Intro to Modern Analysis I

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Transcribed by Ron Wu

This is an advanced undergraduate course, offered in Summer 2013 at Columbia University. Course textbook is Rudin, *Principles of Mathematical Analysis.* Recommended books for set theory is Halmos, *Naive Set Theory.* Grading: assignments from the book 10%; two midterms 30%, 15%; final 45%. Office hours: T,R 5:00-6:00.

Contents

Course Overview

Lecture 1 (5/28/13)

This course will cover Rudin chapters 1-6.

Plan

Chapter 1

(1) Definition of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and their algebraic properties.

(2) Order relation and the least upper bound property; ordering property (Archimedean property). Some of them will be taken as granted.

Chapter 2: most difficult chapter

(3) Topology: notations of open, closed, compact, connected. Notice Rudin didn't define topology space. He used metric space.

(4) Cardinality: finite, countable, uncountable (the distinguish between countable and uncountable becomes extremely important as we move to measure theory.)

Chapter 3: Many topics are covered in Calculus class. We will do everything in metric space. That is because convergence can be defined in top space but then the limit is not necessary unique. But keep in mind that many important space studied in geometry are top but not metric.

(5) sequence in a metric space, series over $\mathbb{R}$ or $\mathbb{C}$. (In our class we will be able to see any distinguish between $\mathbb{R}$ and $\mathbb{C}$.)

Chapters 4-6

(6) Functions over metric space: particular continuity $\epsilon - \delta$ definition and top properties.

(7) differentiability over $\mathbb{R}$ (analysis II will study differential from $\mathbb{R}^m$ to $\mathbb{R}^n$) Mean value theorem, L'Hopital, Taylor.

(8) Riemann-Stieltjes over $\mathbb{R}$. This is going towards the direction of measure theory. The contribution of Stieltjes to Riemann is that consider the following Riemann integral

$$\int f(x)dx$$

and Riemann-Stieltjes integral

$$\int f(x)d\alpha(x)$$

where $\alpha(x)$ is an increasing function.

Whether $f(x)$ has a jump discontinuity at x_0 makes no difference in the Riemann integral, but it will make difference in Riemann-Stieltjes if $\alpha(x)$ has too a jump discontinuity at x_0.

Chapter 7: most advanced chapter after chapter 2, if time permitted

(9) uniform convergence, this is going towards functional analysis. We will see that once metric space is developed, for free we can deal with space of functions in topology. That is the reason why Rudin used metric space at very beginning.

1 Foundations

1.1 Relations

We take the concept of a "set" as a primitive principle. If A is a set, $a \in A$ is an element of A. If A, B are sets, $A \subset B$ means $\forall a \in A$, $a \in B$. This includes the possibility that $A = B$. If A, B are sets, we can make Cartesian product

$$A \times B = \{(a, b); a \in A, b \in B\}$$

Definition 1. A relation between two sets A, B is a subset

$$R \subset A \times B$$

If $(a, b) \in R$, we say that a is related to b.

Definition 2. A function f from set A to set B is a relation s.t. $\forall\, a \in A$, $\exists\,!\; b \in B$ s.t. $(a, b) \in f$. We call A the domain of f and B the codomain.

Note that we're usually only interested in codomain, not range. That is because 1) finding range is hard; codomain is easily seen; 2) we're working with sets. Specially in geometry we may have some structures over sets. When we shrink the set from codomain to range, we may break some structures. In practice codomain is better to work with.

Definition 3. $f : A \to B$ is injective iff

$$f(a_1) = f(a_2) \implies a_1 = a_2$$

Definition 4. $f : A \to B$ is surjective iff

$$\forall\, b \in B \quad \exists\, a \in A \text{ s.t. } f(a) = b$$

Definition 5. $f : A \to B$, $C \subset A$. $f(C) = \{f(x); x \in C\}$ is called image of C.

Remark 6. $f : A \to B$ is surjective iff $f(A) = B$. The range is the co domain.

Definition 7. $f : A \to B$, $D \subset B$. $f^{-1}(D) = \{x \in A; f(x) \in D\}$ called pre image of D.

This is always well-defined. It could be $\varnothing$, a point, A, etc.

Definition 8. If f is injective and surjective, then it's called bijective or invertible.

Remark 9. $\forall\, b \in B$, $\exists\, !a \in A$ s.t. $f(a) = b$, we denote a with $f^{-1}(b)$ and the function f^{-1} has the property

$$f \circ f^{-1} = f^{-1} \circ f = \text{indentity}$$

If the function is only injective or surjective, the it has only left inverse or right inverse.

1.2 Natural Numbers

The natural numbers were defined only 100 years ago. The definition uses several axioms, but doesn't tell us how to find $\mathbb{N}$ explicitly.

Definition 10. (Peono, Dedekind) $(\mathbb{N}, S, 0)$ is a triple, $\mathbb{N}$ is a st, $0 \in \mathbb{N}$

$$S : \mathbb{N} \to \mathbb{N}$$

a function s.t.

1) S is injective;

2) $\nexists\, n \in \mathbb{N}$ s.t. $S(n) = 0$;

3) If $k \in \mathbb{N}$ s.t. that $0 \in K$ and $n \in K$ implies $S(n) \in K$, then $K = \mathbb{N}$.

Remark 11. S is the successor of a number.

At this stage addition is yet defined. 3) in the definition gives a way of recursive searching for $\mathbb{N}$. One model for $\mathbb{N}$ could the be set of

$$\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\ldots$$

where $0 = \varnothing$.

However there are many other models. E.g.

$$\{0, 2, 4, 6...\}$$

works for $S(n) = n + 2$. The following proposition shows they are equivalent in what sense.

Proposition 12. $(\mathbb{N}, S, 0)$, $(M, T, \tilde{0})$ *satisfying Peono, then* $\exists!\ f : \mathbb{N} \to M$ *s.t.* f *is bijective and the following diagram commutes, i.e.* $f \circ S = T \circ f$

$$\begin{array}{ccccc} & \mathbb{N} & \xrightarrow{S} & \mathbb{N} & \\ f & \downarrow & & \downarrow & f \\ & M & \xrightarrow{S} & M & \end{array}$$

Definition 13. (addition) Define the operation

$$\begin{aligned} + : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (n, m) &\mapsto n + m \end{aligned}$$

with

$$\begin{cases} n + 0 = n \\ n + S(m) = S(n + m) \end{cases}$$

Note 14. For the above definition to make sense, we have a little bit to show. We need to show $+$ defines a function recursively. E.g

$$\begin{aligned} 1 &\equiv S(0) \\ 1 + 1 &= 1 + S(0) = S(1 + 0) = S(1) \equiv 2 \\ 1 + 2 &= 1 + S(1) = S(1 + 1) = S(2) \equiv 3 \end{aligned}$$

Lecture 2 (5/30/13)

In the same way, we define product

Definition 15. (product) Define the operation

$$\begin{aligned} \cdot : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (n, m) &\mapsto n \cdot m \end{aligned}$$

with

$$\begin{cases} n \cdot 0 = 0 \\ n \cdot S(m) = n \cdot m + n \end{cases}$$

$(\mathbb{N}, +)$ is monoid, i.e. it satisfies

1) associative $(n+m)+p = n+(m+p)$.

2) commutative $n+m = m+n$.

3) neutral element $n+0 = 0+n = n$

Proof. (of 1) fix $n, m \in \mathbb{N}$, let $K = \{p \in \mathbb{N} \text{ s.t. } 1) \text{ is true.}\}$ want to show $K = \mathbb{N}$.

First prove that $0 \in K$

$$n+m = (n+m)+0$$

and

$$n+m = n+(m+0)$$

Then prove that if $p \in K$ then $S(p) \in K$

$$(n+m)+S(p) = S((n+m)+p) = S(n+(m+p)) = n+S(m+p) = n+(m+S(p))$$

$\square$

By the same spirit, we can show $+$, $\times$ have many familiar algebraic properties.

1.3 Order Relations

Definition 16. A is a set, a relation $\leq$ of A with itself is a partial order iff

1) $a \leq a \; \forall a \in A$

2) if $a \leq b$ and $b \leq c$, then $a \leq c$ (known as transitivity)

3) if $a \leq b$ and $b \leq a$, then $a = b$

Definition 17. A is ordered (sometimes called total ordered) if it has a partial order and $\forall a, b \in A \; a \leq b$ or $b \leq a$.

We can show $\mathbb{N}$ is ordered. Since $n \leq S(n)$ or $n \leq m$ iff $\exists k \in \mathbb{N}$ s.t. $n+k = m$.

Definition 18. An ordered set A, $(A, \leq)$. $B \subset A \; b \in B$ is the minimum of B if $b \leq x \; \forall x \in B$.

Definition 19. An ordered set A is well-ordered iff every non empty subset has a minimum.

Proposition 20. $(\mathbb{N}, \leq)$ *is well-ordered.*

Proof. $A \subset \mathbb{N}$, $A \neq \varnothing$. Prove that it has a minimum.

Consider $B = \{n \in \mathbb{N} \text{ s.t. } n \leq A\}$ B contains at least 0. $\exists\, b \in B$ s.t. $S(b) \notin B$. If not true, by induction, $B = \mathbb{N} \implies A = \varnothing$, because $\mathbb{N}$ has no upper bound. Now $S(b) \notin B \implies \exists\, a_0 \in A$ s.t.

$$b \leq a_0 < S(b) \implies a_0 = b$$

□

The last line in the proof requires a little bit of justification: $\nexists n \in \mathbb{N}$ s.t. $b < n < S(b)$, hence $S(n)$ is truly the successor of n.

1.4 Integers

Definition 21. a set A and $\sim$ is a relation of A with A itself is an equivalence iff

1) $a \sim a \;\forall\, a \in A$

2) $a \sim b$ and $b \sim c \implies a \sim c$

3) $a \sim b \implies b \sim a$ (known as symmetric property)

Definition 22. $a \in A$ equivalence class of a

$$[a] = \{b \in A, b \sim a\}$$

Proposition 23. *1) $[a] \neq \varnothing$; 2) $[a] = [b]$ iff $a \sim b$; 3) if a is not related to b, $[a] \cap [b] = \varnothing$.*

Definition 24. $A/\sim$, the quotient of A by th equivalent relation $\sim$ is the set of all the equivalence classes

$$A/\sim = \{[a]; a \in A\}$$

Consider the following map

$$\begin{aligned} \pi : A &\to A/\sim \\ a &\mapsto [a] \end{aligned}$$

is surjective. If we employ axiom of choice, we can make

$$\begin{aligned} S : A/\sim &\to A \\ [a] &\mapsto a \end{aligned}$$

called a sector, and S is a one-sided inverse, $\pi \circ S = I$.

In measure theory, measurability depends on axiom of choice in some sense. It is good not to use axiom of choice unless one has no other choices.

We want to have negative numbers. Suppose we take $(n, m) \in \mathbb{N} \times \mathbb{N}$ and interpret it as the integer $n - m$.

Definition 25. An equivalence on $\mathbb{N} \times \mathbb{N}$

$$(n, m) \sim (p, q) \iff n + q = m + p$$

One can check that it is an equivalence.

Definition 26. $\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim$.

Example 27. $0 = [(a, a)]$.

We want to put some algebraic structures on $\mathbb{Z}$. First we put them on $\mathbb{N} \times \mathbb{N}$

$$\begin{aligned} (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) &\xrightarrow{+} \mathbb{N} \times \mathbb{N} \\ (n, m); (p, q) &\mapsto (n + p, m + q) \end{aligned}$$

Then try on $\mathbb{Z}$

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z} &\xrightarrow{+} \mathbb{Z} \\ [(n, m)]; [(p, q)] &\mapsto [(n + p, m + q)] \end{aligned}$$

we need to check well-defined, i.e.

$$\begin{cases} (n', m') \sim (n, m) \\ (p', q') \sim (p, q) \end{cases} \implies (n' + p', m' + q') \sim (n + p, m + q)$$

$n + m' = n' + m$, $p + q' = p' + q \implies n + p + m' + q' = m + q + n' + p'$, hence

$$(n' + p', m' + q') \sim (n + p, m + q)$$

In other words, diagram ceasing

$$\begin{array}{ccc} (\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{N}) & \xrightarrow{+} & \mathbb{N}\times\mathbb{N} \\ \downarrow(\pi,\pi) & & \downarrow\pi \\ \mathbb{Z}\times\mathbb{Z} & \overset{\oplus}{\dashrightarrow} & \mathbb{Z} \end{array}$$

namely, $\oplus$ is defined in the following sense, for given $a, b \in \mathbb{Z}$, to find $a \oplus b$, first pick a representation in $(\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{N})$, then follow the counterclockwise path to $\mathbb{Z}$.

In the same spirit, define product on $(\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{N})$ as

$$(n, m)\cdot(p, q) = (np + mq, nq + mp)$$

then descends to the quotient space $\mathbb{Z}$.

1.5 Rationals

Exact same procedures. We take $(m, n) \in \mathbb{Z}\times(\mathbb{N}-\{0\})$, and interpret it as

$$\frac{m}{n}$$

then build the equivalence class

$$(m, n) \sim (p, q) \iff mq = np$$

Now define $\mathbb{Q} = \mathbb{Z}\times(\mathbb{N}-\{0\})/\sim$.

Define $+$ on $\mathbb{Z}\times(\mathbb{N}-\{0\})$ first by

$$(m, n) + (p, q) = (mq + np, nq)$$

check this can be descended to quotient.

Define product on $\mathbb{Z}\times(\mathbb{N}-\{0\})$ by

$$(m, n)\cdot(p, q) = (mq, np)$$

Proposition 28. $(\mathbb{Q}, +)$ *forms a commutative group, i.e.*

1) $m + (n + p) = (m + n) + p$

2) neutral element $n + 0 = 0 + n = n$

3) commutative $m + n = n + m$

4) $\forall\, n \in \mathbb{Q}$, $\exists\, m \in \mathbb{Q}$ *s.t.* $n + m = 0$

4) makes $(\mathbb{Q},+)$ a monoid, $(\mathbb{N},+)$, to be group.

Proposition 29. *$(\mathbb{Q}-\{0\},\cdot)$ is a commutative group.*

Proposition 30. *$(\mathbb{Q},+,\cdot)$ has distributivity $a\cdot(b+c)=a\cdot b+a\cdot c$.*

Remark 31. $(K,+,\cdot)$ s.t. $(K,+)$ is an abelian group, and $(K-\{0\},\cdot)$ is also an abelian group and the product is distributive, we call it a field.

Proposition 32. *(cancellation) $x+y=z+y \implies x=z$.*

Proof.

$$\begin{aligned} x &= x+0 = x+(y+(-y)) = (x+y)+(-y) = (z+y)+(-y) \\ &= z+(y+(-y)) = z+0 = z \end{aligned}$$

where $(-y)$ is the additive inverse of y. $\square$

Furthermore we want $\leq$ to work with $+,\cdot$ with some common sense properties:

Definition 33. $(K,+,\cdot)$ field with an order $\leq$, we call it an ordered field iff

1) if $x\leq y \implies x+z\leq y+z\ \forall z\in K$

2) if $x\geq 0,\ y\geq 0 \implies xy\geq 0$

Proposition 34. *$(K,+,\cdot)$ ordered field.*

1) $x\geq 0 \implies (-x)\leq 0$;

2) $x^2\geq 0$.

1.6 Reals

Lecture 3 (6/4/13)

Definition 35. $(A,\leq)$ an ordered set $B\subset A$, $\alpha\in A$ is the supremum of B iff

1) $\alpha\geq B$

2) $\forall a\in A,\ a<\alpha \implies a$ is not an upper bound of B. (i.e. $\exists b\in B$ s.t. $a<b$)

We denote it with

$$\alpha = \sup B$$

Infimum is too defined by swapping the order relation.

Definition 36. $(A, \leq)$ ordered set. A has the least upper bound property (LUP) iff $\forall\, B \subset A$, $B \neq \varnothing$ and bounded from above $\implies$ B has a supremum.

Remark 37. A has LUP $\forall\, B \subset A$, $B \neq \varnothing$ and bounded from below $\implies \exists$ infimum.

Proof. (How to prove this remark? We cannot just take negative, because we don't what negative of an element mean.)

$C = \{a \in A; a \leq B\}$ C is not empty because B is bounded from below. C is bounded from above because $B \neq \varnothing$.

$$\gamma = \sup C$$

Prove that γ is a lower bound of B. If it's not, $\exists\, b \in B$ s.t. $b < \gamma$, $b \geq C \implies \gamma$ cannot be the supremum.

Prove that γ is the greatest lower bound. $\exists\, \gamma' \in C$ lower bound of B s.t. $\gamma' > \gamma$, which contradicts the fact that γ is an upper bound of C. Hence

$$\gamma = \inf B$$

$\square$

We now show

Proposition 38. $(\mathbb{Q}, \leq)$ *doesn't have LUP.*

Proof. Consider

$$A = \{p \in \mathbb{Q}, p > 0, p^2 < 2\}$$
$$B = \{p \in \mathbb{Q}, p > 0, p^2 > 2\}$$

(noting special about the prime number 2, any other prime number works too.)

If $p^2 = 2 \implies p \in \mathbb{Q}$. That is because $p = m/n$ $m, n \in \mathbb{Z}$ (One can use axiom of choice to pick such m, n or more sophisticate uses unique representation of prime factorization.) Assume that one of m, n is not even

$$\frac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies m^2 \text{ even} \implies m \text{ even}$$

$$m = 2k, k \in \mathbb{Z} \implies \frac{4k^2}{n^2} 2 \implies 2k^2 = n^2 \implies n\,\text{even}$$

We'll show if $p \in A$, $\exists\, p' \in A$ s.t. $p' > p$; if $p \in B$, $\exists\, p'' \in B$ s.t. $p'' < p$. We do two in one shot.

Fix p, put

$$q = p - \frac{p^2 - 2}{p + 2}$$

if $p \in A$, $q > p$; if $p \in B$, $q < p$. Check

$$q^2 - 2 = \frac{(2p+2)^2}{(p+2)} - 2 = 2\frac{p^2 - 2}{(p+2)^2} \implies q \in \begin{cases} A & p \in A \\ B & p \in B \end{cases}$$

If $q, p > 0$, $q^2 < p^2$ iff $q < p \implies A$ is the set of positive lower bounds of B and B is upper bounds of A. There is no greatest element in A nor smallest element in B, hence $\nexists \sup A$ nor $\inf B$. $\square$

We would like to extend $\mathbb{Q}$ so "compatible" with addition, multiplication of $\mathbb{Q}$ as well as the order field property of $\mathbb{Q}$.

Definition 39. $(K, +, \cdot)$ a field $(K', \oplus, \odot)$ another field, $i : K \mapsto K'$ injective. i is a field extension iff

1)

$$\begin{array}{ccc} K \times K & \xrightarrow{+} & K \\ \downarrow (i,i) & & \downarrow i \\ K' \times K' & \xrightarrow{\oplus} & K' \end{array}$$

is a commutative diagram.

2)

$$\begin{array}{ccc} K \times K & \xrightarrow{\cdot} & K \\ \downarrow (i,i) & & \downarrow i \\ K' \times K' & \xrightarrow{\odot} & K' \end{array}$$

is a commutative diagram.

Definition 40. $(K, \leq)$, $(K', \lesssim)$ two ordered sets. $i : K \to K'$ injective. $\lesssim$ extends $\leq$ iff

$$\forall\, a, b \in K, \text{ s.t. } a \leq b \implies i(a) \lesssim i(b)$$

Theorem 41. *There exists an ordered field* $(\mathbb{R}, +, \cdot)$ *that has the LUP and s.t.*

1) it's a field extension of $\mathbb{Q}$

2) it extends the order relation of $\mathbb{Q}$

3) it's unique in the sense that if $(\tilde{\mathbb{R}}, +, \cdot)$ satisfies the same condition. $\exists!$

$$f : \mathbb{R} \to \tilde{\mathbb{R}} \textit{ isomorphism}$$

s.t.

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "i"] \arrow[dr, "j"'] & \mathbb{R} \arrow[d, "f"] \\ & \tilde{\mathbb{R}} \end{tikzcd}$$

commutes

We will not prove this now. The proof in Rudin uses Dirichlet cut, which is extremely tedious and only works from $\mathbb{Q}$ to $\mathbb{R}$. Later when we are ready, we will prove the theorem using Cauchy sequence, and it's a better proof.

Proposition 42. *(Archimedean) $x, y \in \mathbb{R}$ $x > 0$ $\exists\, n \in \mathbb{N}$ s.t. $nx > y$.*

Proof. Suppose it's false $\forall\, n \in \mathbb{N}$ $nx \leq y$

$$B = \{nx, n \in \mathbb{N}\}$$

then y is an upper bound of $B \implies \exists\, \alpha \in \mathbb{R}$ s.t.$\alpha = \sup B$.

$x > 0 \implies \alpha - x < \alpha$ and $\alpha - x$ is not an upper bound of B, then $\exists\, \bar{n} \in \mathbb{N}$ s.t. $\bar{n}x > \alpha - x$, then we reach contradictions

$$(\bar{n} + 1)x > \alpha \text{ and } (\bar{n} + 1)x \in B$$

□

Proposition 43. *($\mathbb{Q}$ is dense in $\mathbb{R}$) $\forall\, x, y \in \mathbb{R}$, $x < y$ $\exists\, q \in \mathbb{Q}$ s.t. $x < q < y$.*

Proof. WLOG assume $0 < x < y$ $\exists\, n \in \mathbb{N}$ s.t. $n(y - x) > 1$ by Archimedean. Let

$$B = \{m \in \mathbb{N}, nx < m\}$$

$B \subset \mathbb{N}$, $\mathbb{N}$ is well-ordered so $\exists\, \bar{m} \in B$ s.t.

$$\bar{m} = \min B$$

then $\bar{m} - 1 \leq nx < \bar{m}$, then

$$nx < \bar{m} \leq nx + 1 < ny$$

(notice in Rudin, the existence of $\bar{m}$ natural between two reals that are 1 unit apart is taken as granted. That is one reason why Rudin didn't want to define $\mathbb{N}$ from axioms)

Therefore

$$x < \frac{\bar{m}}{n} < y$$

□

Theorem 44. *(nth root of a real) $x \in \mathbb{R}$ $x > 0$ fix $n \in \mathbb{N}$ $n \geq 1$ $\exists!$ $y \in \mathbb{R}$ $y > 0$ s.t. $y^n = x$.*

we denote y with $\sqrt[n]{x}$ or $x^{1/n}$.

Idea of proof

$$A = \{t \in \mathbb{R}, t > 0 \text{ and } t^n < x\}$$

then show

$$\alpha = \sup A$$

is the nth root by using the third axiom of order relation, definition 16, and the following lemma.

Lemma 45. $x, y \in \mathbb{R}$ $x, y > 0$

1) $x^n - y^n < 0$ iff $x < y$

2) $x^n - y^n = 0$ iff $x = y$

3) if $x > y$ then $x^n - y^n \leq n(x-y)x^{n-1}$

Proof. $x^n - y^n = (x-y)\underbrace{(x^{n-1} + x^{n-2}y + ... + y^{n-1})}_{>0} \implies$ 1) and 2) are true.

$x > y \implies x^k > y^k \implies x^{n-1-k}y^k < x^{n-1}$, then summing over k, and the right is dependent of k, we get

$$x^{n-1} + x^{n-2}y + ... + y^{n-1} = \sum_{k=0}^{n-1} x^{n-1-k}y^k < nx^{n-1}$$

3) is proven. □

We now prove theorem 44.

Proof. $A = \{t \in \mathbb{R}, t > 0 \text{ s.t. } t^n < x\}$

First show A is bounded from above by $x+1$. By lemma above

$$x > 0 \implies x+1 > 1 \implies (x+1)^{n-1} > 1$$

then

$$(x+1)^n > x+1 > x > t^n \implies x+1 > t$$

Prove $A \neq \varnothing$. By lemma

$$\left(\frac{x}{x+1}\right)^n < \frac{x}{x+1} < x \implies \frac{x}{x+1} \in A$$

By LUP, $\exists!\, \alpha \in \mathbb{R}$ s.t. $\alpha = \sup A$. Prove that $\alpha^n = x$

Lecture 4 (6/6/13)

1) Suppose $\alpha^n < x$, we want to find $\epsilon \in \mathbb{R}$ $\epsilon > 0$ s.t.

$$(\alpha + \epsilon)^n < x$$

Our goal is to find such ϵ. If we succeed, then α is not upper bound of A, contradiction is reached.

