Quantum Field Theory I

Lam Hui

Transcribed by Ron Wu

This is a graduate course. Offered in Spring 2014 at Columbia University. Required Course textbooks: Srednicki, *Quantum Field Theory*; Zee, *Quantum Field Theory in a Nutshell.* Particularly Useful: *Coleman's physics 251a notes*. Office hours: Wed 3:15-4:15.

Contents

Part I.
Scalar Field Theory

Lecture 1 (1/22/14)

For the most part of the course, we will do scalar field theory, meaning no spin, meaning not so physical, except some Higgs. We will study renormalization, loops, perturbation theory, Feynman diagram. Towards the end we will do spin 1/2, and see applications to condensed matter.

Adopt $c = \hbar = 1$

1. Free Fields

1.1. Second Quantization

First we show an example of why naive application of quantum mechanics is wrong.

We compute the probability amplitude of events 1 $(t_1, \vec{x}_1)$ and 2 $(t_2, \vec{x}_2)$ that are outside of the light cone (spacelike separation). We assume $|\vec{x}_1\rangle$, $|\vec{x}_2\rangle$ are localized position states, so

$$\left\langle \vec{k} \middle| \vec{x} \right\rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}}$$

then

$$\begin{aligned} \left\langle \vec{x}_2 \middle| e^{-iH(t_2-t_1)} \middle| \vec{x}_1 \right\rangle &= \int d^3k \left\langle \vec{x}_2 \middle| \vec{k} \right\rangle \left\langle \vec{k} \middle| e^{-iH(t_2-t_1)} \middle| \vec{x}_1 \right\rangle \\ &= \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot\vec{x}_2} e^{-iE_k(t_2-t_1)} e^{-i\vec{k}\cdot\vec{x}_1} \\ &= \frac{1}{(2\pi)^2} \int k^2 dk \int d(\cos\theta) e^{ik\Delta x \cos\theta} e^{-iE_k \Delta t} \end{aligned}$$

set $\Delta\vec{x} = \vec{x}_2 - \vec{x}_1$, $\Delta t = t_2 - t_1$. use spherical coordinate and choose $\hat{z}$ along $\Delta\vec{x}$.

Continue

$$\begin{aligned} \langle \vec{x}_2 | e^{-iH(t_2-t_1)} | \vec{x}_1 \rangle &= \frac{1}{(2\pi)^2} \int_0^\infty k^2 dk \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{ik\Delta x} e^{-iw_k \Delta t} \\ &= \frac{1}{(2\pi)^2 i\Delta x} \int_{-\infty}^\infty k dk e^{ik\Delta x} e^{-iw_k \Delta t} \end{aligned} \tag{1.1}$$

because $E_k = w_k$ is made of k^2. Use relativistic energy $w_k = \sqrt{\vec{k}^2 + m^2}$. Because of $\Delta x > \Delta t > 0$, we want $e^{ik\Delta x}$ to vanish, we should choose contour of upper half, which will cross one of the two branch cuts $[im, i\infty)$ and $(-i\infty, -im]$ for the square root. Why are $[im, i\infty)$ and $(-i\infty, -im]$ branch cuts? We know the principle branch for $\sqrt{z}$ is the negative real axis, i.e.

$$k^2 + m^2 < 0 \implies k \in [im, i\infty) \text{ or } (-i\infty, -im]$$

so we will have to deform the half circle to around the $[im, i\infty)$ branch.

Set $k = iz \pm \epsilon$

$$(1.1) \to \lim_{\epsilon \to 0^+} - \int_\infty^m (iz+\epsilon) i dz e^{i(iz+\epsilon)\Delta x} e^{+\sqrt{z^2-m^2}\Delta t} - \int_m^\infty (iz-\epsilon) i dz e^{i(iz-\epsilon)\Delta x} e^{-\sqrt{z^2-m^2}\Delta t}$$

so

$$\langle \vec{x}_2 | e^{-iH(t_2-t_1)} | \vec{x}_1 \rangle = \frac{1}{(2\pi)^2 \Delta x} \int_m^\infty z dz e^{-z\Delta x} \sinh \sqrt{z^2 - m^2} \Delta t \tag{1.2}$$

The integrand is positive, hence the probability of a particle at $(t_1, \vec{x}_1)$ making to $(t_2, \vec{x}_2)$ is non-zero, violating causality. If we use $E_k = \frac{k^2}{2m}$ instead in (1.1), we will get non-zero too.

$$\langle \vec{x}_2 | e^{-iH(t_2-t_1)} | \vec{x}_1 \rangle \sim \int_{-\infty}^\infty k dk e^{-ia(k-b)^2} e^{-ic(\Delta x)^2} \sim \int_{-\infty}^\infty (k+b) dk e^{-ik^2} \sim \int_{-\infty}^\infty b dk e^{-ik^2} \neq 0$$

What causes the problem? In nonrelativistic qm, we can localize the particle in an arbitrarily small region, as long as we pay the price of having an arbitrarily large uncertainty in its momentum, which corresponds to have the probability of having huge energy. Experiments tell us such energy will create particles and antiparticles pairs (because of charge conservation). If the region (atomic physics) is about Compton $\lambda_C \sim 20$fm, nonrelativistic qm works fine. But for bound quarks inside of proton 1fm say, we need multiparticle theory. In addition, recall Einstein solved the causality problem of Newton gravity by inducing 4d spacetime and showed that gravitational wave propagates at speed of light. We will use that idea to study qm in field language and in 4d spacetime.

Define

$$|0\rangle = \text{vacuum (no particles)}$$

For now assume vacuum is unique, and

$$a_k |0\rangle = 0 \qquad \langle 0|0\rangle = 1$$

Later we'll see there are other vacuums and our universe has jumped from one vacuum to other. In fact 1998 experiment found vacuum energy was related to cosmological constant. But most experiments, we don't care about it, since we only care about transition energy.

What is k? what is a_k? From above discussion eigenstate of position

$$|\vec{x}\rangle$$

is bad, however $|k\rangle$ is good. We can measure $|k\rangle$ with arbitrary accuracy and take advantage of energy-momentum conservation. Define

$$\begin{aligned} |k\rangle &= \text{1 particle state with 4 momentum } k \\ &= a_k^+ |0\rangle \end{aligned}$$

a_k^+ creation operator. a_k annihilation operator. They look very much like the ladder operators for SHO. Indeed we will show later that the Hamiltonian of the field theory look like the Hamiltonian of SHO

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}w^2x^2$$

2-particle state

$$|k_1k_2\rangle = a_{k_1}^+ a_{k_2}^+ |0\rangle$$

meaning that $a_{k_1}^+ a_{k_2}^+$ created two particles. The advantage of representing $|k_1k_2\rangle$

by a_k's is that the manipulation becomes much simpler. Later we'll show any operators can be written in terms of a_k's.

For now we do free theory, no interaction, only one kind of particle, spin 0 Boson, with mass m.

$$|k_1 k_2\rangle = |k_2 k_1\rangle = a^+_{k_2} a^+_{k_1} |0\rangle \tag{1.3}$$

so

$$[a^+_{k_2}, a^+_{k_1}] = 0 \tag{1.4}$$

Likewise

$$[a_{k_2}, a_{k_1}] = 0 \tag{1.5}$$

We make conventional choice

$$[a_{k_1}, a^+_{k_2}] = (2\pi)^3 2w_{k_1} \delta^3(\vec{k}_1 - \vec{k}_2) \tag{1.6}$$

The factor $(2\pi)^3$ comes from Feynman rule.

Check (1.6) make sense

$$\begin{aligned} \delta^4(k_1 - k_2) \propto \langle k_1 | k_2 \rangle &= \big\langle 0 \big| a_{k_1} a^+_{k_2} \big| 0 \big\rangle = \big\langle 0 \big| [a_{k_1}, a^+_{k_2}] - a^+_{k_2} a_{k_1} \big| 0 \big\rangle \\ &= \big\langle 0 \big| [a_{k_1}, a^+_{k_2}] \big| 0 \big\rangle = (2\pi)^3 2w_{k_1} \delta^3(\vec{k}_1 - \vec{k}_2) \end{aligned}$$

close to what we want.

1.2. Free Propagator

Now we show (1.6) is Lorentz invariant, consider the following (proper) Lorentz invariant quantity

$$\int \frac{d^4k}{(2\pi)^4} (2\pi) \delta(k^2 + m^2) \theta(k^0) \tag{1.7}$$

Why above integrand is Lorentz invariant? Because it only involves d^4k volume element, k^2 length, proper Lorentz transformation no sign change for k^0. Since

$$k^2 + m^2 = -k^{0^2} + |\vec{k}|^2 + m^2 = -k^{0^2} + w_k^2 \tag{1.8}$$

The δ function forces $w_k = k^0$ called on-shell mass.

Integrating over k^0, and $\Theta(k^0)$ forces $k^0 > 0$

$$\begin{aligned}(1.7) \quad &= \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(-k^{0^2}+w_k^2)\Theta(k^0) \\ &= \int \frac{d^3\vec{k}}{(2\pi)^4}(2\pi)\frac{1}{2w_k} \qquad (1.9)\end{aligned}$$

so integrand of (1.9) is Lorentz invariant. Now consider

$$\int \frac{d^3\vec{k}_1}{(2\pi)^4}(2\pi)\frac{1}{2w_{k_1}}\underbrace{(2\pi)^3 2w_{k_1}\delta^3(\vec{k}_1-\vec{k}_2)}_{(1.6)} = 1 \qquad (1.10)$$

1 is Lorentz invariant, so (1.6) is Lorentz invariant. Notice without the $1/w_k \sim 1/E \sim t$, the measure $d^3\vec{k}$ alone is not Lorentz invariant.

Lecture 2 (1/27/14)

Recall for SHO

$$a_k^+ a_k = \sqrt{n}\sqrt{n} = n$$

for us it means $a_k^+ a_k$ count the number of particles. For more see Quantum Mechanics II notes "Second Quantization of Boson". The Hamiltonian of the multiparticle system is

$$H = \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2w_k}}_{=(1.9)} w_k(a_k^+ a_k + c)$$

c would be 1/2 if it were SHO. For us c gives the vacuum energy

$$\langle 0|H|0\rangle = c\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2}$$

the integral is clearly divergent, but later we will learn how to regulate divergent integrals, e.g. put a physical cutoff of upper bound of the integral at Planck scale. But for reason we have discussed before, we will neglect vacuum energy, so we put $c = 0$.

One can check that by (1.3), (1.6) and (1.10)

$$H\,|k_1k_2\rangle = (w_{k_1}+w_{k_2})\,|k_1k_2\rangle$$

so indeed free particles.

1.3. Klein-Gordon Equation

Next we want to derive the expansions of Lagrangian and of wave function. Recall SHO, set mass = 1

$$S = \int dtL = \int dt\frac{1}{2}\dot{x}^2 - \frac{1}{2}w^2x^2 \tag{1.11}$$

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$$

so Hamiltonian

$$H = p\dot{x} - L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}w^2x^2 \tag{1.12}$$

eom, EL-eq

$$\ddot{x} + w^2x = 0 \tag{1.13}$$

$$x \sim e^{\pm iwt}$$

Now we go to QM of SHO, and work in Heisenberg picture, state independent of time, operator depends of time, e.g. $\hat{x}(t)$. Elevate $\hat{x}(t)$ to a second quantized operator in Heisenberg picture

$$\hat{x}(t) = \frac{1}{\sqrt{2w}}(ae^{-iwt} + a^+e^{iwt}) \tag{1.14}$$

the $\sqrt{2w}$ factor is there so that

$$[\hat{x}, \dot{\hat{x}}] = i \tag{1.15}$$

recall

$$[a, a^+] = 1$$

One way to see (1.14) makes sense is to plug (1.14) into (1.12)

$$H = w(a^+a + \frac{1}{2})$$

We want to apply this analog to field theory. We guess the wave function

$$\phi(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3 2w_k}(a_k e^{ik\cdot x} + a_k^+ e^{-ik\cdot x}) \tag{1.16}$$

which is real, like a superposition of many SHO each with momentum k, where

$$e^{ik\cdot x} = e^{-ik^0 t + i\vec{k}\cdot\vec{x}} \quad k^0 = \sqrt{\vec{k}^2 + m^2} = \sqrt{w_k^2 + m^2}$$

consistent with (1.8). For (1.16) to look like plane wave, $a_k^\pm$ have to be time independent. Alternative interpretation of (1.16): since $a^\pm$ are operators in the momentum space, (1.16) is like a Fourier transform of $a^\pm$. So we can invert (1.16) to get

$$\int d^3x e^{-ikx}\phi(x) = \frac{1}{2w_k}a_k + \frac{1}{2w_k}e^{2iw_k t}a_{-k}^+$$

$$\int d^3x e^{-ikx}\partial_0\phi(x) = -\frac{i}{2}a_k + \frac{i}{2}e^{2iw_k t}a_{-k}^+$$

combine these two to get

$$a_k = i\int d^3x \left(e^{-ikx}\partial_t\phi - \phi\partial_t e^{-ikx}\right) \tag{1.17}$$

$$a_k^+ = -i\int d^3x \left(e^{ikx}\partial_t\phi - \phi\partial_t e^{ikx}\right)$$

Now we see that $a_k^\pm$ are time independent

$$\partial_t a_k^\pm = 0 \implies \partial_{tt}\phi = -k^{0^2}\phi$$

We can check that similar to the derivation of (1.15), we get by (1.6)

$$[\phi(t,\vec{x}), \partial_t\phi(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}') \tag{1.18}$$

Notice the t in the argument of ϕ and $\partial\phi$ has to be the same; also notice the whole equation is Lorentz invariant, while the RHS and LHS alone is not. Also by (1.5),

(1.4)

$$[\phi(t,\vec{x}),\phi(t,\vec{x}')] \;\; = [\partial_t\phi(t,\vec{x}),\partial_t\phi(t,\vec{x}')] = \;\; 0 \tag{1.19}$$

What kind of eom has (1.16) as its solution? We guess it should look like (1.13), but in field language

$$\partial_t^2\phi - \nabla^2\phi + m^2\phi = 0$$

or in compact form

$$(-\Box + m^2)\phi = 0 \tag{1.20}$$

This is known as Klein-Gordon equation.

Indeed if we plug in (1.16), we get

$$\int \frac{d^3\vec{k}}{(2\pi)^3 2w_k} a_k^{\pm}[-k^{0^2} + \vec{k}^2 + m^2]e^{\pm ikx} = 0$$

What action associated with Klein-Gordon equation

$$S = \int d^4x\mathcal{L}$$

Clearly here $\mathcal{L}$ is Language density. We guess it should look like (1.11),

$$\begin{aligned} S &= \int d^4x\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\underbrace{(\nabla\phi)^2}_{\vec{\nabla}\phi\cdot\vec{\nabla}\phi} - \frac{1}{2}m^2\phi^2 \qquad (1.21)\\ &= \int d^4x\left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) \end{aligned}$$

Indeed

$$\begin{aligned} \delta S &= \int d^4x\left(-\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi - m^2\phi\delta\phi\right) \\ &= \int d^4x\left(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi\right)\delta\phi \end{aligned}$$

so

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0 \implies (1.20)$$

The conjugate momentum

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi \tag{1.22}$$

agrees what we have done (1.18).

Lecture 3 (1/29/14)

What is the free theory ϕ? One can say that it is a quantum theory of Higgs field, which is a scale field. When $m = 0$, we get

$$\Box\phi = 0 \implies \phi =^{\pm i(-wt+\vec{k}\cdot\vec{x})} \tag{1.23}$$

wave travel at speed of light. We should not try to push the classical analogy too far, comparing (1.23) to a sound wave traveling in the air. Its action is

$$S = \int d^4x \frac{1}{2c_s^2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2$$

which is not Lorentz invariant, because c_s is not Lorentz invariant.

1.4. Lorentz Transformation

Just to point out some notations we have been carried out in the calculation.

Metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

so e.g.

$$\begin{aligned} k^2 &= \eta_{\mu\nu}k^\mu k^\nu = -k^{0^2} + k^{1^2} + k^{2^2} + k^{3^2} = -E^2 + \vec{p}^2 \\ &= k_\nu k^\nu \end{aligned}$$

$$k \cdot x = \eta_{\mu\nu}k^\mu x^\nu = -k^0x^0 + k^1x^1 + k^2x^2 + k^3x^3 = -wt + \vec{k}\cdot\vec{x}$$

Inverse metric, and contra variant form

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

check

$$\eta_{\mu\nu}\eta^{\nu\alpha} = \delta^{\alpha}_{\mu}$$

Derivative

$$\eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

For conjugate momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

only time component, not usually written in lower or upper way.

Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

s.t. proper time is independent of frame under Lorentz transformation, i.e.

$$\eta_{\alpha\beta} \underbrace{dx'^\alpha dx'^\beta}_{\Lambda^\alpha{}_\mu dx^\mu \Lambda^\beta{}_\nu dx^\nu} = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 = -d\tau^2$$

so

$$\Lambda^\alpha{}_\mu \eta_{\alpha\beta} \Lambda^\beta{}_\nu = \eta_{\mu\nu} \tag{1.24}$$

or

$$\Lambda^T \eta \Lambda = \eta$$

so

$$(\det \Lambda)^2 = 1$$

drop $\det \Lambda = -1$, which means improper Lorentz transform, including parity inversion. Claim: there are 6 independent Lorentz transformation: 3 rotations, 3 boots. Proof, consider a small Lorentz transformation

$$\Lambda^{\alpha}{}_{\mu} = \underbrace{\delta^{\alpha}{}_{\mu}}_{I} + \delta w^{\alpha}{}_{\mu} \tag{1.25}$$

Plug in (1.24), and keep linear order

$$\delta w^{\alpha}{}_{\mu} \eta_{\alpha\nu} + \eta_{\mu\beta} \delta w^{\beta}{}_{\nu} = 0$$

so

$$\delta w_{\nu\mu} + \delta w_{\mu\nu} = 0 \tag{1.26}$$

hence δw is antisymmetric, so 6 independent components. QED.

The inverse Lorentz transformation

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\mu}{}^{\nu}$$

because

$$\Lambda_{\mu}{}^{\nu} \Lambda^{\alpha}{}_{\nu} = \eta_{\mu\beta} \eta^{\gamma\nu} \Lambda^{\beta}{}_{\gamma} \Lambda^{\alpha}{}_{\nu} = \eta_{\mu\beta} \eta_{\alpha\beta} = \delta^{\alpha}_{\mu}$$

Since we are in flat space, these are all we need.

1.5. Conserved Currents

Let's examine some symmetries of (1.21).

First we find (1.21) is invariant under

$$\phi \to -\phi$$

this is a discrete symmetry. For now we only study continuous symmetry.

$$\phi(x) \to \phi(x - a) \tag{1.27}$$

and

$$\phi(x) \to \phi(\Lambda^{-1} x) \tag{1.28}$$

by setting $\tilde{x} = \Lambda^{-1} x$ and the Jacobian is 1.

Why do we care about symmetry?

Theorem. *(Noether) For every continuous symmetry there is a conserved current.*

Suppose S is invariant under small change

$$\phi \to \phi + \delta\phi \tag{1.29}$$

Why do we need $\delta\phi$ to be small? Because in the derivation of Noether current, we Taylor expand. This is why Noether works only for continuous symmetry. (1.29) is certainly true for (1.28) by considering (1.25). It is also true for (1.27).

$$\phi(x-a) \sim \phi(x) - a^\mu \partial_\mu \phi$$

and require $a^\mu \partial_\mu \phi$ to be small.

If instead of (1.29), we do

$$\phi \to \phi + \xi(x)\delta\phi \tag{1.30}$$

where $\xi(x)$ is any function of spacetime. It implies

$$\delta S = \int d^4x j^\mu \partial_\mu \xi \tag{1.31}$$

where j is some function of $\delta\phi$, $\partial\delta\phi$, ϕ and $\partial\phi$. Why? because if ξ is constant, clearly $\delta S = 0$, so no ξ involved in (1.31). If higher derivatives of ξ involve in (1.31), we can always use integration by parts to absorb in j and reduce to 1st derivative.

So we can derive j from (1.31). Why is j conserved? Because (1.31) is also a variance, i.e. (1.31) $= 0$ for all ξ, so

$$(1.31) = \int d^4x (\partial_\mu j^\mu)\xi = 0$$

or

$$\partial_\mu j^\mu = 0 \tag{1.32}$$

so

$$\int d^3x(\partial_0 j^0 + \nabla \cdot \vec{j}) = 0$$

so

$$\partial_0 \int d^3x j^0 = -\int d^2\vec{S} \cdot \vec{j} = 0$$

by pushing the surface to ∞, hence

$$\int d^3x j^0 \text{ is conserved.} \tag{1.33}$$

which is sort of total "charges".

Lecture 4 (2/3/14)

We now do a more restricted version of Noether. We assume as

$$\phi \to \phi + \delta\phi$$

$\mathcal{L}$ is invariant, i.e.

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \delta\phi = 0 \qquad (1.34) \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) \because \text{emo} \end{aligned}$$

so put

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \tag{1.35}$$

i.e. we are back to (1.32).

To bridge the more restricted version and less restricted version, we suppose $\mathcal{L}$ is under changing

$$\mathcal{L} \to \mathcal{L} + \partial_\mu F^\mu$$

while as S is invariant, when

$$\phi \to \phi + \delta\phi.$$

Then we modify (1.34)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) = \partial_\mu F^\mu$$

so if we put

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - F^\mu \tag{1.36}$$

we are back to (1.32).

Notice that not all conserved currents are from eom. There are topological conserved quantity, e.g.

$$j^\nu = \partial_\mu F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

so

$$\partial_\nu j^\nu = 0$$

Example. Massless free scale field

$$S = \int d^4x - \frac{1}{2}(\partial\phi)^2$$

$\mathcal{L}$ is invariant under

$$\phi \to \phi + c$$

so by (1.35)

$$j^\mu = -c\partial^\mu \phi$$

we normally forget about c, since j is subject to some overall normalization

$$j^\mu = -\partial^\mu \phi$$

Indeed

$$\Box\phi = 0 \implies \partial_\mu j^\mu = 0$$

on the EOM for $L = -\frac{1}{2}(\partial\phi)^2$.

Consider

$$\phi \to \phi + b_\mu x^\mu$$

Is this a symmetry?

$$\delta S = \int d^4x(-\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi) = \int d^4x(-\eta^{\mu\nu}\partial_\mu\phi b_\nu) = \int d^4x \partial_\mu \underbrace{-\eta^{\mu\nu}b_\nu\phi}_{j^\mu} = 0$$

Yes, if j^μ vanishes at the boundary, which is indeed true, for b_ν is constant and ϕ decays at ∞.

Example. Now we study a special case of (1.27)

$$a^\mu = (a, 0, 0, 0)$$

time translation.

$$\phi(x) \to \phi(x-a) = \phi(x) - a\partial_0\phi \tag{1.37}$$

Clearly

$$S = \int d^4x - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$$

is invariant, $\mathcal{L} \to \mathcal{L} + \partial_\mu F^\mu$ and

$$\partial_\mu F^\mu = a\partial_0(\partial\phi)^2 + am^2\phi\partial_0\phi$$

so

$$F^\mu = a\eta^{0\mu}\left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right)$$

so by (1.36)

$$j^\mu = \partial^0\phi\partial^\mu\phi - \eta^{0\mu}\left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right)$$

forgetting about a.

From (1.33)

$$H = \int d^3x j^0 \tag{1.38}$$

claim: Noether current associated with time translation gives Hamilton, or Hamilton density

$$\mathcal{H} = j^0$$

In our case

$$\begin{aligned}
\mathcal{H} &= (\partial^0\phi)^2 + \left(-\frac{1}{2}(\partial^0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right) \\
&= \frac{1}{2}(\partial^0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2
\end{aligned}$$

which is indeed correct if we use conjugate momentum (1.22)

$$\mathcal{H} = (\partial^0\phi)^2 - \mathcal{L} \tag{1.39}$$

In sum, we found symmetry of time translation, then we found conserved current, then found $\mathcal{H}$. Later we will show that $\mathcal{H}$ generates symmetry and $\mathcal{H}$ also gives the time evolution, just like

$$i\hbar\frac{\partial}{\partial t}|\phi\rangle = H|\phi\rangle \tag{1.40}$$

This somehow conflicts some old books saying that Schrodinger equation is out when QM meets GR. In fact (1.40) is correct as long as H is Lorentz invariant and ϕ is expressed in the term of Heisenberg picture, because ϕ itself is an operator. (cf Problem Set 4 problem 1)

1.6. Symmetry Transformation

Lecture 5 (2/5/14)

Now we study symmetry of (1.28) Lorentz transformation. Recall in Schrodinger picture

$$e^{iH\Delta t}|\psi(t)\rangle = |\psi(t-\Delta t)\rangle$$

in Heisenberg picture

$$e^{-iH\Delta t}\hat{O}(t)e^{iH\Delta t} = \hat{O}(t-\Delta t)$$

We do the similar thing for our field operator $\hat{\phi}(t,\vec{x})$, e.g. momentum operator

$$e^{ip_\mu x^\mu}\hat{\phi}(x)e^{-ip_\mu x^\mu} = \hat{\phi}(x^\mu - a^\mu)$$

How states change under Lorentz transformation?

$$U^{-1}(\Lambda)\hat{\phi}(x)U(\Lambda) = \hat{\phi}(\Lambda^{-1}x) \tag{1.41}$$

by choosing this way as we'll see

$$U(\Lambda)|k\rangle = |\Lambda k\rangle \tag{1.42}$$

Consider small Lorentz transformation

$$\Lambda = I + \delta w \tag{1.43}$$

then

$$U(I + \delta w) = I + \frac{i}{2}\delta w_{\mu\nu} M^{\mu\nu}$$

that is because U must obey

$$U(\Lambda)U(\Lambda') = U(\Lambda\Lambda') \tag{1.44}$$

that is linearity in δw

$$U(I + \delta w)U(I + \delta w') = U(I + \delta w + \delta w')$$

For Lorentz transformation, $\delta w_{\mu\nu}$ is antisymmetric see (1.26), the symmetric part of M is useless, so M is too antisymmetric with respect to $\mu\nu$. The 1/2 factor will gives e.g.

$$\frac{\delta w_{12} M^{12} + \delta w_{21} M^{21}}{2} = \delta w_{12} M^{12}$$

so that in

$$M^{\mu\nu} = \begin{pmatrix} 0 & -K_1 & -K_2 & -K_3 \\ & 0 & J_3 & -J_2 \\ & & 0 & J_1 \\ & & & 0 \end{pmatrix} \tag{1.45}$$

K_i, J_i's are generators of the boots and rotations. Or in short notation

$$\begin{aligned} M^{ij} &= \epsilon^{ijk} J_k \\ M^{i0} &= K_i \end{aligned}$$

e.g. $J_3 = M_{12}$ is the angular momentum operator along $\hat{z}$ and $\delta w_{12} = \theta$ is the rotation angle of the z axis. Not only J_i are not commute, J and K are not commute. Combining with (1.44), they give Lie algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma}) - i(\eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho}) \tag{1.46}$$

From there we get some useful

$$
\begin{cases}
[J_i, J_j] & = i\epsilon_{ijk}J_k \\
[J_i, K_j] & = i\epsilon_{ijk}K_k \\
[K_i, K_j] & = -i\epsilon_{ijk}J_k
\end{cases}
\tag{1.47}
$$

The last line shows why boosts in two direction result a rotation. One can extend the 6 generators of the Lorentz transformation to 10 generators of Poincare group, including 3 space and 1 time translation, then

$$
\begin{aligned}
[J_i, H] &= 0 \\
[J_i, P_j] &= i\epsilon_{ijk}P_k \\
[K_i, H] &= iP_i \\
[K_i, P_j] &= i\delta_{ij}H
\end{aligned}
$$

It is also true from (1.44)

$$
U(\Lambda^{-1}) = U(\Lambda)^{-1} \tag{1.48}
$$

Now look at what operator does to the wave function (1.16).

$$
\hat{\phi}(\Lambda^{-1}x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2w_k}(a_k e^{ik\cdot\Lambda^{-1}x} + h.c.)
$$

$h.c.$ = hermitian conjugate. Claim:

$$
k\cdot(\Lambda^{-1}x) = (\Lambda k)\cdot x
$$

Indeed

$$
k\cdot(\Lambda^{-1}x) = \eta_{\mu\nu}k^\mu {\Lambda_\alpha}^\nu x^\alpha = \Lambda_{\alpha\mu}k^\mu x^\alpha = \eta_{\alpha\beta}{\Lambda^\beta}_\mu k^\mu x^\alpha = (\Lambda k)\cdot x
$$

Therefore let $k' = \Lambda k$, since $\int \frac{d^3\vec{k}}{(2\pi)^3 2w_k}$ is Lorentz invariant measure.

$$\hat{\phi}(\Lambda^{-1}x) = \int \frac{d^3\vec{k}'}{(2\pi)^3 2w_{k'}}(a_{\Lambda^{-1}k'}e^{ik'x} + h.c.)$$

on the other hand

$$\hat{\phi}(\Lambda^{-1}x) = U^{-1}(\Lambda)\hat{\phi}(x)U(\Lambda) = \int \frac{d^3\vec{k}}{(2\pi)^3 2w_k}(U^{-1}(\Lambda)a_k U(\Lambda)e^{ikx} + h.c.)$$

Hence

$$a_{\Lambda^{-1}k} = U^{-1}(\Lambda)a_k U(\Lambda)$$

and

$$a^+_{\Lambda^{-1}k} = U^{-1}(\Lambda)a^+_k U(\Lambda)$$

or by (1.48)

$$a^+_{\Lambda k} = U(\Lambda)a^+_k U(\Lambda^{-1})$$

Let us infer how $U(\Lambda)$ acts on states

$$U(\Lambda)\,|k\rangle = U(\Lambda)a^+_k\,|0\rangle = \underbrace{U(\Lambda)a^+_k U(\Lambda^{-1})}_{a^+_{\Lambda k}}\underbrace{U(\Lambda^{-1})\,|0\rangle}_{|0\rangle} = |\Lambda k\rangle$$

proving (1.42). Here we used a fact that Lorentz transform of $|0\rangle$ is $|0\rangle$, so in particular boosts vacuum results constant energy. Later we will come back to this point, and show that in the case of spontaneous symmetry breaking, where some symmetry operations can change vacuum to a different vacuum. This happens when H has certain symmetry that the vacuum doesn't.

