cdot(\rho\vec{v}) - \rho\vec{v}\cdot\nabla(\frac{1}{2}v^2 + w) + \rho T\vec{v}\cdot\nabla s$$

(2) Internal energy

Recall

$$d(\rho\epsilon) = \epsilon d\rho + \rho d\epsilon$$

so

$$d\epsilon = \frac{dU}{\rho V} = T\frac{dS}{\rho V} - \frac{p}{\rho}\frac{dV}{V} = Tds + \frac{p}{\rho^2}d\rho$$

because $d(\rho V) = 0 \implies \frac{dV}{V} = -\frac{d\rho}{\rho}$.

So

$$\frac{\partial \epsilon}{\partial t} = T\frac{\partial s}{\partial t} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial t}$$

so

$$\frac{\partial}{\partial t}(\rho\epsilon) = (\epsilon + \frac{p}{\rho})\frac{\partial \rho}{\partial t} + \rho T\frac{\partial s}{\partial t} = -w\nabla\cdot(\rho\vec{v}) + \rho T\frac{\partial s}{\partial t}$$

because $w = \epsilon + \frac{p}{\rho}$ and $\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{v})$.

Now add the two energies

$$\begin{aligned}
\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2 + \rho\epsilon) &= -(w + \frac{1}{2}v^2)\nabla\cdot(\rho\vec{v}) - \rho\vec{v}\cdot\nabla(\frac{1}{2}v^2 + w) + \rho T[\frac{\partial s}{\partial t} + \vec{v}\cdot\nabla s] \\
&= -\nabla\cdot[(\rho\vec{v})(w + \frac{1}{2}v^2)]
\end{aligned}$$

this gives familiar scheme: change of energy is divergent of something.

Lecture 13 (3/6/13)

We get conservation of energy in stokes form

$$\frac{\partial}{\partial t}\int_V (\frac{1}{2}\rho v^2 + \rho\epsilon)d^3x = -\int d\vec{S}\cdot[\rho\vec{v}(\epsilon + \frac{1}{2}v^2) + \vec{v}p] \tag{5.11}$$

the first term on the right is flux of energy in particle motion and internal energy; the second term is flux of energy due to work done by pressure.

Momentum Tensor

We are now going to use conservation of momentum. This will help us when we do viscous fluid and gives Navier stoke equation. By continuity and Euler,

$$\frac{\partial}{\partial t}(\rho v_i) = v_i\frac{\partial \rho}{\partial t} + \rho\frac{\partial v_i}{\partial t} = -v_i\nabla_j(\rho v_j) - \rho(v_j\nabla_j)v_i - \nabla_i p$$

Define momentum tensor

$$\Pi_{ij} = \rho v_i v_j + \delta_{ij}p$$

(it is hard to get feeling for $\rho v_i v_j$, which is coming from the non-linear in Euler, and it ultimately comes from the fact that we have introduced convective derivation, which allows us to change our perspectives between velocity fields for fixed time and location and velocity vectors moving along the particles; the another term is clear, since ∇p is naturally the force.) Then we arrive

$$\frac{\partial}{\partial t}(\rho v_i) = -\nabla_j\cdot\Pi_{ij} \tag{5.12}$$

which is equivalent to Euler's equation.

So by Stokes

$$\frac{\partial}{\partial t}\int_V \rho v_i d^3x = -\int_S dS_j\Pi_{ij}$$

saying that the rate of change ith component of momentum is equal to the flow of the ith component of momentum through the surface element dS_j, so physically Π_{ij} is the flux of the ith component of momentum in the direction j, but since Π_{ij} is symmetric, we can equally say that it is the flux of the jth component of momentum in the direction i.

5.7 Problem Set 4 (due 3/12/13)

10)

Suppose one has an incompressible ideal fluid in the region $-\infty < z < 0$, $-\infty < x, y < \infty$. Now insert a spherical ball of radius R and mass M into the fluid with the center of the ball at $z = z_0$. Give the force on the ball due to the fluid.

11)

Now suppose that an ideal gas at temperature T fills the region $-\infty < z < 0$, $-\infty < x, y < \infty$.The equation of state of the gas is $p = nT$ with p the pressure and n the number density of the gas which is made up of particles (molecules) of mass m so that $\rho = mn$. Use Euler's equation to determine p as a function of z. Now insert a ball in the gas as in the previous problem. At what value of z_0 will there be no force on the ball due to the fluid? You may assume $mgR/T \ll 1$. Also suppose $4\pi R^3 m p_0/3T \ll M$.

12)

Recall the problem of a spherical ball moving at constant velocity thorough an ideal irrotational an incompressible fluid. What is the net force of the fluid on the ball?

6 Viscous Fluids

6.1 Viscosity and the Navier Stoke Equation

Our starting point is equation (5.12). We know some kinetic energy goes into thermal energy because of dissipation. However linear momentum is still conserved. Imaginary fluid at rest initially. Apply force on the surface, because of the viscosity or stickiness, it will pull some fluid underneath with it . And the these fluids will pull the fluids under it. So we shall see some transfer of velocities down the layers, i.e. velocity gradient. Over long time, applied force continue and keep the surface at $\vec{v}$, the discrepancy of velocity gradient will disappear, so all fluid move at $\vec{v}$. And no viscous force.

We have seen gradient in velocity is the key for viscosity to have effects, and we also expect to write equation similar to (5.12). So naturally we try to add $\nabla_j v_i$ term to our original Π_{ij}

Two possibilities: symmetric $\nabla_i v_j + \nabla_j v_i$ or antisymmetric $\nabla_i v_j - \nabla_j v_i$. We can rule out the antisymmetric term by the following argument.

Suppose the fluid is concentrically rotating the z axis, $\vec{w} = w\hat{z}$

$$\vec{v} = \vec{w} \times \vec{x} = w(-y\hat{x} + x\hat{y})$$

There are certainly velocity gradient, but since the whole fluid moves stably, we don't expect to have viscous force between fluid at different radius, (although there are inward pressure in the fluid due to centrifugal pressure, but that is not viscous force.) So we find

$$\nabla_i v_j + \nabla_j v_i = 0$$

but

$$\nabla_i v_j - \nabla_j v_i = -2w \neq 0$$

Try putting

$$\Pi_{ij} = \rho v_i v_j + p\delta_{ij} - \sigma'_{ij}$$

where

$$\sigma'_{ij} = \eta[\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}\nabla_k v_k] + \xi\delta_{ij}\nabla_k v_k$$

is called viscous stress tensor, η is shear viscosity (or dynamic viscosity) due to different velocity gradients; ξ bulk viscosity due to compress fluids.

One often also write

$$\Pi_{ij} = \rho v_i v_j - \sigma_{ij}$$

where $\sigma_{ij} = \sigma'_{ij} - p\delta_{ij}$, and σ is called stress tensor.

So we arrive an equation of the dynamics

$$\frac{\partial}{\partial t}(\rho v_i) = -\nabla_j \Pi_{ij}$$

this works for velocity gradient is not too big, which is a fundamental assumption of all fluid dynamics. We will also assume η, ξ constant. Overall we have added more force to Euler

$$\rho[\frac{\partial v_i}{\partial t} + v_j \nabla_j v_i] = -\nabla_i p + \eta[\nabla_i \nabla_j v_j + \nabla^2 v_i - \frac{2}{3}\nabla_i \nabla_j v_j] + \xi \nabla_i \nabla_j v_j$$

Now write in vector notation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \vec{v} + \frac{1}{\rho}(\xi + \frac{\eta}{3})\nabla(\nabla \cdot \vec{v})$$

Assuming incompressibility, we get Navier-Stokes

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \vec{v}$$

also set $\eta/\rho = \nu$, η dynamic viscosity; ν kinetic viscosity.

If we don't care about non-linear term, it looks a lot like diffusion equation.

$$\dim \nu = \dim \eta/\rho = \frac{(\text{length})^2}{\text{time}}$$

$$\dim \eta = \frac{\text{mass}}{\text{length} \cdot \text{time}}$$

Those dimensions will be important for us later when we discuss Reynolds numbers.

Boundary Condition for N-S

Lecture 14 (3/11/13)

Definition of mathematical fluid at boundary where fluid cannot penetrate, $v_{tanget} = 0$. Fluid along the surface of the wall is constantly receiving frictional forces to slow it down. Microscopically continuous attempt to stop the fluid molecule and recall Π_{ij} is interpreted as the flux of the ith component of momentum in direction j. Suppose the boundary wall is normal i direction, then the force perpendicular to the wall should be

$$p_i = \Pi_{ij}\hat{n}_j = \rho v_i v_j \hat{n}_j + p\delta_{ij}\hat{n}_j - \sigma'_{ij}\hat{n}_j \tag{6.1}$$

the first term on the right is 0 because $v_i = 0$, and the second term is p_i, so we naturally require the third term to be 0, hence at the boundary $v_{tang} = 0$, because

$$\frac{\partial v_{tangent}}{\partial n_{normal\,to\,wall}} = F_{normal} = 0$$

we don't want to have additional normal force from the stress tensor. If the boundary is moving too, when the fluid should have the same velocity as the boundary at the boundary.

