Facultatea de Matematica si Informatica 25 Ianuarie 2022
Universitatea din Bucuresti

Examen final

Disciplina: Ecuatii cu derivate partiale
Tipul examinarii: Examen
Nume student: ____
Seria 31: Grupele 311, 312 ________________________________
Timp de lucru : 3 ore si 30 min (incluzand atasarea rezolvarilor pe email sau pe Moodle)

Acest examen contine 5 probleme (toate obligatorii).

Examenul este individual. Nu uitati sa va salvati foile cu rezolvarile subiectelor intr-un singur fisier de tip PDF in timp util astfel incat sa va incadrati in cele 3 ore si 30 minute pentru incarcarea fisierului pe email sau pe platforma Moodle.

Salvati fisierul PDF creat cu numele vostru (Nume_Prenume_Grupa.pdf).

Pentru elaborarea lucrarii scrise puteti folosi orice materiale ajutatoare.

Pentru redactare tineti cont de urmatoarele sugestii:

- Daca folositi o teorema fundamentala, rezultat cunoscut, etc **indicati** acest lucru si explicati cum se poate aplica rezultatul respectiv.
- **Organizati-va munca** intr-un mod coerent pentru a avea toti de castigat ! Incercati ca la crearea fisierului PDF fiecare problema sa fie redactata in ordinea aparitiei pe foaia cu subiecte. Ideal ar fi ca si subpunctele sa fie redactate in ordine. Daca nu stiti a rezolva un anumit subpunct scrieti numarul subpunctului si lasati liber.
- Raspunsurile corecte dar argumentate incomplet (din punct de vedere al calculelor/explicatiilor) vor primi punctaj partial.

Barem: P1 (2p) + P2 (1.5p)+ P3 (2p) +P4 (1.5p)+P5 (2p) + 1p oficiu= **10p** (Plus eventual BONUS acolo unde este cazul in functie de activitatea/temele din timpul semestrului).

Pentru orice nelamuriri scrieti-mi la adresa cristian.cazacu@fmi.unibuc.ro, sau lasati un mesaj pe chat-ul grupei creat pe Microsoft Teams.

Rezultatele finale vor fi postate pe Moodle si Microsoft Teams in cel mai scurt timp posibil.

Problema 1. *(2p).*

1). Calculati $\operatorname{div}(|x|^3 \cdot \nabla v(x))$*, unde* $v : \mathbb{R}^5 \setminus \{0\} \to \mathbb{R}$, $v(x) := |x|^{-\frac{7}{3}}$.

2). Sa se determine pentru ce valori $p \geq 1$ *are loc* $|v|^p \in L^1(B_1(0))$*, unde* $B_1(0)$ *este bila unitate din* $\mathbb{R}^5$.

3). Sa se determine pentru ce valori $p \geq 1$ *are loc* $\frac{|v(x)|^p}{|x|^3+1} \in L^1(\mathbb{R}^5 \setminus \overline{B_1(0)})$.

4). Dati exemplu de o functie strict superarmonica ($-\Delta u > 0$*) pe* $\mathbb{R}^2$ *care sa se anuleze pe dreapta* $x - 2y = 0$.

5). Consideram functia $u : B_1(0) \setminus \{0\} \to \mathbb{R}$ *data de*

$$u(x) = \left(\ln \frac{2}{|x|}\right)^{\frac{1}{2}}, \quad x = (x_1, x_2),$$

unde $B_1(0)$ *este bila unitate din* $\mathbb{R}^2$ *centrata in origine. Aratati ca*

$$-\Delta u(x) = \frac{u(x)}{4|x|^2 \ln^2(\frac{2}{|x|})}, \quad \forall x \in B_1(0) \setminus \{0\}.$$

Problema 2. *(1.5p). Se considera problema la limita*

$$(1) \qquad \begin{cases} u_{xx}(x,y) + 3u_{yy}(x,y) = 0, & (x,y) \in (0,1) \times (0,1) \\ u(x,0) = u(x,1) = 0, & x \in (0,1),\ y \in (0,1) \\ u(0,y) = \sin(3\pi y),\ u(1,y) = e^{-3\sqrt{3}\pi} \sin(3\pi y), & y \in (0,1). \end{cases}$$

1). Determinati solutia problemei (1) cautand-o in variabile separate sub forma $u(x,y) = A(x)B(y)$.

*2). * Aratati (folosind eventual metoda energetica) ca (1) are cel mult o solutie de clasa* C^2.

Problema 3. *(2p). Consideram urmatoarea problema de tip "unde"*

$$(2) \qquad \begin{cases} u_{tt}(x,t) + u_{tx}(x,t) - 6u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

unde $f, g \in C^2(\mathbb{R})$ *sunt functii date.*

1). Aratati ca daca $u = u(x,t)$ *este o functie de clasa* C^2 *atunci* u *verifica*

$$(\partial_t + 3\partial_x)(u_t(x,t) - 2u_x(x,t)) = u_{tt}(x,t) + u_{tx}(x,t) - 6u_{xx}(x,t),$$

pe domeniul sau de definitie.

2). Rezolvati problema cu valori initiale satisfacuta de u *in (2) (scrieti forma generala a lui* u*) reducand-o la rezolvarea a doua ecuatii de transport (una omogena si alta neomogena).*

3). Folosind conditiile la $t = 0$ *deduceti solutia* u *a problemei (2) in cazul particular* $f(x) = \cos x$ *si* $g(x) = \sin^2 x$.

Problema 4. *(1.5p). Consideram problema Cauchy*

$$
\left\{ \begin{array}{ll} u_t(x,t) - u_{xx}(x,t) + \frac{e^{2t}}{e^{2t}+1} u(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = e^{-4x^2}, & x \in \mathbb{R}. \end{array} \right. \tag{3}
$$

1). Gasiti o functie $\phi : \mathbb{R} \to \mathbb{R}$ astfel incat functia $v(x,t) := u(x,t)\phi(t)$ sa verifice ecuatia caldurii

$$
v_t(x,t) - v_{xx}(x,t) = 0, \quad \forall x \in \mathbb{R},\ \forall t > 0. \tag{4}
$$

2). Scrieti problema Cauchy verificata de v si determinati explicit solutia problemei (3).