2. LSZ Interaction

2.1. LSZ Formula

This formula relates the probability of scattering

$$|_{out}\,\langle k_1' k_2' | k_1 k_2\rangle_{in}|^2$$

to correlation function

$$\left\langle 0 \middle| T\hat{\phi}(x_2')\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_1) \middle| 0 \right\rangle$$

where $|k_1 k_2\rangle_{in}$ is incoming state and $|k_1' k_2'\rangle_{out}$ is outgoing state, and T is time order, i.e.

$$T\hat{O}_1\hat{O}_2 = \hat{O}_1(t_1)\hat{O}_2(t_2)\theta(t_1 > t_2) + \hat{O}_2(t_2)\hat{O}_1(t_1)\theta(t_2 > t_1)$$

The formula is

$$\begin{aligned} {}_{out}\langle k_1' k_2' | k_1 k_2 \rangle_{in} &= i^4 \int d^4x_2' e^{-ik_2'x_2'}(-\Box_{x_2'} + m^2) \int d^4x_1' e^{-ik_1'x_1'}(-\Box_{x_1'} + m^2) \\ &\quad \int d^4x_2 e^{ik_2x_2}(-\Box_{x_2} + m^2) \int d^4x_1 e^{ik_1x_1}(-\Box_{x_1} + m^2) \\ &\quad \left\langle 0 \middle| T\hat{\phi}(x_2')\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_1) \middle| 0 \right\rangle \end{aligned} \tag{2.1}$$

This formula is true for any interaction theory of scalar particles ϕ and time independent H (i.e. stable particles), not necessary 2 in 2 out, and the mass of the particles don't have to be the same.

Proof. (problem set 2 problem 7) Start from (1.17), where we assumed $a_k^\pm$ was time independent, but in general it can depend on t, so we use another notation to denote $a_k^\pm$

$$\tilde{\phi}_k(t) = i \int d^3x \left(e^{-ikx}\partial_t\phi - \phi\partial_t e^{-ikx} \right) \tag{2.2}$$

$$\tilde{\phi}_k^+(t) = -i \int d^3x \left(e^{ikx}\partial_t\phi - \phi\partial_t e^{ikx} \right) \tag{2.3}$$

so we say

$$|k\rangle_{in} = \lim_{t\to-\infty} \tilde{\phi}_k^+(t)\,|0\rangle \qquad |k_1k_2\rangle_{in} = \lim_{t\to-\infty} \tilde{\phi}_{k_2}^+(t)\tilde{\phi}_{k_1}^+(t)\,|0\rangle$$

$$|k'\rangle_{out} = \lim_{t\to\infty} \tilde{\phi}_{k'}^+(t)\,|0\rangle \qquad |k_1'k_2'\rangle_{out} = \lim_{t\to\infty} \tilde{\phi}_{k_2'}^+(t)\tilde{\phi}_{k_1'}^+(t)\,|0\rangle$$

We don't distinguish a state of the vacuum in the far past and far future. Because in the proof we need to use (2.6), we should assume

$$|k\rangle_{in} \qquad |k\rangle_{out} \tag{2.4}$$

at far distance from the collisions are plane waves.

Similarly a state of a particle of momentum k in the far past should remain in the same state in the far future, i.e.

$$\lim_{t\to\infty} |0\rangle = \lim_{t\to-\infty} |0\rangle$$

$$|k\rangle_{in} = |k'\rangle_{out} \quad \text{if } k = k'$$

for single particle, but

$$|k_1 k_2\rangle_{in} \neq |k_1 k_2\rangle_{out} \tag{2.5}$$

(read Weinberg volume I, 3.1 to find out more about the distinctions between in, out and free states.)

Now compute the RHS of (2.1), pay attention only to terms involving x_2'

$$i\int d^4x_2' e^{-ik_2'x_2'}(-\Box_{2'}+m^2)\left\langle 0\middle| T\hat{\phi}(x_2')\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle$$

$$\begin{aligned}
&= \left\langle 0\middle|\int d^4x_2' i e^{-ik_2'x_2'}(-\Box_{2'}+m^2)[T...\phi(x_2')...]\middle|0\right\rangle \\
&= \left\langle 0\middle|T...\int d^4x_2' i\left(e^{-ik_2'x_2'}\partial^2_{t_2'}\phi(x_2') - \phi(x_2')\partial^2_{t_2'}e^{-ik_2'x_2'}\right)...\middle|0\right\rangle
\end{aligned}$$

by integration by part in space twice, and use

$$\vec{k}^2 + m^2 = w_k^2 = k^{0^2} \tag{2.6}$$

By (2.2)

$$\begin{aligned}
&= \left\langle 0\middle|T...\int d^4x_2' i\partial_{t_2'}\left(e^{-ik_2'x_2'}\partial_{t_2'}\phi(x_2') - \phi(x_2')\partial_{t_2'}e^{-ik_2'x_2'}\right)...\middle|0\right\rangle \\
&= \left\langle 0\middle|T...\left(\tilde{\phi}_{k_2'}(\infty) - \tilde{\phi}_{k_2'}(-\infty)\right)...\middle|0\right\rangle \\
&= \left\langle 0\middle|\tilde{\phi}_{k_2'}(\infty)T\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle
\end{aligned}$$

because

$$\tilde{\phi}_{k_2'}(-\infty)|0\rangle = 0$$

Next integrate

$$i\int d^4x_1' e^{-ik_1'x_1'}(-\Box_{1'}+m^2)\left\langle 0\middle|\tilde{\phi}_{k_2'}(\infty)T\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle$$

$$\begin{aligned}&= \left\langle 0\middle|\tilde{\phi}_{k_2'}(\infty)\tilde{\phi}_{k_1'}(\infty)T\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle\\ &= {}_{out}\left\langle k_2',k_1'\middle|T\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle\end{aligned}$$

Next integrate as if(2.6)

$$i\to -i$$

$$-i\int d^4x_2 e^{ik_2x_2}(-\Box_2+m^2)\,{}_{out}\left\langle k_2',k_1'\middle|T\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle$$

$$\begin{aligned}&= {}_{out}\left\langle k_2',k_1'\middle|T...\left(\tilde{\phi}^+_{k_2}(\infty)-\tilde{\phi}^+_{k_2}(-\infty)\right)...\middle|0\right\rangle\\ &= {}_{out}\left\langle k_2',k_1'\middle|\tilde{\phi}^+_{k_2}(\infty)\hat{\phi}(x_1)\middle|0\right\rangle - {}_{out}\left\langle k_2',k_1'\middle|\hat{\phi}(x_1)\tilde{\phi}^+_{k_2}(-\infty)\middle|0\right\rangle\end{aligned}$$

Lastly integrate

$$-i\int d^4x_1 e^{ik_1x_1}(-\Box_1+m^2)\left({}_{out}\left\langle k_2',k_1'\middle|\tilde{\phi}^+_{k_2}(\infty)\hat{\phi}(x_1)\middle|0\right\rangle - {}_{out}\left\langle k_2',k_1'\middle|\hat{\phi}(x_1)\tilde{\phi}^+_{k_2}(-\infty)\middle|0\right\rangle\right)$$

$$\begin{aligned}= {}& {}_{out}\left\langle k_2',k_1'\middle|\hat{\phi}^+_{k_1}(\infty)\tilde{\phi}^+_{k_2}(\infty)\middle|0\right\rangle - {}_{out}\left\langle k_2',k_1'\middle|\hat{\phi}^+_{k_2}(\infty)\tilde{\phi}^+_{k_1}(-\infty)\middle|0\right\rangle\\ &-{}_{out}\left\langle k_2',k_1'\middle|\tilde{\phi}^+_{k_1}(\infty)\hat{\phi}^+_{k_2}(-\infty)\middle|0\right\rangle + {}_{out}\left\langle k_2',k_1'\middle|\hat{\phi}^+_{k_2}(-\infty)\tilde{\phi}^+_{k_1}(-\infty)\middle|0\right\rangle\end{aligned}$$

The first 3 terms are equal to 1, because all have at least one out-k in the bra, e.g.

$${}_{out}\left\langle k_2',k_1'\middle|\hat{\phi}^+_{k_2}(\infty)\tilde{\phi}^+_{k_1}(-\infty)\middle|0\right\rangle = {}_{out}\left\langle k_2',k_1'|k_2\right\rangle_{out}|k_1\rangle_{in} = {}_{out}\langle k_1'|k_1\rangle_{in}\,\delta(k_2'-k_1)$$

so to get non-zero 2 point function, $k_2'=k_1$, so no collision, so ${}_{out}\langle k_1'|k_1\rangle_{in}=1$, too.

Therefore the more precise statement of LSZ, is to add 1 to (2.1)

$$\begin{aligned} {}_{out}\langle k_1'k_2'|k_1k_2\rangle_{in} -' 1' &= i^4 \int d^4x_2' e^{-ik_2'x_2'}(-\Box_{x_2'}+m^2)\int d^4x_1' e^{-ik_1'x_1'}(-\Box_{x_1'}+m^2) \\ &\quad \int d^4x_2 e^{ik_2x_2}(-\Box_{x_2}+m^2)\int d^4x_1 e^{ik_1x_1}(-\Box_{x_1}+m^2) \\ &\quad \left\langle 0\middle|T\hat{\phi}(x_2')\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle \\ &= i^4(k_1'^2+m^2)(k_2'^2+m^2)(k_1^2+m^2)(k_2^2+m^2)\int d^4x_1'd^4x_2'd^4x_1d^4x_2 \\ &\quad e^{i(-k_1'x_1-k_2'x_2+k_1x_1+k_2x_2)}\left\langle 0\middle|T\hat{\phi}(x_2')\hat{\phi}(x_1')\hat{\phi}(x_2)\hat{\phi}(x_1)\middle|0\right\rangle \end{aligned}$$

where

$$'1' = (2\pi)^6 4w_{k_1}w_{k_2}(\delta^3(\vec{k}_1-\vec{k}_1')\delta^3(\vec{k}_2-\vec{k}_2')+\delta^3(\vec{k}_1-\vec{k}_2')\delta^3(\vec{k}_2-\vec{k}_1'))$$

QED

Because we expect total momentum to be conserved, we put

$$ {}_{out}\langle k_1'k_2'|k_1k_2\rangle_{in} -' 1' = (2\pi)^4\delta^4(k_1+k_2-k_1'-k_2')i\mathcal{M} \tag{2.7}$$

$\mathcal{M}$ is the scattering amplitude.

Because of (2.5), it is convenient to define S matrix

$$ {}_{out}\langle k_1'k_2'|k_1k_2\rangle_{in} = {}_{free}\langle k_1'k_2'|S|k_1k_2\rangle_{free}$$

so

$$ {}_{out}\langle k_1'k_2'|k_1k_2\rangle_{in} -' 1' = {}_{free}\langle k_1'k_2'|S-I|k_1k_2\rangle_{free}$$

2.2. Path Integrals

Lecture 6 (2/10/14)

To use LSZ, we need to compute correlation function.

We will prove two claims:

(1)

$$\langle q'', t''|q', t'\rangle = \int Dq e^{iS} \tag{2.8}$$

(2)

$$\langle q'', t''|T\hat{q}(t_a)\hat{q}(t_b)|q', t'\rangle = \int Dq q(t_a)q(t_b)e^{iS} \tag{2.9}$$

where

$$S = \int_{t'}^{t''} dt L(q(t), \dot{q}(t))$$

with two ends are fixed

$$q(t') = q' \quad q(t'') = q''$$

$$Dq = \lim_{N\to\infty} dq(t_1)dq(t_2)...dq(t_N)$$

In classical mechanics (1) becomes

$$\langle q'', t''|q', t'\rangle = \int Dq e^{iS/\hbar} \quad \hbar \to 0$$

so the path is dominated by extremal value of S hence classical path. Also notice that in the Lagrangian $q(t), \dot{q}(t)$ are numbers not operators.

For (2), in the LHS of (2.9) $\hat{q}(t_a)\hat{q}(t_b)$ are operators and they are time ordered, whiles in the RHS $q(t_a)q(t_b)$ are numbers and their order don't matter.

First ask what $|q', t'\rangle$ is? $|q', t'\rangle$ is a Heisenberg eigenstate. Recall in Schrodinger picture

$${}_s\left\langle q''\middle|e^{-iH(t''-t')}\middle|q'\right\rangle_s = \text{transition amplitude}$$

or

$$\underbrace{{}_s\langle q''|\, e^{-iHt''}}_{\langle q'',t''|}\underbrace{e^{iHt'}\, |q'\rangle_s}_{|q',t'\rangle} \tag{2.10}$$

think $e^{iHt'}|q'\rangle$ as backward in time, so q' in $|q', t'\rangle$ is not $q'_s(t')$. More precisely, suppose we have a Schrodinger state

$$|q(t)\rangle_s$$

and a Schrodinger operator

$$\hat{q}_s$$

s.t.

$$\hat{q}_s\,|q(t)\rangle_s = q(t)\,|q\rangle_s$$

The counter part of $\hat{q}_s$ in Heisenberg picture is

$$\hat{q}(t) = e^{iHt}\hat{q}_s e^{-iHt} \tag{2.11}$$

Then take our $|q', t'\rangle$

$$\hat{q}(t')|q', t'\rangle = e^{iHt'}\hat{q}_s e^{-iHt'} e^{iHt'}|q'\rangle_s = e^{iHt'}q'|q'\rangle_s = q'|q', t'\rangle$$

that is $|q', t'\rangle$ is a Heisenberg eigenstate.

We now prove (2.8),

$$\langle q'', t''|q', t'\rangle = \int dq_N dq_{N-1}...dq_1 \langle q'', t''|q_N, t_N\rangle \langle q_N, t_N|q_{N-1}, t_{N-1}\rangle ... \langle q_1, t_1|q', t'\rangle$$

Consider one of them

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle$$

Let

$$\delta t = t_{j+1} - t_j$$

By (2.10)

$$\begin{aligned} \langle q_{j+1}, t_{j+1}|q_j, t_j\rangle &= {}_s\left\langle q_{j+1}\middle|e^{-i\hat{H}\delta t}\middle|q_j\right\rangle_s \\ &= \int dp_j\, {}_s\langle q_{j+1}|p_j\rangle_s\, {}_s\left\langle p_j\middle|e^{-i\hat{H}\delta t}\middle|q_j\right\rangle_s \end{aligned}$$

$${}_s\langle q_{j+1}|p_j\rangle_s = \frac{1}{\sqrt{2\pi}}e^{ip_j q_{j+1}}$$

For ${}_s\left\langle p_j\middle|e^{-i\hat{H}\delta t}\middle|q_j\right\rangle_s$, since $H = p^2/2m + V(q)$, apply momentum to the left and apply position to the right

$${}_s\left\langle p_j\middle|e^{-i\hat{H}\delta t}\middle|q_j\right\rangle_s = e^{-iH\delta t}\, {}_s\langle q_j|p_j\rangle_s = e^{-iH\delta t}\frac{1}{\sqrt{2\pi}}e^{-ip_j q_j}$$

so

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle = \int \frac{dp_j}{2\pi} e^{i\left[-H(p_j, q_j) + p_j\frac{q_{j+1}-q_j}{\delta t}\right]\delta t}$$

so

$$\langle q'', t''|q', t'\rangle = \int dq_N \frac{dp_N}{2\pi}...dq_1 \frac{dp_1}{2\pi} e^{i\left[\sum_{j=0}^{N} p_j \frac{q_{j+1}-q_j}{\delta t} - H(p_j, q_j)\delta t\right]}$$

with

$$q_0 = q' \qquad q_{N+1} = q''$$

$$\begin{aligned} \langle q'', t''|q', t'\rangle &= \int DqDp \frac{1}{(2\pi)^N} e^{i\int_{t'}^{t''} dt(p\dot{q} - H(p,q))} \\ &= \int DqDp \frac{1}{(2\pi)^N} e^{i\int_{t'}^{t''} dt(p\dot{q} - \frac{p^2}{2m} - V(q))} \end{aligned}$$

Integrate over dp by putting imaginary time $t \to it$ we get a Gaussian integral, then analytic continuation back to real time. Thus

$$\begin{aligned} \langle q'', t''|q', t'\rangle &= \int DqDp \frac{1}{(2\pi)^N} e^{i\int_{t'}^{t''} dt\left[-\frac{1}{2m}(p - m\dot{q})^2 + \frac{1}{2}m\dot{q}^2 - V(q)\right]} \\ &= \int Dq e^{i\int_{t'}^{t''} dt\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]} \end{aligned}$$

absorb any proportionality into Dq, proving claim (1).

Now we prove claim (2), WLOG, assume $t'' > t_a > t'$, by (2.11)

$$\begin{aligned} \langle q'', t''|\hat{q}(t_a)|q', t'\rangle &= \left\langle q'', t'' \middle| e^{i\hat{H}t_a} \hat{q}_s e^{-i\hat{H}t_a} \middle| q', t' \right\rangle \\ &= \int dq_x dq_{xx} \underbrace{{}_s\left\langle q'' \middle| e^{-i\hat{H}(t''-t_a)} \middle| q_{xx} \right\rangle_s}_{I} \underbrace{{}_s\langle q_{xx}|\hat{q}_s|q_x\rangle_s}_{q(t_a)\delta(q_x - q_{xx})} \underbrace{{}_s\left\langle q_x \middle| e^{-i\hat{H}(t_a - t')} \middle| q' \right\rangle_s}_{II} \end{aligned}$$

by (2.10) I and II are path integrals from t_a to t'' and t' to t_a with fixed end points

$$q'' = q(t'') \qquad q' = q(t') \qquad q_{xx} = q(t_a)$$

therefore by claim (1)

$$\langle q'', t''|\hat{q}(t_a)|q', t'\rangle = \int Dq e^{i\int_{t_a}^{t''} dtL + t\int_{t'}^{t_a} dtL} q(t_a) = \int Dq e^{i\int_{t'}^{t''} dtL} q(t_a)$$

For $\langle q'', t''|T\hat{q}(t_a)\hat{q}(t_b)|q', t'\rangle$, same thing WLOG assume $t'' > t_a > t_b > t'$, then insert complete states in between, we get

$$\langle q'', t''|T\hat{q}(t_a)\hat{q}(t_b)|q', t'\rangle = \int Dqe^{i\int_{t'}^{t''} dtL} q(t_a)q(t_b)$$

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or in general

$$\langle q'', t''|T\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)|q', t'\rangle = \int Dqe^{i\int_{t'}^{t''} dtL} q(t_1)q(t_2)...q(t_N)$$

It is not so obvious that how to compute the path integral unless it looks like Gaussian. Luckily almost always the problems we study will be able to convert to Gaussian integrals.

2.3. 2 Point Function

A few formulas we should remember

1D

$$\int dxe^{-\frac{1}{2}Mx^2} = \sqrt{\frac{2\pi}{M}}$$

$$\int dxe^{-\frac{1}{2}Mx^2+Jx} = \sqrt{\frac{2\pi}{M}}e^{\frac{J^2}{2M}}$$

If we need to compute the moments

$$\langle x^2\rangle = \int dxx^2\frac{e^{-\frac{1}{2}Mx^2}}{\sqrt{\frac{2\pi}{M}}} = \frac{\partial^2}{\partial J^2}\bigg|_{J=0}\int dx\frac{e^{-\frac{1}{2}Mx^2+Jx}}{\sqrt{\frac{2\pi}{M}}} = \frac{1}{M}$$

which is a technique of generating function.

A useful formula

$$\langle x^n\rangle = \begin{cases} 0 & n \text{ odd} \\ \#\left\langle x^{n/2}\right\rangle^{n/2} & n \text{ even} \end{cases}$$

where # is the number of Wick pairings. E.g.

$$\left\langle x^4\right\rangle = 3\left\langle x^2\right\rangle^2$$

because

$$\langle x, y, z, w\rangle = \langle x, y\rangle \langle z, w\rangle + \langle x, z\rangle \langle y, w\rangle + \langle x, w\rangle \langle y, z\rangle \tag{2.12}$$

This is called Gaussian random field. Later we will show that the idea is that any n-point function can be expressed as product of 2-pont functions.

Gaussian integral in n dimensions

$$\int d^n x e^{-\frac{1}{2}x^T M x} = \sqrt{\frac{(2\pi)^n}{\det M}} \tag{2.13}$$

$$\int d^n x e^{-\frac{1}{2}x^T M x + J^T x} = \sqrt{\frac{(2\pi)^n}{\det M}} e^{\frac{1}{2}J^T M^{-1} J} \tag{2.14}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad J = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_n \end{pmatrix} \qquad M = n \times n \text{ matrix}$$

We can similarly compute moments

$$\langle x_i x_j\rangle = \int d^n x x_i x_j \frac{e^{-\frac{1}{2}x^T M x}}{\sqrt{\frac{(2\pi)^n}{\det M}}} = \frac{\partial^2}{\partial J_i \partial J_j}\bigg|_{J_i, J_j = 0} \int d^n x \frac{e^{-\frac{1}{2}x^T M x + J^T x}}{\sqrt{\frac{(2\pi)^n}{\det M}}} = (M^{-1})_{ij} \tag{2.15}$$

One can prove (2.12) by taking derivatives of (2.14) just like (2.15) and induction.

More generally if we have

$$\int d^n x P(x) e^{J^T x} = \text{generating function}$$

where $P(x)$ is probability density, i.e.

$$\int d^n x P(x) = 1$$

Now observe claims (2.8) and (2.9), say we want to compute (2.9). We can start from (2.8)

Let

$$Z_f = \int Dq e^{i\int dt(L+f(t)q(t))}$$

where we pretend that

$$\int Dq e^{i\int dtL} \to 1 \tag{2.16}$$

probability amplitude is $\int d^n x P(x)$ and $f(t) \to J$ and $q \to x$, therefore

$$\langle q'', t''|T\hat{q}(t_a)\hat{q}(t_b)|q', t'\rangle = \frac{\delta}{\delta i f(t_a)} \frac{\delta}{\delta i f(t_b)}\bigg|_{f=0} Z_f$$

where $\frac{\delta}{\delta i f(t_a)}$ is kind of functional derivative, e.g.

$$\frac{\delta f(t')}{\delta f(t)} = \delta(t'-t)$$

which is analogous to

$$\frac{\partial J_i}{\partial J_j} = \delta_{ij}$$

Now we want to connect (2.9)

$$\langle q'', t''|T\hat{q}(t_1)\hat{q}(t_2)|q', t'\rangle$$

to the correlation function from LSZ

The trick is to give H a small negative imaginary part

$$|q', t'\rangle = e^{iHt'}|q'\rangle_s$$

inserting a complete set of energy eigenstates

$$|q', t'\rangle = \sum |n\rangle \langle n| e^{iHt'}|q'\rangle_s$$

Change

$$H \to H - i\epsilon H$$

and let

$$t \to -\infty$$

so that the lowest energy dominates

$$\lim_{t' \to -\infty} |q', t'\rangle = |0\rangle \langle 0| e^{iHt' + H\epsilon t'} |q'\rangle_s$$

projecting onto ground state, or in short

$$\lim_{t' \to -\infty} |q', t'\rangle \propto |0\rangle$$

Similar argument shows

$$\lim_{t'' \to \infty} \langle q'', t''| \propto \langle 0|$$

Therefore

$$\langle 0|T\hat{q}(t_1)\hat{q}(t_2)|0\rangle = \int Dqe^{iS}q(t_1)q(t_2) \tag{2.17}$$

$$S = \int_{-\infty}^{\infty} dtL$$

absorb any proportionality into Dq. With the normalization

$$1 = \langle 0|0\rangle = \int Dqe^{iS}$$

agreeing (2.16).

