Examen 311.

Problema 1:

$f: \mathbb{R}^4 \to \mathbb{R}$, $f(x) = \ln(|x|^2+1)$, $x = (x_1, \dots, x_4)$

1). f radială $\Rightarrow f(x) = g(|x|) = g(r) = \ln(r^2+1)$

$$\Delta f(x) = g''(r) + \frac{N-1}{|r|} g'(r).$$

2) Cum $\Delta f(x) = g''(r) + \frac{N-1}{|r|} g'(r)$

$$g(r) = \ln(|r|^2+1) \Rightarrow g'(r) = \frac{1}{r^2+1} \cdot 2r = \frac{2r}{r^2+1}.$$

$$g''(r) = \left(\frac{2r}{r^2+1}\right)' = \frac{2(r^2+1) - 2r(2r)}{(r^2+1)^2} =$$

$$= \frac{2r^2+2-4r^2}{(r^2+1)^2} = \frac{-2r^2+2}{(r^2+1)^2} = \cancel{-2(r^2+1)}$$

$$\Delta f(x) = \frac{2-2r^2}{(r^2+1)^2} + \frac{3}{\cancel{r}} \cdot \frac{2\cancel{r}}{r^2+1} = \frac{2-2r^2+6(r^2+1)}{(r^2+1)^2} = \frac{4r^2+8}{(r^2+1)^2} = \frac{4(r^2+2)}{(r^2+1)^2}.$$

$u: B_1(0) \setminus \{0\} \to \mathbb{R}$; $u(x) = |x|^{-\frac{3}{5}}$, $x = (x_1 \dots x_5)$

3) $\Delta(x \cdot \nabla u) = \alpha \frac{u}{|x|^2}$.

$$\nabla u = \sum_{i=1}^{5} \frac{\partial u}{\partial x_i} e_i \dots$$

$$\frac{\partial u}{\partial x_i} = -\frac{3}{5} x_i |x|^{-\frac{3}{5}-2} = -\frac{3}{5} x_i |x|^{-\frac{13}{5}}$$

$$x_i \nabla u = -\frac{3}{5} x_1^2 |x|^{-\frac{13}{5}} + \dots + -\frac{3}{5} x_5^2 |x|^{-\frac{13}{5}} = -\frac{3}{5}|x|^{-\frac{13}{5}}(x_1^2 + \dots + x_5^2) =$$

$$= -\frac{3}{5}|x|^{-\frac{13}{5}} \cdot |x|^2 = -\frac{3}{5} \cdot |x|^{-\frac{3}{5}}.$$

Fie $g(x) = -\frac{3}{5}|x|^{-\frac{3}{5}} = h(|x|) = h(r) = -\frac{3}{5} r^{-\frac{3}{5}}$

$$\Delta g = h''(r) + \frac{N-1}{|r|} \cdot h'(r).$$

$h'(r) = -\frac{3}{5}\cdot -\frac{3}{5}\cdot r^{-\frac{3}{5}-1} = \frac{9}{25} r^{-\frac{8}{5}}$.

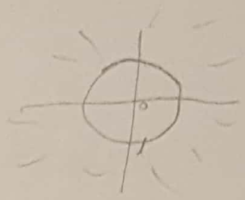
$h''(r) = \left(\frac{9}{25} r^{-\frac{8}{5}}\right)' = \frac{9}{25}\cdot -\frac{8}{5}\cdot r^{-\frac{13}{5}} = -\frac{72}{125} r^{-\frac{13}{5}}$.

$\Delta g(x) = -\frac{72}{125} r^{-\frac{13}{5}} + \frac{4}{r}\cdot \frac{9}{25} r^{-\frac{8}{5}}$.