6.2 Dissipation of Energy

In ideal fluid, energy transfer between kinetic and potential are reversible, but it is not the case for viscous fluids or at least at the macroscopic level.

Here we focus on incompressible fluid. If the fluid is compressible, then we will take ξ into account. For compressible fluid, pressure will heat up the fluid and dissipation will cool it down.

$$E_{kin}(V) = \int_V d^3x \frac{1}{2}\rho\vec{v}^2 = \text{kinectic energy in volume } V$$

$$\frac{d}{dt}E_{kin} = \int d^3x \frac{\partial}{\partial t}(\frac{1}{2}\rho v^2)$$

$$\begin{aligned}
\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2) &= \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho v_j \frac{\partial}{\partial t}v_j \\
&= -\frac{1}{2}v^2\nabla_i(\rho v_i) + \rho v_j[-(v_i\nabla_i)v_j - \frac{1}{\rho}\nabla_j p + \frac{1}{\rho}\nabla_j\sigma'_{ij}] \\
&= -\frac{1}{2}v^2\nabla_i(\rho v_i) - \frac{\rho}{2}(v_i\nabla_i)v_j^2 - v_i\nabla_i p + \nabla_i(v_j\sigma'_{ij}) - \sigma'_{ij}\nabla_i v_j \\
&= -\nabla_i[(\rho v_i)(\frac{1}{2}v^2 + \frac{p}{\rho}) - \sigma'_{ij}v_j] - \sigma'_{ij}\nabla_i v_j
\end{aligned}$$

The last term (for incompressible fluid)

$$\sigma'_{ij}\nabla_i v_j = -\frac{\eta}{2}(\nabla_i v_j + \nabla_j v_i)^2$$

is the dissipation term, because it is not the divergent of something, the process is not reversible. More precisely,

$$\frac{dE_{kin}}{dt} = -\frac{\eta}{2}\int_{\text{all space}} d^3x(\nabla_i v_j + \nabla_j v_i)^2$$

the minus manifests energy lost, and if $\eta = 0$, no energy lost and we are back to equation (5.11). But before when we do energy flux for ideal fluids, we consider kinetic energy and internal energy. The reason we didn't do it that way here, because energy is always conserved, if we consider both terms, we would not see the dissipation (non-reversible) of kinetic energy into internal energy (heat).

6.3 Three simple examples

All are incompressible, and the last example is classical. Follow Landau notation: v velocity of fluid, u velocity of other thing. All three use Navier-Stokes and all three have $\frac{\partial \vec{v}}{\partial t} = 0$ (steady states), and all neglect non-linear terms for symmetry reasons.

1)

A fixed ∞ plate is placed at xz plane. 2nd ∞ plate at $y = h$ moves at constant $\vec{u} = u\hat{x}$. Find $\vec{v}$ of viscous fluid between plates and force per unit area of fluid on plates. No applied pressure.

Solve: we argue $v_z = 0$ because of symmetry in z direction, and $v_y = 0$ because the problem should be the same if $y = h$ hold fixed and $y = 0$ plate moves with $-\vec{u}$. Therefore the non-linear term is

$$(v_x \frac{\partial}{\partial x})v_x = 0$$

v_x cannot depend on x. Hence we get

$$\nabla p = \eta \nabla^2 \vec{v}$$

And by symmetry, it is also clear that p depends only on y. So

$$\frac{dp}{dy} = \eta \nabla^2 v_y = 0 \implies p \text{ is constant}$$

So

$$\nabla^2 v_x = 0$$

or

$$\frac{dv_x^2}{dy^2} = 0$$

with boundary conditions $v_x(y = 0) = 0$ and $v_x(h) = u$.

So we get

$$v_x = y\frac{u}{h}$$

Now find force. By symmetry, force on the two plates should be the same. Let's calculate force on plate at $y = h$. The normal direction is clearly p. and there is also drag force in x direction (cf equation (6.1))

$$p_x = -\sigma'_{xy} = -\eta\frac{u}{h}$$

because all other $\sigma'_{xx} = \sigma'_{xz} = 0$.

2)

Two plates fixed, one at $y = 0$ and the other at $y = h$. We push fluid between the plates in x direction so as reach a steady state velocity. Find $\vec{v}$ and force on plates.

Solve: As before, $\vec{v}$ has only v_x and only depends on y, so non-linear term drop. We expect p is linear in x, just like pressure in fluid under gravity (we don't consider gravity in this problem) is linear in z,

So

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\eta}\frac{dp}{dx} \implies v_x = -\frac{1}{2\eta}\frac{dp}{dx}(h - y)y$$

Force on plate $y = h$, normal direction is p, in x direction is $\sigma'_{xy} = \eta\frac{dv_x}{dy} = \frac{h}{2}\frac{dp}{dx}$.

3)

Lecture 15 (3/13/13)

Flow through a long l pipe at constant velocity. A pressure p_0 is exerted at one end $z = 0$ of the pipe. Calculate velocity profile and the rate of fluid flow. Total force see problem set 6 problem 16.

Solve: By symmetry $\vec{v} = v(r)\hat{z}$ long pipe no z dependence, so non-linear term $= 0$. Hence

$$\frac{\partial p}{\partial x} = 0 \text{ and } \frac{\partial p}{\partial y} = 0$$

so $p = p(z)$, and we assume p is linear in z, i.e.

$$p = \frac{p_0(l - z)}{l}$$

then

$$\frac{d}{dz}p(z) = \eta \nabla^2 v(r)$$

so

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}v = -\frac{p_0}{l\eta} \implies r\frac{dv}{dr} = -\frac{p_0}{2l\eta}r^2 + a$$

then

$$v = -\frac{p_0}{4l\eta}r^2 + a\ln r + b$$

take $a = 0$ to avoid singularity at $r = 0$. That $v = 0$ at $r = R$ gets

$$v(r) = \frac{p_0}{4l\eta}(R^2 - r^2)$$

If $\eta \to 0$, we have to have $p_0 \to 0$, so ideal fluid needs no pressure.

One can also calculate Q mass discharge, rate at which mass flow through the pipe

$$Q = \int 2\pi r dr \rho v(r) = \frac{p_0 \pi \rho}{4l\eta}\int_0^{R^2} d(r^2)(R^2 - r^2) = \frac{p_0\pi}{8l\nu}\rho R^4$$

This is the poiseuille's formula

6.4 Problem Set 5 (due 4/9/13)

13)

Consider the Navier-Stokes for an incompressible viscous fluid. Show for flow where $\vec{v} = \nabla\phi$ the N-S equation reduces to the Euler equation. Now show that this means that $\nabla \times \vec{v} \neq 0$ in order to have viscous effects.

14)

We found $\vec{v}$ for the flow of an incompressible fluid between the two plates. Evaluate $\nabla \times \vec{v}$. Can you give a closed path, for which $\oint \vec{v} \cdot d\vec{x} \neq 0$.

15)

Write $\sigma'_{ij} = \eta(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_l}{\partial x_l})$ in spherical coordinates. Evaluate σ'_{ab} where $a, b = r, \theta, \phi$ for the situation where there is no ϕ dependence and v has no ϕ component. Your answer should be in terms of r and θ derivatives of v_r and v_θ.

6.5 Reynolds Number and Similarity

We will show in fluid dynamics problem there is a dimensionless quantity called Reynolds number, R, that sets the scale so that if R is very big, no analytic solution to Navier-Stoke. And the method of perturbation breaks down and we will get turbulence flow. Later we'll study Kolmogorov theory, where no use of dynamics only similarity (rescaling).

Imagine we have an infinite amount of incompressible viscous fluid. Suppose some solid body of some fixed shape and with general size l, presuming the variation of the shape is much smaller than l. Let it move at velocity u in the medium, then we define the Reynolds number

$$R = \frac{lu\rho}{\eta} = \frac{lu}{\nu}$$

(here we define Reynolds number for incompressible fluid, if one wants do it for compressible fluid, then there is similar R which will incorporate bulk viscous into a new dimensionless quantity.)

First let's check R is dimensionless, i.e. we know from before

$$\text{dim}\nu = \frac{(\text{length})^2}{\text{time}}$$

Rule of thumb

At small R flow is regular (stable); at large R, flow is turbulence (chaotic) so the deterministic equation of Navier-Stoke become un useful, because of the non-linear term is not negligible

Similarity

For steady state

$$\vec{v}(\vec{x}) = u\vec{f}(\frac{\vec{x}}{l}, R_{rey})$$

$$p(\vec{x}) = \rho u^2 g(\frac{\vec{x}}{l}, R_{rey})$$

where $\vec{f}$, g are dimensionless parts. In doing so, we can the exact the same problem at different scale.

6.6 Stokes Formula

Einstein used it in his problem of Brownian motion, we'll study later when we discuss diffusion.

Consider a solid ball of radius R moving at constant velocity $\vec{u} = u\hat{z}$ through a viscous incompressible fluid. (We will see if assume the fluid is irrotational, then the solution will be almost the same as in the ideal fluid case, so let's not assume that.)