How to compute the path integrals? To ensure (2.17) is well-defined, we often do the following two tricks.

Example. SHO

$$H = \frac{1}{2}\dot{q}^2 + \frac{1}{2}w^2q^2$$

One way is to deform H

$$H \to H - i\epsilon H = \frac{1}{2}(1 - i\epsilon)\dot{q}^2 + \frac{1}{2}(w^2 - i\epsilon)q^2$$

at the level of action

$$\int Dqe^{i\int_{-\infty}^{\infty} dt\left(\frac{1}{2}\dot{q}^2 + \frac{1}{2}(w^2 - i\epsilon)q^2\right)}$$

then compute Gaussian in dq.

Another way is to go to Euclidean time (Wick rotation)

$$t \to t_E = it$$

$$\int Dq e^{-\int_{-\infty}^{\infty} dt_E \left(\frac{1}{2}(\frac{dq}{dt_E})^2 + \frac{1}{2}w^2q^2\right)}$$

we too get Gaussian in dq.

2.4. Feynman Propagator

Now we elevate path integral for qm to path integral for field theory, so we replace (2.17) by

$$\left\langle 0 \middle| T\hat{\phi}(x_1)...\hat{\phi}(x_n) \middle| 0 \right\rangle = \int D\phi e^{iS} \phi(x_1)...\phi(x_n)$$

Normalization

$$1 = \langle 0|0 \rangle = \int D\phi e^{iS}$$

Generating function, denoted

$$\langle 0|0 \rangle_J = Z[J] = \int D\phi e^{iS + i\int d^4x J(x)\phi(x)} \tag{2.18}$$

The short hand notation for $\left\langle 0 \middle| T\hat{\phi}(x_1)...\hat{\phi}(x_n) \middle| 0 \right\rangle$ is

$$\left\langle \hat{\phi}(x_1)...\hat{\phi}(x_n) \right\rangle \tag{2.19}$$

Consider 2-point function

$$\left\langle \hat{\phi}(x_1)\hat{\phi}(x_2) \right\rangle = \int D\phi e^{iS} \phi(x_1)\phi(x_2) \tag{2.20}$$

and

$$
\begin{aligned}
iS &= \int id^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right) \\
&= \int id^4x \left(\frac{1}{2}\phi\Box\phi - \frac{1}{2}m^2\phi^2\right) \\
&= -\frac{1}{2}\int id^4x\phi\left(-\Box + m^2\right)\phi \\
&= -\frac{1}{2}\int id^4xd^4y\phi(x)\underbrace{\delta(x-y)\left(-\Box + m^2\right)}_{\sim M}\phi(y)
\end{aligned}
$$

We are hoping this looks like (2.13). Although here M may not be positive define, we may need to put in $i\epsilon$. We also hope (2.20) looks like (2.15).

$$M = i\delta(x-y)\left(-\Box + m^2\right)$$

what is M^{-1}? Discretize it

$$M_{ab} = i\delta_{ab}\left(-\Box_b + m^2\right)$$

and

$$\sum_b M_{ab}M^{-1}_{bc} = \delta_{ac}$$

so

$$i\delta_{ab}\left(-\Box_b + m^2\right)M^{-1}_{bc} = \delta_{ac}$$

or

$$\left(-\Box_a + m^2\right)iM^{-1}_{ac} = \delta_{ac} = \delta(x_a - x_c) \tag{2.21}$$

Let

$$iM^{-1}_{ac} = \Delta(x_a - x_c)$$

we get

$$\left\langle\hat\phi(x_a)\hat\phi(x_c)\right\rangle = \frac{1}{i}\Delta(x_a - x_c)$$

and $\Delta(x_a - x_c)$ is given by (2.21), or better

$$\left(-\Box_a + m^2 - i\epsilon\right) \Delta(x_a - x_c) = \delta(x_a - x_c) \tag{2.22}$$

If we Fourier transform above

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \tag{2.23}$$

This is the Feynman propagator, a Green's function for the Klein-Gordon equation for the free theory.

Lecture 8 (2/17/14)

It is possible to get 2-point correlation function from (2.22), not putting in

$$m^2 \to m^2 - i\epsilon$$

but using the Euclidean time

$$t \to t_E = it$$

So (2.22) becomes

$$\left(-\Box_E + m^2\right) \Delta_E(x-y) = \delta(x-y)$$

where

$$\Box_E = \frac{\partial^2}{\partial t_E^2} + \nabla^2$$

then

$$\Delta_E(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2} \qquad k^2 = k^{0^2} + \vec{k}^2$$

Instead of doing path integral and generating function, we can obtain the same result of the 2-point correlation function from creation and annihilation operators (1.16) and (1.6). The easiest way to show two methods give same result is to start from (2.23) obtained from path integral methods

$$\left\langle \hat{\phi}(x_1)\hat{\phi}(x_2) \right\rangle = \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x_1-x_2)}}{-k^{0 2} + \vec{k}^2 + m^2 - i\epsilon} \tag{2.24}$$

Write not exactly equal but close to the order of accuracy.

$$-k^{0^2} + \vec{k}^2 + m^2 - i\epsilon = -[k^0 + (\sqrt{\vec{k}^2 + m^2} - i\epsilon)][k^0 - (\sqrt{\vec{k}^2 + m^2} - i\epsilon)]$$

Integrate (2.24) over k^0, there are two simple poles, which is enclosed depending on the exponents

$$e^{-iw_k(t_1-t_2)}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}$$

$t_1 > t_2$ or $t_1 < t_2$, so after contour integral,

$$\left\langle \hat{\phi}(x_1)\hat{\phi}(x_2) \right\rangle = \int \frac{d^3k}{(2\pi)^3 2w_k} \left(\theta(t_1 > t_2)e^{ik(x_1-x_2)} + \theta(t_2 > t_1)e^{-ik(x_1-x_2)}\right) \quad (2.25)$$

one can show that using (1.16) and (1.6), one can get (2.25) too.

(Problem Set 3 problem 3) Similar to the complex manipulation in (1.2), one can further write (2.25) for space-like separation

$$\begin{aligned} \left\langle \hat{\phi}(x_1)\hat{\phi}(x_2) \right\rangle &= \theta(t_1 > t_2)\frac{1}{(2\pi)^2 x_{12}} \int_m^\infty \frac{zdz}{\sqrt{z^2 - m^2}} e^{-zx_{12}} \cosh(\sqrt{z^2 - m^2}t_{12}) \\ &\quad + \theta(t_1 < t_2)\frac{1}{(2\pi)^2 x_{21}} \int_m^\infty \frac{zdz}{\sqrt{z^2 - m^2}} e^{-zx_{21}} \cosh(\sqrt{z^2 - m^2}t_{21}) \end{aligned}$$

where

$$x_{12} = |\vec{x}_1 - \vec{x}_2| = x_{21} \qquad t_{12} = t_1 - t_2 = -t_{21}$$

then one finds a surprising result that two-point function does not in general vanish for space-like separated points. Lorentz invariance does not demand it to be zero. What has to be zero is that operators (observables) should commute at space-like separations. For instance, we expect

$$[\hat{\phi}(x_1), \hat{\phi}(x_2)] = 0$$

for a space-like separation.

Something interesting about the generating function (2.18), it allows to compute interacting field theory, e.g. LSZ, from a free theory, i.e. $\int D\phi e^{iS}$ is free path integral and (2.23) is free propagator. Why? What is J? The new S for the interacting field theory is

$$S = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi \right) \tag{2.26}$$

classical eom

$$(-\Box + m^2)\phi = J \tag{2.27}$$

J is an external source for ϕ, or some coupling, or current.

For simplicity assume J is static, i.e.

$$J = J(\vec{x})$$

then Fourier transforms (2.27)

$$\phi(x) = \int d^4y J(\vec{y}) \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2}$$

first pass dy^0 into the integral

$$\int dy^0 e^{ik^0y^0} = 2\pi\delta(k^0)$$

then integrate over k^0

$$\phi(x) = \int d^3y J(\vec{y}) \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{\vec{k}^2 + m^2}}_{V = \frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}} = \int d^3y J(\vec{y}) V(\vec{x} - \vec{y}) \tag{2.28}$$

We get $V(\vec{x} - \vec{y})$ Yukawa potential, a model for strong interaction.

We can find the energy associated with the field configuration sourced by J, by (1.39)

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - J\phi$$

So energy is

$$E = \int d^3x \mathcal{H}(\vec{x})$$

Plug (2.28) in

$$E = \int d^3x \frac{1}{2}\phi(\Box + m^2)\phi - J\phi = -\frac{1}{2}\int d^3x J\phi = -\frac{1}{2}\int d^3x d^3y J(\vec{x})J(\vec{y})V(\vec{x}-\vec{y})$$

The minus sign shows the two "charges" interact attractively, and ϕ mediates the attractive force, and from (2.28) the force is short range

$$\frac{1}{m} = \text{Compton wavelength} \tag{2.29}$$

3. Cubic Interaction

3.1. Renormalization Factors

Lecture 9 (2/19/14)

We are going to extend the idea of (2.26). Suppose we add a cubic interaction term

$$\left\langle \hat{\phi}(x_1)...\hat{\phi}(x_n)\right\rangle = \int D\phi\phi_1(x_1)...\phi(x_n)e^{i\int d^4x(\mathcal{L}_0+\mathcal{L}_1)}$$

where

$$\mathcal{L}_0 = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \qquad \mathcal{L}_1 = \frac{1}{3!}g\phi^3$$

assuming g is small, we are essentially doing perturbation around free theory. Later we will apply this ϕ^3 theory to QED with $g = \frac{1}{137}$.

Then

$$e^{i\int d^4x\mathcal{L}_1} \sim 1 + i\int d^4x\mathcal{L}_1 + \frac{i^2}{2}\left(\int d^4x\mathcal{L}_1\right)^2 + ...$$

and

$$\begin{aligned}\left\langle \hat{\phi}(x_1)...\hat{\phi}(x_n)\right\rangle &= \int D\phi\phi_1(x_1)...\phi(x_n)\left(1 + i\int d^4x\mathcal{L}_1 + ...\right)e^{i\int d^4x\mathcal{L}_0} \\ &= \left\langle \hat{\phi}(x_1)...\hat{\phi}(x_n)\right\rangle_{free} + \frac{ig}{6}\int d^4x\left\langle \hat{\phi}(x_1)...\hat{\phi}(x_n)\hat{\phi}^3(x)\right\rangle_{free} + O(g^2)\end{aligned}$$

There is another formal way of doing perturbation, using generating function. Let

$$Z[J] = \int D\phi e^{iS[\phi]+i\int d^4xJ\phi} = \int D\phi e^{iS_J} \tag{3.1}$$

expand $Z[J]$ in power of the interaction, then take $\frac{\delta}{\delta J}$ to get $\left\langle \hat{\phi}(x_1)...\hat{\phi}(x_n) \right\rangle$.

Consider the following

$$S_J = \int d^4\mathcal{L} + J\phi$$

where

$$\mathcal{L} = -Z_\phi \frac{1}{2}(\partial\phi)^2 - Z_m \frac{1}{2} m^2 \phi^2 + \frac{1}{6} Z_g g \phi^3 + Y\phi + \Lambda \tag{3.2}$$

where Z_ϕ,..., Λ are called renormalization factors, and Λ is also called the cosmological constant. Z_m is there because we want m to be physical mass. Intuitively as particle swim through the vacuum it interacts with itself, so the physical mass changes. Put in Z_g so that g is the physical coupling. Need Y so that

$$\left\langle 0 \middle| \hat{\phi}(x) \middle| 0 \right\rangle = 0 \tag{3.3}$$

i.e. quantum fluctuation around 0 is 0. Need Λ so that

$$\langle 0|H|0\rangle = 0$$

vacuum has 0 energy. Need Z_ϕ so that

$$\left\langle k \middle| \hat{\phi}(x) \middle| 0 \right\rangle = e^{-ikx} \tag{3.4}$$

that is because in proving LSZ, we assumed (2.6) and see remark after (2.4). It is saying that $\hat{\phi}$ satisfies Klein-Gordon, so $\hat{\phi}$ is (1.16), so (3.4) is true. One can rewrite (3.4)

$$\left\langle k \middle| \hat{\phi}(x) \middle| 0 \right\rangle \quad = \quad \left\langle k \middle| e^{-i\hat{p}x} \hat{\phi}(0) e^{i\hat{p}x} \middle| 0 \right\rangle$$

$\hat{p}$ momentum operator acts on $|0\rangle$ is 0.

$$\left\langle k \middle| \hat{\phi}(x) \middle| 0 \right\rangle = e^{-ikx} \left\langle k \middle| \hat{\phi}(0) \middle| 0 \right\rangle$$

so (3.4) is equivalent to

$$\left\langle k \middle| \hat{\phi}(0) \middle| 0 \right\rangle = 1$$

Back to (3.2), write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

where

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \\
\mathcal{L}_1 &= -(Z_\phi - 1)\frac{1}{2}(\partial\phi)^2 - (Z_m - 1)\frac{1}{2}m^2\phi^2 + \frac{1}{6}Z_g g\phi^3 + Y\phi + \Lambda
\end{aligned}$$

It turns out $Z_\phi - 1$, $Z_m - 1$, Z_g, Y scales as power of g, so we can safely do perturbation.

3.2. Feynman Diagrams

The full $Z[J]$ depends on the full $\mathcal{L}$, for now let's focus on cubic interaction

$$Z_1[J] = \int D\phi e^{i\int d^4x \mathcal{L}_0 + \frac{1}{6}Z_g g\phi^3 + J\phi} \tag{3.5}$$

Free generating function

$$Z_0[J] = \int D\phi e^{i\int d^4x \mathcal{L}_0 + J\phi}$$

Check that

$$Z_1[J] = e^{\left[\frac{i}{6}Z_g g\int d^4x\left(\frac{\delta}{\delta iJ(x)}\right)^3\right]} Z_0[J] \tag{3.6}$$

Since

$$\langle\phi_y\phi_z\rangle_{free} = \frac{\delta}{\delta iJ_y}\frac{\delta}{\delta iJ_z}\bigg|_{J=0} Z_0[J] = \left(\frac{1}{i}\Delta_{yz}\right)_{free}$$

there is another way to express $Z_0[J]$

$$Z_0[J] = e^{\frac{i}{2}\int d^4y d^4z J_y J_z \Delta_{yz}} \tag{3.7}$$

we add $\frac{1}{2}$ to Z_0 is for the later step, (3.8) where x will be y then z and z then y.

Combining (3.6), (3.7), expending exponents relaying on the assumption of small g

$$Z_1[J] = \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{i}{6} Z_g g \int d^4x \left(\frac{\delta}{i\delta J(x)} \right)^3 \right]^V \sum_{P=0}^{\infty} \frac{1}{P!} \left[\frac{1}{2} \int d^4y d^4z \, (iJ_y) \, (iJ_z) \left(\frac{1}{i} \Delta_{yz} \right) \right]^P \tag{3.8}$$

$$J = \#\text{ legs sticking out} = 2P - 3V$$

$$V = \#\text{ vertex} \qquad P = \#\text{ propagator}$$

Let us organize (3.8) terms by terms

terms with no J	e.g. $V = 2$ $P = 3$	
$1J$	e.g. $V = 1$ $P = 2$	$\rightarrow$ 1-point function
$2J$		$\rightarrow$ 2-point function
$\vdots$		

Why does $1J$ give 1-point function? and $2J$ gives 2-point function, because

$$\left\langle \hat{\phi}_1 \hat{\phi}_2 \right\rangle = \left. \frac{\delta}{\delta i J_1} \frac{\delta}{\delta i J_2} \right|_{J=0} Z_1[J] \tag{3.9}$$

Let's look closely e.g. what are the terms of $V = 2$ $P = 3$?

$$\frac{1}{2!} \left[\frac{iZ_g g}{6} \int d^4x \left(\frac{\delta}{i\delta J(x)} \right)^3 \right] \left[\frac{iZ_g g}{6} \int d^4y \left(\frac{\delta}{i\delta J(y)} \right)^3 \right] \frac{1}{3!} \left[\frac{1}{2} \int d^4a d^4a' \, (iJ_a) \, (iJ_{a'}) \left(\frac{1}{i} \Delta_{aa'} \right) \right]$$

$$\left[\frac{1}{2} \int d^4b d^4b' \, (iJ_b) \, (iJ_{b'}) \left(\frac{1}{i} \Delta_{bb'} \right) \right] \left[\frac{1}{2} \int d^4c d^4c' \, (iJ_c) \, (iJ_{c'}) \left(\frac{1}{i} \Delta_{cc'} \right) \right]$$

We can study them one at a time, e.g. there is a term of the form

$$\left(\frac{1}{12} \right) (iZ_g g)^2 \int d^4x \int d^4y \left(\frac{1}{i} \Delta_{xy} \right)^3 \tag{3.10}$$

$\frac{1}{12}$ is called symmetric factors, it combines the factors in (3.8) as well as contributions from terms that have the same form. Feynman invented a graphical way to represent this term which is illustrated on page 60 Srednicki, the second one of figure 9.1. We can label the two vertices by x and y, and

$$iZ_g g$$

gives the strength of the vertices, and

$$\frac{1}{i}\Delta_{xy}$$

is the propagator between x and y. Because $J = 0$, no legs sticking out. Notice in ϕ^3 theory, all vertices are connected by 3 internal lines.

E.g. Another term of the form

$$\left(\frac{1}{8}\right)(iZ_g g)^2 \int d^4x \int d^4y \left(\frac{1}{i}\Delta_{xy}\right)\left(\frac{1}{i}\Delta_{xx}\right)\left(\frac{1}{i}\Delta_{yy}\right) \tag{3.11}$$

This term is illustrated on page 60 Srednicki, the first one of figure 9.1.

For larger V, P, it is possible to get disconnected diagrams, e.g. $V = 3$, $P = 6$

$$\sim (iZ_g g)^3 \int d^4x \int d^4y \int d^4z \left(\frac{1}{i}\Delta_{xy}\right)\left(\frac{1}{i}\Delta_{xx}\right)\left(\frac{1}{i}\Delta_{yy}\right) \int d^4a\,(iJ_a)\left(\frac{1}{i}\Delta_{az}\right)$$
$$\int d^4b\,(iJ_b)\left(\frac{1}{i}\Delta_{bz}\right) \int d^4c\,(iJ_c)\left(\frac{1}{i}\Delta_{cz}\right)$$

which consists of two disconnected diagrams, one is (3.11) and the other one is a simple tree with 3 legs and z is the vertex at the center.

There is a theorem to simplify the situation:

Theorem.

$$e^{\textit{connected diagrams}} = \textit{all diagrams}$$

symbolically

$$e^{iW[J]} = Z[J]$$

Lecture 10 (2/24/14)

So we only have to deal with connected diagrams. In our previous notation, from (3.5)

$$Z_1[J] = e^{i\{\text{all no } J \text{ connected diagrams}\} + i\{1\ J\} + i\{2\ J\} + \ldots}$$

all no J connected diagrams are called bubble diagrams, they are not related to interaction as long as gravity is not involved, so we want to neglect them.

3.3. Renormalization Factors (continued)

We add Λ to $\mathcal{L}$

$$Z_1[J] = \int D\phi e^{i\int d^4x\mathcal{L}_0 + \frac{1}{6}Z_g g\phi^3 + J\phi + \Lambda} \tag{3.12}$$

so that now

$$Z_1[J] = e^{i\{1 \ J\} + i\{2 \ J\} + \dots} \tag{3.13}$$

or

$$Z_1[J = 0] = 1 \tag{3.14}$$

It is interesting to point out that $\Lambda = \infty$, each bubble diagram e.g. (3.10) is ∞

$$\begin{aligned}
(3.10) \quad &\sim \int d^4x \int d^4y \left(\frac{1}{i}\Delta_{xy}\right)^3 \\
&= \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} d^4x d^4y \frac{e^{i(k_1+k_2+k_3)(x-y)}}{(k_1^2+m^2-i\epsilon)(k_2^2+m^2-i\epsilon)(k_3^2+m^2-i\epsilon)} \\
&\sim \underbrace{\int d^4k_1 d^4k_2 \frac{1}{k_1^2+m^2-i\epsilon}\frac{1}{k_2^2+m^2-i\epsilon}\frac{1}{(k_1+k_2)^2+m^2-i\epsilon}}_{I}\underbrace{\int d^4x}_{II}
\end{aligned}$$

$II \to \infty$ but it is harmless, it is the same kind of divergence in (3.12)

$$\int d^4x\Lambda = \Lambda \int d^4x$$

while as $I \to \infty$, because both k_1, k_2 have 4 power upstairs and downstairs, called logarithmic divergence, which is bad, because I is the Lagrangian density. So $I \to \infty$ even before integrating over spacetime. Later we will study how to regulate theory to get a finite answer.

Consider 1-point function

$$\left\langle \hat{\phi}(x) \right\rangle = \left. \frac{\delta}{\delta(iJ(x))} \right|_{J=0} Z[J] \tag{3.15}$$
$$= \underbrace{e^{iW[J]}}_{1} \left. \frac{\delta W[J]}{\delta J(x)} \right|_{J=0}$$

by (3.13)

$$\left\langle \hat{\phi}(x) \right\rangle = \left. \frac{\delta\{1J \text{ terms connected diag}\}}{\delta J(x)} \right|_{J=0}$$

Claim: we can add $Y\phi$ to (3.12) to enforce

$$\left\langle \hat{\phi}(x) \right\rangle = 0 \tag{3.16}$$

Why do we want (3.16)? It turns out in the derivation of LSZ, we assume (3.16). Similar to the argument we gave following (3.4). We explicitly assume $\hat{\phi}$ is a plane wave, i.e. (2.3) creates prefect eigenstate, e.g.

$$\langle 0|k\rangle_{in} = 0$$

They essentially say that the vacuum fluctuation around 0 is 0 cf (3.3). If we don't enforce (3.16), all of our formulas of 1pt, 2pt... functions would have to be replaced by e.g.

$$\langle [\phi_1 - \langle\phi_1\rangle][\phi_2 - \langle\phi_2\rangle]\rangle$$

How to choose Y to accomplish (3.16)? After adding $Y\phi$, (3.6) becomes

$$Z_1[J] = e^{i\int d^4x Y\left(\frac{\delta}{\delta iJ(x)}\right)} e^{\left[\frac{i}{6} Z_g g \int d^4x \left(\frac{\delta}{\delta iJ(x)}\right)^3\right]} Z_0[J] \tag{3.17}$$

then by (3.8)

$$Z_1[J] = \sum_{V=0}^{\infty} \frac{1}{T!} \left[i \int d^4x Y \left(\frac{\delta}{\delta iJ} \right) \right]^T \sum_{V=0}^{\infty} \frac{1}{V!} \left[\frac{i}{6} Z_g g \int d^4x \left(\frac{\delta}{i\delta J} \right)^3 \right]^V$$
$$\sum_{P=0}^{\infty} \frac{1}{P!} \left[\frac{1}{2} \int d^4y d^4z \, (iJ_y) \, (iJ_z) \left(\frac{1}{i} \Delta_{yz} \right) \right]^P$$

Similar to (3.14), we want

$$Z_1[J=1]=1$$

We compute Y in the order of g. First put

$$T=0, V=1, P=2$$

we get one J with g order of 1

$$\sim Z_g g \int d^4x d^4y (iJ_y) \frac{1}{i} \Delta_{xy} \frac{1}{i} \Delta_{xx} \tag{3.18}$$

Then put

$$T=1, V=0, P=1$$

we get one J with Y order of 1

$$\sim Y \int d^4x d^4y i J_y \frac{1}{i} \Delta_{xy} \tag{3.19}$$

So choose Y so that

$$(3.18)+(3.19)=0$$

The next order is g^3, put

$$T=0, V=3, P=5 \tag{3.20}$$

the counter term is

$$T=1, V=2, P=4 \tag{3.21}$$

we get one J with g order of 3. Making sure them canceled, we find correction of Y to the order of g^3. The procedure is to choose Y order by order of g to cancel all one J terms.

This has additional consequence: this procedure removes all tadpoles, a diagram that becomes two separated pieces if one can remove one internal line and one of the remaining pieces has no J.

One can see that (3.20) has 3 different connected diagrams, on page 61 Srednicki, figure 9.4. What about (3.21)?

$$V = 2, P = 4$$

are illustrated on page 61 Srednicki, figure 9.6. The second one is a tadpole. Removing (3.21) from (3.17) is equivalently to remove tadpoles $V = 2, P = 4$ from (3.13) before putting $Y\phi$.

So after putting Λ, $Y\phi$,

$$Z_1[J] = e^{i\sum(\text{connected, non-bubble, non-tadpoles, } J>1\text{diagrams})}$$

For connected, non-bubble, non-tadpoles $J > 1$ diagrams see page 68 Srednicki, figure 9.13. Among them there are UV divergent terms. That is why we finally add in Z_ϕ and Z_m in (3.12)

$$Z_1[J] = \int D\phi e^{i\int d^4x\mathcal{L}_0+\frac{1}{6}Z_g g\phi^3-(Z_\phi-1)\frac{1}{2}(\partial\phi)^2-(Z_m-1)\frac{1}{2}m^2\phi^2+J\phi+Y\phi+\Lambda}$$

Like before we can do integration by parts on $-(Z_\phi - 1)\frac{1}{2}(\partial\phi)^2 - (Z_m - 1)\frac{1}{2}m^2\phi^2$

$$\frac{1}{2}\phi\left((Z_\phi - 1)\Box - (Z_m - 1)m^2\right)\phi$$

The lowest order of this in $Z_1[J]$ is

$$\sim \int d^4x d^4y d^4z \left((Z_\phi - 1)\Box_x - (Z_m - 1)m^2\right)(iJ_y)(iJ_z)(\frac{1}{i}\Delta_{xy})(\frac{1}{i}\Delta_{xz}) \tag{3.22}$$

Later we will show this is mass renormalization term. It is there is to cancel the second diagram of figure 9.13, on page 68 Srednicki, which is a self interacting term. Although both terms are UV divergent, the sum is finite and the sum give the right physical mass. In other words, self interaction is manifested in mass renormalization.

