problem there are n boys and n girls in a room. each person is friends with exactly k people of the other gender, and friendship is always mutual. the girls noticed that any pair of them have precisely c common boy friends. show that then any pair of boys have precisely c common girl friends.

<u>solution</u> let $A \in \{0,1\}^{n\times n}$ be the matrix $A_{ij} = \begin{cases} 1 & \text{girl } i \text{ is friends with boy } j \\ 0 & \text{else} \end{cases}$. then the conditions are $Aj = kj$ and $j^t A = kj^t$ where j is the all ones vector, as well as $AA^t = \begin{bmatrix} k & c & \cdots & c \\ c & k & \ddots & \vdots \\ \vdots & \ddots & \ddots & c \\ c & \cdots & c & k \end{bmatrix} = cJ + (k-c)I$ where J is the all ones matrix. applying $AJ = kJ$ yields $A(A^t - \dfrac{c}{k}J) = (k-c)I$. it follows[1] that $A^{-1} = \frac{1}{k-c}(A^t - \dfrac{c}{k}J)$, which implies $(A^t - \dfrac{c}{k}J)A = (k-c)I$. since $JA = kJ$, we have $A^tA = cJ + (k-c)I$, implying any pair of boys have precisely c common girl friends, as we wanted.

[1]if $c = k$ then all the girls are friends with all the boys and the problem is trivial.