Topology

Elliott Stein

Transcribed by Ron Wu

This is an advanced undergraduate course. Offered in Fall 2013 at Columbia University. Course textbook is Munkres, *Topology*. Office hours: TR 5:00-6:00.

Contents

General Topology

Lecture 1 (9/3/13)

We will follow the book very closely, and spend first half of the semester study general topology and second half study algebraic topology.

1 Set Theory

1.1 Set Algebra

$a \in S$, a is an element of set S. Denote $\varnothing$ set with no element, empty set. A, B sets

Definition 1. Union of two sets

$$A \cup B = \{a : a \in A \text{ or } a \in B\} = B \cup A$$

Intersection of two sets

$$A \cap B = \{a : a \in A \text{ and } a \in B\} = B \cap A$$

We can do more general things

$$\alpha \in J \text{ index set}$$

J is countable or uncountable. S_α family of sets, we can have

$$\bigcup_{\alpha \in J} S_\alpha \qquad \bigcap_{\alpha \in J} S_\alpha$$

Definition 2. Difference of two sets $A, B \subset C$,

$$B - A = \{b : b \in B \ \text{ and } b \notin A\}$$

C here acts like an universal set although there is not one.

Theorem 3. *(deMorgan's Law)*

$$(A - B) \cup (B - C) = A - (B \cap C)$$

$$(A - B) \cap (A - C) = A - (B \cup C)$$

S set, we define the power set

$$P(S) = \{A : A \subset S\}$$

$$A \in P \iff A \subset S$$

Here we didn't use lower letter to represent element of $P(S)$, we are going to be absolutely rigid.

Definition 4. Cartesian product

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

E.g $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Note 5. Technically $(A \times B) \times C$ is not the same as $A \times (B \times C)$, but for our purpose we won't make a big deal of it.

$A_1, ..., A_k$ countable finite many of sets,

$$A_1 \times A_2 \times ... \times A_k = \{(\underbrace{a_{1,} a_2, ..., a_k}_{k\text{-tuple}}) : a_i \in A_i \forall\, i\}$$

Or $A_1, A_2 ...\ i \in \mathbb{Z}_+$ countable or uncountable infinitely many of sets

$$A_1 \times A_2 \times ... \times = \{(\underbrace{a_{1,} a_2, ...}_{\omega\text{-tuple}}) : a_i \in A_i \forall\, i\}$$

ω-tuple is really a function

$$f : \mathbb{Z}_+ \to \bigcup_{i \in \mathbb{Z}_i} A_i$$

$$\prod_{i \in \mathbb{Z}_+} A_i = \{f : \mathbb{Z}_+ \to \bigcup_{i \in \mathbb{Z}_i} A_i \text{ s.t. } f(i) \in A_i\}$$

hence f is a choice function. Or more general for a family $\{S_\alpha : \alpha \in J\}$

$$\prod_{\alpha \in J} S_\alpha = \{f : J \to \bigcup_{\alpha \in J} S_\alpha \text{ s.t. } f(i) \in S_\alpha\}$$

Clearly if one of $A_i = \varnothing$, then $\prod_{i \in \mathbb{Z}_+} A_i = \varnothing$.

Question 6. *If all $A_i \neq \varnothing$, does it imply $\prod_{i \in \mathbb{Z}_+} A_i \neq \varnothing$?*

We cannot prove this within our logic system. One has to introduce axiom of choice.

1.2 Functions

A, B sets,

$$f : A \to B$$

A domain, B codomain. Image of A

$$\{f(a), a \in A\} \subset B$$

Definition 7. f is 1-1 $f(x) = f(y) \implies x = y$.

f is onto $\forall\, y \in B$, $\exists\, x \in A$ s.t. $f(x) = y$. In this case image = range.

f is one to one correspondence if it is both 1-1 and onto.

synonyms for these: injective/subjective/bijective.

Pre-image of C

$$f^{-1}(C) = \{a \in A : f(a) \in C\}$$

this doesn't assume f to be invertible.

1.3 Relations (binary)

Relation R on S is subset of $S \times S$, denote aRb for $(a, b) \in R$.

We will only talk about 2 relations: equivalence relation, and total ordering.

Definition 8. R is equivalence $a \sim b$ iff

1) $a \sim a \;\forall\, a \in S$ (reflexive)

2) $a \sim b \implies b \sim a \;\forall\, a, b \in S$ (symmetric)

3) $a \sim b, b \sim c \implies a \sim c$ (transitive)

Example 9. on $\mathbb{Z}$ $m \sim n$ iff

$$\frac{m-n}{p} \in \mathbb{Z}$$

for some fixed $p \neq 0$.