3.4. Scattering

Lecture 11 (2/26/14)

Let us summarize what we have done. We want to compute the 4 point function

$$\langle \phi_1 \phi_2 \phi_{1'} \phi_{2'} \rangle$$

we follow the generating function approach, for a direct method see problem set 5 problem 1. From (3.9), the 2 point function is given by

$$\begin{aligned}
\langle \phi_1 \phi_2 \rangle &= \frac{\delta}{\delta i J_1} \frac{\delta}{\delta i J_2} \bigg|_{J=0} Z_1[J] \\
&= \frac{\delta}{\delta i J_1} \frac{\delta i W[J]}{\delta i J_2} e^{iW[J]} \bigg|_{J=0} \\
&= \frac{\delta^2 i W[J]}{\delta i J_1 \delta i J_2} e^{iW[J]} \bigg|_{J=0} + \frac{\delta i W[J]}{\delta i J_1} \frac{\delta i W[J]}{\delta i J_2} e^{iW[J]} \bigg|_{J=0}
\end{aligned}$$

$$\frac{\delta i W[J]}{\delta i J_1} \bigg|_{J=0} = 0 \tag{3.23}$$

because of removal of tadpoles.

$$\langle \phi_1 \phi_2 \rangle = \frac{\delta^2 i W[J]}{\delta i J_1 \delta i J_2} \bigg|_{J=0}$$

So $iW[J]$ =connected, non-bubble, non-tadpoles $J > 1$ diagrams including $Z_\phi - 1$, $Z_m - 1$ terms. The first term in $iW[J]$ which is the first diagram of figure 9.13, on page 68 Srednicki, is

$$\frac{1}{2} \int d^4x d^4y i J_x i J_y \frac{1}{i} \Delta_{xy}$$

and $\frac{\delta^2}{\delta i J_1 \delta i J_2}$ of it is

$$\frac{1}{i} \Delta_{xy}$$

i.e. free 2 point function.

So we can think

$$\langle \phi_1 \phi_2 \rangle \equiv \frac{1}{i} \Delta_{full} = \frac{1}{i} \Delta_{xy} + ... \tag{3.24}$$

The first term on the right is from the first diagram of figure 9.13, on page 68 Srednicki, and the second is from the second diagram,... Very often the symmetric factor cancel away.

Following the same steps and by (3.23), we find 4 point function

$$\begin{aligned}
\langle\phi_1\phi_2\phi_{1'}\phi_{2'}\rangle &= \frac{\delta}{\delta iJ_1}\frac{\delta}{\delta iJ_2}\frac{\delta}{\delta iJ_{1'}}\frac{\delta}{\delta iJ_{2'}}\bigg|_{J=0} e^{iW[J]} \\
&= \frac{\delta iW[J]}{\delta iJ_1\delta iJ_2}\frac{\delta iW[J]}{\delta iJ_{1'}\delta iJ_{2'}}e^{iW[J]}\bigg|_{J=0} + \frac{\delta iW[J]}{\delta iJ_1\delta iJ_{1'}}\frac{\delta iW[J]}{\delta iJ_2\delta iJ_{2'}}e^{iW[J]}\bigg|_{J=0} \\
&\quad + \frac{\delta iW[J]}{\delta iJ_1\delta iJ_{2'}}\frac{\delta iW[J]}{\delta iJ_{1'}\delta iJ_2}e^{iW[J]}\bigg|_{J=0} + \frac{\delta iW[J]}{\delta iJ_1\delta iJ_{1'}\delta iJ_2\delta iJ_{2'}}e^{iW[J]}\bigg|_{J=0} \\
&= \langle\phi_1\phi_2\rangle\langle\phi_{1'}\phi_{2'}\rangle + \langle\phi_1\phi_{1'}\rangle\langle\phi_2\phi_{2'}\rangle + \langle\phi_1\phi_{2'}\rangle\langle\phi_{1'}\phi_2\rangle \\
&\quad + \langle\phi_1\phi_2\phi_{1'}\phi_{2'}\rangle^c
\end{aligned}$$

As long as on shell (1.8), only $\langle\phi_1\phi_2\phi_{1'}\phi_{2'}\rangle^c$ contribute the LSZ calculation, because take $\langle\phi_1\phi_2\rangle\langle\phi_{1'}\phi_{2'}\rangle$ e.g.

$$i^4\int d^4x_2'x_1'x_2x_1e^{-ik_2'x_2'-ik_1'x_1'+ik_2x_2+ik_1x_1}(-\Box_1+m^2)(-\Box_2+m^2)(-\Box_{1'}+m^2)(-\Box_{2'}+m^2)\frac{1}{i}\Delta_{12}\frac{1}{i}\Delta_{1'2'} \tag{3.25}$$

by (2.22)

$$\int d^4x_2'x_1'x_2x_1e^{-ik_2'x_2'-ik_1'x_1'+ik_2x_2+ik_1x_1}\delta(x_1-x_2)\delta(x_{1'}-x_{2'})\sim\delta(k_1-k_2)\delta(k_{1'}-k_{2'})$$

so momentum don't change, i.e. no scattering occurred, so we don't care.

See the last five diagrams of figure 9.13, on page 68 Srednicki. One is $O(g^2)$ and the other 4 are $O(g^4)$. For now we study $O(g^2)$, called tree diagram, no loop. Suppose we label the four legs as a,b,c,d and the two vertices as y,z, we get

$$iW[J]_{tree}\sim(iZ_gg)^2\int d^4ad^4bd^4cd^4dd^4yd^4ziJ_aiJ_biJ_ciJ_d\frac{1}{i}\Delta_{ya}\frac{1}{i}\Delta_{yb}\frac{1}{i}\Delta_{zd}\frac{1}{i}\Delta_{zc}\frac{1}{i}\Delta_{yz}$$

take

$$\frac{\delta iW[J]}{\delta iJ_1\delta iJ_{1'}\delta iJ_2\delta iJ_{2'}}\sim(iZ_gg)^2\int d^4yd^4z\frac{1}{i}\Delta_{y1}\frac{1}{i}\Delta_{y2}\frac{1}{i}\Delta_{z1'}\frac{1}{i}\Delta_{z2'}\frac{1}{i}\Delta_{yz} \tag{3.26}$$

Since we can permute the labels on the 4 legs as 1,2,1',2', we have 3 different channels: s, t, u. We will always write the incoming particles on the left legs and outgoing on the right legs. This results for t channel the internal line is vertical, and for u channel the outgoing particles are switch comparing to t channel.

Now compute LSZ for s channel, as we showed before cf (3.25), plug (3.26) into LSZ

$$(iZ_g g)^2 \int d^4y d^4z e^{-ik'_2 z - ikz + ik_2 y + ik_1 y} \frac{1}{i} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(y-z)}}{p^2 + m^2 - i\epsilon} \tag{3.27}$$

Integral over y

$$k_1 + k_2 = -p$$

Integral over z

$$k_{1'} + k_{2'} = -p$$

therefore

$$s \text{ channel} = (iZ_g g)^2 \frac{1}{i} \frac{1}{(k_1 + k_2)^2 + m^2 - i\epsilon} \delta(k_1 + k_2 - k_{1'} - k_{2'})$$

the δ function agrees energy momentum conservation. Thus $s + t + u$ channels is

$$(iZ_g g)^2 \frac{4\pi}{i} \left(\frac{1}{(k_1 + k_2)^2 + m^2 - i\epsilon} + \frac{1}{(k_1 - k_{1'})^2 + m^2 - i\epsilon} + \right. \tag{3.28}$$

$$\left. \frac{1}{(k_1 - k_{2'})^2 + m^2 - i\epsilon} \right) \delta(k_1 + k_2 - k_{1'} - k_{2'})$$

One can interpret the internal line carrying momentum $k_1 + k_2$, etc, the virtual particles being created in the momentum space. This concludes Feynman rule for 2→2 process, i.e. scattering amplitude is product of δ which enforces momentum conservation and green function of the internal propagators.

3.5. Cross Section

Lecture 12 (3/3/14)

One can simplify (3.28) a little bit by introducing Mandelstam variables

$$\begin{aligned} s &= -(k_1 + k_2)^2 \\ t &= -(k_1 - k_{1'})^2 \\ u &= -(k_1 - k_{2'})^2 \end{aligned}$$

If all masses are the same, let θ be the scattering angle, i.e. the angle wrt to the incoming particles, assuming two incoming particles are head-on and θ is measured in the center of momentum (COM) frame, one can show

$$\begin{aligned} t &= -\frac{1}{2}(s-4m^2)(1-\cos\theta) \\ u &= -\frac{1}{2}(s-4m^2)(1+\cos\theta) \end{aligned}$$

In terms of scattering amplitude (2.7)

$$\mathcal{M} = (g)^2 \left(\frac{1}{m^2-s} + \frac{1}{m^2-t} + \frac{1}{m^2-u} \right) \tag{3.29}$$

so if $s \gg m^2$ (extreme relativistic limit)

$$|\mathcal{M}|^2 \sim \frac{g^2}{s^2 \sin^2\theta} \tag{3.30}$$

One can generalize this to 2→2 process with different masses,

$$s+t+u = m_1^2 + m_2^2 + m_{1'}^2 + m_{2'}^2$$

by assuming they are on shell and $k_1 + k_2 = k_{1'} + k_{2'}$.

Meaning of s?

In COM

$$\vec{k}_1 + \vec{k}_2 = 0$$

so

$$s = (k_1^0 + k_2^0)^2 = (E_1 + E_2)^2$$

called center of mass energy squared.

Find differential cross section for 2→N process

$$d\sigma(\text{incoming flux}) = \text{prob of having a scattering/unit time}$$

$$d\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2|(2E_1)(2E_2)} |\mathcal{M}|^2 (2\pi)^4 \delta(k_{1'} + k_{2'} + \ldots + k_N - k_1 - k_2) \frac{d^3\vec{k}_{1'}}{(2\pi)^3 2E_{1'}} \frac{d^3\vec{k}_{2'}}{(2\pi)^3 2E_{2'}} \cdots \frac{d^3\vec{k}_{N'}}{(2\pi)^3 2E_{N'}} \tag{3.31}$$

The expression is Lorentz invariant, and true in any frame.

Instead of proving this expression, we do a dimensional analysis. Inserting V and $1/V$, ∼density, to the denominator

$$|\vec{v}_1 - \vec{v}_2|\frac{1}{V} = \frac{1}{\text{time(length)}^2} = \text{mass}^3 = \text{flux}$$

in fact [length] = [time] because velocity∼ c =dimensionless.

Assume $N = 2$, $\mathcal{M}$ has dimension of g

$$\int d^4x g\phi^3 = L_{int} = \hbar = \text{dimensionless} = \int d^4x - \frac{1}{2}(\partial\phi)^2 \tag{3.32}$$

so

$$\phi = \frac{1}{\text{length}} = \text{mass} = \frac{1}{\text{time}}$$

for the ϕ^3 theory, because

$$[x,p] = \hbar = 1 \implies \text{(length)(velocity)(mass)} = 1 = \text{velocity}=\frac{\text{length}}{time}$$

so everything is in term of power of mass

$$g = \text{mass}$$

so

$$\mathcal{M} = \text{dimensionless}$$

for $N = 2$. It is not dimensionless if $N \neq 2$.

Back to (3.31)

$$\left[\frac{1}{E^2V}\mathcal{M}^2\frac{k^6}{E^2}\delta\right] = \frac{\text{mass}^3}{\text{mass}^2}\frac{\text{mass}^6}{\text{mass}^2}\frac{1}{\text{mass}^4} = \text{mass}$$

therefore

$$[d\sigma] = \text{mass}^2 = \frac{1}{\text{length}^2}$$

We can rewrite (3.31) for 2→2 process in COM. So $\vec{k}_1 = -\vec{k}_2$

$$\begin{aligned} |\vec{v}_1 - \vec{v}_2|(2E_1)(2E_2) &= |\frac{\vec{k}_1}{E_1} - \frac{\vec{k}_2}{E_2}|(2E_1)(2E_2) \\ &= 4(E_1 + E_2)\left|\vec{k}_1\right| = 4\sqrt{s}\left|\vec{k}_1\right|_{COM} \end{aligned}$$

$$d\sigma = \frac{1}{4\sqrt{s}\left|\vec{k}_1\right|_{COM}} \left|\mathcal{M}\right|^2 (2\pi)^4 \delta(E_{1'} + E_{2'} - E_1 - E_2) \frac{d^3\vec{k}_{1'}}{(2\pi)^3 2E_{1'}} \frac{d^3\vec{k}_{2'}}{(2\pi)^3 2E_{2'}}$$

Since $\vec{k}_{1'} = -\vec{k}_{2'}$, we can ignore one δ function

$$d\sigma = \frac{1}{4\sqrt{s}\left|\vec{k}_1\right|_{COM}} \left|\mathcal{M}\right|^2 (2\pi) \delta(E_{1'} + E_{2'} - E_1 - E_2) \frac{d^3\vec{k}_{1'}}{(2\pi)^3 2E_{1'}} \frac{1}{2E_{2'}}$$

and

$$d^3\vec{k}_{1'} = d\Omega \left|\vec{k}_{1'}\right|^2 d\left|\vec{k}_{1'}\right| \quad d\Omega = \sin\theta d\theta d\phi$$

where θ is the angle of scattering. So

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\sqrt{s}\left|\vec{k}_1\right|_{COM}} \left|\mathcal{M}\right|^2 \frac{2\pi}{(2\pi)^3 2E_{1'} 2E_{2'}} \left|\vec{k}_{1'}\right|^2 \delta(E_{1'} + E_{2'} - E_1 - E_2) d\vec{k}_{1'}$$

Integrating out the δ

$$\frac{\partial}{\partial \vec{k}_{1'}} \delta(E_{1'} + E_{2'} - \sqrt{s}) = \frac{\left|\vec{k}_{1'}\right|}{E_{1'}} + \frac{\left|\vec{k}_{1'}\right|}{E_{2'}} = \frac{\left|\vec{k}_{1'}\right|\sqrt{s}}{E_{1'}E_{2'}}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \left|\mathcal{M}\right|^2 \frac{\left|\vec{k}_{1'}\right|}{\left|\vec{k}_1\right|}$$

If 4 masses are the same, $\left|\vec{k}_{1'}\right| = \left|\vec{k}_1\right|$, and we have (3.30). The $\sigma \sim 1/s^2$ relation is not universal. For QED it's $\sigma \sim s$.

4. Renormalization

4.1. Spectral of Full Propagator

Consider

$$S = \int d^d x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{6}g\phi^3 \right)$$

from (3.32), we know know to find

$$[\phi] = \text{mass}^{\frac{d-2}{2}}$$

Here d can be odd, later d can be fractional. Later e^- has dimension 3/2. Think extra dimensions are spatial dimension, time is always 1 dimension

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$$

Similarly find

$$[g] = \text{mass}^{-\frac{d}{2}+3} \tag{4.1}$$

notice g is dimensionless when $d = 6$. Later we will show that it implies the theory is renormalizable.

Lecture 13 (3/5/14)

To understand why we need renormalization, we need to understand where ∞ comes from and what type of ∞ they are. In (3.24) we mentioned the full propagator and its first term is the free propagator.

Consider

$$\left\langle 0 \middle| \hat{\phi}_x \hat{\phi}_y \middle| 0 \right\rangle$$

in some interacting theory (not yet T ordered) , inserting complete set of states

$$\left\langle 0 \middle| \hat{\phi}_x \hat{\phi}_y \middle| 0 \right\rangle = \left\langle 0 \middle| \hat{\phi}_x \sum |n\rangle \langle n| \, \hat{\phi}_y \middle| 0 \right\rangle$$

In Heisenberg picture

$$\hat{\phi}_x = e^{-i\hat{p}x}\hat{\phi}_0 e^{i\hat{p}x} \quad \hat{\phi}_y = e^{-i\hat{p}y}\hat{\phi}_0 e^{i\hat{p}y}$$

and $e^{i\hat{p}y}|0\rangle = 0$, $e^{i\hat{p}x}|n\rangle = e^{ik_nx}|n\rangle$,

$$\begin{aligned}\left\langle 0\middle|\hat{\phi}_x\hat{\phi}_y\middle|0\right\rangle &= \sum_n e^{ik_n(x-y)}\left|\left\langle 0\middle|\hat{\phi}_0\middle|n\right\rangle\right|^2 \\ &= \int d^dp e^{ip(x-y)}\sum_n \left|\left\langle 0\middle|\hat{\phi}_0\middle|n\right\rangle\right|^2 \delta^{(d)}(p-k_n)\end{aligned}$$

Define

$$\underbrace{\sum_n \left|\left\langle 0\middle|\hat{\phi}_0\middle|n\right\rangle\right|^2 \delta^{(d)}(p-k_n)}_{\text{spectral of density}} := \frac{1}{(2\pi)^{d-1}}\theta(p^0>0) \qquad \underbrace{\rho_{tot}(-p^2)}_{\int_0^\infty d\mu^2\rho_{tot}(\mu^2)\delta(\mu^2+p^2)}$$

what can we say about $\rho_{tot}(-p^2)$? Its argument is p^2, so it is Lorentz invariant, and $\rho_{tot}\geq 0$, because the LHS≥ 0. Does $\theta(p^0>0)$ make sense? Yes, since $k_n^0 = E_n \geq 0$ so the δ function require $p^0>0$. By the same reasoning of (1.7) thus

$$\begin{aligned}\left\langle 0\middle|\hat{\phi}_x\hat{\phi}_y\middle|0\right\rangle &= \int_0^\infty d\mu^2\rho_{tot}(\mu^2)\int\frac{d^d\rho}{(2\pi)^{d-1}}e^{ip(x-y)}\theta(p^0>0)\delta(p^2+\mu^2) \\ &= \int_0^\infty d\mu^2\rho_{tot}(\mu^2)\int\frac{d^{d-1}\rho}{(2\pi)^{d-1}2w_p}e^{ip(x-y)}\end{aligned}$$

integrating over p^0.

$$w_p = \sqrt{\vec{p}^2+\mu^2}$$

$$\left\langle 0\middle|\hat{\phi}_y\hat{\phi}_x\middle|0\right\rangle = \int_0^\infty d\mu^2\rho_{tot}(\mu^2)\int\frac{d^{d-1}\rho}{(2\pi)^{d-1}2w_p}e^{-ip(x-y)}$$

So comparing to (2.25)

$$\left\langle 0 \middle| T\hat{\phi}_x\hat{\phi}_y \middle| 0 \right\rangle = \int_0^\infty d\mu^2 \rho_{tot}(\mu^2) \underbrace{\theta(t_x > t_y)\int \frac{d^{d-1}\rho e^{ip(x-y)}}{(2\pi)^{d-1}2w_p} + \theta(t_y > t_x)\int \frac{d^{d-1}\rho e^{-ip(x-y)}}{(2\pi)^{d-1}2w_p}}_{\text{free propagator}}$$

$$= \int_0^\infty d\mu^2 \rho_{tot}(\mu^2)\frac{1}{i}\int \frac{d^d k}{(2\pi)^d}\frac{e^{ip(x-y)}}{p^2+\mu^2-i\epsilon}$$

Two possible extreme:

1) $|n\rangle$ represents one particle state.

$$p^2 = -m^2$$

evaluated in the rest frame.

2) $|n\rangle$ represents two (non-interacting) particle state.

$$\begin{aligned} p^2 &= (p_1+p_2)(p_1+p_2) \\ &= -(\sqrt{\vec{k}_1^2+m^2}+\sqrt{\vec{k}_2^2+m^2})^2 + (\vec{k}_1+\vec{k}_2)^2 \end{aligned}$$

which reaches maximum

$$p^2 = -4m^2$$

when

$$\vec{k}_1 = \vec{k}_2$$

For 2 particle bound state, the maximum is higher than $-4m^2$, around

$$-4m^2 < p^2 < -2m^2$$

So for $\delta(p^2+\mu^2)$ to be non-zero

$$\mu^2 \gtrsim 4m^2$$

So the full propagator is the sum of the two cases

$$\left\langle 0\middle|T\hat{\phi}_x\hat{\phi}_y\middle|0\right\rangle = \frac{1}{i}\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip(x-y)}}{p^2+m^2-i\epsilon}+\int_{\sim 4m^2}^{\infty}d\mu^2\rho_{tot}(\mu^2)\frac{1}{i}\int\frac{d^dk}{(2\pi)^d}\frac{e^{ip(x-y)}}{p^2+\mu^2-i\epsilon} \tag{4.2}$$

The first term on the right is the free propagator in (3.24). It has an isolated pole $m^2 < \mu^2$.

4.2. Loop Corrections

As we have seen that change d from 4 to another value doesn't alter many formulas. Back to (3.24), we understand the first term on the right is the first diagram of figure 9.13, on page 68 Srednicki, is from the free theory

$$S = \int d^dx(\underbrace{-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2\phi^2}_{I}+\underbrace{\frac{1}{6}g\phi^3}_{II}-\underbrace{\frac{1}{2}(Z_\phi-1)(\partial\phi)^2-\frac{1}{2}(Z_m-1)m^2\phi}_{III}$$

we will no longer write Λ, $Y\phi$ terms with the understanding that we throw out bubble tadpoles.

The second is the second diagram of figure 9.13, on page 68 Srednicki. III has lowest diagram given by (3.22), which has two legs y, z and x in the middle as a ϕ^2 order vertex.

We mentioned that II diagram is UV divergent. Here is why. Write the propagator in Fourier transform

$$\Delta_{xy\,full} = \int\frac{d^dk}{(2\pi)^d}\tilde{\Delta}_{full}(k)e^{ik(x-y)}$$

In this way, by (4.2) and

$$\frac{1}{i}\tilde{\Delta}_{full}(k) \quad = \quad \frac{1}{i}\tilde{\Delta}_{free}(k)+\frac{1}{i}\tilde{\Delta}_{II\text{ one loop correction}}(k)+\frac{1}{i}\tilde{\Delta}_{III}(k)+.. \tag{4.3}$$

The II is called the loop corrections to the propagator. It is computed similarly to (3.26), (3.27), if we label the two vertices as a, b there will be two propagators

$$\Delta_{ab} \quad \Delta_{ba}$$

if we denote k to the incoming momentum then k has to be the outgoing momentum. Within the loop, we can say from a to b the momentum is $k+l$ and from b to a the momentum is l. One can follow the same steps in (3.27), to show that by Feynman rule: only internal lines count. So

$$\frac{1}{i}\tilde{\Delta}_{II \text{ one loop correction}}(k) = \left(\frac{1}{i}\tilde{\Delta}_{free}(k)\right)^2 \frac{(ig)^2}{2} \underbrace{\int \frac{d^dl}{(2\pi)^d}\frac{1}{i}\frac{1}{(k+l)^2+m^2-i\epsilon}\frac{1}{i}\frac{1}{l^2+m^2-i\epsilon}}_{\text{diverge}} \tag{4.4}$$

1/2 is the correct symmetry factor, which can be obtained from the generating function like in (3.26). The integral is clearly diverge. It is logarithmic divergent for $d=4$ and power divergent for $d>4$.

Similarly

$$\frac{1}{i}\tilde{\Delta}_{III}(k) = \left(\frac{1}{i}\tilde{\Delta}_{free}(k)\right)^2 \frac{1}{i}[(Z_\phi-1)k^2+(Z_m-1)m^2] \tag{4.5}$$

thus for this to cancel (4.4), we need

$$Z_\phi - 1 \sim Z_m - 1 = O(g^2)$$

There are higher loop corrections to the propagator. See the third, fourth, fifth figure 9.13, on page 68 Srednicki. They too have counter parts from III to make them finite.

Because of the steps in (3.27), they all have similar factors like $\left(\frac{1}{i}\tilde{\Delta}_{free}(k)\right)^2$ comparing (4.4) and (4.5). It turns out we can group loop corrections to the propagator by the number of vertices, 2,4,6,... and miraculously they all have the same ratio

$$\frac{1}{i}\tilde{\Delta}_{full}(k) = \frac{1}{i}\tilde{\Delta}_{free}(k) + \left(\frac{1}{i}\tilde{\Delta}_{free}(k)\right)^2 i\Pi(k^2) + \left(\frac{1}{i}\tilde{\Delta}_{free}(k)\right)^3 \left(i\Pi(k^2)\right)^2 + ...$$

Each term is given by those grouped corrections, from (4.4) and (4.5), one gets

$i\Pi(k^2)$ (see problem set 6 problem 3)

$$i\Pi(k^2) = \frac{1}{2}(ig)^2 \int \frac{d^d l}{(2\pi)^d} \frac{1}{i}\tilde{\Delta}_{free}(l^2)\frac{1}{i}\tilde{\Delta}_{free}((l+k)^2) + \frac{1}{i}[(Z_\phi - 1)k^2 + (Z_m - 1)m^2] \tag{4.6}$$

called 1 particle-irreducible diagrams (1PI). Therefore

$$\begin{aligned} \frac{1}{i}\tilde{\Delta}_{full}(k) &= \frac{1}{i}\tilde{\Delta}_{free}(k)\frac{1}{1-\tilde{\Delta}_{free}(k)\Pi(k^2)} \\ &= \frac{1}{i}\frac{1}{k^2+m^2-\Pi(k^2)-i\epsilon} \end{aligned} \tag{4.7}$$

4.3. On Shell Renormalization

We will choose Z_ϕ and Z_m based on physical conditions so that the propagator (4.7) has pole at the right location, i.e.

$$\begin{cases} \Pi(k^2 = -m^2) = 0 \\ \Pi'(k^2 = -m^2) = 0 \end{cases} \tag{4.8}$$

The first one forces m to be the physical mass, i.e. on shell (OS), and the second one imposes condition on the amplitude of the 1 particle state.

Lecture 14 (3/10/14)

(4.8) implies that in expanding $\Pi(k^2)$ around $-m^2$,

$$\begin{aligned} \Pi(k^2) &= \Pi(k^2=-m^2) + \Pi'(k^2=-m^2)(k^2+m^2) + \frac{1}{2}\Pi''(k^2+m^2)^2 + \ldots \\ &= \frac{1}{2}\Pi''(k^2+m^2)^2 + \ldots \end{aligned} \tag{4.9}$$

Let us compute $\Pi(k^2)$ from (4.6)

$$i\Pi(k^2) = (ig)^2\left(\frac{1}{2}\right)\frac{1}{i^2}\underbrace{\int \frac{d^d l}{(2\pi)^d}\frac{1}{(k+l)^2+m^2-i\epsilon}\frac{1}{l^2+m^2-i\epsilon}}_{:=II_a} + \frac{1}{i}[(Z_\phi - 1)k^2 + (Z_m - 1)m^2] \tag{4.10}$$

Feynman's trick

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{(xb+(1-x)a)^2}$$

proof using

$$\frac{1}{a} = \int_0^\infty e^{-at}dt \quad \frac{1}{b} = \int_0^\infty e^{-bs}ds$$

then change variables

$$\begin{cases} t = (1-x)y \\ s = xy \end{cases} \quad \text{with Jacobian}=y$$

integrate $0 < x < 1$ and $0 < y < \infty$.

Using Feynman's trick, we get (4.10)

$$\begin{aligned} II_a &= \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[x((k+l)^2 + m^2 - i\epsilon) + (1-x)(l^2 + m^2 - i\epsilon)\right]^2} \\ &= \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{\left[(l+xk)^2 + k^2 x(1-x) + m^2 - i\epsilon\right]^2} \\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 + D]^2} \end{aligned}$$

where

$$q = l + xk \ D = k^2 x(1-x) + m^2 - i\epsilon \tag{4.11}$$

q is dummy variable, so overall II_a is a function of k^2,

$$II_a(k^2)$$

Because of $-i\epsilon$, we can rotate the real axis without touching the pole.

