definition + notation we say that a sequence (x_n) weakly converges to x if for all y we have $\lim \langle x_n, y\rangle = \langle x, y\rangle$. we write this as $x_n \xrightarrow{w} x$.

example any orthonormal sequence (in particular, say, e_n in ℓ^2) weakly converges to 0.

exercises
i. weak convergence is linear.
ii. $x_n \longrightarrow x$ implies $x_n \xrightarrow{w} x$. namely regular convergence implies weak convergence.
iii. moreover, $x_n \longrightarrow x$ iff $x_n \xrightarrow{w} x$ and $\lim \|x_n\| = \|x\|$
iv. in finite dimensional Hilbert spaces, weak and regular convergence are equivalent.
v. if $\lim \langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in \overline{\mathrm{span}(x_n, x)}$ then $x_n \xrightarrow{w} x$.
vi. $x_n \xrightarrow{w} x$ implies $x_{n_k} \xrightarrow{w} x$ for any subsequence n_k.
vii. $x_n \xrightarrow{w} x$ implies $\|x\| \leq \liminf \|x_n\|$.

weak sequential compactness theorem any bounded sequene has a weakly convergent subsequence.

theorem (Banach-Saks) if x_n weakly converges to x then it has a subsequence x_{n_k} which is Cesàro convergent to x. in particular, a convex closed set in a Hilbert space is closed under weak convergence.

definition we say that a sequence (x_n) is weakly Cauchy if for all y the scalar sequence $\langle x_n, y\rangle$ is Cauchy (ie convergent).

proposition a sequence is weakly convergent iff it is weakly Cauchy.

lemma any weakly Cauchy sequence is bounded.

proof of lemma let $V_m = \{y : |\langle x_n, y\rangle| \le m \ \ \forall n\}$ so that each V_m is closed and their union is the entire Hilbert space. applying the Baire category theorem, fix $B(y,\varepsilon) \subseteq V_m$. this yields that $\|x_n\| = \frac{1}{\varepsilon}\left|\left\langle x_n, \varepsilon\frac{x_n}{\|x_n\|} + y - y\right\rangle\right| \le \frac{2m}{\varepsilon}$ is bounded.

proof of proposition fix x_n weakly Cauchy. let $\phi(y) = \lim \langle x_n, y\rangle$. then ϕ is a linear functional, which is bounded because x_n is. applying Riesz theorem, there exist x so that $\lim \langle x_n, y\rangle = \langle x, y\rangle$ for all y.

proof of weak sequential compactness let $\|x_n\| \le M$ be any bounded sequence. applying v, we assume H is seperable, and let (d_n) be a dense sequence. we let $x_{0,n} = x_n$ and generally $x_{k,n}$ be a subsequence of $x_{k-1,n}$ so that $\langle x_{k,n}, d_k\rangle$ converges. let $x_n^* = x_{n,n}$ be the diagonal subsequence. then $\langle x_n^*, d_k\rangle$ converges for all k. we show that x_n^* is weakly Cauchy. fix y and ε, let k be st $\|y - d_k\| \le \varepsilon/M$ and let N be such that $|\langle x_n^* - x_m^*, d_k\rangle| \le \varepsilon$ for all $n, m \ge N$. then we have $|\langle x_n^* - x_m^*, y\rangle| \le 3\varepsilon$ for all $n, m \ge N$ as we wanted.

proof of Banach Saks by linearity we assume $x = 0$. applying the lemma, let $\|x_n\| \le M$. now, $n_1 = 1$, and generally let n_k be such that $|\langle x_{n_k}, x_{n_m}\rangle| \le 1/k$ for $m = 1, \ldots, k-1$. then

$$\left\|\frac{1}{k}\sum_{j=1}^{k} x_{n_j}\right\|^2 = \frac{1}{k^2}\left(\sum_{j=1}^{k}\left\|x_{n_j}\right\|^2 + 2\sum_{i<j}\Re\left\langle x_{n_i}, x_{n_j}\right\rangle\right) \le \frac{M^2}{k} + \frac{2}{k^2}\left(\frac{1}{2} + \frac{2}{3} + \cdots + \frac{k-1}{k}\right)$$

converges to 0 as $k \longrightarrow \infty$.