proposition there exists a 12×12 Hadamard matrix (an orthogonal matrix with ± 1 entries) proof sketch first, we notice that the following operations preserve Hadamard matrices : flipping the signs of all entries of some row (or column), and permuting the rows (or columns) of the matrix. with this in mind, the first three rows of our Hadamard matrix are wlog

$$\begin{matrix} + & + & + & + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & - & - & - & - & - & - \\ + & + & + & - & - & - & + & + & + & - & - & - \end{matrix}$$

now let $(x_1, \ldots, x_{12})$ be any of the remaining 9 rows. we divide it into 4 triplets $(x_1, x_2, x_3), (x_4, x_5, x_6), \ldots$, and let a, b, c, d be the sum of the 4 triplets. (so $a, b, c, d \in \{\pm 1, \pm 3\}$). then the condition of orthogonality with the first 3 rows is just $a = d = -b = -c$. thus $(a, b, c, d) \in \{\pm(3, -3, -3, 3), \pm(1, -1, -1, 1)\}$. if we had, say, the fourth be $+++------+++$, then it would not be possible for the fifth row to be orthogonal to the sum of the first 4 rows. wlog, this yields $(a, b, c, d) = (1, -1, -1, 1)$ for the remaining 9 rows. so we have to construct 9 rows, where each triplet has 3 possibilities (either $(++-), (+-+), (-++)$ or their negations). let us find out the condition for two of these remaining rows to be orthogonal. taking the dot product, if two corresponding triplets are equal, their dot product is 3. if they are different, their dot product is -1. that we are able to do this is our remaining task.

exercise there exists a 2 dimensional subspace V of $\mathbf{F}_3^4$ so that for any (distinct) $u, v \in V$ their Hamming distance is 3, namely they have 1 coordinate in agreement, and 3 coordinates differing.

exercise solution pick $V = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}_3^4 : x_1 + x_2 + x_3 = 0,\ \ x_4 = x_1 - x_2\}$. we can't have a Hamming distance of 4, because if all x_1, x_2, x_3 change and their sum remains 0, they must all increase by the same amount, causing x_4 to remain fixed. however, once two coordinates stay fixed, the remaining two coordinates are algebraically determined to do the same. explicitly,

$$\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 \\
0 & 2 & 1 & 1 \\
1 & 0 & 2 & 1 \\
1 & 1 & 1 & 0 \\
1 & 2 & 0 & 2 \\
2 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
2 & 2 & 2 & 0
\end{array}$$

proposition proof adding all the pieces up, we have a 12×12 Hadamard matrix

$$\begin{array}{cccccccccccc}
+ & + & + & + & + & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & - & - & - & - & - & - \\
+ & + & + & - & - & - & + & + & + & - & - & - \\
+ & + & - & - & - & + & - & - & + & + & + & - \\
+ & + & - & - & + & - & + & - & - & - & + & + \\
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+ & - & + & + & - & - & - & - & + & - & + & + \\
- & + & + & - & - & + & - & + & - & - & + & + \\
- & + & + & - & + & - & - & - & + & + & - & + \\
- & + & + & + & - & - & + & - & - & + & + & -
\end{array}$$