

# Regression

Hung-yi Lee

李宏毅

# Regression: Output a scalar

- Stock Market Forecast

$$f(\text{Image of stock market charts}) = \text{Dow Jones Industrial Average at tomorrow}$$

- Self-driving Car

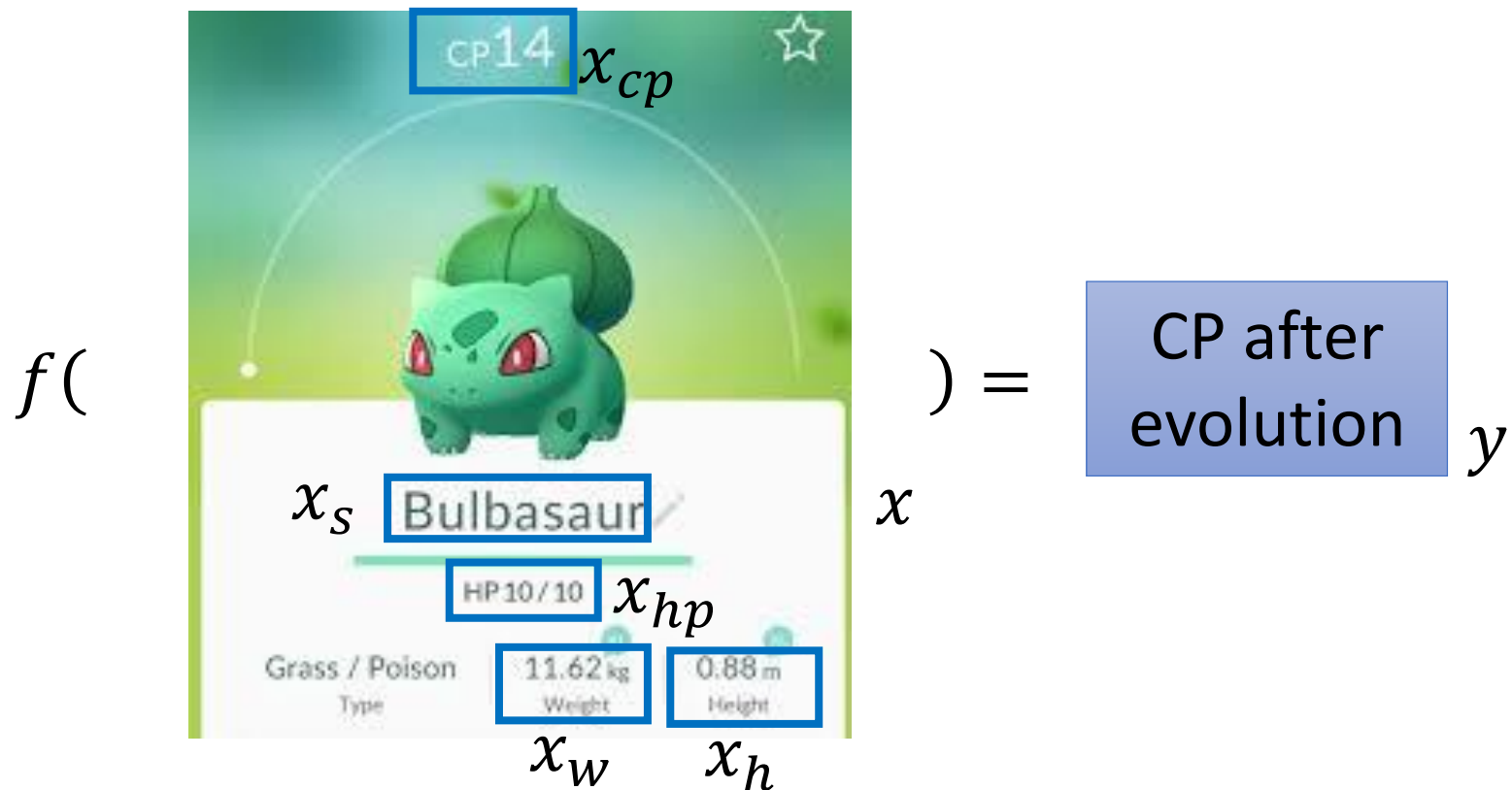
$$f(\text{Image of a self-driving car on a road}) = \text{方向盤角度}$$

- Recommendation

$$f(\text{使用者 A} \quad \text{商品 B}) = \text{購買可能性}$$

# Example Application

- Estimating the Combat Power (CP) of a pokemon after evolution



# Step 1: Model

$$y = b + w \cdot x_{cp}$$

A set of  
function

Model

$f_1, f_2 \dots$

w and b are parameters  
(can be any value)

$$f_1: y = 10.0 + 9.0 \cdot x_{cp}$$

$$f_2: y = 9.8 + 9.2 \cdot x_{cp}$$

$$f_3: y = -0.8 - 1.2 \cdot x_{cp}$$

..... infinite

$f($



$x)$

CP after  
evolution

$y$

Linear model:

$$y = b + \sum w_i x_i$$

$x_i: x_{cp}, x_{hp}, x_w, x_h \dots$

feature

$w_i$ : weight, b: bias

# Step 2: Goodness of Function

$$y = b + w \cdot x_{cp}$$

A set of  
function

Model

$f_1, f_2 \dots$

Training  
Data

function  
input:

function  
Output (scalar):



# Step 2: Goodness of Function

Training Data:  
10 pokemons

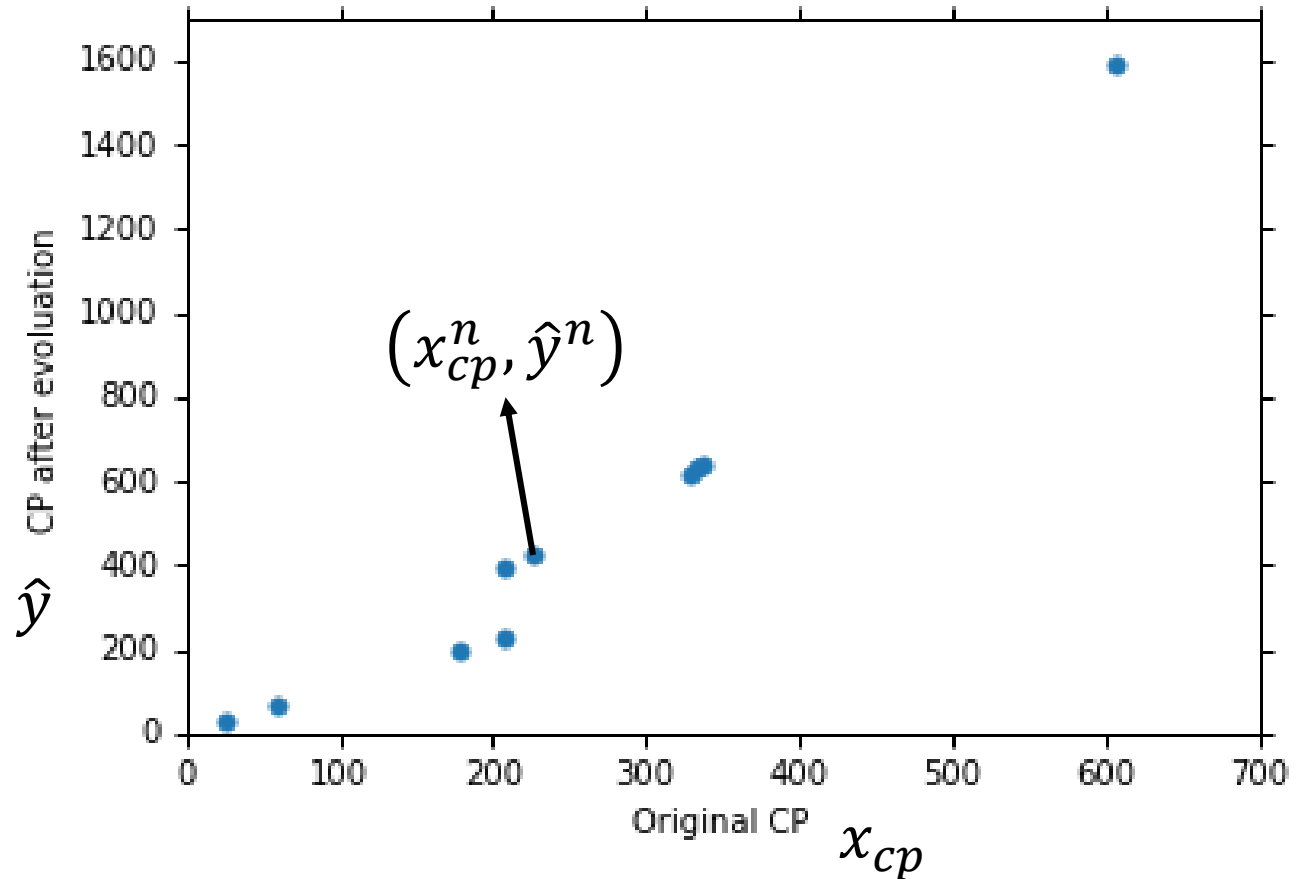
$$(x^1, \hat{y}^1)$$

$$(x^2, \hat{y}^2)$$

⋮

$$(x^{10}, \hat{y}^{10})$$

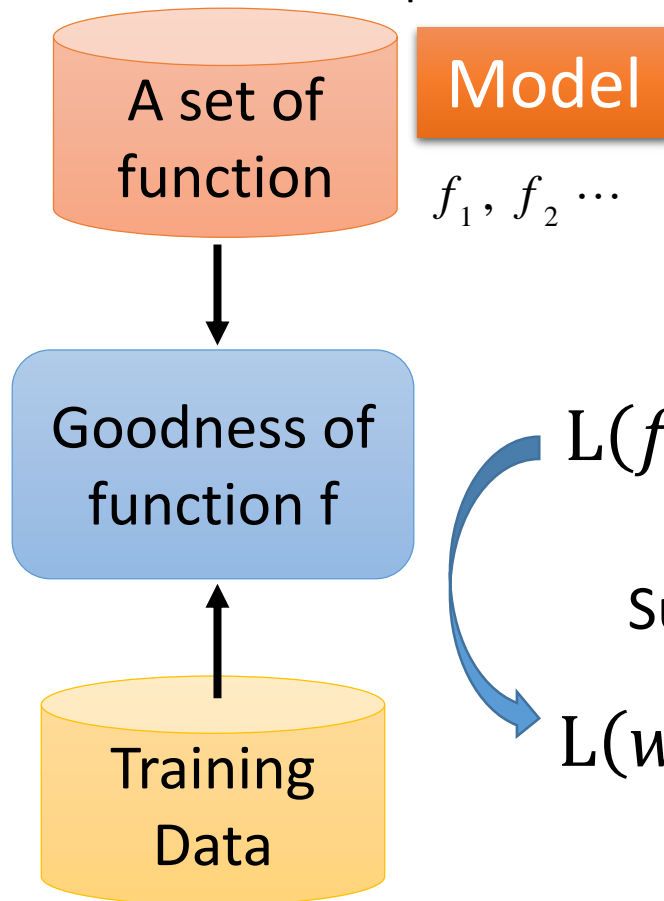
This is real data.