First notice $\frac{\partial v}{\partial t} = 0$ for steady state, now let's estimate the size of non-linear term

$$\frac{(\vec{v} \cdot \nabla)\vec{v}}{\nu \nabla^2 \vec{v}} \sim \frac{ul}{\nu} = R_{rey}$$

Hence if $R_{rey} \ll 1$, we will forget the non-linear term, and after we obtain the solution, we will come back and check R_{rey} is indeed small. so the perturbation is right.

$$\nabla p = \eta \nabla^2 \vec{v}$$

Apply $\nabla\times$ to above, we get

$$\nabla^2(\nabla \times \vec{v}) = 0 \tag{6.2}$$

and with

$$\nabla \cdot \vec{v} = 0 \tag{6.3}$$

we are going to solve the problem based on the two above equations.

One way to solve them is to use vector spherical harmonics, which is too much for us. We will rather follow a more physical approach, as in Landau Lifshitz.

From (6.3),

$$\vec{v} = \nabla \times \vec{A}(\vec{x})$$

We claim

$$\begin{aligned} \vec{A} &= (\nabla f(r)) \times \vec{u} \qquad (6.4) \\ &= \nabla \times f\vec{u} \text{ because } \vec{u} \text{ is constant} \end{aligned}$$

then plugging into (6.2), we get

$$\nabla^2\{\nabla \times [\nabla \times (\nabla \times f\vec{u})]\} = 0 \tag{6.5}$$

which is not much harder then ideal fluid, and next lecture we will solve it for f.

Now let's show the claim. The argument is quite tricky. We notice $\vec{A}$ is an axial vector, because both $\vec{v}$ and ∇ are odd under coordinate parity. So $\vec{A}$ is even under $x \to -x$.

More generally at any instant in time take a moving coordinate system with origin at the center of the ball and change $\vec{u} \to -\vec{u}$, $\vec{x} \to -\vec{x}$, the new solution should be the same as the old solution with

$$\vec{v}(\vec{x}, t) \to -\vec{v}(-\vec{x}, t)$$

We also expect $\vec{v}$ is linear in $\vec{u}$ so $\vec{A}$ must be linear in $\vec{u}$ too. So

$$\vec{A} = f'(r)\hat{n} \times \vec{u}$$

where the derivative f' is with respect to the moving coordinate, and $\hat{n} = \vec{x}/r$, so (6.4) gives all the right properties of $\vec{A}$.

Lecture 16 (4/1/13)

Now solve for (6.5).

$$\nabla^2 \nabla \times [\nabla(\vec{u} \cdot \nabla f) - \vec{u}\nabla^2 f] = 0$$

So

$$\vec{u} \times [(\nabla^2)^2 \nabla f] = 0 \implies$$

$$\nabla[(\nabla^2)^2 f] = 0$$

so

$$(\nabla^2)^2 f = \text{const}$$

and we shall choose the constant to be 0, because otherwise v won't go to 0 at $r \to \infty$.

So

$$\frac{1}{r^2}\frac{d}{dt}r^2\frac{d}{dr}(\nabla^2 f) = 0 \implies r^2\frac{d}{dr}\nabla^2 f = -2a$$

for some constant a.

$$\nabla^2 f = \frac{2a}{r}$$

then

$$r^2\frac{d}{dr}f = ar^2 - b$$

for some constant b. Then

$$f = ar + \frac{b}{r} + c$$

where $c = 0$ by choice. So

$$\nabla f = a\frac{\vec{x}}{r} - b\frac{\vec{x}}{r^3} \quad \nabla^2 f = \frac{2a}{r}$$

$$\begin{aligned}
\vec{v} &= \nabla(\vec{u}\cdot\nabla f) - \vec{u}\nabla^2 f \\
&= \nabla[a\frac{\vec{x}\cdot\vec{u}}{r} - b\frac{\vec{x}\cdot\vec{u}}{r^3}] - \frac{2a}{r}\vec{u} \\
&= -a\frac{\vec{u}}{r} - a\frac{\vec{n}(\vec{u}\cdot\vec{n})}{r} - b\frac{\vec{u}}{r^3} + b\frac{3\vec{n}(\vec{u}\cdot\vec{n})}{r^3} \\
&= -\frac{a}{r}[\vec{n}(\vec{u}\cdot\vec{n}) + \vec{u}] + \frac{b}{r^3}[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}]
\end{aligned}$$

Matching conditions $r = R$, $\vec{v} = \vec{u}$ (namely at the boundary fluid moves at the 0 velocity with respect to the ball) for all $\hat{n}$

$$\begin{cases} -\frac{a}{R} - \frac{b}{R^3} = 1 \\ -\frac{a}{R} + \frac{3b}{R^3} = 0 \end{cases} \implies \begin{cases} a = -\frac{3R}{4} \\ b = -\frac{R^3}{4} \end{cases}$$

That is

$$\vec{v} = \frac{3R}{4r}[\vec{n}(\vec{u}\cdot\vec{n}) + \vec{u}] - \frac{R^3}{4r^3}[3\vec{n}(\vec{u}\cdot\vec{n}) - \vec{u}]$$

The second part is similar to ideal fluid except the minus.

Now we want to find the drag force, which was a famous problem in 19 century calculated be Stokes.

Recall in ideal fluid, no net force for ball to move, here we need to put in energy at certain rate and the rate is exactly equal to the rate at what energy dissipates. The dissipation doesn't happen right at the surface, because there is no relative motion. The dissipation is in the gradient of velocity.

By symmetry, net force will be on z direction

$$\frac{dF_i}{dA} = -pn_i + \sigma'_{ij}n_j$$

and in spherical $\sigma'_{ij} = \sigma'_{rr}$, $\sigma'_{r\theta}$, $\sigma'_{r\phi} = 0$ (no ϕ dependence)

$$F_z = \iint dA(-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)$$

That the sine and cosine added in the integral is to project the force to z direction.

Find p,

$$\begin{aligned}\nabla p &= \eta\nabla^2\vec{v}\\ &= \eta\nabla^2[\nabla(\vec{u}\cdot\nabla f)-\vec{u}\nabla^2 f]\\ &= \eta\nabla^2[\nabla(\vec{u}\cdot\nabla f)]\text{ second term doesn't contribute }(\nabla^2)^2 f=0\\ &= \eta\nabla[(\vec{u}\cdot\nabla)\nabla^2 f]\end{aligned}$$

So

$$\begin{aligned}p &= \eta(\vec{u}\cdot\nabla)\nabla^2 f+p_0\\ &= \frac{3}{2}\eta\frac{\vec{u}\cdot\vec{n}}{r^2}R\end{aligned}$$

leave out p_0 because it doesn't contribute to the integration.

Find σ', we get v in spherical

$$v_r=\vec{n}\cdot\vec{v}=\frac{Ru}{2r}\cos\theta(3-\frac{R^2}{r^2})$$

$$v_\theta=-\sin\theta\frac{Ru}{4r}(3+\frac{R^2}{r^2})$$

Now calculate (see problem set 5 (15))

$$\sigma'_{rr}=2\eta\frac{\partial v_r}{\partial r}=0$$

$$\sigma'_{r\theta}=\eta(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial\theta}{\partial r}-\frac{1}{r}v_\theta)=\frac{3u\eta\sin\theta}{2R}$$

So

$$F_z=\int_0^\pi\int_0^{2\pi}(R\sin\theta d\phi)Rd\theta(-\frac{3}{2}\eta\frac{u\cos^2\theta}{R}+0-\frac{3u\eta\sin^2\theta}{2R})=-6\pi Ru\eta$$

which is called Stokes formula. We will use this in discussion of diffusion mobility. That it is linear in u and η is natural, but one may expect it is quadratic in R, since drag force may depend on cross section.

6.7 Laminar Flow

Lecture 17 (4/3/13)

Incompressible viscous fluid, put in a rigid thin half plate $x > 0$ at the $y = 0$. Suppose a fluid over all space, and fluid moves mainly in the x direction, and $\vec{v}$ is independent of z, and $\vec{v}(x, y) \to u_0 \hat{x}$ at large y as if there were no plate, and $\vec{v} = 0$ when $y = 0$.

For large Reynolds number, we get to consider non-linear term in the Navier-Stoke and it results many may solutions. And the situation is chaotic. We have no hope to solve the problem. But instead we will get a general scaling law.

Equations are

$$v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_x = -\frac{1}{\rho}\frac{\partial}{\partial x} p + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_x \tag{6.6}$$

$$v_x \frac{\partial}{\partial x} v_y + v_y \frac{\partial}{\partial y} v_y = -\frac{1}{\rho}\frac{\partial}{\partial y} p + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_y \tag{6.7}$$

$$\frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y = 0 \tag{6.8}$$

We look at around some not too small value of x_0 because we don't want to go to $x \to 0$ region. It's too hard. We suppose v_y small compare to v_x. We won't be able to get the most general solution anyway, so we drop y equation (6.7). We will see this dropping of y agrees with our solution for large Reynolds number.

