exercise $\max\{\|T(a+b)\|, \|T(a-b)\|\} \geq \max\{\|Ta\|, \|Tb\|\}$

corollary let $T : X \longrightarrow Y$ be a bounded operator. then $\forall B = B(x_0, r)$ we have $\|T\| r \leq \sup\limits_{x \in B} \|Tx\|$.

uniform boundedness theorem let $\mathcal{F} \subseteq \mathfrak{B}(X,Y)$ be a family of bounded operators. if $\sup\limits_{T \in \mathcal{F}} \|T\| = \infty$ then there exists x_0 for which $\sup\limits_{T \in \mathcal{F}} \|Tx_0\| = \infty$.

proof (Sokal) pick a sequence $T_n \in \mathcal{F}$ with $\|T_n\| \geq 4^n$. let $p_0 = 0$, let $r_n = 3^{-n}$ and choose $p_n \in B(p_{n-1}, r_n)$ such that $\frac{2}{3}\|T_n\| r_n \leq \|T_n p_n\|$. let $x_0 = \lim p_n$, for which $\|x_0 - p_n\| \leq \frac{1}{2}3^{-n}$. in total, $\|T_n x_0\| \geq \frac{1}{6}\|T_n\| 3^{-n}$ tends to infinity.