Problema 5. *(2p). Fie functia $f : [0,3] \to \mathbb{R}$, $f(x) = |x^2 - 2|$.*

1). Explicitati functia f si schitati graficul functiei f.

2). Sa se determine punctele de derivabilitate ale lui f pe intervalul $(0,3)$.

3). Argumentati ca $f \in H^1(0,3)$ si calculati norma lui f in $H^1(0,3)$ (precizati inainte norma cu care lucrati).

4). Determinati $\alpha \in \mathbb{R}$ astfel incat functia $z : (0,1) \to \mathbb{R}$, $z(x) = x^\alpha$ sa apartina lui $H^1(0,1)$.

*5). * Determinati $\alpha \in \mathbb{R}$ astfel incat functia $z : (1,\infty) \to \mathbb{R}$, $z(x) = \frac{x^\alpha}{1+x^2}$ sa apartina lui $H^1(1,\infty)$.*

25/01/2022

DRAGOMIR ELENA ALEXANDRA
GRUPA 312.

Examen EDP

Problema 1

1) $\mathrm{div}(|x|^3 \cdot \nabla v(x))$, $v: \mathbb{R}^5 \setminus \{0\} \to \mathbb{R}$, $v(x) = |x|^{-\frac{7}{3}}$

$\nabla v(x) = \nabla |x|^{-\frac{7}{3}} = -\frac{7}{3} x \cdot |x|^{-\frac{7}{3}-2}$

$\mathrm{div}(|x|^3 \cdot -\frac{7}{3} \cdot x |x|^{-\frac{7}{3}-2}) = -\frac{7}{3} \mathrm{div}(x \cdot |x|^{-\frac{7}{3}+1})$

Avem $\mathrm{div}(x|x|^{\lambda}) = (n+\lambda)|x|^{\lambda}$

$\Rightarrow \mathrm{div}(x \cdot |x|^{-\frac{4}{3}}) = (5 - \frac{4}{3})|x|^{-\frac{4}{3}} = \frac{11}{3}|x|^{-\frac{4}{3}}$

$\Rightarrow \mathrm{div}(|x|^3 \cdot \nabla v(x)) = -\frac{7}{3} \cdot \frac{11}{3} |x|^{-\frac{4}{3}} = -\frac{77}{9}|x|^{-\frac{4}{3}}$

2) pt. a $p \geq 1$ $|v|^p \in L^1(B_1(0))$

$|v|^p \in L^1(B_1(0)) \Leftrightarrow \int_{B_1(0)} ||v|^p| \, dx < \infty$

$\int_{B_1(0)} |(|x|^{-\frac{7}{3}})^p| \, dx = \int_{B_1(0)} |x|^{-\frac{7}{3}p} \, dx \overset{\text{F.coarii}}{=} \int_0^1 \int_{\partial B_r(0)} r^{-\frac{7}{3}p} \, d\sigma \, dr$

$= \int_0^1 r^{-\frac{7}{3}p} \int_{\partial B_r(0)} d\sigma \, dr = \int_0^1 r^{-\frac{7}{3}p} \cdot \text{aria}(\partial B_r(0)) \, dr =$

$= \int_0^1 r^{-\frac{7}{3}p} \cdot r^{5-1} \omega_5 \, dr = \omega_5 \int_0^1 r^{-\frac{7}{3}p+4} \, dr. := I_1$

$-\frac{7}{3}p + 4 = -1 \Rightarrow I_1 = \omega_5 \cdot \ln r \Big|_0^1 = \omega_5(\ln 1 - \ln 0) = \infty$ $\not<$

$-\frac{7}{3}p + 4 < -1 \Rightarrow I_1 = \omega_5 \cdot \frac{r^{-\frac{7}{3}p+5}}{-\frac{7}{3}p+5}\Big|_0^1 = \omega_5\left(\frac{1}{-\frac{7}{3}p+5} - \underbrace{\frac{0^{-\frac{7}{3}p+5}}{-\frac{7}{3}p+5}}_{\infty}\right)$ $\not<$

$-\frac{7}{3}p + 4 > -1 \Rightarrow I_1 = \omega_5 \cdot \left(\frac{1}{-\frac{7}{3}p+5} - 0\right) < \infty$ ✓.

$\Rightarrow$ Pt ca $I_1 < \infty$ vrem $-\frac{7}{3}p + 4 > -1 \Leftrightarrow -\frac{7}{3}p > -5$

$\left.\begin{array}{l} \Leftrightarrow p < \frac{15}{7} \\ p \geq 1 \end{array}\right\} \Rightarrow p \in [1, \frac{15}{7})$.

3). pt. ce $p \geq 1$. $\dfrac{|u(x)|^p}{|x|^3+1} \in L^1(\mathbb{R}^5 \setminus \overline{B_1(0)})$.

$$\frac{|u(x)|^p}{|x|^3+1} \in L^1(\mathbb{R}^5 \setminus \overline{B_1(0)}) \Leftrightarrow \int_{\mathbb{R}^5 \setminus \overline{B_1(0)}} \frac{|u(x)|^p}{|x|^3+1}\, dx < \infty.$$

$$\int_{\mathbb{R}^5 \setminus B_1(0)} \frac{|x|^{-\frac{7}{3}p}}{|x|^3+1}\, dx \underset{\text{F. coarii}}{=} \int_1^\infty \int_{\partial B_r(0)} \frac{r^{-\frac{7}{3}p}}{r^3+1}\, d\sigma\, dr = \int_1^\infty \frac{r^{-\frac{7}{3}p}}{r^3+1} \cdot \omega_5 \cdot r^4\, dr$$

$$= \omega_5 \int_1^\infty \frac{r^{-\frac{7}{3}p+4}}{r^3+1}\, dr < \omega_5 \int_1^\infty r^{-\frac{7}{3}p+4-3}\, dr =$$

$$= \omega_5 \int_1^\infty r^{-\frac{7}{3}p+1}\, dr =: I_2$$

$-\frac{7}{3}p+1 = -1 \Rightarrow I_2 = \omega_5 \ln|r| \Big|_1^\infty = \infty$. $\cancel{\infty}$