Definition 10. Equivalence class of $a \in S$

$$C_a = \{b \in S : b \sim a\}$$

Example 11. $p = 5$,

$$C_2 = \{2, 7, 12, ..., -3, -8, ...\}$$

Equivalence classes give a partition of S, i.e. S is divided into a collection of disjoint subsets

$$S = \bigcup_{\alpha \in J} C_\alpha \quad C_\alpha \cap C_\beta = \varnothing \text{ if } \alpha \neq \beta$$

Conversely given S and a partition $S = \cup C_\alpha$, one can define an equivalence $a \sim b \iff a, b \in C_\alpha$ for some α.

Example 12. For $p = 5$

$$\mathbb{Z} = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4$$

Definition 13. $(S, <)$ is total order (or called linear order) iff

1) it's not the case that $a < a$ (anti reflexivity)

2) if $a \neq b$ then $a < b$ or $b < a$ (comparability)

3) $a < b, b < c \implies a < c$ (transitivity)

Definition 14. Given $(S, <)$, M is the maximum element if $a \leq M$ for all $a \in S$. m is the minimum element if $m \leq a$ for all $a \in S$.

Example 15. $(\mathbb{R}, <)$ (usual $<$) 2 is an upper bound for $(0, 1)$ open interval.

Definition 16. Given $(A, <)$, $(B, <)$ one can define dictionary ordering on $A \times B$

$$(a_1, b_1) < (a_2, b_2) \iff a_1 < a_2 \text{ or } \begin{cases} a_1 = a_2 \\ b_1 < b_2 \end{cases}$$

Example 17. In $\mathbb{R}^2$ with the dictionary ordering, the open interval between two points (a, b), where $a, b \in \mathbb{R}^2$ is all vertical lines between a, b and ray from a up and ray from b down.

1.4 Countability

Definition 18. An infinite set is a set that is not finite.

Definition 19. A set S is countable iff S is finite or S can be put in $1-1$ correspondence with $\mathbb{Z}_+$.

S is countable infinite iff S is in $1-1$ correspondence with $\mathbb{Z}_+$.

Example 20. $\mathbb{Z}_+ : 1, 2, 3, 4, \ldots$ $\mathbb{Z} : 0, \pm 1, \pm 2, \ldots$ One can find a $1-1$ correspondence between $\mathbb{Z}_+$ and $\mathbb{Z}$.

Example 21. $\mathbb{Z}_+ \times \mathbb{Z}_+$ is too countable. Use diagonal lines connect the grid.

Proposition 22. *The following statements are equivalent*

1) S is countable

2) There is a function $\mathbb{Z}_+ \to S$ that is onto

3) There is a function $S \to \mathbb{Z}_+$ that is 1-1

The proof requires axiom of choice.

Proposition 23. *A, B constable then $A \times B$ is countable.*

Proposition 24. *$A_1, .., A_k$ countable*

$$A_1 \times A_2 \times \ldots \times A_k$$

is countable

Notice the finite product about, if we do

Example 25. $S = \{0, 1\}$

$$S \times S \ldots = \{0, 1\}^\omega$$

is uncountable.

But union is find

Proposition 26. *A_i is countable, then*

$$\bigcup_{i \in \mathbb{Z}_+} A_i$$

is countable.

2 Topological Spaces

2.1 Topology

Lecture 2 (9/5/13)

Recall from advanced calculus. $\mathbb{R}^n$ euclidean space

$$x = (x_1, ..., x_n) \in \mathbb{R}^n$$

we have distance $x, y \in \mathbb{R}^n$

$$\|x - y\| = [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2}$$

If $n = 2$, it is Pythagorean. From there we have notion of an open ball centered at x radius ϵ

$$B(x, \epsilon) = \{y : \|y - x\| < \epsilon\}$$

Definition 27. $U \subset \mathbb{R}^n$ open if $\forall\, x \in U\ \exists\, \epsilon > 0$ s.t. $B(x, \epsilon) \subset U$.

E.g. x axis is not open in $\mathbb{R}^2$.

Notion of distance is very important. It tells about whether f is continuous.

We do these in a much more general settings.

Definition 28. X set, a topology $\mathcal{J}$ on X is a subset of $P(X)$ satisfying

1) $\varnothing \in \mathcal{J}$, $X \in \mathcal{J}$

2) $A_1, A_2 \in \mathcal{J} \implies A_1 \cap A_2 \in \mathcal{J}$

3) A_α,$\alpha \in J$ with $A_\alpha \in \mathcal{J} \implies \bigcup_\alpha A_\alpha \in \mathcal{J}$.

Claim 29. Check the collection of open sets in $\mathbb{R}^n$ is a topology on $\mathbb{R}^n$.

$\varnothing$ is open vacuously.