By lemma

$$(\alpha+\epsilon)^n - \alpha^n < n\epsilon(\alpha+\epsilon)^{n-1}$$

We want

$$n\epsilon(\alpha+\epsilon)^{n-1} < x - \alpha^n$$

this gives us a way to solve for ϵ. (In doing analysis, we don't have to be too sophisticate, i.e. not necessary to be the optimal ϵ) so assume $\epsilon < 1$, then

$$n\epsilon(\alpha+\epsilon)^{n-1} < n\epsilon(\alpha+1)^{n-1} < x - \alpha^n$$

hence we pick

$$\epsilon < \min\{1, \frac{x-\alpha^n}{n(\alpha+1)^{n-1}}\}$$

2) suppose $\alpha^n > x$. Find $\epsilon > 0$ s.t. $(\alpha-\epsilon)^n > x$. Similar we construct

$$\alpha^n - (\alpha-\epsilon)^n < n\epsilon\alpha^{n-1} < \alpha^n - x$$

the right inequality holds when $\epsilon < \frac{\alpha^n - x}{n\alpha^{n-1}}$. Hence

$$(\alpha-\epsilon)^n > x > t^n$$

implies that $\alpha - \epsilon$ is an upper bound of A, which contradicts that $\alpha = \sup A$.

Now prove uniqueness.

Suppose that $\exists\, y_1, y_2 \in \mathbb{R}$, $y_1, y_2 > 0$ and $y_1^n = x$, $y_2^n = x$. Then by lemma $y_1^n - y_2^n = 0 \iff y_1 = y_2$. $\square$

Exercise 46. Show $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$.

Solve: By the theorem, $\exists\, \alpha, \beta$ s.t.

$$\alpha^n = x \qquad \beta^n = y$$

then using the induction and commutativity of field

$$\alpha^n \beta^n = (\alpha\beta)^n \implies (\alpha\beta)^n = xy \implies \alpha\beta = \sqrt[n]{xy}.$$

1.7 Complex Numbers

We now define $+$, $\cdot$ component wise on $\mathbb{R}^2$: operation $(\mathbb{R}^2, +)$

$$(x, y) + (u, v) = (x + u, y + v) \tag{1.1}$$

define $(\mathbb{R}^2, +, \cdot)$ product so that it's a field (most importantly invertible)

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu) \tag{1.2}$$

Theorem 47. $(\mathbb{R}^2, +, \cdot)$ *is a commutative field.*

The only thing to check is multiplicative inverse. If $(x, y) \neq (0, 0)$, $\exists\, (u, v)$ s.t.

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

this is certainly true since $\det = x^2 + y^2 \neq 0$.

Note 48. It has been proven that $\mathbb{R}^n$ has multiplicative inverse only $n = 2, 4, 8$. With the invertible multiplication $n = 4$ is not commutative, Quaternion; $n = 8$ is neither commutative nor associative, Octonion.

Note 49. For $\mathbb{R}^2$ the invertible multiplication rule (1.2) is fixed, if $+$ is chosen to be (1.1).

Question 50. *Consider a map $K : \mathbb{R} \to \mathbb{R}^2$. Find K s.t. it is a field extension.*

Solve: Suppose

$$K(t) = (a + bt, c + dt)$$

Preserve addition

$$K(t_1 + t_2) = K(t_1) + K(t_2)$$

(+ on the left operates on $\mathbb{R}$ and + on the right operates on $\mathbb{R}^2$.) Then

$$(a + b(t_1 + t_2), c + d(t_1 + t_2)) = (a + bt_1, c + dt_1) + (a + bt_2, c + dt_2)$$

So

$$a = c = 0$$

Preserve multiplication

$$K(t_1t_2) = K(t_1) \cdot K(t_2)$$

So

$$(bt_1t_2, dt_1t_2) = (b^2t_1t_2 - d^2t_1t_2, 2bdt_1t_2)$$

Hence

$$\begin{cases} b = b^2 - d^2 \\ d = 2bd \end{cases} \implies \begin{cases} b = 1 \\ d = 0 \end{cases}$$

That is

$$\begin{aligned} K : \mathbb{R} &\to \mathbb{R}^2 \\ \alpha &\mapsto (\alpha, 0) \end{aligned}$$

We denote $(0, 1) = i$, so $(a, b) = a + ib$.

Note 51. There are some common functions: e.g. Conjugate

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z = a + ib &\mapsto \bar{z} = a - bi \end{aligned}$$

and some other

$$\begin{aligned} \mathbb{C} &\to \mathbb{R} \\ z = za + ib &\mapsto \sqrt{a^2 + b^2} = |z| \\ z = a + ib &\mapsto a = \Re z \\ z = a + ib &\mapsto b = \mathrm{Im} z \end{aligned}$$

All four above are not differentiable.

Proposition 52. *(Properties of $\mathbb{C}$)*

1) $|z|^2 = z\bar{z}$

2) $\Re z = \frac{z+\bar{z}}{2}$, $Imz = \frac{z-\bar{z}}{2i}$

3) $z^{-1} = \frac{\bar{z}}{|z|^2}$

4) $|z| \geq 0$ *and* $|z| = 0$ *iff* $z = 0$

5) $|\Re z| \leq |z|$, $|Imz| \leq |z|$

6) $\bar{\bar{z}} = z$, $\overline{zw} = \bar{z}\bar{w}$, $\overline{z+w} = \bar{z} + \bar{w}$

7) $||z| - |w|| \leq |z+w| \leq |z| + |w|$

Proof. We prove the most right inequality in 7)

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w = z\bar{z} + w\bar{w} + z\bar{w} + \overline{z\bar{w}} \\
&= z\bar{z} + w\bar{w} + 2\Re(z\bar{w}) \\
&\leq |z|^2 + |\bar{w}|^2 + 2|z\bar{w}| = (|z| + |\bar{w}|)^2 \\
&= (|z| + |w|)^2
\end{aligned}$$

□

2 Abstract Spaces and Properties

2.1 Vector Spaces

Definition 53. K is a field, V is a set with 2 operations

$$\begin{aligned}
V \times V &\xrightarrow{+} V \\
a, b &\mapsto a + b
\end{aligned}$$

$$\begin{aligned}
K \times V &\xrightarrow{\cdot} V \\
\alpha, v &\mapsto \alpha v
\end{aligned}$$

satisfying

1) $(V, +)$ commutative group;

2) distributive $\alpha, \beta \in K$, $v, v_1, v_2 \in V$

$$(\alpha + \beta)v = \alpha v + \beta v$$

$$\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$$

3) associative of the product

$$(a\beta)v = \alpha(\beta v)$$

4) $1v = v$, $1 \in K$ neutral element of K

$(V, +, \cdot)$ is a $K-$vector space or $K-$linear space.

Lecture 5 (6/11/13)

Theorem 54. *If V is finite dimensional of dimension $n \in \mathbb{N}$, $\exists$ an isomorphism between V and K^n.*

We will only work with vector spaces isomorphic to $\mathbb{R}^n$, $\mathbb{C}^n$.

Definition 55. V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, an inner product is a function

$$\begin{aligned} V \times V &\to K \\ v, u &\mapsto \langle v, u \rangle \end{aligned}$$

(K is $\mathbb{R}$ or $\mathbb{C}$) satisfying

1) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

2) $\langle v, w \rangle = \overline{\langle w, v \rangle}$

3) linear in the 1st argument

$$\langle \alpha v_1 + \beta v_2, w \rangle = \alpha \langle v_1, w \rangle + b \langle v_2, w \rangle$$

$\alpha, \beta \in \mathbb{C}$.

The definition above implies inner product is anti-linear in the 2nd argument

$$\langle v, \alpha w_1 + \beta w_2 \rangle = \bar{\alpha} \langle v, w_1 \rangle + \bar{b} \langle v, w_2 \rangle$$

because

$$\langle v, \alpha w \rangle = \overline{\langle \alpha w, v \rangle} = \overline{\alpha \langle w, v \rangle} = \bar{\alpha} \langle v, w \rangle$$

Example 56. $V = C^n$ $v_1 = (z_1, ..., z_n)$, $v_2 = (u_1, ..., u_n)$, $\langle v_1, v_2 \rangle = \sum_{i=1}^n z_i \bar{u}_i$

Example 57. V is $n-$dimensional vector space over $\mathbb{C}$, $e_1, ...e_n$ is a basis,

$$v = \sum_{i=1}^n a_i e_i \quad u = \sum_{i=1}^n b_i e_i$$

$$\langle v, u \rangle = \sum_{i=1}^n a_i \left\langle e_i, \sum_{j=1}^n b_j e_j \right\rangle = \sum_{i,j=1}^n a_i \bar{b}_j \langle e_i, e_j \rangle$$

This example suggests we can set matrix $C_{ij} = \langle e_i, e_j \rangle$. Because $\langle e_i, e_j \rangle \in \mathbb{C}$ and $c_{ij} = \bar{c}_{ji}$, C is hermitian.

$$\langle v, u \rangle = v^T C \bar{u}$$

where v^T is row vector, u is column vector. The eigenvalue of C is strictly positive, since C is positive definite.

It is very common to work without fixing a basis.

Proposition 58. *(Cauchy Schwartz) V is vector space with inner product*

$$|\langle v, u\rangle| \leq \|v\| \, \|u\|$$

Notation 59. Norm $\|v\| = \sqrt{\langle v, v\rangle}$

Proof. Assume $v \neq 0$, consider

$$\begin{aligned} 0 \leq \left\| u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v \right\|^2 &= \left\langle u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v, u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v \right\rangle \\ &= \langle u, u\rangle - \frac{\overline{\langle u, v\rangle}}{\|v\|^2} \langle u, v\rangle - \frac{\langle u, v\rangle}{\|v\|^2} \langle v, u\rangle + \frac{\langle u, v\rangle \overline{\langle u, v\rangle}}{\|v\|^2} \\ &= \|u\|^2 - 2\frac{|\langle u, v\rangle|^2}{\|v\|^2} + \frac{|\langle u, v\rangle|^2}{\|v\|^2} = \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2} \end{aligned}$$

□

Proposition 60. *(Triangle inequality) V with inner product*

$$\|u + v\| \leq \|u\| + \|v\|$$

Proof. $\|u + v\|^2 = \langle u + v, u + v\rangle = \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2$,

$$\langle u, v\rangle + \langle v, u\rangle = 2\Re \langle u, v\rangle \leq 2|\langle u, v\rangle| \leq 2\|u\| \, \|v\|$$

Hence

$$\|u + v\|^2 \leq (\|u\| + \|v\|)^2$$

□

In above proof we don't have to assume $\exists$ basis nor it is finite dimensional.

In $\mathbb{R}^n$, $u = (u_1, ..., u_n)$, $v = (v_1, ..., v_n)$,

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$$

and geometrically, by Pythagorean

$$\|u - v\|$$

is the distance between two points.

We want to have a notion of distance in abstract sense.

2.2 Metric Spaces

Definition 61. X is a set, $d : X \times X \to [0, \infty) \subset \mathbb{R}$ satisfying

1) $d(x, y) = 0$ iff $x = y$

2) $d(x, y) = d(y, x)$

3) $d(x, y) \leq d(x, z) + d(z, y)\ \forall\, z \in X$

(X, d) is called metric space, and d a distance.

There are cases where we want d to be range in $\mathbb{Q}$ not in all $\mathbb{R}$.

Example 62. V is a vector space with inner product $\|v\| = \sqrt{\langle v, v\rangle}$. Define $d(u, v) = \|u - v\|$, (V, d) is a metric space. Here V is coming from a vector space, so it has addition operations of vectors. In general metric space X may not have such operation.

There are examples of distance far from our intuition.

Example 63. (Discrete Topology)X is a set with at least 2 elements

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \tag{2.1}$$

we'll later from topological view, this type of metric spaces have very wired properties.

Definition 64. (X, d) and $x \in X$ and $r \in R$, $r > 0$

$$B_{x,r} = \{y \in X; d(x, y) < r\}$$

This is a ball with center x radius r.

For example 63

$$B_{x,r} = \begin{cases} \{x\} & r < 1 \\ X & r \geq 1 \end{cases}$$

Definition 65. $U \subset X$ is open (in X) iff $\forall\, u \in U\ \exists\, r > 0$ s.t. $B_{u,r} \subset U$.

Definition 66. $U \subset X$ and $u \in U$ has the property $\exists\, r > 0$ s.t. $B_{u,r} \subset U$, we call u interior point of U.

In example 63, every subset is open. In particular every single point set is open. Later we'll see in metric space, every single point set is closed, cf corollary 73, i.e. in example 63, every subset is both open and closed.

Definition 67. (X, d) $E \subset X$. $p \in X$ is a limit point of E iff $\forall r > 0$ $(B_{p,r} \cap E) - \{p\} \neq \varnothing$.

Meaning that if $p \notin E$, then p is arbitrary close to E. Why is it called limit point? Somehow relate to limit. Later we will study limit of a function $f : E \to \mathbb{R}$, E domain of f

$$\lim_{x \to x_0} f(x)$$

where x_0 needs not to be in E, because taking above limit doesn't evaluate f at x_0, but x_0 has to be close to E, i.e. x_0 is a limit point of E.

Definition 68. $E \subset X$ is closed (in X) iff every limit point of E is contained in E.

Notice this doesn't say every point of a closed set is a limit point.

Later we will the definitions of open and closed given here are equivalent to the definitions given in Topology.

Example 69. In $\mathbb{R}$, interval $[a, b)$ is not closed, for limit point b is not included.

Example 70. In $\mathbb{R}^2$, the line segment $(-1, 1)$ on the x axis is not open; whiles the line segment $[-1, 1]$ on the x axis is closed.

Proposition 71. *A ball $B_{p,r}$ is open.*

Proof. Fix $q \in B_{p,r}$, let $\rho = r - d(p, q)$, prove that $B_{q,\rho} \subset B_{p,r}$.

$$x \in B_{q,\rho} \iff d(q, x) < \rho$$

$$d(p, x) \leq d(p, q) + d(q, x) < d(p, q) + \rho = r \implies x \in B_{p,r}$$

$\square$

Proposition 72. *(X, d) $E \subset X$, p is a limit point of E. Then $\forall r > 0$ $B_{p,r}$ contains infinitely many points of E.*

Proof. Suppose $\exists r > 0$ s.t. $B_{p,r} \cap E - \{p\} = \{p_1, ..., p_n\}$, put $r_i = d(p, p_i)$, $r_i > 0$

$$\tilde{r} = \min\{r_i\}$$

(notice minimum always exists, for finite set is bijective to a subset of $\mathbb{N}$ and $\mathbb{N}$ is well-ordered. This doesn't work if ∞ replaces finite) then

$$B_{p,\tilde{r}} \cap E - \{p\} = \varnothing$$

contradicts p is a limit point. $\square$

Corollary 73. *(X, d) metric space, every finite set $E \subset X$ is closed.*

Proof. If E is finite, it doesn't have limit points at all. By the definition of closed, the set of all limit points of E is $\varnothing$, and $\varnothing \subset E \implies E$ is closed. $\square$

Let's see how open and closed are related.

Proposition 74. *(X, d) $E \subset X$ is open iff E^c is closed.*

Proof. Suppose that E is open and E^c is not closed, that is $\exists\, p \in X$ limit point of E^c and $p \notin E^c \implies p \in E$. Then $\forall\, r > 0$ $B_{p,r} \cap E^c \neq \varnothing \implies \nexists\, r > 0$ s.t. $B_{p,r} \subset E \implies p$ is not interior, so E isn't open.

Suppose that E^c is closed but E is not open. So $\exists\, p \in E$ s.t. p not interior so $\forall\, r > 0$ $B_{p,r} \not\subset E$, or $\forall\, r > 0$ $\exists\, q \in E^c$ s.t. $q \in B_{p,r}$ that is p is a limit point of E^c. Since $p \in E$, E^c is not closed. $\square$

Back example 63. Since every subset is open, by complement, every subset is closed. This is actually the case where maximal number of both open and closed subsets are found. Late we will show the other extreme: minimal number of both open and closed subsets i.e. only $\varnothing$ and X themselves. This happens when the set is connected.

2.3 Topological Spaces

Lecture 6 (6/13/13)

Proposition 75. *(X, d) metric space,*

1) U_α, $\alpha \in A$ is a collection of open sets, then $\cup_{\alpha \in A} U_\alpha$ is open

2) F_α, $\alpha \in A$ is a collection of closed sets, then $\cap_{\alpha \in A} U_\alpha$is closed

3) U_1,..., U_N is a finite collection of open sets, then $\cap_{i=1}^N U_i$ is open

4) F_1, ..., F_N is a finite collection of closed sets, then $\cup_{i=1}^N U_i$ is closed

5) X, $\varnothing$ are both open and closed.

Note 76. A is not necessary countable.

Proof. 1) Let $x \in \cup_{\alpha \in A} U_\alpha$ prove x is an interior. $\exists\, \bar{\alpha} \in A$ s.t. $x \in U_{\bar{\alpha}}$ x interior in $U_{\bar{\alpha}} \implies \exists\, r > 0$ s.t.

$$B_{x,r} \subset U_{\bar{\alpha}} \subset \cup_{\alpha \in A} U_\alpha$$

3) Let $x \in \cap_{i=1}^N U_i \iff x \in U_\alpha\ \forall\, \alpha = 1, ..., N\ \exists\, r(\alpha) > 0\ B_{x,r(\alpha)} \subset U_\alpha$ all balls with the same center. Pick

$$r = \min\{r(1), ..., r(N)\}$$

$r > 0$ implies

$$B_{x,r} \subset B_{x,r(\alpha)} \subset \cap_{i=1}^N U_i$$

□

Lemma 77. *V_α, $\alpha \in A$ a collection of sets $V_\alpha \subset X$.*

$$(\cup_{\alpha \in A} V_\alpha)^c = \cap_{\alpha \in A} V_\alpha^c$$

Proof. $x \in (\cup_{\alpha \in A} V_\alpha)^c \iff x \notin \cup_{\alpha \in A} V_\alpha \iff \forall\, \alpha \in A\ x \notin V_\alpha \iff \forall\, \alpha \in A,\ x \in V_\alpha^c \iff x \in \cap_{\alpha \in A} V_\alpha^c$. □

Continue proof of proposition

Proof. 2) F_α, $\alpha \in A$ closed sets.

$$\cap_{\alpha \in A} F_\alpha = \cap_{\alpha \in A} F_\alpha^{cc} = (\cup_{\alpha \in A} F_\alpha^c)^c$$

F_α closed $\implies F_\alpha^c$ is open, so by 1) $\cup_{\alpha \in A} F_\alpha^c$ is open, then $(\cup_{\alpha \in A} F_\alpha^c)^c$ is closed.

4) very similar to 2).

5) trivial. □

Remark 78. Examples of finite is dropped in 3) and 4):

$$\bigcup_{n \in \mathbb{N} - \{0\}} [\frac{1}{n}, 1] = (0, 1]$$

not closed. (the interval is determined by Archimedean)

$$\bigcap_{n \in \mathbb{N} - \{0\}} (\frac{1}{n} + 1, 2) = [1, 2)$$

not open.

In measure theory, they are called Borel sets.

In mathematics, it is common to take the structure of some known stuff, and remove the stuff to define new things.

Definition 79. X is a set. $\mathcal{F}$ is a collection of subsets of X. $\mathcal{F}$ satisfies

1) $X, \varnothing \in \mathcal{F}$

2) $U_\alpha \in \mathcal{F}$,$\alpha \in A \implies \cup_{\alpha \in A} U_\alpha \in \mathcal{F}$

3) U_1,..., $U_N \in \mathcal{F} \implies \cap_{i=1}^N U_i \in \mathcal{F}$

We call $(X, \mathcal{F})$ a topology (or topological space) and every element $U \in \mathcal{F}$ is called open.

So the notion of open is not from a metric anymore.

Remark 80. The definitions impose topology satisfy 3 of the properties in proposition 75. We don't need to impose the other two, for they are automatically satisfied, because the way closed sets are defined in topology.

Example 81. (X, d) metric space and

$$\mathcal{F} = \{U \subset X; U \text{ is open according to the metric}\}$$

then $(X, \mathcal{F})$ is a topology.

Hence a metric always defines a topology. Hard question: can topology always turn into metric space? At what circumstance it can be turned into a metric space?

Definition 82. $(X, \mathcal{F})$ a topology $U \in \mathcal{F}$ (U is of course open), we call U^c closed.

Example 83. X has more than 1 point. $\mathcal{F} = \{X, \varnothing\}$, the smallest (coarse) topology can be made, then for $x \in X$, $\{x\}$ is not closed, for its complement is not open in $\mathcal{F}$. By corollary 73, $(X, \mathcal{F})$ can not be made into a metric space.

Note the finest topology $\mathcal{F}$ consist all subsets of X, power set, is metrizable.

Question: in study analysis, when should we use metric space? when should we use topological space?

Pro & Con

Metric space doesn't work well with quotient space. Let (X, d) be metric space, $\sim$ equivalence relation, π continuous map

$$X \xrightarrow{\pi} X/\sim$$

$X/\sim$ is not necessary metric space. But if X is metric space or topological space, then $X/\sim$ is always a topology.

However the problem in topology is that if the space is not Hausdorff, the limit of a function may not be unique. In metric space limit is always unique.

Definition 84. (X, d) metric space $E \subset X$. $\bar{E} = E \cup \{p \in X; p \text{ limit points of } E\}$. We call $\bar{E}$ the closure of E.

Proposition 85. *$E \subset X$*

1) $\bar{E}$ is closed

2) $E = \bar{E}$ iff E is closed

3) $\bar{E}$ is the smallest closed set containing E

Proof. 1) $E' = \{\text{limit points of } E\}$. Let $x \in X$ be a limit point of E'. $\forall\, r > 0\ \exists\, x' \in E'$ s.t. $x' \in B_{x,r}$. Take

$$\delta = r - d(x, x')$$

$\exists\, x'' \in E$ s.t. $x'' \in B_{x',\delta} \subset B_{x,r} \implies x$ is a limit point of E.

2) Obvious

3) Small wrt inclusion. Recall inclusion defines a partial order.

We want to show if $E \subset F \subset \bar{E}$ and F is closed, then $F = \bar{E}$.

Suppose $F \subsetneq \bar{E}$, $x \in \bar{E}$, $x \notin F \implies x \notin E$, x is a limit point of E, so x is a limit point of F, hence F is not closed. $\square$

Exercise 86. $\mathbb{R}$ euclidean. $E \subset \mathbb{R}$ bounded from above, then $\sup E \in \bar{E}$.

Proof. $\alpha = \sup E$, prove that α is a limit point. We know $\alpha \geq x\ \forall\, x \in E$ and $\forall\, \epsilon > 0\ \exists\, x_\epsilon \in E$ s.t.

$$\alpha - \epsilon < x_\epsilon$$

(this step can only be done if we work with a field, and this is why we are in $\mathbb{R}$ now)

Then

$$\alpha - \epsilon < x_\epsilon \leq \alpha < \alpha + \epsilon$$

so

$$x_\epsilon \in B_{\alpha,\epsilon}$$

$\square$

Definition 87. (X, d) metric $Y \subset X$, from

$$d : X \times X \to [0, \infty)$$

by restricting

$$d : Y \times Y \to [0, \infty)$$

we get (Y, d), which is also a metric space, called submetric space.

Example 88. $B_{x,r}$ ball in X, $Y \subset X$. If $x \in Y$, a ball in Y is

$$B^Y_{x,r} = \{y \in Y; d(x, y) < r\}$$

Remark 89. $B^Y_{x,r} = B_{x,r} \cap Y$.

