Mathematical Method of Physics

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This is an advanced undergraduate course. Offered in Spring 2014 at Columbia University. Required textbook: Hassani, *Mathematical Physics : a Modern Introduction to Its Foundations*.

Contents

Course Overview

Lecture 1 (1/21/14)

Topics we will cover

1. Hilbert space
2. Operator theory
3. Distribution
4. Fourier Transform
5. Green's Functions
6. Complex Analysis
7. Representation Theory
8. Differential Geometry
9. Probability & Statistics

They are relation to fields of physics approximately

QM →1. 2. 3. 4. 5. 6. 9.

QFT → 1. 2. 3. 4. 5. 6. 9.

Gravity → 4. 5. 6. 9. 8.

Cosmology → 4. 5. 6. 8. 9.

Particles Physics → 2. 4. 5. 6. 7. 9.

Solid State → 2. 3. 4. 5. 6. 7. 9.

Fluid Dynamics → 4. 5. 6. 8. 9.

EM → 3. 4. 5. 6. 9.

1 Linear Algebra

It means finite dimensional.

1.1 Vector Space

Definition 1. Vector space V over $\mathbb{C}$ s.t.

1) $|a\rangle, |b\rangle \in V$ implies $|a\rangle + |b\rangle \in V$

2) $0 \in V$, s.t. $|a\rangle + 0 = |a\rangle \quad \forall |a\rangle \in V$

3) $|a\rangle \in V$ implies $\exists\, |-a\rangle \in V$ s.t. $|a\rangle + |-a\rangle = 0$

4) $|a\rangle \in V$, $\alpha \in \mathbb{C}$ implies $\alpha |a\rangle \in V$ (sometmes we write $\alpha |a\rangle = |\alpha a\rangle$)

We don't use $|0\rangle$ for the 0 vector, because $|0\rangle$ means vacuum state.

Example 2. of vector spaces

usual 2D or 3D arrows on the plane in space; $\mathbb{C}^n$; $\{$polynomals of x of degree $\leq 2\}$

Definition 3. Linearly independent vector $|a_1\rangle, |a_2\rangle, ..., |a_N\rangle$

$$\sum_{n=1}^{N} \alpha_n |a_n\rangle = 0 \iff \alpha_n = 0 \ \ \forall n$$

Definition 4. Subspace of V is alos a vector space.

Definition 5. Basis of V, set of linearly indenpendent vectors

$$\{|a_1\rangle, |a_2\rangle, ..., |a_N\rangle\}$$

that spans all of V, i.e. any $|b\rangle \in V$ can be expressed as

$$|b\rangle = \alpha_1 |a_1\rangle + ... + \alpha_N |a_N\rangle$$

Theorem 6. *A vector space has ∞ many basis, but they all have the same number of element, which is the* $\dim V$.

Definition 7. Scalar Product $|a\rangle, |b\rangle \in V$

$$\langle a|b\rangle \in \mathbb{C}$$

such that

1) $\langle a|b\rangle = \langle b|a\rangle^*$
2) $\langle a| (\beta |b\rangle + \gamma |c\rangle) = \beta \langle a|b\rangle + \gamma \langle a|c\rangle$
3) $\langle a|a\rangle \geq 0$, $=$ is obtained iff $|a\rangle = 0$
4) $\langle \alpha a + \beta b|c\rangle = \alpha^* \langle a|c\rangle + \beta^* \langle b|c\rangle$ anti-linear in bra

Properity 2) + 4) called qusilinear.

The form of inner product is not unique. The common ones are

Example 8. $a, b \in \mathbb{C}^n$

$$\langle a|b\rangle = \sum a_i^* b_i$$

Example 9. $|p\rangle = a_0 + a_1 x + a_2 x^2, |q\rangle = b_0 + b_1 x + b_1 x^2$ polynomails of degree ≤ 2

Not to common to use

$$\langle p|q\rangle = a_0^* b_0 + a_1^* b_1 + a_2^* b_2$$

because if we want to extend this to polynomails of all degrees, there is convergence issue. This is common to use

$$\langle p|q\rangle = \int_{-1}^{1} p^*(x) q(x) dx = a_0 b_0 + \frac{2}{3}(a_0 b_2 + a_1 b_1 + a_2 b_0) + ...$$

Definition 10. Orthonormal basis of V, $\{|e_1\rangle, |e_2\rangle, ..., |e_N\rangle\}$

$$\langle e_i|e_j\rangle = \delta_{ij}$$

Definition 11. Norm of a vector

$$\||a\rangle\| = \sqrt{\langle a|a\rangle}$$

One can also define define inner product from norm, known as the polarization identity

$$\langle a|b\rangle = \frac{1}{4}\left(\|a+b\|^2 - \|a-b\|^2\right)$$

for the real vector space

$$\langle a|b\rangle = \frac{1}{4}\left(\|a+b\|^2 - \|a-b\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

Lecture 2 (1/23/14)
Lecture 3 (1/28/14)
Lecture 4 (1/30/14)
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