Source: <https://www.openintro.org/stat/data/?data=pokemon>

# Step 2: Goodness of Function

$$y = b + w \cdot x_{cp}$$



Loss function  $L$ :

Input: a function, output: how bad it is, 损失函数实际上就是一个函数的函数

$$L(f) = \sum_{n=1}^{10} \left( \hat{y}^n - \underbrace{f(x_{cp}^n)}_{\text{Estimated } y \text{ based on input function}} \right)^2$$

Sum over examples

Estimation error

$$L(w, b) = \sum_{n=1}^{10} \left( \hat{y}^n - (b + w \cdot x_{cp}^n) \right)^2$$

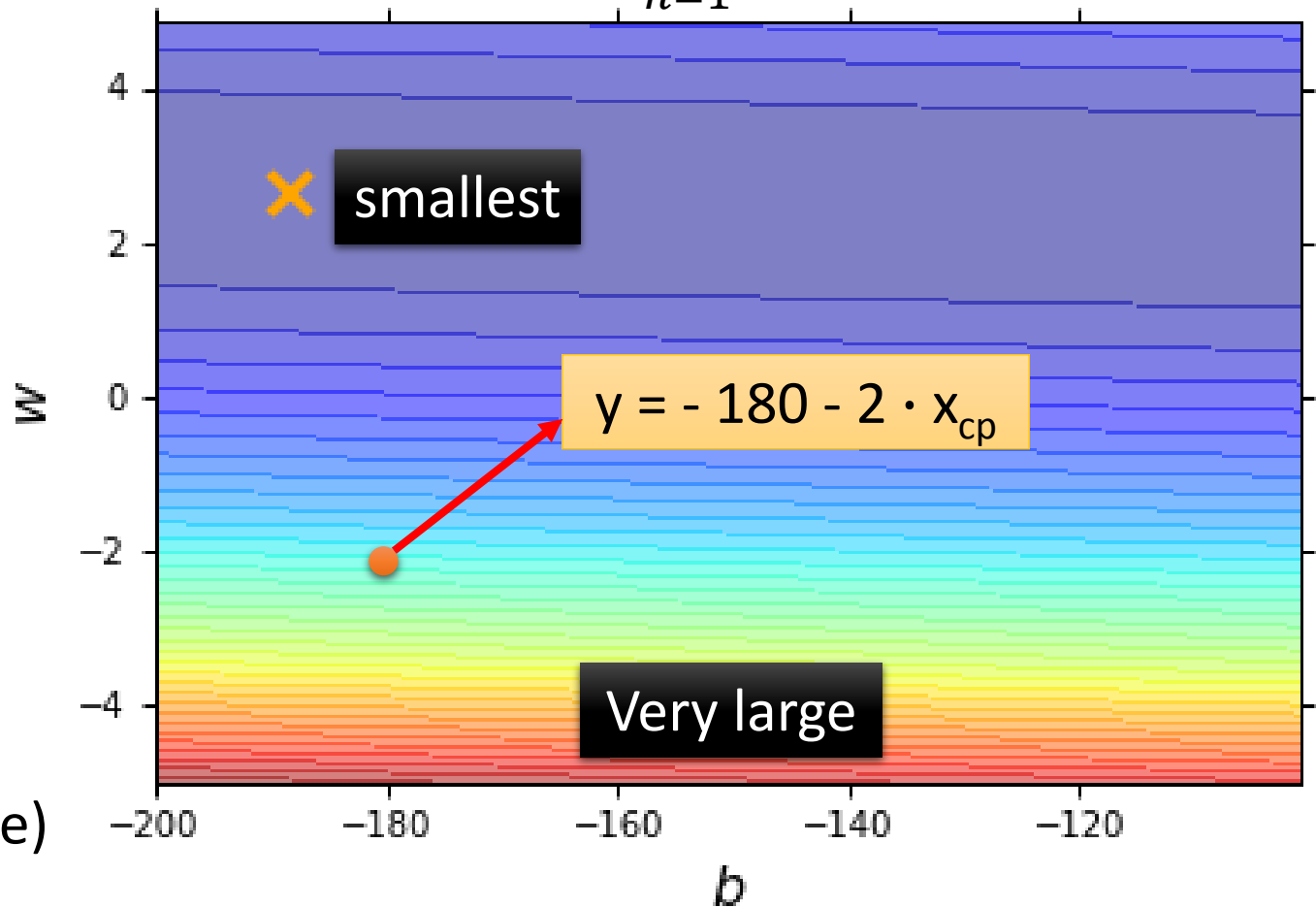
# Step 2: Goodness of Function

$$L(w, b) = \sum_{n=1}^{10} \left( \hat{y}^n - (b + w \cdot x_{cp}^n) \right)^2$$

- Loss Function

Each point in the figure is a function

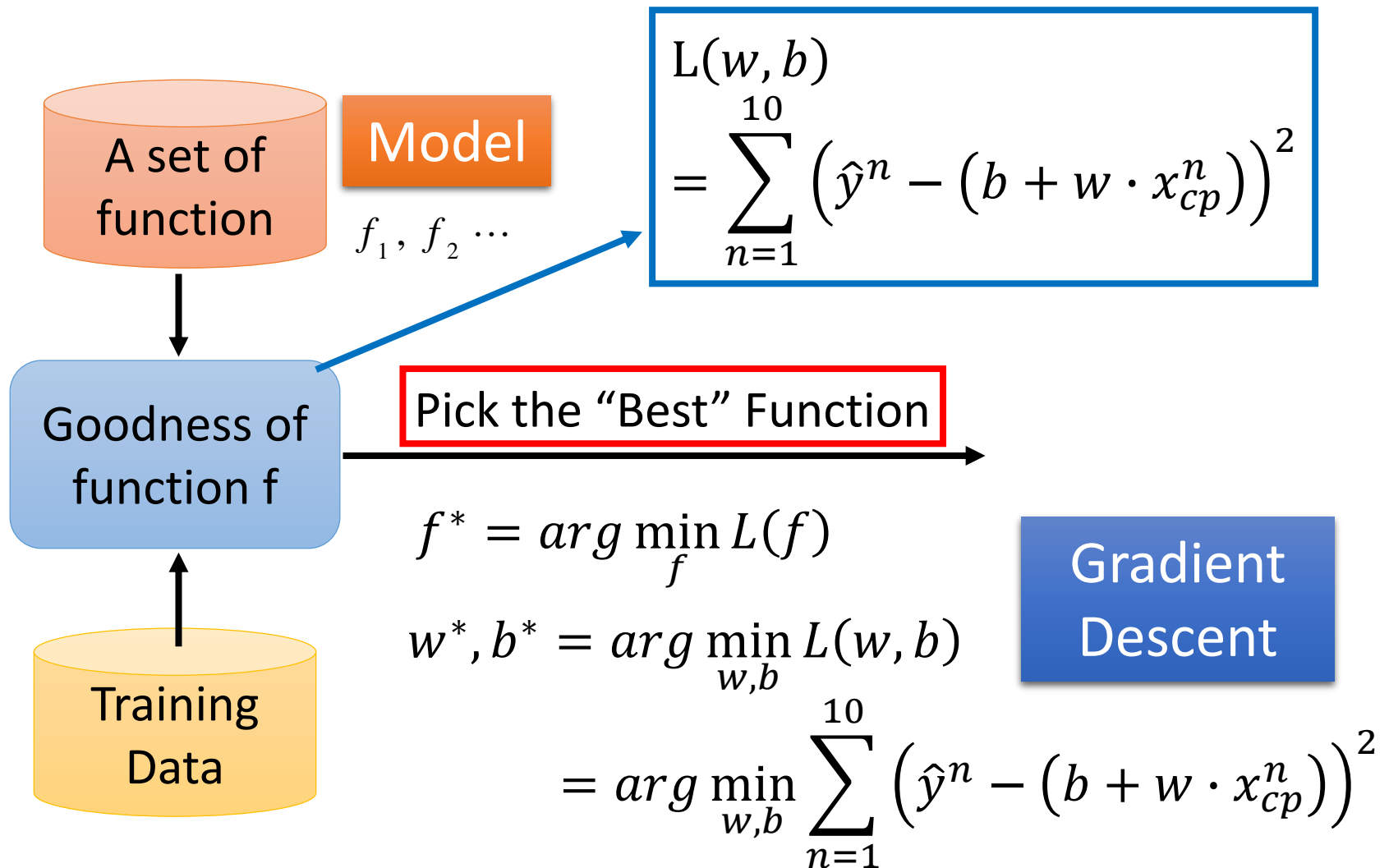
The color represents  $L(w, b)$ .



(true example)



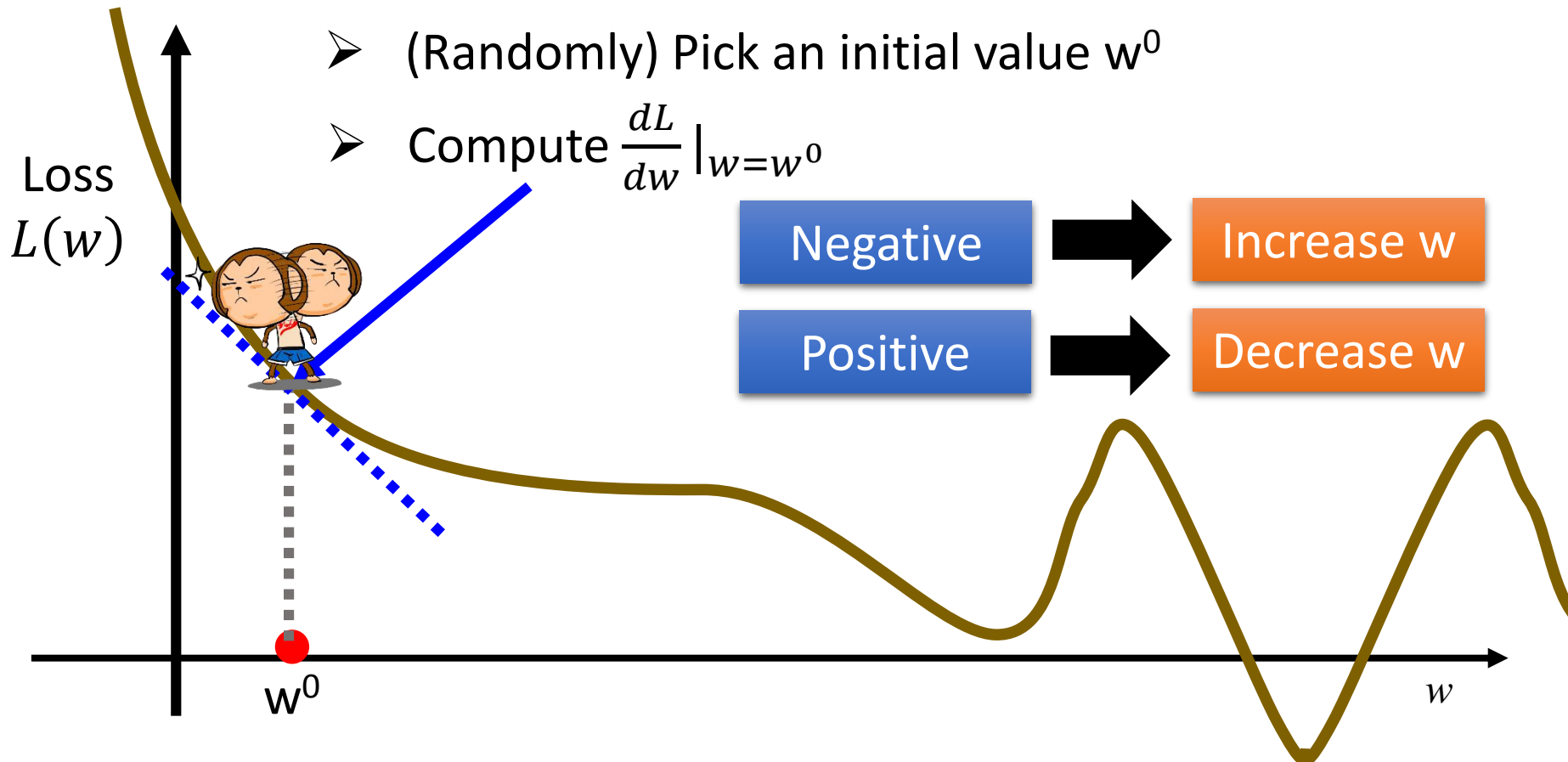
# Step 3: Best Function



# Step 3: Gradient Descent

$$w^* = \arg \min_w L(w)$$

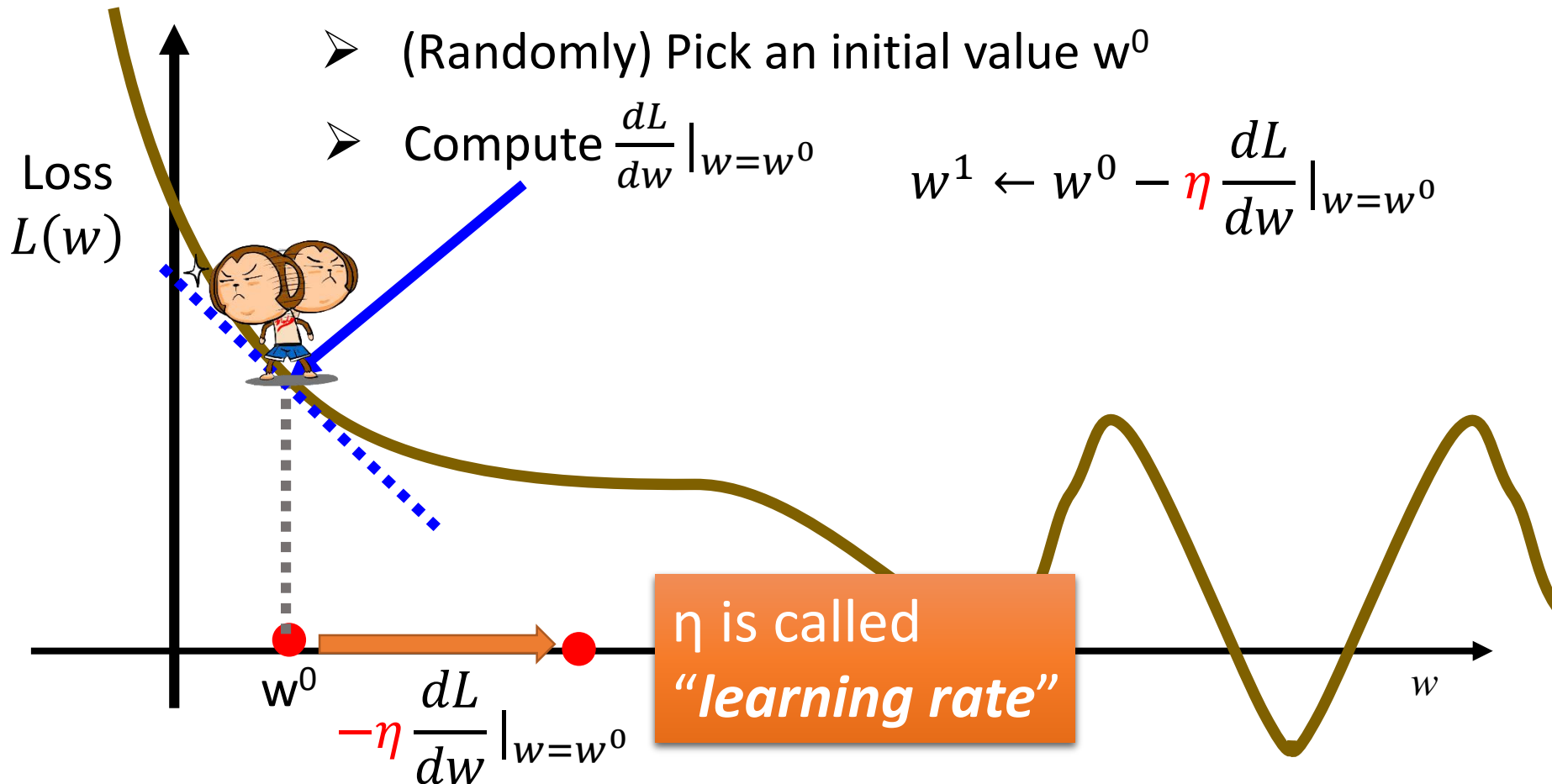
- Consider loss function  $L(w)$  with one parameter  $w$ :