$\mathbb{R}^n$ is open since for $x \in \mathbb{R}^n$ $B(x, 1) \subset \mathbb{R}^n$.

A_1, A_2 open, let $x \in A_1 \cap A_2$, $\exists\, \epsilon_1, \epsilon_2$ s.t.

$$B(x, \epsilon_1) \subset A_1 \qquad B(x, \epsilon_2) \subset A_2$$

Let $\epsilon = \min(\epsilon_1, \epsilon_2)$, then

$$B(x, \epsilon) \subset A_1 \text{ and } A_2 \implies B(x, \epsilon) \subset A_1 \cap A_2$$

A_α open $x \in \bigcup A_\alpha$ then $x \in A_{\alpha_0}$ for some particular α_0

$$B(x, \epsilon) \subset A_{\alpha_0} \subset \bigcup A_\alpha \implies \bigcup A_\alpha \text{ is open}$$

Definition 30. $(X, \mathcal{J})$ topological space. The elements of $\mathcal{J}$ are called the open sets in the topology.

So we define openness without referring to distance.

Example 31. Indiscrete topology X, $\mathcal{J} = \{\varnothing, X\}$.

Example 32. Discrete topology X, $\mathcal{J} = P(X)$.

Example 33. Cofinite (or finite complement topology) $X = \mathbb{R}$, $A \in \mathcal{J}$ iff $\mathbb{R} - A$ is a finite set or $A = \varnothing$.

check it is a topology. $\varnothing, \mathbb{R} \in \mathcal{J}$, $A_1, A_2 \in \mathcal{J}$ then

$$A_1 = \mathbb{R} - F_1 \qquad A_2 = \mathbb{R} - F_2$$

where $F_{1,2}$ are finite

$$A_1 \cap A_2 = (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2) = \mathbb{R} - (F_1 \cup F_2) \in \mathcal{J}$$

since $F_1 \cup F_2$ is finite.

$A_\alpha = \mathbb{R} - F_\alpha \bigcup A_\alpha = \bigcup \mathbb{R} - F_\alpha = \mathbb{R} - \bigcap F_\alpha \in \mathcal{J}$, since $\bigcap F_\alpha \subset F_{\alpha_0}$ that is $\bigcap F_\alpha$ is finite.

Hence we have see 4 different topologies on $\mathbb{R}$.

Definition 34. $(X, \mathcal{J}_1)$,$(X, \mathcal{J}_2)$. We say $\mathcal{J}_1$ is finer than $\mathcal{J}_2$ if $\mathcal{J}_2 \subset \mathcal{J}_1$ (or say $\mathcal{J}_2$ is coarser than $\mathcal{J}_1$)

We say that $\mathcal{J}_1$ is strictly finer than $\mathcal{J}_2$ if $\mathcal{J}_2 \subsetneqq \mathcal{J}_1$.

The reason it is called finer, because we think in $\mathbb{R}^n$ standard topology, $\mathcal{J}_2 \subset \mathcal{J}_1$ means $\mathcal{J}_1$ contains some smaller sets that may not be in $\mathcal{J}_2$.

Definition 35. $\mathcal{J}_1$,$\mathcal{J}_2$ are compatible if

$$\mathcal{J}_2 \subset \mathcal{J}_1 \text{ or } \mathcal{J}_1 \subset \mathcal{J}_2$$

Claim 36. $X = \mathbb{R}$

$$\mathcal{J}_{\text{indiscrete}} \subsetneqq \mathcal{J}_{\text{cofinite}} \subsetneqq \mathcal{J}_{\text{euclidean}} \subsetneqq \mathcal{J}_{\text{discrete}}$$

We only need to check the middle

$$U \in \mathcal{J}_{\text{cofinite}} \implies U \in \mathcal{J}_{\text{euclidean}}$$

$U = \mathbb{R} - F$, $F = \{t_1, ..., t_n\}$ fix $x \in U$, $|x - t_i| > 0\ \forall\, i$.

let $\epsilon = \min\{|x - x_i|, i = 1, .., n\} \implies t_i \notin B(x, \epsilon) \implies B(x, \epsilon) \subset U$, so U is open in $\mathcal{J}_{\text{euclidean}}$.

2.2 Basis and subbasis

The role of open ball in euclidean is very useful.

Definition 37. A basis $\mathcal{B}$ on X is a subset of $P(X)$ satisfying

1)

$$\bigcup_{A \in \mathcal{B}} A = X$$

2) $A_1, A_2 \in \mathcal{B}$, $x \in A_1 \cap A_2$, then

$$\exists\, A_3 \in \mathcal{B} \text{ with } x \in A_3 \subset A_1 \cap A_2$$

Note 38. one can have a basis without knowing the Topology. In the definition no where mention the kind of topology.

