that $x^3 - 2$ is irreducible over $\mathbf{Q}$ implies this is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbf{Q}$ and thus 1, $\sqrt[3]{2}$, $\sqrt[3]{4}$ are linearly independent over $\mathbf{Q}$. let me now give a high-school-olympiad-style proof of this result.
suppose $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$ where wlog $a, b, c \in \mathbf{Z}$ are not all zero. multiplying by $\sqrt[3]{2}$ and by $\sqrt[3]{4}$ we have

$$\begin{aligned}(a, b, c) \cdot (1, \sqrt[3]{2}, \sqrt[3]{4}) &= 0\\ (2c, a, b) \cdot (1, \sqrt[3]{2}, \sqrt[3]{4}) &= 0\\ (b, 2c, a) \cdot (1, \sqrt[3]{2}, \sqrt[3]{4}) &= 0\end{aligned}$$

from which it follows that

$$\begin{vmatrix} a & b & c \\ 2c & a & b \\ 2b & 2c & a \end{vmatrix} = a^3 + 2b^3 + 4c^3 - 6abc = 0 \qquad (\star)$$

now, for any integer solution (a, b, c) of $\star$ we clearly must have a even. furthermore, $(b, c, a/2)$ is also an integer solution of $\star$. reapplying this principle, if (a, b, c) is an integer solution of $\star$ then all the coordinates are even and $(a/2, b/2, c/2)$ is too an integer solution of $\star$. since only zero is infinitely divisible by 2, there are no nonzero solutions of $\star$ in integers, and 1, $\sqrt[3]{2}$, $\sqrt[3]{4}$ are linearly independent over $\mathbf{Q}$.

<u>exercise</u> use the above method to show in general that $x^n - p$ is irreducible over $\mathbf{Q}$, or equivalently that $1, p^{\frac{1}{n}}, \ldots, p^{\frac{n-1}{n}}$ are linearly independent over $\mathbf{Q}$ for any $n \geq 1$ and any prime number p.