Example 90. $X = \mathbb{R}^2$, $Y = \{(x, x)\}$ Ball in Y becomes open interval on the line $x = y$.

Now openness becomes a relative notion.

Proposition 91. *(X, d), (Y, d) metric spaces $U \subset Y \subset X$, U is open in Y iff $\exists U \subset X$ open in X s.t.*

$$U = V \cap Y$$

Remark 92. The statement is true if we replace open with closed.

Proof. Assume that U is open in Y, find $V \subset X$, $x \in U$ is interior according to the metric in Y iff $\exists\, r(x) > 0$ s.t.

$$B^Y_{x,r(x)} \subset U \text{ and } B^Y_{x,r(x)} = B_{x,r(x)} \cap Y$$

Let $V = \cup_{x \in U} B_{x,r(x)}$, then

$$V \cap Y = \cup_{x \in U} \left(B_{x,r(x)} \cap Y\right) = \cup_{x \in U} B^Y_{x,r(x)} = U$$

Conversely $V \subset X$ is open in X, let $x \in V \cap Y$, $\exists\, r > 0$ s.t. $B_{x,r} \subset V$, then

$$B_{x,r} \cap Y \subset V \cap Y$$

that is $B^Y_{x,r(x)} \subset V \cap Y$, hence $V \cap Y$ is open in Y. $\square$

Example 93. $X = \mathbb{R}^2$, $Y = \{2 \text{ non-parallel lines}\}$, $U = \{\text{one of the two lines}\}$. We know U is closed in X, is U open in Y? No because the intersection point of the two lines is not an interior point of U , for any $B^{\mathbb{R}^2}_{p,r} \cap Y \not\subset U$.

For completeness, we define relative topology.

Definition 94. $(X, \mathcal{F})$ is a topology $Y \subset X$ we define a topology on Y

$$\mathcal{F}^Y = \{U \cap Y; U \in \mathcal{F}\}$$

2.4 Compactness

Lecture 7 (6/18/13)

Definition 95. (X, d) metric $Y \subset X$ is an open cover if Y (in X) is a collection of V_α $\alpha \in A$, V_α open in X, s.t.

$$Y \subset \bigcup_{\alpha \in A} V_\alpha$$

Remark 96. A is not necessary countable.

Definition 97. $K \subset X$ is compact iff for every open cover V_α, $\alpha \in A$ $K \subset \cup_{\alpha \in A} V_\alpha$, there is a finite subcover: $\alpha_1, ..., a_n \in A$ s.t.

$$K \subset \bigcup_{i=1}^{n} V_{\alpha_i}$$

Remark 98. To prove that K is not compact, one finds a cover V_α, $\alpha \in A$ that doesn't have a finite subcover.

Example 99. $(0, 1) \subset \mathbb{R}$ euclidean. Try cover

$$\cup_{n=2}^{\infty} (\frac{1}{n}, 1) = (0, 1)$$

suppose $\{(\frac{1}{n_1}), (\frac{1}{n_2}), ..., (\frac{1}{n_N})\}$ is a cover, let $M = \max\{n_1, n_2, ..., n_N\}$, then

$$\cup_{k=1}^{N} (\frac{1}{n_k}, 1) = (\frac{1}{M}, 1)$$

then by Archimedean, $\exists x \in (0, 1)$ s.t. $x < 1/M$. Hence no finite subcover.

If we change $(0, 1)$ to $[0, 1)$ or $(0, 1]$, similar arguments show they are not compact.

As from the examples above, it is easier to prove something not compact. Later we shall show a big theorem: Borel in $\mathbb{R}^n$ euclidean, compact $\iff$ closed bounded.

Remark 100. In the definition of compactness, no where mention specific metric, only involving openness, so one may think this definition of compact may make work on topological spaces, but it turns out that there are top spaces like

$$\mathcal{F} = \{X, \varnothing\}$$

X is the only open cover, so wired that every subset is compact. Therefore, when applying our definition of compactness to top space, we call it quasi-compact, and if the space is Hausdorff, we drop quasi.

Proposition 101. *(X, d) metric space $Y \subset X$, $K \subset Y$ is compact in Y iff K is compact in X.*

Proof. Assume that K is compact in Y, let $V_\alpha \subset X$, $\alpha \in A$ be open cover of K, $K \subset \cup_{\alpha \in A} V_\alpha$, so

$$K = K \cap Y \subset \cup_{\alpha \in A}(V_\alpha \cap Y)$$

$V_\alpha \cap Y$ is open in Y, that K is compact in Y implies $\exists\, \alpha_1, ..., \alpha_N$ s.t.

$$K \subset \bigcup_{i=1}^{N} (V_{\alpha_i} \cap Y) \subset \bigcup_{i=1}^{N} V_{\alpha_i}$$

so K is compact in X.

Conversely K is compact in X. Let $K \subset \cup_{\alpha \in A} W_\alpha$, open cover in Y, $\exists\, V_\alpha$ open in X s.t.

$$W_\alpha = V_\alpha \cap Y$$

so

$$K \subset \bigcup_{\alpha \in A} W_\alpha = \bigcup_{\alpha \in A} V_\alpha \cap Y \subset \bigcup_{\alpha \in A} V_\alpha$$

That K is compact in X implies $\exists\, \alpha_1, ..., \alpha_N$ s.t.

$$K \subset \bigcup_{i=1}^{N} V_{\alpha_i} \implies K \subset \bigcup_{i=1}^{N} V_{\alpha_i} \cap Y = \bigcup_{i=1}^{N} W_\alpha$$

so K is compact in Y. □

This shows compactness is an absolute notion.

Remark 102. (X, d) metric, where X itself can be compact or not. E.g. $\mathbb{R}$ is not compact.

Proposition 103. *(X, d) metric $K \subset X$. K compact $\implies K$ is closed.*

Proof. The proof is very standard, and most of proofs of compactness look just like this.

Pick $p \in X - K$, prove that p is not a limit point of K, i.e. find $r > 0$ s.t. $B_{p,r} \cap K = \varnothing$.

Take $r_q = \frac{1}{2} d(p, q)$, hence

$$B_{p,r_q} \cap B_{q,r_q} = \varnothing$$

And $K \subset \cup_{q\in K} B_{q,r_q}$, K compact, so $\exists\, q_1, ..., q_N \in K$ s.t.

$$K \subset \bigcup_{i=1}^{N} B_{q_i, r_{q_i}}$$

Consider $\cap_{i=1}^{N} B_{p,r_{q_i}} = B_{p,\tilde{r}}$, where $\tilde{r} = \min\{r_{q_1}, ..., r_{q_N}\}$. Then

$$B_{p,\tilde{r}} \cap \bigcup_{i=1}^{N} B_{q_i, r_{q_i}} = \varnothing \implies B_{p,\tilde{r}} \cap K = \varnothing$$

so p is not a limit point. □

Remark 104. What if we change above to top space: $(X, \mathcal{F})$ top $K \subset X$. K compact $\overset{?}{\implies}$ K is closed. Thing of

$$\mathcal{F} = \{X, \varnothing\}$$

every subset is quasi-compact, but every proper subset of X is not closed.

Although the proof above relays on the notion of distance, the proof will be the same if we replace metric space by top space and require it to be Hausdorff.

Definition 105. $(X, \mathcal{F})$ a top s.t. $\forall\, p, q \in X$, $p \neq q$, $\exists\, V \in \mathcal{F}$ and $W \in \mathcal{F}$ s.t. $p \in V$; $q \in W$ and $V \cap W = \varnothing$. We call $(X, \mathcal{F})$ a Hausdorff space or T2.

Here "T2" classifies how well points can be separated. "2" refers to 2 open sets separating points.

Definition 106. If $(X, \mathcal{F})$ is Hausdorff and quasi-compact, we call it compact top space.

Remark 107. 1) If $K \subset (X, \mathcal{F})$ is T2 compact, then it is closed.

2) If the space is Hausdorff, the limit is unique.

3) Just like general top space, Hausdorff space may not be a metric space.

Proposition 108. *$K \subset X$ compact, $F \subset K$ closed in X, then F is compact.*

Proof. $F \subset \cup_{\alpha\in A} V_\alpha$, V_α open in X. F^c is open in X,

$$K \subset \bigcup_{\alpha\in A} V_\alpha \cup F^c$$

so $\exists\, \alpha_1, ..., a_n$ s.t.

$$F \subset K \subset \bigcup_{i=1}^{n} V_{\alpha_i} \cup F^c \implies F \subset \bigcup_{i=1}^{n} V_{\alpha_i}$$

F is compact. □

Remark 109. Above proposition works in a top space.

Corollary 110. *$K \subset X$ compact, $F \subset X$ closed, then $K \cap F$ is compact.*

Remark 111. Above corollary may not work in a top space unless it's T2.

Proposition 112. *K is compact and $E \subset K$ is an infinite set, then $\exists\, p \in K$ limit point of E.*

Remark 113. The converse is also true, but significantly harder to prove. We'll do later.

Proof. Suppose that $\forall\, p \in K$ p is not a limit point of E. $\exists\, r_p > 0$ s.t.

$$B_{p,r_p} \cap E = \{p\} \text{ or } \varnothing$$

Since K compact

$$K \subset \bigcup_{p\in K} B_{p,r_p} \implies E \subset K \subset \bigcup_{i=1}^{N} B_{p_i,r_{p_i}} \subset \{p_1, ..., p_N\}$$

□

Proposition 114. *$K_1 \supset K_2 \supset ...$, $K_N \supset K_{N+1}$, K_n is compact and not empty, then*

$$\bigcap_{i=1}^{\infty} K_i \neq \varnothing$$

Later we'll do a refinement of above show if $|K_n| \to 0$ then

$$\bigcap_{i=1}^{\infty} K_i = \{\text{one point}\}$$

We will prove a more general statement:

Proposition 115. *K_α, $\alpha \in A$ a collection of compact sets in X s.t. every finite subcollection has non empty intersection (symbolically $\forall\, B \subset A$ finite, $\cap_{\beta \in B} K_\beta \neq \varnothing$), then*

$$\bigcap_{\alpha \in A} K_\alpha \neq \varnothing$$

Proof. Suppose that

$$\bigcap_{\alpha \in A} K_\alpha = \varnothing$$

choose $\bar{\alpha} \in A$, $\forall\, x \in K_{\bar{\alpha}}$, $\exists\, \alpha(x) \in A$ s.t.

$$x \notin K_{\alpha(x)} \iff x \in K^c_{\alpha(x)}$$

$K^c_{\alpha(x)}$ is open, $K_{\bar{\alpha}} \subset \cup_{x \in K} K^c_{\alpha(x)}$ open cover, so

$$K_{\bar{\alpha}} \subset \bigcup_{i=1}^{N} K^c_{\alpha(x_i)} = \left(\bigcap_{i=1}^{N} K_{\alpha(x_i)} \right)^c$$

That is

$$K_{\bar{\alpha}} \cap \left(\bigcap_{i=1}^{N} K_{\alpha(x_i)} \right) = \varnothing$$

contraindicating that every finite subcollection has non empty intersection. □

Lecture 8 (6/20/13)

Example 116. Top space

$$\mathcal{F} = \mathcal{P}(X)$$

power set, hence all subsets are open. What are the compact sets in this top?

Since $Y \subset \cup_{y \in Y} \{y\}$ open cover, Y is compact iff it's finite.

Definition 117. (X, d) metric space, $Y \subset X$ is bounded in X iff $\exists\, p \in X$ and $r > 0$ s.t.

$$Y \subset B_{p,r}$$

Theorem 118. *(Heine Borel) $\mathbb{R}^n$ euclidean, $K \subset \mathbb{R}^n$, the following are equivalent*

1) K is closed and bounded

2) K is compact

3) $\forall\, E \subset K$ infinite subset, E has a limit point in K

Remark 119. One can replace above $\mathbb{R}^n$ by $\mathbb{C}^n$.

Remark 120. 2) $\implies$ 3): cf proposition 112, true for any metric, not just euclidean. 3) $\implies$ 2) is also true for any metric. 1) $\implies$ 2) requires euclidean, e.g.

$$A = \{p \in \mathbb{Q}, p > 0, 2 < p^2 < 3\}$$

is closed and bounded in $(\mathbb{Q}, \text{euclidean})$, but A is not compact in $(\mathbb{Q}, \text{euclidean})$, because it's not compact in $\mathbb{R}$ euclidean. Another simpler example, $X = [0, 1)$ euclidean, X is closed and bounded but not compact.

To prove 1) $\implies$ 2), it suffices to prove that the closure of ball is compact. Then

$$Y \subset B \subset \bar{B}$$

Y is closed in the compact so it is compact. But we will instead to prove closed cell is compact, which is a little bit easier. After we prove the following theorem, 1) $\implies$ 2) is automatic: if K is closed and bounded $\exists\, I \subset \mathbb{R}^n$ a cell s.t. $K \subset I$, I is compact $\implies K$ is also compact.

Theorem 121. *Every $n-$cell is compact in $\mathbb{R}^n$ euclidean.*

Remark 122. We have the assumption "$\mathbb{R}^n$ euclidean". This is the only place in the entirely proof of Heine Borel where $\mathbb{R}^n$ euclidean is required.

First prove cell in $\mathbb{R}$.

Proposition 123. *$I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$, $I_n = [a_n, b_n]$, cell in $\mathbb{R}$, $I_n \neq \varnothing\ \forall\, n$, then*

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing$$

Proof. $I_n \supset I_{n+1} \implies a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$, let

$$A = \{a_n, n \in \mathbb{N}\}$$

It is bounded from above by any b_n, $\alpha = \sup A$

$$a_n \leq \alpha \leq b_n\, \forall\, n \in \mathbb{N} \implies \alpha \in \bigcap_{n \in \mathbb{N}} I_n$$

□

Proposition 124. *$I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$, $I_n = [a_{1,n}, b_{1,n}] \times \ldots \times [a_{k,n}, b_{k,n}]$, cell in $\mathbb{R}^k$, $I_n \neq \varnothing\ \forall\, n$, then*

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing$$

Proof. use the previous proof on each component. □

Proposition 125. *$I \subset \mathbb{R}^k$ a cell, then I is compact in the euclidean metric.*

Proof. Suppose the opposite. U_α, $\alpha \in A$ an open cover of I that doesn't have a finite subcover.

Cut every interval, $\{[a_1, b_1], ..., [a_k, b_k]\}$ in half, we obtain 2^k cells. Of the 2^k cells, there is one that doesn't have a finite subcover, we call it I_2. Repeat the procedure we obtain a sequence of cells I_n s.t.

$$I = I_1,\ I_n \supset I_{n+1}$$

and

$$I_n \text{ doesn't have a finite subcover}$$

So

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing$$

then $\exists\, x \in \bigcap_{n\in\mathbb{N}} I_n$, so $\exists\, \bar{\alpha}$ s.t. $x \in U_{\bar{\alpha}}$. Since $U_{\bar{\alpha}}$ is open, $\exists\, r > 0$ s.t.

$$B_{x,r} \subset U_{\bar{\alpha}}$$

We know the length of the edge of I_n is $\forall\, x, y \in I_n$

$$d(x, y) \leq \frac{\sqrt{\sum_{i=1}^{k} (b_i - a_i)^2}}{2^n}$$

Hence we can choose proper $m \in \mathbb{N}$ s.t.

$$d(x, y) \leq r$$

$\forall\, x, y \in I_m$. Therefore

$$I_m \subset B_{x,r} \subset U_{\bar{\alpha}}$$

hence $U_{\bar{\alpha}}$ is a finite subcover of I_m. We have contradiction. □

Note 126. The idea of the proof also used in proving Goursat theorem in complex analysis.

Now prove Heine Borel 3) $\Longrightarrow$ 1).

Proof. Assume K is not bounded, we can construct a sequence x_n as following: $\forall\, n \in \mathbb{N}\ \exists\, x_n \in K$ s.t.

$$\|x_n\| > n$$

Suppose that $\{x_n, n \in \mathbb{N}\}$ is finite, then $\exists\, x \in K$ and $A \subset \mathbb{N}$ infinite s.t.

$$x_n = x\ \forall\, n \in A$$

then $\|x\| > n\ \forall\, n \in A$, which contradicts that A is infinite subset of $\mathbb{N}$, i.e. A is unbounded.

So $E = \{x_n, n \in \mathbb{N}\}$ is infinite. By assumption E has a limit point $y \in K$.

Choose N s.t. $\forall\, n \geq N$

$$\|x_n\| > 2\,\|y\|$$

then

$$\|y - x_n\| \geq |\|y\| - \|x_n\|| \geq \|x_n\| - \|y\| > \|y\|$$

That is to say $B_{y,\|y\|}$ contains at most x_1,...,x_{N-1} $\implies$ y is not a limit point.

Assume K is not closed. $\exists\, x \in \mathbb{R}^k - K$ s.t. x is a limit point of K, then we can construct a sequence $x_n \in K$, $n \in \mathbb{N}$ s.t.

$$\|x_n - x\| < \frac{1}{n}$$

Let $E = \{x_n\}$, then E is infinite because x is a limit point. By assumption, E has a limit point $y \in K$.

Choose $N \in \mathbb{N}$ s.t. $\forall\, n \geq N$

$$\|x_n - x\| \leq \frac{\|x - y\|}{2}$$

then

$$\|x_n - y\| \geq |\|x_n - x\| - \|x - y\|| = \frac{\|x - y\|}{2}$$

This says $B_{y,\frac{\|x-y\|}{2}}$ contains at most $x_1, ..., x_{N-1}$, so y is not a limit point. $\square$

This completes Heine-Borel.

Theorem 127. *(Weierstrass theorem) $E \subset \mathbb{R}^n$ a bounded infinite set, then E has a limit point.*

Proof. E contains in a cell K, and K is compact. By H-B 2) $\implies$ 3). $\square$

2.5 Countability

Definition 128. Two sets A, B have same cardinality iff $\exists$ a bijection $f : A \to B$.

We will do cardinality on the ground that distinguish countable and uncountable. We won't do much more.

Definition 129. A set B is countable if it has the cardinality of $\mathbb{N}$, i.e.$f : \mathbb{N} \to B$ bijective.

Notice that B is a set, so no repeated elements. In our class, we distinguish countable and finite, and call both at most countable.

Definition 130. $I_n = \{0, 1, ..., n\}$, $n \in \mathbb{N}$ B is finite if it's in bijection with I_n for some $n \in \mathbb{N}$.

Definition 131. A set B is at most countable if $\exists\, f : B \to \mathbb{N}$ injective.

Definition 132. If B is not countable nor finite, then B is uncountable.

Proposition 133. *E is a countable set and $A \subset E$ is an infinite set, then A is countable.*

Proof.

$$\begin{array}{ccl} \mathbb{N} & \to & E \text{ bijection} \\ n & \mapsto & x_n \end{array}$$

Using well-ordering, we find $n_1 \in \mathbb{N}$ the smallest natural s.t. $x_{n_1} \in A$. Recursively $n_k \in \mathbb{N}$ is chosen, $n_{k+1}$is the smallest natural s.t.

$$n_{k+1} > n_k \text{ and } x_{n_{k+1}} \in A$$

What have we created?

$$\begin{array}{ccccc} \mathbb{N} & \to & \mathbb{N} & \to & A \\ k & \mapsto & n_k & \mapsto & x_{n_k} \end{array}$$

Show $k \mapsto x_{n_k}$ is bijective.

Injectivity is trivial, for $n_{k+1}$is strictly $> n_k$.

Show surjectivity. Suppose not. $\exists\, x \in A$ s.t. $\nexists\, k \in \mathbb{N}$ s.t. $x_{n_k} = x$. And $\exists!\bar{n}$ s.t. $x = x_{\bar{n}}$. so $\nexists\, k$ s.t.$n_k = \bar{n}$, so $\exists!\bar{k} \in \mathbb{N}$ s.t.

$$n_{\bar{k}} < \bar{n} < n_{\bar{k}+1}$$

then $x_{n_{\bar{k}}}$, $x_{\bar{n}}$, $x_{n_{\bar{k}+1}} \in A$, so $n_{\bar{k}+1}$is not the smallest natural after $n_{\bar{k}}$ that $x_{n_{\bar{k}+1}} \in A$. $\square$

Lecture 9 (6/25/13)

Proposition 134. *U_α, $\alpha \in A$. A a countable set and U_α is countable. $\forall\, \alpha \in A$, then*

$$\bigcup_{\alpha \in A} U_\alpha \text{ is countable}$$

The proof in Rudin has a little flaws. He actually proved the following statement.

Notation 135. Disjoint union

$$\coprod_{\alpha \in A} U_\alpha = \{(x, a); x \in U_\alpha, \alpha \in A\}$$

(The same symbol $\coprod$ is used for coproduct)

If U_α are disjoint

$$\begin{array}{ccc} \coprod_{\alpha \in A} U_\alpha & \to & \bigcup_{\alpha \in A} U_\alpha \\ (x, \alpha) & \mapsto & x \end{array}$$

is bijective.

Proposition 136. *A is countable and $U_\alpha$is countable $\forall\, a \in A$, then*

$$\coprod_{\alpha \in A} U_\alpha \text{ is countable}$$

We now prove the two propositions together.

Proof. WLOG assume that A is $\mathbb{N}$. $U_1, ..., U_n, ...$. For $\forall\, n \in \mathbb{N}$,

$$\begin{array}{rcl} \exists\, f_n : \mathbb{N} & \to & U_n \\ k & \mapsto & x_k^{(n)} \end{array}$$

bijection. Graphically

$$\begin{array}{cccccccccc}
U_1: & x_1^{(1)} & , & x_2^{(1)} & , & \dots & , & x_k^{(1)} & , & \dots \\
 & & \swarrow & & \swarrow & & \swarrow & & & \\
U_2: & x_1^{(2)} & , & x_2^{(2)} & , & \dots & , & x_k^{(2)} & , & \dots \\
\vdots & & \swarrow & & & & & & & \\
\vdots & & \swarrow & & \swarrow & & & & \swarrow & \\
U_n: & x_1^{(n)} & , & x_2^{(n)} & , & \dots & , & x_k^{(n)} & , & \dots \\
\vdots & & & & & & & & &
\end{array}$$

Following the arrow: $x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_3^{(1)}, ...$, one gets

$$\begin{aligned}(g,h):\mathbb{N} &\rightarrow \mathbb{N}\times\mathbb{N}\\ n &\rightarrow (g(n),h(n))\end{aligned}$$

bijection. Hence

$$\begin{aligned}\mathbb{N} &\rightarrow \coprod_{n\in\mathbb{N}} U_n\\ n &\mapsto x_{h(n)}^{g(n)}\end{aligned}$$

bijection. This proves proposition 136.

Since

$$\begin{aligned}\bigcup_{\alpha\in A} U_\alpha &\rightarrow \coprod_{\alpha\in A} U_\alpha\\ x &\mapsto (x,\alpha(x))\end{aligned}$$

is injective, where $\alpha(x)$ is a choice of an α s.t. $x\in U_\alpha$.

