proposition $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\mathrm{SL}_2(\mathbf{Z})$.

proof firstly note that $S^2 = -\mathrm{id}$ and $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for all $n \in \mathbf{Z}$. thus we have

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$$

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

$$ST^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a+nc & b+nd \end{pmatrix}$$

fixing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, we'll show that $A \in \langle T, S \rangle$ by induction on $|c|$.
base if $c = 0$ then $ad = 1 \implies a = d = \pm 1 \implies A = \pm T^{\pm d} \in \langle T, S \rangle$.
step if $c \neq 0$ then we pick n for which $|a + nc| < |c|$. by the induction hypothesis $ST^n A \in \langle T, S \rangle$ and so $A \in \langle T, S \rangle$.

programming exercise in your favourite language write a function which takes a general $\mathrm{SL}_2(\mathbf{Z})$ matrix as input and outputs a string in $T^{\pm}, S^{\pm}$ that equals the input.