that x^3-2 is irreducible over $\mathbf{Q}$ implies this is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbf{Q}$ and thus 1, $\sqrt[3]{2}$, $\sqrt[3]{4}$ are linearly independent over $\mathbf{Q}$. let me now give a high-school-olympiad-style proof of this result.
suppose $a+b\sqrt[3]{2}+c\sqrt[3]{4}=0$ where wlog $a,b,c\in\mathbf{Z}$ are not all zero. multiplying by $\sqrt[3]{2}$ and by $\sqrt[3]{4}$ we have

$$\begin{aligned}(a,b,c)\cdot(1,\sqrt[3]{2},\sqrt[3]{4})&=0\\(2c,a,b)\cdot(1,\sqrt[3]{2},\sqrt[3]{4})&=0\\(2b,2c,a)\cdot(1,\sqrt[3]{2},\sqrt[3]{4})&=0\end{aligned}$$

from which it follows that

$$\begin{vmatrix}a&b&c\\2c&a&b\\2b&2c&a\end{vmatrix}=a^3+2b^3+4c^3-6abc=0\qquad(\star)$$

now, for any integer solution (a,b,c) of $\star$ we clearly must have a even. furthermore, $(b,c,a/2)$ is also an integer solution of $\star$. reapplying this principle, if (a,b,c) is an integer solution of $\star$ then all the coordinates are even and $(a/2,b/2,c/2)$ is too an integer solution of $\star$. since only zero is infinitely divisible by 2, there are no nonzero solutions of $\star$ in integers, and 1, $\sqrt[3]{2}$, $\sqrt[3]{4}$ are linearly independent over $\mathbf{Q}$.

<u>exercise</u> use the above method to show in general that x^n-p is irreducible over $\mathbf{Q}$, or equivalently that $1,p^{\frac{1}{n}},\ldots,p^{\frac{n-1}{n}}$ are linearly independent over $\mathbf{Q}$ for any $n\geq 1$ and any prime number p.