(6.8) suggests that y dependence is stronger than x dependence, so drop $\frac{\partial^2}{\partial x^2} v_x$ in (6.6), so we end up

$$\begin{cases} v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_x - \nu \frac{\partial^2}{\partial y^2} v_x = -\frac{1}{\rho}\frac{\partial}{\partial x} p \\ \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y = 0 \end{cases} \tag{6.9}$$

To eliminate p, we use Bernoulli equation

$$\frac{1}{2}\rho u^2 + p = \text{const} \tag{6.10}$$

where u is velocity on streamline, and this only works for y not too small, because y is small viscous effect is big, and Bernoulli actually assume no viscosity. So

(6.10) gives

$$u\frac{du}{dx} = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

Look around x_0, call x_0 to be the scale size l of the problem, that is important, because unlike for the spherical go through viscous fluid there was a nature size, i.e. the radius of the ball. Here we have to find a scale, which complicates the problem, then

$$R_{rey} = \frac{u_0 l}{\nu}$$

Now do rescaling $u = u_0 u'$,

$$x = lx' \quad v_x = u_0 v'_x \tag{6.11}$$

$$y = ly'/\sqrt{R} \quad v_y = u_0 v'_y/\sqrt{R} \tag{6.12}$$

so that the prime terms are dimensionless. In addition, because $v'_x \sim v'_y \sim 1$, R_{rey} large means $v_x \gg v_y$. That agrees with our assumption.

Now rescale (6.9),

$$\begin{cases} \frac{u_0^2}{l}(v'_x\frac{\partial}{\partial x'}v'_x + v'_y\frac{\partial}{\partial y'}v'_x) - \nu\frac{u_0 R}{l^2}\frac{\partial^2}{\partial y'^2}v'_x = \frac{u_0^2}{l}u'\frac{du'}{dx'} \\ \frac{u}{l}(\frac{\partial}{\partial x'}v'_x + \frac{\partial}{\partial y'}v'_y) = 0 \end{cases}$$

simply

$$\begin{cases} v'_x\frac{\partial}{\partial x'}v'_x + v'_y\frac{\partial}{\partial y'}v'_x - \frac{\partial^2}{\partial y'^2}v'_x = u'\frac{du'}{dx'} \\ \frac{\partial}{\partial x'}v'_x + \frac{\partial}{\partial y'}v'_y = 0 \end{cases}$$

This ensures that our guess of R_{rey} and rescaling (6.11), (6.12) are right. The solution to above has no scale. For example as y goes from 0 to ∞, over the region $\delta y' \sim 1$, v'_x goes from 0 to 1.

The scale δy over which v_x goes from 0 to u_0 must be

$$\delta y \sim \frac{l}{\sqrt{R}} \sim \sqrt{\frac{\nu x_0}{u_0}} \tag{6.13}$$

Hence if x_0 or ν is large, we have to get to far away from the plate to see $\vec{v} \approx \vec{u}_0$, however if ν goes down, fix x_0, R_{rey} goes up, then δy becomes very small, but the solution is not valid at all and turbulence will be produced, because (6.13) is saying at very small y closed to the plate $\vec{v} \approx \vec{u}_0$, such rapid transition from $v = 0$ to u_0 won't occur in nature.

6.8 Problem Set 6 (due 4/18/13)

16)

We found $\vec{v}$ profile of fluid flow through a pipe. Evaluate the total force that the fluid exerts on the pipe.

17)

Consider a incompressible fluid flowing between two fixed half-plane plates $-\infty < z < \infty$, $x > 0$ placed at $y = 0$ and $y = h$. $\frac{dp}{dx}$ is constant. In class we found

$$v_x = -\frac{1}{2\eta}\frac{dp}{dx}[(\frac{h}{2})^2 - (\frac{h}{2} - y)^2]$$

for full plates. For sufficiently large x this should be the case here also. Estimate how large x must be for the above formula to apply.

18)

Suppose a fluid has a velocity profile $v_y(z)$ that is the fluid moves in the y-direction and the velocity only depends on z. Consider a small area element located at $x_0, y_0 z_0$ and parallel to the xy plane. Show that the viscosity is equal to

$$\eta = -\frac{\delta p_y}{\delta t \delta A}\frac{1}{\left.\frac{\partial v_y(z)}{\partial z}\right|_{z_0}}$$

where δp_y is the amount of y component of the momentum which flows downward through the area element in a time δt.

7 Sound Waves

We now look for irrotational motion in a compressible fluid, say a gas. We are looking for compressible waves, i.e. sound waves. This is the only time we do compressible fluid.

Consider

$$\rho = \rho_0 + \rho' \qquad p = p_0 + p'$$

where ρ_0, p_0 are constant in $\vec{x}$, t. $p_0 = NkT$ for ideal gas. Additionally, we assume

$$\frac{\rho'}{\rho_0}, \frac{p'}{p_0} \ll 1$$

so small amplitude waves, small perturbation.

The underlining dynamics is adiabatic, so Euler is valid, and non-linear term is dropped because speed of molecules motion is small compare to 1/air density, so

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho}\nabla p$$

we have continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Putting perturbation, we have

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho_0}\nabla p' = 0 \tag{7.1}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0 \tag{7.2}$$

Fortunately here $\nabla \cdot \vec{v} \neq 0$, otherwise we got ρ' constant, no an interesting solution.

There are 5 variables: $\vec{v}$, ρ', p' but only 4 equations, we need one more: equation of state $U(S, V)$

$$dU = TdS - pdV$$

so

$$p = \left(\frac{\partial U}{\partial V}\right)_S = p(S, V) = p(S, \rho)$$

last equality is because $\rho V =$ constnat, and so

$$dp = \left(\frac{\partial p}{\partial S}\right)_\rho dS + \left(\frac{\partial p}{\partial \rho}\right)_S d\rho$$

For adiabatic motion $dS = 0$,

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_S d\rho$$

or in our case

$$p' = \left(\frac{\partial p_0}{\partial \rho_0}\right)_S \rho'$$

For irrotational $\vec{v} = \nabla\phi$, we get from (7.1)

$$\nabla\frac{\partial \phi}{\partial t} + \frac{1}{\rho_0}\nabla p' = 0$$

so

$$\frac{\partial \phi}{\partial t} + \frac{1}{\rho_0}p' = 0$$

from (7.2),

$$\frac{\partial p'}{\partial t} + \left(\frac{\partial p_0}{\partial \rho_0}\right)_S \rho_0 \nabla^2\phi = 0$$

combine the two

$$\frac{\partial^2 \phi}{\partial t^2} - c^2\nabla^2\phi = 0$$

where $c^2 = \left(\frac{\partial p_0}{\partial \rho_0}\right)_S$

Lecture 18 (4/8/13)

We get an equation of wave. So let

$$\phi = \phi_0 e^{i(\vec{k}\cdot\vec{x} - wt)}$$

$kc = w$. where c is the phase velocity, not the v before. Clearly $v \ll c$ because

We know

$$\vec{v} = \nabla\phi = i\vec{k}\phi$$

use (7.2),

$$-iw\rho' + \rho_0 ikv = 0 \implies v = \frac{w}{k}\frac{\rho'}{\rho_0} = c\frac{\rho'}{\rho_0} \implies \frac{v}{c} = \frac{\rho'}{\rho_0} \ll 1$$

Evaluate c for ideal gas.

Assume air is made of one type of molecule Nitrogen. ($T = kT$) From $pV = NT$,

$$p = \rho\frac{T}{m}$$

Then

$$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{T}{m}$$

So

$$\begin{aligned}
c^2 &= \left(\frac{\partial p_0}{\partial \rho_0}\right)_S \\
&= \left(\frac{\partial p}{\partial \rho}\right)_S \quad \text{drop }_0\text{ because we look for states in equilibrium} \\
&= \frac{\partial(p,S)}{\partial(\rho,S)} = \frac{\partial(p,T)}{\partial(\rho,T)} \cdot \frac{\partial(p,S)}{\partial(p,T)} \cdot \frac{1}{\frac{\partial(\rho,S)}{\partial(\rho,T)}} \\
&= \frac{\partial(p,T)}{\partial(\rho,T)}\frac{C_p}{C_V} = \left(\frac{\partial p}{\partial \rho}\right)_T \frac{C_p}{C_V} = \frac{5}{3}\frac{T}{m}
\end{aligned}$$

People initially thought speed of sound would be lower at high temperature, since thermal motion disturb the wave motion, but that was a mistake.

8 Turbulence

We will describe turbulence from a phenomenological point of view, because the actual dynamics is too hard. Then we will do one calculation that is one of the few exactly solvable problems in turbulence. In the calculation we will see the condition for stability. Finally we will study Kolmogorov. Generally what essence of turbulence was, has to do with how energy distributes and dissipates. Looking into Kolmogorov equation, we find all numbers, no use any dynamics, just like in thermodynamics.

We will assume incompressibility.

8.1 Stability Analysis

When R_{rey} is big, non-linear term becomes important, and turbulence always have to do with non-linear term. Before technical analysis, let's do a classical discussion. Imagine we start a big wave by sticking big object into the fluid, and move the object drastically, that creates a big wave. Although the fluid looks like not in equilibrium, but it is in local equilibrium. The big shape motion has big R_{rey}, but the fluid is very uncomfortable, because dissipation happens only at atomic distance (relative motion of neighboring molecules). Big motion itself cannot effectively dissipates energy, so many many small eddies are created, and small eddies create smaller eddies. Kolmogorov found there were many modes where energy is stored.

