proposition let L be a regular language. let $\ell_n = \#$words of length n in L. then $\sum_{n\geq 0} \ell_n z^n$ is a rational function of z.

proof let M be a dfa for L with state space Q, and let A be the matrix whose q_1, q_2 entry is the number of letters σ that transition q_1 into q_2. then ℓ_n equals the sum of entries of A^n whose row is the start state and whose columns are the accepting states, namely $\ell_n = \chi_{q_0}^t A^n \chi_{\text{acc}}$. thus $\sum \ell_n z^n = \chi_{q_0}^t \sum z^n A^n \chi_{\text{acc}} = \chi_{q_0}^t (I - zA)^{-1} \chi_{\text{acc}}$ equals a sum of (the same) entries of $(I - zA)^{-1}$. these entries are rational functions of z. in fact, using the adjugate, we may take the denominator to be $\det(I - zA)$, which has degree $|Q|$.