Fourier analysis

definition. given $f : \mathbf{R} \longrightarrow \mathbf{C}$ integrable, we let $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-2\pi i \xi x} \mathrm{d}x$.

exercise 1. $\hat{f} : \mathbf{R} \longrightarrow \mathbf{C}$ is a well defined uniformly continuous function. moreover $\hat{f}(0) = \int_{\mathbf{R}} f$ and $||\hat{f}||_\infty \leq \|f\|_1$.

claim 2. we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \hat{f}(\xi) = \int_{\mathbf{R}} -2\pi i x f(x) e^{-2\pi i \xi x} \mathrm{d}x$$

assuming $xf(x)$ is integrable.

more generally,

$$\frac{\mathrm{d}^k}{\mathrm{d}\xi^k} \hat{f}(\xi) = (-2\pi i)^k (\widehat{x^k f(x)})(\xi)$$

assuming $x^k f(x)$ is integrable.

exercise 3. $f = \chi_{[0,1]} \implies \hat{f}(\xi) = \dfrac{1 - e^{-2\pi i \xi}}{2\pi i \xi}$ (with $\hat{f}(0) = 1$).

claim 4. $f(x) = e^{-\pi x^2} \implies \hat{f}(\xi) = e^{-\pi \xi^2}$

proof sketch. we have $\int_{\mathbf{R}} e^{-\pi(x+a)^2} \mathrm{d}x = 1$ for all real a. analytically continuing this for complex a, one simply plugs $a = i\xi$ to get the desired identity.

proof 1. let $g(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}} e^{-\pi x^2 - 2\pi i \xi x} \mathrm{d}x$. then $g'(\xi) = -2\pi i \int_{\mathbf{R}} x e^{-\pi x^2 - 2\pi i \xi x} \mathrm{d}x$. let $u' = xe^{-\pi x^2}$ and $v = e^{-2\pi i \xi x}$ so that $g'(\xi) = 2\pi \xi g(\xi)$. it remains to see $g(0) = 1$, but indeed $\int_{\mathbf{R}} e^{-\pi x^2} \mathrm{d}x = 1$.

proof 2. apply $\oint_{\partial M} e^{-\pi z^2} \mathrm{d}z = 0$ where M is the rectangle $[-R, R] \times [0, \xi]$ with R tending to infinity.

claim 5. $\hat{f} \equiv 0$ implies $f = 0$ a.e.

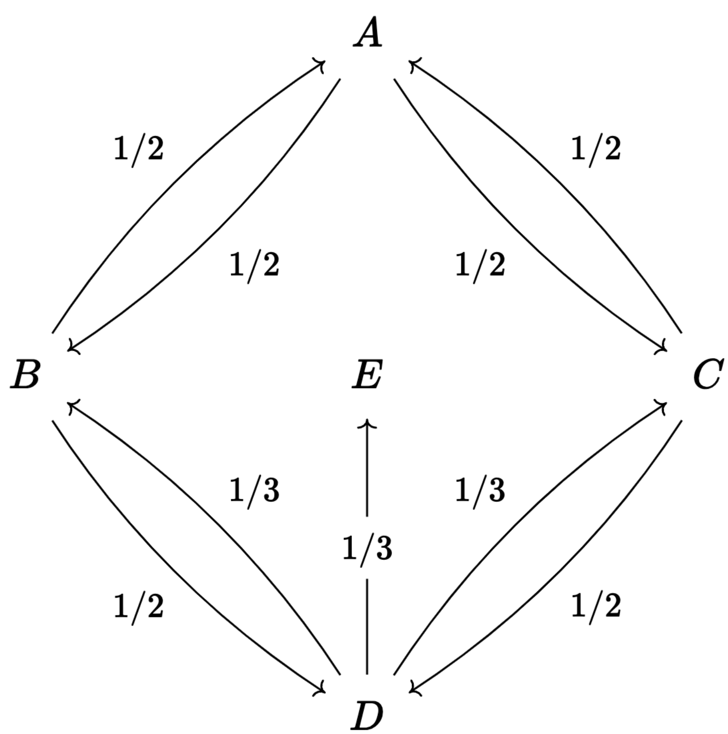
proof. see [1]

claim 6. $\lim_{|\xi| \longrightarrow \infty} \hat{f}(\xi) = 0$.

