theorem there exists an outer automorphism of S_6.

<u>lemma</u> let $\mathbf{F}$ be a field, $\mathbf{P} = \mathbf{F} \cup \{\infty\}$ the projective line over $\mathbf{F}$. then $\text{Möb}(\mathbf{F})$ acts sharply three transitively on $\mathbf{P}$.

<u>proof of theorem</u> let $\mathbf{P} = \mathbf{F}_5 \cup \{\infty\} = \{0, 1, 2, 3, 4, \infty\}$ be our six element set. we have an inclusion $\text{Möb}(\mathbf{F}_5) \leq S_{\mathbf{P}}$ of index $\dfrac{6!}{(5^2-1)(5^2-5)/(5-1)} = 6$. therefore the coset space $C = S_{\mathbf{P}}/\text{Möb}(\mathbf{F}_5)$ is another six element set on which $S_{\mathbf{P}}$ naturally acts. we have thus defined a homomorphism $\rho : S_{\mathbf{P}} \to S_C$. we'll show it is bijective by showing $\ker \rho$ is trivial. indeed, $\ker \rho$ is a normal subgroup of $S_{\mathbf{P}}$, so if it were not trivial it would be either the whole $S_{\mathbf{P}}$ or the alternating group of index two. however, when G acts on G/H, the kernel of the action is a subgroup of H and so has index at least $[G : H]$. now we claim $\text{id}, (12), (13), (23), (123), (132)$ is a set of representatives for the coset space C. indeed, if $g, h \in S_{\mathbf{P}}$ fix each of $0, 4, \infty$ then so does $g^{-1}h$, and so $g\text{Möb}(\mathbf{F}_5) = h\text{Möb}(\mathbf{F}_5)$ implies $g = h$. now, consider the action of the three cycle $g = (123)$ on C via these representatives. we have

$$g : \begin{array}{cccccc} \text{id} & (12) & (13) & (23) & (123) & (132) \\ (123) & (13) & (23) & (12) & (132) & \text{id} \end{array}$$

which as a permutation of C decomposes as two three cycles. thus

$$S_{\mathbf{P}} \xrightarrow{\rho} S_C \xrightarrow{C \cong P} S_{\mathbf{P}}$$

gives an automorphism of $S_{\mathbf{P}}$ where (123) gets sent to $(0, 4, \infty)(123)$ - but any inner automorphism of S_n preserves the cycle decomposition structure of permutations, and so such an automorphism must be outer.