$-\frac{7}{3}p+1 < -1 \Rightarrow I_2 = \omega_5 \dfrac{r^{\overbrace{-\frac{7}{3}p+2}^{<0}}}{-\frac{7}{3}p+2} \Big|_1^\infty = 0 - \omega_5 \cdot \dfrac{1}{-\frac{7}{3}p+2} < \infty$ ✓

$-\frac{7}{3}p+1 > -1 \Rightarrow I_2 = \omega_5 \dfrac{r^{\overbrace{-\frac{7}{3}p+2}^{>0}}}{-\frac{7}{3}p+2} \Big|_1^\infty = \infty - \omega_5 \cdot \dfrac{1}{-\frac{7}{3}p+2} = \infty$ $\cancel{\infty}$

$\Rightarrow$ Pt. ca $I_2 < \infty$ vrem $-\frac{7}{3}p+1 < -1 \Leftrightarrow -\frac{7}{3}p < -2$

$\left.\begin{array}{l} \Leftrightarrow p > \frac{6}{7} \\ p \geq 1 \end{array}\right\} \Rightarrow p \in [1, +\infty)$.

4). $-\Delta u > 0$ ai $u(x,y) = 0$ pe dreapta $x - 2y = 0$.

Vrem $u(2x, x) = 0$

Fie $u(x,y) = -(x-2y)^2 = -(x^2+4y^2-4xy)$ $\Rightarrow$ $\left\{\begin{array}{l} u_x = -2x+4y. \Rightarrow u_{xx} = -2 \\ u_y = -8y+4x \Rightarrow u_{yy} = -8. \end{array}\right.$

$\Rightarrow u_{xx} + u_{yy} = -2-8 = -10.$ $\Rightarrow -\Delta u = 10. > 0.$

$\Rightarrow u$ superarmonică și $u(2x, x) = -(2x-2x)^2 = 0$.

5). $u: B_1(0)\setminus\{0\}\to\mathbb{R}$

$$u(x) = \left(\ln \frac{2}{|x|}\right)^{\frac12}, \quad x=(x_1,x_2).$$

$$-\Delta u = \frac{u(x)}{4|x|^2 \ln^2\left(\frac{2}{|x|}\right)} \quad //$$

$$u_{x_1} = \frac{d}{dx_1}\left(\ln\frac{2}{|x|}\right)^{\frac12} = \frac{1}{2\sqrt{\ln\frac{2}{|x|}}}\cdot\frac{d}{dx_1}\left(\ln\frac{2}{|x|}\right)$$

$$= \frac12 \left(\ln\frac{2}{|x|}\right)^{-\frac12}\cdot\frac{1}{\frac{2}{|x|}}\cdot\frac{d}{dx_1}\left(2|x|^{-1}\right) =$$

$$= \frac12\left(\ln\frac{2}{|x|}\right)^{-\frac12}\cdot\frac{|x|}{\cancel{2}}\cdot\cancel{2}\cdot\frac{-1}{|x|^2}\cdot\frac{d}{dx_1}(|x|) =$$

$$= \frac{-1}{2}\left(\ln\frac{2}{|x|}\right)^{-\frac12}\cdot\frac{1}{|x|}\cdot\frac{x_1}{|x|}$$

$$u_{x_1x_1} = \frac{-1}{2}\cdot\frac{d}{dx_1}\left(\ln\left(\frac{2}{|x|}\right)^{-\frac12}\cdot\frac{x_1}{|x|^2}\right) =$$

$$= \frac{-1}{2}\left(\frac{-1}{2}\cdot\ln\left(\frac{2}{|x|}\right)^{-\frac32}\cdot\frac{d}{dx_1}\left(\frac{2}{|x|}\right)\cdot\frac{x_1}{|x|^2} + \ln\left(\frac{2}{|x|}\right)^{-\frac12}\cdot\frac{|x|^2 - x_1\cdot 2\cancel{|x|}\cdot\frac{x_1}{\cancel{|x|}}}{|x|^4}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{2}\ln\left(\frac{2}{|x|}\right)^{-\frac32}\cdot 2\cdot\frac{-1}{|x|^2}\cdot\frac{x_1}{|x|}\cdot\frac{x_1}{|x|^2} + \ln\left(\frac{2}{|x|}\right)^{-\frac12}\cdot\frac{|x|^2-2x_1^2}{|x|^4}\right)$$

$$-(u_{x_1x_1}+u_{x_2x_2}) = \frac12\left(\frac{-1}{\cancel{2}}\cdot u(x)^{-3}\cdot\cancel{2}\cdot\frac{-x_1^2}{|x|^4} + u(x)^{-1}\left(\frac{1}{|x|^2}-\frac{2x_1^2}{|x|^4}\right) - \frac{1}{\cancel{2}}u(x)^{-3}\cdot\cancel{2}\cdot\frac{-x_2^2}{|x|^4} + u(x)^{-1}\left(\frac{1}{|x|^2}-\frac{2x_2^2}{|x|^4}\right)\right)$$

$$= \frac12\left(u(x)^{-3}\left(\frac{\overbrace{x_1^2+x_2^2}^{|x|^2}}{|x|^4}\right)+u(x)^{-1}\left(\frac{1}{|x|^2}+\frac{1}{|x|^2}-\frac{2(\overbrace{x_1^2+x_2^2}^{|x|^2})}{|x|^4}\right)\right)$$

$$= \frac12\left(u(x)^{-3}\cdot\frac{1}{|x|^2}+u(x)^{-1}\left(\frac{2}{|x|^2}-\frac{2}{|x|^2}\right)\right)$$

$$= \frac12\left(u(x)^{-3}\cdot\frac{1}{|x|^2}\right.$$

$$= \frac{u(x)^{-3}}{2|x|^2}$$

$$\frac{u(x)}{4|x|^2\cdot\ln^2\left(\frac{2}{|x|}\right)} = \frac{u(x)}{4|x|^2\cdot u(x)^4} = \frac{u(x)^{-3}}{4|x|^2}.$$

$\rightarrow$?