[1] https://matthewhr.wordpress.com/2012/02/20/elegant-proof-of-fourier-uniqueness/

proof. this holds for the characteristic function $\chi_{[0,1]}$, and generally for all simple step functions. since these step functions are dense in L^1, and we have the bound $||\hat{h}||_\infty \leq \|h\|_1$, we have $\lim_{|\xi| \longrightarrow \infty} \hat{f}(\xi) = 0$ for arbitrary integrable f.

exercise.[2,3] a scientist puts a rat suffering from amnesia inside a maze. the entrance is position A, and the exit is at position E. at any point the rat travels to a new position by randomly selecting an edge that's available at its current state (independently of how he got to that state). assuming the rat travels at a constant speed of 1 edge per second, how long should we expect the rat's journey to take?



problem. place a knight randomly on a chessboard. do the following 10^{10} times: move the knight to a position it can legally move to, chosen at random independently of all previous choices.

in total, about how many times should we expect the knight occupied the upper right corner square?

what about any other square?

what if the knight started at some fixed position instead?

problem. consider a stochastic chain with two states a, b and transition probabilities $P_{aa} = p$, $P_{ab} = 1-p$, $P_{bb} = q$, $P_{ba} = 1-q$. find $\lim_{n \longrightarrow \infty} P(X_n = a)$.

[2]taken from columbia.edu/~ks20/stochastic-I/stochastic-I-MCI.pdf
[3]it's fun to simulate this exercise on a computer

suppose we place a dot at 0, and each second we move it one place up or down randomly and independently of all previous moves. we call this process a random walk on the integers.

exercise.

i. show that with probability 1 we return to the origin at some point in the future.

ii. how many seconds does it take, on average, until we return to the origin for the first time.

solution. the probability we first hit 0 at time $2k$, $k \geq 1$ is $2\frac{W_k}{2^{2k}}$ where $W_k = \frac{(2k-2)!}{(k-1)!^2 k}$ is the number of ± 1 sequences of length $2k$ whose sum is 0 and any proper prefix sum is positive. thus the expected number of seconds till we hit 0 is

$$\sum_{j \geq 0} \frac{(2j)!}{j!^2 4^j} = \infty$$

(using $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$)

setting. a stochastic chain is described by:

a nonempty finite set of states Q,

for any pair of states $q, q' \in Q$, a probability $P_{qq'}$ to transition to q' from q

(ie a matrix $P \in [0,1]^{Q \times Q}$ such that $\sum_{q'} P_{qq'} = 1 \; \forall q \in Q$),

a starting state probability distribution σ on Q.

and admits a sequence of random variables $X_0, X_1, X_2, \ldots$ on Q such that

$$\Pr(X_0 = q_0, X_1 = q_1, \ldots, X_n = q_n) = \sigma_{q_0} p_{q_0 q_1} \cdots p_{q_{n-1} q_n}$$

called the stochastic chain.

given a stochastic chain, we may ask questions such as the following:

what is the probability the 3rd state we visit is q_0?

what is the expected number of steps until we visit q_0 for the first time?

what is the expected number of steps until each state was visited?

exercise. what is the probability distribution of the nth state we visit, X_n? (for $n = 0$ the answer is σ)

solution. σP^n (where we think of σ as a row vector).

setting. given states q, q', we write $q \longrightarrow q'$ if for some $n \geq 0$ there is a positive probability to transition from q to q' after n steps. that is, if $P^n_{qq'} > 0$ for some n. we write $q \longleftrightarrow q'$ if $q \longrightarrow q'$ and $q' \longrightarrow q$, in which case we call q, q' connected.

exercise. $\longleftrightarrow$ is an equivalent relation on Q.

setting. the chain is called irreducible if any two states are connected.

setting. a state q is called recurrent if the expected number of steps to hit q is finite, assuming we start at q.

proposition. q is recurrent iff $\sum_n P^n_{qq} = \infty$

exercise. solve in integers

$$a + b + c = 100$$
$$abc = 2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11$$
$$1 \leq a \leq b \leq c$$

solution. considering mods 2, 3, and 4 we find that $a, b, c \not\equiv_4 2$ and $a, b, c \not\equiv_3 1$. the factors of abc that are divisible by 7 and under 100 are $7, 14, 21, 28, 35, 42, 49, 70, 77, 84, 98$. after complience mods 3 and 4, we're left with $21, 35, 77, 84$. two members of a, b, c are among this short list. since $84 + 21 > 100$, we can't have $84 \in \{a, b, c\}$. similarly, if $77 \in \{a, b, c\}$ then $21 \in \{a, b, c\}$ meaning $\{a, b, c\} = \{2, 21, 77\}$ and this can't be. thus $a = 21, b = 35, c = 44$.

exercise. $I = \int_0^{\pi/2} \log(\sin x)\mathrm{d}x$

solution. note that $I = \int_0^{\pi/2} \log(\cos x)\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \log(\sin x)\mathrm{d}x$. using $\sin 2x = 2\sin x \cos x$ we get $I = -\pi \log 2$.

proposition. let $A \in \mathbf{F}_2^{n\times n}$ be a symmetric matrix. then $(a_{11}, a_{22}, \ldots, a_{nn}) = \operatorname{diag}(A)$ belongs to the row span of A.

