

that $x^3 - 2$ is irreducible over \mathbf{Q} implies this is the minimal polynomial of $\sqrt[3]{2}$ over \mathbf{Q} and thus $1, \sqrt[3]{2}, \sqrt[3]{4}$ are linearly independent over \mathbf{Q} . let me now give a high-school-olympiad-style proof of this result.

suppose $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$ where wlog $a, b, c \in \mathbf{Z}$ are not all zero. multiplying by $\sqrt[3]{2}$ and by $\sqrt[3]{4}$ we have

$$\begin{aligned}(a, b, c) \cdot (1, \sqrt[3]{2}, \sqrt[3]{4}) &= 0 \\ (2c, a, b) \cdot (1, \sqrt[3]{2}, \sqrt[3]{4}) &= 0 \\ (b, 2c, a) \cdot (1, \sqrt[3]{2}, \sqrt[3]{4}) &= 0\end{aligned}$$

from which it follows that

$$\begin{vmatrix} a & b & c \\ 2c & a & b \\ 2b & 2c & a \end{vmatrix} = a^3 + 2b^3 + 4c^3 - 6abc = 0 \quad (\star)$$

now, for any integer solution (a, b, c) of \star we clearly must have a even. furthermore, $(b, c, a/2)$ is also an integer solution of \star . reapplying this principle, if (a, b, c) is an integer solution of \star then all the coordinates are even and $(a/2, b/2, c/2)$ is too an integer solution of \star . since only zero is infinitely divisible by 2, there are no nonzero solutions of \star in integers, and $1, \sqrt[3]{2}, \sqrt[3]{4}$ are linearly independent over \mathbf{Q} .

exercise show that $x^n - 2$ is irreducible over \mathbf{Q} , or equivalently that $1, 2^{\frac{1}{n}}, \dots, 2^{\frac{n-1}{n}}$ are linearly independent over \mathbf{Q} for any n .