# Step 3: Gradient Descent

$$w^* = \arg \min_w L(w)$$

- Consider loss function  $L(w)$  with one parameter  $w$ :



# Step 3: Gradient Descent

$$w^* = \arg \min_w L(w)$$

- Consider loss function  $L(w)$  with one parameter  $w$ :

➤ (Randomly) Pick an initial value  $w^0$

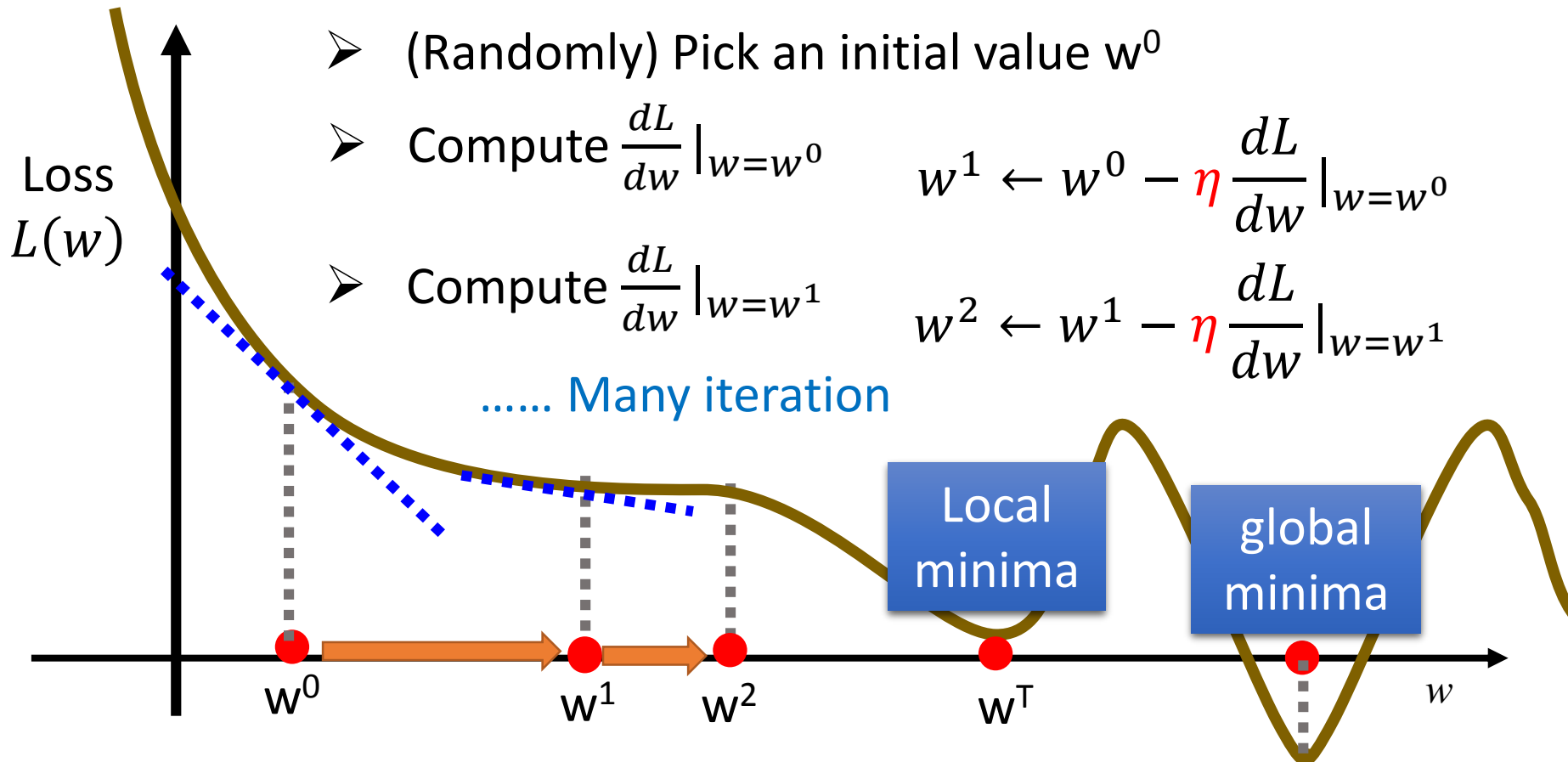
➤ Compute  $\frac{dL}{dw} \big|_{w=w^0}$

$$w^1 \leftarrow w^0 - \eta \frac{dL}{dw} \big|_{w=w^0}$$

➤ Compute  $\frac{dL}{dw} \big|_{w=w^1}$

$$w^2 \leftarrow w^1 - \eta \frac{dL}{dw} \big|_{w=w^1}$$

..... Many iteration



# Step 3: Gradient Descent

$$\begin{bmatrix} \frac{\partial L}{\partial w} \\ \frac{\partial L}{\partial b} \end{bmatrix} \text{gradient}$$

- How about two parameters?  $w^*, b^* = \arg \min_{w, b} L(w, b)$

➤ (Randomly) Pick an initial value  $w^0, b^0$

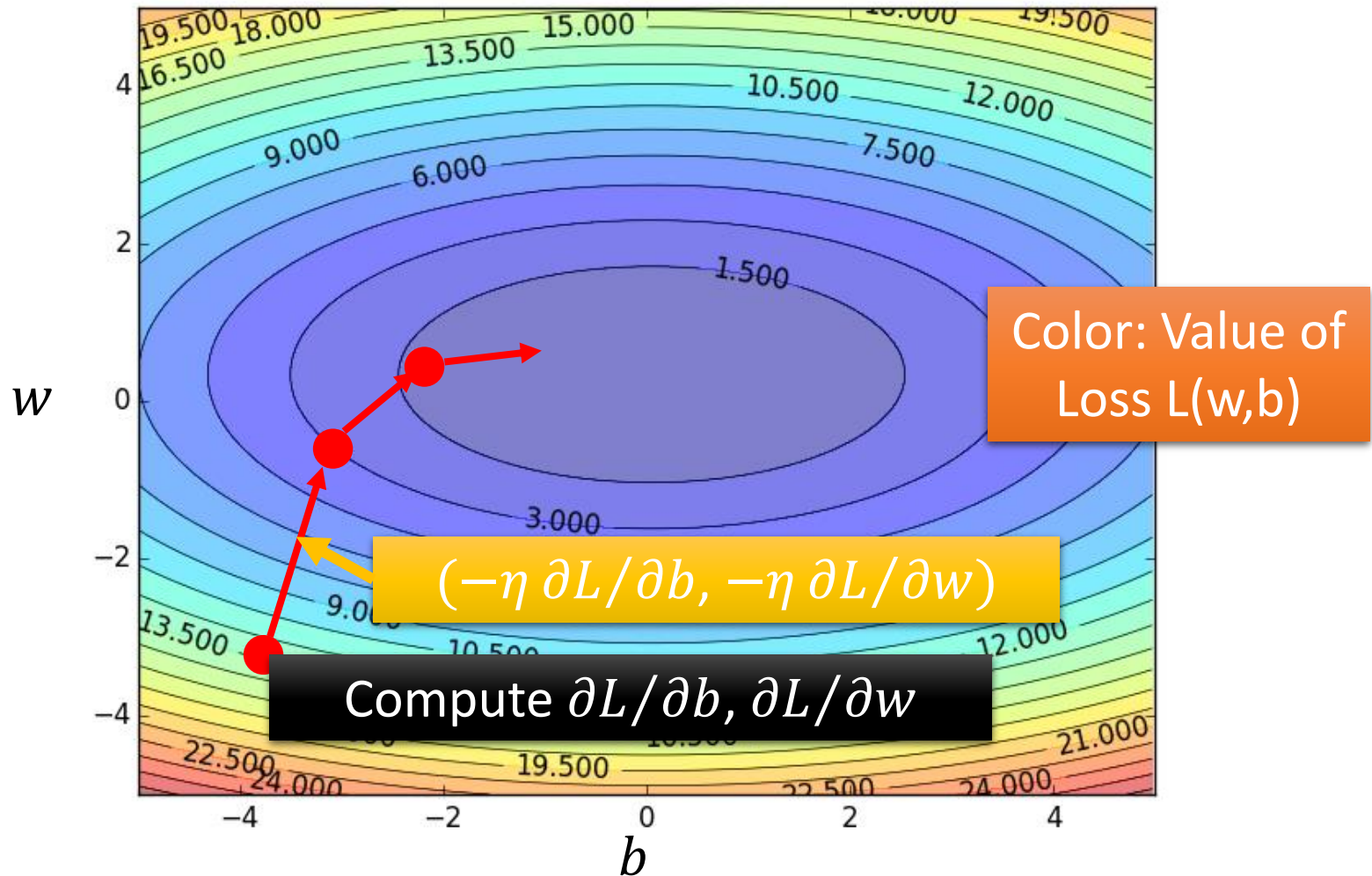
➤ Compute  $\frac{\partial L}{\partial w} \big|_{w=w^0, b=b^0}, \frac{\partial L}{\partial b} \big|_{w=w^0, b=b^0}$

$$w^1 \leftarrow w^0 - \eta \frac{\partial L}{\partial w} \big|_{w=w^0, b=b^0} \quad b^1 \leftarrow b^0 - \eta \frac{\partial L}{\partial b} \big|_{w=w^0, b=b^0}$$

➤ Compute  $\frac{\partial L}{\partial w} \big|_{w=w^1, b=b^1}, \frac{\partial L}{\partial b} \big|_{w=w^1, b=b^1}$

$$w^2 \leftarrow w^1 - \eta \frac{\partial L}{\partial w} \big|_{w=w^1, b=b^1} \quad b^2 \leftarrow b^1 - \eta \frac{\partial L}{\partial b} \big|_{w=w^1, b=b^1}$$

