proposition (Euler) if x is the sum of four squares and y is the sum of four squares, so is xy.

exercise if $2n$ is the sum of four squares then n is the sum of four squares.

claim given any prime p, there exists $1 \leq m < p$ such that $mp - 1$ is the sum of two squares.

proposition any prime p is the sum of four squares.

corollary (Lagrange) each nonnegative integer is the sum of four squares.

Euler's identity

$$(a+b+c+d)^2(x+y+z+w)^2 = (ax+by+cz+dw)^2+(ay-bx+cw-zd)^2+(az-cx-bw+dy)^2+(aw-dx+bz-cy)^2$$

shows that the product of two sums of four squares is a sum of four squares.

proof of claim we may take p odd. let $A = \{x^2 \mid x \in \mathbf{F}_p\}$ and $B = \{-1 - y^2 \mid y \in \mathbf{F}_p\}$. then $|A| = |B| = \dfrac{p+1}{2}$, which implies $A \cap B$ is nonempty. let us have then $x^2 \equiv -1 - y^2 \mod p$. picking representatives $[-\frac{p-1}{2}, \frac{p-1}{2}]$ we have $x^2 + y^2 + 1 = mp$ for some $0 < mp < 2(\dfrac{p}{2})^2 + 1 < p^2$, namely $1 \leq m < p$.

proof of proposition let k be the minimal positive integer for which kp is the sum of four squares. we need to show that $k = 1$. assuming the contrary and applying the claim and exercise, $1 < k < p$ is odd. write $kp = x_1^2+x_2^2+x_3^2+x_4^2$ as well as $x_i = kq_i+r_i$ with $r_i \in [-\frac{k-1}{2}, \frac{k-1}{2}]$. let $nk = r_1^2+r_2^2+r_3^2+r_4^2$. since $k \nmid p$ we must have $n > 0$. since $|r_i| < \frac{k}{2}$ we have $n < k$. it remains to show np is the sum of four squares. indeed, $k^2np = (x_1^2+x_2^2+x_3^2+x_4^2)(r_1^2+r_2^2+r_3^2+r_4^2) = (x_1r_1+x_2r_2+x_3r_3+x_4r_4)^2+(x_1r_2-x_2r_1+x_3r_4-x_4r_3)^2+(x_1r_3-x_3r_1-x_2r_4+x_4r_2)^2+(x_1r_4-x_4r_1+x_2r_3-x_3r_2)^2$. we see that each of the four terms being squared is a multiple of k, and deduce np is the sum of four squares.