proposition for any real symmetric matrix $A \in \mathbf{R}^{n \times n}$ there exists an orthonormal basis of $\mathbf{R}^n$ diagonalizing A.

proof by induction. first, we find an eigenvector for A, as follows. let v be a minimizer of the function $f(x) = x^t Ax$ on the unit sphere $\{x \in \mathbf{R}^n \mid g(x) = x^t x = 1\}$. by the method of Lagrange multipliers, there exists some real λ for which $\nabla f|_v = \lambda \nabla g|_v$, which reads $Av = \lambda v$. to finish, we note the following.

exercise let A be a real symmetric matrix, v a unit norm eigenvector. then $v^\perp$ is A-stable. futhermore, $A' = A|_{v^\perp} : v^\perp \to v^\perp$ is self adjoint.