Partial Differential Equations

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This is an undergraduate course. Offered in Spring 2014 at Columbia University. Textbook: Strauss, *Partial Differential Equations: an introduction.*

Contents

1 Transport Equation

Lecture 1 (1/21/14)

A general partial differential equation involve partial differentiation

$$F(u_x, u_y, u_{xx}, ...) = f$$

for given F,f, solve for $u(x, y, ...)$.

Example 1. First order

$$au_x + bu_y = 0$$

$a, b \in \mathbb{R}$ given. This is called transport equation.

Example 2. Second order Laplace equation

$$u_{xx} + u_{yy} = 0$$

which is the simplest elliptic equation. One can define Laplace operator

$$\Delta : C^2(\mathbb{R}^n) \to C^0(\mathbb{R}^n)$$

$$\Delta u := u_{xx} + u_{yy}$$

Example 3. Heat equation

$$u_t = u_{xx}$$

$u(x, t)$ describes temperature at position x and time t.

In 2D it becomes

$$u_t = \Delta_{(x,y)} u$$

Example 4. Non-linear porous medium equation

$$u_t = (u^m)_{xx} \quad m \geq 0$$

$u(x, t)$ describes density of a gas in porous medium.

Examples 3 & 4 are examples of parabolic equations.

Example 5. Wave equation

$$u_{tt} = u_{xx}$$

usually $u(x, t)$ describes vertical displacement of a longitudinal wave propagating on a string. Compare to heat equation, we will show that the speed of wave propagating is finite while the speed of heat equation propagating is ∞.

In high dimension

$$u_t = \Delta u$$

which is example of a hyperbolic pde.

Example 6. Wave equation with interaction

$$u_{tt} = u_{xx} + u^p$$

Example 7. Schrodinger equation

$$u_t = iu_{xx}$$

examples 4 & 6 are non-linear pde.

We will only study linear pde. Given a pde we want to ask

1) does solution exists?

2) satisfies the boundary or initial conditions?

3) is the solution unique?

4) Is it regular? Smooth?

5) If not regular, what kind of singularities are they?

E.g. Naive stokes equation in 2D solutions are regular. In 3D it is yet unknown whether the solution is regular or not.

1.1 Constant Coefficients

If the pde of $u(x, y)$ is as simple as

$$u_x = 0 \tag{1.1}$$

then we know solution

$$u(x, y) = f(y)$$

Consider the following first order constant coefficients

$$au_x + bu_y = 0 \tag{1.2}$$

Method One

Introduce new variables so that pde becomes like (1.1). Put

$$\begin{cases} \tilde{x} = ax + by \\ \tilde{y} = bx - ay \end{cases} \tag{1.3}$$

such coordinate transformation has to have non degenerated Jacobian, so

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial u}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = au_{\tilde{x}} + bu_{\tilde{y}}$$

similarly

$$u_y = bu_{\tilde{x}} - au_{\tilde{y}}$$

thus (1.2) becomes

$$(a^2 + b^2)u_{\tilde{x}} = 0$$

thus

$$u = f(\tilde{y}) = f(bx - ay) \tag{1.4}$$

for any function f. Hence u is constant long a characteristic line

$$bx - ay = \text{const}$$

with slope b/a which is also the $\tilde{x}$ axis in the xy plane.

Method Two

Solving (1.2). consider the vector $\vec{v} = (a, b)$, (1.2) becomes

$$\nabla u \cdot \vec{v} = 0$$

this is saying the directional derivative along $\vec{v}$ is 0, hence u is constant along $\vec{v}$.

