claim let α be irrational and n be a positive integer. then there exists an integers p, q for which $\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq}$ and $1 \leq q \leq n$.

corollary (Dirichlet) for any irrational number α there exists infinitely many integer solutions to $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.

theorem (Pell) let A be a positive integer that is not a perfect square. then there exists infinitely many integer solutions to the equation $x^2 = Ay^2 + 1$.

proof of claim let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. consider the $n+1$ points $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ in $[0,1)$. divide it into n intervals $[0, \frac{1}{n}) \cup [\frac{1}{n}, \frac{2}{n}) \cup \ldots \cup [\frac{n-1}{n}, 1)$ so that by the pigeonhole principle, two of the points fall in the same interval. let these be $|\{j\alpha\} - \{i\alpha\}| < \frac{1}{n}$ for $0 \leq i < j \leq n$. write $q = j - i$ so that $|q\alpha - p| < \frac{1}{n}$ for $p = \lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor$, as we wanted.

proof of Pell's theorem let α be the irrational number $\sqrt{A}$. we have infinitely many solutions (p,q) to Dirichlet's equation, and for each we have $|p^2 - Aq^2| = |q\sqrt{A} - p| \cdot |q\sqrt{A} + p| < |\sqrt{A} + \frac{p}{q}| < 2\sqrt{A} + 1$. this gives a finite range for the whereabouts of the integer $p^2 - Aq^2$. by the infinite piegonhole principle, there exists $|k| < 2\sqrt{A} + 1$ with infinitely many solutions (p,q) such that $p^2 - Aq^2 = k$. since A is not a perfect square, $k \neq 0$. applying the infinite piegonhole principle yet again, we pick two solutions $(p_1, q_1), (p_2, q_2), (p_3, q_3), \ldots$ with $p_1 \equiv p_2 \equiv p_3 \equiv \ldots$ and $q_1 \equiv q_2 \equiv q_3 \equiv \ldots$ modulo $|k|$. one sees that $\frac{p_n + q_n\sqrt{A}}{p_1 + q_1\sqrt{A}}$ has the form $x_n + y_n\sqrt{A}$ for x_n, y_n integers with $x_n^2 = Ay_n^2 + 1$.