introduction fix $\alpha = \sqrt[3]{6}$. numerically, we may think of α as an infinite expansion of digits 1.25992.... however, algebraically we understand α as a solution to the equation

$$\alpha^3 = 6$$

(the real one).

pick something a little more complicated, say $\beta = 1 + \sqrt{3} = 2.73205....$ what algebraic equation does β solve? well, we have $(\beta - 1)^2 = 3$, i.e.

$$\beta^2 - 2\beta - 2 = 0$$

stepping up, let us fix $\gamma = \sqrt{2} + \sqrt{3}$. what polynomial does γ solve?
again, an algebraic manipulation yields $\gamma^2 = 2 + 3 + 2\sqrt{2}\sqrt{3} \implies (\gamma^2 - 5)^2 = 24$, namely

$$\gamma^4 - 10\gamma^2 + 1 = 0$$

problem what polynomial (monic with integer coefficients) does $\lambda = \sqrt{5} + \sqrt[3]{2}$ solve?

magic solution let us consider the principal vector $v = (1, \sqrt{5}, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{5}\sqrt[3]{2}, \sqrt{5}\sqrt[3]{4})$ as well as the product λv. it is evident each of the coordinates of λv can be given as linear integer combinations of the coordinates of v. explicitly we have

$$\lambda v = (\sqrt{5} + \sqrt[3]{2})(1, \sqrt{5}, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{5}\sqrt[3]{2}, \sqrt{5}\sqrt[3]{4}) = v \begin{pmatrix} 0 & 5 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

it follows that λ is an eigenvalue of this integer matrix - a root of its characteristic polynomial.
explicitly,

$$\det(\lambda I - M) = \lambda^6 - 15\lambda^4 - 4\lambda^3 + 75\lambda^2 - 60\lambda - 121 = 0$$

motivation when calculating the powers $1, \lambda, \lambda^2, \lambda^3, \ldots$ we can express each number as a linear combination with integer coefficients in $(1, \sqrt{5}, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{5}\sqrt[3]{2}, \sqrt{5}\sqrt[3]{4}) = (a, b, c, d, e, f)$. assuming we know how to express λ^k as such a combination, to do the same for λ^{k+1} we must know the multiplication table - namely to express $\lambda a, \lambda b, \ldots, \lambda f$ in linear terms of $a, b, \ldots, f$ with integer coefficients.
we find

$$\begin{aligned} \lambda a &= b + c \\ \lambda b &= 5a + e \\ \lambda c &= e + d \\ \lambda d &= 2a + f \\ \lambda e &= 5c + f \\ \lambda f &= 5d + 2b \end{aligned}$$

finally either one notice this means λ is an eigenvalue of this multiplication table (as we did in the solution), or one notices that the seven numbers $1, \lambda, \lambda^2, \ldots, \lambda^6$ when expressed as linear combinations with integer coefficients in the six variables $a, b, \ldots f$ must admit a linear dependence. this simply means that there's a polynomial in λ of degree six with rational coefficients which is evaluates to zero.

exercises

- find a polynomial of degree 3 with integer coefficients for which $1 + \sqrt[3]{2} + \sqrt[3]{4}$ is a root.
- find a polynomial of degree 4 with integer coefficients for which $\sqrt[4]{3} - 2\sqrt{3}$ is a root.