# Step 3: Gradient Descent



# Step 3: Gradient Descent

- When solving:

$$\theta^* = \arg \max_{\theta} L(\theta) \quad \text{by gradient descent}$$

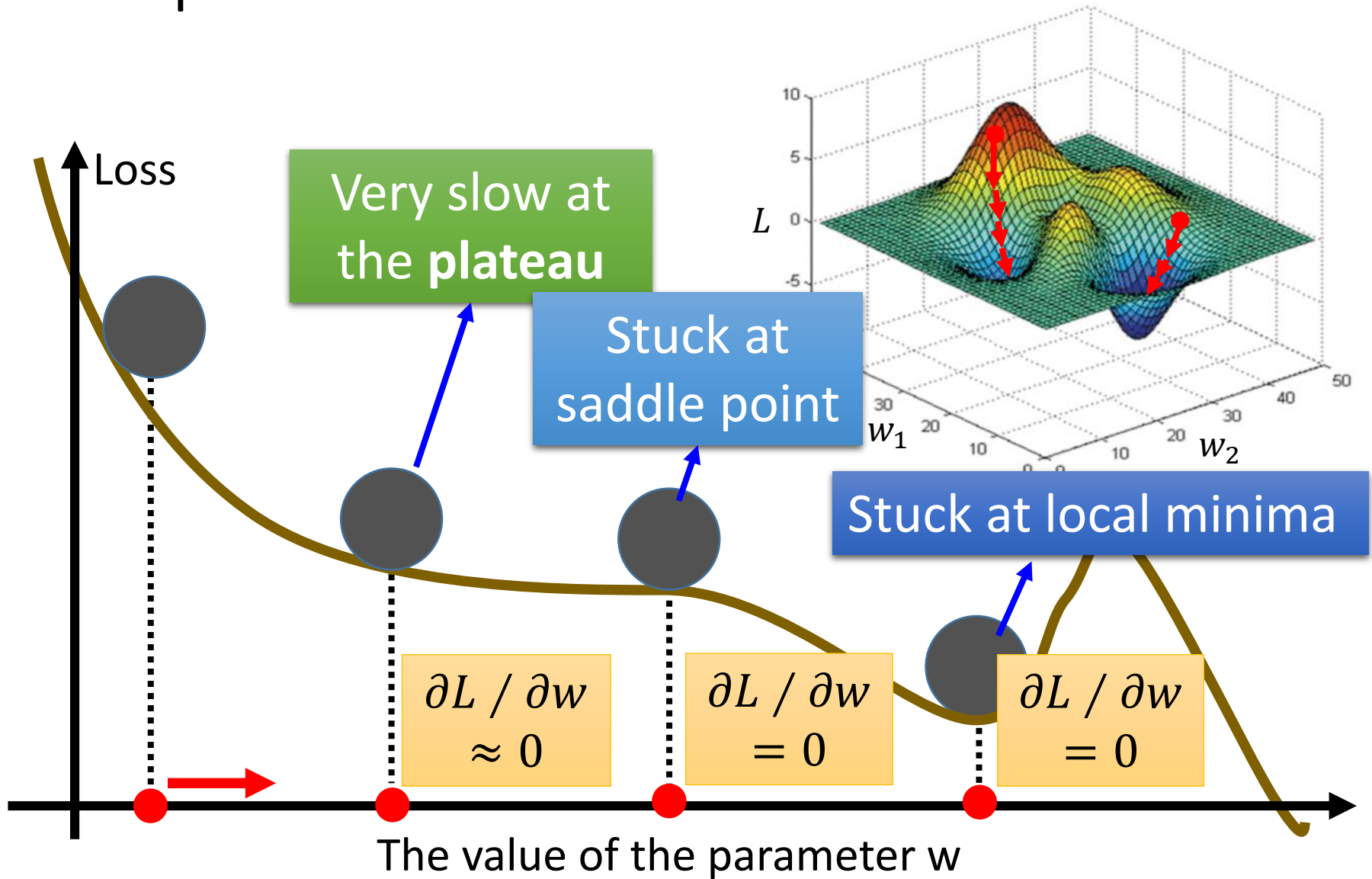
- Each time we update the parameters, we obtain  $\theta$  that makes  $L(\theta)$  smaller.

$$L(\theta^0) > L(\theta^1) > L(\theta^2) > \dots$$

Is this statement correct?

**Wrong!** 如果学习率选择不合适, 损失函数反而会升高。

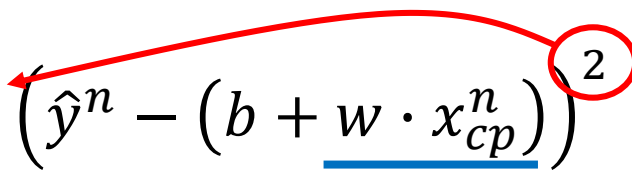
# Step 3: Gradient Descent





# Step 3: Gradient Descent

- Formulation of  $\partial L / \partial w$  and  $\partial L / \partial b$


$$L(w, b) = \sum_{n=1}^{10} \left( \hat{y}^n - (b + \underline{w \cdot x_{cp}^n}) \right)^2$$


$$\frac{\partial L}{\partial w} = ? \sum_{n=1}^{10} 2 \left( \hat{y}^n - (b + w \cdot x_{cp}^n) \right)$$

$$\frac{\partial L}{\partial b} = ?$$

# Step 3: Gradient Descent

- Formulation of  $\partial L / \partial w$  and  $\partial L / \partial b$

$$L(w, b) = \sum_{n=1}^{10} \left( \hat{y}^n - (\underline{b} + w \cdot x_{cp}^n) \right)^2$$


$$\frac{\partial L}{\partial w} = ? \sum_{n=1}^{10} 2 \left( \hat{y}^n - (b + w \cdot x_{cp}^n) \right) (-x_{cp}^n)$$

$$\frac{\partial L}{\partial b} = ? \sum_{n=1}^{10} 2 \left( \hat{y}^n - (b + w \cdot x_{cp}^n) \right)$$

## Step 3: Gradient Descent

# How's the results?

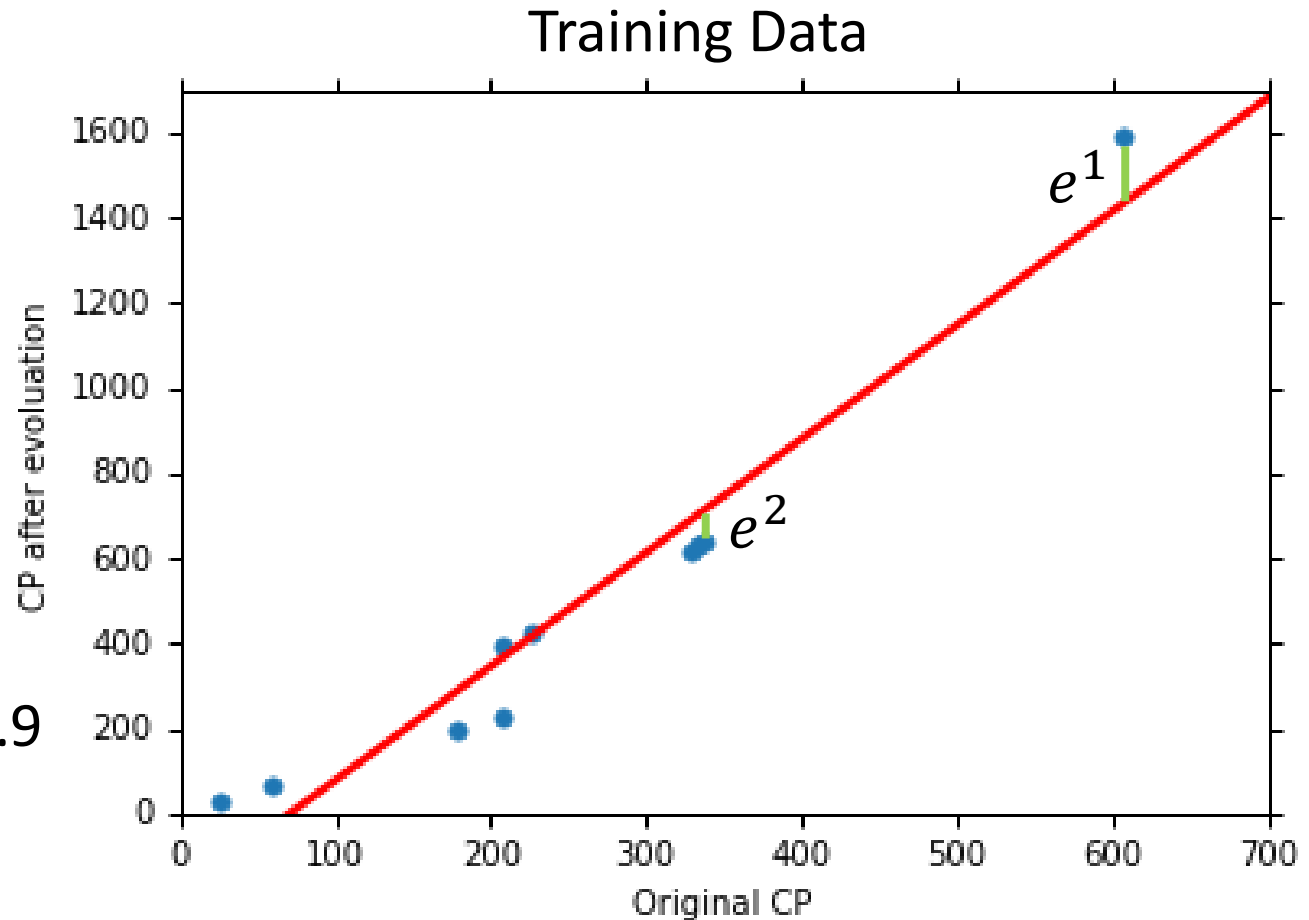
$$y = b + w \cdot x_{cp}$$

$$b = -188.4$$

$$w = 2.7$$

Average Error on  
Training Data

$$= \frac{1}{10} \sum_{n=1}^{10} e^n = 31.9$$



# How's the results?

- Generalization

What we really care about is the error on new data (testing data)

$$y = b + w \cdot x_{cp}$$

$$b = -188.4$$

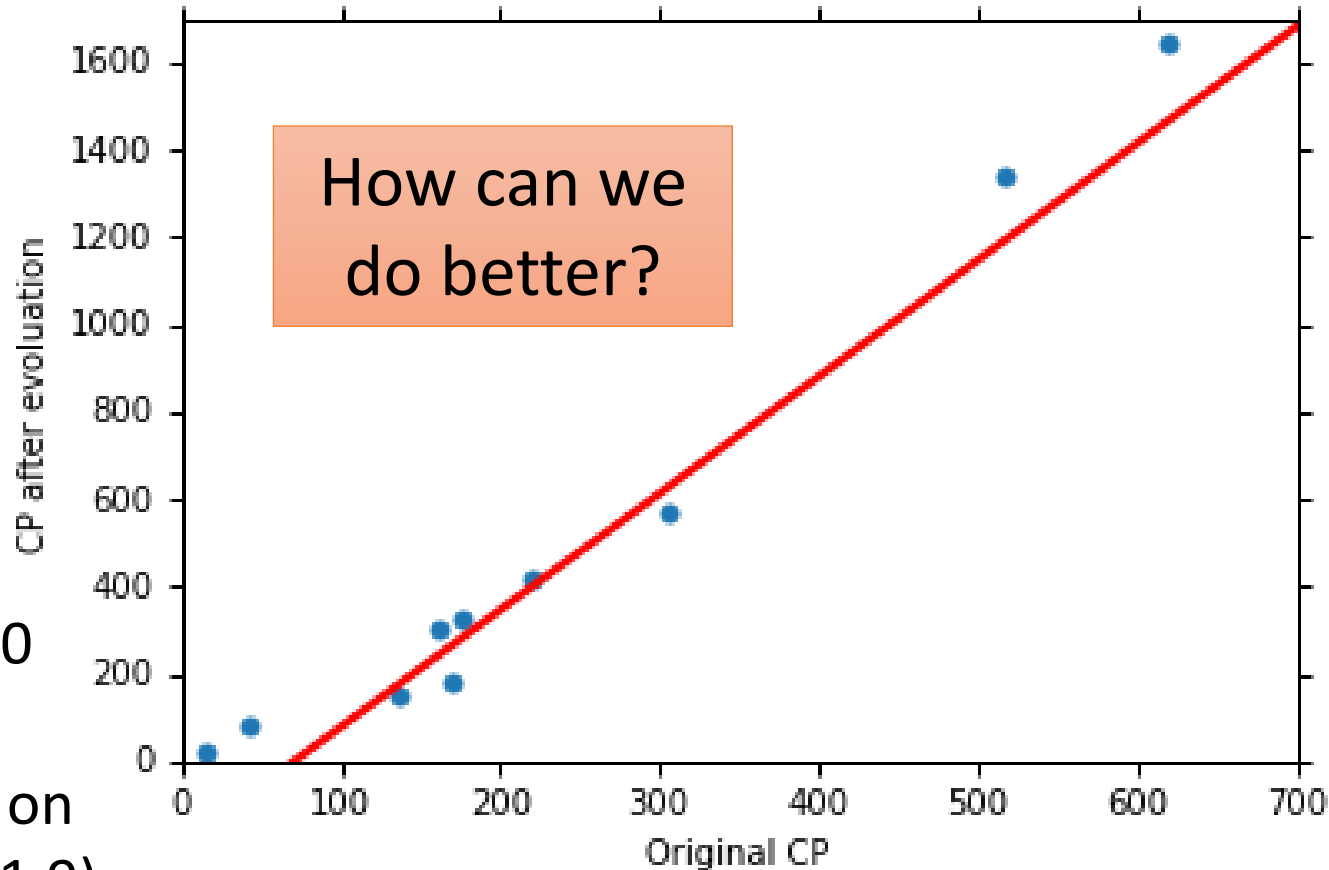
$$w = 2.7$$

Average Error on  
Testing Data

$$= \frac{1}{10} \sum_{n=1}^{10} e^n = 35.0$$

> Average Error on  
Training Data (31.9)