$= r^{-\frac{13}{5}}\left(\frac{36}{25} - \frac{72}{125}\right) = r^{-\frac{13}{5}}\left(\frac{180-72}{125}\right) = r^{-\frac{13}{5}}\cdot \frac{108}{125}$

Știm $\alpha\cdot\frac{r^{-\frac{3}{5}}}{r^2} = \alpha\cdot r^{-\frac{13}{5}} \Rightarrow \boxed{\alpha = \frac{108}{125}}$

4). $u\in L^p(\mathbb{R}^5\setminus\overline{B(0,1)})$; ce valori ia p?



$\int_{B(0,1)} |u|^p < \infty$

$\int_{B(0,1)} |u|^p dx = \int_{B(0,1)} |x|^{-\frac{3p}{5}} dx \overset{\text{co-arie}}{=\!=} \int_0^1\left(\int_{\partial B(0,r)} r^{-\frac{3p}{5}} d\sigma(x)\right) dr =$

$= \int_0^1 r^{-\frac{3p}{5}}\cdot|\partial B(0,r)|\, dr = \int_0^1 \omega_N\cdot \underbrace{r^{N-1}}_{r^4}\cdot r^{-\frac{3p}{5}} dr = \omega_N\cdot\frac{r^{5-\frac{3p}{5}}}{5-\frac{3p}{5}}\Big|_0^1 =$

$= \omega_N\cdot\frac{1}{5-\frac{3p}{5}} < \infty \iff 5-\frac{3p}{5} > 0 \iff 5 > \frac{3p}{5} \Rightarrow 25 > 3p \Rightarrow \frac{25}{3} > p.$

Deci $p\in[1,\frac{25}{3})$ $p\in\mathbb{R}$.

5. $\operatorname{div}\left(\frac{x}{|x|^2}\right) = \frac{6}{|x|^2}$.

$\operatorname{div}\left(\frac{x}{|x|^2}\right) = \operatorname{div}\left(\underbrace{\frac{x_1}{|x|^2}}_{u_1},\dots,\underbrace{\frac{x_8}{|x|^2}}_{u_8}\right) = \sum_{i=1}^{8}\frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{8}\frac{8|x|^2-2|x|^2}{|x|^4} = \frac{6\cancel{|x|^2}}{\cancel{|x|^4}} = \frac{6}{|x|^2}$

$\left(\frac{x_i}{|x|^2}\right)'_{x_i} = \frac{|x|^2-2x_ix_i}{|x|^4} =$

Problema 2:

$$\begin{cases} u_{tt}(x,t) - 3u_{xx}(x,t) = \cos t, & x\in\mathbb{R}, t>0 \\ u(x,0) = f(x), & x\in\mathbb{R} \\ u_t(x,0) = g(x), & x\in\mathbb{R} \end{cases}$$

Fie $v(x,t) = u(x,t) + \cos t$.

1) $v_t(x,t) = u_t(x,t) - \sin t$ $\qquad v(x,0) = u(x,0) = f(x)$

$v_{tt}(x,t) = u_{tt}(x,t) - \cos t$ $\qquad v_t(x,0) = u_t(x,0) = g(x)$.

$v_x(x,t) = u_x(x,t)$

$v_{xx}(x,t) = u_{xx}(x,t)$

Deci $v_{tt}(x,t) + \cancel{\cos t} - 3v_{xx}(x,t) = \cancel{\cos t}$.

$$\begin{cases} v_{tt}(x,t) - 3v_{xx}(x,t) = 0 \\ v(x,0) = f(x) \\ v_t(x,0) = g(x) \end{cases}$$ ec. satisf. de v.

2) $w_{tt} - 3w_{xx} = \left(\frac{\partial}{\partial t} + \sqrt{3}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \sqrt{3}\frac{\partial}{\partial x}\right) w$.

$\neq$

$\left(\frac{\partial}{\partial t} - \sqrt{3}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \sqrt{3}\frac{\partial}{\partial x}\right) w =$

$\left[\frac{\partial^2}{\partial t^2} + \cancel{\sqrt{3}\frac{\partial^2}{\partial t\partial x}} - \cancel{\sqrt{3}\frac{\partial^2}{\partial x\partial t}} - 3\frac{\partial^2}{\partial x^2}\right] w = w_{tt} - 3w_{xx}$.

