theorem [von Neumann] let $A : \mathscr{H} \to \mathscr{H}$ be a unit norm operator. then $\forall x \in \mathscr{H}$ the sequence $A^n x$ Cesàro converges to the fixed point of A closest to x. that is,

$$\frac{1}{N+1} \sum_{k=0}^{N} A^k x \longrightarrow \mathrm{Proj}_M x$$

where $M = \{s \in \mathscr{H} : As = s\}$.

lemma let A be a unit norm operator on a Hilbert space. then A and A^* have the same fixed points.

the lemma is an immediate consequence of the Cauchy Schwarz inequality.

proof of theorem let $B_N = \frac{1}{N+1} \sum_{k=0}^{N} A^k$. the claim is immediate for $x \in M$ a fixed point of A. so let us fix $x \in M^\perp$ and show $B_N(x) \longrightarrow 0$. we fix $\varepsilon > 0$. since $M = \mathrm{fix}_A = \mathrm{fix}_{A^*} = \ker(A^* - I)$ we have $x \in M^\perp = \overline{\mathrm{img}(A - I)}$ and thus $\|x - y\| \leq \varepsilon$ for some $y = Az - z$. now $B_N(y) = \frac{A^{N+1}z - z}{N+1} \longrightarrow 0$ since $\|A\| = 1$. moreover, $B_N(x - y)$ has norm $\leq \varepsilon$ since $\|B_N\| \leq 1$. thus $\|B_N(x)\| \leq 2\varepsilon$ for sufficiently large N.