Another 10 pokemons as testing data



## Selecting another Model

$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2$$

### Best Function

$$b = -10.3$$

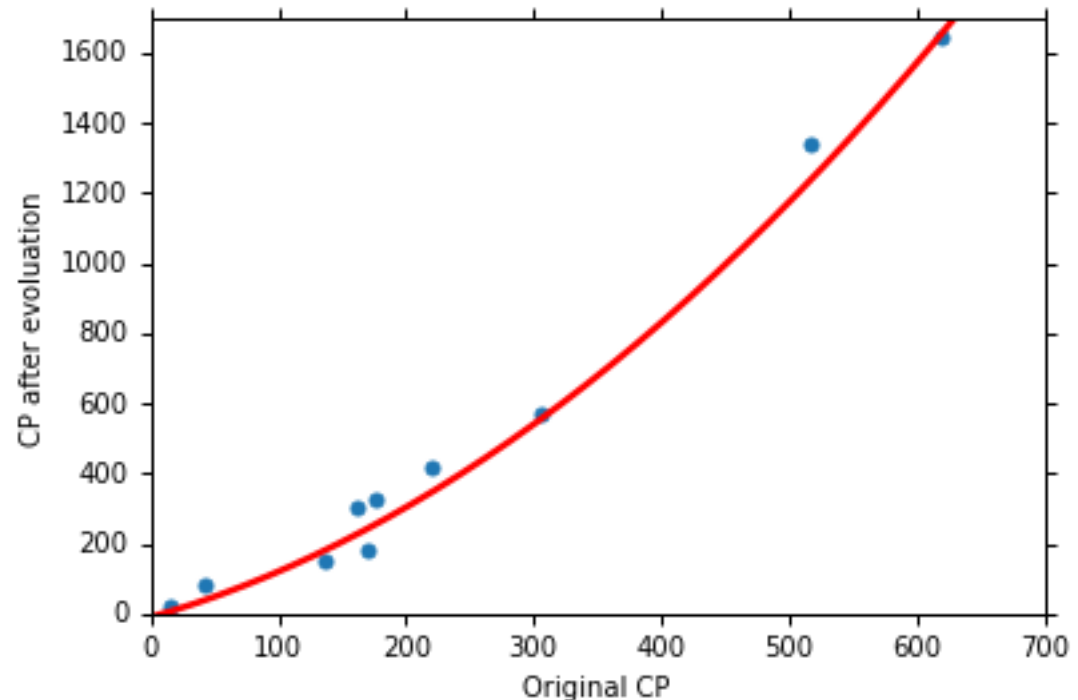
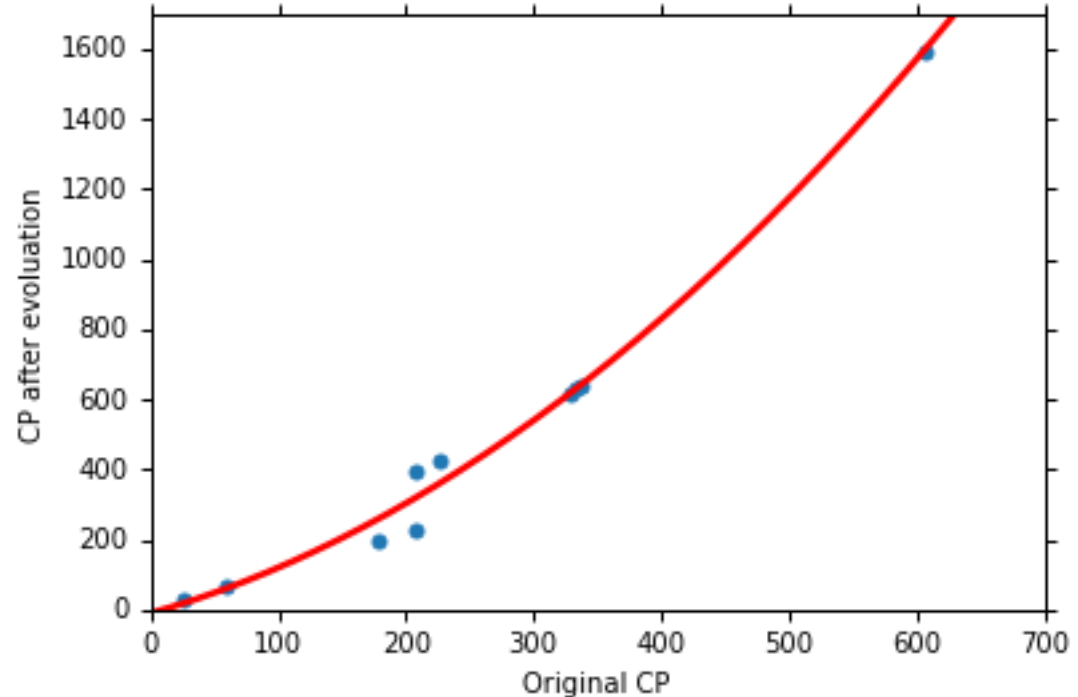
$$w_1 = 1.0, w_2 = 2.7 \times 10^{-3}$$

Average Error = 15.4

### Testing:

Average Error = 18.4

Better! Could it be even better?



## Selecting another Model

$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3$$

### Best Function

$$b = 6.4, w_1 = 0.66$$

$$w_2 = 4.3 \times 10^{-3}$$

$$w_3 = -1.8 \times 10^{-6}$$

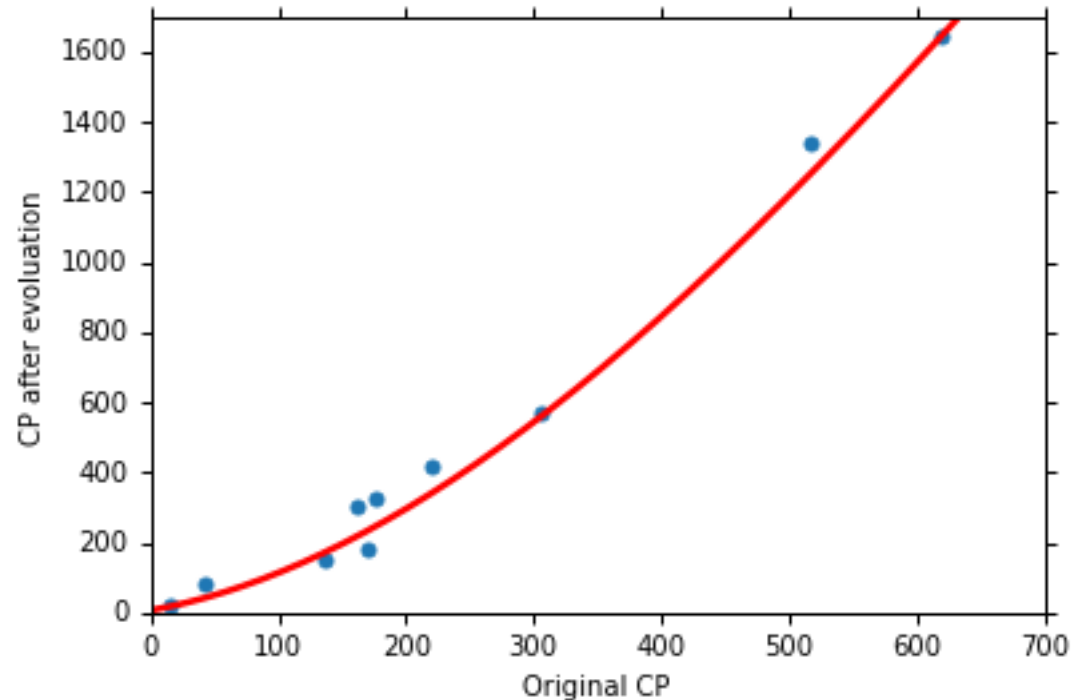
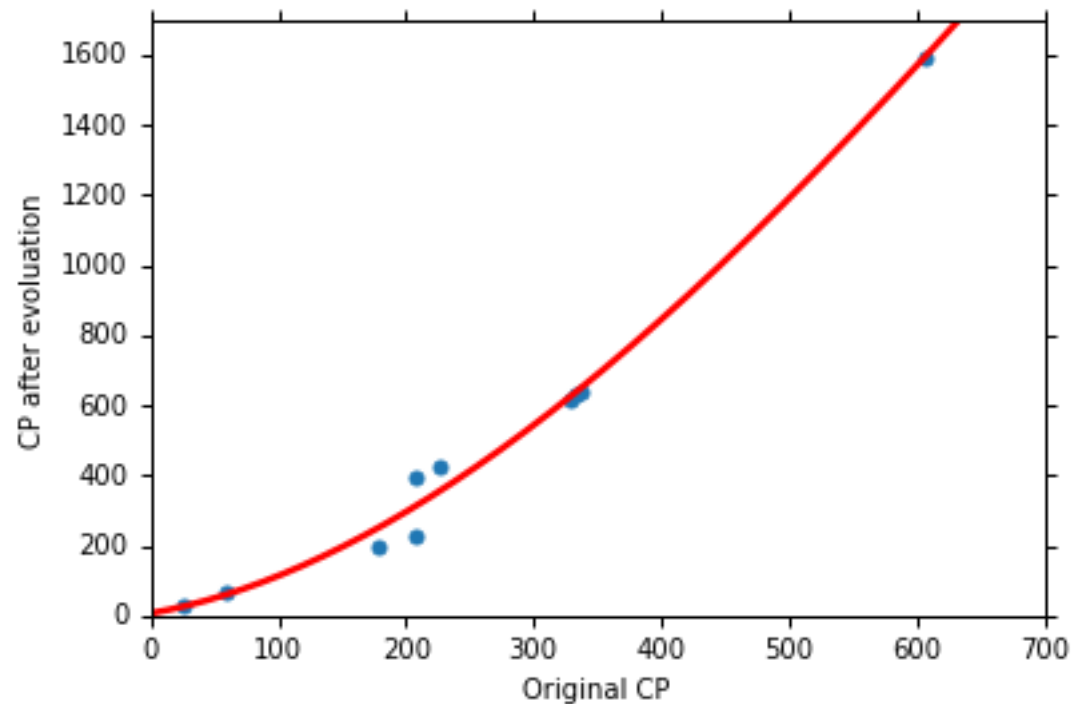
Average Error = 15.3

### Testing:

Average Error = 18.1

Slightly better.

How about more complex model?



## Selecting another Model

$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4$$

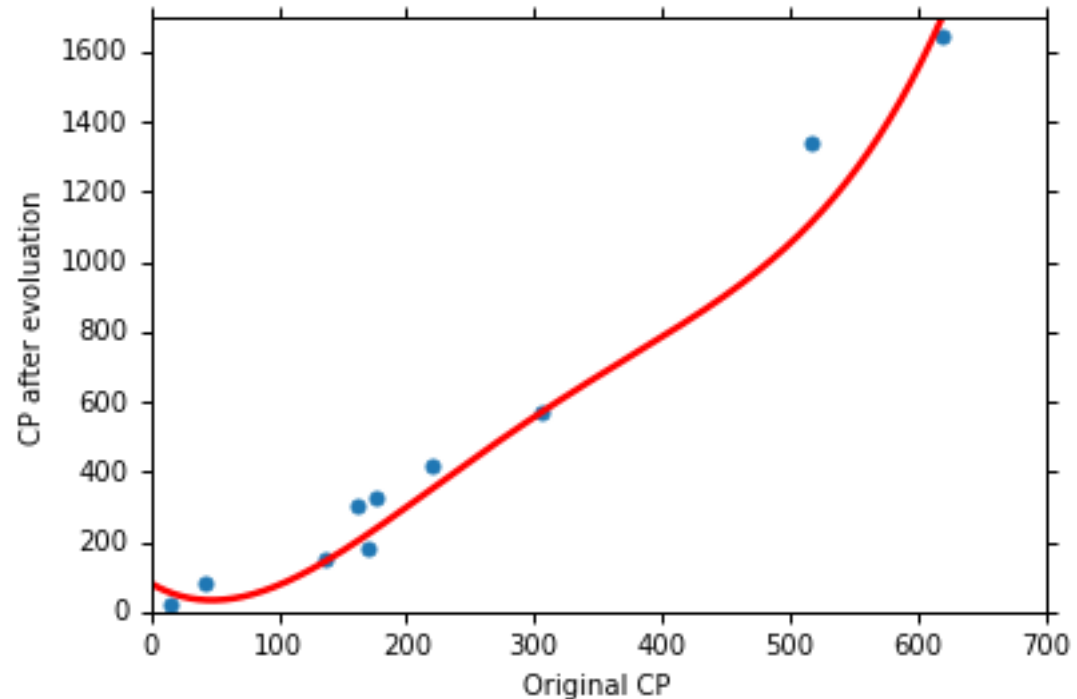
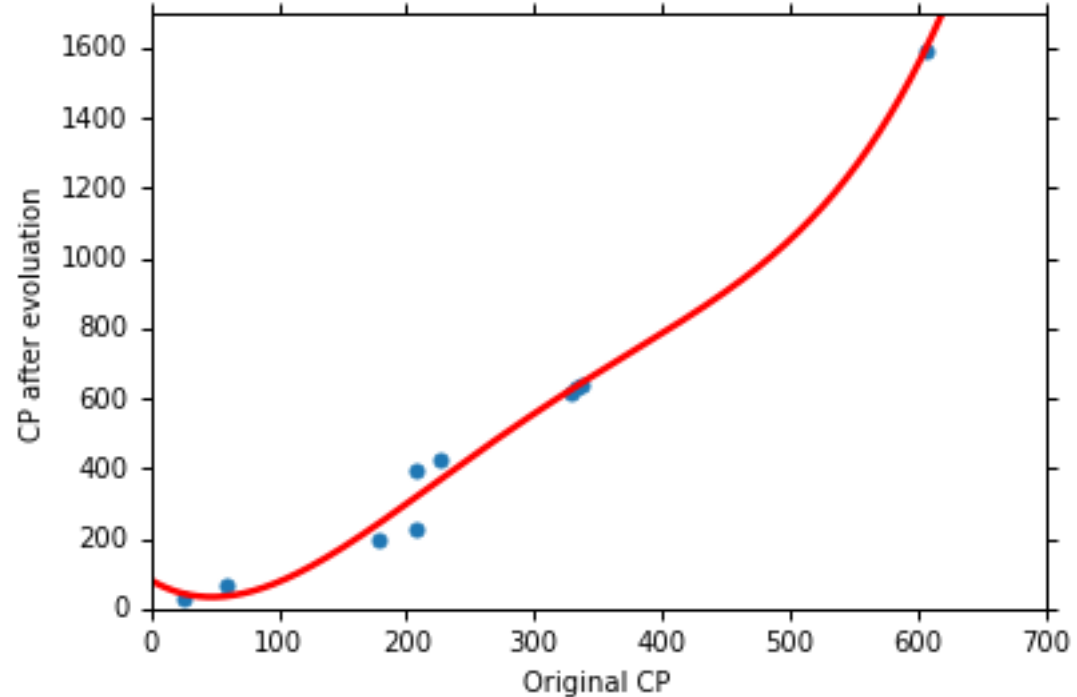
## Best Function