3) $z(x,t) = \frac{\partial v}{\partial t} + \sqrt{3}\frac{\partial v}{\partial x}$. $\Rightarrow z(x,t) = \left(\frac{\partial}{\partial t} + \sqrt{3}\frac{\partial}{\partial x}\right) v$

Știm că $v_{tt} - 3v_{xx} = \left(\frac{\partial}{\partial t} - \sqrt{3}\frac{\partial}{\partial x}\right)\underbrace{(\ldots)}_{z(x,t)} = 0 \Rightarrow \left(\frac{\partial}{\partial t} - \frac{\sqrt{3}\partial}{\partial x}\right)\cdot z =$

$z_t - \sqrt{3}z_x = 0$.

$z_t \mp \sqrt{3}z_x = 0$.

$z(x,0) = v_t + \sqrt{3}v_x = g(x) - \underset{0}{\sin 0} + \sqrt{3}f'(x)$.

Deci $\begin{cases} z_t - \sqrt{3}z_x = 0 \\ z(x,0) = g(x) + \sqrt{3}f'(x) \end{cases}$

Ω :=

$\begin{cases} -\Delta u(x,y) = 3\cos y & \Omega \\ u(x,y) = 0 & \partial\Omega \end{cases}$

1) sol unică:

u_1 & u_2 sol $\Rightarrow u = u_1 - u_2$ sol $\Rightarrow \Delta u = 0$ în Ω $\Rightarrow u$ arm $\Rightarrow$
$u = 0$ pe $\partial\Omega$

$\Rightarrow \max_{\Omega} = \max_{\partial\Omega} = 0$
$\min$ — $\underline{\quad}$ $= 0$ $\Rightarrow u \equiv 0 \Rightarrow u_1 = u_2$.

2). $v(x,y) = C(x^2+y^2)$

$v_x = 2Cx \Rightarrow v_{xx} = 2C$
$v_y = 2Cy \Rightarrow v_{yy} = 2C$ $\Bigg| \Rightarrow -\Delta v = 4C = -3 \Rightarrow C = -\frac{3}{4}$.

3) Fie $v(x,y) = -\frac{3}{4}(x^2+y^2)$ și $U = u - v$

$\Delta U = \Delta u - \Delta v = 0$
$U|_{\partial} = u|_{\partial} - v|_{\partial} = \frac{3}{4}(x^2+y^2) = 3$ $\Bigg| \Rightarrow$ a arm $\Rightarrow \max = 3 = \min \Rightarrow U = 3$.

$\Rightarrow u = U + v \Rightarrow u = 3 - \frac{3}{4}(x^2+y^2)$.

Sol ph cu P maxim $\Rightarrow$ se ia v ant și u sol $\dots$ și $U = u - v$
~~$\Delta U = 0$, $U|_{\partial} = $~~

$f(x) = x^2$
$f'(0) = ?$
$f'(x) = 2x \Rightarrow x = 0$.
$(f(0))' = 0$.

$u(x,0) = f(x)$
$u(x,t)\big|_{t=0} = f(x)$

$u_x(x,t) = \dots = \big|_{t=0}$.

4) forma gen a lui z.

Curbe caracteristice:

$\frac{d}{dt}[z(x(s), t(s))] = 0 \Leftrightarrow z_x x'(s) + z_t t'(s) = 0$

$\begin{cases} x'(s) = \sqrt{3} \\ t'(s) = 1 \end{cases} \Rightarrow \begin{cases} x(s) = \sqrt{3}s + c_1 \Rightarrow c_1 = x(s) + \sqrt{3}s \\ t(s) = s + c_2 \Rightarrow c_2 = t(s) + s \end{cases}$

$z(x(s), t(s)) = z(\sqrt{3}s + c_1, s + c_2) = k$ (derivata $= 0$), $k \in \mathbb{R}$.