$\bigcup_{\alpha\in A} U_\alpha$ is isomorphic to a subset of a countable set and $U_\alpha$is infinite, so is $\bigcup_{\alpha\in A} U_\alpha$, so $\bigcup_{\alpha\in A} U_\alpha$is countable. □

In en route, we proved $\mathbb{N}\times\mathbb{N}$ is countable. More generally is true

Proposition 137. *X is countable, $n\in\mathbb{N}$,*

$$X^n = \{(x_1,x_2,...,x_n); x_i\in X\} \textit{ is countable}$$

Proof. By induction if $n=1$, $X^1=X$ is countable. Assume that X^n is countable, then

$$X^{n+1} = \bigcup_{x\in X}\{(y,x); y\in X^n\}$$

is countable union of two countable sets, so it is countable. □

Proposition 138. *$\mathbb{Q}$ is countable.*

Proof.

$$\begin{aligned}\mathbb{Z}\times(\mathbb{N}-\{0\}) &\rightarrow \mathbb{Q}\\ (m,n) &\mapsto \left[\frac{m}{n}\right]\end{aligned}$$

surjective. We can find a injective map from $\mathbb{Q}\rightarrow\mathbb{Z}\times(\mathbb{N}-\{0\})$ by axiom of choice, then $\mathbb{Q}$ is isomorphic to a subset of $\mathbb{Z}\times(\mathbb{N}-\{0\})$, which is countable, so $\mathbb{Q}$ is countable. □

Example 139. (Cantor) Consider $x \in E$ if x is a sequence of $\{0, 1\}$, $x : \mathbb{N} \to \{0, 1\}$, show E is uncountable

Proof. Show that any $A \subset E$ a countable subset is strictly contained in E. List the elements of A

$$
\begin{array}{cccccccc}
\vdots & & & & & & & \\
x^{(n)} & 0 & 0 & 1 & ... & \boxed{1} & 0 & ... \\
\vdots & & & & \nearrow & & & \\
\vdots & & & \nearrow & & & & \\
x^{(2)} & 1 & \boxed{1} & 0 & 1 & 0 & 1 & ... \\
x^{(1)} & \boxed{0} & 0 & 1 & 0 & 0 & 1 & ...
\end{array}
$$

We make a sequence $y : \mathbb{N} \to \{0, 1\}$ s.t.

$$
y_n = \begin{cases} 1 & \text{if the } n\text{th entry of } x^{(n)},\ x_n^{(n)} = 0 \\ 0 & \text{if the } n\text{th entry of } x^{(n)},\ x_n^{(n)} = 1 \end{cases}
$$

complementing the diagonal. So $y \neq x_n \forall\, n$ because at least the nth digit is different, $y \notin A$. ☐

Since the elements of E are infinitely long, E is just a subset of $\mathbb{R}$ if we think of binary numbers. This proves $\mathbb{R}$ is uncountable. Next we will prove this in some topological methods.

Definition 140. (X, d) metric $U \subset X$ is dense iff $\bar{U} = X$

This is just fancy way of saying every point in X is either a point of U or a limit point of U.

Definition 141. $E \subset X$ is perfect if it's closed and every point of E is a limit point.

The following proposition will prove interval $[0, 1]$ and cells in $\mathbb{R}^k$ are uncountable in one shot.

Proposition 142. *$P \subset \mathbb{R}^k$ is perfect and $P \neq \varnothing$, then P is uncountable.*

Proof. P is infinite, for it contains its own limit points. Suppose P is countable.

$$
\begin{aligned}
\mathbb{N} &\to P \\
n &\mapsto x_n
\end{aligned}
$$

bijection. Choose V_1 any ball centered at x_1, then

$$V_1 \cap P$$

is infinite, for x_1 is a limit point.

Find $x_2 \neq x_1$, $x_2 \in V_1 \cap P$, and find V_2 ball centered at x_2 s.t.

$$\bar{V}_2 \subset V_1 \text{ and } x_1 \notin \bar{V}_2$$

then

$$V_2 \cap P$$

is infinite, for x_2 is a limit point. Find x_3, V_3... continue on

By Heine-Borel, $\bar{V}_n$ are compact sets,

$$\bar{V}_1 \supset \bar{V}_2 \supset ... \supset \bar{V}_n \supset ...$$

Then

$$K_n = P \cap \bar{V}_n$$

is also compact, for P is closed and K_n is closed in a compact.

$$K_1 \supset K_2 \supset ... \supset K_n \supset ... \text{ and } K_n \neq \varnothing \,\forall\, n$$

so

$$\bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$$

But $x_n \notin \bar{V}_{n+1} \implies x_n \notin K_{n+1}$, so

$$\left(\bigcap_{n\in\mathbb{N}} K_n\right) \cap P = \varnothing$$

Since $\bigcap_{n\in\mathbb{N}} K_n \subset P$,

$$\bigcap_{n\in\mathbb{N}} K_n = \varnothing$$

□

Remark 143. In the proof above, where did we use the hypothetical assumption P is countable? Strictly speaking not in $\bigcap K$ (except the nested part), because we can take intersection or union of uncountable items. That is because

$$\bigcap_{\alpha\in A} K_\alpha \text{ really means } \{k; k \in K_\alpha \,\forall\, \alpha \in A\}$$

It doesn't require to physically compute

$$K_1 \cap K_2$$

then

$$(K_1 \cap K_2) \cap K_3$$

etc.

Countability was used when we made the constructive argument, building circles one by one.

2.6 Connectedness

Definition 144. (X, d) metric space $C \subset X$, C is not connected iff $\exists\, A, B \subset C$ s.t.

1) A, B are open in C

2) $A \cup B = C$ and $A \cap B = \varnothing$

3) A, $B \neq \varnothing$

Remark 145. 1) connect iff it is not not connected.

2) In the definition, no reference to X, only about openness i.e. metric is considered. (attention $A, B \subset C$ Not $A, B \subset X$) This allows to say the space X is connected or not.

3) A, B are also closed in C, for $A^c = B$. Therefore C is connected iff the only subsets of C that are both open and closed are C and $\varnothing$.

4) Proving something is connected is a lot harder.

Example 146. $(\mathbb{Q}, \text{euclidean})$ is not connected. Because it is not connected if embed in $\mathbb{R}$, connectedness is absolute. Alternatively we can find $A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$, and $B = (\sqrt{2}, \infty) \cap \mathbb{Q}$.

The only connected component of $(\mathbb{Q}, \text{euclidean})$ are points, and they are closed

Example 147. $X = \mathbb{R}^2$, $E = \{\text{two parallel lines}\}$, each of the two lines, connected component, is both open and closed in E, E is clearly not connected.

Example 148. discrete topology (2.1) is not connected. The connected components are points, and they are closed and open.

Lecture 10 (7/2/13)

Proposition 149. *$\mathbb{R}$ euclidean $I \subset \mathbb{R}$ is connected iff I is an interval.*

Intervals are

$$I = (a,b),\ (a,b],\ [a,b) \text{ or } [a,b]$$

$a, b \in \mathbb{R} \cup \{\pm\infty\}$.

Exercise 150. Show $I \subset \mathbb{R}$ interval iff it has property:

$$\forall\, x, y \in I, x < y \ \text{ if } z \in \mathbb{R} \text{ s.t. } x < z < y \implies z \in I. \tag{2.2}$$

Proof. (of proposition) By the exercise above, we will show I is connected iff it has the property above.

Suppose I is connected but it has no such property (2.2).

$\exists\, x, y \in I$ s.t. $x < y$ and $\exists\, z \in \mathbb{R}$ s.t. $x < z < y$ and $z \in I$. Let

$$A = (-\infty, z) \cap I \quad B = (z, \infty) \cap I$$

Since $x \in A$, $y \in B$, $A, B \neq \varnothing$. A,B are open in I, $A \cap B = \varnothing$, and $A \cup B = I$. Hence I is not connected.

Suppose I has such property (2.2) but is not connected.

$\exists\, U_1$,U_2 open in $\mathbb{R}$ s.t.

$$A = I \cap U_1,\ B = I \cap U_2$$

A, B disjoint, not empty and $I = A \cup B$.

$\exists\, x \in A$, $y \in B$, we can assume $x < y$ (otherwise switch the role of A, B), then

$$[x, y] \subset I$$

because I has property (2.2).

Consider $S_1 = [x, y] - U_2 = [x, y] \cap U^c$, S_1 is compact, for U^c is closed and $[x, y]$ is compact. Then

$$\alpha = \sup S_1 = \max S_1$$

so $\alpha \in U_1 \implies x \leq \alpha < y$, (here $\alpha \neq y$, for $\alpha \notin U_2$)

Consider $S_2 = [\alpha, y] - U_1 = [\alpha, y] \cap U_1^c$ is compact, so

$$\beta = \inf S_2 = \min S_2$$

so $\beta \in U_2 \implies x \leq \alpha < \beta \leq y$. Then $\exists\, \gamma$ s.t. $\alpha < \gamma < \gamma$, and $\gamma \notin I$, because $\gamma \notin U_1$, for α is the maximum element in I; $\gamma \notin I$, either.

So I doesn't have property (2.2). $\square$

Example 151. (Cantor set) Let $K_0 = [0, 1]$, $K_1 = [0, 1/3] \cup [2/3, 1]$,...

Given K_n, make K_{n+1} by cutting each segment in 3 equal pieces and remove the middle,

$$K_1 \supset K_2 \supset ... \supset K_n \supset ...$$

Define

$$K = \bigcap_{n \in \mathbb{N}} K_n \neq \varnothing$$

Because it's the intersection of compact sets and it contains the end points of any interval in any K_n, K is closed and bounded, i.e. K is compact.

We call K the Cantor set.

Question 152. *Does K contain only end points?*

No, we'll show K is uncountable by showing it's perfect, but the set of all end points is countable.

Proposition 153. *The Cantor set is perfect.*

Proof. $x \in K$, prove x is a limit point of K. Fix $\epsilon > 0$, $S = (x - \epsilon, x + \epsilon)$

$$x \in K \implies x \in K_n \, \forall \, n \in \mathbb{N}$$

Denote with I_n, the interval in K_n s.t. $x \in I_n$. By Archimedean, $\exists \, I_n$ s.t. $I_n \subset S$. The two end points of I_n, $\alpha, \beta \neq x$, and $\alpha, \beta \in S \, \forall \, n$. $\square$

In analysis II, we will learn that countable sets have Lebesgue measure 0. Cantor set shows even uncountable set can have 0 Lebesgue measure. Cantor set also gives many applications in Stochastic, Brownian motion, etc.

3 Sequences and Series

3.1 Sequences

Definition 154. Sequences in a metric space (X, d) is a function

$$\begin{array}{rcl} \mathbb{N} & \rightarrow & X \\ n & \mapsto & a_n \end{array}$$

Definition 155. $a \in X$, a_n converges to a iff $\forall\, \epsilon > 0$, $\exists\, N_\epsilon \in \mathbb{N}$ s.t.

$$\forall\, n \geq N_\epsilon \implies d(a_n, a) < \epsilon$$

The following proposition is not trivial.

Proposition 156. *If a limit exists in a metric space, it is unique.*

Proof. $\exists\, a, a' \in X$ s.t. $\forall\, \epsilon > 0\ \exists\, N_\epsilon$s.t. $n \geq N_\epsilon$, then $\forall\, \epsilon > 0\ \exists\, N_\epsilon$ s.t. $n \geq N_\epsilon$

$$d(a_n, a) < \epsilon$$

and $\exists\, M_\epsilon$ s.t. $n \geq M_\epsilon$

$$d(a_n, a') < \epsilon$$

Fix $\epsilon > 0$, for $n \geq \max\{N_\epsilon, M_\epsilon\}$

$$0 \leq d(a, a') \leq d(a_n, a) + d(a_n, a') < 2\epsilon$$

so

$$d(a, a') = 0 \implies a = a'$$

□

As mentioned before, limit may not be unique in Top. First of all how to define limit, if we don't have distance?

Consider $(X, \mathcal{F})$ top space, an sequence a_n in X, $a \in X$, what does

$$a_n \to a, \text{ as } n \to \infty$$

mean?

Definition 157. $U \in \mathcal{F}$ is a neighborhood of $x \in X$ iff $x \in U$.

Denote $\mathcal{U}_x$ the set of neighborhood of x.

Definition 158. $a_n \to a$ iff $\forall\, U \in \mathcal{U}_x$, $\exists\, N \subset \mathbb{N}$ s.t.

$$\forall\, n \geq N \implies a_n \in U$$

Example 159. Consider the coarsest topology $(X, \mathcal{F})$, $\mathcal{F} = \{X, \varnothing\}$, let a_n be a sequence in X, then a_n converges to anything in X, i.e.

$$\forall\, b \in X,\ a_n \to b$$

because $\mathcal{U}_b = \{X\}$.

Proposition 160. *$(X, \mathcal{F})$ T2, the limit is unique.*

So it's not necessary to have metric to have uniqueness of limit.

Proposition 161. *(X, d)*

1) $a_n \to a$ iff every ball $B_{a,\epsilon}$ contains all but finitely many points of $\{a_n\}$

2) If a is limit point of E, then there is a sequence in E that converges to a

3) convergence implies boundedness

The statement "contains all but finitely many" in 1) is very precise. It is not true to replace it with "contains infinitely many", because "contains infinitely many" may mean that there are infinitely many outside the ball, but "contains all but finitely many" means only finitely many outside and there are not necessarily infinitely many inside either. E.g. convergent sequence that has constant tail

$$a_n = a_{n+1} = ... = a$$

Proof. 1) Forward direction is obvious. We show the converse. Suppose $a_{n_1}, a_{n_2}, ..., a_{n_k} \notin B_{a,\epsilon}$, let $N_\epsilon = \max\{n_1, ..., n_k\}$. then

$$n > N_\epsilon \implies a_n \in B_{a,\epsilon}$$

2) a is a limit point of E. $\exists\, a_1 \in E$ s.t. $a_1 \in B_{a,1}$, $\exists\, a_2 \in E$ s.t. $a_2 \in B_{a,\frac{1}{2}}$... So let $N = \lceil \frac{1}{\epsilon} \rceil$,

$$n \geq N \implies d(a, a_n) \leq \frac{1}{n} < \epsilon$$

3) $a_n \to a \implies \exists\, N$ s.t. $n \geq N$ $d(a_n, a) < 1$

$$R = \max\{1, d(a, a_1), ..., d(a, a_{N-1})\}$$

then $a_n \in B_{a,R}\ \forall\, n$. □

Let us now move from generic metric space to $\mathbb{R}$,$\mathbb{C}$ euclidean, so familiar algebraic structures will be applied. $\mathbb{R}$ and $\mathbb{C}$ behave the same.

Proposition 162. *$a_n \to a$, $b_n \to b$*

1) $(a_n + b_n) \to a + b$;

2) $c \in \mathbb{C}$ $ca_n \to ca$;

3) $a_n b_n \to ab$;

4) if $b \neq 0$, $\frac{a_n}{b_n} \to \frac{a}{b}$.

Proof. 1) fix $\epsilon > 0$, if $n \geq N_\epsilon$, $|a_n - a| < \epsilon$, if $n \geq M_\epsilon$, $|b_n - b| < \epsilon$, so

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < 2\epsilon$$

if $n \geq \max\{N_\epsilon, M_\epsilon\}$.

(the end 2ϵ is okay, not necessary to get a clean ϵ. E.g. $\sqrt{\epsilon}$, ϵ^2, $\epsilon/3$ are too okay, but $\epsilon - 1$ is not)

4) assume we have proven 3), so it suffices to prove

$$\frac{1}{b_n} \to \frac{1}{b}$$

so

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|bb_n|} < \frac{\epsilon}{|b||b_n|}$$

if $n > N_\epsilon$. We need an estimate of b_n,

$$|b_n| = |b_n - b + b| \geq ||b_n - b| - b|$$

$\exists\, M$ s.t. $n \geq M \implies |b_n - b| < |b|/2$, so

$$|b_n| > |b|/2$$

Hence

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| < \frac{2\epsilon}{|b|^2}$$

□

Lecture 11 (7/5/13)

Definition 163. a_n a sequence in X,

$$\begin{array}{ccc} \mathbb{N} & \to & X \\ n & \mapsto & a_n \end{array}$$

We make a subsequence a_{n_k} of a_n

$$\begin{array}{ccccc} \mathbb{N} & \to & \mathbb{N} & \to & X \\ k & \mapsto & n_k & \mapsto & a_{n_k} \end{array}$$

if n_k is an increasing sequence ($n_k < n_{k+1}$).

Remark 164. A sequence a_n is convergent iff any subsequence a_{n_k} convergent to the same limit.

Example 165. $a_n = (-1)^n$. Subsequences $a_{2n} = 1$, $a_{2n+1} = -1$

Exercise 166. Prove a sequence a_n converges to a iff $a_{2n} \to a$ and $a_{2n+1} \to a$.

Proposition 167. *(K, d) a compact metric space and $a_n \in K$ is a sequence. $\exists\, n_k$ s.t. a_{n_k} is convergent in K.*

Proof. Let $E = \{a_n; n \in \mathbb{N}\}$. If E is finite,

$$\begin{array}{rcl} \mathbb{N} & \to & E \cong \{0, 1, ..., N\} \\ k & \mapsto & a_k \end{array}$$

There is an element in E that has infinitely many preimages. Using this element defines a constant subsequence.

If E is infinite, E has a limit point $p \in K$. Find n_1, smallest natural s.t. $d(a_{n_1}, p) < 1$.

Suppose n_k is known and $d(a_{n_k}, p) < \frac{1}{k}$ and $n_k > n_{k-1}$ find n_{k+1}, the smallest natural s.t. $n_{k+1} > n_k$ and $d(a_{n_{k+1}}, p) < \frac{1}{k+1}$, then $a_{n_k}$is a sequence that converges to p. $\square$

Corollary 168. *Every bounded sequence in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a convergent subsequence.*

Definition 169. a_n is a sequence in (X, d), let E be the set of sub sequential limits. That is $p \in E$ iff there is a subsequence that converges to P.

Example 170. $a_n = n$, $E = \varnothing$.

Example 171. $a_n = \begin{cases} 0 & n \text{ even} \\ n & n \text{ odd} \end{cases}$, $E = \{0\}$

Both a_n above are unbounded. One can prove if a_n is bounded in $\mathbb{R}^n$ or $\mathbb{C}^n$, then $E \neq \varnothing$.

Example 172. Let a_n be$\mathbb{N} \to \mathbb{Q}$ bijection. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ euclidean, $E = \mathbb{R}$.

Proposition 173. *(X, d) a_n a sequence, E the set of sub sequential limits, then E is closed.*

Proof. p is a limit point of E, prove $p \in E$ or equivalently find $a_{n_k} \to p$.

Find $p_1 \in E$ s.t. $d(p_1, p) < 1$, and $\exists$ a subsequence a_{n_k} converges to p_1. Find n_1 smallest natural in that subsequence s.t.

$$d(a_{n_1}, p_1) < 1 \implies d(a_{n_1}, p) < 2$$

Constructively, $n_k > n_{k-1}$ s.t. $d(a_{n_k}, p) < \frac{2}{k}$ is known (e.g. above showed how to obtain n_1), find $p_{k+1} \in E$ s.t.

$$d(p_{k+1}, p) < \frac{1}{k+1}$$

there is another subsequence a_{n_k} that converges to p_{k+1}, find the smallest natural n_{k+1}, s.t. $n_{k+1} > n_k$ and

$$d(a_{n_{k+1}}, p_{n+1}) < \frac{1}{k+1} \implies d(a_{n_{k+1}}, p) < \frac{2}{k+1}$$

Hence we found a subsequence goes to p. □

3.2 Completeness

Definition 174. (X, d) metric space, a sequence in X, a_n is Cauchy iff $\forall\, \epsilon > 0$, $\exists\, N_\epsilon \in \mathbb{N}$ s.t. $\forall\, n, m \geq N_\epsilon$, $d(a_n, a_m) < \epsilon$.

Proposition 175. *convergent* $\implies$ *Cauchy*

Proof. Given $\epsilon > 0$ find N_ϵ s.t. $d(a_n, a) < \epsilon$ if $n > N_\epsilon$, so if $n, m > N$

$$d(a_n, a_m) \leq d(a_n, a) + d(a_m, a) < 2\epsilon$$

□

Remark 176. Cauchy $\not\Longrightarrow$ convergent. E.g. $X = \mathbb{R} - \{0\}$ $a_n = \frac{1}{n}$ Cauchy but not convergent in X. Another example, $X = \mathbb{Q}$, $a_n = \left(1 + \frac{1}{n}\right)^n$ is Cauchy but not convergent in $\mathbb{Q}$.

Later we will show these are the worst scenario. There is always a way to make Cauchy $\implies$ convergent.

Theorem 177. *(K, d) compact metric space, $a_n \in K$ is Cauchy iff it's convergent.*

Exercise 178. Show Cauchy $\implies$ bounded (similar proof as in convergent $\implies$ bounded).

Definition 179. (X, d) a metric space s.t. every Cauchy is convergent. We call X a complete space.

Remark 180. compact $\Longrightarrow$ completeness; but not the other way. $\mathbb{R}^n$, $\mathbb{C}^n$ are complete but not compact.

Corollary 181. *$a_n \in \mathbb{R}^n$ or $\mathbb{C}^n$ is convergent iff it is Cauchy.*

In analysis, complete is the second best thing after compact. Sometimes compact is too much. There is only one way to make space complete, while compactification is not unique.

Definition 182. Tail of a sequence a_n is

$$E_N = \{a_n; n \geq N\}$$

Definition 183. (X, d) metric space, $A \subset X$,

$$diamA = \sup\{d(x, y); x, y \in A\}$$

Remark 184. a_n is a sequence and E_n are the tails of a_n,

$$a_n \text{ is Cauchy } \Longleftrightarrow diamE_n \to 0 \text{ as } n \to \infty$$

Proof. $\forall \epsilon > 0 \; \exists N_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\epsilon$ $diamE_n = \sup\{d(a_k, a_m); k, m > n\} < \epsilon$.

Conversely $d(a_k, a_m) \leq diamE_n < \epsilon$ if $k, m \geq n$. □

Lemma 185. *$diamE = diam\bar{E}$*

Proof. $E \subset \bar{E} \implies diamE \leq diam\bar{E}$.

$p, q \in \bar{E}$, fix $\epsilon > 0$, $\exists p', q' \in E$ s.t.

$$d(p, p') < \epsilon \text{ and } d(q, q') < \epsilon$$

So

$$d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < 2\epsilon + d(p', q') \leq 2\epsilon + diamE$$

Take sup for $p, q \in \bar{E}$,

$$diam\bar{E} \leq 2\epsilon + diamE \implies diam\bar{E} \leq diamE$$

□

Lemma 186. *K_n compact not empty $\forall\, n$, $K_1 \supset ... \supset K_n \supset K_{n+1} \supset ...$ and $diam K_n \to 0$, then*

$$\bigcap_{n\in\mathbb{N}} K_n = \{one\ point\}$$

Proof. Clearly $\cap K_n \neq \varnothing$. Suppose that it contains at least two points, x, y.

$$diam\,(\cap K_n) \geq d(x,y) > 0$$

Since $K_n \supset (\cap K_n)\ \ \forall\, n$, take diameter

$$diam K_n \geq diam\,(\cap K_n) > 0\ \forall\, n$$

take limit $n \to \infty$, $diam K_n \to 0$, contradiction. □

We now prove theorem 177.

Proof. a_n is Cauchy in K, E_n tails

$$E_n \supset E_{n+1} \qquad \bar{E}_n \supset \bar{E}_{n+1}$$

$\bar{E}_n$ is closed in compact K, because K is complete, then $\bar{E}_n$ is compact. By the previous remark and lemma

$$\text{Cacuhy} \iff diam E_n = diam \bar{E}_n \to 0$$

and

$$\bigcap_{n\in\mathbb{N}} \bar{E}_n = \{p\}$$

Therefore $\forall\, \epsilon > 0\ \exists\, N_\epsilon \in \mathbb{N}$ s.t. $n \geq N_\epsilon$

$$diam \bar{E}_n < \epsilon$$

that is

$$d(a_n, p) \leq diam \bar{E}_n < \epsilon$$

Hence

$$a_n \to p$$

□

Definition 187. If X is not complete, we can complete (X, d) to $(\tilde{X}, \tilde{d})$, s.t.

1) It is complete;

2) $\exists\, i$ injective $i : X \to \tilde{X}$

3) $i(X)$ is dense in $\tilde{X}$

4) i is an isometry, i.e. $d(x, y) = \tilde{d}(i(x), i(y))\ \forall\, x, y \in X$

$(\tilde{X}, \tilde{d}, i)$ satisfying 1-4 is a completion of (X, d).

Remark 188. Completion is unique.

Suppose we have a second completion $(\bar{X}, \bar{d}, j)$, then $\exists!\quad f : \tilde{X} \to \bar{X}$ isometry and

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"'] & \tilde{X} \arrow[d, "f"] \\ & \bar{X} \end{tikzcd}$$

commute.

Remark 189. How to make completion?