Average Error = 14.9

## Testing:

Average Error = 28.8

The results become worse ...





## Selecting another Model

$$y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4 + w_5 \cdot (x_{cp})^5$$

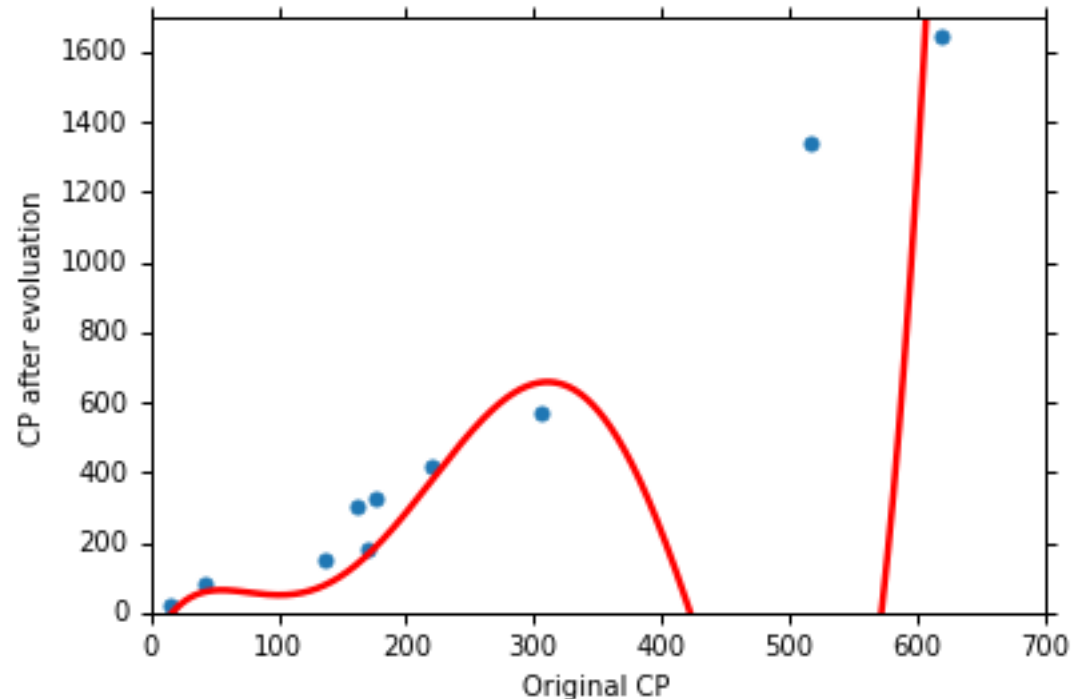
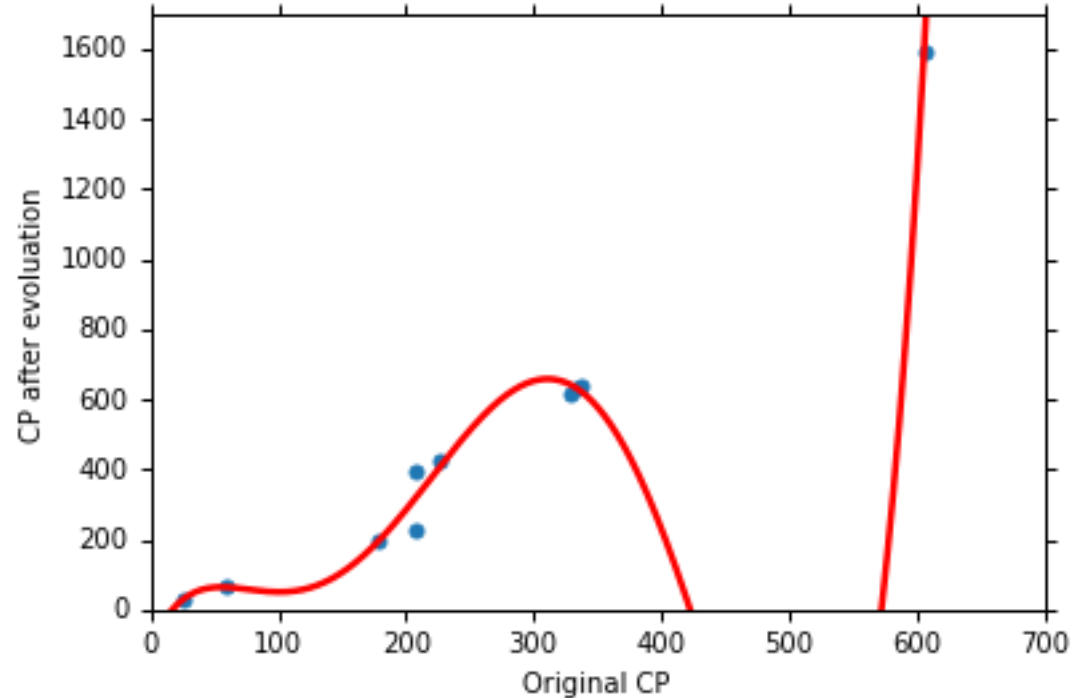
## Best Function

Average Error = 12.8

## Testing:

Average Error = 232.1

The results are so bad.



# Model Selection

1.  $y = b + w \cdot x_{cp}$

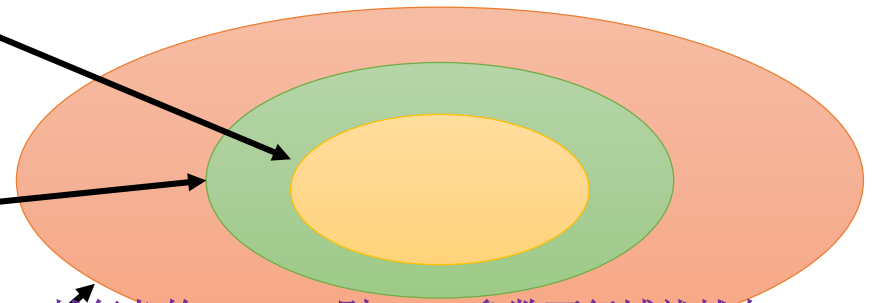
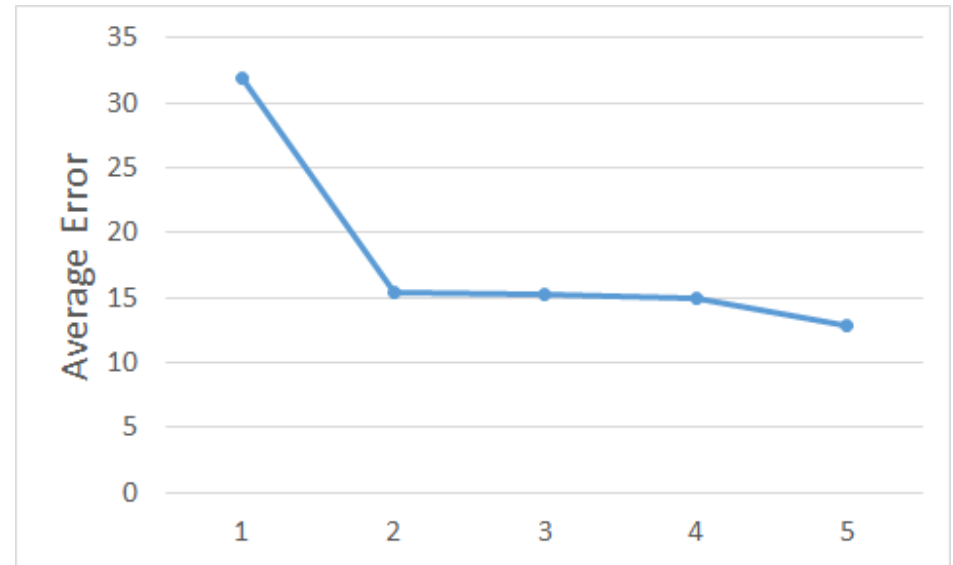
2.  $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2$

3.  $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3$

4.  $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4$

5.  $y = b + w_1 \cdot x_{cp} + w_2 \cdot (x_{cp})^2 + w_3 \cdot (x_{cp})^3 + w_4 \cdot (x_{cp})^4 + w_5 \cdot (x_{cp})^5$

Training Data

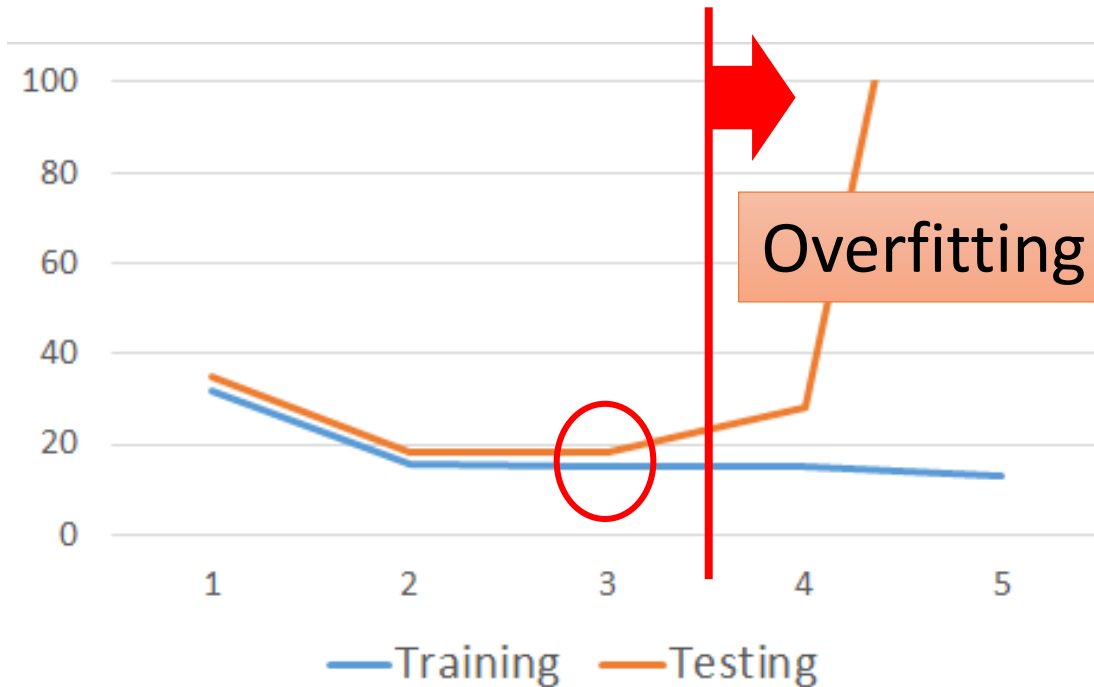


越复杂的model, 则model参数可行域就越大, 从而就更容易找到适合training data的参数。

A more complex model yields lower error on training data.