$s \to -c_2 \Rightarrow z(+\sqrt{3}c_2 + c_1, 0) = g(+\sqrt{3}c_2 + c_1) + \sqrt{3}f'(+\sqrt{3}c_2 + c_1)$.

$z(x,t) = g(+\sqrt{3}(t-s) + x + \sqrt{3}s) + \sqrt{3}f'(+\sqrt{3}(t-s) + x + \sqrt{3}s)$

$= g(+\sqrt{3}t + \cancel{\sqrt{3}s} + x + \cancel{\sqrt{3}s}) + \sqrt{3}f'(+\sqrt{3}t + \cancel{\sqrt{3}s} + x - \cancel{\sqrt{3}s})$

$= g(+\sqrt{3}t + x) + \sqrt{3}f'(+\sqrt{3}t + x)$.

5. $\frac{\partial u}{\partial t} + \sqrt{3}\frac{\partial u}{\partial x} = g(+\sqrt{3}t + x) + \sqrt{3}f'(+\sqrt{3}t + x) \Rightarrow$

$\Rightarrow \begin{cases} u_t + \sqrt{3}u_x = g(\sqrt{3}t + x) + \sqrt{3}f'(\sqrt{3}t + x) \\ u_x(x,0) = f'(x) \\ u_t(x,0) = g(x) \end{cases}$ $\leftarrow$ forma gen a lui u

6). Aleg $w(s) = u(x + \sqrt{3}s, t + s)$.

$w'(s) = u_t(\underbrace{x+\sqrt{3}s}_{x}, \underbrace{t+s}_{t}) + \sqrt{3}u_x(x + \sqrt{3}s, t + s)$

$= g(\sqrt{3}(t+s) + x + \sqrt{3}s) + \sqrt{3}f'(x + \sqrt{3}s + \sqrt{3}(t+s))$

$= g(2\sqrt{3}s + \sqrt{3}t + x) + \sqrt{3}f'(2\sqrt{3}s + \sqrt{3}t + x). \quad \Big| \int_{-t}^{0} ds \Rightarrow$

$\Rightarrow \underset{\underset{u(x,t) - u(x+\sqrt{3}t, 0)}{\parallel}}{w(0)} - w(-t) = \int_{-t}^{0} g + \int_{-t}^{0} f'\sqrt{3}$

$f('' x - \sqrt{3}t)$

$$\Rightarrow v(x,t) - f(x-\sqrt{3}t) = \int_{t}^{0} g + \int_{-t}^{0} \sqrt{3} f'$$

$\Rightarrow$

$$= \int_{-t}^{0} g\, ds + \sqrt{3} \cdot f(\sqrt{3}s + \sqrt{3}t + x) \Big|_{s=-t}^{s=0}$$

$$= \int_{-t}^{0} g\, ds + \sqrt{3} f(\sqrt{3}t + x) + \sqrt{3} \cdot f(x - \sqrt{3}t) =$$

$$\cancel{= \int_{}^{0} g\, ds} =$$

$$\Rightarrow u(x,t) = \int_{-t}^{0} g\, ds + \sqrt{3} f(\sqrt{3}t + x) + \sqrt{3} f(x - \sqrt{3}t) + f(x - \sqrt{3}t).$$

$$= \int_{-t}^{0} g\, ds + \sqrt{3} f(\sqrt{3}t + x) + (\sqrt{3}+1) f(x - \sqrt{3}t).$$

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Problema B.

$$\begin{cases} u_{xx}(x,y)+u_{yy}(x,y)=0, & (x,y)\in(0,1)\times(0,1). \iff \Delta u=0. \\ u(x,0)=u(x,1)=u(1,y)=0, & x\in(0,1),\ y\in(0,1). \\ u(0,y)=\sin(2\pi y). & y\in(0,1). \end{cases}$$