Problema 2

$$(1)\begin{cases} u_{xx}(x,y)+3u_{yy}(x,y)=0 & ,(x,y)\in(0,1)\times(0,1). \\ u(x,0)=u(x,1)=0 & ,x\in(0,1),\ y\in(0,1) \\ u(0,y)=\sin(3\pi y) & \\ u(1,y)=e^{-3\sqrt{3}\pi}\sin(3\pi y) & \end{cases}\Big\}\ y\in(0,1).$$

1). Căutăm soluție $u(x,y)$ sub forma $u(x,y)=A(x)B(y)$:

$u_x=A'(x)B(y) \Rightarrow u_{xx}=A''(x)B(y)$

$u_{yy}=A(x)B''(y).$

Avem sistemul:

$$\begin{cases} A''(x)B(y)+3A(x)B''(y)=0. & /:\ A(x)B(y) \\ A(x)B(0)=A(x)B(1)=0 & \Rightarrow B(0)=B(1)=0. \\ \text{(iii)}\ A(0)B(y)=\sin(3\pi y). & \\ \text{(iv)}\ A(1)B(y)=e^{-3\sqrt{3}\pi}\sin(3\pi y). & \end{cases}$$

Prima relație este echivalentă cu:

$$\frac{A''(x)}{A(x)}=-3\frac{B''(y)}{B(y)}=\lambda\in\mathbb{R}\text{ ct. independentă de } x,y.$$

$$(2)\begin{cases} B''(y)+\frac{\lambda}{3}B(y)=0 & ,y\in(0,1) \\ B(0)=B(1)=0. \end{cases}$$

Rezolvăm problema de valori proprii dată de (2):

Rezolvăm soluțiile netriviale ale lui $B(y)$

Căutăm soluțiile netriviale ale lui $B(y)$

Avem ecuația caracteristică:

$r^2+\frac{\lambda}{3}=0.$

- $\frac{\lambda}{3}=0 \Rightarrow r_{1,2}=0 \Rightarrow B(y)=0$ $\cancel{}$
- $\frac{\lambda}{3}<0 \Rightarrow r_{1,2}=\pm\sqrt{\frac{-\lambda}{3}} \Rightarrow B(y)=c_1e^{\sqrt{\frac{-\lambda}{3}}y}+c_2e^{-\sqrt{\frac{-\lambda}{3}}y}.$

$B(0)=B(1)=0 \overset{(3)}{\Rightarrow}\begin{cases} c_1+c_2=0 \\ c_1e^{\sqrt{\frac{-\lambda}{3}}}+c_2e^{-\sqrt{\frac{-\lambda}{3}}}=0 \end{cases}$ $\begin{vmatrix} 1 & 1 \\ e^{\sqrt{\frac{-\lambda}{3}}} & e^{-\sqrt{\frac{-\lambda}{3}}} \end{vmatrix}=$
$=e^{-\sqrt{\frac{-\lambda}{3}}}-e^{\sqrt{\frac{-\lambda}{3}}}<0.$

$\Rightarrow$ sistemul (3) nu are soluții neomogene $\Rightarrow c_1 = c_2 = 0 \Rightarrow B(y) = 0$. $\cancel{\#}$

- $\frac{\lambda}{3} > 0 \Rightarrow r_{1,2} = \pm i\sqrt{\frac{\lambda}{3}}$

$\Rightarrow B(y) = c_1 \cos\left(\sqrt{+\frac{\lambda}{3}}\, y\right) + c_2 \sin\left(\sqrt{+\frac{\lambda}{3}}\, y\right)$.

$$\begin{cases} B(0) = 0 \Rightarrow \cancel{\cos\sqrt{\frac{\lambda}{3}}}\ c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0. \\ B(1) = 0 \Rightarrow 0 = c_2 \cdot \sin\sqrt{\frac{\lambda}{3}} \end{cases}$$

Dacă $c_2 = 0 \Rightarrow B(y) = 0$ $\cancel{\#}$

$c_2 \neq 0 \Rightarrow \sin\sqrt{\frac{\lambda}{3}} = 0 \Rightarrow \sqrt{\frac{\lambda}{3}} = n \cdot \pi,\ n \geq 1. \Rightarrow$

$\Rightarrow \lambda = 3n^2\pi^2 \qquad \Rightarrow B(y) = c_2 \cdot \sin(n\pi y)$.

~~$\Rightarrow B(y) =$~~

Acum rezolvăm problema de valori proprii dată de A:

$$\frac{A''(x)}{A(x)} = \lambda = 3n^2\pi^2$$

$\Rightarrow A''(x) - 3n^2\pi^2 A(x) = 0$

Ec. caracteristică asociată: $r^2 = 3n^2\pi^2 > 0 \Rightarrow r = \pm n\pi\sqrt{3}$.

$\Rightarrow A(x) = c_3 e^{n\pi\sqrt{3}x} + c_4 e^{-n\pi\sqrt{3}x}$.

Ne uităm la condițiile date de iii, iv)

- $A(0)B(y) = \sin 3\pi y. \quad \Leftrightarrow (c_3 + c_4) \cdot c_2 \sin(n\pi y)$.