How to know the system is stable or not? We can put in some perturbation, the solution is stable if perturbation converges, if perturbation grows, we have an unstable situation.

Suppose $\vec{v}_0(\vec{x}, t)$ satisfies Navier-Stokes with p_0, write

$$\vec{v}(\vec{x}, t) = \vec{v}_0(\vec{x}, t) + \vec{v}_1(\vec{x}, t)$$

$$p = p_0 + p_1$$

then we have

$$\frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_0 \cdot \nabla)\vec{v}_1 + (\vec{v}_1 \cdot \nabla)\vec{v}_0 = -\frac{1}{\rho}\nabla p_1 + \nu \nabla^2 \vec{v}_1 \tag{8.1}$$

we have dropped $(\vec{v}_1 \cdot \nabla)\vec{v}_1$ because it is second order small.

(8.1) looks like Schrodinger equation but no i here, so (8.1) is not hermitian, so we don't expect to have all eigenvalue to be real.

Look for solution

$$\vec{v}_1(\vec{x}, t) = A(t)\vec{v}_1(\vec{x}) \tag{8.2}$$

where

$$A(t) = ce^{-iw_1 t + \gamma_1 t} \tag{8.3}$$

(For Schrodinger equation, $A(t)$ would be $e^{-iEt/\hbar}$ and $E \in \mathbb{R}$, here "E" $= w_1 - i\gamma_1$ not assume to be real)

Claim: There exists a rough boarder line $R_{critical}$ such that if $R_{rey} < R_{critical}$, so R_{rey} is very small, γ_1 is always less than 0, hence the system is stable. And (8.2), (8.3) is the solution. R_{rey} small means l is small so small scale of motion, so no need to create smaller eddies. R small could also come from small u, so slow varying motion.

On the other hand, if $R_{rey} > R_{cr}$, γ_1 will be positive. Then we believe

$$\frac{d}{dt}\left|A\right|^2 = 2\gamma_1\left|A\right|^2 + \text{high order terms}$$

whose high order terms come in , because $\gamma_1 > 0$ causes $A(t)$ to grow exponentially, but physically there is no ∞ amount energy source, so all unstable growth will be cut off at some point. The cutoffs are from the high order terms. We are not going to analyze $R_{rey} > R_{cr}$, analytically.

We will suppose γ_1 is function of R_{rey}, phenomenologically say when R is close to but still larger than R_{cr},

$$\gamma_1(R) = \gamma_1(R_{cr}) + \gamma_1'(R_{cr})(R - R_{cr}) \tag{8.4}$$

and

$$\gamma_1(R_{cr}) = 0$$

The follows much like second order phase transition (see Landau)

$$\frac{d}{dt}A^*A = 2\gamma_1 A^*A + (\quad)(AA^*A^* + AAA^*) + (\quad)(AAA^*A^* + ...) \tag{8.5}$$

(In quantum field theory all Lagrangian terminates at some higher term, because of theoretical normalization imposition. But all other physical phenomenological areas: physical fluid, condense material, etc. There are always higher terms beyond A^*A.)

Now do time average of (8.5) over times large than the period$\sim 1/w_1$, $(AA^*A^* + AAA^*)$ gives 0, then we arrive the famous Landau order parameter, phenomenological expression for R is close to but still larger than R_{cr},

$$\frac{d}{dt}\left|A\right|^2 = 2\gamma_1 \left|\bar{A}\right|^2 - \alpha_1 \left|\bar{A}\right|^4 + ... \tag{8.6}$$

where $\alpha_1 > 0$ in all well-known cases and $\alpha_1 \sim \gamma_1'$. Then as $|A|$ starts to grow, $\left|\bar{A}\right|^4$ becomes more and more comparable to the first term, and after the two match, $|A|$ reaches its asymptomatic value. We can get the value by using (8.4), and (8.6),

$$2\gamma'(R - R_{cr})\left|\bar{A}\right|^2 \sim \gamma'\left|\bar{A}\right|^4 \implies \left|\bar{A}\right|_{t\gg 1} \sim \sqrt{R - R_{cr}}$$

Hence we have shown when $R_{rey} < R_{cr}$, the system is stable as there is one equilibrium point for the energy. If $R_{rey} > R_{cr}$, the system is unstable and equilibrium point itself will disintegrate into many different points (2 pints for second order transition.)

Many people have tried to apply knowledge of order parameters to turbulence, but none is quite successful, because the dynamics is too hard.

8.2 The Ansatz of Turbulence

Lecture 19 (4/10/13)

The case here is analogous to cyclotron radiation in EM, (see EM note page 68) when the charge particle moving at frequency w, the radiation field E have frequency w, $2w$, $3w$, ...

So long as $\{(w_1, \gamma_1)\}$ are the only unstable eigenvalue of NS, we can write

$$\vec{v}(\vec{x}, t) = \sum_{n_1=1}^{\infty} c_{n_1} e^{(-iw_1 t + \gamma)n_1} \vec{v}_{n_1}(\vec{x})$$

Now increase R_{rey} so we have N unstable eigenvalue of NS, i.e. N modes $\{(w_1, \gamma_2), ..., (w_N, \gamma_N)\}$, the solution is chaotic, they all come from NS, but

$$\vec{v}(\vec{x}, t) = \sum_{n_1} ... \sum_{n_N} c_{n_1} e^{\sum(-iw_1 t + \gamma)n_i} \vec{v}_{n_i}(\vec{x})$$

is too complicated so NS is hopeless. We say that the turbulence is fully developed.

8.3 An example

Before do fully developed turbulence, we will solve a motion of fluid exactly, we can see that the motion become unstable under what conditions.

Consider 2 concentric cylinders of radius R_1 and R_2 moving with angular velocities Ω_1, Ω_2. Put in an incompressible viscous fluid between them, we will not drop non-linear term, because they gives turbulence. But the problem is very hard to solve, one way is to do numerical analysis, but we do it via very cleaver dimensional analysis.

We run the system over long time, so reach steady state. There should be no z dependence because z is ∞ long. $\vec{v}$ should have no r component because over long time fluid cannot pile up. So in cylindrical coordinate

$$\vec{v} = \hat{\phi} v(r)$$

no z dependence,

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi},$$

$$\begin{aligned}
\nabla^2 &= (\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}) \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r}\frac{\partial}{\partial r}
\end{aligned}$$

because

$$\frac{\partial}{\partial \phi}\hat{r} = \hat{\phi} \qquad \frac{\partial}{\partial r}\hat{\phi} = \frac{\partial}{\partial r}\hat{r} = 0 \qquad \frac{\partial}{\partial \phi}\hat{\phi} = -\hat{r} \tag{8.7}$$

NS gives

$$(v\frac{\partial}{r\partial \phi})(v\hat{\phi}) = -\frac{\hat{r}}{\rho}\frac{dp}{dr} + \frac{\eta}{\rho}(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r}\frac{\partial}{\partial r})\hat{\phi} v$$

use (8.7) again, we simplify to

$$-\frac{v}{r}\hat{r}v = -\frac{\hat{r}}{\rho}\frac{dp}{dr} + \frac{\eta}{\rho}\hat{\phi}(\frac{d^2}{dr^2} - \frac{1}{r^2} + \frac{1}{r}\frac{d}{dr})v$$

So in $\hat{r}$ direction

$$\rho\frac{v^2}{r} = \frac{dp}{dr}$$

which is the centrifugal force

In $\hat{\phi}$ direction

$$(\frac{d^2}{dr^2}-\frac{1}{r^2}+\frac{1}{r}\frac{d}{dr})v=0 \implies v=ar+\frac{b}{r}$$

Match at $r=R_1,\ R_2$

$$aR_1+\frac{b}{R_1}=R_1\Omega_1 \quad aR_2+\frac{b}{R_2}=R_2\Omega_2$$

so

$$r\dot{\phi}=v(r)=\frac{\Omega_2R_2^2-\Omega_1R_1^2}{R_2^2-R_1^2}r+\frac{(\Omega_1-\Omega_2)(R_1R_2)^2}{(R_2^2-R_1^2)r}$$

Now test stability, usually people test stability by putting some perturbation, and to see if it converges, but this method is too sophisticated for us, we will use a physical argument instead.

Imagine move a small volume ΔV of fluid having mass $m=\rho\Delta V$ from its equilibrium position r_0 to r. The angular momentum of ΔV of fluid at r is

$$\delta L_z=\mu(r)=mr^2\dot{\phi}$$

In the original problem in order to have no radial velocity, the centrifugal force has to be balanced by the pressure, namely

$$\rho\frac{v^2}{r}=\frac{dp}{dr}$$

That is equivalent to

$$\frac{\mu^2(r)}{mr^3}=\Delta V\frac{dp}{dr} \tag{8.8}$$

If we can push ΔV radially out just a little bit without distributing the fluid too much, so our initial moving doesn't itself generate instability. To do that, we need to keep our force radially too, so the torque on ΔV due to us is 0, and if R_{rey} is big, i.e. viscosity is small, the neighboring viscous force from another fluid to ΔV is small too, we can reasonably believe that angular momentum is constant. (If R_{rey} is small the system is always stable, and when we push ΔV some neighbor will go with it, so it becomes a different problem.)