1) Fie u_1, u_2 soluții $\Rightarrow u=u_1-u_2$ sol $\Rightarrow \begin{cases}\Delta u=0 \Rightarrow \text{armonica} \rightarrow \\ u\big|_{\partial}=0\end{cases}$

$$\begin{cases}\max\limits_{\bar\Omega}=\max\limits_{\partial\Omega}=0 \\ \min\limits_{\bar\Omega}=\min\limits_{\partial\Omega}=0\end{cases} \Rightarrow u\equiv 0 \Rightarrow u_1=u_2.$$

2). $u(x,y)=A(x)B(y)$.

$A''B+AB''=0 \Rightarrow \frac{A''}{A}+\frac{B''}{B}=0 \Rightarrow \frac{A''}{A}=-\frac{B''}{B}=\lambda \Rightarrow \begin{cases}A''-\lambda A=0. \\ B''+\lambda B=0.\end{cases}$

$A(x)B(0)=A(x)B(1)=A(1)B(y)=0. \Rightarrow B(0)=B(1)=A(1)=0.$

$A(0)B(y)=\sin(2\pi y).$

Pt $\lambda=0 \Rightarrow B''=0 \Rightarrow B'(y)=c \Rightarrow B(y)=cy+d$

$$\left.\begin{array}{l} B(0)=0. \Rightarrow d=0 \\ B(1)=0 \Rightarrow c=0 \end{array}\right\} \Rightarrow B\equiv 0 \Rightarrow u\equiv 0.$$

$\lambda>0 \Rightarrow B''-\lambda B=0.$

$t^2-\lambda=0 \Rightarrow t=\pm\sqrt{\lambda} \Rightarrow B=c_1e^{\sqrt{\lambda}y}+c_2e^{-\sqrt{\lambda}y}$

$B(0)=c_1+c_2=0 \Rightarrow c_1=-c_2.$

$B(1)=c_1e^{\sqrt{\lambda}}+c_2e^{-\sqrt{\lambda}}=0 \Rightarrow c_2(\underbrace{e^{-\sqrt{\lambda}}-e^{\sqrt{\lambda}}}_{\neq 0})=0 \Rightarrow$

$\Rightarrow c_2=0 \Rightarrow c_1=0 \Rightarrow B\equiv 0 \Rightarrow u\equiv 0.$

$\lambda<0 \Rightarrow B''+\lambda B=0.$

$t^2+\lambda=0 \Rightarrow t=\pm i\sqrt{\lambda} \Rightarrow B=c_1\cos(\sqrt{-\lambda}y)+c_2\sin(\sqrt{-\lambda}y)$

$B(0)=c_1=0$

$B(1)=c_2\sin(\sqrt{-\lambda})=0 \underset{c_2\neq 0}{\Rightarrow} \sin(-\sqrt{\lambda})=0$

$\Rightarrow \sqrt{\lambda}=k\pi \Rightarrow -\lambda=(k\pi)^2 \Rightarrow \boxed{\lambda=-(k\pi)^2} \Rightarrow B(y)=c\cdot\sin(-k\pi y)$

$A'' - \lambda A = 0 \Longleftrightarrow A'' - (k\pi)^2 A = 0$

$t^2 - (k\pi)^2 = 0 \Rightarrow \boxed{t = \pm k\pi}$

$A(x) = c_1 e^{k\pi x} + c_2 e^{-k\pi x}$

$A(1) = c_1 e^{k\pi} + c_2 e^{-k\pi} = 0.$

$A(0) = c_1 + c_2 \longrightarrow A(0) = \dfrac{\sin(2\pi y)}{c \cdot \sin(-k\pi y)} = c_1 + c_2$. ?