Claim 39. The open balls in $\mathbb{R}^n$ are a basis.

Clearly

$$\bigcup_{\substack{x \in \mathbb{R}^n \\ \epsilon > 0}} B(x, \epsilon) = X$$

For given $B(x_1, \epsilon_1)$, $B(x_2, \epsilon_2)$, $x_3 \in B(x_1, \epsilon_1) \cap B(x_2, \epsilon_2)$, the ball

$$B(x_3, \min\{\epsilon_1 - \|x_1 - x_3\|, \epsilon_2 - \|x_2 - x_3\|\}) \subset B(x_1, \epsilon_1) \cap B(x_2, \epsilon_2)$$

If $\mathcal{B}$ is a basis on X, it defines a topology $\mathcal{J}$ as follows:

Definition 40. $U \subset X$ is open iff $\forall\, x \in U$, $U \in \mathcal{J}$, $\exists\, B_x \in \mathcal{B}$ with $x \in B_x \subset U$.

It turns out all elements of $\mathcal{J}$ can be expressed via a basis, so basis gives a simpler way to characterize the topology.

Proposition 41. *Suppose $\mathcal{B}$ is a basis for $\mathcal{J}$ on X, $U \in \mathcal{J}$ iff*

$$U = \bigcup_{\alpha \in J} B_\alpha$$

for some J.

Proof. Assume U is an arbitrary open set, $x \in U \implies \exists\, B_x$ with

$$x \in B_x \subset U$$

so for any y, $y \in \bigcup_{x \in U} B_x$, so $U \subset \bigcup_{x \in U} B_x$. On the other hand, each $B_x \subset U$, so $\bigcup_{x \in U} B_x \subset U$, thus

$$U = \bigcup B_x$$

Conversely B_α is open in $\mathcal{J} \implies \bigcup_{\alpha \in J} B_\alpha \in \mathcal{J}$. □

Example 42. $X = \mathbb{R}$, $\mathcal{B} = \{[a, b)\}$ is a basis.

Check it.

$$\mathbb{R} = \bigcup_{B \in \mathcal{B}} B$$

Moreover instead of proving $\exists\, B \in \mathcal{B}$, s.t.

$$B \subset [a_1, b_1) \cap [a_2, b_2)$$

we find

$$[a_1, b_1) \cap [a_2, b_2) \in \mathcal{B}$$

The $\mathcal{J}$ generated by $\mathcal{B}$ is called the lower limit topology on $\mathbb{R}$, denoted $\mathbb{R}_l$.

Question 43. *Is this compatible to standard topology in* $\mathbb{R}$*?*

Yes,

$$(a, b) = \bigcup_{n \in \mathbb{Z}_+} [a + \frac{\epsilon}{n}, b)$$

chose a proper ϵ *so that* $a + \epsilon < b$.

In other words, the basis of standard topology can be expressed by basis of $\mathbb{R}_l$*, so according to the proposition below* $\mathbb{R}_l$ *is finer. In fact it is strictly finer.*

Proposition 44. $\mathcal{J}_1$,$\mathcal{J}_2$ *on* X *with* $\mathcal{J}_1$ *associated with* $\mathcal{B}_1$ *and* $\mathcal{J}_2$ *associated with* $\mathcal{B}_2$ *then* $\mathcal{J}_1$ *is finer than* $\mathcal{J}_2$ *iff*

$$\forall\, x \in B_2 \in \mathcal{B}_2 \quad \exists\, B_1 \in \mathcal{B}_1 \textit{ with } x \in B_1 \subset B_2$$

Proof. We prove the converse direction.

Suppose U is open in $\mathcal{J}_2$, need to show that U is open in $\mathcal{J}_1$, i.e. $\forall\, x \in U$, $\exists\, B_x \in B_1$ with $B_x \subset U$.

Since U is open in $\mathcal{J}_1$, $\exists\, B'_x \in \mathcal{B}_2$ with $x \in B'_x \in \mathcal{B}_2$, by assumption $\exists\, B_x \in \mathcal{B}_1$ with $x \in B_x \in B'_x \subset U$, so U is open in $\mathcal{J}_1$. □

Definition 45. Let $\mathcal{B}'$ be any collection of subsets of X with the properties that $X = \bigcup_{B' \in \mathcal{B}} B'$

$$\mathcal{B} = \{\text{all finite intersections of elements of } \mathcal{B}'\}$$

If $\mathcal{B}$ becomes a basis, then $\mathcal{B}'$ is a subbasis for the topology generated by $\mathcal{B}$.

Subbasis is even better than basis, because it contains less sets.

Example 46. $\{(-\infty, b), (a, \infty)\}$ subbasis for $\mathbb{R}$ standard topology.

Lecture 3
(9/10/13)