Let

$$q^0 = i\bar{q}^d$$

and

$$q^j = \bar{q}^j \quad j = 1, 2, ..., d-1$$

so

$$II_a = i \int_0^1 dx \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{[\bar{q}^2 + D]^2}$$

where $\bar{q}^2$ is Euclidean norm

$$\bar{q}^2 = \bar{q}_1^2 + ... + \bar{q}_d^2$$

therefore

$$\Pi(k^2) = \frac{1}{2}g^2 II_a(k^2) - [(Z_\phi - 1)k^2 + (Z_m - 1)m^2] \tag{4.12}$$

To be consistent with (4.9), expend around $-m^2$

$$\begin{aligned} \Pi(k^2) \quad = \quad & \frac{1}{2}g^2 II_a(-m^2) - [(Z_\phi - 1)(-m^2) + (Z_m - 1)m^2] && (4.13) \\ & + \left(\frac{1}{2}g^2 \; II'_a|_{-m^2} - (Z_\phi - 1)\right)(k^2 + m^2) \\ & + \left(\frac{1}{2}g^2 \; II''_a|_{-m^2}\right)(k^2 + m^2)^2 + ... && (4.14) \end{aligned}$$

By (4.8)

$$\begin{cases} \frac{1}{2}g^2 II_a(-m^2) - [(Z_\phi - 1)(-m^2) + (Z_m - 1)m^2] = 0 \\ \frac{1}{2}g^2 \; II'_a|_{-m^2} - (Z_\phi - 1) = 0 \end{cases} \tag{4.15}$$

We mentioned II_a diverged, so does II'_a

$$II'_a(k^2) = i\int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{-2x(1-x)}{[\bar{q}^2 + D]^3}$$

diverges for $d \geq 6$.

However II''_a or higher are finite

$$II''_a(k^2) = i\int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{6x^2(1-x)^2}{[\bar{q}^2 + D]^4}$$

for $d = 6$.

So the two ∞ in (4.13) are gone by (4.15). But in practice, it is not so easy to compute $\Pi(k^2)$ by the Taylor expansion (4.14) unless k is very close to m. So we develop another method.

4.4. Dimension Regulation

This uses another integral trick

$$\int \frac{d^d\bar{q}}{(2\pi)^d}\frac{(\bar{q}^2)^a}{(\bar{q}^2+D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{d}{2})}D^{-(b-a-\frac{d}{2})} \tag{4.16}$$

which is proved by writing

$$d^d\bar{q} = |\bar{q}|^{d-1}\, d\,|\bar{q}|\, d\Omega_{d-1}$$

$$\Omega_{d-1} = \int d\Omega_{d-1} = \frac{(2\pi)^{d/2}}{\Gamma(\frac{d}{2})}$$

(4.16) allows d to be even fractional.

For us $a=0$ $b=2$ and if we put $d=6$

$$(4.16) = \frac{D}{(4\pi)^3}\Gamma(2-\frac{d}{2}) = \frac{D}{(4\pi)^3}\Gamma(-1)$$

which is a pole. For any negative integer $-n$,

$$\Gamma(-n+\delta) = \frac{(-1)^n}{n!}\left(\frac{1}{\delta}-\gamma+1+\frac{1}{2}+...+\frac{1}{n}+O(\delta)\right)$$

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant. Let

$$d = 6-\epsilon$$

so

$$\int \frac{d^d\bar{q}}{(2\pi)^d}\frac{(\bar{q}^2)^a}{(\bar{q}^2+D)^b} = -\frac{D}{(4\pi)^3}\left(\frac{2}{\epsilon}+\ln\frac{4\pi}{e^{\gamma-1}}-\ln D+O(\epsilon)\right)$$

the additional $\ln 4\pi$ and $\ln D$ are from

$$(4\pi)^{3-\epsilon/2} = (4\pi)^3\left(1-\frac{\epsilon}{2}\ln 4\pi+O(\epsilon^2)\right)$$

Therefore from (4.12)

$$\Pi(k^2) = -\frac{1}{2}\frac{g^2}{(4\pi)^3}\left(\frac{2}{\epsilon} + \ln\frac{4\pi}{e^{\gamma-1}} + O(\epsilon)\right)\int_0^1 dx D + \frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx D\ln D - [(Z_\phi - 1)k^2 + (Z_m - 1)m^2] \tag{4.17}$$

We now understand what kind of ∞ we have, which will be useful when we study renormalization group. The only ∞ is from $2/\epsilon$, and

$$\int_0^1 dx D = \frac{1}{6}k^2 + m^2$$

so choose

$$\begin{cases} Z_\phi - 1 & = -\frac{1}{6}\frac{g^2}{(4\pi)^3}\frac{1}{\epsilon} + \text{finite} \\ Z_m - 1 & = -\frac{g^2}{(4\pi)^3}\frac{1}{\epsilon} + \text{finite} \end{cases} \tag{4.18}$$

to cancel ∞ and give

$$\Pi(k^2) = \frac{1}{2}\frac{g^2}{(4\pi)^3}\left(-\frac{1}{6}(k^2 + m^2) + \int_0^1 dx D\ln\frac{D}{D_0}\right) \tag{4.19}$$

where D is in (4.11) and

$$D_0 = -x(1-x)m^2 + m^2$$

agreeing (4.8). All of them look arbitrary, but they are all justified by applications e.g. Lamb shift which agrees amazingly well to the experiments.

Lecture 15 (3/12/14)

We have mentioned that when $d = 6$, g =dimensionless, cf (4.1). It is possible to replace g by

$$g(\tilde{\mu})^{\epsilon/2}$$

so that in the $6 - \epsilon$ dimension, g is still dimensionless and $\tilde{\mu}$ has some mass scale. So

$$(\tilde{\mu})^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln\tilde{\mu} + O(\epsilon^2)$$

so

$$g^2(\tilde{\mu})^\epsilon = g^2(1 + \epsilon\ln\tilde{\mu})$$

this will be later interpreted as CM. The $\epsilon\ln\tilde{\mu}$ term is there because in the product in (4.17)

$$\frac{g^2}{(4\pi)^3}\left(1+\epsilon\ln\tilde{\mu}\right)\left(\frac{2}{\epsilon}+...\right)$$

After all

$$(4.17)=\frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dxD\ln D+ak^2+bm^2$$

a, b are chosen so that (4.8) is satisfied and g has no dimension. One can further simplify putting

$$D_0=D(k^2=-m^2)$$

$$\begin{aligned}\Pi_{OS}(k^2) &= \frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dxD\ln\frac{D}{D_0}+a(k^2+m^2)\\ &= -\frac{1}{2}\alpha\left(\frac{1}{6}(k^2+m^2)+\int_0^1 dxD\ln\frac{D}{D_0}\right)\end{aligned}\tag{4.20}$$

where

$$a=-\frac{1}{12}\frac{g^2}{(4\pi)^3}\qquad \alpha=\frac{g^2}{(4\pi)^3}$$

notice D/D_0 is not necessarily positive, so branch cuts are needed. Same as (4.19).

In fact there are 3 other independent methods (see problem set 7 problem 1) to get (4.19): (1) differentiating (4.10) twice with respect to k^2; (2) perform the 6D integral (4.10) in 5-spherical coordinate; (3) do Pauli-Villars regularization on (4.10).

4.5. Minimal Subtraction

We actually have done minimal subtraction (MS). It is just to pick (4.18) with the finite terms attached to them.

$$\begin{cases} Z_\phi-1 &= -\frac{1}{6}\frac{g^2}{(4\pi)^3}\frac{1}{\epsilon}\\ Z_m-1 &= -\frac{g^2}{(4\pi)^3}\frac{1}{\epsilon}\end{cases}$$

This will get rid of ∞ but mass is not the physical mass, i.e.

$$\Pi(k^2=-m^2)\neq 0$$

This will complicate thing when gravity has mass. So it is more popular to do modified minimal subtraction ($\overline{\text{MS}}$), getting rid of ∞ and 4π and e^{γ}

$$\begin{cases} Z_\phi - 1 & = -\frac{1}{6}\frac{g^2}{(4\pi)^3}\left(\frac{1}{\epsilon}+\frac{1}{2}\ln\frac{4\pi}{e^\gamma}\right) \\ Z_m - 1 & = -\frac{g^2}{(4\pi)^3}\left(\frac{1}{\epsilon}+\frac{1}{2}\ln\frac{4\pi}{e^\gamma}\right) \end{cases} \tag{4.21}$$

Then

$$\begin{aligned} \Pi_{\overline{MS}}(k^2) &= \frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dxD\ln\frac{D}{\mu^2} - \frac{1}{2}\frac{g^2}{(4\pi)^3}(\frac{k^2}{6}+m^2) \\ &= -\frac{1}{2}\alpha\left(\frac{k^2}{6}+m^2+\int_0^1 dxD\ln\frac{\mu^2}{D}\right) \end{aligned} \tag{4.22}$$

where we put

$$\mu^2 = \frac{4\pi\tilde{\mu}^2}{e^\gamma} \tag{4.23}$$

4.6. Loop Correction (continued)

Now we discuss another type of ∞, happens at the vertex. This explained the renormalization coupling Z_g. See the third diagram on the second raw and second diagram on the third raw figure 9.13 Srednicki page 68. For a ϕ^3 theory each vertex has 3 legs. Now we add a loop to it. See figure 16.1 Srednicki page 112. Before we compute loop correction to propagator, recall propagator is directly related to 2 point function and scattering amplitude, now we similarly define vertex function of cubic vertex

$$iV_3(k_1,k_2,k_3)$$

where k_1, k_2, k_3 are the momenta of the three legs.

$$iV_3(k_1,k_2,k_3) = \underbrace{iZ_g g}_{\text{tree}} + (ig)^3\frac{1}{i^3}\underbrace{\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2+m^2-i\epsilon}\frac{1}{(l+k_2)^2+m^2-i\epsilon}\frac{1}{(l-k_1)^2+m^2-i\epsilon}}_{II} \tag{4.24}$$

and

$$k_1+k_2+k_3=0 \tag{4.25}$$

We don't have to put Z_g in the loop correction, because

$$Z_g \sim 1 + O(g^2) \tag{4.26}$$

putting Z_g in $(ig)^3$ will not contribute to g^3 order. Why is (4.26)? That is because at tree level, each vertex should still look

$$(Z_g g)^3 \sim g^3$$

and to cancel the ∞ from II, need $Z_g g$ from tree to contain some g^3. If our Lagrangian contains ϕ^4 or higher terms, we don't need more Z_g. Only one Z_g suffices, because loop correction to e.g. fourth power vertex converges

$$\int d^6 l \frac{1}{l^2 + \ldots} \frac{1}{l^2 + \ldots} \frac{1}{l^2 + \ldots} \frac{1}{l^2 + \ldots} < \infty$$

On Shell Renormalization

One way to pick Z_g is to set

$$g = V_3(0,0,0)$$

which has calculation advantages, or more practical

$$g = V_3(k_1', k_2', k_3') \tag{4.27}$$

for small k_1', k_2', k_3'. So one can measure g very well, and this method forces g to be the physical coupling in contrast to the other two renormalization schemes we will discuss afterward.

Lecture 16 (3/22/14)

Similarly apply Feynman's trick to (4.24) with

$$q = l - x_1 k_1 - x_2 k_2$$

$$dF_3 = 2 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1)$$

and x_1, x_2, x_3 is between 0 and 1.

$$II = \int dF_3 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^3}$$

where

$$D = x_3x_1k_1^2 + x_3x_2k_2^2 + x_1x_2k_3^2 + m^2 - i\epsilon$$

In the expression x_1, x_2, x_3 don't look like freely interchangeable, but they are because of (4.25). Then use Gamma function and dimension regulation and

$$g \to g\tilde{\mu}^{\epsilon/2}$$

we get

$$V_3 = Z_g g + \frac{1}{2}\frac{g^3}{(4\pi)^3}\left(\frac{2}{\epsilon} + \int dF_3 \ln \frac{4\pi\tilde{\mu}^2}{e^\gamma D}\right) \tag{4.28}$$

so plugging in (4.27), solve for Z_g. We get

$$V_3 = g + \frac{1}{2}\frac{g^3}{(4\pi)^3}\left(\int dF_3 \ln \frac{D_0}{D}\right) \tag{4.29}$$

where $D_0 = D(k_1', k_2', k_3')$.

So we get a finite expression for V_3 without the $1/\epsilon$ divergence. We can also show that

$$Z_g = 1 + cg^2$$

$$c = -\frac{1}{2}\frac{1}{(4\pi)^3}\left(\frac{2}{\epsilon} + \int dF_3 \ln \frac{4\pi\tilde{\mu}^2}{e^\gamma D_0}\right) \tag{4.30}$$

where c contains $1/\epsilon$, so to be consistent with our early discussion (4.26), we require g goes to 0 faster than ϵ. Although all of them sound fishy, they are all justified by applications e.g. Lamb shift.

Modified Minimal Subtraction

Put

$$Z_{\tilde{g}} = 1 + A\tilde{g}^2 \tag{4.31}$$

getting rid of ∞ and 4π and e^γ, where

$$A = -\frac{1}{2}\frac{1}{(4\pi)^3}\left(\frac{2}{\epsilon} + \int dF_3 \ln \frac{4\pi}{e^\gamma}\right)$$

so that plugging into (4.28)

$$V_3 = \tilde{g} + \frac{1}{2}\frac{\tilde{g}^3}{(4\pi)^3}\left(\int dF_3 \ln\frac{\tilde{\mu}^2}{D}\right) \tag{4.32}$$

Comparing (4.29), (4.32), if we say the two V_3 are equal at k_1', k_2', k_3', then

$$V_3(k_1', k_2', k_3') = \tilde{g} + \frac{1}{2}\frac{\tilde{g}^3}{(4\pi)^3}\left(\int dF_3 \ln\frac{\tilde{\mu}^2}{D_0}\right) = g$$

the two g are not the same. Let's invert g in the lowest order

$$g - \frac{1}{2}\frac{g^3}{(4\pi)^3}\left(\int dF_3 \ln\frac{\tilde{\mu}^2}{D_0}\right) = \tilde{g}$$

plugging in (4.32), we get (4.29). So the two methods are the same. Usually we use modified minimal subtraction, because in some cases where particles is massless on-shell renormalization runs into trouble, because in (4.20), when $m \to 0$, $D_0 \to 0$. While as (4.22) doesn't have this problem.

Bare Coupling

This doesn't introduce Z at all. So (4.28)

$$V_3 = g_b + \frac{1}{2}\frac{g_b^3}{(4\pi)^3}\left(\frac{2}{\epsilon} + \int dF_3 \ln\frac{\mu^2}{D}\right) \tag{4.33}$$

Comparing (4.29), (4.33), if we say the two V_3 are equal at k_1', k_2', k_3', then

$$V_3(k_1', k_2', k_3') = g_b + \frac{1}{2}\frac{g_b^3}{(4\pi)^3}\left(\frac{2}{\epsilon} + \int dF_3 \ln\frac{\mu^2}{D_0}\right) = g \tag{4.34}$$

Then invert g to get g_b in term of g. Then plug in (4.33), we will get (4.29). Notice g_b contains $1/\epsilon$.

4.7. Renormalizability

Consider

$$\mathcal{L} = -\frac{1}{2}Z_\phi(\partial\phi)^2 - \frac{1}{2}Z_m m^2\phi^2 - \sum_{n=3}^{\infty}\frac{1}{n!}Z_n g_n \phi^n$$

beyond cubic interaction. So vertex can have more then 3 legs. Consider a general Feynman diagram

$$\begin{aligned}
d &= \text{dimension} \\
E &= \#\text{ external lines} = J \\
I &= \#\text{ internal lines} \\
L &= \#\text{ loops} \\
w_n &= \#\text{ vertices with } n \text{ legs} \\
D &= \text{superficial degree of divergence}
\end{aligned}$$

Then if we want to compute similar objects like Π, V_3, for each L, we have to introduce a new l integral and for each I, there will be $1/l^2$

$$\int d^d l_1 \int d^d l_2 ... \frac{1}{l_1^2 + ...}\frac{1}{l_2^2 + ...}$$

so the important quantity is

$$D = dL - 2I$$

that is because

$$\text{If } D \geq 0, \text{ divergent} \tag{4.35}$$

in general the inverse is not always true, that is why called superficial. Our early example: one loop correction to propagator

$$d = 6, L = 1, I = 2, \text{ divergent}$$

one loop correction to a cubic vertex

$$d = 6, L = 1, I = 3, \text{ divergent}$$

However if one uses (4.35) crudely, e.g. the first diagram of figure 18.2 Srednicki. We known it diverges because of the one loop correction to propagator, but

$$D = (6)(2) - (2)(7) = -2$$

What is the dimension of g_n? Recall

$$S = \int d^d x m^2 \phi + g_n \phi^n + ...$$

is dimensionless, so

$$[\phi] = \text{mass}^{\frac{d-2}{2}}$$

so

$$[g_n] = \text{mass}^{d-n\frac{d-2}{2}} \tag{4.36}$$

Mass dimension of g is negative if $n > \frac{2d}{d-2}$. As we'll see negative mass dimension of g means that the theory is not renormalizable. For $d = 6$, $n > 3$ is not renormalizable, i.e. it needs ∞ number of counter terms, which are too ∞. For $d = 4$, $n > 4$ is not renormalizable.

Lecture 17 (3/24/14)

What is the mass dimension of a general Feynman diagram with E external legs?

Two ways of counting

1) e.g.

$$(ig_3)^2 \int d^d l_1 \int d^d l_2 ... \frac{1}{l_1^2 + ...} \frac{1}{l_2^2 + ...} ...$$

i.e.

$$[\text{diagram}] = D + \sum_{n=3}^{\infty} w_n [g_n]$$

2) treating the whole thing as one object, i.e. a tree with E legs

$$[\text{diagram}] = [g_E]$$

What does (4.35) imply? By (4.36)

$$[g_2] = d - n\frac{d-2}{2} = 2 > 0$$

independent of d. For others, $d = 6$ gives

$$\begin{aligned}
[g_3] &= 0 \\
[g_4] &= -2 \\
[g_5] &= -4 \\
&\vdots \\
\vdots &< 0
\end{aligned}$$

Theorem. *A theory with coupling of negative mass dimension need ∞ number of counter terms, i.e. non-renormalizable.*

We won't prove it, but the idea is clear.

$$D = [g_E] - \sum_{n=3}^{\infty} w_n [g_n]$$

If the theory has negative mass dimension, it will appear in the sum for higher corrections for instance, so D will be positive. Such negative mass dimension coupling can happen ∞ times in the sum, so ∞ number of renormalization are needed. However non-renormalizable theory doesn't mean it cannot make prediction. In fact we use non-renormalizable theory all the time, e.g. gravity. Later we will learn effective field theory, we will learn that a non-renormalizable theory must break down at high energy, e.g. gravity breaks down at Planck energy. Because say gravity

$$\mathcal{L} \sim g_3 \phi^3 + g_4 \phi^4$$

and

$$g_4 \sim \frac{1}{\Lambda^2}$$

so if energy is $\ll \Lambda$, we can forget about $g_4 \phi^4$.

As we mentioned before $D \geq 0$ divergence, but $D < 0$ doesn't mean convergence. So for ϕ^3 theory does it converge? The answer is yes. We can do perturbation to all orders by following these steps:

1) work out $i\Pi(k^2)$ of 1 particle irreducible diagram cf (4.7), (4.17), etc

2) work out iV_3, for sub diagrams with 3 legs and the internal line uses the

full propagator $i\Pi(k^2)$

3) work out iV_4, for sub diagrams with 4 legs and the internal line uses the full propagator $i\Pi(k^2)$ and internal vertex uses iV_3.

4) work out iV_5,

......if necessary.

These are called skeleton diagram. In such a way all orders of perturbations are included.

Finally in computing scattering amplitude for E number of total particles, i.e. E external legs, we write down all possible even "false" tree diagrams, e.g. $E = 4$ (2$\rightarrow$ 2 scattering) There are two possible tree diagrams: one is the normal one; the other is a vertex with 4 legs, which is not a legitimate tree diagram for ϕ^3 theory, but we include it anyway. Then we compute scattering amplitude, use $i\Pi(k^2)$ for the internal line, iV_3 for vertex with 3 legs, iV_4, for vertex with 4 legs, etc.

4.8. Renormalization Group

Modified Minimal Subtraction

Lecture 18 (3/31/14)

In summary we have show in the modified minimal subtraction scheme cf (4.21), (4.31), let

$$\alpha = \frac{g^2}{(4\pi)^3} \tag{4.37}$$

then

$$\begin{cases} Z_\phi - 1 & = \alpha(-\frac{1}{6\epsilon} + \text{finite const}) + O(\alpha^2) \\ Z_m - 1 & = \alpha(-\frac{1}{\epsilon} + \text{finite const}) + O(\alpha^2) \\ Z_g - 1 & = \alpha(-\frac{1}{\epsilon} + \text{finite const}) + O(\alpha^2) \end{cases} \tag{4.38}$$

for the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}Z_\phi(\partial\phi)^2 - \frac{1}{2}Z_m m^2\phi^2 + \frac{1}{6}Z_g g\tilde{\mu}^{\epsilon/2}\phi^3 \tag{4.39}$$

of dimension $d = 6-\epsilon$. As mentioned before modified minimal subtraction is better than OS scheme. But it has one extra degree of freedom, an extra parameter μ cf

(4.23). We need to eliminate this freedom, so this actually imposes constraint to the theory: m_{ph} or other physical observable quantities should be independent of μ. Surprisingly this trick will infer that how good our overall perturbation is.

Before do that, we ask what μ^2 is? From (4.34)

$$\mu^2 \sim D_0(k_1', k_2', k_3') \sim k^2 \tag{4.40}$$

The key idea of $\overline{\text{MS}}$ is that propagator has pole at $k^2 = -m_{ph}^2$, so we first expand

$$\Pi(k^2) = \Pi(-m_{ph}^2) + \Pi'(-m_{ph}^2)(k^2 + m_{ph}^2) + ...$$

From (4.7)

$$\tilde{\Delta}_{full}(k^2) = \frac{1}{k^2 + m^2 - \Pi(-m_{ph}^2) - \Pi'(-m_{ph}^2)(k^2 + m_{ph}^2) - i\epsilon}$$

To have a pole, we have to have

$$m_{ph}^2 = m^2 - \Pi(-m_{ph}^2) \tag{4.41}$$

By the way now

$$\tilde{\Delta}_{full}(k^2) = \underbrace{\frac{1}{1 - \Pi'(-m_{ph}^2)}}_{:=R} \frac{1}{k^2 + m_{ph}^2 - i\epsilon}$$

In other words, in the modified minimal subtraction scheme, propagator is no longer unit residue (or unit amplitude $R \neq 1$), so in computing scattering amplitude for instance, we would have to include an exact normalization factor. See problem set 8 problem 4.

Back to (4.41), use approximation and use (4.22)

$$
\begin{aligned}
m_{ph}^2 &= m^2 - \Pi(-m^2) + \text{high order correction}(\alpha) \\
&= m^2 + \frac{1}{2}\alpha\left(-\frac{m^2}{6} + m^2 + \int_0^1 dx(1-x+x^2)m^2 \ln\frac{\mu^2}{(1-x+x^2)m^2}\right) + O(\alpha^2) \\
&= m^2 + \frac{1}{2}\alpha[\underbrace{\int_0^1 dx(1-x+x^2)}_{\frac{5}{6}} m^2 \ln\frac{\mu^2}{m^2} + \underbrace{\frac{m^2}{6} - \int_0^1 dx(1-x+x^2)m^2\ln(1-x+x^2)]}_{\text{const}\cdot m^2} + ... \\
&= m^2\left(1 + \frac{1}{2}\alpha\left(\frac{5}{6}\ln\frac{\mu^2}{m^2} + \text{const}\right)\right) + ...
\end{aligned}
$$

Therefore for m_{ph} to be independent of μ, we need m, α depend on μ in such a way that μ dependence cancel out. Since

$$
\ln\left[1 + \frac{1}{2}\alpha\left(\frac{5}{6}\ln\frac{\mu^2}{m^2} + \text{const}\right)\right] \approx \frac{1}{2}\alpha\left(\frac{5}{6}\ln\frac{\mu^2}{m^2} + \text{const}\right)
$$

then

$$
0 = \frac{d\ln m_{ph}^2}{d\ln\mu} = \frac{d\ln m^2}{d\ln\mu} + \frac{d\frac{1}{2}\alpha\left(\frac{5}{6}\ln\frac{\mu^2}{m^2} + \text{const}\right)}{d\ln\mu}
$$

thus

$$
2\frac{d\ln m}{d\ln\mu} = -\frac{5\alpha}{12}2 - \frac{5}{12}\alpha\frac{d\ln m^2}{d\ln\mu} + \frac{d\alpha}{d\ln\mu}(\frac{5}{12}\ln\frac{\mu^2}{m^2} + \text{const})
$$

Later we will prove

$$
\frac{d\alpha}{d\ln\mu} = O(\alpha^2), \tag{4.42}
$$

so

$$
\gamma_m = \frac{d\ln m}{d\ln\mu} = -\frac{5\alpha}{12} + O(\alpha^2) \tag{4.43}
$$

which is call anomalous dimension of mass.

What is the μ dependence on α or g? i.e. show (4.42). We look at bare coupling.

Bare Coupling

Comparing to (4.39), Lagrangian for bare field

$$\mathcal{L} = -\frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{6}g_0\phi_0^3$$

so the two theories are exchangeable if we relate

$$\begin{aligned} \phi_0 &= Z_\phi^{1/2}\phi \\ m_0 &= mZ_m^{1/2}Z_\phi^{-1/2} \\ g_0 &= gZ_gZ_\phi^{-3/2}\tilde{\mu}^{\epsilon/2} \end{aligned} \tag{4.44}$$

then coupling is also charge from (4.37) to

$$\alpha_0 = \frac{g_0^2}{(4\pi)^3} = Z_g^2Z_\phi^{-3}\tilde{\mu}^\epsilon\alpha$$

In the bare theory there is no μ, so a_0 should not depend on μ

$$\frac{d\ln\alpha_0}{d\ln\mu} = 0$$

Using (4.38)

$$\begin{aligned} \ln\alpha_0 &= \ln\alpha + 2\ln Z_g - 3\ln Z_\phi + \epsilon\ln\mu + \epsilon\text{const} \\ &= \ln\alpha - \frac{3}{2\epsilon}\alpha + \text{const}\alpha + \epsilon(\ln\mu + \text{const}) \end{aligned}$$

$$0 = \frac{d\ln\alpha_0}{d\ln\mu} = \frac{1}{\alpha}\frac{d\alpha}{d\ln\mu} - \frac{3}{2\epsilon}\frac{d\alpha}{d\ln\mu} + \text{const}\frac{d\alpha}{d\ln\mu} + \epsilon + O(\alpha^2)$$

Since eventually as $\epsilon \to 0$ (not for now), the 2nd term dominates the third term, we ignore the third term

We guess the solution

$$\frac{d\alpha}{d\ln\mu} = A\alpha + B\alpha^2$$

so

$$A + B\alpha = \frac{3}{2\epsilon}(A\alpha + B\alpha^2) - \epsilon$$

Group other of α,

$$A = -\epsilon \qquad B = -\frac{3}{2}$$

Now take $\epsilon \to 0$

$$\frac{d\alpha}{d\ln\mu} = -\frac{3}{2}\alpha^2 \equiv \beta$$

proving (4.42). This equation is very important. It says as $\mu \uparrow$, $\alpha \downarrow$, so by (4.40) the coupling becomes weaker at higher energy, which theory is called asymmetrically free, e.g. QCD. However QED is opposite, its coupling goes up at high energy, so perturbation breaks down. We need better theory, could be supper symmetry.