$\Rightarrow$ luăm $n = 3$ și $(c_3 + c_4)c_2 = 1$.

- $A(1)B(y) = e^{-3\sqrt{3}\pi} \sin(3\pi y) \quad \Leftrightarrow$

$\Leftrightarrow \left(c_3 e^{3\sqrt{3}\pi} + c_4 e^{-3\sqrt{3}\pi}\right)(c_2 \sin 3\pi y) = e^{-3\sqrt{3}\pi} \sin(3\pi y)$.

$$\Leftrightarrow \begin{cases} c_2 c_3 = 0. \\ c_2 c_4 = 1 \end{cases} \Rightarrow c_3 = 0,\ c_4 = \frac{1}{c_2}$$

Deci avem:

$$\begin{cases} A(x) = \frac{1}{c_2} \cdot e^{-3\sqrt{3}\pi x} \\ B(y) = c_2 \cdot \sin(3\pi y). \end{cases} \Rightarrow u(x,y) = e^{-3\sqrt{3}\pi x} \cdot \sin(3\pi y).$$

2) Fie u_1, u_2 soluții ale sistemului (1). și fie $U := u_1 - u_2$.

Atunci U verifică:

$$(1) \begin{cases} U_{xx} + 3U_{yy} = u_{1xx} + 3u_{1yy} - u_{2xx} - 3u_{2yy} = 0. \\ U(x,0) = u_1(x,0) - u_2(x,0) = 0 \\ U(x,1) = 0 \\ U(0,y) = u_1(0,y) - u_2(0,y) = \sin(3\pi y) - \sin(3\pi y) = 0 \\ U(1,y) = 0. \end{cases}$$

Vrem $U \equiv 0$. Înmulțim prima relație cu U și integrăm:

$$0 = \int_{(0,1)^2} U \cdot (U_{xx} + U_{yy}) + U \cdot 2U_{yy} \, dx dy =$$

$$= \int_{(0,1)^2} U \cdot \Delta U \, dx dy + 2\int_{(0,1)^2} U \cdot U_{yy} \, dx dy =$$

$$\overset{G1}{=} \underbrace{\int_{\partial(0,1)^2} U \cdot \frac{\partial U}{\partial \nu} \, d\sigma}_{0 \text{ pe frontieră}} - \int_{(0,1)^2} \nabla U \cdot \nabla U \, dx dy + 2\int_{(0,1)^2} U \cdot U_{yy} \, dx dy$$

$$\Rightarrow \int_{(0,1)^2} U \cdot U_{yy} \, dx dy = \int_{(0,1)^2} |\nabla U|^2 \, dx dy \geq 0. \quad (2)$$

$$\int_{(0,1)^2} U (U_y)_y \, dx dy = \underbrace{\int_{\partial(0,1)^2} U \cdot U_y \cdot \nu_2 \, d\sigma}_{0 \text{ pe frontieră}} - \int_{(0,1)^2} U_y \cdot U_y \, dx dy.$$

$$= -\underbrace{\int_{(0,1)} |U_y|^2 \, dx dy}_{\geq 0} \overset{(2)}{\geq} 0. \Rightarrow \int_{(0,1)^2} |U_y|^2 \, dx dy = 0$$

$$\overset{(2)}{\Rightarrow} \int_{(0,1)^2} |\nabla U|^2 \, dx dy = 0 \Rightarrow \nabla U = 0 \text{ pe } (0,1)^2 \Rightarrow$$

$$\left.\begin{matrix} \Rightarrow U \text{ ct. în } \overline{\Omega} \\ U|_{\partial\Omega} \equiv 0 \end{matrix}\right\} \Rightarrow U = 0 \text{ în } \overline{\Omega} \Rightarrow u_1 = u_2.$$

Problema 3

$$(1)\begin{cases} u_{tt}+u_{tx}-6u_{xx}=0 & , x\in\mathbb{R}, t>0 \\ u(x,0)=f(x) & , x\in\mathbb{R} \\ u_t(x,0)=g(x). & , x\in\mathbb{R} \end{cases}$$

$f, g\in C^2(\mathbb{R})$ date.

1) $u=u(x,t)\overset{C^2(\mathbb{R})}{\Rightarrow} (\partial_t+3\partial_x)(u_t-2u_x)=u_{tt}+u_{tx}-6u_{xx}.$

$$(\partial_t+3\partial_x)(u_t-2u_x)=\frac{\partial}{\partial t}u_t(x,t)-2\frac{\partial}{\partial t}u_x(x,t)$$
$$+3\frac{\partial}{\partial x}u_t(x,t)-6\frac{\partial}{\partial x}u_x(x,t).$$

$$=u_{tt}(x,t)-2\cdot u_{tx}(x,t)+3u_{xt}(x,t)-6u_{xx}(x,t)\overset{*}{=}$$

cum $u\in C^2(\mathbb{R}) \overset{\text{Lema lui Schwarz}}{\Rightarrow} u_{tx}=u_{xt}. \Rightarrow$

$$\overset{*}{=} u_{tt}+u_{\cancel{xt}}-6u_{xx} \quad \square.$$

2). Notăm cu $v(x,t)=(\partial_t-2\partial_x)u=u_t-2u_x$. Avem că $v(x,t)$ verifică ec. de transport omogenă cu date inițiale:

$$(3)\begin{cases} v_t(x,t)+3v_x(x,t)=0 & , x\in\mathbb{R}, t>0 \\ v(x,0)=u_t(x,0)-2u_x(x,0)=g(x)-2f'(x). \end{cases}$$

Avem că:
$$\underbrace{(v_{\cancel{x}}, v_{\cancel{t}})}_{\nabla v}\underbrace{(3,1)}_{\bar a}=0 \Rightarrow \nabla v\cdot\bar a=\frac{dv}{d\bar a}=0 \Rightarrow v \text{ ct. pe direcția } \bar a$$

Scriem (x,t) ca: $(x,t)=t(3,1)+(x-3t,0)$. e.i.