For now we know when ΔV reaches r, its angular momentum is still $\mu(r_0)$.

So the system is stable if centrifugal force < pressure at r, hence ΔV will be pushed back by the fluid. Stability condition is

$$\frac{\mu^2(r_0)}{mr^3} < \Delta V \frac{dp}{dr} \tag{8.9}$$

If centrifugal force > pressure at r, ΔV will run away radially.

We can combine (8.8), (8.9),

$$\frac{\mu^2(r) - \mu^2(r_0)}{mr^3} > 0$$

or

$$\frac{\mu^2(r) - \mu^2(r_0)}{r} > 0$$

Therefore we get

$$\mu \frac{d\mu}{dr} > 0 \tag{8.10}$$

for large R_{rey} or small η.

This result is better than (8.8), (8.9), because it takes case the case if we actually push ΔV inward initial instead of outward. $d\mu/dr$ will take care of $\pm$.

Lecture 20 (4/15/13)

Now we're back to the example.

For our example (8.10) says

$$\mu \frac{d\mu}{dr} = 2m^2 r^3 \dot{\phi} \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} > 0$$

for stability.

Discussion

(1) Suppose Ω_1,Ω_2 have opposite signs. But $\dot{\phi}$ varies continuously, so between R_1,R_2, $\dot{\phi}$ flips signs, so unstable.

(2) Suppose both Ω_1,Ω_2 postive. (or both negative) Now $\dot{\phi}$ must be positive too, so the system is stable if

$$\Omega_2 R_2^2 > \Omega_1 R_1^2$$

then intuitively if $\Omega_1 = 0$ and $\Omega_2 \neq 0$ stable; if $\Omega_1 \neq 0$ and $\Omega_2 = 0$ unstable.

8.4 Fully Developed Turbulence

Once upon a time when German put Russian scientists in the camp, and in the camp, Heisenberg discovered that 5 years before Kolmogorov, a mathematician, bravely solved turbulence, using general rule of scaling. Such method was highly controversial, since there was no mathematical rigorous, but agreed with experiments.

We have learned that when R_{rey} gets sufficiently big, many many modes produced. In the NS equation the non-linear term becomes very big, but all physical system has eventually become stable, so later linear term will stabilize the solution.

Suppose we create a motion in a fluid at rest. Let l be size of the motion (e.g. wave created by hand, l is size of hand) and u the velocity. We suppose R_{rey} is big, so energy cannot dissipate. But we know the fluid will eventually come to rest, because the fluid will do whatever it can to dissipate energy. The way it does is to create small eddy of size $\lambda_0 \ll l$ at which energy can dissipate.

So we can imagine the fluid first create eddies of size $l/2$ then these eddies break down to $l/4$, $l/8$ ... , if the size is still large than λ_0, it will continue break down, and before it gets to λ_0, we assume no energy dissipated, so the whole process before gets to order of λ_0 is adiabatic and at each stage the system is in steady state, moreover the flow of energy is constant through different sizes. These are the fundamental assumptions made by Kolmogorov, not what he was able to prove, but agree experiments.

The same idea is employed in quantum field theory, called renormalization group. People have tried to apply renormalization group concept to solve turbulence, but it turns out not to be useful.

Let's give more precise meaning of energy

$$\epsilon = \frac{\text{flow of energy}}{\text{mass}} = \frac{\text{energy}}{\text{mass} \cdot \text{time}} = \frac{(\Delta u)^2}{l} \tag{8.11}$$

where Δu =variation of velocity of original motion. We get this from dimensional analysis. Later chapters of Landau Lifshitz gave more technical meaning of Δu. But we won't do that.

The scaling Law

Look at scales of size λ, the dynamics of scale λ is determined solely by matter varying on that scale. View turbulence as a field, interaction only happen between objects of their own sizes. In other words, interactions are local in size. Alternatively we say that dynamics looks the same at each scale.

To interact with different size objects, it has to cascade down size in steps as shown before. Let v_λ be the variation of velocity on scale λ, then

$$\epsilon = \frac{v_\lambda^3}{\lambda}$$

or

$$v_\lambda \sim (\epsilon\lambda)^{1/3} \tag{8.12}$$

This is the Kolmogorov law.

For practical application, we express this in another way, in term of energy stored in different wave number (or energy stored per mode), so called the spectrum of energy.

$$\frac{dE}{dk} \text{ v.s. } k$$

This has been measured for many different liquid. Here E is energy/mass.

From (8.11)

$$\epsilon = \frac{dE}{dk}\dot{k} \tag{8.13}$$

$\dot{k}$ describes how long will take for wave number from one value to a different value. Clearly $\dot{k} = \dot{\lambda}/\lambda^2$.

Claim:

$$\dot{\lambda} \sim v_\lambda$$

Show: By dimensional analysis

$$\dot{v}_\lambda \sim \frac{v_\lambda^2}{\lambda}$$

From (8.12),

$$\dot{v}_\lambda \sim \epsilon^{1/3}\frac{\dot{\lambda}}{\lambda^{2/3}}$$

Putting the two together

$$\dot{\lambda} \sim \frac{v_\lambda^2}{\epsilon^{1/3}\lambda^{1/3}} \sim v_\lambda$$

Complete the claim.

Now put the claim into (8.13)

$$\epsilon = \frac{dE}{dk}\frac{v_\lambda}{\lambda^2} \sim \frac{dE}{dk}\frac{(\epsilon\lambda)^{1/3}}{\lambda^2}$$

So finally

$$\frac{dE}{dk} \sim \frac{\epsilon^{1/3}}{k^{5/3}}$$

The $k^{-5/3}$ agrees amazingly with the plot

$$\frac{dE}{dk} \text{ v.s. } k$$

This is magnificent. $k^{-5/3}$ law appears in so many other places, like density occupation, thermal modes, etc. Kolmogorov figured that out in 1941, since then people have tried to prove this rigorously, not many progress has been made.

8.5 Problem Set 7 (due 4/25/13)

19)

For turbulent motion one can determine the scale $\lambda_d = 2\pi/k_d$ at which dissipation sets in by requiring the $\nu\nabla^2\vec{v}$ term in N-S

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v}$$

be comparable to the $\frac{\partial\vec{v}}{\partial t}$ and $\vec{v}\cdot\nabla\vec{v}$ terms. Use $v_\lambda \sim (\epsilon\lambda)^{1/3}$ to determine λ_d.

20)

In class we used Euler equation and continuity equation, we found sound waves in an ideal fluid. Now use the N-S equation assuming the $\nu\nabla^2\vec{v}$ term is small to evaluate the damping rate. This is just an estimate since our derivation of N-S assumed incompressibility.

9 Diffusion Equation and Brownian Motion

9.1 The diffusion equation

Suppose we have a fluid consisting of two types of molecules with molecule numbers N_1, N_2 with mass density $\rho_1(\vec{x},t)$,$\rho_2(\vec{x},t)$ and total density $\rho = \rho_1 + \rho_2$. We will see the character of diffusion is different than flow. In diffusion things move with $x \sim t^2$ while in flow $x \sim t$.

Lecture 21 (4/17/13)

Define concentration of molecule 1

$$c = \frac{\rho_1}{\rho} \tag{9.1}$$

this definition turns out to be very flexible, later we will put c as the ratio of N_1 v.s. N_2, or sometimes size of 1 v.s. 2, or sometimes mass of 1 v.s. 2.

We have continuity $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$ for the whole fluid, but we have

$$\frac{\partial c\rho}{\partial t} + \nabla \cdot (c\rho\vec{v}) \neq 0$$

because here $\vec{v}$ is misguided. This suggests to define a diffusion current density $\vec{i}$ so that

$$\rho\frac{\partial c}{\partial t} + \nabla \cdot \vec{i} = 0 \tag{9.2}$$

Let's explain what causes the current. It is the chemical potential difference which is due to directly non-uniform distribution of molecules 1 and 2. Hence

$$\vec{i} = -\alpha \nabla \mu$$

for some proportionality α.

Recall

$$dU = TdS - pdV + \mu_1 dN_1 + \mu_2 dN_2$$

Since $dN_2 = -dN_1$, we get

$$dU = TdS - pdV + \mu dN_1$$

where $\mu = \mu_1 - \mu_2$. Let's introduce Gibbs function.

$$dG = -SdT + Vdp + \mu dN_1$$

and divide everything by V, get

$$dg = -sdT + dp + \mu dc$$

here $c = N_1/V$

So μ is function of p,T, c. Suppose p and T are uniform and constant, then

$$\nabla\mu = (\frac{\partial u}{\partial c})_{p,T}\nabla c$$

so (9.2) says

$$\rho\frac{\partial c}{\partial t} = \nabla\cdot[\alpha(\frac{\partial u}{\partial c})_{p,T}\nabla c]$$

Or

$$\frac{\partial c}{\partial t} = D\nabla^2 c \tag{9.3}$$

if we set $\rho D = \alpha(\frac{\partial u}{\partial c})_{p,T}$, where D is called diffusion constant.