If we can truly find the best function

# Model Selection

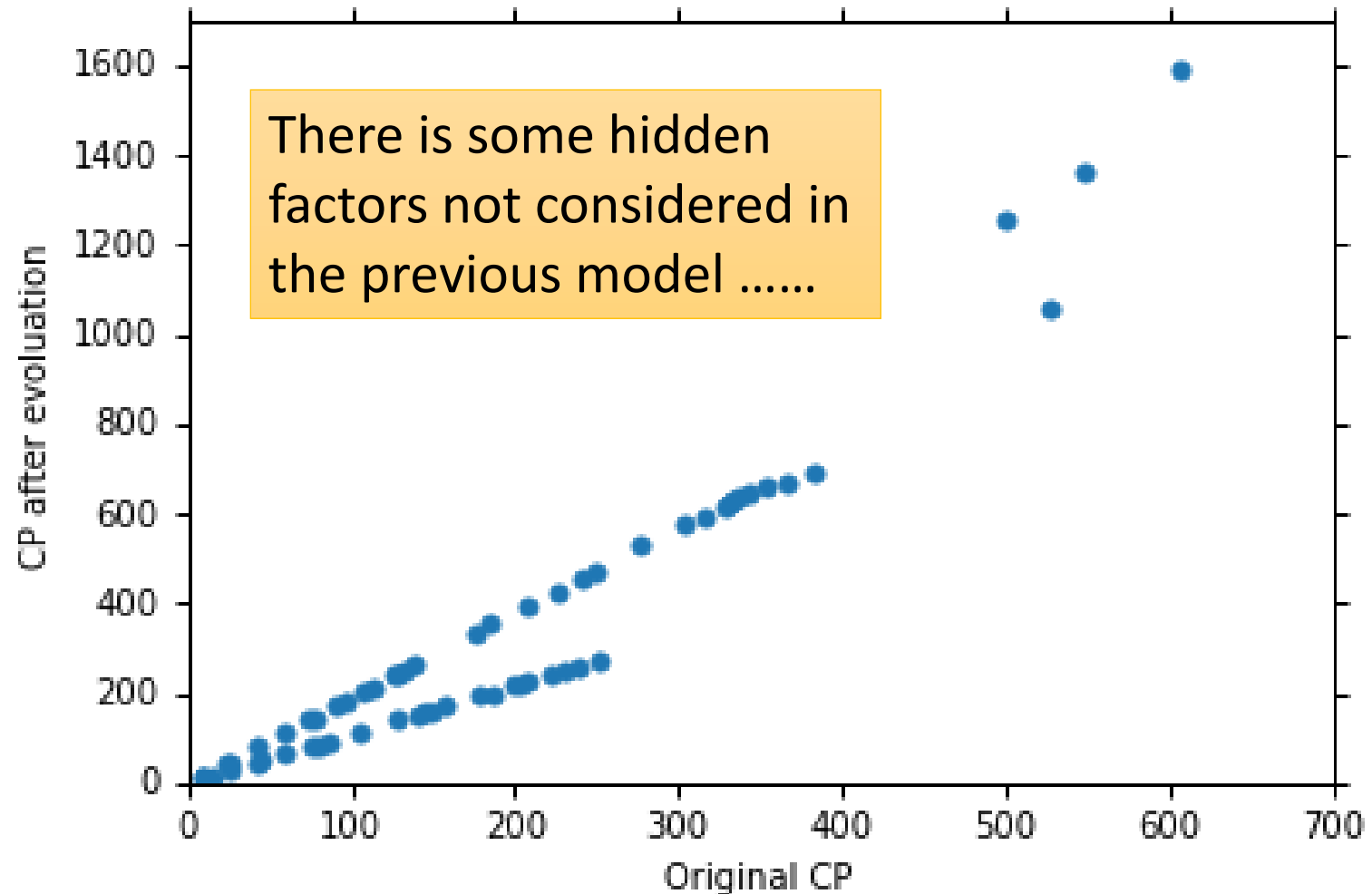


	Training	Testing
1	31.9	35.0
2	15.4	18.4
3	15.3	18.1
4	14.9	28.2
5	12.8	232.1

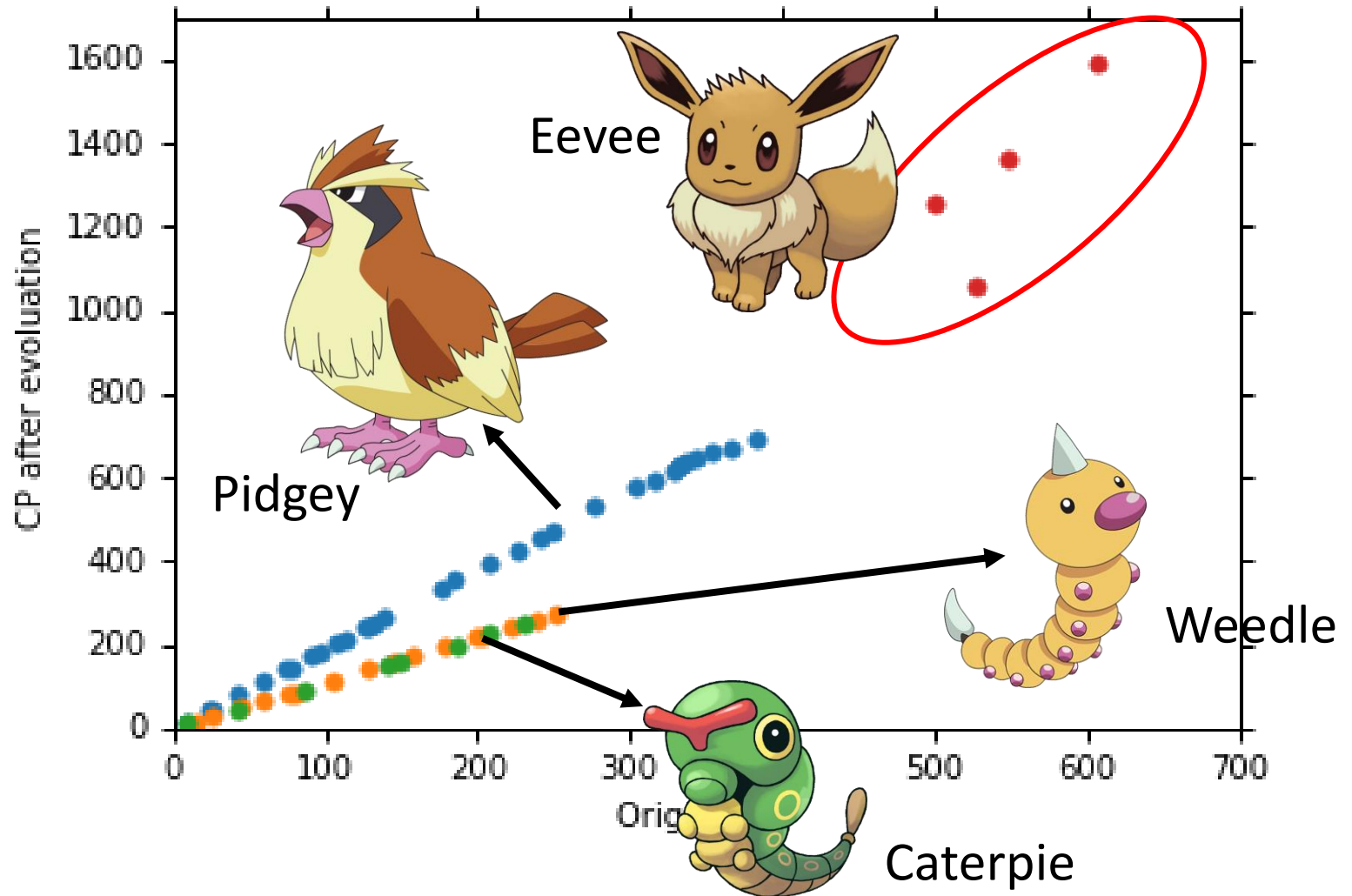
A more complex model does not always lead to better performance on testing data.

This is Overfitting.  Select suitable model

# Let's collect more data



# What are the hidden factors?



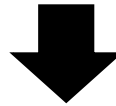
# Back to step 1: Redesign the Model

$$y = b + \sum w_i x_i$$

Linear model?

$x_s$  = species of  $x$

$x$



If  $x_s = \text{Pidgey}$ :

$$y = b_1 + w_1 \cdot x_{cp}$$

If  $x_s = \text{Weedle}$ :

$$y = b_2 + w_2 \cdot x_{cp}$$

If  $x_s = \text{Caterpie}$ :

$$y = b_3 + w_3 \cdot x_{cp}$$

If  $x_s = \text{Eevee}$ :

$$y = b_4 + w_4 \cdot x_{cp}$$



$y$

# Back to step 1: Redesign the Model

$$y = b + \sum w_i x_i$$

Linear model?

$$\begin{array}{l} y = b_1 \cdot \boxed{1} \\ + w_1 \cdot \boxed{1} \quad x_{cp} \\ + b_2 \cdot \boxed{0} \\ + w_2 \cdot \boxed{0} \\ + b_3 \cdot \boxed{0} \\ + w_3 \cdot \boxed{0} \\ + b_4 \cdot \boxed{0} \\ + w_4 \cdot \boxed{0} \end{array}$$

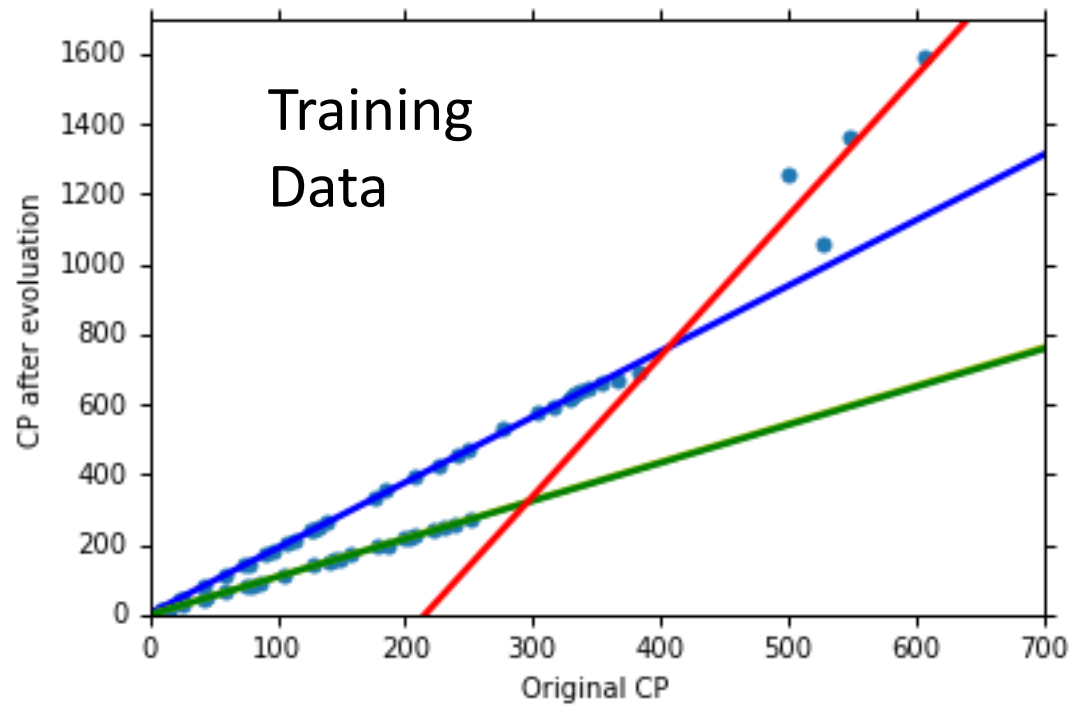
$$\delta(x_s = \text{Pidgery})$$

$$\begin{cases} =1 & \text{If } x_s = \text{Pidgery} \\ =0 & \text{otherwise} \end{cases}$$