Make X' the set of all Cauchy sequence in X. Define equivalent relation $\sim$: if $\{p_n\}$,$\{q_n\}$ are elements in X'

$$\{p_n\} \sim \{q_n\} \iff d(p_n, q_n) \to 0 \tag{3.1}$$

Then the completion

$$\tilde{X} = X'/\sim$$

Remark 190. Application of completion

$$\mathbb{Q} \to \mathbb{R}$$

However there are some logical problem of this approach if it is not handled properly. We need a notion of distant in $\mathbb{R}$ for evaluating (3.1), but $\mathbb{R}$ itself is not yet defined.

3.3 Limsup & Liminf

Lecture 12 (7/9/13)

Sometime we extend the limit to include $\pm\infty$. It becomes convenient to do so in this section.

Definition 191. $\lim_{n\to\infty} a_n = \infty(-\infty)$ iff

$$\forall\, M \in \mathbb{R}, \exists\, N \in \mathbb{N} \text{ s.t.} \forall\, n \geq N \; a_n > M \; (a_n < M)$$

Definition 192. a_n is a real sequence it's increasing iff $a_n \leq a_{n+1}$ or decreasing iff $a_n \geq a_{n+1}$.

Proposition 193. *a_n a real sequence bounded from above and increasing then a_n is convergent.*

Remark 194. One can replace increase with decreasing and bounded above with bounded below.

Proof. $\sup\{a_n; n \in \mathbb{N}\} = \alpha$ fix $\epsilon > 0$ $\exists\, N_\epsilon \in \mathbb{N}$ s.t.

$$\alpha - \epsilon < a_{N_\epsilon} \leq \alpha < \alpha + \epsilon$$

a_n increasing implies if $n \geq N_\epsilon$,

$$a_{N_\epsilon} \leq a_n \leq \alpha$$

that is

$$\alpha - \epsilon < a_n < \alpha + \epsilon$$

□

We will give three definition of limsup, liminf.

Exercise 195. Show they are equivalent.

Consider a_n a real sequence, let $b_n = \sup_{m\geq n} a_m$, then b_n is decreasing because the set $\{m \geq n\}$ is shrinking. The limit of b_n always exists in $\mathbb{R} \cup \{\pm\infty\}$.

Definition 196.

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \left(\sup_{m\geq n} a_m \right)$$

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \left(\inf_{m\geq n} a_m \right)$$

Although limit of sequence may not exist, but limsup, liminf always exist and they satisfy the same algebra of limits. The advantage of this definition is that it gives a way to calculate limsup, liminf.

The way Rudin defined is the following. cf proposition 173, we can extend it to

a_n real sequence and E is the set of sequential limit (including limit goes to $\pm\infty$)

Definition 197.

$$\limsup_{n\to\infty} a_n := \sup E$$

$$\liminf_{n\to\infty} a_n := \inf E$$

This definition suggests there is a subsequence converges to limsup, liminf. The following proposition can be taken as the third definition

Proposition 198. *$\alpha = \limsup a_n$ iff $\exists\, a$ subsequence $a_{n_k} \to \alpha$ and $\forall\, \epsilon > 0 \exists\, N_\epsilon \in \mathbb{N}$ s.t. $\forall\, n \geq N$, $a_n < \alpha + \epsilon$.*

The last part of above, $a_n < \alpha + \epsilon$, looks very like definition of a limit, expect it is now only one-sided.

Proposition 199. *1) $a_n \to \alpha$, $\alpha \in \mathbb{R} \cup \{\pm\infty\}$ iff $\limsup a_n = \liminf a_n = \alpha$*

2) $\liminf a_n \leq \limsup a_n$

3) $\limsup(a_n + b_n) = \limsup a_n + \limsup b_n$, $\liminf(a_n + b_n) = \liminf a_n + \liminf b_n$

3.4 Examples of Sequences

Exercise 200. Show

$$a_n = \frac{1}{n^p} \to 0 \text{ as } n \to \infty$$

$p \in \mathbb{R}$, $p > 0$.

Solve: $\frac{1}{n^p} < \epsilon \iff n > \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$ choose $N_\epsilon = \left\lceil \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}} \right\rceil$

Exercise 201. Show

$$p^{\frac{1}{n}} \to 1 \text{ as } n \to \infty$$

$p \in \mathbb{R}$ $p > 0$.

Solve: let

$$x_n = p^{\frac{1}{n}} - 1$$

If $p > 1$, $x_n \geq 0$

$$p = (x_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k \geq n x_n$$

So

$$0 \leq x_n \leq \frac{p}{n} \to 0$$

If $0 < p < 1 \implies \frac{1}{p} > 0$, then by above

$$\left(\frac{1}{p}\right)^{\frac{1}{n}} \to 1 \implies \frac{1}{p^{\frac{1}{n}}} \to 1 \implies p^{\frac{1}{n}} \to 1$$

Exercise 202. Show

$$n^{\frac{1}{n}} \to 1 \text{ as } n \to \infty$$

Solve:

$$x_n = n^{\frac{1}{n}} - 1 \iff n = (x_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k$$

so

$$n \geq x_n^2 \binom{n}{2} = x_n^2 \frac{n(n-1)}{2}$$

Hence

$$0 \leftarrow -\left(\frac{2}{n-1}\right)^{1/2} \leq x_n \leq \left(\frac{2}{n-1}\right)^{1/2} \to 0$$

Later we will use continuity to solve three exercises above.

Recall in Calculus we are told based on how fast function growth

$$\log\log x \ll \log x \ll n^\alpha \ll p^n \ll n! \ll n^n$$

Exercise 203. Show

$$\frac{n^\alpha}{(1+p)^n} \to 0 \text{ as } n \to \infty$$

$\alpha \in \mathbb{R}$, $p > 0$.

Solve

$$\begin{aligned}
(1+p)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} p^k &\geq \begin{pmatrix} n \\ k \end{pmatrix} p^k \\
&= \frac{n(n-1)...(n-k+1)}{k!} p^k \\
&= \frac{n^k p^k}{k!} \left(1 - \frac{1}{n}\right) ... \left(1 - \frac{k-1}{n}\right) \\
&\geq \frac{n^k p^k}{k!} \left(1 - \frac{k-1}{n}\right)^{k-1}
\end{aligned}$$

for any $0 < k < n$.

Fix k, pick $n > 2(k-1) \implies 1 - \frac{k-1}{n} > \frac{1}{2}$, then

$$0 \leq \frac{n^\alpha}{(1+p)^n} \leq \frac{2^{k-1} k!}{p^k} n^{\alpha - k}$$

If $\alpha < k$, then $n^{\alpha-k} \to 0$

Exercise 204. Show

$$x^n \to 0 \text{ as } n \to \infty$$

if $|x| < 1$ $x \in \mathbb{C}$.

Solve: By previous exercise with $\alpha = 0$,

$$|x^n| = |x|^n \to 0 \implies x^n \to 0$$

3.5 Series

Definition 205. a_n a sequence over $\mathbb{R}$ or $\mathbb{C}$, define the sequence of partial sums

$$S_n = \sum_{i=0}^{n} a_i$$

We call S_n a series and if it converges, we denote its limit with

$$\sum_{i=0}^{\infty} a_i$$

Series is just a sequence, so we can use what we have leaned and apply t series.

Remark 206. A series is converging iff it's Cauchy (because we're in $\mathbb{R}$, $\mathbb{C}$), i.e.

$$\forall\, \epsilon > 0\, \exists\, N_\epsilon \in \mathbb{N} \text{ s.t. } \forall\, p \geq q \geq N_\epsilon \; \left| \sum_{n=q}^{p} a_n \right| < \epsilon$$

Proposition 207. *If* $\sum a_n$ *is convergent, then* $a_n \to 0$.

Proof. Suppose $S_n = \sum_{k=0}^{n} a_k \to L$,

$$a_n = S_n - S_{n-1}$$

Take the limit

$$a_n \to L - L = 0$$

□

Generally if all terms have the sign, the series is much easier to handle.

Proposition 208. $a_n \in R$ *and* $a_n \geq 0$, S_n *is a bounded sequence, then* S_n *convergent.*

Proof. $a_n \geq 0$, S_n increasing + bounded $\implies$ convergent. □

3.6 Convergent Tests

Proposition 209. $a_n \in \mathbb{C}$ *or* $\mathbb{R}$ *and* $b_n \in \mathbb{R}$ *and* $\exists\, N \in \mathbb{N}$ *s.t.* $\forall\, n \geq N$ $|a_n| \leq b_n$ *and* $\sum_{n=0}^{\infty} b_n < \infty$*, then*

$$\left| \sum_{n=0}^{\infty} a_n \right| < \infty$$

Proof. $\sum b_n$ is convergent $\iff \forall\, \epsilon > 0\, \exists\, N_\epsilon$ s.t. $\forall\, p \geq q \geq N_\epsilon$

$$\sum_{n=q}^{p} b_n < \epsilon$$

Let $p, q \geq \max\{N, N_\epsilon\}$, then

$$\left| \sum_{n=0}^{\infty} a_n \right| \leq \sum_{n=q}^{p} |a_n| \leq \sum_{n=q}^{p} b_n \leq \epsilon$$

By Cauchy, it is convergent. □

Similarly one has

Proposition 210. *a_n, $b_n \in \mathbb{R}$ and $a_n \leq b_n$ if $n \geq N$ and $\sum_{n=0}^{\infty} a_n = \infty$, then*

$$\sum_{n=0}^{\infty} b_n = \infty$$

Example 211.

$$\sum_{n=0}^{\infty} x^n$$

$x \in \mathbb{C}$.

$$S_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

assume $x \neq 1$. If $|x| < 1$, $x^{n+1} \to 0 \implies S_n \to \frac{1}{1-x}$. If $|x| \geq 1$, divergent.

Proposition 212. *Sequence $a_n \geq 0$ decreasing. $S_n = \sum_{k=0}^{n} a_k$, $T_n = \sum_{k=0}^{n} 2^k a_{2^k}$*

S_n is convergent iff T_n is convergent.

This will be used analogously as the integral test in Calculus.

Example 213. $p-$series

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

This is a classical example that ratio or root tests are inclusive about it.

By the proposition this is convergent iff

$$\sum_{k=0}^{\infty} \frac{2^k}{(2^k)^p} = \sum 2^{k(1-p)}$$

hence convergent iff

$$\left|2^{1-p}\right| < 1$$

or

$$p > 1$$

Lecture 13 (7/11/13)

Proof. $S_n = a_1 + a_2 + ... + a_n$, $T_k = a_1 + 2a_2 + 4a_4 + ... + 2^k a_{2^k}$

Prove that S_n is bounded iff T_k is bounded.

Let $2^k \leq n$

$$\begin{aligned}
S_n = a_1 + a_2 + ... + a_n &\geq a_1 + a_2 + ... + a_{2^k} \\
&= a_1 + (a_2) + (a_3 + a_4) + ... + \left(a_{2^{k-1}+1} + ... + a_{2^k}\right) \\
&\geq a_1 + a_2 + 2a_4 + ... + 2^{k-1}a_{2^k} \\
&\geq \frac{1}{2}\left(a_1 + 2a_2 + 4a_4 + ... + 2^k a_{2^k}\right) \\
&= \frac{1}{2}T_k
\end{aligned}$$

Hence if $2^k \leq n$, then $2S_n \geq T_k$

Consider $2^k > n$

$$\begin{aligned}
S_n = a_1 + a_2 + ... + a_n &\leq a_1 + a_2 + ... + a_{2^{k+1}-1} \\
&= a_1 + (a_2 + a_3) + (a_4 + ... + a_7) + ... + \left(a_{2^k} + ... + a_{2^{k+1}-1}\right) \\
&\leq a_1 + 2a_2 + 4a_4 + ... + 2^k a_{2^k} \\
&= T_k
\end{aligned}$$

Hence if $2^k > n$, then $S_n \leq T_k$.

Assume that S_n is convergent, then $\exists\, M$ s.t. $S_n \leq M \ \forall\, n \in \mathbb{N}$. Fix k find n s.t. $2^k \leq n$ then

$$T_k \leq 2S_n \leq 2M \quad \forall\, k$$

so T_k is convergent.

Similar way to prove the other way. □

Example 214. Recall $p-$series divergent if $p \leq 1$, what if we do

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^p n}$$

just raise a little bit.

This is same to check

$$\sum \frac{2^k}{(\log 2^k)^p 2^k} = \frac{1}{(\log 2)^p} \sum \frac{1}{k^p}$$

convergent if $p > 1$. So

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)n}$$

is too enough to make it convergent.

Example 215. (Euler number) Two definitions

$$e := \sum_{n=0}^{\infty} \frac{1}{n!} \text{ and } \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

Are they the same?

This is the first application of $\limsup, \liminf$.

Let

$$S_n = \sum_{k=0}^{n} \frac{1}{k!}$$

which is clearly convergent by comparison $S_n \leq \sum \frac{1}{2^k}$.

$$T_n = \left(1 + \frac{1}{n}\right)^n \quad = \quad \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$$

We don't know if T_n converges,

$$\limsup_{n\to\infty} T_n \leq \lim_{n\to\infty} S_n =: e$$

because the products $\left(1 - \frac{1}{n}\right), \dots, \left(1 - \frac{k-1}{n}\right) < 1$.

On the other hand, if $m < n$, the truncated sum

$$T_n \geq \sum_{k=0}^{m} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$$

Fix m, take $n \to \infty$

$$\liminf_{n\to\infty} T_n \geq \sum_{k=0}^{m} \frac{1}{k!} = S_m$$

Then take $m \to \infty$, and notice now LHS is independent of m, yield

$$\liminf_{n\to\infty} T_n \geq \lim_{m\to\infty} S_m = e$$

Since $\liminf_{n\to\infty} T_n \leq \limsup_{n\to\infty} T_n$,

$$\limsup_{n\to\infty} T_n = \liminf_{n\to\infty} T_n = e$$

or

$$T_n \to e$$

For completeness, we show $e \notin \mathbb{Q}$.

Lemma 216.

$$e - S_n \leq \frac{1}{n!n}$$

Proof.

$$\begin{aligned} e - S_n \leq \sum_{k=n+1}^{\infty} \frac{1}{k!} &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + ...\right) \\ &\leq \frac{1}{(n+1)!}\sum_{k=0}^{\infty}\left(\frac{1}{n+1}\right)^k = \frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+1}} = \frac{1}{n!n} \end{aligned}$$

□

Proposition 217. $e \notin \mathbb{Q}$.

Proof. Suppose $e \in \mathbb{Q}$,

$$e = \frac{p}{q}$$

By lemma $0 < e - S_q \leq \frac{1}{q!q}$, i.e.

$$\frac{p}{q} - \left(1 + 1 + \frac{1}{2!} + ... + \frac{1}{q!}\right) \leq \frac{1}{q!q}$$

Simplify to

$$0 < \underbrace{(q-1)!p - q!\left(1 + 1 + \frac{1}{2!} + ... + \frac{1}{q!}\right)}_{\in\mathbb{N}} \leq \underbrace{\frac{1}{q}}_{<1}$$

contradiction. □

Definition 218. A series $\sum a_n$ over $\mathbb{C}$ is absolutely convergent iff $\sum_{n=0}^{\infty} |a_n| < \infty$.

Proposition 219. *Absolute convergent $\implies$ convergent.*

Proof. if $q, p \geq N_\epsilon$

$$\left|\sum_{n=q}^{p} a_n\right| \leq \sum_{n=q}^{p} |a_n| < \epsilon$$

So convergent by Cauchy. □

Root test is very important because of its usage in power series.

Proposition 220. *(root test) A complex series $\sum_{n=0}^{\infty} a_n$, let*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} =: L \in [0, \infty]$$

1) If $L < 1$, then the series is absolute convergent

2) If $L > 1$, then the series is not convergent

3) If $L = 1$, the test is inconclusive

Remark 221. (i) That the series is not convergent include both possibilities: series goes to $\pm\infty$ or no limit exists. (ii) If $\sum a_n$ converges but not absolute convergent, then it always falls into inclusive, because it cannot be either 1) or 2). But $\exists$ case that is absolute convergent but still fall into inclusive, e.g. $p-$series.

Proof. Suppose $L < 1$, pick $\epsilon > 0$ s.t. $L + \epsilon < 1$ $\exists N_\epsilon \in \mathbb{N}$ s.t.

$$\forall n \geq N_\epsilon,\ \sqrt[n]{|a_n|} < L + \epsilon$$

(here we used one of the definition of limsup, and this is the only place in the proof using limsup, so notice limsup is enough, no need to have lim)

Then $\forall n \geq N_\epsilon$

$$|a_n| < (L + \epsilon)^n$$

$\sum (L + \epsilon)^n < \infty$ because $L + \epsilon < 1$. Then by comparison $\sum |a_n| < \infty$.

Suppose $L > 1$. $\exists a_{n_k}$ s.t.

$$\sqrt[n_k]{|a_{n_k}|} \to L$$

Pick $\epsilon > 0$ s.t. $L - \epsilon > 1$, find K_ϵ s.t.

$$\forall k \geq K_\epsilon,\ \sqrt[n_k]{|a_{n_k}|} > L - \epsilon > 1$$

That is

$$|a_{n_k}| > 1$$

if $k > K_\epsilon$, so a_n doesn't converge to 0, hence $\sum a_n$ doesn't converge.

(In this step we see the test is not optimal, there are many a_n does go to 0, yet $\sum a_n$ not converge.) $\square$

Proposition 222. *(ratio test) A complex series $\sum_{n=0}^{\infty} a_n$, let*

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =: L \in [0,\infty]$$

1) If $L < 1$, then the series is absolute convergent

2) If $L > 1$, then the series is not convergent

3) If $L = 1$, the test is inconclusive

Proof. More of less the same. □

The following proposition shows root test is a little bit better than ratio test. Sometimes $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ so inclusive, but $\limsup_{n\to\infty}\sqrt[n]{|a_n|} < 1$.

Proposition 223. *A complex series $\sum_{n=0}^{\infty} a_n$,*

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty}\sqrt[n]{|a_n|} \le \limsup_{n\to\infty}\sqrt[n]{|a_n|} \le \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Proof. see Rudin. □

3.7 Power Series

For now let's only deal with complex. Later we will show some strange things can happen for $\mathbb{R}$ power series.

Definition 224. $c_n, a \in \mathbb{C}$, $D \subset \mathbb{C}$ domain: $z \in D$ iff

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

converges. This defines a function

$$\begin{aligned} f : D &\to \mathbb{C} \\ z &\mapsto \sum_{n=0}^{\infty} c_n (z-a)^n \end{aligned}$$

To find D, we'll use root test, for it is finer than ratio test.

Check

$$\limsup_{n\to\infty} \sqrt[n]{|c_n||z-a|^n} = |z-a| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = |z-a|\alpha$$

where $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \in [0, \infty]$.

Root test says

$$\begin{cases} \text{if } |z-a|\alpha < 1 & \text{absolutely convergent} \\ \text{if } |z-a|\alpha > 1 & \text{divergent} \\ \text{if } |z-a|\alpha = 1 & \text{inconclusive} \end{cases}$$

If $\alpha = 0$, $D = \mathbb{C}$, the series always absolutely converge.

If $\alpha = \infty$, $D = \varnothing$, the series diverges (except $z = a$ where $f(z) = c_0$)

Other cases

$$D = |z-a| < \frac{1}{\alpha} := R$$

where R is called the radius of convergence. That is the series absolutely converge inside of disc D, and diverge outside.

What about on the boundary of the disc?

To do that, we need the following theorem.

Theorem 225. *$a_n \in \mathbb{C}$, $b_n \in \mathbb{R}$, b_n decreasing and $b_n \to 0$, partial sum $A_N = \sum_{n=1}^{N} a_n$, $|A_N| \leq M \ \forall N$, then the series*

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Notice one of the conditions of the theorem says partial sum is uniformly bounded, but doesn't require to be convergent. The conclusion of the theorem says $\sum a_n b_n$ converges, not necessarily absolutely converges.

Example 226. (alternating series)

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

where $b_n \to 0$ decreasing and since $\left|\sum_{n=1}^{N} (-1)^n\right| \leq 1$, by the theorem, alternating series converges.

Example 227.

$$\sum_{n=0}^{\infty} (\sin n) b_n$$

where $b_n \to 0$ decreasing and since $\left|\sum_{n=0}^{N} \sin n\right| = \left|\frac{\sin 1 - \sin(N+1) + \sin N}{2 - \cos 1}\right| \le \frac{3}{1}$, by the theorem, the series converges.

Lecture 14 (7/16/13)

Proof. Want to show

$$\left|\sum_{n=q}^{p} a_n b_n\right| < \epsilon \quad \text{if } p > q \ge N$$

$A_n = \sum_{k=1}^{n} a_k$, $a_n = A_n - A_{n-1}$, so

$$\begin{aligned}
\sum_{n=q}^{p} a_n b_n &= \sum_{n=q}^{p} (A_n - A_{n-1}) b_n \\
&= \sum_{n=q}^{p} A_n b_n - \sum_{n=q}^{p} A_{n-1} b_n \\
&= \sum_{n=q}^{p} A_n b_n - \sum_{n=q-1}^{p-1} A_n b_{n+1} \\
&= \sum_{n=q}^{p-1} A_n (b_n - b_{n+1}) + A_p b_p - A_{q-1} b_q
\end{aligned}$$

Then

$$\begin{aligned}
\left|\sum_{n=q}^{p} a_n b_n\right| &\le \sum_{n=q}^{p-1} |A_n||b_n - b_{n+1}| + |A_p b_p| + |A_{q-1} b_q| \\
&= \sum_{n=q}^{p-1} |A_n|(b_n - b_{n+1}) + |A_p| b_p + |A_{q-1}| b_q \\
&\le M \left[\sum_{n=q}^{p-1} (b_n - b_{n+1}) + b_p + b_q\right] \\
&= 2M b_q
\end{aligned}$$

Since $b_q \to 0$, $\exists N_\epsilon$s.t. $b_q < \epsilon$ if $q \ge N_\epsilon$.

That is if $p > q \ge N_\epsilon$, then

$$\left|\sum_{n=q}^{p} a_n b_n\right| < \epsilon$$

□

We now discuss what happen on the boundary

Corollary 228. *Complex power series*

$$\sum_{n=0}^{\infty} c_n z^n$$

$c_n \in \mathbb{R}$, $z \in \mathbb{C}$, $c_n \to 0$ *decreasing, then the series is convergent for* $|z| = 1$ *minus at most the point* $|z| = 1$.

Example 229. $z \in \mathbb{C}$

$$\sum_{n=0}^{\infty} \frac{z^n}{n}$$

converges on $D = \{|z| \leq 1\} - \{z = 1\}$, and converge absolutely on $\{|z| < 1\}$.

Proof. Applying the theorem, check for given z

$$\left|\sum_{n=0}^{N} z^n\right| = \left|\frac{1 - z^{N+1}}{1 - z}\right|$$

(this is still true even in complex, because the proof of geometric series involves only properties of a field.)

Then

$$\left|\sum_{n=0}^{N} z^n\right| \leq \frac{2}{|1 - z|} =: M(z)$$

assume $z \neq 1$, where M is same for all N. □

Let's look one application of power series.

Suppose we want to do

$$\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n$$

What are c_n? Does RHS converge?

We write them in power series

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{m=0}^{\infty} b_m z^m = \sum_{k=0}^{\infty} c_k z^k$$

Assume no problem of convergence,

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{m=0}^{\infty} b_m z^m = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m z^{n+m}$$

Later we will prove: we can change from 2 indices (n, m) to index k as long as we too cover the whole (n, m) plane.

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m z^{n+m} = \sum_{k=0}^{\infty} \underbrace{\sum_{n=0}^{k} a_n b_{k-n} z^k}_{\equiv c_k}$$

so we cover the whole plane like

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
(0,3) & & & & & & \\
& \searrow & & & & & \\
(0,2) & & (1,2) & & & & \\
& \searrow & & \searrow & & & \\
(0,1) & & (1,1) & & (2,1) & & \\
\uparrow & \searrow & & \searrow & & \searrow & \\
(0,0) & & (1,0) & & (2,0) & & \cdots
\end{array}$$

However if both $\sum a_n$, $\sum b_n$ converge but not absolutely converge, then c_n is not always convergent. See Rudin examples.