Using the same trick above one can deduce (4.43) by (4.44), and compute

$$\frac{dZ_\phi}{d\ln\mu} \equiv 2\gamma_\phi$$

called anomalous dimension of the field. See problem set 9 problem 1.

Recall that $\overline{\text{MS}}$ propagator is not unity. Same happens to bare propagator

$$\tilde{\Delta}_0(k^2) = Z_\phi \tilde{\Delta}_{full}(k^2)$$

Using the same trick one can obtain Callan-Symanizk equation for the propagator

$$0 = \frac{d\tilde{\Delta}_0(k^2)}{d\ln\mu} \implies \left(2\gamma_\phi + \frac{\partial}{\partial\ln\mu} + \beta\frac{\partial}{\partial\alpha} + m\gamma_m\frac{\partial}{\partial m}\right)\tilde{\Delta}_{full}(k^2) = 0$$

5. Applications

5.1. Effective Field Theory

Lecture 19 (4/2/14)

We use result from a HW problem (problem set 5 problem 2 and problem set 7 problem 2).Consider 2→2 scattering amplitude of the ϕ^4 theory

$$S = \int d^4x - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

For its Feynman diagram see Peskin figure (4.101) page 112. In the HW we found the first three diagrams

$$\mathcal{M} \sim Z_\lambda \lambda + (...)\lambda^2 \int \frac{d^4 l}{(l^2 + m^2)((l+k)^2 + m^2)}$$

where (...) includes the symmetric factor and $k = k_1 - k_2$ for the second diagram and $k = k_1 + k_2$ for the third diagram. Then use dimensional regulation $d = 4 - \epsilon$

$$\mathcal{M} \sim Z_\lambda \lambda + \lambda^2 \left(\frac{(...)}{\epsilon} + (...) \ln \frac{\mu^2}{k^2} \right) \tag{5.1}$$

Modified Minimal Substitution

Put

$$Z_\lambda - 1 \sim \lambda \frac{1}{\epsilon}$$

to cancel the $1/\epsilon$ in $\mathcal{M}$. So

$$\mathcal{M} \sim \lambda + (...)\lambda^2 \ln \frac{\mu^2}{k^2}$$

We used the trick before

$$0 = \frac{\partial \mathcal{M}}{\partial \ln \mu} \implies \frac{\partial \lambda}{\partial \ln \mu} \sim -2\lambda^2 \tag{5.2}$$

On Shell

Pick an arbitrary k_0 and set

$$\mathcal{M}(k_0) = \lambda_{ph}$$

then solve for Z_λ then plug Z_λ back to (5.1), we get

$$\mathcal{M} \sim \lambda_{ph} + (...)\lambda_{ph}^2 \ln \frac{k_0^2}{k^2}$$

We used the trick again

$$0 = \frac{\partial \mathcal{M}}{\partial \ln k_0} \implies \frac{\partial \lambda_p}{\partial \ln k_0} \sim -2\lambda^2 \tag{5.3}$$

So we see in some sense (5.2) and(5.3) are the same because of (4.40). This restates the fact the μ is kind of proper energy scale to prob the system. The next point of view is very different.

Bare Coupling

We try to get a finite M by imposing a UV cutoff, Λ, which is related to ϵ. The smaller the ϵ is, the large Λ can be.

$$\mathcal{M} \sim \lambda_b + (...)\lambda_b^2 \int^{\Lambda} \frac{d^4 l}{(l^2 + m^2)((l+k)^2 + m^2)} \tag{5.4}$$

In this way we can compute the integral directly without first resolving to dimension regulation. Assuming $m \ll l$

$$\mathcal{M} \sim \lambda_b + (...)\lambda_b^2 \ln \frac{\Lambda^2}{k^2}$$

We used the trick again

$$0 = \frac{\partial \mathcal{M}}{\partial \ln \Lambda} \implies \frac{\partial \lambda_b}{\partial \ln \Lambda} \sim -2\lambda_b^2 \tag{5.5}$$

This looks mathematically alike (5.2) and (5.3), but physically it is very different from them. Λ is unrelated to μ.

Kenneth Wilson noticed the following if one has a theory

$$S_{\Lambda_0} = \int d^4x - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{5.6}$$

is valid for energy around Λ_0 (Ultraviolet scale) and Λ_0 is large enough, then extend the theory beyond Λ_0, higher ϕ power is negligible due to (5.5). But what about for energy less than $\Lambda < \Lambda_0$. Then higher power ϕ will kick in and the theory becomes non-renormalizable, while as S_{Λ_0} is renormalizable. This is why gravity is so hard.

So let's say we assume S_{Λ_0} is correct for all energy scale $|k| < \Lambda_0$, then using the notation from (3.1) with $J=0$, we have

$$Z = \int D\phi e^{iS_{\Lambda_0}}$$

if we distinguish the path integral with energy $\Lambda_0 > |k| > \Lambda$ and energy $|k| < \Lambda$, we get

$$Z = \int D\phi_L \int D\phi_H e^{iS_\Lambda}$$

and

$$\phi = \phi_L + \phi_H$$

why is this true? Because otherwise $D\phi$ cannot be $D\phi_L D\phi_H$ in the path integral.

On the other hand we can write

$$Z = \int D\phi_L e^{iS_{eff}(\Lambda)}$$

where $S_{eff}(\Lambda)$ means the action for the effective field theory when we only study energy less than Λ. Equating the two above, we get

$$e^{iS_{eff}(\Lambda)} = \int D\phi_H e^{iS_{\Lambda_0}}$$

Why is $\int D\phi_H e^{iS_{\Lambda_0}}$ simply a phase? It is easier to see if we first euclideanize it (forget about i). Then put i back.

Claim: $S_{eff}(\Lambda)$ is not renormalizable.

E.g. it contains $g\phi^6$ term. Where does it come from? S_{Λ_0} contains

$$(\phi_L + \phi_H)^4 \text{and it contains } 6\phi_L^2\phi_H^2$$

which could look like figure 29.1 in Srednicki page 179. It has 3 pairs of external lines. After integrating over ϕ_H, the dash lines coalesce, so it becomes 6 legs vertex, hence it becomes

$$\phi_L^6$$

One can show

$$S_{eff}(\Lambda) = \int d^4x - \frac{1}{2}(\partial\phi_L)^2 - \frac{1}{2}m_L^2\phi_L^2 + \frac{\lambda_L}{4!}\phi_L^4 + g_L\phi_L^6$$

The ϕ_L^4 is from a similar diagram of figure 29.1 in Srednicki page 179 with 2 pairs of external lines. So computing the effective coupling λ_L will be almost identical to the bare coupling (5.4). So

$$\lambda_L(\Lambda) \sim \lambda(\Lambda_0) + (...)\lambda^2(\Lambda_0)\ln\frac{\Lambda_0^2}{\Lambda^2} \tag{5.7}$$

where $\lambda(\Lambda_0)$ is the coupling in (5.6).

Since ϕ_L^6 is from figure 29.1 in Srednicki page 179 that has 3 pairs of external lines. So we compute similar to (4.24)

$$g_L(\Lambda) \sim g(\Lambda_0) + (...)g^6(\Lambda_0)\int_\Lambda^{\Lambda_0}\frac{d^4l}{(l^2)(l-k_1)^2(l+k_2)^2}$$

So

$$g_L(\Lambda) \sim g(\Lambda_0) + (...)\lambda^3(\Lambda_0)(\Lambda_0^{-2} - \Lambda^{-2}) \tag{5.8}$$

where $g(\Lambda_0)$ is the coupling in (5.6).

Similarly ϕ_L^2 is from a similar diagram of figure 29.1 in Srednicki page 179 that has 1 pairs of external lines. So we compute

$$m_L(\Lambda) \sim m(\Lambda_0) + (...)m^2(\Lambda_0)\int_\Lambda^{\Lambda_0}\frac{d^4l}{l^2}$$

So

$$m_L(\Lambda) \sim m(\Lambda_0) + (...)\lambda(\Lambda_0)(\Lambda_0^2 - \Lambda^2) \tag{5.9}$$

Compare (5.7), (5.8), (5.9), we see that m_L is most sensitive to the difference between Λ and Λ_0 and g_L is least sensitive.

We call

$$
\begin{array}{rl}
 & \text{the associated operator} \\
m_L \text{ relevant coupling or positive dimension} & \phi^2 \text{ relevant operator} \\
\lambda_L \text{ marginal coupling or dimensionless} & \phi^4 \text{ marginal operator} \\
g_L \text{ irrelevant coupling or negtive dimension} & \phi^6 \text{ irrelevant operator}
\end{array} \tag{5.10}
$$

Why do we care about whether couplings are sensitive to UV physics? They turn out to be more important than renormalizable. They require knowledge beyond Standard Model.

E.g. $\Lambda \ll \Lambda_0$ Higgs

$$m_L^2 \sim 125\text{GeV}$$

however

$$m^2(\Lambda_0) \sim 10^{16}\text{GeV}$$

so by (5.9), we need a term $\lambda(\Lambda_0)(\Lambda_0^2 - \Lambda^2) \sim 10^{16}\text{GeV}$ to cancel $m^2(\Lambda_0)$ and give the precisely right mass. This is called fine tuning problem, solved by supersymmetry.

5.2. General Relativity

In what sense a non-renormalizable theory acts as an effective theory? Let's look at general relativity.

$$S \sim \int d^4x\sqrt{-g}m_p^2 R + \underbrace{\alpha R^2 + \ldots\ldots\ldots\ldots}_{\text{beyond GR terms}}$$

$$m_p^2 = \frac{1}{8\pi G} = \text{Planck scale}$$

$$\alpha = \text{dimensionless coupling (see (5.10))}$$

$$R = \text{Racci curvature}$$

Now let's find EOM, we do a variance on $g_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} \ll 1$$

$\eta_{\mu\nu}$ Minkowski metric. $h_{\mu\nu}$ =gravitational wave or gravitational potential. For simplicity, let's pretend that we are working with scalar not tensor

$$\Delta S \sim \int d^4x \underbrace{-(\partial h)^2 m_p^2 + (\partial^2 h^3) m_p^2 + ...}_{\text{from GR}} + \underbrace{\alpha\left(\partial^4 h^2 + \partial^4 h^3 + ...\right)}_{\text{beyond GR}} + h\rho$$

ρ =energy density. And the last term on the right should be $h_{\mu\nu}T^{\mu\nu}$, coupling sources. EOM

$$\underbrace{m_p^2 \partial^2 h + m_p^2 \partial^2 h^2 + ...}_{\text{from GR}} + \underbrace{\alpha\left(\partial^4 h + \partial^4 h^2 + ...\right)}_{\text{beyond GR}} = -\rho$$

The first term in GR

$$m_p^2 \partial^2 h = -\rho$$

is equivalent to Newton gravity, because

$$h \sim \frac{\rho r^3}{m_p^2 r} \sim \frac{M}{m_p^2 r} \sim \frac{GM}{r}$$

or in more familiar form

$$\nabla^2 \phi = -4\pi G \rho$$

and the correction Δh to Newton from GR, plug

$$h = \frac{GM}{r} + \Delta h$$

into

$$m_p^2 \partial^2 h + m_p^2 \partial^2 h^2 = -\rho$$

we get

$$m_p^2\partial^2\Delta h + m_p^2\partial^2\left(\frac{GM}{r}\right)^2 \sim 0$$

so

$$\Delta h \sim \left(\frac{GM}{r}\right)^2$$

E.g. for earth around the sun

$$\frac{GM_{sun}}{r} \sim 10^{-7}$$

The correction to GR

$$\alpha\partial^4 h + ...$$

are even smaller, because

$$\frac{m_p^2\partial^2 h}{\partial^4 h} \sim \frac{m_p^2\partial^2 h}{\frac{\partial^2}{m_p^2}(\partial^2 h m_p^2)} \sim \frac{m_p^2}{\partial^2} \sim \frac{l_p^2}{L^2}$$

at Planck length scale. Before QFT, people thought these corrections were due to solar wind.

5.3. Critical Phenomena

Lecture 20 (4/7/14)

First let's look at some classical statistical mechanics. Consider a system with field $\phi(\vec{x})$,

$$E = \int d^3x \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

put $\beta = 1/T$, the partition function

$$Z = \int D\phi e^{-\beta\int d^3x \frac{1}{2}(\nabla\phi)^2 + V(\phi)}$$

where $d^3x = dx^0dx^1dx^2$. If we set

$$dx^0 \to d(it) \tag{5.11}$$

then

$$(\nabla\phi)^2 = \left(\frac{\partial\phi}{\partial x^0}\right)^2 + \left(\frac{\partial\phi}{\partial x^1}\right)^2 + \left(\frac{\partial\phi}{\partial x^2}\right)^2 = -\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x^1}\right)^2 + \left(\frac{\partial\phi}{\partial x^2}\right)^2$$

so

$$Z = \int D\phi e^{i\beta \overbrace{\int dtdx^1dx^2 \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x^1}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x^2}\right)^2 - V\right]}^{=S\ \because(1.21)}}$$

In other words

Euclidean QFT in d spacetime dimension =

classical stat mech in d spatial dimension

The interpretation of above is

QM fluctuation = classical thermal fluctuation

Later we will study Hawking radiation, we will see *quantum* stat mechanics is different, not just (5.11), it requires time to be periodic.

We apply this to critical phenomena. E.g. ferromagnet.

$$\phi = \text{magnetization}$$

In the present of external field $\vec{B}$, the magnetic moments will align with $\vec{B}$, but not 100% due to thermal fluctuation. With 0 external field. and at $T = 0$, clearly all spins of the magnet should align so that the groundstate is truly the lowest energy state. However what direction it aligns is completely arbitrary and it is not known a priori. This is called spontaneous symmetric breaking.

However experiments tell us there is a Curie temperature T_c. For $\vec{B} = 0$, if $T < T_c$

$$\text{completely align within a domain, so } \langle\phi\rangle \neq 0 \tag{5.12}$$

and if $T > T_c$

$$\langle\phi\rangle = 0$$

Why smooth change of T will led to abrupt change in $\langle\phi\rangle$? This is called second phase transition.

Landau and Ginzburg figured it out. In term of notation we develop above.

$$E = \int d^3x \frac{1}{2} (\nabla_{3D}\phi)^2 + V(\phi)$$

where

$$V(\phi) = a\phi^2 + b\phi^4 \tag{5.13}$$

so $a\phi^2$ looks like a mass term and $b\phi^4$ looks like an interaction term. Suppose $b(T) > 0$ for all T because we want potential to be bounded below and

$$a(T) = a(T_c) + a'(T_c)(T - T_c) + ...$$

with $a(T_c) = 0$ and $a'(T_c) > 0$, so

$$\begin{cases} a > 0 & T > T_c \\ a < 0 & T < T_c \end{cases}$$

One can draw a graph of (5.13) to see that for $T > T_c$ the potential minimum is at the origin hence ϕ should stay near the origin i.e.

$$\langle\phi\rangle = 0$$

For $T < T_c$ the potential has two minimums not at the origin, so

$$\langle\phi\rangle \neq 0$$

but which one it choose is completely arbitrary, but it cannot be superposition of both. Superposition only happens in QM; in QFT there are too many degree of freedom already, so no superposition.

We can find

$$\frac{\partial V}{\partial \phi} = 0 \implies 2a\phi + 4b\phi^3 = 0 \implies \langle \phi \rangle = \pm\sqrt{-\frac{a}{2b}} \propto \sqrt{|T - T_c|}$$

which is verified by experiment.

What do we mean the domain in (5.12)? It means the domain of correlation. Recall in Yukawa potential discussion (2.28) and (2.29) we say the correlation length is $1/m$, so

$$\langle \phi(\vec{x})\phi(\vec{y}) \rangle \sim e^{-|\vec{x}-\vec{y}|\sqrt{|a|}}$$

and

$$1/\sqrt{|a|} \propto 1/\sqrt{|T - T_c|}$$

which is too verified by experiment.

Idea of associating

$$m^2 \leftrightarrow a \qquad \lambda \leftrightarrow b \tag{5.14}$$

is very power. It allows to apply what we lean about renormalization group to change in potential. Of course the former one is due to quantum loops and the latter one is due to thermal loops or thermal fluctuation. One will learn more about them in condense matter course.

In QFT II, we will learn Higgs mechanism. It works the same as Landau and Ginzburg theory. At low temperature the Higgs potential has $a < 0$, so W, Z particles gain masses.

5.4. Continuous Symmetry

Continue our discussion of symmetry from section 1.5 conserved current. There are discrete symmetry, continuous symmetry, spacetime symmetry and internal (i.e. not spacetime) symmetry. Examples are below

	internal	spacetime
discrete	$\phi \to -\phi$	$\phi(t,\vec{x}) \to \phi(t,-\vec{x})$ parity $\phi(t,\vec{x}) \to \phi(-t,\vec{x})$ time reversal
continuous	$\phi \to \phi + c\,\forall c$ mixing for index field $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$	$\phi(x) \to \phi(x-a)\,\forall a$ $\phi(x) \to \phi(\Lambda^{-1}x)$

Next semester we will add non-abelian symmetry.

We study Ward identity, which gives quantum implication of conserved current and its connection to correlation function. Recall conserved current is from continuous symmetry so Ward identity only works for continuous symmetry.

Theorem. *(Ward identity)*

$$\begin{aligned} i\,\partial_\mu\big|_{x=x_0} \langle j^\mu(x_0)\phi(x_1)\phi(x_2)...\phi(x_n)\rangle &= \delta(x_0-x_1)\langle\delta\phi(x_1)\phi(x_2)...\phi(x_n)\rangle \\ &+\delta(x_0-x_2)\langle\phi(x_1)\delta\phi(x_2)...\phi(x_n)\rangle \\ &+.... \\ &+\delta(x_0-x_n)\langle\phi(x_1)\phi(x_2)...\delta\phi(x_n)\rangle \end{aligned} \tag{5.15}$$

where $\delta\phi$ and j^μ are related according to (1.29) *and* (1.31).

Proof. Start from (1.30), let

$$\phi' = \phi + \xi(x)\delta\phi$$

change of variables in ϕ to ϕ' will not change the path integral because we treat ϕ as dummy variable. That is

$$0 = \int D\phi'\phi'_1\phi'_2...\phi'_n e^{iS[\phi']} - \int D\phi\phi_1\phi_2...\phi_n e^{iS[\phi]} \tag{5.16}$$

also assume

$$D\phi = D\phi'$$

hence

$$D(\xi(x)\delta\phi) = 0$$

we will not impose this condition when we study quantum anomaly next semester. More

$$e^{iS[\phi']} = e^{i(S[\phi]+\delta S)}$$

where δS is given by (1.31). Since $\delta\phi$ small implies j^μ small, and (1.31) implies δS is small. So

$$e^{iS[\phi]}(1 + i\delta S)$$

Therefore keeping up to the 1st order in $\delta\phi$, δS,

$$\begin{aligned}
(5.16) \quad = \quad & \int D\phi\phi_1\phi_2...\phi_n(i\delta S)e^{iS[\phi]} \\
& + \int D\phi\xi(x)\delta\phi_1\phi_2...\phi_n e^{iS[\phi]} \\
& + \int D\phi\phi_1\xi(x)\delta\phi_2...\phi_n e^{iS[\phi]} \\
& + ... \\
& + \int D\phi\phi_1\phi_2...\xi(x)\delta\phi_n e^{iS[\phi]}
\end{aligned}$$

Plugging in (1.31), putting

$$\xi(x) = \delta(x - x_0)$$

and applying integration by parts to the first term on the right, we get Ward identity. QED

Lecture 21 (4/9/14)

The right hand side of (5.15) are called contact terms. They arise from QM. We can get a very important result from Ward identity. Apply both side of Ward with one field

$$\int_{t_b-\epsilon}^{t_b+\epsilon} dt_a \int_{\text{all space}} d^{d-1}x_a \left[i\, \partial_\mu|_{x_a} \langle j^\mu(x_a)\phi(x_b)\rangle = \delta^{(d)}(x_a - x_b)\, \langle \delta\phi(x_b)\rangle \right]$$

We get

$$RHS = \langle \delta\phi(x_b)\rangle = \left\langle 0 \middle| \delta\hat{\phi} \middle| 0 \right\rangle \text{ by (2.19)} \tag{5.17}$$

$$LHS = i\int_{t_b-\epsilon}^{t_b+\epsilon} dt_a \int_{\text{all space}} d^{d-1}x_a \left(\frac{\partial}{\partial t_a}\bigg|_{x_a} \left\langle j^0(x_a)\phi(x_b)\right\rangle + \frac{\partial}{\partial \vec{x}_a}\left\langle \vec{j}(x_a)\phi(x_b)\right\rangle \right)$$

Assume

$$\langle j^\mu(x_a)\phi(x_b)\rangle \to 0 \text{ as } x_a \to \pm\infty$$

saying correlation is 0 at ∞ apart. So by Stokes,

$$LHS = i\int_{t_b-\epsilon}^{t_b+\epsilon} dt_a \int_{\text{all space}} d^{d-1}x_a \, \frac{\partial}{\partial t_a}\bigg|_{x_a} \left\langle j^0(x_a)\phi(x_b)\right\rangle$$

Set

$$Q(t_a) = \int_{\text{all space}} d^{d-1}x_a j^0(x_a)$$

$$\begin{aligned} LHS &= i\int_{t_b-\epsilon}^{t_b+\epsilon} dt_a \, \frac{\partial}{\partial t_a}\bigg|_{x_a} \langle Q(t_a)\phi(x_b)\rangle && (5.18)\\ &= i\left(\langle Q(t_b+\epsilon)\phi(x_b)\rangle - \langle \phi(x_b)Q(t_b-\epsilon)\rangle\right) \text{ taking } \epsilon \to 0 && (5.19)\\ &= i\left\langle 0\middle|[\hat{Q},\hat{\phi}]\middle|0\right\rangle && (5.20)\end{aligned}$$

Recall in the path integral order doesn't matter, so in (5.18) we first express it in term of path integral, then in (5.19) we write it back to time ordered correlation function, so order does matter.

Equating (5.17) and (5.20), we have shown

$$i[\hat{Q},\hat{\phi}] = \delta\hat{\phi} \tag{5.21}$$

is true when sandwiched between $|0\rangle$, but in fact it is true for any states because we can insert extra operators (in far past or future) to represent states we want to create out of the vacuum. For an alternative proof of (5.21) see problem set 10 problem 1.

When there is a time translation symmetry, we can use (1.38),

$$i[H,\hat{\phi}] = \delta\hat{\phi}$$

What is $\delta\hat{\phi}$? Since $\delta\hat{\phi}$ is generated by j^μ, so we are back to (1.37) and forgetting about a as we did in (1.38),

$$i[H,\hat{\phi}] = \partial_t\hat{\phi} \tag{5.22}$$

which is well-known in QM. Likewise we can use some kind of

$$\text{charge} = \text{momentum}$$

to generate spatial translation symmetry

$$\delta\phi = \text{shift in space}$$

There is a related identity to Ward identity, called Dyson Schwinger equation, which does not require symmetry

Theorem. *(Dyson Schwinger)*

$$\begin{aligned}
i\left\langle \frac{\delta S}{\delta\phi(x_0)}\phi(x_1)\phi(x_2)...\phi(x_n)\right\rangle &= -\delta(x_0-x_1)\left\langle\phi(x_2)...\phi(x_n)\right\rangle \\
&\quad -\delta(x_0-x_2)\left\langle\phi(x_1)\phi(x_3)...\phi(x_n)\right\rangle \\
&\quad ... \\
&\quad -\delta(x_0-x_n)\left\langle\phi(x_1)\phi(x_2)...\phi(x_{n-1})\right\rangle
\end{aligned}$$

Proof. Consider total derivative

$$\begin{aligned}
0 &= \int D\phi\frac{\delta}{\delta\phi(x_0)}\left[e^{iS}\phi(x_1)\phi(x_2)...\phi(x_N)\right] \\
&= ie^{iS}\frac{\delta S}{\delta\phi(x_0)}\phi(x_1)\phi(x_2)...\phi(x_n) \\
&\quad +e^{iS}\delta(x_0-x_1)\left\langle\phi(x_2)...\phi(x_n)\right\rangle + e^{iS}\delta(x_0-x_2)\left\langle\phi(x_1)\phi(x_3)...\phi(x_n)\right\rangle \\
&\quad +... + e^{iS}\delta(x_0-x_n)\left\langle\phi(x_1)\phi(x_2)...\phi(x_{n-1})\right\rangle
\end{aligned}$$

QED.

One can use Dyson-Schwinger to show another well-known result

$$[\hat{x},\hat{p}] = i$$

see problem set 10 problem 4.

Discrete Symmetry

We showed (5.21) from continuous symmetry. One can easy get (see problem set 10 problem 2)

$$(5.21) \iff e^{i\hat{Q}}\hat{\phi}e^{-i\hat{Q}} = \hat{\phi} + \delta\hat{\phi}$$

However we can rewrite the right hand side in a more general form

$$\hat{U}^{-1}\hat{\phi}\hat{U} = \hat{\phi}' \tag{5.23}$$

where

$$\hat{\phi}' = \text{transformed } \hat{\phi}$$

Now (5.23) holds for both continuous and discrete symmetry.

Example.

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{5.24}$$

with discrete symmetry

$$\phi \to -\phi$$

One can show for such a theory (5.24), 2ϕ in 3ϕ out is not possible, so no Feynman diagrams with odd legs. We give a crude argument. It is easier to show in Hamilton quantum mechanics than in field language. There is an unitary U such that

$$U^{+}\phi U = -\phi$$

(Not all symmetries are given by unitary U, for time reversal U is anti unitary, see problem set 10 problem 3)

Put

$$|i\rangle = \phi_1\phi_2 \qquad |f\rangle = \phi_1'\phi_2'\phi_3'$$

then

$$U\,|i\rangle = |i\rangle \qquad U\,|f\rangle = -\,|f\rangle$$

Clearly even potential from (5.24) implies

$$[U, H] = 0 \tag{5.25}$$

(This is of course true in general, but not so easy to prove in field language, how invariant action for discrete symmetry (so no help from conserved current) leads to commuting with H in Hamilton mechanics? That is why we avoid field theory for the moment.)