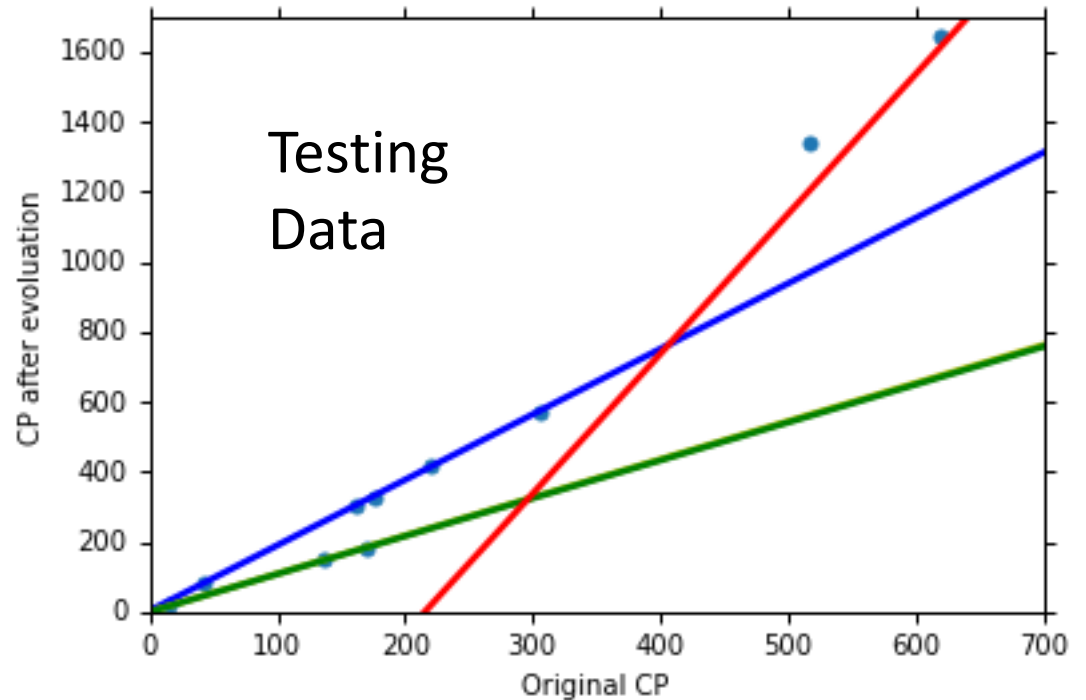
If  $x_s = \text{Pidgery}$

$$y = b_1 + w_1 \cdot x_{cp}$$

Average error  
= 3.8

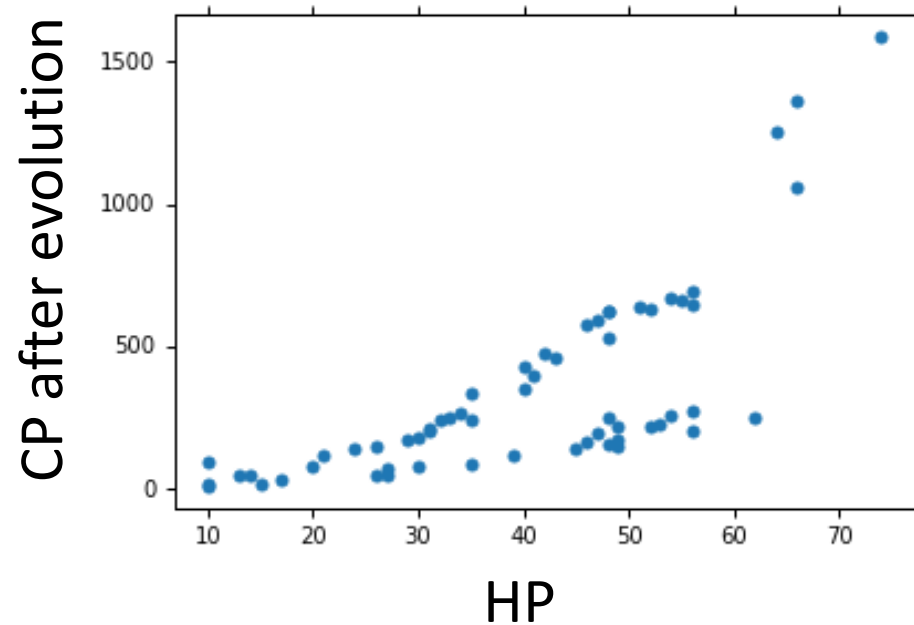
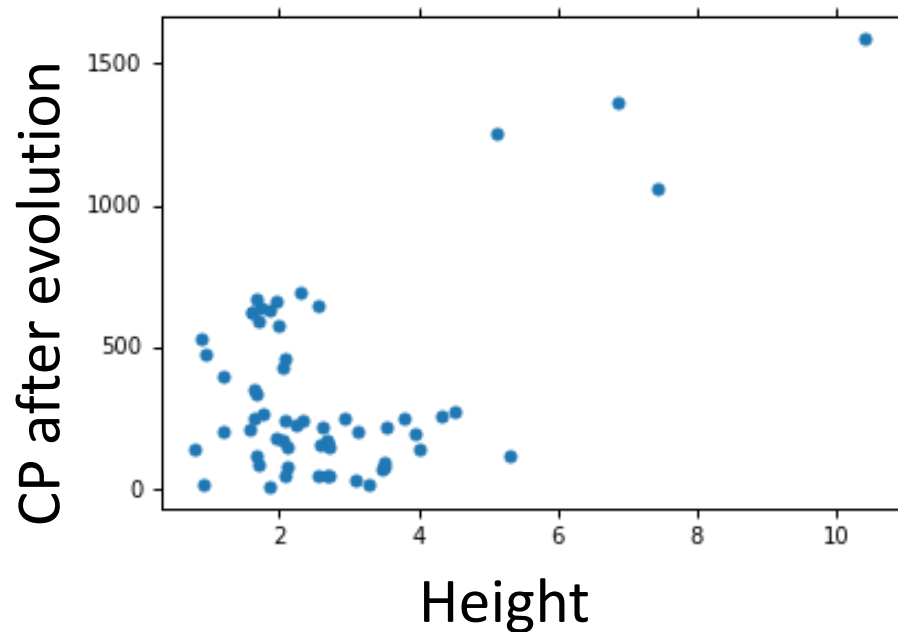
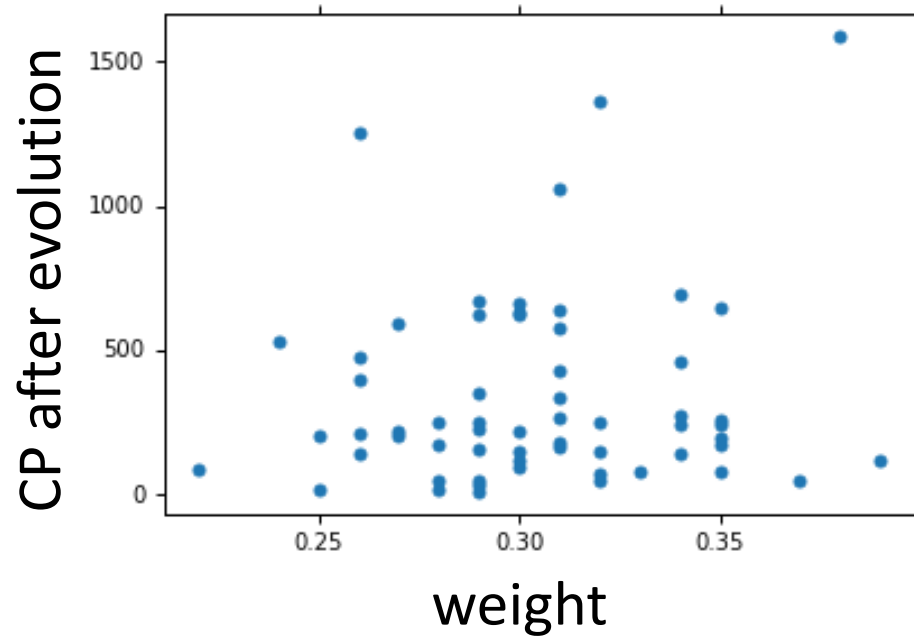


Average error  
= 14.3

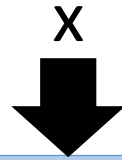




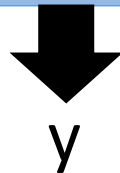
Are there any other hidden factors?



# Back to step 1: Redesign the Model Again



$$\begin{aligned} \text{If } x_s = \text{Pidgey:} \quad & y' = b_1 + w_1 \cdot x_{cp} + w_5 \cdot (x_{cp})^2 \\ \text{If } x_s = \text{Weedle:} \quad & y' = b_2 + w_2 \cdot x_{cp} + w_6 \cdot (x_{cp})^2 \\ \text{If } x_s = \text{Caterpie:} \quad & y' = b_3 + w_3 \cdot x_{cp} + w_7 \cdot (x_{cp})^2 \\ \text{If } x_s = \text{Eevee:} \quad & y' = b_4 + w_4 \cdot x_{cp} + w_8 \cdot (x_{cp})^2 \\ & y = y' + w_9 \cdot x_{hp} + w_{10} \cdot (x_{hp})^2 \\ & + w_{11} \cdot x_h + w_{12} \cdot (x_h)^2 + w_{13} \cdot x_w + w_{14} \cdot (x_w)^2 \end{aligned}$$



Training Error  
= 1.9

Testing Error  
= 102.3

Overfitting!

# Back to step 2: Regularization

$$y = b + \sum w_i x_i$$

$$L = \sum_n \left( \hat{y}^n - \left( b + \sum w_i x_i \right) \right)^2$$

The functions with  
smaller  $w_i$  are better

$$+ \lambda \sum (w_i)^2$$

➤ Smaller  $w_i$  means ...

smoother

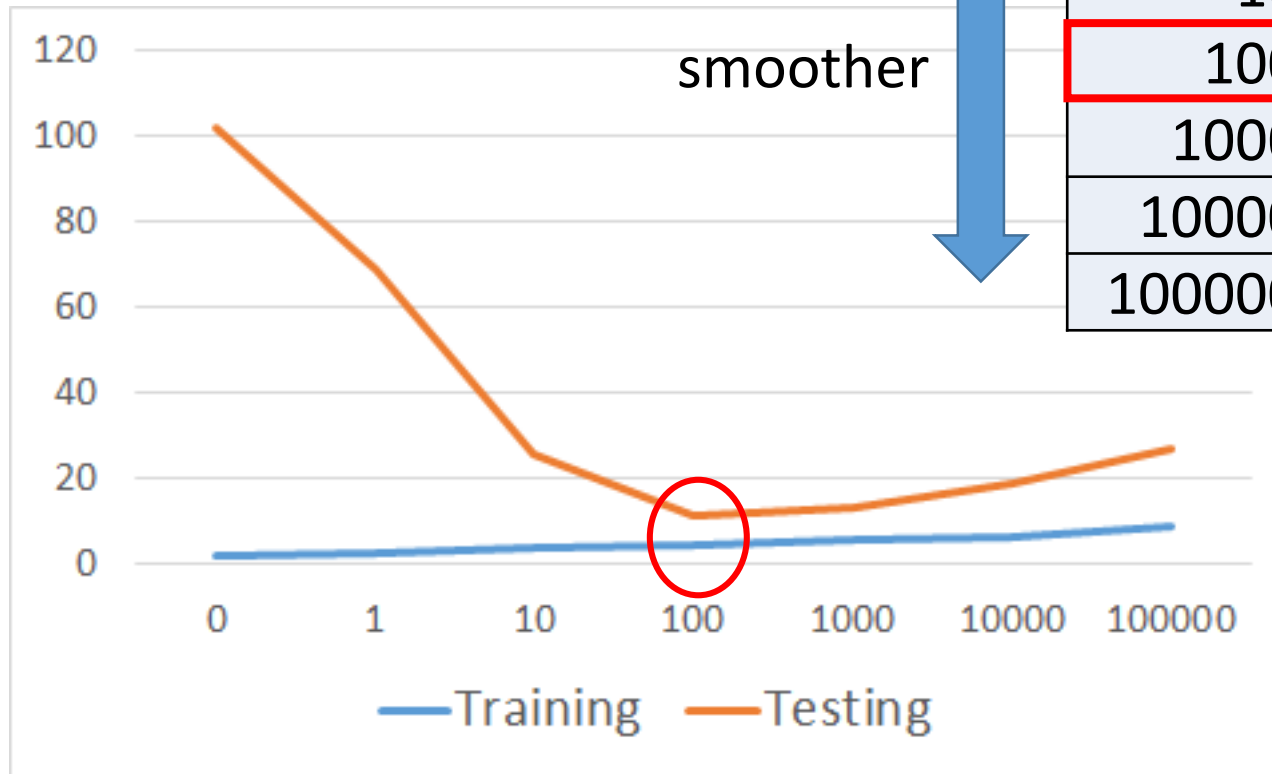
$$y = b + \sum w_i x_i$$

$$y + \sum w_i \Delta x_i = b + \sum w_i (x_i + \Delta x_i)$$

➤ We believe smoother function is more likely to be correct

Do you have to apply regularization on bias?

# Regularization



$\lambda$	Training	Testing
0	1.9	102.3
1	2.3	68.7
10	3.5	25.7
100	4.1	11.1
1000	5.6	12.8
10000	6.3	18.7
100000	8.5	26.8

How smooth?

Select  $\lambda$  obtaining the best model

- Training error: larger  $\lambda$ , considering the training error less
- We prefer smooth function, but don't be too smooth.

# Conclusion

- Pokémon: Original CP and species almost decide the CP after evolution
  - There are probably other hidden factors
- Gradient descent
  - More theory and tips in the following lectures
- We finally get average error = 11.1 on the testing data
  - How about new data? Larger error? Lower error?
- Next lecture: Where does the error come from?
  - More theory about overfitting and regularization
  - The concept of validation

# Reference

- Bishop: Chapter 1.1

# Acknowledgment

- 感謝 鄭凱文 同學發現投影片上的符號錯誤
- 感謝 童寬 同學發現投影片上的符號錯誤
- 感謝 黃振綸 同學發現課程網頁上影片連結錯誤的符號錯誤