proof. one has to show that $\operatorname{diag}(A) \cdot x = 0$ for all $x \in \ker A$. indeed, $Ax = 0 \implies x^t Ax = 0$, and over $\mathbf{F}_2$ we have $x^t Ax = \sum_i a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_ix_j = \operatorname{diag}(A) \cdot x$.

proposition. let $f : [0,1] \times [0,1] \longrightarrow \mathbf{R}$ be continuous in each separate variable. if $f = 0$ on a dense subset of $[0,1]^2$ then $f \equiv 0$.

proof. let (x_0, y_0) be an interior point with $f(x_0, y_0) = r_0 > 0$. fix a closed interval $I = [x_0 - \varepsilon, x_o + \varepsilon] \times \{y_0\}$ such that $f(I) \subseteq [r/2, 3r/2]$. let $F_n = \{(x, y_0) \in I : f(x, y) \in [r/3, 6r/3]\ \forall |y - y_0| < \frac{1}{n}\}$ so that $\bigcup F_n = I$. applying the category theorem, we have some $\overline{F_m}$ containing some open interval $J \times \{y_0\}$. thus for each $x \in J$ and each $|y - y_0| < \frac{1}{m}$ we have $f(x, y) = \lim f(x_n, y) \geq r/3$ for some $(x_n, y_0) \in F_m$, $\lim x_n = x$. this contradicts the assumption that $f = 0$ on a dense subset of the square.

exercise. deduce Liouville's theorem from the Schwarz lemma.

exercise. demonstrate the Cauchy Schwarz inequality using invariance.

solution. wlog $\|x\| = \|y\| = 1$. it remains to show $|\sum x_i y_i| \leq 1$. this follows from $|x_i y_i| \leq \dfrac{x_i^2 + y_i^2}{2}$.

proposition. the only integer matrix of determinant one that is a kth power of an integer matrix for all k is the identity matrix.

proof. fix a prime p. then A is invertible mod p, and taking $k = |\mathrm{GL}_n(p)|$ yields $A = X^k \equiv I \mod p$. since there's infinitely many primes, we have $A = I$.

problem.[4] Hat Trick. The audience has a bottomless supply of hats in ten different colors. They arrange ten people in a line and put one of the hats on each person. Then the magician's assistant comes in and removes a hat from one of the ten people. After that, the magician appears and, abracadabra, guesses the color of the removed hat. The magician and the assistant agreed on a strategy beforehand. What is it?

solution. match the colors with the residues mod 10. if the sum of the 10 hats is x, we remove the hat of the person at position x.

[4]taken from blog.tanyakhovanova.com

problem.[5] let a, b be positive integers and let A, B be finite sets of integers for which $A \sqcup B = (A + a) \sqcup (B - b)$. (for example, $A = [1, n]$, $B = [n + 1, m]$, $a = m - n$, $b = n$ is such a configuration). show that $\frac{|A|}{|B|} = \frac{b}{a}$.

solution. let $f = \chi_A$ and $g = \chi_B$ so that

$$f(s) + g(s) = f(s - a) + g(s + b) \qquad (\star)$$

we may extend this identity to the domain of real numbers by setting $h(s + t) = h(s)$ for $t \in [0, 1)$, $h \in \{f, g\}$. since we have $|A| = \int f$ and $|B| = \int g$ we're led to apply the Fourier transform to $(\star)$ and set the input to 0. the transformed identity

$$\hat{f}(\xi) + \hat{g}(\xi) = \hat{f}(\xi)e^{-2\pi i a\xi} + \hat{g}(\xi)e^{2\pi i b\xi}$$

will yield nothing at $\xi = 0$. however, deriving this identity and setting $\xi = 0$ does yield $a\hat{f}(0) = b\hat{g}(0)$.

[5] taken from dgrozev.wordpress.com

exercise. let M be a finite semigroup of size n. then $\forall m \in M$ the element $m^{\mathrm{lcm}(1,2,\ldots,n)}$ is idempotent.

solution. by the pigeonhole principle, $m^i = m^j$ for some $1 \leq i < j \leq n+1$. thus $m^\alpha = m^{\alpha+j-i}$ for all $\alpha \geq i$, and so $m^\alpha = m^{\alpha+k(j-i)}$ for all $\alpha \geq i$, $k \geq 0$. taking $\alpha = \mathrm{lcm}(1,2,\ldots n)$ and $k = \dfrac{\alpha}{j-i}$ yields m^α is idempotent.

corollary. if $A \in \mathbf{Z}^{n\times n}$ is a kth power of an integer matrix for all $k \geq 1$, then A is an idempotent.

proposition. let S be a linear space of bounded measurable functions on $[0, 1]$. if S is closed in $L^2[0, 1]$, then S is finite dimensional.

Theorem. (Kakutani) Let G be a group operating by isometries on a Banach space X, leaving invariant a compact convex set K. Then G has a fixed point in K.