Consider

$$\begin{aligned} \langle f | e^{-iHt} | i \rangle &= \langle f | U^{+} U e^{-iHt} | i \rangle \\ &= \langle f | U^{+} e^{-iHt} U | i \rangle = - \langle f | e^{-iHt} | i \rangle \end{aligned}$$

thus

$$\langle f | e^{-iHt} | i \rangle = 0$$

Such U operator can do another task. It can flip the groundstate of spontaneous symmetry breaking potential. Jumping from one minimum to the other for the discrete case.

5.5. Spontaneous Symmetry Breaking

Discrete Case

We mentioned the groundstate cannot be the superposition of the two minimums. Here is why. Using (2.8) path integral for field theory with 2 end fixed

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi)$$

V, potential density, is given by (5.13),

$$_{out}\langle 0+ | 0- \rangle_{in} = \int D\phi e^{-S_E}$$

where

$$|0\pm\rangle \text{ means one of the two minimums of (5.13)}$$

Because we renormalize the theory so that the groundstate has 0 energy, i.e. the cosmological constant is 0, $V(\phi) \geq 0$ everywhere, thus

$$S_E = \int d\tau d^3x \left(\frac{1}{2}(\partial\phi)^2 + V(\phi)\right) \text{ with } \tau = it$$

is always positive. Since the spatial volume integral $\int d^3x \to \infty$, (in other words the kind of potential will be used in QM

$$\int d^3x V(\phi) = \infty$$

Physically it means to flip all of the spins requires ∞ energy, as jumping from one minimum to the other. this is one of the places where qft is very different from qm. In qm no $V = \infty$) so

$$S_E \to \infty \implies e^{-S_E} \to 0$$

thus

$$_{out}\langle 0+|0-\rangle_{in} \to 0 \tag{5.26}$$

Continuous Case

For continuous spontaneous symmetry breaking. There are infinitely many groundstate (and physically they are the same), and there is a way to jump from one groundstate to another with 0 energy input.

Lecture 22 (4/14/14)

Consider a complex scale field

$$\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi \underbrace{-m^2\phi\phi^* - \frac{\lambda}{4}(\phi\phi^*)^2}_{-V(\phi)} \text{ with } (m^2) < 0 \; \lambda > 0 \tag{5.27}$$

m^2 is kind of a in (5.14) so our discussion is more general, so it can be negative. Why is there no 1/2 in front of $\partial^\mu\phi^*\partial_\mu\phi$? see problem set 1 problem 6.

It has symmetry

$$\phi \to e^{-i\alpha}\phi \quad \alpha = \text{const}$$

One can plot $V(\phi)$ v.s. x axis= $\Re\phi$ and y axis $= \Im\phi$, and $V(\phi)$ depends only on

$|\phi|$, looking like a Mexican hat. One can label the groundstate by the azimuthal angle φ. Taking derivative of $V(\phi)$, we find minimums

$$\left\langle \varphi \middle| \hat{\phi} \middle| \varphi \right\rangle = \frac{1}{\sqrt{2}} v e^{-i\varphi}$$

where

$$v = \sqrt{\frac{4\left|m^2\right|}{\lambda}}$$

choose the minimum point on the x axis to be our vacuum universe, then consider fluctuation both radially (ρ) and angularly (χ)

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v} \tag{5.28}$$

hence ρ and χ are real and have the same order of dimension. Clearly from the graph $V(\phi)$, radial fluctuation causes energy extraction, as we will see, it corresponds to gaining in mass, and angular fluctuation causes no energy, hence massless.

Indeed plug (5.28) in (5.27), with $m^2 < 0$

$$\mathcal{L} = -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(1 + \frac{\rho}{v})^2(\partial\chi)^2 - \left|m^2\right|\rho^2 - \frac{1}{2}\sqrt{\lambda}\left|m\right|\rho^3 - \frac{1}{16}\lambda\rho^4 + \frac{|m^2|^2}{\lambda}$$

the last term is independent of x, so we ignore it. We recognize $\left|m^2\right|\rho^2$ as a mass term, so we say ρ is massive. The $\frac{1}{2}(1+\frac{\rho}{v})^2(\partial\chi)^2$ term will not generate a χ^2 term, we say χ is massless, called Nambu-Goldstone Boson. So the idea of particle in qft view is just oscillation around groundstate.

Part II. Spin 1/2 Field Theory

6. Dirac Particle

6.1. Heuristic Discussion

The discover of spin 1/2 particles started with Dirac, who asked himself: can I take square root of Klein-Golden equation?

Let's go to the momentum space. Consider

$$\phi \propto e^{ikx}$$

plugging KG

$$(k^2 + m^2)e^{ikx} = 0$$

If we forget about k^2 is actually $k_\mu k^\mu$, we say

$$k^2 + m^2 = (ik + m)(-ik + m)$$

With that inspiration, Dirac said

$$(-\Box + m^2) = (-\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2) \sim (i\gamma^\mu\partial_\mu + m)(-i\gamma^\nu\partial_\nu + m) = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2$$

where γ is to be determined, not depend on spacetime. Hence

$$\gamma^\mu\gamma^\nu = -\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

It turns out

$$\gamma^\mu \text{ cannot be 4 numbers}$$

because they were

$$\gamma^0 = \pm 1 \qquad \gamma^{1,2,3} = \pm i$$

but

$$\gamma^1\gamma^2 \neq 0$$

So γ^μ has to be matrix, so $\eta^{\mu\nu}$ is really $\eta^{\mu\nu}I$ and KG is really

$$(-\eta^{\mu\nu}I\partial_\mu\partial_\nu + m^2 I)$$

acting on vectors. Write

$$\gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = -\eta^{\mu\nu}I$$

Since $\partial_\mu\partial_\nu$ is symmetric,

$$[\gamma^\mu, \gamma^\nu]\partial_\mu\partial_\nu = 0$$

we want

$$-\eta^{\mu\nu}I = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \tag{6.1}$$

called Clifford algebra. (Some book absorb the -1 to the definition of $\gamma \to i\gamma$. That will change the Dirac equation also.) If we can find such γ satisfies Clifford algebra, we get Dirac equation

$$(-i\gamma^\mu\partial_\mu + m)\psi = 0 \tag{6.2}$$

We will show such ψ satisfies Pauli exclusion principle, i.e. Fermions.

The lowest order solution of (6.1) is 4×4 matrices. And ψ is 4 components vector, called spinor. Clearly if γ^μ satisfies (6.1), so is

$$M^{-1}\gamma^\mu M$$

hence we're only interested in non-equivalent representations.

Definition of representations:

A representation is a set of matrices that satisfy the algebra.

E.g. Pauli matrices is a representation of rotation group of spin 1/2 particles, because

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

or putting $J_i = \sigma_i/2$, we get

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

known as the Lie algebra of rotation. Usually algebra determine the representation upto some topological subtitles. E.g. Both $SU(2)$ and $SO(3)$ have the same Lie algebra, but the representation are not the same.

The one solution of (6.1) we're interested in is Weyl representation.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

where

$$\sigma^\mu = (1, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One check Weyl representation satisfy Clifford algebra (6.1) using

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad i, j = 1, 2, 3$$

It turns out Clifford algebra gives Lorentz algebra

Theorem. *Put*

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{6.3}$$

we claim $\Sigma^{\mu\nu}$ are the generators of Lorentz transformation, clearly Σ are antisym-

metry wrt $\mu\nu$, so it has 6 free components, where

$$\begin{aligned}\Sigma^{0i} &= \textit{3 boots} \\ \Sigma^{ij} &= \textit{3 rotational } i \neq j\end{aligned}$$

that is $\Sigma^{\mu\nu}$ satisfies Lie algebra as shown in (1.46).

In mathematical term, Dirac found a 4×4 representation (6.3) of Lorentz algebra, which has a general form we have shown before (1.45). As we will see that for Dirac spinor space, each $\Sigma^{\mu\nu}$ itself is a 4×4 matrix given by (6.3).

Physically, as we will see we say Dirac found a new kind of particle $e^{\pm}$.

6.2. Transformation

Lecture 23 (4/16/14)

This is the way rational people do field theory (for particles of all spins, not limited to 1/2 spin). We want a field theory that has particles and transformation via Lorentz. Recall for scalar field (specified by momentum and rest mass)

$$|k\rangle \longrightarrow |\Lambda k\rangle$$

cf (1.42).

We now want to add some internal degree of freedom (independent of k), and apply Lorentz

$$|k, a\rangle \longrightarrow {L_a}^b(\Lambda)\, |\Lambda k, b\rangle \tag{6.4}$$

summing over b.

Similar to (1.44), we require matrix L form representations of Lorentz group

$${L_a}^b(\Lambda_1){L_b}^c(\Lambda_2) = {L_a}^c(\Lambda_1\Lambda_2) \tag{6.5}$$

As we will see the simplest representation is 2×2 matrix, simpler than Dirac matrix. In fact using (6.4), (6.5) one can find representations of all particles e.g. spin 2 gravitons, spin 3/2 gravitino.

By (6.5), an element of the group should look like (see quantum mechanics I note equation 2.25)

$$
\begin{aligned}
L_a{}^b(\Lambda) &= \left(e^{\frac{i}{2}\delta w_{\mu\nu}(\Lambda)[S^{\mu\nu}]}\right)_a{}^b & (6.6)\\
&= 1_a{}^b + \frac{i}{2}\delta w_{\mu\nu}(\Lambda)(S^{\mu\nu})_a{}^b + \frac{1}{2}(\frac{i}{2})^2 \delta w_{\mu\nu}(\Lambda)\delta w_{\alpha\beta}(\Lambda)[S^{\mu\nu}, S^{\alpha\beta}]_a{}^b + ...
\end{aligned}
$$

where $S^{\mu\nu}$ matrices, called generators. (6.3) serves as a particular example of 4×4 matrices $S^{\mu\nu}$ and indeed it is choose to be antisymmetric with respect to $\mu\nu$. $\delta w_{\mu\nu}$ is a set of real numbers, parametrizing the transformation Λ. How δw is related to Λ see (1.43).

We already mentioned redundancy of representations due to equivalence classes. Another thing is that after some equivalence transformation, some representation can be brought into block diagonal form, i.e. they are reducible representations. So we only have to study the blocks i.e. irreducible representation.

Normally when people say Lorentz transformation, they mean go to another frame. Suppose we have a classical scalar field

$$\phi(x) \to \phi'(x') \;\&\; x \to x' \tag{6.7}$$

in frame O, the same physics observed by frame O' is $\phi'(x')$, where x' is the same physical point x that is measured in O'. The coordinates of the same points are related by

$$x' = \Lambda x \tag{6.8}$$

Clearly

$$\phi'(x') = \phi(\Lambda^{-1}x') \tag{6.9}$$

so we can rewrite (6.7) as

$$\phi(x) \to \phi(\Lambda^{-1}x') \tag{6.10}$$

Comparing (6.7) and (6.10), their interpretations are different. (6.7) means fixing the same point, going to a different frame (passive picture). (6.10) means in the same frame, measure at a different point (active picture).

If $\hat{\phi}$ is quantum operator, e.g. second quantization operator. active picture = (1.41) ; passive picture = (1.42).

If we are talking about a classical vector field $\phi_a(x)$ whose a is index for component of vector. (6.9) becomes

$$\phi'_a(x') = L_a{}^b \phi_b(\Lambda^{-1}x')$$

the effect of $L_a{}^b$ is just like (6.8) to convert coordinate in O to coordinate in O'.

Likewise we say for quantum operator field with components, active picture

$$\hat{\phi}_a(x) \to U^{-1}(\Lambda)\hat{\phi}_a(x)U(\Lambda) = L_a{}^b \hat{\phi}_b(\Lambda^{-1}x) \tag{6.11}$$

passive picture = (6.4).

6.3. Representation of Lorentz Group

Recall in (1.47) we showed the Lie algebra of 2 irreducible nonequivalent representation of Lorentz transformation, J and K.

Wigner figured that if we define

$$\begin{aligned} N_i &= \frac{1}{2}(J_i - iK_i) \\ \tilde{N}_i &= \frac{1}{2}(J_i + iK_i) \end{aligned}$$

then (this trick only works for 2×2 dimension representation, because the only 2×2 representation of rotation is $SU(2)$.)

$$[N_i, N_j] = i\epsilon_{ijk}N_k \tag{6.12}$$

$$[\tilde{N}_i, \tilde{N}_j] = i\epsilon_{ijk}\tilde{N}_k \tag{6.13}$$

$$[N_i, \tilde{N}_j] = 0$$

Hence N_i, $\tilde{N}_j$ form 2 separated $SU(2)$, called $SU(2)_L$ and $SU(2)_R$ respectively.

Lecture 24 (4/21/14)

Since (6.12), (6.13) look like Lie algebra for angular momentum, we may just borrow the idea from it. We can find irreducible representation of Lorentz group in this way, starting from the simplest.

Scalar or Singlet

The simplest representation: $n = 0$, $\tilde{n} = 0$, where $n, \tilde{n}$ are the analogies of j which appears in the eigenvalue $j(j+1)$ of J^2. So there are $2j+1$ states for each j. Here $n = 0$, $\tilde{n} = 0$, we have a

$$(1, 1) \text{ representation}$$

so $N_i,\ \tilde{N}_i = 0$, so $J_i,\ K_i = 0$, so ${L_a}^b = 1$, so (6.11) says

$$\hat{\phi}_a(x) \to \hat{\phi}_a(\Lambda^{-1}x)$$

so a is not observable, nothing can change it. So indeed we have a scalar field.

(2,1) Representation

Next put

$$n = \frac{1}{2}, \tilde{n} = 0$$

then

$$\tilde{n}_i = 0 \implies J_i = -iK_i \implies K_i = iJ_i \tag{6.14}$$
$$n_i = \frac{1}{2} \implies N_i = \frac{1}{2}\sigma_i$$

no other choice. the only $SU(2)$ representation is Pauli and by (6.14)

$$J_i = \frac{1}{2}\sigma_i \qquad K_i = \frac{1}{2}i\sigma_i$$

Therefore we get a non-trivial representation for $SU(2)_L$. It is 2 dimensional, i.e.

$$\psi_a(x) \quad a = 1, 2 \tag{6.15}$$

called left-handed Weyl spinor. They are not equivalent to the Dirac spinor we studied before (6.3).

Although K_i is not hermitian, $S_L^{\mu\nu}$ is hermitian, because from (1.45)

$$\begin{aligned} S_L^{ij} &= \frac{1}{2}\epsilon^{ijk}\sigma_k \\ S_L^{i0} &= \frac{1}{2}i\sigma_i \end{aligned}$$

so ${L_a}^b$ is unitary.

(1,2) Representation

Next put

$$\tilde{n} = \frac{1}{2}, n = 0$$

then

$$N_i = 0 \implies J_i = iK_i \implies K_i = iJ_i$$

so we could have adopted the same thing in (6.14) with exchange $J \leftrightarrow k$. But we want J to be hermitian.

We choose

$$\tilde{N}_i = -\frac{1}{2}\sigma_i^*$$

where σ_i^* is complex conjugate of σ_i, and in fact

$$(\sigma_2)^{-1}(-\sigma_i^*)(\sigma_2) = \sigma_i$$

in other words

$$-\sigma_i^* \text{ and } \sigma_i \text{ are equivalent}$$

then

$$J_i = -\frac{1}{2}\sigma_i^* \qquad K_i = \frac{1}{2}i\sigma_i^*$$

And

$$\begin{aligned} S_R^{ij} &= -\frac{1}{2}\epsilon^{ijk}\sigma_k^* \\ S_R^{i0} &= \frac{1}{2}i\sigma_i^* \end{aligned}$$

thus

$$S_R^{\mu\nu} = -(S_L^{\mu\nu})^* \tag{6.16}$$

We get a non-trivial representation for $SU(2)_R$.

$$\phi_a(x) \quad a = 1, 2$$

called right-handed Weyl spinor. Notice the book uses

$$\phi_{\dot{a}}(x) \quad \dot{a} = 1, 2$$

to distinguish it from (6.15), because the two are living in two completely different spaces. Also the transformation (6.11) gain dots too

$$\hat{\phi}_{\dot{a}}(x) \to L_{\dot{a}}{}^{\dot{b}} \hat{\phi}_{\dot{b}}(\Lambda^{-1}x) \tag{6.17}$$

This kind of notation will be useful later when we do supersymmetry, but we won't use the dots notation very often if it is clear from the contexts.

Lastly we ask how the transformation of (6.11) and (6.17) are related. Starting from (6.6)

$$\psi_a \to \left(\delta_a{}^b + \frac{i}{2} \delta w_{\mu\nu}(\Lambda) (S_L^{\mu\nu})_a{}^b \right) \psi_a$$

take complex conjugation

$$\begin{aligned} \psi_a^* &\to \left(\delta_a{}^b - \frac{i}{2} \delta w_{\mu\nu}(\Lambda) (S_L^{\mu\nu *})_a{}^b \right) \psi_a^* \because (6.16) \\ &= \left(\delta_a{}^b + \frac{i}{2} \delta w_{\mu\nu}(\Lambda) (S_R^{\mu\nu})_a{}^b \right) \psi_a^* \end{aligned} \tag{6.18}$$

hence we say ψ_a^* (or compactly $\psi^\dagger$) transfers as a right-handed Weyl.

$$(\psi_a)^\dagger = \psi_{\dot{a}}^\dagger$$

By the way one can continue the process, e.g. putting

$$n = 1, \tilde{n} = 0$$

in this way one will discover photons spin 1 vector particles.

6.4. Lagrangian

We want to find a Lagrangian and from it to get canonical quantization. Since Lagrangian is a scale and spin is not, our goal is to construct Lorentz scale out of spinor. Recall in QM, one can form singlet, total spin 0, that is invariant under rotation. Say two spin 1/2 particles

$$\begin{cases} |\psi\rangle & = \psi_1 |\uparrow\rangle + \psi_2 |\downarrow\rangle \\ |\chi\rangle & = \chi_1 |\uparrow\rangle + \chi_2 |\downarrow\rangle \end{cases} \tag{6.19}$$

we get by changing of bases

$$\begin{aligned} |\psi\rangle \otimes |\chi\rangle &= \psi_1\chi_1 |\uparrow\uparrow\rangle + \psi_1\chi_2 |\uparrow\downarrow\rangle + \psi_2\chi_1 |\downarrow\uparrow\rangle + \psi_2\chi_2 |\downarrow\downarrow\rangle \\ &= \frac{(\psi_1\chi_2 - \psi_2\chi_1)}{\sqrt{2}} \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} + \frac{(\psi_1\chi_2 + \psi_2\chi_1)}{\sqrt{2}} \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \\ &\quad + \psi_1\chi_1 |\uparrow\uparrow\rangle + \psi_2\chi_2 |\downarrow\downarrow\rangle \end{aligned}$$

in such basis

$$\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = \text{singlet}$$

And the scalar quantity

$$\psi_1\chi_2 - \psi_2\chi_1$$

is invariant under rotations. The mathematical explanation is that it is the determinant of the matrix (6.19) in the basis $|\uparrow\rangle$ and $|\downarrow\rangle$ so invariant.

Notation we write

$$\psi\epsilon\chi = \psi_a\epsilon_{ab}\chi_b = \psi_1\chi_2 - \psi_2\chi_1 \tag{6.20}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2$$

It has a special case when $\psi = \chi$, $\psi\epsilon\chi = \psi_1\psi_2 - \psi_2\psi_1 = 2\psi_1\psi_2$ Grassmann variables. If one wishes to prove $\psi\epsilon\chi$ is a Lorentz scalar rigorously, he can use (6.18). The expression (6.20) suggests one can think of ϵ_{ab} as the metric in spinor space,

$$\psi\epsilon\chi = \langle\psi|\epsilon|\chi\rangle = (-\psi-)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} | \\ \chi \\ | \end{pmatrix} \tag{6.21}$$

but we will try to avoid that because if one uses it crudely with usual metric manipulation get confused $-$ signs.

E.g.

$$\begin{aligned} \psi_a\chi^a = \psi_a\epsilon^{ab}\chi_b &= \epsilon^{ab}\psi_a\chi_b \ \ (\epsilon^{ab}\text{is real number, switching no harm}) \\ &= -\epsilon^{ba}\psi_a\chi_b \ (\epsilon^{ab} \text{ is antisymm}) \\ &= -\psi^b\chi_b \end{aligned}$$

very confusing notation. We will instead writing out ϵ explicitly and stick with notation in (6.20) no matrix multiplication, no raising/lowing indices, only downstairs indices.

Likewise

$$\psi^\dagger\epsilon\chi^\dagger = \psi_a^*\epsilon_{ab}\chi_b^* \tag{6.22}$$

is also a scalar.

If we put $\psi = \chi$, (6.20) and (6.22) look like some kind of square, so they are good candidates for mass term in the Lagrangian. In addition we want the Lagrangian to be real, so we put

$$\frac{1}{2}m(\psi\epsilon\psi - \psi^\dagger\epsilon\psi^\dagger)$$

Unlike before it is not m^2 so we will get Dirac equation which is the square root of KG. The reason why is m not m^2 is discussed later.

For the kinetic term, one may try something like

$$\partial_\mu\psi\epsilon\partial^\mu\psi + h.c.$$

they don't work. Either $\mathcal{L}$ is not Lorentz invariant, or $\mathcal{H}$ is not bounded below.

What works is this

Claim.

$$i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi \tag{6.23}$$

which is a real and Lorentz scalar.

problem set 12 problem 1 shows

$$\psi^\dagger\bar{\sigma}^\mu\psi$$

is a Lorentz 4-vector. Check (6.23) is real.

$$\begin{aligned}(i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi)^\dagger &= -i(\partial_\mu\psi^\dagger)\underbrace{\bar{\sigma}^{\mu\dagger}}_{\bar{\sigma}^\mu}\psi \\ &= i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi \end{aligned} \tag{6.24}$$

The last step uses integration by parts. The rule of IBP for Grassmann variables is different than ordinary numbers. However it gives the same answer here.

Therefore we find the $\mathcal{L}$ for a single (left-handed) 2 component Weyl spinor ψ

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + \frac{1}{2}m(\psi\epsilon\psi - \psi^\dagger\epsilon\psi^\dagger) \tag{6.25}$$

actually $\psi^\dagger$ is for right-handed spinor (so Srednicki put $\psi^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu\psi_b$ for the kinetic term to emphasize it is interaction between left and right handed spinors, but we don't need that sophistication) because the derivation assumes $\psi = \chi$, the left-handed and right-handed are the particles with same 'spin', see (6.19). But when we get to Dirac spinor, we do have to be more careful, because the top and bottom components of Dirac spinor are not related. Top is left-handed and bottom is right-handed. What is the big deal of left v.s right? They have different rotation and boost transformations cf (6.16).

6.5. Majorana Spinor

Lecture 25 (4/23/14)

The action

$$S = \int d^4x i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + \frac{1}{2}m(\psi\epsilon\psi - \psi^\dagger\epsilon\psi^\dagger)$$

As mentioned Wigner trick works only for $d = 4$. This will be a difficult task for dimension regulation which we will study next semester.

Clearly from the kinetic term ψ has mass dimension 3/2, that is the reason why in the mass term we have m not m^2.

We will do variance on S. First vary $\psi^\dagger$ then vary ψ.

Claim: varying $\psi^\dagger$ gives

$$\delta S = \int d^4x \delta\psi^\dagger \left(i\bar{\sigma}^\mu \partial_\mu \psi - m\epsilon\psi^\dagger\right)$$

Let's check the second term

$$\begin{aligned}
\delta(\psi^\dagger\epsilon\psi^\dagger) &= \delta(\psi^\dagger_a \epsilon_{ab} \psi^\dagger_b) \\
&= \delta\psi^\dagger_a \epsilon_{ab} \psi^\dagger_b + \underbrace{\underbrace{\psi^\dagger_a \epsilon_{ab} \delta\psi^\dagger_b}_{-\delta\psi^\dagger_b \epsilon_{ab} \psi^\dagger_a}}_{\delta\psi^\dagger_b \epsilon_{ba} \psi^\dagger_a} \\
&= 2\delta\psi^\dagger_a \epsilon_{ab} \psi^\dagger_b
\end{aligned}$$

QED.

We get EOM

$$-i\bar{\sigma}^\mu \partial_\mu \psi + m\epsilon\psi^\dagger = 0 \tag{6.26}$$

If one gets confused the dimension above, it seems to say a square matrix multiplies a column matrix plus a square matrix multiplies a row matrix is equal to 0, one can either forget about the matrix multiplication, go back to the indices cf the steps in (6.28), or multiply $\phi^\dagger$ any arbitrary row vector to the left of (6.26), then

$$\phi^\dagger(6.26)$$

has the right dimension because whenever we see sandwich between ϵ, we mean inner product cf (6.21).

Now varying ψ, similarly use IBP (6.24)

$$\delta S = \int d^4x[\underbrace{i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\delta\psi}_{-i(\partial_\mu\psi^\dagger)\bar{\sigma}^\mu\delta\psi} - m\psi\epsilon\delta\psi]$$

We get EOM

$$-i(\partial_\mu\psi^\dagger)\bar{\sigma}^\mu + m\psi\epsilon = 0 \tag{6.27}$$

which can be obtained also from taking † of (6.26) and use

$$\sigma_i^t = \epsilon\sigma_i\epsilon \qquad (\bar{\sigma}^\mu)^t = -\epsilon\sigma^\mu\epsilon$$

see problem set 12 problem 2.

We rewrite (6.27) a little bit. Putting indices to (6.27)

$$-i(\partial_\mu\psi_a^\dagger)\underbrace{\bar{\sigma}^\mu_{ab}}_{-(\epsilon\sigma^\mu\epsilon)_{ba}} + m\psi_a\underbrace{\epsilon_{ab}}_{\epsilon^t_{ba}} = 0 \tag{6.28}$$

or

$$i(\epsilon\sigma^\mu\epsilon)_{ba}(\partial_\mu\psi_a^\dagger) + m\epsilon^t_{ba}\psi_a = 0$$

or

$$i\epsilon\sigma^\mu\epsilon\partial_\mu\psi^\dagger + m\epsilon^t\psi = 0$$

Since $\epsilon\epsilon^t = 1,\ \epsilon^2 = -1$, (6.27) becomes

$$-i\sigma^\mu\epsilon\partial_\mu\psi^\dagger + m\psi = 0 \tag{6.29}$$

Combining (6.26) and (6.29)

$$\begin{pmatrix} m & -i\sigma^\mu\partial_\mu \\ -i\bar{\sigma}^\mu\partial_\mu & m \end{pmatrix}\begin{pmatrix} \psi \\ \epsilon\psi^\dagger \end{pmatrix} = 0$$

As we discussed before when sandwiched between ϵ we get inner product, so in

$$\begin{pmatrix} \psi \\ \epsilon\psi^\dagger \end{pmatrix}$$

we will be better off if we think $\epsilon\psi^\dagger$ as a two-component column vector.