Problema 4:

$$\begin{cases} -(e^{-x} u'(x))' = \cos x \quad \forall x \in (0,1). \\ u(0) = u(1) = 0. \end{cases}$$

1). $a(\cdot,\cdot): H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$, $a(u,v) = \int_0^1 e^{-x} u'(x) v'(x)\, dx$

2). Poincaré pt fct. test din $C_c^1(0,1)$: $\int_0^1 |u'(t)|^2 \geq \pi^2 \int_0^1 |u(t)|^2$.

3). ....

4). a continuă:

$|a(u,v)| \leq \int_0^1 \underbrace{|e^{-x}|}_{ct} |u'(x)| |v'(x)| dx \leq \int_0^1 |u'(x)| |v'(x)| dx \overset{C.S.}{\leq} \|u'(x)\|_{L^2} \cdot \|v'(x)\|_{L^2} \leq$

$\leq \|u'\|_{L^2} \cdot \|v'\|_{L^2} + \|u\|_{L^2} \cdot \|v\|_{L^2} \leq (\|u'\|_{L^2} + \|u\|_{L^2})(\|v'\|_{L^2} + \|v'\|_{L^2}) \leq \|u\|_{H_0^1} \cdot \|v\|_{H_0^1}$ $\Rightarrow a$ ct

. a coercivă

$|a(u,u)| \geq \left|\int_0^1 e^{-x} u'(x)^2 dx\right| \geq \frac{1}{e} \int_0^1 u'^2 \geq \frac{1}{e} \underbrace{\int_0^1 |u'|^2}_{\geq \pi^2} \geq \frac{1}{e} \|u'\|_{L^2} \geq \frac{\pi}{e} \|u\|_{H_0^1}$.

5). simetrie:

$a(u,v) = \int_0^1 e^{-x} u' v' dx = \int_0^1 e^{-x} v' u' dx = a(v,u) \Longrightarrow$

6). Inmulțim cu $v \overset{H_0^1}{\in}$ și integrăm:

$-\int_0^1 (e^{-x} u')' v = \int_0^1 \cos x \, v \, dx \Longleftrightarrow$

$-e^{-x} u' \cdot v |_0^1 + \int_0^1 (e^{-x} u')' v' = \int_0^1 v \cos x \Longleftrightarrow$

$v(1) = v(0) = 0$.

$$\begin{cases} \int_0^1 (e^{-x} u') v' = \int_0^1 v \cos x \, dx. \text{ dc.} \\ v \in H_0^1(0,1) \end{cases} \to a \text{ sol slabă}$$

dc. verif. sistemul.

7) u e sol clasică => u e sol slabă
$\not\Leftarrow$

Nu!

8). T. Riesz : a prod scalar => $\exists! \, u \in H_0^1$ sol slabă.

$\langle F_1, v\rangle = \int_0^1 \cos x \, v \, dx$ - liniară.

$|\langle F_1, v\rangle| = |\int_0^1 \cos x \, v(x) \, dx| \le \int_0^1 |\cos x| \cdot |v(x)| \, dx \le \|v\|_{L^2} \le \|v\|_{H_0^1}$

=> cont.

T L-M : $\langle F_1 v\rangle$ lin. cont + coerc => $\exists! \, u \in H_0^1$ sol slabă.

$$\int_0^1 u'v' + \int_0^1 uv = \int_0^1 fv, \quad \forall v \in \ldots$$

$$\begin{cases} -u'' + u = f. \\ u(0) = u(1) = 0 \end{cases}$$

$$\int_0^1 u' \cdot v' \, dx = \cancel{u \cdot v'} \Big|_0^1 - \int_0^1 u \cdot v''.$$

$$\cancel{= - \int_0^1 u \cdot v''}$$

$$= \cancel{vu'} \Big|_0^1 - \int_0^1 v \cdot u''$$

$$= - \int_0^1 v u''$$

$$\int_0^1 uv - \int_0^1 v u'' = \int_0^1 v(u - u'') = \int_0^1 fv$$