$$\Rightarrow v(x,t)=v\big(t(3,1)+(x-3t,0)\big)\overset{v\text{ ct. pe } \bar a}{=}v(x-3t,0)=$$
$$=g(x-3t)-2f'(x-3t).$$

$$\Rightarrow v(x,t)=g(x-3t)-2f'(x-3t)$$

Acum rezolvăm ecuația de transport neomogenă verificată de u:

$$(4)\begin{cases} u_t - 2u_x = v(x,t) = g(x-3t) - 2f'(x-3t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Fixăm x și t și fie $w: \mathbb{R} \to \mathbb{R}$ ai

$w(s) = u(x-2s, t+s)$, $s \in \mathbb{R}$.

$$w'(s) = u_x(x-2s, t+s)\cdot(-2) + u_t(x-2s, t+s)\cdot(+1)$$
$$= u_t(x-2s, t+s) - 2u_x(x-2s, t+s)$$
$$\overset{(4)}{=} v(x-2s, t+s) = g(x-2s-3(t+s)) - 2f'(x-2s-3(t+s))$$
$$= g(x-5s-3t) - 2f'(x-5s-3t). \quad (5)$$

Știm $w(-t) = u(x+2t, t-t) = u(x+2t, 0) \overset{(4)}{=} f(x+2t)$. (6).

și vrem să aflăm $w(0) = u(x,t)$. Integrăm relația (5) de la $-t$ la 0 în funcție de s:

$$\int_{-t}^{0} w'(s)\,ds = \int_{-t}^{0} g(x-3t-5s) - 2f'(x-3t-5s)\,ds =$$
$$= \int_{-t}^{0} g(x-3t-5s)\,ds + \frac{2}{5}\int_{-t}^{0} f'(x-3t-5s)\cdot(-5)\,ds.$$
$$= \int_{-t}^{0} g(x-3t-5s)\,ds + \frac{2}{5}\, f(x-3t-5s)\Big|_{s=-t}^{s=0} =$$
$$= \frac{2}{5}\left[f(x-3t) - f(x+2t)\right] + \int_{-t}^{0} g(x-3t-5s)\,ds.$$

Dar $\int_{-t}^{0} w'(s)\,ds = w(0) - w(-t) \overset{(6)}{=} w(0) - f(x+2t) =$
$= u(x,t) - f(x+2t)$

$$\Rightarrow u(x,t) = f(x+2t) + \frac{2}{5}\left[f(x-3t) - f(x+2t)\right] + \int_{-t}^{0} g(x-3t-5s)\,ds. \quad (7).$$

3). $\begin{cases} f(x) = \cos x \\ g(x) = \sin^2 x \end{cases}$

Din (*) avem că: $u(x,t) = f(x+2t) + \frac{2}{5}\left[f(x-3t) - f(x+2t)\right] + \int_{-t}^{0} g(x-3t-5s)\, ds$

$$\Rightarrow u(x,t) = \cos(x+2t) + \frac{2}{5}\left[\cos(x-3t) - \cos(x+2t)\right] + \underbrace{\int_{-t}^{0} \sin^2(x-3t-5s)\, ds}_{:= I}.$$

Facem schimbarea de variabile $x-3t-5s := a \Rightarrow -5ds = da$

$s = -t \Rightarrow a = x+2t$

$s = 0 \Rightarrow a = x-3t$

$$\Rightarrow I = \int_{x+2t}^{x-3t} \sin^2(a) \cdot \frac{-1}{5}\, da = \frac{-1}{5}\int_{x+2t}^{x-3t} \sin^2(a)\, da =$$

$$= \frac{-1}{5}\int_{x+2t}^{x-3t} \frac{1-\cos(2a)}{2}\, da = \frac{-1}{10} \cdot a \Big|_{x+2t}^{x-3t} + \frac{1}{10}\int_{x+2t}^{x-3t} \cos(2a)\, da$$

$$= \frac{-1}{10}(\cancel{x}-3t-\cancel{x}-2t) + \frac{1}{10} \cdot \frac{\sin(2a)}{2}\Big|_{x+2t}^{x-3t}$$

$$= \frac{1}{2}t + \frac{1}{20}\left[\sin(2x-6t) - \sin(2x+4t)\right]$$

$$\Rightarrow u(x,t) = \cos(x+2t) + \frac{2}{5}\left[\cos(x-3t) - \cos(x+2t)\right] + \frac{1}{2}t + \frac{1}{20}\left[\sin(2x-6t) - \sin(2x+4t)\right].$$

Problema 4

$$(3).\begin{cases} u_t(x,t) - u_{xx}(x,t) + \dfrac{e^{2t}}{e^{2t}+1}\, u(x,t) = 0 & , x\in\mathbb{R}, t>0 \\ u(x,0) = e^{-4x^2} & , x\in\mathbb{R} \end{cases}$$

1) $\phi:\mathbb{R}\to\mathbb{R}$ ai $v(x,t) = u(x,t)\phi(t)$ să verifice

$$v_t(x,t) - v_{xx}(x,t) = 0 \quad \forall x\in\mathbb{R}, \forall t>0.$$

$$v_t(x,t) = u_t(x,t)\,\phi(t) + u(x,t)\,\phi'(t)$$
$$v_x(x,t) = u_x(x,t)\,\phi(t)$$
$$v_{xx}(x,t) = u_{xx}(x,t)\phi(t)$$

$$\Rightarrow v_t - v_{xx} = u_t\phi(t) + u\cdot\phi'(t) - u_{xx}\cdot\phi(t) =$$
$$= (u_t - u_{xx})\cdot\phi(t) + u\cdot\phi'(t) =$$
$$\overset{(3)}{=} -\frac{e^{2t}}{e^{2t}+1}\cdot u\cdot\phi(t) + u\cdot\phi'(t) =$$
$$= \left[\frac{-e^{2t}}{e^{2t}+1}\phi(t) + \phi'(t)\right]\cdot u \overset{\text{ip.}}{=} 0 \quad \overset{u\neq 0}{\Rightarrow}$$