The matrix

$$\begin{pmatrix} m & -i\sigma^\mu\partial_\mu \\ -i\bar{\sigma}^\mu\partial_\mu & m \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\partial_\mu + m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -i\gamma^\mu\partial_\mu + m$$

hence we get Dirac equation for the particle that has left handed = right handed spin, and

$$\begin{pmatrix} \psi \\ \epsilon\psi^\dagger \end{pmatrix} = \text{Majorana spinor}$$

In other words the 4 components spinor has only 2 independent components.

6.6. Dirac Equation

To get 2 independent spin wave function, we need some extra structure. Consider

$$\mathcal{L} = \sum_{i=1}^{2} i\psi^{(i)\dagger}\bar{\sigma}^\mu\partial_\mu\psi^{(i)} + \frac{1}{2}m(\psi^{(i)}\epsilon\psi^{(i)} - \psi^{(i)\dagger}\epsilon\psi^{(i)\dagger}) \tag{6.30}$$

i labels 2 different (left handed) Weyl spinor. Consider some internal symmetry (no change on spacetime)

We do

$$\begin{aligned} \chi &= \frac{1}{\sqrt{2}}(\psi^{(1)} + i\psi^{(2)}) \\ \xi &= \frac{1}{\sqrt{2}}(\psi^{(1)} - i\psi^{(2)}) \end{aligned}$$

Plugging into (6.30) we get interaction in the mass term, while the kinetic term is still separable.

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi + m(\chi\epsilon\xi - \xi^\dagger\epsilon\chi^\dagger) \tag{6.31}$$

In this way if

$$\chi = \xi \tag{6.32}$$

called Majorana condition, then

$$\psi^{(2)} = 0$$

so we reduce to (6.30) with $i = 1$ that is (6.25).

Now do variance (6.31) on $\chi^\dagger$ to get EOM

$$-i\bar{\sigma}^\mu \partial_\mu \chi + m\epsilon\xi^\dagger = 0 \tag{6.33}$$

do variance on $\xi^\dagger$ get EOM

$$i\bar{\sigma}^\mu \partial_\mu \xi - m\epsilon\chi^\dagger = 0$$

or rewritten as

$$-i\sigma^\mu \partial_\mu \epsilon\xi^\dagger + m\chi = 0 \tag{6.34}$$

(6.33) & (6.34) are two independent equations for 2 Weyl spinor. Combine them

$$\begin{pmatrix} m & -i\sigma^\mu\partial_\mu \\ -i\bar{\sigma}^\mu\partial_\mu & m \end{pmatrix} \begin{pmatrix} \chi \\ \epsilon\xi^\dagger \end{pmatrix} = 0$$

$$\Psi = \begin{pmatrix} \chi \\ \epsilon\xi^\dagger \end{pmatrix} = \text{Dirac spinor}$$

Comparing Dirac spinor and Majorana spinor is like comparing real scalar field and complex scalar field. See problem set 1 problem 6. Both real scalar field and complex scalar field satisfy KG equation. Complex scalar field has two different kinds of annihilation operators. Both Dirac spinor and Majorana spinor satisfy Dirac equation. As we'll see Dirac spinor has two different kinds of annihilation operators, and it contains particles & antiparticles.

We can rewrite $\mathcal{L}$ (6.31) in terms of Dirac spinor Ψ (see problem set 12 problem 3)

$$\mathcal{L}_{Dirac} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \tag{6.35}$$

where

$$\bar{\Psi} = \Psi^\dagger\gamma^0$$

(since γ^μ will appear many places, some people put slash $\gamma^\mu\partial_\mu = \not{\partial}$, but we are not a big fan of it), then the Dirac equation become trivially seen by doing variance on (6.35) on $\bar{\Psi}$. One can also do variance on Ψ and get

$$\bar{\Psi}(i\gamma^\mu\partial_\mu + m) = 0 \tag{6.36}$$

which is alternative form of Dirac equation. The derivative is understood to be acting to its left. We can also try to move $\bar{\Psi}$ to the right of ∂_μ but that requires to make some change to γ^μ because they don't commute. At the end we will just get the original form of Dirac equation. One may ask that it seems like $\mathcal{L}_{Dirac} \equiv 0$ because

$$\mathcal{L}_{Dirac} = \bar{\Psi}\underbrace{[i\gamma^\mu\partial_\mu - m]\Psi}_{\text{Dirac equa}} \tag{6.37}$$

it is actually 0 on the eom but not 0 everywhere.

Problem set 12 problem 4 shows an alternative way to get $\mathcal{L}_{Dirac}$ using (6.3).

Like problem set 1 problem 6, we ask what kind of symmetry $\mathcal{L}_{Dirac}$ has, and what the associated Noether current is.

$\mathcal{L}_{Dirac}$ is invariant under $U(1)$ globe symmetry

$$\Psi \to e^{i\alpha}\Psi$$

so by (1.35)

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\delta\Psi = (i\bar{\Psi}\gamma^\mu)(i\alpha\Psi) \propto \bar{\Psi}\gamma^\mu\Psi \tag{6.38}$$

which is of course conserved on EOM, i.e. Dirac equation.

What is j^μ (6.38) for Majorana spinor?

$$\bar{\Psi}\gamma^\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\chi - \xi^\dagger\bar{\sigma}^\mu\xi$$

By (6.32)

$$j^\mu = 0$$

Lecture 26 (4/28/14)

Lastly we show a way to go from Dirac spinor to top and bottom Weyl spinor by projection. Define chirality operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that defining

$$P_L = \frac{1-\gamma^5}{2} \quad P_R = \frac{1+\gamma^5}{2}$$

we have

$$P_L\Psi = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \qquad P_R\Psi = \begin{pmatrix} 0 \\ \epsilon\xi^\dagger \end{pmatrix}$$

and

$$\gamma^5 \begin{pmatrix} \chi \\ 0 \end{pmatrix} = -\begin{pmatrix} \chi \\ 0 \end{pmatrix} \qquad \gamma^5 \begin{pmatrix} 0 \\ \epsilon\xi^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon\xi^\dagger \end{pmatrix}$$

so we say P_L and P_R project Dirac spinor onto left and right handed parts of Ψ, which are chirality eigenstates with eigenvalues ± 1.

6.7. Solutions to Dirac Equation

We get inspiration from KG solution (1.16), we guess solution to Dirac

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2w_p} \sum_{s=\pm} [u_s(p)e^{ip\cdot x}b_s(p) + v_s(p)e^{-ip\cdot x}d_s^+(p)] \tag{6.39}$$

where

$$s = \pm \iff \text{spin up/down along } z$$

We will later (6.49), (6.53) show operators

$$\begin{aligned} b_s(p) &= \text{annihilate a } e^-\text{with spin } s \\ d_s^\dagger(p) &= \text{create a } e^+\text{with spin } s \end{aligned}$$

and

$$u_s, v_s$$

are 4 components vectors obey the following by Dirac equation (6.2) or (6.36),

$$
\begin{aligned}
(\gamma^\mu p_\mu + m)u_s(p) &= 0 && (6.40)\\
(-\gamma^\mu p_\mu + m)v_s(p) &= 0 && (6.41)\\
\bar{u}_s(p)(\gamma^\mu p_\mu + m) &= 0\\
\bar{v}_s(p)(-\gamma^\mu p_\mu + m) &= 0
\end{aligned}
$$

only the 2 of them are independent, where

$$
\bar{u}_s(p) = u^\dagger\gamma^0 \qquad \bar{v}_s(p) = v^\dagger\gamma^0
$$

By (6.3), (1.45), the spin angular momentum operator is

$$
\vec{S} = (\Sigma^{23}, -\Sigma^{13}, \Sigma^{12}) = \frac{1}{2}\left(\begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}, \begin{pmatrix} \sigma_2 & \\ & \sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & \\ & \vec{\sigma} \end{pmatrix}
$$

Non-Relativistic Limit

In the rest frame $\vec{p} = 0$, $p_0 = -m$, (6.40) becomes

$$
m\begin{pmatrix} 1 & & -1 & \\ & 1 & & -1 \\ -1 & & 1 & \\ & -1 & & 1 \end{pmatrix} u_s = 0 \implies u_+ \sim \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_- \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}
$$

(6.41) becomes

$$
m\begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix} v_s = 0 \implies v_+ \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v_- \sim \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
$$

and they satisfy

$$
\begin{cases} S_z u_\pm = \pm\frac{1}{2}u_\pm \\ S_z v_\pm = \mp\frac{1}{2}v_\pm \end{cases} \tag{6.42}
$$

we choose such relations because we want $b^\dagger_\pm$, $d^\dagger_\pm$ to create particle with spin up/down along the z axis, see Srednicki chapter 38.

It is kind of surprising to learn that u, v have non-zero components in both top and bottom. So the two unrelated Weyl spinors in a Dirac spinors contains both particles and antiparticles.

Intermediate Limit

Problem set 12 problem 5 shows if a particle is moving in the z direction, compute

$$u_s(p) = e^{\phi\begin{pmatrix} -\vec{\sigma}\cdot\hat{p} & \\ & \vec{\sigma}\cdot\hat{p}\end{pmatrix}/2} u_s(0)$$

where

$$\vec{\sigma}\cdot\hat{p} = \text{helicity operator}$$

Since in the formula $\vec{p}$ is the momentum of the particle wrt the lab frame. If our lab is moving fast than the particle, $\vec{p}$ will become $-\vec{p}$, so helicity will flip, thus the concept of spin up/down, helicity means no more than just the eigenvalue of $\vec{\sigma}\cdot\hat{p}$ operator, i.e. it is not an internal property of the particle. If the particle is massless like neutrino, certain helicity will stay the same.

We get

$$u_+ \sim \begin{pmatrix} \sqrt{E+m} - \sqrt{E-m} \\ 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \end{pmatrix} \quad u_- \sim \begin{pmatrix} 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \\ \sqrt{E+m} - \sqrt{E-m} \end{pmatrix}$$
$$v_+ \sim \begin{pmatrix} 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \\ -\sqrt{E+m} + \sqrt{E-m} \end{pmatrix} \quad v_- \sim \begin{pmatrix} -\sqrt{E+m} + \sqrt{E-m} \\ 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \end{pmatrix}$$

They satisfy (6.42). So we can say by (6.42)

$$u_+, v_- \text{ spin up or helicity } +1 \text{ or right-handedness}$$

u_{-}, v_{+} spin down or helicity -1 or left-handedness

They also obey orthonormality

$$\begin{aligned}\bar{u}_{s'}(p)u_s(p) &= 2m\delta_{ss'} \\ \bar{v}_{s'}(p)v_s(p) &= -2m\delta_{ss'} \\ \bar{u}_{s'}(p)v_s(p) &= \bar{v}_{s'}(p)u_s(p) = 0\end{aligned}$$

They obey some sort of completeness

$$\begin{aligned}\sum_s u_s(\vec{p})\bar{u}_s(-\vec{p}) &= -\gamma^\mu p_\mu + m \\ \sum_s v_s(p)\bar{v}_s(p) &= -\gamma^\mu p_\mu - m\end{aligned}$$

They obey Gordon identities (Srednicki equation 38.18)

$$\begin{aligned}2m\bar{u}_{s'}(p')\gamma^\mu u_s(p) &= \bar{u}_{s'}(p')[(p'+p)^\mu - 2i\Sigma^{\mu\nu}(p'-p)_\nu]u_s(p) \\ -2m\bar{v}_{s'}(p')\gamma^\mu v_s(p) &= \bar{v}_{s'}(p')[(p'+p)^\mu - 2i\Sigma^{\mu\nu}(p'-p)_\nu]v_s(p)\end{aligned}$$

Special case when $p = p'$

$$\bar{u}_{s'}(p)\gamma^\mu u_s(p) = 2p^\mu\delta_{ss'} \tag{6.43}$$

$$\bar{v}_{s'}(p)\gamma^\mu v_s(p) = 2p^\mu\delta_{ss'} \tag{6.44}$$

another special case when $p^0 = p^{0'}$ and $\vec{p} = -\vec{p}'$

$$\bar{u}_{s'}(p)\gamma^0 v_s(p') = 0 \tag{6.45}$$

$$\bar{v}_{s'}(p)\gamma^0 u_s(p') = 0 \tag{6.46}$$

Relativity Limit

$E \gg m$, (6.40) becomes

$$u_+ \sim \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_- \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_+ \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad v_- \sim \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

They satisfy (6.42).

Lastly we recall (6.38)

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi$$

since we have an expression of Ψ, we can compute the conserved charge (1.33)

$$\bar{\Psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2w_p} \sum_{s=\pm} [\bar{u}_s(p)e^{-ip\cdot x}b_s^\dagger(p) + \bar{v}_s(p)e^{ip\cdot x}d_s(p)] \tag{6.47}$$

$$\begin{aligned} Q &= \int d^3x \int \frac{d^3p}{(2\pi)^3 2w_p} \sum_s \int \frac{d^3p'}{(2\pi)^3 2w_{p'}} \sum_{s'} \left(\bar{u}_{s'}(p')e^{-ip'\cdot x}b_{s'}^\dagger(p') + \bar{v}_{s'}(p')e^{-ip'\cdot x}d_s(p')\right) \\ &\quad \gamma^0 \left(u_s(p)e^{ip\cdot x}b_s(p) + v_s(p)e^{-ip\cdot x}d_s^+(p)\right) \end{aligned} \tag{6.48}$$

Using (6.45), (6.46), we can show the two cross terms give 0. E.g. to tackle

$$\left(\bar{u}_{s'}(p')e^{-ip'\cdot x}b_{s'}^\dagger(p')\right)\gamma^0\left(v_s(p)e^{-ip\cdot x}d_s^+(p)\right)$$

first do x integral

$$\int d^3x e^{-i(p'+p)x} = (2\pi)^3\delta(\vec{p}' + \vec{p})e^{-i(p^{0'}+p^0)t}$$

The other terms are evaluated by (6.43), (6.44),

$$Q = \int \frac{d^3p}{(2\pi)2w_p} \sum_s b_s^\dagger(p)b_s(p) + \underbrace{d_s(p)d_s^\dagger(p)}_{-d_s^\dagger(p)d_s(p)} \tag{6.49}$$

$b_s^\dagger(p)b_s(p)$ counts the number e^- and $d_s^\dagger(p)d_s(p)$ counts e^+. Hence eQ = total charge

6.8. Canonical Quantization

From (6.35), we get momentum

$$\Pi_\alpha = \frac{\partial \mathcal{L}_{Dirac}}{\partial(\partial\dot{\Psi}_\alpha)} = i(\bar{\Psi}\gamma^0)_\alpha$$

For Dirac, we need anti commutator, if one uses commutator instead, it will be illogical, as we will see e.g. adding more particle energy goes down. Similar to scalar field commutators (1.18), (1.19), one can show

$$\{\Psi_\alpha(\vec{x},t), \Pi_\beta(\vec{y},t)\} = i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{6.50}$$

$$\{\Psi_\alpha(\vec{x},t), \Psi_\beta(\vec{y},t)\} = \{\Pi_\alpha(\vec{x},t), \Pi_\beta(\vec{y},t)\} = 0 \tag{6.51}$$

However for Majorana particle

$$\{\Psi_\alpha(\vec{x},t), \Psi_\beta(\vec{y},t)\} \neq 0$$

Lecture 27 (4/30/14)

We want to get anti commutation relation between b_s and d_s. Rewrite (6.50)

$$\{\Psi_\alpha(\vec{x},t), \bar{\Psi}_\epsilon(\vec{y},t)\}\gamma^0_{\epsilon\beta} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$

they are numbers, so take out $\gamma^0_{\epsilon\beta}$ does no harm. Multiply both sides by $\gamma^0_{\beta\sigma}$ and sum over β. Since $(\gamma^0)^2 = 1$

$$\{\Psi_\alpha(\vec{x},t), \bar{\Psi}_\sigma(\vec{y},t)\} = \gamma^0_{\alpha\sigma}\delta^3(\vec{x}-\vec{y}) \tag{6.52}$$

Using (6.39), (6.51), (6.52), one can show

$$\{b_s(p), b^\dagger_{s'}(p')\} = \{d_s(p), d^\dagger_{s'}(p')\} = \delta_{ss'}(2\pi)^3 2w_p \delta^3(\vec{p} - \vec{p}')$$

are the only non-zeros. All others like

$$\{b, b\} = \{b^\dagger, b^\dagger\} = ... = 0$$

In particular $\{b^\dagger, b^\dagger\} = 0 \implies$

$$b^\dagger_s(p)b^\dagger_s(p) + b^\dagger_s(p)b^\dagger_s(p) = 2b^\dagger_s(p)b^\dagger_s(p) = 0 \implies b^\dagger_s(p)b^\dagger_s(p)\left|0\right\rangle = 0$$

i.e. no 2 particle can be created in the same state.

$$b^\dagger_s(p)b^\dagger_{s'}(p') + b^\dagger_{s'}(p')b^\dagger_s(p) = 0 \implies b^\dagger_s(p)b^\dagger_{s'}(p')\left|0\right\rangle = -b^\dagger_{s'}(p')b^\dagger_s(p)\left|0\right\rangle$$

$$\implies \left|p, s; p's'\right\rangle = -\left|p', s'; p, s\right\rangle$$

i.e. exchanging 2 Fermions induce $-$.

6.9. Hamiltonian

From (6.35), we get

$$\begin{aligned} \mathcal{H} &= \Pi_\alpha \dot{\Psi}_\alpha - \mathcal{L} \\ &= i\bar{\Psi}\gamma^0\partial_0\Psi - i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + m\bar{\Psi}\Psi \\ &= i\bar{\Psi}\gamma^0\partial_0\Psi \end{aligned}$$

assume Ψ is on the EOM, see (6.37).

Plugging (6.39), (6.47),

$$i\gamma^0\partial_0\Psi = \int \frac{d^3p}{(2\pi)^3 2w_p} \sum_s \underbrace{\underbrace{-\gamma^0 p_0}_{\gamma^0 p^0}}_{w_p\gamma^0} u_s(p)e^{ip\cdot x}b_s(p) + \underbrace{\gamma^0 p_0}_{-w_p\gamma^0} v_s(p)e^{-ip\cdot x}d^+_s(p)$$

then the rest is very similar to steps in (6.48)

$$
\begin{aligned}
H &= \int d^3x\mathcal{H} \\
&= \int \frac{d^3p}{(2\pi)^3 2w_p} \sum_s w_p \left(b_s^\dagger(p) b_s(p) - d_s(p) d_s^\dagger(p) \right)
\end{aligned}
\tag{6.53}
$$

If we had some kind of trivial commutator relation between

$$[d_s(p), d_s^\dagger(p)] = 1 \text{ or something}$$

then

$$H = \int \frac{d^3p}{(2\pi)^3 2w_p} \sum_s w_p \left(b_s^\dagger(p) b_s(p) - d_s(p)^\dagger d_s(p) + 1 \right)$$

so $d_s(p)^\dagger d_s(p)$ counts particles so that adding more particles energy decease. Bad. By the way that may be the reason why it was called antiparticle, hence it was initially thought to carry negative energy. But now we know from (6.49), antiparticle is just particle with opposite charge. And for Majorana spinor we put

$$b_s(p) = d_s(p)$$

in (6.39), (6.53), etc, which is analogous to going from complex scalar field to real scalar field, by putting

$$a_k = b_k$$

see problem set 1 problem 6.

7. Applications

7.1. QED

We have done free theory. Free theory are nice but not interesting. One can add an external source. Recall in problem set 11 problem 2, we say that by adding

$$A_\mu J^\mu$$

to the free photon theory

$$S = \int d^4x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$$

we get correct Maxwell equations with source J. Now we add in Fermions, we get photon e^- interaction, and the J is given by (6.38), thus

$$\begin{aligned} S &= \int d^4x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi + e\bar{\Psi}\gamma^\mu\Psi A_\mu \\ &= \int d^4x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma^\mu(\partial_\mu - ieA_\mu)\Psi - m\bar{\Psi}\Psi \end{aligned}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The term

$$e\bar{\Psi}\gamma^\mu\Psi A_\mu$$

is a cubic interaction: 2 fermion and 1 photon, and e is the coupling. After computing the propagator (problem set 12 problem 6), we can use that to compute scattering of 2 fermions in 2 fermions out (Coulomb scattering) or a photon strikes a e^-(Compton scattering). Notice the μ index in $A_\mu$is spin index as well as spacetime index, related to gauge symmetry, which we will study next semester. This is special for photon: massless spin 1 particle.

So we are done with QED.

7.2. Hawking Radiation

Lecture 28 -Last Lec- (5/5/14)

Imagine we do now field theory in a box. Suddenly make the size of box $\to \infty$ (inflation) What happens to the vacuum state? In the black hole content, this is the Hawking radiation.

If the potential change slowly, by adiabatic approximation, groundstate stays groundstate, so no particle creation. If sudden change occurs, then state doesn't change, but no longer groundstate, so it has particle created. And how much being created depends on acceleration.

Consider an expanding universe

$$ds^2 = -dt^2 + a(t)[dx^2 + dy^2 + dz^2]$$

$a(t)$ is scalar factor. Assume

$$a(t) = \begin{cases} 1 & t > 0 \\ L & t < 0 \end{cases} \sim \sqrt{\det g^{\alpha\beta}}$$

a sudden change at $t = 0$.

$$S_{\text{free scalar field}} = \int dt d^3x a^3(t)[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2a^2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2]$$

The $(\nabla/a)^2$ gives the physical gradient.

$$\delta S = \int dt d^3x \underbrace{a^3\partial_t\phi\partial_t\delta\phi}_{-\partial_t(a^3\partial_t\phi)\delta\phi} + a\nabla^2\phi\delta\phi - a^3m^2\phi\delta\phi$$

Get EOM

$$-\partial_t(a^3\partial_t\phi) + a\nabla^2\phi - a^3m^2\phi = 0 \tag{7.1}$$

If $a = 1$, we get KG.

We say $\phi(t, \vec{x})$ is continuous at $t = 0$ by physical sense. And

$$a^3\partial_t\phi$$

is continuous at $t = 0$, because otherwise

$$\partial_t(a^3\partial_t\phi) \sim \delta$$

but there is no δ elsewhere in the equation (7.1).

And

$$a^3\partial_t\phi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \Pi(t, \vec{x}) \tag{7.2}$$

So we do canonical quantization

$$[\phi(t,\vec{x}), a^3\partial_t\phi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}) \tag{7.3}$$

Consider plane wave solution $\phi = e^{ikx}$ to (7.1).

For $t<0$

$$L^3\nu_k^2 = L\vec{k}^2 + L^3m^2 \implies \nu_k^2 = \frac{\vec{k}^2}{L^2} + m^2$$

For $t>0$

$$w_k^2 = \vec{k}^2 + m^2 \implies w_k^2 = \vec{k}^2 + m^2$$

Now we construct wave function for $t<0$

$$\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2\nu_k L^{3/2}}[b_k e^{ikx} + b_k^\dagger e^{-ikx}]$$

why is there $L^{3/2}$? because we want (7.3)

$$[\phi(t,\vec{x}), L^3\partial_t\phi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})$$

the derivation is same as (1.18), and using b_k satisfies (1.6).

For $t>0$

$$\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2w_k}[a_k e^{ikx} + a_k^\dagger e^{-ikx}]$$

Claim

$$a_k \neq b_k$$

Compute Π by (7.2).

For $t<0$

$$\Pi = \int \frac{d^3k L^{3/2}}{(2\pi)^3 2\nu_k}[-i\nu_k b_k e^{ikx} + i\nu_k b_k^\dagger e^{-ikx}]$$

For $t>0$

$$\Pi = \int \frac{d^3k}{(2\pi)^3 2w_k}[-iw_k a_k e^{ikx} + iw_k a_k^\dagger e^{-ikx}]$$

By the continuity condition for ϕ and by Fourier

$$\frac{1}{\nu_k L^{3/2}}(b_k + b_{-k}^\dagger) = \frac{1}{w_k}(a_k + a_{-k}^\dagger)$$

and by the continuity condition for Π and by Fourier

$$L^{3/2}(-b_k + b^\dagger_{-k}) = (-a_k + a^\dagger_{-k})$$

solve for a_k

$$a_k = \frac{1}{2}\left[(\frac{w_k}{\nu_k L^{3/2}} + L^{3/2})b_k + (\frac{w_k}{\nu_k L^{3/2}} - L^{3/2})b^\dagger_{-k}\right]$$

This implies particle production. Because $t<0$ vacuum state

$$b_k \left|0\right\rangle_{t<0} = 0$$

but for $t>0$

$${}_{t<0}\left\langle 0\middle|a^\dagger_k a_k\middle|0\right\rangle_{t<0} \propto (\frac{w_k}{\nu_k L^{3/2}} - L^{3/2})^2 \neq 0$$

Before we go to study black holes, we need some preliminary quantum statistical mechanics.

Quantum partition function

$$Z_{stat} = \sum_n \left\langle n\middle|e^{-\beta\hat{H}}\middle|n\right\rangle \quad \beta = 1/T$$

Recall path integral (2.18)

$$Z_{qft} = \langle q_f, t_f | q_i, t_i\rangle = \int Dq e^{iS}$$

so we can relate Z_{stat} to Z_{qft}, by going to Euclidean time and the path evolutes back to the same state.

$$Z_{stat} = \int dq_i \left\langle q_i\middle|e^{-\beta\hat{H}}\middle|q_i\right\rangle \quad \beta = i(t_f - t_i)$$

Or more fancy way saying is that temperature is imaginary and periodic. This is part of finite temperature quantum field theory.

Now go to black holes, we use Schwarzschild metric

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + ...) \tag{7.4}$$

Classically nothing comes out beyond BH horizon

$$r = 2GM$$

Quantum mechanically something comes out. First put

$$it = t_E$$

so (7.4) becomes

$$ds^2 = (1 - \frac{2GM}{r})dt_E^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + \text{angular terms} \tag{7.5}$$

Do a coordinate transformation

$$r = 2GM + R^2$$

then

$$dr^2 = 4R^2 dR^2$$

Assume R is small, so

$$1 - \frac{2GM}{r} = \frac{R^2}{2GM + R^2} \approx \frac{R^2}{2GM}$$

(7.5) becomes

$$ds^2 = \frac{R^2}{2GM}dt_E^2 + 8GMdR^2 + \text{angular terms}$$

Define

$$\tilde{R} = \sqrt{8GM}R \quad \alpha = \frac{t_E}{4GM}$$

then

$$ds^2 = \tilde{R}^2 d\alpha^2 + d\tilde{R}^2$$

looking like polar coordinates, so α has to be periodic of 2π, so

$$t_E \text{ has period } 8\pi GM$$

which is equal to β. So

$$\text{Temp of BH} = \frac{1}{8\pi GM}$$

such temp is indeed in form of Hawking radiation.

For large mass BH, we have no hope of detecting the temperature, but for primordial black holes we may have some luck.