We will solve diffusion equation (9.3) for initial $c(\vec{x}, t = 0) = \delta(\vec{x})$, this will allows us to build general solutions for arbitrary initial c.

Fourier

$$c(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}}\tilde{c}(\vec{k}, t)e^{i\vec{k}\cdot\vec{x}}$$

so (9.3) gives

$$\frac{\partial\tilde{c}}{\partial t} = -Dk^2\tilde{c} \implies \tilde{c}(\vec{x}, t) = e^{-Dk^2t}\tilde{c}(\vec{k}, 0)$$

initial

$$c = \delta(\vec{x}) \implies \tilde{c} = \frac{1}{(2\pi)^{3/2}}$$

so

$$c(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{-Dk^2t} e^{i\vec{k}\cdot\vec{x}}$$

complete the square of the exponent

$$i\vec{k}\cdot\vec{x} - Dk^2t = -Dt(\vec{k} - i\frac{\vec{x}}{2Dt})^2 - \frac{x^2}{4Dt}$$

we get

$$c(\vec{x},t) = e^{-\frac{x^2}{4Dt}} \frac{1}{(2\pi)^3} (\sqrt{\frac{\pi}{Dt}})^3 = \left(\frac{1}{4\pi Dt}\right)^{3/2} e^{-\frac{x^2}{4Dt}}$$

for the delta initial.

General solution

$$c(\vec{x},t) = \int d\vec{x}_0^3 \frac{e^{-\frac{(\vec{x}-\vec{x}_0)^2}{4D(t-t_0)}}}{[4\pi D(t-t_0)]^{3/2}} c(\vec{x}_0, t_0)$$

9.2 Mobility and Einstein Relation

Put in a single large particle into a fluid, suppose the particle to be spherical. We suppose there is a constant force $\vec{f}$ exerting on the particle, we don't care what kind of the force is. It could be electric force, if the fluid is electric neutral, and we situate some charge (not to small in size) at some location, then put in an external E, this will produce the force we want.

The particle will reach a velocity $\vec{v}$ with

$$\vec{v} = b\vec{f}$$

where b is called the mobility. Here $\vec{f}$ is balanced by collisions.

Recall as a sphere flows through viscous fluid, the drag force is

$$\vec{F}_{drag} = -6\pi\eta R\vec{v}$$

Since $\vec{F} = -\vec{f}$,

$$b = \frac{1}{6\pi\eta R}$$

for the special problem.

Now put a concentration c of large particles in a fluid. c is very small so no interactions between large particles themselves.

Put on an external force $\vec{f}$. The diffusion current of the large particles is

$$\vec{i} = c\rho b\vec{f} - \alpha\nabla\mu = c\rho b\vec{f} - \rho D\nabla c \tag{9.4}$$

That ρ won't be uniform now, because of external force, but if we wait long enough until equilibrium sets in, then $\vec{i} = 0$. After that, the concentration should follow Boltzmann law

$$c \sim e^{-\frac{V}{kT}}$$

and

$$\vec{f} = -\nabla V$$

so (9.4) says

$$0 = c\rho b\vec{f} - \rho D(-\nabla\frac{V}{kT}c)$$

Therefore

$$D = bkT$$

This is Einstein's relation.

Later we will do Boltzmann equation. We will do the collision term, then we will see scatters that change momentum. Then we will relook Einstein relation, we will see how Boltzmann bring system into equilibrium.

9.3 The Boltzmann Equation in Diffusion Approximation

Lecture 22 (4/22/13)

After examining diffusion aggregately, we now go to microscopic view. Suppose a small concentration of heavy particles in a thermal medium. As we will see, this allows us to say the temperature is set by the solute, and not effected by the heavy particles. Now imagine that the spatial distribution is uniform, but the energy distribution is not thermal. Let $f(\vec{p}, t)$ be the distribution of heavy particles.

From Boltzmann

$$\frac{\partial f(\vec{p}, t)}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f(\vec{p}, t) + \dot{\vec{v}} \cdot \nabla_{\vec{v}} f(\vec{p}, t) = C(f)$$

the second term is 0 because uniform in $\vec{x}$, and third term on the left is 0, because no applied force.

The collision term can be view as in terms of scattering which give a "gain" term, i.e. some portion of heavy particles with initial momentum $\vec{p}+\vec{q}$ becomes $\vec{p}$ after scattering, and there is also a "lost" term, i.e. some portion of heavy particles with initial momentum $\vec{p}$ becomes $\vec{p}-\vec{q}$ after scattering.

$|\vec{p}| \gg |\vec{q}|$ change of momentum of heavy particles is very small.

And rate for which gain term happens is

$$\frac{d(\text{rate})}{d^3q} = w(\vec{p}+\vec{q},\vec{q})$$

and for lost term

$$\frac{d(\text{rate})}{d^3q} = w(\vec{p},\vec{q})$$

In quantum mechanics, one would look into scattering amplitude and put in flux to find the rate, and it highly depends on process, but we'll not go there.

Now put things together, expand $w(\vec{p}+\vec{q},\vec{q})$, $f(\vec{p}+\vec{q},t)$ about $\vec{p}$, we get the rate of collision is

$$\begin{aligned}\frac{\partial f}{\partial t} &= \int d^3q[w(\vec{p}+\vec{q},\vec{q})f(\vec{p}+\vec{q},t) - w(\vec{p},\vec{q})f(\vec{p},t)] \qquad (9.5)\\ &= \int d^3q[q_i\nabla_{p_i}(w(\vec{p},\vec{q})f(\vec{p},t)) + \frac{1}{2}q_iq_j\nabla_{p_i}\nabla_{p_j}(w(\vec{p},\vec{q})f(\vec{p},t)) + ...]\\ &= \nabla_{p_i}[\tilde{A}_if(\vec{p},t) + \nabla_{p_j}B_{ij}(\vec{p})f(\vec{p},t) + ...]\end{aligned}$$

where $\tilde{A}_i = \int d^3qq_iw(\vec{p},\vec{q})$, and $B_{ij} = \frac{1}{2}\int d^3qq_iq_jw(\vec{p},\vec{q})$ are called transport coefficients. We ignore high order terms because $\frac{q^2}{m} \sim T$, high order term

$$q^n \implies (Tm)^{n/2}$$

but m is small.

9.4 Fokker-Planck Equation

So we get Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \nabla_{p_i}[\tilde{A}_i f(\vec{p}, t) + \nabla_{p_j} B_{ij}(\vec{p}) f(\vec{p}, t)] \tag{9.6}$$

It comes right from Boltzmann. This equation is useful to study momentum transfer in momentum space. Say Brownian motion. In momentum space where there exist heavy particles bombarded by small random kicks, one small kick does not affect much. But after long period of time, heavy particles get sizable amount of monument.

We can relate $\tilde{A}_i$ and B_{ij} by the following argument.

In equilibrium

$$C(f) = 0$$

collision term is 0, this can be taken as definition of equilibrium. So it means if the distribution momentum is lower than it could be in equilibrium we shall see more gain terms than loss terms.

Take an equilibrium

$$f \sim e^{-\frac{p^2}{2MT}}$$

so $\partial f/\partial t = 0$, what about the right hand side of (9.6)? We can write (9.6) as

$$0 = \frac{\partial f}{\partial t} = \nabla_{p_i}(-s_i)$$

where s_i momentum current: rate of flow of particle of one momentum into a different momentum region in momentum space. Here $s_i = 0$ in equilibrium, so

$$\tilde{A}_i f + \nabla_{p_j}(B_{ij} f) = f(\tilde{A}_i - B_{ij}\frac{p_j}{MT} + \nabla_{p_j} B_{ij}) = 0$$

so

$$\tilde{A}_i = B_{ij}\frac{p_j}{MT} - \nabla_{p_i} B_{ij}$$

this should work in general. So (9.6) becomes

$$\frac{\partial f}{\partial t} = [B_{ij}(\frac{p_j}{MT} f + \nabla_{p_i} f)]$$

This is as far as we can go. To continue we need to make another approximation.

If we are not to far from equilibrium

$$\frac{p^2}{2M} \sim T \implies v^2 \sim \frac{T}{M} \ll 1$$

This means velocity is very small. It is not obvious but there exists singularity in the transport coefficient. But rate of collision is very smoother than velocity, so let's take $v = 0$, then

$$B_{ij} = \frac{1}{2} \int d^3q q_i q_j w(0, \vec{q}) = \delta_{ij} B$$

We now get B_{ij} from a tensor to a number B. This is a good approximation verified by experiments.

Notice that the assumption of the second approximation here $v_{heavy\, particle} \to 0$ doesn't conflict the assumption of the first approximation $|p| \gg |q|$ in (9.5), because

$$\frac{p^2}{2M} \sim T = \frac{q^2}{2m}$$

so

$$M v_{heavy} \sim m v_{light} \implies v_{heavy} \ll v_{light}$$

and

$$mp^2 \sim Mq^2 \implies p \gg q$$

That is the reason in collison analysis, velocity is more important than momentum.

