proposition let $f : \mathbf{R}^n \longrightarrow \mathbf{C}$ be harmonic. if f admits a finite limit at infinity, then f is a constant.

proof let $\ell = \lim_{x \longrightarrow \infty} f(x)$ and let $p \in \mathbf{R}^n$ be arbitrary. for any given ε, we pick r so that $|f(x) - \ell| \leq \varepsilon$ for all x outside $B_r(p)$. then we get $|f(p) - \ell| = |\operatorname*{ave}_{S_r(p)} f - \ell| \leq \operatorname*{ave}_{S_r(p)} |f - \ell| \leq \varepsilon$.

theorem let $f : \mathbf{R}^n \longrightarrow \mathbf{C}$ be harmonic. if f is bounded, then f is a constant.

proof suppose $|f(x)| \leq M$ for all x. let $p \neq q$ be arbitrary points. then

$$|f(p) - f(q)| = \frac{\left| \int_{B_r(p)} f - \int_{B_r(q)} f \right|}{\operatorname{vol}(B_r)} \leq \frac{M \cdot \operatorname{vol}\left(B_r(p) \triangle B_r(q)\right)}{\operatorname{vol}(B_r)}$$

for all r. letting $r \uparrow \infty$ yields[1] $f(p) = f(q)$.

[1] let $d = \|p - q\|$, so that $B_r(p) \backslash B_r(q) \subseteq \{x \mid r - d \leq \|x - p\| < r\}$, implying $\operatorname{vol}\left(B_r(p) \backslash B_r(q)\right)$ is bounded by a polynomial of degree $n - 1$, and so negligible when compared to $\operatorname{vol}(B_r)$.