Theorem 230. *$\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, and at least one of them is absolutely convergent, and put*

$$c_k = \sum_{n=0}^{k} a_n b_{k-n}$$

then $\sum_{k=0}^{\infty} c_k$ converges to AB.

This is not a very deep result, but the proof technique is useful.

Proof. The proof will stuck with finite sums, so at each stage no issue of swapping infinite sum rises.

Assume $\sum a_n$ converges absolutely. Consider partial sum of c_k

$$\sum_{k=0}^{N} c_k = \sum_{k=0}^{N}\sum_{n=0}^{k} a_n b_{k-n}$$

Graphically on the (k, n) plane it looks like

$$\begin{array}{cccc}
 & & & (3,3) \\
 & & & \uparrow \\
 & & (2,2) & (3,2) \\
 & & \uparrow & \uparrow \\
 & (1,1) & (2,1) & (3,1) \\
 & \uparrow & \uparrow & \uparrow \\
(0,0) & (1,0) & (2,0) & (3,0)
\end{array}$$

We now swap sum

$$\begin{array}{ccccccc}
 & & & & & & (3,3) \\
 & & & & (2,2) & \rightarrow & (3,2) \\
 & & (1,1) & \rightarrow & (2,1) & \rightarrow & (3,1) \\
(0,0) & \rightarrow & (1,0) & \rightarrow & (2,0) & \rightarrow & (3,0)
\end{array}$$

$$\begin{aligned}
\sum_{k=0}^{N}\sum_{n=0}^{k} a_n b_{k-n} &= \sum_{n=0}^{N}\sum_{k=n}^{N} a_n b_{k-n} \\
&= \sum_{n=0}^{N}\sum_{l=0}^{N-n} a_n b_l \text{ set } l = k - n \\
&= \sum_{n=0}^{N} a_n B_{N-n} \text{ set } B_n = \sum_{k=0}^{n} b_k \\
&= \sum_{n=0}^{N} a_n(\beta_{N-n} + B) \text{ set } \beta_n = B_n - B \\
&= \sum_{n=0}^{N} a_n \beta_{N-n} + B\sum_{n=0}^{N} a_n
\end{aligned}$$

The second term goes to BA, which is what we want, so we'd better have the first term goes to 0.

Fix $\epsilon > 0$ if $n \geq N_\epsilon$ $|\beta_n| < \epsilon$, then

$$\sum_{n=0}^{N} a_n \beta_{N-n} = \sum_{n=0}^{N} a_{N-n} \beta_n$$

For $N > N_\epsilon$

$$\left|\sum_{n=0}^{N} a_{N-n}\beta_n\right| \leq \left|\sum_{n=0}^{N_\epsilon - 1} a_{N-n}\beta_n\right| + \sum_{n=N_\epsilon}^{N} |a_{N-n}|\epsilon$$

Taking $N \to 0$, first term on the right goes to 0 because it is sum of finitely many 0's, for $a_{N-n} \to 0$. The second term goes to 0 for $\sum_{n=N_\epsilon}^{N} |a_{N-n}|$ is the tail of absolutely convergent $\sum a_n$, hence it's bounded.

□

Another theorem about product of series. The proof require uniform convergent, so we will postpone the proof to the end of semester.

Theorem 231. *(Abel) $\sum a_n \to A$, $\sum b_n \to B$, $\sum c_n \to C$, and $c_k = \sum_{n=0}^{k} a_n b_{k-n}$, then*

$$C = AB$$

3.8 Rearrangement of Series

Definition 232. rearrangement $\exists$

$$\begin{aligned} \mathbb{N} &\to \mathbb{N} \\ k &\mapsto n_k \end{aligned}$$

bijection, we call $\sum_{k=0}^{\infty} a_{n_k}$ a rearrangement of $\sum_{n=0}^{\infty} a_n$.

Theorem 233. *$a_n \in \mathbb{R}$, $\sum_{n=0}^{\infty} a_n$ convergent but no absolutely. Pick any $\alpha \leq \beta \in [0, \infty]$, $\exists$ a rearrangement n_k s.t.*

$$\limsup_{N\to\infty} \sum_{n=0}^{N} a_n = \beta \text{ and } \liminf_{N\to\infty} \sum_{n=0}^{N} a_n = \alpha$$

The proof is tedious, and not very enlightening.

If pick $\alpha = \beta$, then

Corollary 234. *A convergent but not absolutely series can go to anything.*

An easier theorem

Theorem 235. *If $\sum a_n$ is absolutely convergent, and $\sum a_n \to A$, then $\exists$ a rearrangement n_k s.t.*

$$\sum_{k=0}^{\infty} a_{n_k} \to A$$

4 Functions: Continuity, Differential, and Integral

4.1 Limit of functions

Lecture 15 (7/18/13)

metric spaces (X, d_X), (Y, d_Y), $D \subset X$, f function

$$f : D \to Y$$

Definition 236. p is a limit point of D L is a limit iff $\forall\, \epsilon > 0\ \exists\, \delta_\epsilon > 0$ s.t.

$$\forall\, x \in D\ 0 < d_X(x,p) < \delta_\epsilon \implies d_Y(f(x), L) < \epsilon$$

Remark 237. (i) sometimes we just say $p \in \bar{D}$, i.e. including isolated points. Whether we consider limits at isolated points depend on situations. E.g. in corollary 239 below, not to include isolated points, in the definition of continuity we do include isolated points. (ii) In the definition "$0 < d_X(x,p)$" is important, it says $p \neq x$ hence the limit is never evaluated at the point x.

Proposition 238.

$$\lim_{x \to p} f(x) = L$$

iff $\forall$ p_n sequences in D s.t. $p_n \to p$ and $p_n \neq p$, implies $f(p_n) \to L$.

Here $f(p_n)$ is too a sequence. This proposition is very useful in complex analysis, where it's often convenient to replace the limit by a sequence.

Proof. Assume $\lim_{x \to p} f(x) = L$ and $p_n \to p$ and $p_n \neq p$.

By definition $\exists\, N_\epsilon \in N$ s.t. $d(p_n, p) < \delta_\epsilon$ if $n \geq N_\epsilon$, then $d(f(p_n), L) < \epsilon$, i.e. $f(p_n) \to L$.

Conversely assume that if $p_n \to p$ and $p_n \neq p$, implies $f(p_n) \to L$, but the limit $\lim_{x \to p} f(x)$ is not L. (We say "not L" to mean that the limit is something else or may not exist.)

Hence $\exists\, \epsilon > 0$ s.t. $\forall\, \delta > 0\ \exists\, x_\delta \in D$ s.t.

$$0 < d_X(x_\delta, p) < \delta \text{ but } d_Y(f(x_\delta), L) \geq \epsilon$$

Set $\delta = \frac{1}{n}$ to make x_δ a sequence, i.e.

$$0 < d_X(x_n, p) < \delta \text{ but } d_Y(f(x_n), L) \geq \epsilon$$

so $x_n \to p$ and $x_n \neq p$ but $\lim_{n\to\infty} f(x_n)$ is not L. $\square$

Corollary 239. *1) the limit is unique*

2) if $\lim_{x\to p} f(x) = L$ *and* $\lim_{x\to p} g(x) = M$*, then*

$$\lim_{x\to p} (f+g)(x) = L + M$$

$$\lim_{x\to p} (f \cdot g)(x) = LM$$

$$\lim_{x\to p} \frac{f}{g}(x) = \frac{L}{M} \textit{ if } M \neq 0$$

Proof. use sequential limit. $\square$

In analysis II, we will study limit of functions of higher dimension. In high dimension, it's hard to give an affirmative definition like definition 236. We only have the negative criterion

Definition 240. $f : D \to \mathbb{R}^k$, $D \subset \mathbb{R}^n$, $\vec{a}$ limit point of D. If $\exists\, \vec{r}_1(t)$, $\vec{r}_2(t)$ two continuous curves in D passing through $\vec{a}$ when $t \to t_0$ and s.t.

$$\lim_{t\to t_0} f(\vec{r}_i(t)) = L_i$$

and $L_1 \neq L_2$, then the limit doesn't exist.

4.2 Continuity

Definition 241. $p \in D$ domain of f, including isolated point, f is continuous at p iff $\forall\, \epsilon > 0\ \exists\, \delta_\epsilon > 0$ s.t. $\forall\, x \in D$

$$d_X(x, p) < \delta_{\epsilon,x} \implies d_Y(f(x), f(p)) < \epsilon$$

Remark 242. Notice in the definition "$d_X(x,p) < \delta_\epsilon$" does not say $x \neq p$, which is a subtle difference between the definition of limit to accommodate isolated points. If p is a limit point of D and $p \in D$, then we can say f is continuous at p iff

$$\lim_{x \to p} f(x) = f(p)$$

By our definition f is always continuous at isolated points. Why do we want to include isolated point? Because that will make our $\epsilon - \delta$ definition of continuity from Calculus compatible with the definition of continuity from Topology, i.e. the following proposition.

Proposition 243. *$f : X \to Y$ continuous $\forall\, x \in D$ iff $\forall\, U \subset Y$ open, then $f^{-1}(U)$ is open in X.*

Here no mention of distance.

Suppose we look at f on topology space

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

$$f : D = (\mathbb{R}, d) \to (\mathbb{R}, \text{euclidean})$$

By the proposition above, every function f is continuous everywhere, because every subset of D is open. Interestingly every points in D is isolated. That is why we include isolated points.

Proof. Suppose that $\forall\, U \subset Y$ open, $f^{-1}(U)$ is open in X. Pick $p \in X$, fix $\epsilon > 0$

$$d(f(x), f(p)) < \epsilon \iff f(x) \in B_{f(p),\epsilon}$$

In particular $f^{-1}(B_{f(p),\epsilon})$ is open in X and $p \in f^{-1}(B_{f(p),\epsilon})$, then p is an interior point, so

$$\exists\, \delta_\epsilon > 0 \text{ s.t. } B_{p,\delta_\epsilon} \subset f^{-1}(B_{f(p),\epsilon})$$

That is $x \in B_{p,\delta_\epsilon} \implies f(x) \in B_{f(p),\epsilon}$.

Conversely assume the $\epsilon - \delta$ statement of continuity is true. Let U open, if $f^{-1}(U) = \varnothing$ done. Otherwise pick $p \in f^{-1}(U)$, prove p is interior.

$f(p) \in U$ is interior, $\exists\, \epsilon > 0$ s.t. $B_{f(p),\epsilon} \subset U$. Then $\exists\, \delta_\epsilon > 0$ s.t. $\forall\, x \in B_{p,\delta_\epsilon}, f(x) \in B_{f(p),\epsilon} \subset U \implies x \in f^{-1}(U)$, then

$$B_{p,\delta_\epsilon} \subset f^{-1}(U) \implies p \text{ is interior}$$

$\square$

Next we show composition of continuous functions is continuous.

Proposition 244. *g is continuous at x_0 and f is continuous at $g(x_0)$, then*

$$f(g(x)) \text{ is continuous at } x_0$$

Proof.

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ x_0 & \mapsto & g(x_0) & \mapsto & f(g(x_0)) \end{array}$$

$\forall\, \epsilon > 0$, $\exists\, \delta_{\epsilon>0}$ s.t.

$$d_Y(y, g(x_0)) < \delta_\epsilon \implies d_Z(f(y), f(g(x_0))) < \epsilon$$

$\exists\, \eta_\epsilon > 0$ s.t.

$$d_X(x, x_0) < \eta_\epsilon \implies d_Y(g(x), g(x_0)) < \delta_\epsilon$$

Combining the two. □

Proposition 245. *Sum, product, quotient of continuous functions are continuous.*

Corollary 246. *Polynomials are continuous.*

Proof. first show $f = x$ is continuous, then apply sum, product. □

Proposition 247. *f is continuous iff the per-image of every closed is closed.*

Proof. It follows from

$$f^{-1}(U^c) = \left(f^{-1}(U)\right)^c$$

□

Note 248. Not true $f(U^c) = (f(U))^c$, because although $U \cap U^c = \varnothing$, $f(U^c) \cap (f(U^c))^c \neq \varnothing$. These are true: $K \subset f^{-1}f(K)$, $K \supset ff^{-1}(K)$.

Image of continuous functions of open sets is not necessary open, e.g.

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

$f(\mathbb{R}) = [0, \infty)$ is not open.

If f is continuous and invertible, f^{-1} is not necessary continuous. Because the preimage of f^{-1} is the image of f and $f(\text{open}) \neq \text{open}$. E.g.

$$\begin{array}{ccc} D = [0, 2\pi) & \xrightarrow{f} & S^1 \\ \theta & \mapsto & (\cos\theta, \sin\theta) \end{array}$$

S^1 denotes unit circle. f is invertible when $[0, 2\pi)$ means

$$\mathbb{R}/\sim$$

where $\sim$ defines an equivalence $x, y \in \mathbb{R}$, $x \sim y \iff x - y = 2\pi n$ for some $n \in \mathbb{N}$.

In particular $[0, 1)$ is open in D, but $f[0, 1)$ is not open on C^1.

Furthermore:

Definition 249. If f is invertible and continuous, and f^{-1} is also continuous, we call it homeomorphism.

Not to confuse with homomorphism, which means preserve some algebraic structure.

However image of compact/connected is compact/connected.

Proposition 250. *f continuous, $f : X \to Y$*

1) if $K \subset X$ compact, then $f(K)$ is compact.

2) if $C \subset X$ connected, then $f(C)$ is connected.

Proof. 1) consider $f(K) \subset \cup_{\alpha \in A} V_\alpha$ open cover

$$K \subset f^{-1} f(K) \subset f^{-1}\left(\cup V_\alpha\right) = \cup f^{-1}(V_\alpha)$$

$f^{-1}(V_\alpha)$ is open because f is continuous. K compact, so

$$K \subset \bigcup_{i=1}^{n} f^{-1}(V_\alpha)$$

then

$$f(K) \subset f\left(\bigcup_{i=1}^{n} f^{-1}(V_\alpha)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(V_\alpha)\right) \subset \bigcup_{i=1}^{n} V_\alpha$$

2) Like compactness, connectedness is absolute notion. We can restrict co domain,

$$f : C \quad \to \quad f(C)$$

so f is surjective. Assume $f(C)$ is not connected, i.e. $\exists\, A, B$ open $\neq \varnothing$, $A \cap B = \varnothing$, $f(C) = A \cup B$.

Then $f^{-1}(A)$, $f^{-1}(B) \neq \varnothing$, and $f^{-1}(A) \cup f^{-1}(B) = C$ because f is surjective. $f^{-1}(A), f^{-1}(B)$ open, because f is continuous. $f^{-1}(A), f^{-1}(B)$ disjoint, because A, B are disjoint. So C is not connected. □

The following proposition has two very important consequences.

Proposition 251. *(Extreme Value Theorem) X compact, $f : X \to \mathbb{R}$ continuous, then f has maximum and minimum.*

Proof. $f(K)$ is compact in $\mathbb{R}$, so it is closed and bounded in $\mathbb{R}$. Bounded implies

$$\exists\, \alpha = \sup f(x) \qquad \beta = \inf f(x)$$

Closed implies

$$\alpha, \beta \in \{f(x)\}$$

□

Proposition 252. *(Intermediate Value Theorem) $f : [a, b] \to \mathbb{R}$ continuous, and assume $f(a) < f(b)$, then*

$$\forall\, c, f(a) < c < f(b)\ \exists\, x \in (a, b)\ s.t. f(x) = c$$

Proof. $\{f([a, b])\}$ is connected in $\mathbb{R}$, so it is an interval, so $\forall\, c\ f(a) < c < f(b)$, $\exists\, x$, $f(x) = c$. □

Lecture 16 (7/23/13)

Definition 253. $f : X \to Y$ is uniformly continuous on X iff $\forall\, \epsilon > 0$ $\exists\, \delta_\epsilon$, $\forall\, x, y \in X$

$$d_X(x, y) < \delta_\epsilon \implies d_Y(f(x), f(y)) < \epsilon$$

Remark 254. uniform continuous $\implies$ continuous at $\forall\, x \in X$.

Proposition 255. *X compact and f continuous at $\forall\, x \in X$, then f is uniformly continuous.*

Proof. Fix $\epsilon > 0$ $\forall\, x \in X$ find $\delta_x > 0$ s.t.

$$d(x, y) < \delta_x \implies d(f(x), f(y)) < \epsilon$$

$\forall\, x \in X$ let $B_{x,\frac{\delta_x}{2}} = V_x$, $X \subset \cup_{x\in X} V_x$ compact implies

$$X \subset \bigcup_{i=1}^{n} V_{x_i}$$

Let $\delta = \min\{\frac{\delta_{x_1}}{2}, ..., \frac{\delta_{x_n}}{2}\}$, want to prove $\forall\, x, y\; d(x,y) < \delta \implies d(f(x), f(y)) < \epsilon$.

$\exists\, V_{x_{\bar{i}}}$s.t. $x \in V_{x_{\bar{i}}}$, i.e. $d(x, x_{\bar{i}}) < \frac{\delta_{x_{\bar{i}}}}{2} < \delta_{x_{\bar{i}}} \implies d(f(x), f(x_{\bar{i}})) < \epsilon$. More

$$d(y, x_{\bar{i}}) \leq d(y,x) + d(x, x_{\bar{i}}) < \delta + \frac{\delta_{x_{\bar{i}}}}{2} \leq \delta_{x_{\bar{i}}} \implies d(f(y), f(x_{\bar{x}})) < \epsilon$$

We have shown

$$\forall\, x, y\; d(x,y) < \delta \implies d(f(x), f(y)) < 2\epsilon$$

□

What kind of functions are continuous but not uniformly continuous?

Something with vertical asymptote. E.g.

$$\frac{1}{x}$$

on $(0,1]$ is continuous but not uniformly continuous. While on $[\epsilon, 1]$ $\forall\, 0 < \epsilon < 1$, $\frac{1}{x}$ is uniformly continuous. That is all we will say about uniformly continuous, later we will study again.

4.3 Discontinuities

We now study monotonic function, the simplest non-continuous function, because they have very simple singularities, i.e. jump.

Definition 256. *$f : D \to \mathbb{R}$ $D \subset \mathbb{R}$ f is increasing iff $x < y \implies f(x) \leq f(y)$.*

Notice we allow $f(x) = f(y)$, the reason will become apparent soon.

Definition 257. *a is a limit point of the domain, limit from the right, denote*

$$\lim_{x\to a^+} f(x) = L$$

means $\forall\, \epsilon > 0\; \exists\, \delta_\epsilon > 0$ s.t. $\forall\, x \in D\; a < x < a + \delta_\epsilon \implies |f(x) - L| < \epsilon$.

Definition 258. Limit from the left

$$\lim_{x \to a^-} f(x) = L$$

means $\forall\, \epsilon > 0\ \exists\, \delta_\epsilon > 0$ s.t. $\forall\, x \in D\ a - \delta_\epsilon < x < a \implies |f(x) - L| < \epsilon$.

Remark 259. $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Definition 260. f is right continuous at x_0 iff

$$\lim_{x \to a^+} f(x) = f(x_0)$$

we denote $\lim_{x \to a^+} f(x) =: f(x_0+)$

Similar define left continuous.

Definition 261. f has a discontinuity of first type at x_0 iff

$$f(x_0+) \neq f(x_0-)$$

Definition 262. If $f(x_0+)$ or $f(x_0-)$ doesn't exist, then x_0 is discontinuity of the 2nd type.

Example 263. Characteristic of rationals

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q} \backslash \mathbb{R} \end{cases}$$

discontinuous everywhere of 2nd type

Example 264.

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

continuous at 0, discontinuous everywhere else of 2nd type.

Example 265.

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

continuous everywhere except 0, discontinuous at 0 of 2nd type.

Example 266.

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

continuous everywhere. Later we'll see it is not differentiable at 0, and

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable everywhere.

Proposition 267. *$f(x)$ is increasing, x_0 is a point in the domain, then*

$$\sup_{t<x_0} f(t) = f(x_0-) \leq f(x_0) \leq f(x_0+) = \inf_{t>x_0} f(t)$$

Proof.

$$\alpha = \sup_{t<x_0} f(t)$$

$\forall\, \epsilon > 0$ $\alpha - \epsilon$ is not an upper bounded, so $\exists\, x_\epsilon$ s.t. $x_\epsilon < x_0$

$$\alpha - \epsilon < f(x_\epsilon) \leq \alpha <$$

If $x_\epsilon < x < x_0$, then

$$\alpha - \epsilon < f(x_\epsilon) \leq f(x) \leq \alpha \implies \alpha = f(x_0-)$$

Similar prove the other inequalities. □

Remark 268. 1) if $f(x)$ is monotonic, all the discontinuities are of 1st type.

2) If f is increasing, $x < y \implies f(x+) \leq f(y-)$. Proof

$$f(x+) = \inf_{x<t} f(t) = \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = \sup_{t<y} f(t) = f(y-)$$

Second and second from the last equalities are due to the fact f is increasing.

By the remark, we get the following proposition

Proposition 269. *$f : D \to \mathbb{R}$ monotonic*

$$E = \{x \in D \ s.t. \ f(x-) \neq f(x+)\}$$

is at most countable.

Proof. $\exists\, j$ injective
$$j : E \to \mathbb{Q}$$
by $\forall\, x \in E$ choosing a point $j(x)$ s.t.
$$f(x-) < j(x) < f(x+)$$
$x_1, x_2 \in E$, $x_1 < x_2$, then
$$f(x_1-) < j(x_1) < f(x_1+)$$
$$f(x_2-) < j(x_2) < f(x_2+)$$
By remark
$$j(x_1) < f(x_1+) \leq f(x_2-) < j(x_2)$$
hence j is injective. □

4.4 Derivative

Definition 270. $f : D \to \mathbb{R}$, $x_0 \in D$ x_0 is a limit point of D, the derivative at x_0 if it exists
$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
or equivalently
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Here is some subtlety. As defined in Rudin, derivative will work on the end points of $D = [a, b]$. But more commonly people only take D open.

Remark 271*.* Complex derivative $f : D \to \mathbb{C}$, D open in $\mathbb{C}$ and $z_0 \in D$
$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Remark 272*.* Hence derivative over $\mathbb{R}$, $\mathbb{C}$ are essential the same. But this does not work in dimension higher than 2. Hence $h \in \mathbb{C}$ still works, because $\mathbb{C}$ is a field, i.e. division by complex numbers makes sense. cf note 48.

derivative defines a function. $f : D \to \mathbb{R}/\mathbb{C}$ and $D \in \mathbb{R}/\mathbb{C}$ open and f is differentiable $\forall\, x \in D$,
$$\begin{aligned} f' : D &\to \mathbb{R}/\mathbb{C} \\ x &\mapsto f'(x) \end{aligned}$$

Remark 273. f is differentiable at $x_0 \implies f$ is continuous at x_0. Proof

$$\lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}(x-x_0) = f'(x_0)\cdot 0 = 0$$

Remark 274. If $f(x)$ differentiable at x_0, then $\exists$ a function η s.t.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \eta(x-x_0)$$

and

$$\lim_{x\to x_0}\frac{\eta(x-x_0)}{x-x_0} = 0$$

Proof: $\lim_{x\to x_0}\frac{\eta(x-x_0)}{x-x_0} = \lim_{x\to x_0}\left[\frac{f(x-x_0)}{x-x_0} - f'(x_0)\right] = 0$.

This is 1st order Taylor.

We study some properties of derivatives

Lecture 17 (7/25/13)

Proposition 275. *f, g over $\mathbb{R}$ or $\mathbb{C}$ and differentiable at x_0, then*

1) $(f+g)(x)$ is differentiable at x_0.

2) $(f\cdot g)(x)$ is differentiable at x_0.

3) $\frac{f}{g}(x)$ is differentiable at x_0, if $g(x_0)\neq 0$.