Example 8. Solve

$$\begin{cases} 2u_t - 5u_x = 0 \\ u(x,0) = \sin x \end{cases}$$

using (1.4)

$$u(x,t) = f(2x + 5t)$$

$$u(x,0) = f(2x) = \sin x \implies f = \sin\frac{x}{2}$$

therefore

$$u(x,t) = \sin\frac{2x + 5t}{2}$$

1.2 Variable Coefficients

Lecture 2 (1/23/14)

Example 9. Solve

$$u_x + yu_y = 0$$

this is equivalent to

$$(1, y) \cdot \nabla u = 0$$

Hence u is constant along a curve with slope $y/1$

$$\frac{dy}{dx} = \frac{y}{1} \implies y = ce^x$$

or

$$u(x, y) = f(ye^{-x})$$

Example 10. Solve

$$\begin{cases} xu_x + u_y = y \\ u(x, 0) = x^2 \end{cases}$$

This is inhomogeneous pde. First find characteristic line

$$\frac{dy}{dx} = \frac{1}{x} \implies \frac{dx}{dy} = x \implies x = ce^y$$

so

$$u_h = f(xe^{-y})$$

solves the homogenous pde.

Claim: the general solution is

$$u = u_h + u_p \tag{1.5}$$

Pf: Suppose there is another solution $\tilde{u}$. Then $u - \tilde{u}$ solves

$$x(u - \tilde{u})_x + (u - \tilde{u})_y = 0$$

hence

$$u - \tilde{u} = u_{h'}$$

for some homogenous solution $u_{h'}$. Or

$$\tilde{u} = u - u_{h'} = (u_h - u_{h'}) + u_p$$

so $\tilde{u}$ is too in the form of (1.5).

For our case

$$u(x, y) = f(xe^{-y}) + \frac{y^2}{2}$$

IC

$$u(x,0) = f(x) = x^2$$

therefore

$$u(x,y) = x^2 e^{-2y} + \frac{y^2}{2}$$

We have seen that for some simple looking pde, find characteristic line is the quickest way. What about more complicated problem?

Example 11. Solve

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$$

We try the coordinate transformation (1.3), so that u_y will be dropped.

$$\begin{cases} X = x + 2y \\ Y = 2x - y \end{cases}$$

then

$$\begin{aligned} u_x &= u_X + 2u_Y \\ u_y &= 2u_X - u_Y \end{aligned}$$

Hence

$$5u_X + Yu = XY \tag{1.6}$$

Using integrating factor from ODE. The only distinction is that it is partial now,

$$e^D(u_X + \frac{1}{5}Yu) = e^D \frac{XY}{5}$$

$$\text{LHS} = \frac{\partial}{\partial X}(e^D u) = e^D u_X + \left(\frac{\partial D}{\partial X} e^D\right) u$$

$$\frac{\partial D}{\partial X} = \frac{Y}{5} \implies D = \frac{XY}{5}$$

That is

$$\frac{\partial}{\partial X}(e^{\frac{XY}{5}} u) = e^{\frac{XY}{5}} \frac{XY}{5}$$

so

$$e^{\frac{XY}{5}} u = \int e^{\frac{XY}{5}} \frac{XY}{5} d(\frac{XY}{5}) \frac{5}{Y} + f(Y) = \frac{5}{Y}(\frac{XY}{5} - 1)e^{\frac{XY}{5}} + f(Y) = (X - \frac{5}{Y})e^{\frac{XY}{5}} + f(Y)$$

or

$$\begin{cases} X = x + 2y \\ Y = 2x - y \end{cases}$$

$$\begin{aligned} u &= (X - \frac{5}{Y}) + f(Y)e^{-\frac{XY}{5}} \qquad (1.7) \\ &= (x + 2y - \frac{5}{2x - y}) + f(2x - y)e^{-\frac{(x+2y)(2x-y)}{5}} \end{aligned}$$

Looking at the finial solution (1.7) of (1.6), we see that $u_h = f(Y)e^{-\frac{XY}{5}}$ solves the homogenous part and $u_p = X - \frac{5}{Y}$ is a particular solution, so if we are able to guess u_p, we may not have to use integrating factor.

2 Wave Equation

Lecture 3 (1/28/14)

$$u_{tt} = c^2 u_{xx}$$

As we'll see unlike the heat equation, solutions get smooth out. The wave equation will keep the discontinuity of the initial condition as wave propagates.

One can check that

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

So let

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$$

then wave equation becomes transport equation

$$v_t - cv_x = 0 \implies v = h(x + ct)$$

Lecture 4 (1/30/14)

Lecture 5 (2/4/14)

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