$$\Rightarrow \phi'(t) = \frac{e^{2t}}{e^{2t}+1}\phi(t). \Rightarrow \phi(t) = c\cdot e^{\int\frac{e^{2t}}{e^{2t}+1}dt}$$

$$\int\frac{e^{2t}}{e^{2t}+1}dt = \int\frac{1}{e^{2t}+1}\cdot(e^{2t}+1)'\tfrac{1}{2}dt = \int\tfrac{1}{2}\ln(e^{2t}+1)'\,dt$$
$$= \frac{1}{2}\ln(e^{2t}+1) + \mathcal{C}.$$

$$\Rightarrow \text{putem alege } \phi(t) = e^{\frac{1}{2}\ln(e^{2t}+1)} = \left[e^{\ln(e^{2t}+1)}\right]^{\frac{1}{2}}$$
$$= \sqrt{e^{2t}+1}.$$

2). pb. Cauchy verificată de v:

$$(4).\begin{cases} v_t - v_{xx} = 0 \\ v(x,0) = u(x,0)\cdot\phi(0) = e^{-4x^2}\cdot\sqrt{e^0+1} = \sqrt{2}\,e^{-4x^2}. \end{cases}$$

$$\Rightarrow \begin{cases} v_t - v_{xx} = 0 \\ v(x,0) = \sqrt{2}\,e^{-4x^2} \end{cases}$$

(4) este pb. Cauchy în $\mathbb{R}$ pt. ecuația căldurii. Avem că

$$v(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \cdot \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} \cdot \sqrt{2}\, e^{-4y^2} dy \quad \text{soluție pt. (4).}$$

$$v(x,t) = \frac{1}{2\sqrt{\pi t}} \cdot \sqrt{2} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t} - 4y^2} dy$$

Căutăm soluțiile $v(x,t)$ sub formă de variabile separabile:

$v(x,t) = A(x) B(t)$.

Avem:

$$\begin{cases} A(x)\, B'(t) - A''(x)\, B(t) = 0 & /: A(x) B(t) \\ A(x)\, B(0) = \sqrt{2}\, e^{-4x^2} \end{cases} \Rightarrow A(x) = \sqrt{2}\, e^{-4x^2} \cdot c, \quad c = \frac{1}{B(0)}.$$

$$A'(x) = c\sqrt{2}\, e^{-4x^2} \cdot (-8x).$$

$$A''(x) = c\sqrt{2}\, e^{-4x^2} \cdot (-8x) \cdot (-8x) + c\sqrt{2}\, e^{-4x^2} \cdot (-8) =$$

$$= -8\, c\sqrt{2}\, e^{-4x^2} \left(-8x^2 + 1\right).$$

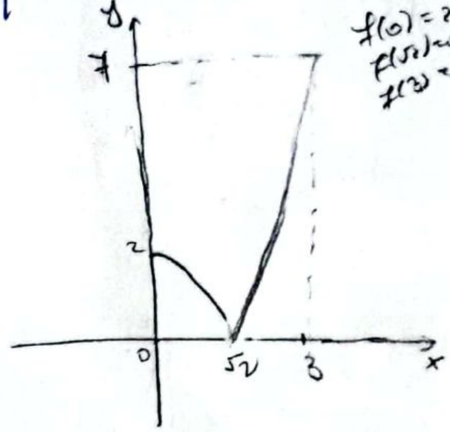
$$\frac{B'(t)}{B(t)} = \frac{A''(x)}{A(x)} = \frac{-8c\sqrt{2}\, \cancel{e^{-4x^2}} (-8x^2+1)}{c\sqrt{2}\, \cancel{e^{-4x^2}}} = 64x^2 - 8.$$

Problema 5

$f : [0,3] \to \mathbb{R}, \; f(x) = |x^2-2|$

1) $x^2-2 \geq 0 \Leftrightarrow x^2 \geq 2 \Leftrightarrow \left.\begin{array}{l}|x| \geq \sqrt{2} \\ x \in [0,3]\end{array}\right\} \Leftrightarrow x \in [\sqrt{2}, 3].$

$$\Rightarrow f(x) = \begin{cases} x^2-2 & , \text{dacă } x \in [\sqrt{2},3] \\ 2-x^2 & , \text{dacă } x \in [0,\sqrt{2}] \end{cases}$$



$f(0)=2$
$f(\sqrt{2})=0$
$f(3)=7$

2) pe $[0,\sqrt{2})$: $f(x) = 2-x^2$, $f'(x) = -2x$.

pe $(\sqrt{2},3)$: $f(x) = x^2-2 \Rightarrow f'(x) = 2x$.

$$f_s'(\sqrt{2}) = \lim_{h\to 0} \frac{f(\sqrt{2}-h) - f(\sqrt{2})}{-h} = \lim_{h\to 0} \frac{2-(\sqrt{2}-h)^2 - 0}{-h} = \lim_{h\to 0} \frac{\cancel{2}-\cancel{2}-h^2+2\sqrt{2}h}{-h} = \lim_{h\to 0}(h-2\sqrt{2})$$

$$= -2\sqrt{2}$$

$$f_d'(\sqrt{2}) = \lim_{h\to 0} \frac{f(\sqrt{2}+h) - f(\sqrt{2})}{h} = \lim_{h\to 0} \frac{\cancel{2}+h^2+2\sqrt{2}h-\cancel{2}-0}{h} =$$

$$= \lim_{h\to 0} (h+2\sqrt{2}) = 2\sqrt{2}. \neq f_d'(\sqrt{2}),$$

$\Rightarrow$ nu există derivata în $\sqrt{2}$ $\Rightarrow f$ derivabilă pe $(0,3) \setminus \{\sqrt{2}\}$.