Proof. 3)

$$\begin{aligned}
\lim_{x\to x_0}\frac{\frac{f(x)}{g(x)}-\frac{f(x_0)}{g(x_0)}}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x)}{(x-x_0)g(x)g(x_0)} \\
&= \lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x_0)+f(x_0)g(x_0)-f(x_0)g(x)}{(x-x_0)g(x)g(x_0)} \\
&= \lim_{x\to x_0}\frac{g(x_0)\frac{f(x)-f(x_0)}{x-x_0}-f(x_0)\frac{g(x)-g(x_0)}{x-x_0}}{g(x)g(x_0)} \\
&= \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}
\end{aligned}$$

we used f, g are differentiable at x_0 and g is continuous at x_0. □

Remark 276. Derivative is linear

$$(af(x)+bf(x))' = af'(x)+bf'(x)$$

Proposition 277. *(chain rule) $g(x)$ is differentiable at x_0 and $f(y)$ is differentiable at $g(x_0)$ then $f(g(x_0))$ is also differentiable at x_0.*

Proof. (There are many ways to prove this. This way we prove will work even in higher dimension, because we start the proof from the following statement which is actually taken as the definition of derivative in higher dimension.)

$$f(y) = f(g(x_0)) + f'(g(x_0))(y - g(x_0)) + \eta(y - g(x_0))$$

where $\lim_{y \to g(x_0)} \frac{\eta(y-g(x_0))}{y-g(x_0)} = 0$, $\forall\, y$

In particular, set $y = g(x)$

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} &= \lim_{x \to x_0} \left[f'(g(x_0)) \frac{g(x) - g(x_0)}{x - x_0} + \frac{\eta(g(x) - g(x_0))}{x - x_0} \right] \\
&= f'(g(x_0))g'(x_0) + \lim_{x \to x_0} \frac{\eta(g(x) - g(x_0))}{g(x) - g(x_0)} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(g(x_0))g'(x_0) + 0 \cdot g'(x_0) \\
&= f'(g(x_0))g'(x_0)
\end{aligned}$$

Because g is continuous at x_0,

$$\lim_{x \to x_0} \frac{\eta(g(x) - g(x_0))}{g(x) - g(x_0)} = \lim_{g(x) \to g(x_0)} \frac{\eta(g(x) - g(x_0))}{g(x) - g(x_0)} = 0$$

$\square$

In higher dimension

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

chain rule looks like

$$f(g(x))'\big|_{x=x_0} = f'(g(x_0))g'(x_0)$$

RHS are Jacobian matrices, not commutative, must be in this order.

4.5 Applications of Derivatives

So far we see $\mathbb{R}/\mathbb{C}$ are the same, but starting now things, more apparently in mean value theorem, only work for $\mathbb{R}$ not $\mathbb{C}$.

Definition 278. $f : X \to \mathbb{R}$ f has a local maximum at $a \in X$ iff $\exists\, \epsilon > 0$ s.t.

$$\forall\, x \in B_{a,\epsilon}, f(x) \leq f(a)$$

Similarly one can define local minimum. Range of f has to in $\mathbb{R}$ to make sense of maximum/minimum.

Proposition 279. *$f : [a,b] \to \mathbb{R}$ at $c \in (a,b)$ f has a local max or min and f is differentiable at c, then $f'(c) = 0$.*

This proposition tells us in searching for local max or min, one can look for points where $f' = 0$ or where f' don't exist (e.g. end points of an interval), and these points are called critical points.

Proof. c is a local max and $f'(c)$ exists. $\exists\, \epsilon > 0$ s.t. $c-\epsilon < x < c+\epsilon \implies f(x) \leq f(c)$. Since $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, consider

$$\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$$

it is ≤ 0 because $x - c > 0$ and $f(x) - f(c) \leq 0$. Similar

$$\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$$

Hence

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$$

□

We now do mean value theorem. It has two important consequences: Taylor and L'hopital Because MVT is false in $\mathbb{C}$, so is L'hopital. Surprisingly Taylor works in $\mathbb{C}$ and it behaves better in $\mathbb{C}$ than in $\mathbb{R}$, but the proof over $\mathbb{C}$ is very different, not coming from MVT.

Theorem 280. *(Rolle) $f : [a,b] \to \mathbb{R}$ f continuous on $[a,b]$ differentiable on (a,b) and $f(a) = f(b)$, then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$.*

Proof. f continuous on compact $\implies$ $\exists$ max/min. If max=min $\implies$ $f(x)$ is constant, so $f'(x) = 0\ \forall\, x$. Otherwise min<max, so a, b can be both min and max, hence $\exists\, c \in (a,b)$ to be min or max, then $f(c) = 0$. □

Theorem 281. *(MVT) f, g continuous on closed interval $[a,b]$ differentiable on (a,b) $\exists\, c \in (a,b)$ s.t.*

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c)$$

When $g(x) = x$, we get the common picture of MTV,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

hence there is a point in between of a, b that has the slope same as the line connecting a, b.

Proof. Let

$$h(t) = (f(b) - f(a))\, g(t) - (g(b) - g(a))\, f(t)$$

then $h(a) = h(b)$, $h(t)$ is continuous and differentiable. So by Rolle, $h'(c) = 0$ for some c. □

Remark 282. $f(x)$ continuous on $[a, b]$ differentiable on (a, b)

1) $f'(x) = 0 \,\forall\, x \in (a, b) \implies f$ is constant.

2) $f'(x) \geq 0 \,\forall\, x \in (a, b) \implies f$ is increasing

3) $f'(x) \leq 0 \,\forall\, x \in (a, b) \implies f$ is decreasing

Proof: 3) Let $a \leq x_1 \leq x_2 \leq b$, $x_1 < c < x_2$

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \leq 0$$

because $f'(c) \leq 0$ and $x_2 - x_1 > 0$. (If we have $f'(x) < 0 \,\forall\, x \in (a, b)$ strict inequality, then f is strictly decreasing.)

Example 283. $f(x)$ is continuous, differentiable, but $f'(x)$ is not continuous

$$f = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

what is f'? For $x \neq 0$, follow the usual scheme, for $x = 0$, use definition. Assume $f'(0)$ exists

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = 0 \end{aligned}$$

So

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f'(x)$ has discontinuity of 2nd type at 0. The actually reflects a general fact: If f continuous and differentiable, then f' cannot have jump of 1st type.

Proposition 284. *$f(x)$ on $D \subset \mathbb{R}$, $[a,b] \subset D$ and f is differentiable on $[a,b]$ and $f'(a) < f'(b)$, then*

$$f'(a) < \lambda < f'(b) \implies \exists\, c \in (a,b) \text{ s.t. } f'(c) = \lambda$$

Proof. Let $h(t) = f(t) - \lambda t$

$$h'(a) = f'(a) - \lambda < 0$$

$$h'(b) = f'(b) - \lambda > 0$$

Our goal is to find c s.t. $h'(c) = 0$. We cannot use IMV directly on $h'(t)$ because we don't know it is continuous. We cannot use Rolle on $h(t)$ either, because we don't know if $\exists\, a', b'$ s.t. $h(a') = h(b')$.

Fix $\epsilon > 0$, s.t. $h'(a) + \epsilon < 0$, then

$$h'(a) = \lim_{x \to a^+} \frac{h(x) - h(a)}{x - a} < 0$$

gives

$$h'(a) - \epsilon < \frac{h(x) - h(a)}{x - a} < h'(a) + \epsilon < 0$$

if $a < x < a + \delta_\epsilon$. (this is equivalently to say that if $h(t)$ is continuous at a and $h(a) < 0$, then $\exists$ neighborhood of a that $h(x) < 0$.)

Hence in the interval $(a, a + \delta_\epsilon)$, h is downward slopping. Similar one shows that h is upward slopping near b. Because h is continuous on a compact, it has absolutely minimum, which cannot be a, b, so $\exists\, c$ being the minimum, i.e.

$$h'(c) = 0$$

$\square$

Proposition 285. *(Taylor) $f(x)$ continuous on $[a,b]$ and differentiable $(n+1)$ times on (a,b). Fix $\alpha \in (a,b)$ $\forall\, x \in [a,b]$ $\exists\, z$ between x, α ($x < z < \alpha$ or $\alpha < z < x$) s.t.*

$$f(x) = P_{n,\alpha}(x) + \frac{f^{(n+1)}(z)}{(n+1)!}(x - \alpha)^{n+1}$$

where

$$P_{n,\alpha}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x - \alpha)^k$$

Remark 286. Here if we further assume $\left|f^{(n+1)}(x)\right| < M \;\forall\, x$, then just like before, cf remark 274 the error term

$$R(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-\alpha)^{n+1}$$

satisfies

$$\lim_{x\to\alpha} \frac{R(x)}{(x-\alpha)^n} = 0$$

We need the additional assumption to do that because z is a function of x, α.

Clearly $\deg P_{n,\alpha} \leq n$

$$\begin{aligned} P_{0,\alpha} &= f(\alpha) \\ P_{1,\alpha} &= f(\alpha) + f'(\alpha)(x-\alpha) \\ &\vdots \end{aligned}$$

and

$$P_{n,\alpha}^{(k)}(\alpha) = f^{(k)}(\alpha) \text{ if } k \leq n \tag{4.1}$$

i.e. $P_{n,\alpha}$ has the same 1st, 2nd, ..., nth derivatives as $f(z)$, so Taylor is a good approximation of $f(x)$ near α.

Proof. Fix α and x, find z,

$$f(x) = P_{n,\alpha}(x) + M(x-\alpha)^{n+1}$$

this defines M.

Put

$$g(t) = f(t) - P_{n,\alpha}(t) - M(t-\alpha)^{n+1}$$

so $g(x) = g(\alpha) = 0$. WLOG assume $x < \alpha$, $\exists\, z$ s.t. $x < z_1 < \alpha$, $g'(z_1) = 0$

By (4.1), $g^{(k)}(\alpha) = 0$ if $k \leq n$, hence $\exists\, z_2$ s.t. $z_1 < z_2 < \alpha$, and $g''(z_2) = 0$, then apply Rolle again, $\exists\, z_3$ s.t. $z_2 < z_3 < \alpha$, and $g'''(z_3) = 0$, continuous on till $\exists\, z_n$ s.t. $z_{n-1} < z_n < \alpha$ and $g^{(n)}(z_n) = 0$, then

$$\exists\, z_{n+1} \text{ s.t. } z_n < z_{n+1} < \alpha \text{ and } g^{(n+1)}(z_{n+1}) = 0$$

$z_{n+1} \equiv z$ the candidate we're looking for, because

$$g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - M(n+1)!$$

so

$$M = \frac{f^{(n+1)}(z_{n+1})}{(n+1)!}$$

$\square$

Lecture 18 (7/30/13)

Last time, we showed Taylor for f that is $(n+1)$th differentiable. What if $f \in C^\infty(D)$, i.e. f can be differentiated arbitrary many times at every $x \in D$, like sine, cosine, exponential, etc.

$$f(x) \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}$$

Is the series convergent? This can be answered by root test (note: many cases radius of convergent is 0)

More important question: If it is convergent, is it equal to $f(x)$?

Example 287. (bump function) it's power series convergent everywhere but not equal to $f(x)$.

$$f(x) = \begin{cases} 0 & |x| \geq 1 \\ e^{\frac{1}{x^2-1}} & |x| < 1 \end{cases}$$

one can check $f(x)$ and $f'(x)$ are continuous at ± 1.

Now Taylor of f at ± 1, by induction

$$f^{(n)}(\pm 1) = 0 \quad \forall n \in \mathbb{N}$$

so

$$\text{Taylor of } f = \sum \frac{0}{n!}(x \mp 1)^n \equiv 0$$

The radius of convergence is ∞.

This kind of bad behavior only happens to Taylor in $\mathbb{R}$. $\mathbb{C}$ functions have Taylor and behave well.

Proposition 288. *$f : \Omega \to \mathbb{C}$ $\Omega \subset \mathbb{C}$ open. The following three are equivalent*

1) f is complex differentiable, i.e. 1st derivative exists

2) nth derivative exists $\forall n \in \mathbb{N}$

3) $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$ within its radius of convergence.

If we replace x by $z \in \mathbb{C}$ for the bump function, i.e.

$$f(z) = \begin{cases} 0 & |z| \geq 1 \\ e^{\frac{1}{z^2-1}} & |z| < 1 \end{cases}$$

then $f(z)$ is not differentiable at ± 1, at which function has the worst kind of singularity: essential singularity.

Now we understand why real and complex analysis courses divorce after some points, because they behave differently.

Proposition 289. *(de L'hopital) $f(x)$, $g(x)$ defined on (a, b) continuous and if for some $\alpha \in (a, b)$*

$$\textit{either (1)} \lim_{x \to \alpha} f(x), \lim_{x \to \alpha} g(x) \to 0 \;\; \textit{or (2)} \;\; \lim_{x \to \alpha} g(x) \to \infty$$

and $\lim_{x \to \alpha} \frac{f'(x)}{g'(x)} = L$, *then*

$$\lim_{x \to \alpha} \frac{f(x)}{g(x)} = L$$

In Calculus we are told that the intermediate form is $0/0$ or ∞/∞, but actually for the second cases we only need $g \to \infty$.

Proof.

$$\lim_{x \to \alpha^+} \frac{f'(x)}{g'(x)} = L$$

(Here we do one side at a time, because order of the interval is important as we're going to apply MVT.)

Fix $\epsilon > 0$ $\exists \, \delta_\epsilon > 0$ s.t.

$$\alpha < t < \alpha + \delta_\epsilon \implies L - \epsilon < \frac{f'(t)}{g'(t)} < L + \epsilon$$

Applying MVT for $\alpha < x < y < \alpha + \delta_\epsilon$, $\exists \, x < t < y$

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)}$$

(notice here we need to use the general form of MVT. If one uses the less general version, i.e.

$$f(x) - f(y) = f'(t)(x - y) \qquad g(x) - g(y) = g'(s)(x - y)$$

then combining to

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(s)}$$

this is not good, because $t \neq s$)

Hence

$$\alpha < x < y < \alpha + \delta_\epsilon \implies L - \epsilon < \frac{f(x)-f(y)}{g(x)-g(y)} < L + \epsilon \tag{4.2}$$

Case 1: $\lim_{x\to\alpha} f(x) = 0$, $\lim_{x\to\alpha} g(x) = 0$

Fix y, take limit of (4.2) as $x \to a^+$

$$L - \epsilon < \frac{f(y)}{g(y)} < L + \epsilon \text{ if } \alpha < y < \alpha + \delta_\epsilon$$

That is

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

(Here we take $x \to a^+$ first, then take $y \to a^+$, so MVT still holds)

Case 2: $\lim_{x\to\alpha} g(x) = \infty$

From (4.2), fix y, $\exists\, c$, $\alpha < c < y$ s.t. if $\alpha < x < c < y$, then

$$g(x) - g(y) > 0 \text{ and } g(x) > 0$$

So the 2nd inequality in (4.2) gives

$$\frac{f(x)}{g(x)} < (L+\epsilon)\frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

$\exists\, d$ s.t. $\alpha < d < c$ s.t. $\alpha < x < d$

$$\left|\frac{g(y)}{g(x)}\right| < \epsilon \quad \left|\frac{f(y)}{g(x)}\right| < \epsilon$$

Then

$$\left|\frac{g(x)-g(y)}{g(x)}\right| = \left|1 - \frac{g(y)}{g(x)}\right| < 1 + \epsilon$$

so

$$\frac{f(x)}{g(x)} < (L+\epsilon)(1+\epsilon) + \epsilon = L + \epsilon(2 + L + \epsilon) \to L$$

use the 1st inequality in (4.2) gives

$$L + \epsilon(2 - L - \epsilon) < \frac{f(x)}{g(x)}$$

Hence

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

Similarly can show

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = L$$

$\square$

Remark 290. de L'hopital doesn't work over $\mathbb{C}$ or in high dimension. E.g. the following function from $\mathbb{R}\backslash\{0\} \to \mathbb{C}$

$$f(x) = \frac{x}{x + x^2 e^{\frac{i}{x^2}}}$$

$$\lim_{x \to 0} f(x) = \frac{1}{1 + xe^{\frac{i}{x^2}}} = 1$$

Apply L'H

$$\lim_{x \to 0} \frac{1}{1 + 2xe^{\frac{i}{x^2}} - \frac{2i}{x}e^{\frac{i}{x^2}}} = \lim_{x \to 0} \frac{x}{x + 2x^2 e^{\frac{i}{x^2}} - 2ie^{\frac{i}{x^2}}}$$

$$\left| \frac{x}{x + 2x^2 e^{\frac{i}{x^2}} - 2ie^{\frac{i}{x^2}}} \right| \leq \frac{|x|}{\left| \left| x + 2x^2 e^{\frac{i}{x^2}} \right| - 2 \right|} \to 0$$

A few words about derivative of functions like above

$$\begin{aligned} f : [a, b] &\to \mathbb{R}^k \\ x &\mapsto (f_1(x), ..., f_k(x)) \end{aligned}$$

derivative at $x_0 \in (a, b)$ is a vector

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = (f_1'(x_0), ..., f_k'(x_0))$$

i.e. $f(x)$ is differentiable iff all components of $f(x)$ are differentiable.

What happens to MTV for such functions?

Proposition 291. $f : [a, b] \to \mathbb{R}^k$ *continuous and differentiable on* (a, b) $\exists c \in (a, b)$ *s.t.*

$$|f(b) - f(a)| \leq |f'(c)| (b - a)$$

That is not an inequity, so MVT has to be modified, so L'H is not true for higher dimension functions.

Proof. let
$$g(t) = f(t) \cdot (f(b) - f(a))$$
(this is a dot product.) Now $g(t) : [a, b] \to \mathbb{R}$, continuous and differentiable. $\exists\, c \in (a, b)$ s.t.
$$g(b) - g(a) = g'(c)(b - a)$$
Then by Cauchy-Schwartz
$$\begin{aligned} |f(b) - f(a)|^2 &= f'(c) \cdot (f(b) - f(a))\,(b - a) \\ &\leq \left|f'(c)\right| |f(b) - f(a)|\,(b - a) \end{aligned}$$
□

Remark 292. This proposition still holds in higher dimension $f : \mathbb{R}^n \to \mathbb{R}^m$, but require additional assumption: domain to be convex.

4.6 Riemann Integral

Consider
$$f : [a, b] \to \mathbb{R}$$
the Riemann integral theory we do works mostly for bounded domain, and bounded function $|f(x)| \leq M\ \forall\, x \in [a, b]$. Although in Calculus we have seen things like
$$\int_0^1 \frac{1}{\sqrt{x}} dx \qquad \int_1^\infty \frac{1}{x^2} dx$$
either f is not bounded or domain is unbounded. These integrals do not come into our definition naturally, we'll later include them with extended definition: improper integral. However they are automatically included in Lebesgue.

First we partition the domain. We don't require partition of intervals with equal size, there are uncountable many partitions, because $[a, b]$ has uncountably many points and countable union of uncountable is uncountable. This says we cannot take limit on partitions directly, as we'll show, only $\limsup$, $\liminf$ are possible operations.

Definition 293. A partition P of $[a, b]$ is a finite ordered collection of points s.t.
$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

$$\Delta x_i = x_i - x_{i-1}$$

Let

$$U(P,f) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)\Delta x_i \qquad L(P,f) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)\Delta x_i$$

Then clearly

$$U(P,f) \geq -M(b-a) \quad L(P,f) \leq M(b-a)$$

so it makes sense to do sup or inf on them and denote upper/lower integral

$$\sup_P L(P,f) =: \underline{\int_a^b} f(x)dx$$
$$\inf_P U(P,f) =: \overline{\int_a^b} f(x)dx$$

Definition 294. f is Riemann integral iff

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

we say $f \in \mathcal{R}$, and denote the integral as $\int_a^b f(x)dx$.

Example 295. Characteristic function of rationals

$$\chi_{\mathbb{Q}}_{a\leq x\leq b} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

fix P,

$$U(P,\chi_{\mathbb{Q}}) = b-a \text{ and } L(P,\chi_{\mathbb{Q}}) = 0$$

They are independent of P, so

$$\overline{\int_a^b} f(x)dx = b-a \text{ and } \underline{\int_a^b} f(x)dx = 0$$

so $\chi_{\mathbb{Q}} \notin \mathcal{R}$.

Interestingly $\chi_{\mathbb{Q}}$ is Lebesgue integrable. There are things not even Lebesgue integrable. In analysis I, we are unable to show that the if and only if relation between continuity v.s. Riemann integrability.

Proposition. *$f \in \mathcal{R}$ (not Riemann-Stieltjes) iff the set of points where f is discontinuous has measure 0.*

Lecture 19 (8/6/13)

Before we do Riemann, we would like to generalize to Riemann-Stieltjes. In analysis Stieltjes is poorly motivated. Its idea lie in measure theory. What measure do, it assigns a number to set of space.

That's X a set

$$\mu : \mathcal{P}(X) \to [0, \infty]$$

we want μ to be increasing in the sense that

$$A \subset B \implies \mu(A) \leq \mu(B) \tag{4.3}$$

or more generally we want

$$\mu(A \cup B) = \mu(A) + \mu(B) \text{ if } A \cap B = \varnothing \tag{4.4}$$

since we eventually want to take integral, we may just require more generally,

$$\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n) \tag{4.5}$$

One can show indeed (4.5) $\implies$ (4.4) $\implies$ (4.3).

When $X = \mathbb{R}$, one can show that choosing $\alpha(x)$ increasing,

$$\mu([x_i, x_{i+1}]) = \alpha(x_{i+1}) - \alpha(x_i) = \Delta\alpha_i$$

defines a measure on $\mathbb{R}$, although this does not include all possible measures.

We will use $\Delta\alpha_i$ as replacement of the base of the rectangle in integration.

Definition 296. $\alpha(x)$ increasing on $[a, b]$

$$U(P, f, \alpha) = \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i-1}]} f(x)\Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=0}^{n-1} \inf_{x \in [x_i, x_{i-1}]} f(x)\Delta\alpha_i$$

Then

$$m(\alpha(b)-\alpha(a)) \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq M(\alpha(b)-\alpha(a))$$

if $m \leq f \leq M$. Then we too can define

$$\sup_P L(P,f,\alpha) =: \underline{\int_a^b} f(x)d\alpha(x)$$

$$\inf_P U(P,f,\alpha) =: \overline{\int_a^b} f(x)d\alpha(x)$$

Similar we say

Definition 297. $f \in \mathcal{R}(\alpha)$ iff

$$\underline{\int_a^b} f(x)d\alpha(x) = \overline{\int_a^b} f(x)d\alpha(x)$$

Now integrability becomes a little bit more delicate. If f is continuous, then does not matter is Riemann integral or Riemann-Stieltjes. If both f and α have jumps and jump together, we need to make sure they combine in a correct way. If α is differentiable, then Stieltjes is not more general than Riemann, as we will later α is differentiable

$$\int_a^b f d\alpha = \int_a^b f\alpha' d\alpha$$

what happen if α jumps? In Riemann individual points of f don't matter. In measure theory perspective, measure of individual points have 0 measure, but intuitively if α jumps, then the measure of jump is not 0.

We will see many possible scenarios next time.

Lemma 298. *f is $\mathcal{R}(\alpha)$ iff $\forall\, \epsilon > 0\ \exists\, P_\epsilon$ s.t.*

$$U(P_\epsilon,f,\alpha) - L(P_\epsilon,f,\alpha) < \epsilon$$

This is a fundamental tool to prove integrability, we need some intermediate lemmas to prove it.

Definition 299. P_1, P_2 and $P_1 \subset P_2$, then P_2 is finer than P_1.

This gives a partial order of partitions.

Definition 300. P_1, P_2 and $P = P_1 \cup P_2$, P is the common refinement of $P_{1,}P_2$.

Lemma 301. *If $P_1 \subset P_2$, then*

$$U(P_1, f, \alpha) \geq U(P_2, f, \alpha)$$

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha)$$

Proof. P_1, P_2 are finite points, add one point to P_1 to make it closer to P_2, then we see one interval is changing, i.e $U \downarrow$ and $L \uparrow$. $\square$

Lemma 302.

$$\underline{\int_a^b} f(x) d\alpha(x) \leq \overline{\int_a^b} f(x) d\alpha(x)$$

Proof. Pick any P_1, P_2 $P = P_1 \cup P_2$

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$$

Fix P_2, take sup in P_1, get

$$\underline{\int_a^b} f(x) d\alpha(x) \leq U(P_2, f, \alpha)$$

take inf in P_2, get

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$$

$\square$

Now prove the fundamental lemma.