Now Fokker-Planck

$$\frac{\partial f}{\partial t} = B \nabla_{\vec{p}} \cdot [\frac{\vec{p}}{MT} f + \nabla_{\vec{p}} f]$$

This looks diffusion equation in momentum space with diffusion constant B.

9.5 Diffusion Constant Relook

Now we want to find connection between diffusion constant in momentum space and its counterpart in coordinate space. To do that, let's put a force $\vec{F}$ on the heavy particles

$$\frac{\partial f}{\partial t} + \vec{F} \cdot \nabla_{\vec{p}} f = B \nabla_{\vec{p}} \cdot [\frac{\vec{p}}{MT} f + \nabla_{\vec{p}} f] \tag{9.7}$$

Suppose $\vec{F}$ is small and in ∞ medium (so uniformly in space no $\vec{x}$ dependence.)

Look for equilibrium solution

$$f = f_0 + \delta f$$

f_0 Maxwell-Boltzmann, δf is due to external $\vec{F}$.

In equilibrium (9.7) reads

$$\vec{F} \cdot \nabla_{\vec{p}} f_0 = B \nabla_{\vec{p}} \cdot [\frac{\vec{p}}{MT} \delta f + \nabla_{\vec{p}} \delta f]$$

The $\frac{\partial f}{\partial t}$ is gone, because $\frac{\partial f_0}{\partial t} = B \nabla_{\vec{p}} \cdot [\frac{\vec{p}}{MT} f_0 + \nabla_{\vec{p}} f_0]$, and we neglect $\vec{F} \cdot \nabla_{\vec{p}} \delta f$.

Since $\vec{F} \cdot \nabla_{\vec{p}} f_0 = \nabla_{\vec{p}} (f_0 \vec{F})$, we have

$$f_0 \vec{F} = B(\frac{\vec{p}}{MT} \delta f + \nabla_{\vec{p}} \delta f) + c$$

where $c = 0$ because as $\vec{p} \to \infty$, f_0, $\delta f \to 0$.

Integrate $\vec{p}$ over all space, gives

$$\vec{F} = \frac{B}{T} \vec{v}$$

because $\int d^3 p f_0 = 1$, $\int d^3 p \frac{\vec{p}}{M} f_0 = \vec{v}$, $\int d^3 p \nabla_{\vec{p}} \delta f = \int ds \delta f = 0$.

But before $\vec{v} = b\vec{F}$, so

$$B = T/b$$

And einstein $D = bT$, so we obtain

$$B = T^2/D$$

So we get some kind of uncertainty principle like form from our statistical analysis. D helps the motion, the larger the D is, the faster the diffusion takes place. B impedes the motion in coordinate space. The large the B, the more random the Brownian motion is.

9.6 Transition into Equilibrium

Lecture 23 -Last Lec- (4/24/13)

Let's recap how we got here. We set up the problem of small concentration of heavy particles in fluid. We assumed (1) In scattering with light particles, we can expand in the momentum transfer (diffusion approximation) (2) In $w(\vec{p}, \vec{q})$, we set $\vec{p} = 0$. These approximations are better than the justification we give, in reality one can use Fokker-Plank for light particles in light solution.

We now want to solve Fokker-Planck (9.7) for some initial f.

First take

$$f(\vec{p}, t = 0) = c(2\pi)^3 \delta(\vec{p}) \tag{9.8}$$

(here c is not the same c concentration) then the second term (diffusion term) on the right of (9.7) is dominated. Indeed the two terms on the right is in same order when

$$\frac{p}{MT} \sim \frac{1}{p} \implies \frac{p^2}{M} \sim T$$

initially $p = 0$, so the second term is much bigger.

Fourier

$$f(\vec{p}, t) = \int d^3x \tilde{f}(\vec{x}, t) e^{-i\vec{p}\cdot\vec{x}}$$

Using integration by parts

$$\vec{p}f = -i \int d^3x (\nabla_{\vec{x}} \tilde{f}) e^{-i\vec{p}\cdot\vec{x}}$$

$$\nabla_{\vec{p}}(\vec{p}f) = -\int d^3x \vec{x} \cdot (\nabla_{\vec{x}} \tilde{f}) e^{-i\vec{p}\cdot\vec{x}}$$

$$\nabla^2_{\vec{p}} f = -\int d^3x x^2 \tilde{f} e^{-i\vec{p}\cdot\vec{x}}$$

therefore (9.7) becomes

$$\frac{\partial}{\partial t}\tilde{f} = -\frac{B}{MT}\vec{x} \cdot \nabla_{\vec{x}} \tilde{f} - Bx^2 \tilde{f}$$

Try solution $\tilde{f} = e^{-\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}} g(\vec{x}, t)$ into above, yielding

$$e^{-\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}} \frac{\partial g}{\partial t} = -Bx^2 e^{-\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}} g$$

so

$$\frac{\partial g}{\partial t} = -Be^{\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}}\vec{x}^2 e^{-\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}} g$$

Use Campbell–Baker–Hausdorff formula from quantum mechanics

$$e^{\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}}\vec{x}^2 e^{-\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}} = x^2 + \frac{Bt}{MT}[\vec{x}\cdot\nabla_{\vec{x}}, \vec{x}^2] + \frac{1}{2!}(\frac{Bt}{MT})^2[\vec{x}\cdot\nabla_{\vec{x}}, \vec{x}^2, [\vec{x}\cdot\nabla_{\vec{x}}, \vec{x}^2, \vec{x}^2]] + ...$$

Lucky $[\vec{x}\cdot\nabla_{\vec{x}}, \vec{x}^2] = 2\vec{x}^2$ so above becomes

$$\frac{\partial g}{\partial t} = -B\vec{x}^2 e^{\frac{2Bt}{MT}} g$$

so take initial (9.8), $\tilde{f}(\vec{x}, t=0) = c$, so

$$g = ce^{-B\vec{x}^2\int_0^t dt' e^{\frac{2Bt}{MT}}} = ce^{-\frac{MT}{2}\vec{x}^2(e^{\frac{2Bt}{MT}}-1)}$$

So

$$\begin{aligned} \tilde{f}(\vec{x}, t) &= ce^{-\frac{Bt}{MT}\vec{x}\cdot\nabla_{\vec{x}}} e^{-\frac{MT}{2}\vec{x}^2(e^{\frac{2Bt}{MT}}-1)} \\ &= ce^{-\frac{MT\vec{x}^2}{2}(1-e^{-\frac{2Bt}{MT}})} \end{aligned}$$

Now

$$f = c\int d^3x e^{-i\vec{p}\cdot\vec{x} - \frac{MT\vec{x}^2}{2}(1-e^{-\frac{2Bt}{MT}})}$$

complete the square of the exponent

$$-i\vec{p}\cdot\vec{x} - \frac{MT\vec{x}^2}{2}(1-e^{-\frac{2Bt}{MT}}) = -\frac{MT}{2}(1-e^{-\frac{2Bt}{MT}})(\vec{x} + \frac{i\vec{p}}{MT(1-e^{-\frac{2Bt}{MT}})})^2 - \frac{p^2}{2MT(1-e^{-\frac{2Mt}{MT}})}$$

so

$$f(\vec{p}, t) = c\left(\frac{2\pi}{MT(1-e^{-\frac{2Bt}{MT}})}\right)^{3/2} e^{-\frac{\vec{p}^2}{2MT(1-e^{-\frac{2Mt}{MT}})}}$$

Discussion

(1) $t \ll \frac{MT}{2B}$,

$$f(\vec{p}, t) = c\left(\frac{\pi}{Bt}\right)^{3/2} e^{-\frac{\vec{p}^2}{4Bt}}$$

diffusion.

(2) $t \gg \frac{MT}{2B}$

$$f(\vec{p},t) = c\left(\frac{2\pi}{MT}\right)^{3/2} e^{-\frac{\vec{p}^2}{2MT}}$$

Maxwell-Boltzmann.

Physically the whole discussion says we put heavy particles uniform in space with some definite momentum, so no $\vec{x}$ dependence, at earlier stage we see diffusion of momentum between heavy and light particles. Later on when it reaches equilibrium, we are back to Maxwell-Boltzmann distribution.

9.7 Problem Set 7 (continued)

21)

Show that the vanishing of the collision term in Fokker-Planck leads to the expected form for $f(\vec{p},t)$.

22)

Consider the Boltzmann equation under the circumstances of problem (21) but where now there is a constant force acting on the heavy particles, $\vec{F}$. In class we write $f = f_0 + \delta f$ and determined δf when $\delta f/f_0 \ll 1$ and f_0 is the equilibrium distribution in the absence of $\vec{F}$. Now do not assume f is close to f_0, and find the equilibrium solution to

$$\frac{\partial f}{\partial t} + \vec{F}\cdot\nabla_{\vec{p}} f = B\nabla_{\vec{p}}\cdot[\frac{\vec{p}}{MT}f + \nabla_{\vec{p}} f]$$

where $f = f(\vec{p},t)$ with no $\vec{x}$ dependence.