Proof. If $\forall \epsilon > 0 \; \exists P_\epsilon$ s.t.

$$U(P_\epsilon, f, \alpha) - L(P_\epsilon, f, \alpha) < \epsilon$$

then

$$L(P_\epsilon, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P_\epsilon, f, \alpha)$$

so

$$0 \leq \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha \leq U - L < \epsilon \; \forall \epsilon$$

so

$$\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$$

Conversely, fix $\epsilon > 0 \, \exists \, P_1 \;\; P_2$ s.t.

$$\overline{\int_a^b} f d\alpha + \epsilon \geq U(P_1, f, \alpha)$$

$$\underline{\int_a^b} f d\alpha - \epsilon \leq L(P_2, f, \alpha)$$

Take $P = P_1 \cup P_2$, then

$$U(P_1, f, \alpha) \geq U(P, f, \alpha) \text{ and } L(P, f, \alpha) \leq L(P_2, f, \alpha)$$

so

$$U(P_1, f, \alpha) \leq \int_a^b f d\alpha + \epsilon \leq L(P_1, f, \alpha) + 2\epsilon$$

so

$$U - L \leq 2\epsilon$$

□

Proposition 303. *If $f(x)$ is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha) \; \forall \, \alpha(x)$ increasing.*

Proof. Fix $\epsilon > 0$ find P_ϵ s.t.

$$U(P_\epsilon, f, \alpha) - L(P_\epsilon, f, \alpha) < \epsilon$$

and let $M_i = \sup_{x \in [x_i, x_{i+1}]} = f(x) = f(t_i)$, $m_i = \inf_{x \in [x_i, x_{i+1}]} f(x) = f(s_i)$,

$$U(P_\epsilon, f, \alpha) - L(P_\epsilon, f, \alpha) = \sum_{i=0}^{n-1} (M_i - m_i) \Delta\alpha_i$$

f is uniformly continuous $\forall \, \epsilon > 0 \; \exists \, \delta_\epsilon > 0$ s.t.

$$|x - y| < \delta_\epsilon \implies |f(x) - f(y)| < \epsilon$$

Make P_ϵ s.t. $\Delta x_i < \delta_\epsilon \forall \, i$ then since $|t_i - s_i| < \delta_\epsilon$

$$U - L = \sum (f(t_i) - f(s_i)) \Delta\alpha_i < \epsilon \sum \Delta\alpha_i = \epsilon(\alpha(b) - \alpha(a)) \to 0$$

Even in the degenerated case $\alpha =$constant, it is still true. □

f is not necessary to be continuous to be integrable.

Proposition 304. *f is increasing and α is continuous and increasing, then*

$$f \in \mathcal{R}(\alpha)$$

Proof. Fix $\epsilon > 0$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum (M_i - m_i)\Delta\alpha_i$$

$\exists\, \delta_\epsilon$ s.t.

$$|x - y| < \delta_\epsilon \implies |\alpha(x) - \alpha(y)| < \epsilon$$

Make P_ϵ s.t. $\Delta x_i < \delta_\epsilon \implies \Delta\alpha_i < \epsilon$, therefore

$$\sum (M_i - m_i)\Delta\alpha_i \leq \epsilon(f(b) - f(a))$$

□

Rudin uses intermediate value theorem to prove above.

Now show both f and α are not continuous.

Proposition 305. *f has finitely many discontinuities*

$$E = \{x \in [a, b]; f \text{ not continuous at } x\}$$

α is continuous at every $x \in E$, α is increasing, then

$$f \in \mathcal{R}(\alpha)$$

The proof will not work for f that has ∞ many discontinuity.

Proof. Fix $\epsilon > 0$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum (M_i - m_i)\Delta\alpha_i$$

$$E = \{x_1, ..., x_n\}$$

choose v_i, u_i s.t.

1) $x_i \in (v_i, u_i)$

2) $(u_i, v_i) \cap (u_j, v_j) = \varnothing$ if $i \neq j$

3) $\Delta\alpha_i = \alpha(v_i) - \alpha(u_i) = (\alpha(v_i) - \alpha(x_i)) + (\alpha(x_i) - \alpha(u_i)) < \epsilon$

Let

$$K = [a,b] - \bigcup_{i=1}^{n} (u_i, v_i)$$

K is compact so f is uniformly continuous on K, $\exists\, \delta_\epsilon > 0$

$$|x-y| < \delta_\epsilon \implies |f(x) - f(y)| < \epsilon$$

Make P_ϵ s.t. $u_i, v_i \in P_\epsilon$ and if $x \in P_\epsilon$, $x \notin (u_i, v_i)\ \forall\, i$. Hence for (x_j, x_{j+1}) with $x_{j+1} \neq v_i$ for any i, $x_{j+1} - x_j < \delta_\epsilon$.

$i = 0, 1, ..., N-1$, $i \in A$ if

$$(x_i, x_{i+1}) \neq (u_j, v_j) \text{ for any } j$$

$i \in B$ if

$$(x_i, x_{i+1}) = (u_j, v_j) \text{ for some } j$$

Now

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i}^{n-1} (M_i - m_i)\Delta\alpha_i \\ &= \sum_{i \in A} + \sum_{i \in B} \end{aligned}$$

$$\sum_{i \in A} (M_i - m_i)\Delta\alpha_i < \epsilon \sum_{i \in A} \Delta\alpha_i \leq \epsilon(\alpha(b) - \alpha(a))$$

$$\sum_{i \in B} (M_i - m_i)\Delta\alpha_i < \epsilon \sum_{i \in A} (M_i - m_i) \leq \epsilon n 2K$$

where $n = |B| = |E|$ (that is why we need finitely many discontinuities), and $|f| < K$. $\square$

Lecture 20 (8/8/13)

Proposition 306. *$f \in \mathcal{R}(\alpha)$ on $[a,b]$ $m \leq f(x) \leq M\ \forall\, x \in [a,b]$ and $\psi(y)$ is a continuous function on $[m,M]$, then*

$$\psi(f) \in \mathcal{R}(\alpha)$$

Proof. Fix $\epsilon > 0$. ψ uniform continuous $\exists\, \delta_\epsilon > 0$ s.t.

$$|u - v| < \delta_\epsilon \implies |\psi(u) - \psi(v)| < \epsilon$$

and find P_ϵ s.t.

$$U(P_\epsilon, f, \alpha) - L(P_\epsilon, f, \alpha) < \delta_\epsilon^2$$

that is

$$\sum_{i=0}^{n-1}(M_i - m_i)\Delta\alpha_i < \delta_\epsilon^2$$

where

$$M_i = \sup_{x\in[x_i,x_{i+1}]} f(x) \qquad m_i = \inf_{x\in[x_i,x_{i+1}]} f(x)$$

Prove that

$$U(P_\epsilon, \psi(f), \alpha) - L(P_\epsilon, \psi(f), \alpha) < \epsilon$$

let

$$M_i^* = \sup_{x\in[x_i,x_{i+1}]} \psi(f(x)) \qquad m_i^* = \inf_{x\in[x_i,x_{i+1}]} \psi(f(x))$$

case 1 if $M_i - m_i < \delta_\epsilon$, call such $i \in A$, then $M_i^* - m_i^* < \epsilon$

case 2 if $M_i - m_i \geq \delta_\epsilon$, call such $i \in B$, then

$$\delta_\epsilon \sum_{i\in B} \Delta\alpha_i \leq \sum_{i\in B}(M_i - m_i)\Delta\alpha_i \leq \sum_{i=0}^{n-1}(M_i - m_i)\Delta\alpha_i < \delta_\epsilon^2$$

that is

$$\sum_{i\in B} \Delta\alpha_i < \delta_\epsilon$$

so

$$\sum_{i=0}^{n-1}(M_i^* - m_i^*)\Delta\alpha_i < \epsilon(\alpha(b) - \alpha(a)) + 2K\delta_\epsilon$$

$K = \sup_{y\in[m,M]} |\psi(y)|$.

If $\delta_\epsilon > \epsilon$, replace δ_ϵ with ϵ in the uniform continuous of ψ, and any following δ_ϵ becomes ϵ. □

Remark 307. f is integrable, fix $\epsilon > 0$ find P s.t.

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

1) if Q is finer than P,

$$U(Q, f, \alpha) - L(Q, f, \alpha) < \epsilon$$

2) any $t_i, s_i \in [x_i, x_{i+1}]$, $x_i \in P$

$$\sum |f(s_i) - f(t_i)|\Delta\alpha_i < \epsilon$$

because

$$|f(s_i) - f(t_i)| \leq M_i - m_i$$

3) Recall in Calculus we take right or left end point $[x_i, x_{i+1}]$ in approximate the integral. In fact for any $t_i \in [x_i, x_{i+1}]$

$$\left| \sum_{i=0}^{n-1} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \epsilon$$

because

$$L(P, f, \alpha) \leq \sum_{i=0}^{n-1} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha)$$

$$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha)$$

Proposition 308. *1) Linearity $f, g \in \mathcal{R}(\alpha)$, $c, d \in \mathbb{R}$, then $cf + dg \in \mathcal{R}(\alpha)$, and*

$$\int_a^b (cf + dg) d\alpha = c \int_a^b f d\alpha + d \int_a^b g d\alpha$$

2) $f \leq g \implies \int_a^b f d\alpha \leq \int_a^b g d\alpha$

3) $m \leq f \leq M \implies m(b - a) \leq \int_a^b f d\alpha \leq M(b - a)$

4) if f is integrable on $[a, b]$ then f is integrable on $[a, c]$ and $[c, b]$, $a < c < b$,

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

(which f is integrable on any closed interval contained in $[a, b]$

5) linearity in α. If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $\forall c, d \in \mathbb{R}$, then $f \in R(c\alpha_1 + d\alpha_2)$ and

$$\int_a^b f d(c\alpha_1 + d\alpha_2) = c \int_a^b f d\alpha_1 + d \int_a^b f d\alpha_2$$

(this comes from the fact that $d\alpha$, "differential" has linearity.)

6) $f \in \mathcal{R}(\alpha) \implies |f| \in \mathcal{R}(\alpha)$ and

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$$

7) $f, g \in \mathcal{R}(\alpha) \implies f \cdot g \in \mathcal{R}(\alpha)$

Proof. 6)

$$\left|\int_a^b f d\alpha\right| = c\int_a^b f d\alpha = \int_a^b c f d\alpha \leq \int_a^b |f| d\alpha$$

where $c = \pm 1 = \operatorname{sign}(\int_a^b f d\alpha)$

7) Take $\psi(x) = x^2$ by the previous proposition 306, $f^2 \in \mathcal{R}(\alpha)$, so is

$$f \cdot g = \frac{(f+g)^2 - (f-g)^2}{4}$$

□

Proposition 309. $\alpha(x) = \theta(x-a) = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases}$, *$f(x)$ is continuous at a, $a \in (c, d)$*

$$\int_c^d f(x) d\alpha = f(a)$$

If think of $\delta = d\alpha/dx$ or $d\alpha = \delta dx$, then it says $\int_c^d f(x)\delta dx = f(a)$.

Proof. P a partition s.t. $a \in P$, then $\Delta\alpha_i = 0$ unless for i s.t. $x_{i+1} = a$, in that case $\Delta\alpha_i = 1$, so

$$U(P, f, \alpha) = \sup_{x\in[x_i,a]} f(x), \quad L(P, f, \alpha) = \inf_{x\in[x_i,a]} f(x)$$

f is continuous at a, fix $\epsilon > 0$, $\exists\, \delta_\epsilon > 0$ s.t.

$$|x - a| < \delta_\epsilon \implies |f(x) - f(a)| < \epsilon$$

that is

$$f(a) - \epsilon < f(x) < f(a) + \epsilon$$

Pick P s.t. $a - x_i < \delta_\epsilon$. Then

$$f(a) - \epsilon \leq \sup_{x\in[x_i,a]} f(x) \leq f(a) + \epsilon$$

so

$$|U(P, f, \alpha) - f(a)| < \epsilon \;\forall\, \epsilon > 0$$

so it is true for any Q finer than P, take inf in P, hence

$$\left|\overline{\int_a^b} f d\alpha - f(a)\right| < \epsilon$$

Since we know $f \in \mathcal{R}(\alpha)$, we don't have to do $L(P, f, \alpha)$. □

Proposition 310. *on $[a, b]$, s_n is a sequence in (a, b) $c_n \geq 0$ $\sum_{n=1}^{\infty} c_n < \infty$*

$$\alpha(x) = \sum_{n=1}^{\infty} c_n \theta(x - s_n)$$

$f(x)$ continuous on $[a, b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

Here clearly α is increasing and jumps like a stair. This is second worst possible situation for α increasing: countable many jumps and flat line in between. The 1st worst α is countable jumps and continuous curves in between.

Proof. Fix $\epsilon > 0$ find $N \in \mathbb{N}$ s.t. $\sum_{n=N+1}^{\infty} c_n < \epsilon$,

$$\alpha_1(x) = \sum_{n=1}^{N} c_n \theta(x - s_n) \qquad \alpha_2(x) = \sum_{n=N+1}^{\infty} c_n \theta(x - s_n)$$

hence $\alpha = \alpha_1 + \alpha_2$, so

$$\int_a^b f d\alpha = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$\int_a^b f d\alpha_1 = \sum_{n=1}^{N} c_n f(s_n)$$

by linearity of finite sum.

$$\begin{aligned} \left| \int_a^b f d\alpha_2 \right| \leq \int_a^b |f| \, d\alpha_2 &\leq M \int_a^b f d\alpha_2 \\ &= M(\alpha(b) - \alpha(a)) \\ &= M\epsilon \end{aligned}$$

because $\theta(a - s_n) = 0$, $\theta(b - s_n) = 1 \; \forall \, n$

$$\alpha(b) = \sum_{n=N+1}^{\infty} c_n$$

Hence

$$\left| \int_a^b f d\alpha - \sum_{n=1}^{N} c_n f(s_n) \right| < M\epsilon \quad \forall \, \epsilon$$

then take $N \to \infty$, get the result. □

Proposition 311. *α is differentiable on (a,b) and $\alpha' \in \mathcal{R}(\alpha)$, then $f \in \mathcal{R}(\alpha)$ iff*

$$f\alpha' \in \mathcal{R} \text{ and } \int_a^b f d\alpha = \int_a^b f\alpha' dx$$

Proof. P is a partition and any $s_i \in [x_i, x_{i+1}]$, by

$$\Delta\alpha_i = \alpha(x_{i+1}) - \alpha(x_i) = \alpha'(t_i)\Delta x_i$$

$$\left|\sum_{i=0}^{n-1} f(s_i)\Delta\alpha_i - \sum_{i=0}^{n-1} f(s_i)\alpha'(s_i)\Delta x_i\right| = \left|\sum_{i=0}^{n-1} f(s_i)(\alpha'(t_i) - \alpha'(s_i))\Delta x_i\right| \leq M\epsilon$$

because fix $\epsilon > 0$ find P s.t.

$$U(P, \alpha') - L(P, \alpha') < \epsilon$$

If $u_i, v_i \in [x_i, x_{i+1}]$,

$$\sum \left|\alpha'(u_i) - \alpha'(v_i)\right| \Delta x_i < \epsilon$$

so

$$\sum f(s_i)\alpha'(s_i)\Delta x_i - M\epsilon < \sum f(s_i)\Delta x_i < \sum f(s_i)\alpha'(s_i)\Delta x_i + M\epsilon$$

Take $\sup_{s_i}$

$$U(P, f\alpha') - M \leq U(P, f, \alpha) \leq U(P, f\alpha') + M$$

hence

$$\left|U(P, f, \alpha) - U(P, f\alpha')\right| < M$$

The inequality is true if we replace P with a finer partition

$$\inf_P \left(\left|U(P, f, \alpha) - U(P, f\alpha')\right| < M\epsilon\right)$$

so

$$\left|\overline{\int} f d\alpha - \overline{\int} f\alpha' dx\right| \leq M\epsilon \quad \forall\, \epsilon > 0$$

Same argument with $L(P, f, \alpha)$, $L(P, f\alpha')$ gives

$$\left|\underline{\int} f d\alpha - \underline{\int} f\alpha' dx\right| \leq M\epsilon \quad \forall\, \epsilon > 0$$

so

$$\overline{\int} f d\alpha = \overline{\int} f\alpha' dx \qquad \underline{\int} f d\alpha = \underline{\int} f\alpha' dx$$

$\square$

From here we get change of variables for free.

Proposition 312.

$$\begin{array}{ccccc} & \psi & & f & \\ [A,B] & \to & [a,b] & \to & \mathbb{R} \end{array}$$

ψ increasing and invertible $f \in R(\alpha) \implies f(\psi) \in \mathcal{R}(\alpha(\psi))$ and

$$\int_a^b f d\alpha = \int_A^B f(\psi) d(\alpha(\psi))$$

Note: strictly increasing implies one to one, but here we just say increasing, so additional assumption invertible is needed.

Proof.

$$C = \{Q, \text{ partitions of } [A,B]\}$$
$$D = \{P, \text{ partitions of } [a,b]\}$$

$$\begin{array}{rcl} \tilde{\psi} : C & \to & D \\ \{x_0, ..., x_n\} & \mapsto & \{\psi(x_0), ..., \psi(x_n)\} \end{array}$$

$\tilde{\psi}$ is bijection

$$U(Q, f(\psi), \alpha(\psi)) = U(\tilde{\psi}(Q), f, \alpha)$$
$$U(P, f, \alpha) = U(\tilde{\psi}^{-1}(P), f(\psi), \alpha(\psi))$$
$$U(P, f, \alpha) = U(\psi^{-1}(P), f(\psi), \alpha(\psi)) \geq \overline{\int_A^B} f(\psi) d\alpha(\psi)$$

Take inf in P,

$$\overline{\int_a^b} f d\alpha \geq \overline{\int_A^B} f(\psi) d\alpha(\psi)$$

exchange of role of $[a,b]$, $[A,B]$,

$$\overline{\int_a^b} f d\alpha \leq \overline{\int_A^B} f(\psi) d\alpha(\psi)$$

□

Proposition 313.

$$[A,B] \xrightarrow{\psi} [a,b]$$

invertible increases differentiable, $\psi' \in \mathcal{R}$, then

$$\int_a^b f(x) dx = \int_A^B f(\psi(x)) \psi'(x) dx$$

Proof. set $\alpha(x) = x$ in the previous proposition,

$$\int_a^b f(x)dx = \int_{\psi(A)}^{\psi(B)} f(x)dx = \int_A^B f(\psi(x))d\psi(x) = \int_A^B f(\psi(x))\psi'(x)dx$$

□

4.7 Fundamental Theorem of Calculus

Lecture 21 -Last Lec- (8/13/13)

We are now back to Riemann integral. We will study the relation between integration and differentiation.

Proposition 314. *$f(x) \in \mathcal{R}$ on $[a,b]$. Let*

$$F(x) = \int_a^x f(t)dt$$

1) F is continuous on $[a,b]$

2) if f is continuous at $x_0 \in (a,b)$ then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

In one sentence, starting out integrable function, integration makes it continuous. Starting out continuous, integration make it differentiable.

Proof. 1) Prove

$$\lim_{x \to x_0} F(x) = F(x_0)$$

Assume $x_0 < x$

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_a^x f(t)dt - \int_a^{x_0} f(t)dt \right| = \left| \int_{x_0}^x f(t)dt \right| \\ &\leq \int_{x_0}^x |f(t)|\, dt \leq \sup_{t\in[a,b]} |f(t)|\, |x - x_0| = M|x - x_0| < \epsilon \end{aligned}$$

if $|x - x_0| < \delta_\epsilon = \epsilon/M$.

What we have actually proven is that F is uniformly continuous.

2) Assume $x > x_0$

$$\begin{aligned}\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| &= \left|\frac{1}{x-x_0}\int_{x_0}^{x} f(t)dt - f(x_0)\right| \\ &= \left|\frac{1}{x-x_0}\int_{x_0}^{x} \left(f(t)-f(x_0)\right)dt\right| \\ &< \frac{x-x_0}{x-x_0}\epsilon\end{aligned}$$

because f is continuous at x_0, take $x < \delta_\epsilon$, i.e. $x_0 < t < x < \delta_\epsilon + x_0$, then $|f(t) - f(x_0)| < \epsilon$

so

$$\lim_{x \to x_0^+} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$$

□

Proposition 315. *(fundamental theorem of calculus) $f \in \mathcal{R}$ on $[a, b]$, $\exists\, H(x)$ s.t. $H'(x) = f(x)\ \forall\, x \in (a, b)$, then*

$$\int_a^b f(x)dx = H(b) - H(a)$$

This is better version than what is taught in Calculus class. Here we don't need f to be continuous.

Remark 316. If $f(x)$ is continuous, then $\int_a^x f(t)dt = F(x)$ is differentiable and $F'(x) = f(x)$. How to find $F(b)$? The calculus II way

$$H'(x) = F'(x) \implies (H - F)'(x) = 0$$

so by mean value

$$H(x) - F(x) = c$$

put $x = a$, $F(a) = 0$

$$H(a) - F(a) = c \implies c = H(a)$$

so

$$H(b) - F(b) = c = H(a)$$

we find

$$F(b) = H(b) - H(a)$$

This argument doesn't work here, for we don't assume F is differentiable. We use general mean value.

Proof. $f \in \mathcal{R}\ \forall\, \epsilon > 0\ \exists\, P$ s.t.

$$U(P, f) - L(P, f) < \epsilon$$

$P = \{x_0, ..., x_n\}$, so $\exists\, t_i \in [x_i, x_{i+1}]$

$$\left| \int_a^b f(t)dt - \sum_{i=0}^{n-1} f(t_i)\Delta x_i \right| < \epsilon$$

$$H(x_{i+1}) - H(x_i) = H'(s_i)\Delta x_i$$

for some $s_i \in (x_i, x_{i+1})$, put $f(s_i) = H'(s_i)$

$$\sum \left(H(x_{i+1}) - H(x_i)\right) = \sum_{i=0}^{n-1} f(s_i)\Delta x_i$$

then

$$\left| \int_a^b f(t)dt - (H(b) - H(a)) \right| < \epsilon$$

$\square$

Lastly we do integration of vector functions

$$\begin{aligned} \vec{f} : [a, b] &\rightarrow \mathbb{R}^n \\ t &\mapsto (f_1(t), ..., f_n(t)) \end{aligned}$$

clearly $\vec{f}$ is Riemann iff $f_i \in \mathcal{R}\ \forall\, i$, and

$$\int_a^b \vec{f}(t)dt = \left(\int_a^b f_1(t)dt, ..., \int_a^b f_n(t)dt \right)$$

Proposition 317.

$$\left| \int_a^b \vec{f}(t)dt \right| \leq \int_a^b \left| \vec{f}(t) \right| dt$$

This is not the same proof as before. Before we have absolutely value, here is norm of vector.

Standard trick. Use Cauchy-Schwartz. Manipulate LHS to make it in position so that CS is ready to apply.

Proof. Let $L_i = \int_a^b f_i(t)dt$

$$\begin{aligned}
\left|\int_a^b \vec{f}(t)dt\right|^2 &= \sum_{i=1}^{n}\left(\int_a^b f_i(t)dt\right)^2 \\
&= \sum_{i=1}^{n}\int_a^b L_i f_i(t)dt \\
&= \int_a^b \sum_{i=1}^{n} L_i f_i(t)dt \\
&\leq \int_a^b \left|\vec{f}(t)\right| \sqrt{\sum L_i^2}dt \\
&= \sqrt{\sum L_i^2}\int_a^b \left|\vec{f}(t)\right| dt
\end{aligned}$$

and

$$\sum L_i^2 = \sum\left(\int_a^b f_i(t)dt\right) = \left|\int_a^b \vec{f}(t)dt\right|$$

□