3). $f \in H^1(0,3)$. , $\|f\|_{H^1(0,3)}$.

Fie $g : (0,3) \to \mathbb{R}$, $g(x) = \begin{cases} -2x & , x \in (0,\sqrt{2}) \\ 2x & , x \in (\sqrt{2},3) \\ c & , x = \sqrt{2}. \end{cases}$

Fie $\varphi \in C_c^\infty(0,3) \Rightarrow \varphi \equiv 0$ în vecinătățile lui $0, 3$.

$$\int_0^3 f\varphi'\,dx = \int_0^{\sqrt{2}} (2-x^2)\varphi'(x)\,dx + \int_{\sqrt{2}}^3 (x^2-2)\varphi'(x)\,dx =$$

$$= (2-x^2)\varphi(x)\Big|_0^{\sqrt{2}} - \int_0^{\sqrt{2}} (-2x)\varphi(x)\,dx + (x^2-2)\varphi(x)\Big|_{\sqrt{2}}^3 - \int_{\sqrt{2}}^3 (2x)\cdot\varphi(x)\,dx$$

$$= -2\varphi(0) + 7\varphi(3) - \left(\int_0^{\sqrt{2}} g(x)\varphi(x)\,dx + \int_{\sqrt{2}}^3 g(x)\varphi(x)\,dx\right).$$

$$= 0 - \int_0^3 g(x)\varphi(x)\,dx \Rightarrow g = f'_{slab}.$$

$$\|f\|_{H^1(0,3)} = \|f\|_{L^2(0,3)} + \|f'\|_{L^2(0,3)}$$
$$= \left(\int_0^3 |f(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_0^3 |f'(x)|^2 dx\right)^{\frac{1}{2}}$$

$$|f'(x)| = |g| = \begin{cases} 2x & , x \in (0,3) \setminus \{\sqrt{2}\} \\ |c| & , x = \sqrt{2} \end{cases} \Rightarrow |g| = 2x \text{ apt.}$$

$$\Rightarrow \|f\|_{H^1(0,3)} = \left[\int_0^3 (x^4 - 2x^2 + 4)\, dx\right]^{1/2} + \left[\int_0^3 2x\, dx\right]^{\frac{1}{2}} =$$
$$= \left[\left(\frac{x^5}{5} - \frac{2x^3}{3} + 4x\right)\Big|_0^3\right]^{\frac{1}{2}} + \left[x^2\Big|_0^3\right]^{\frac{1}{2}} =$$
$$= \left(\frac{243}{5} - 18 + 12\right)^{\frac{1}{2}} + 9^{\frac{1}{2}} =$$
$$= \left(\frac{243 - 30}{5}\right)^{\frac{1}{2}} + 3 < \infty. \Rightarrow f \in H^1(0,2).$$

4). $\alpha \in \mathbb{R}$ ai $z : (0,1) \to \mathbb{R}$, $z(x) = x^\alpha \in H^1(0,1)$

$$H^1(0,1) = W^{1,2}(0,1) = \left\{ f \in L^2(0,1) \mid \frac{df}{dx}\Big|_{slab} \in L^2(0,1) \right\}.$$

Deci avem $z(x) \in H^1(0,1) \Leftrightarrow \begin{cases} z \in L^2(0,1) \\ z'_{slab} \in L^2(0,1). \end{cases}$

$$z(x) = x^\alpha \in L^2(0,1) \Leftrightarrow \int_0^1 |x^\alpha|^2 dx < \infty.$$

$$\xrightarrow{x>0} \int_0^1 |x^\alpha|^2 dx = \int_0^1 |x^{2\alpha}| dx =: I.$$

$2\alpha = -1 \Rightarrow I = -\ln x \Big|_0^1 = \infty$ $\not<$

$2\alpha < -1 \Rightarrow I = -\dfrac{x^{2\alpha+1}}{2\alpha+1}\Big|_0^1 = -\dfrac{1}{2\alpha+1} + \dfrac{0^{(2\alpha+1)<0}}{2\alpha+1} = \infty.$ $\not<$

$2\alpha > -1 \Rightarrow I = \dfrac{x^{2\alpha+1}}{2\alpha+1}\Big|_0^1 = \dfrac{1}{2\alpha+1} < \infty$ ✓.

Deci pt. $\alpha > -\frac{1}{2}$ avem $z(x) \in L^2(0,1)$.

cum $\overset{x>0}{\cancel{\alpha > -\frac{1}{2} \Rightarrow \alpha+1}}$ $\Rightarrow (x^\alpha)' = \alpha x^{\alpha-1}$ $\forall x \in (0,1)$.

$\Rightarrow z'_{slab} = z'_{tare} = \alpha x^{\alpha-1}$.

Verificăm dacă $\alpha x^{\alpha-1} \in L^2(0,1)$.

Analog $x^\alpha \in L^2(0,1)$, obținem. $\int_0^1 |\alpha x^{\alpha-1}|^2 dx = \alpha^2 \int_0^1 x^{2\alpha-2} dx < \infty$.

$\Leftrightarrow 2\alpha - 2 > -1 \Leftrightarrow \alpha > \frac{3}{2}$.

Deci, pt. $\alpha \in \frac{3}{2}$ avem $z(x) = x^\alpha \in H^1(0,1)$.

5) $z : (1,\infty) \to \mathbb{R}$, $z(x) = \frac{x^\alpha}{1+x^2} \in H^1(1,\infty)$.

$z(x) \in H^1(1,\infty) \Rightarrow \begin{cases} z(x) \in L^2(1,\infty) \\ z' \text{slab} \in L^2(1,\infty). \end{cases}$

$z(x) = \frac{x^\alpha}{1+x^2} \in L^2(1,\infty)$. $\Leftrightarrow \int_1^\infty \frac{x^{2\alpha}}{(1+x^2)^2} dx < \infty$.

Dacă $\frac{x^{2\alpha}}{(1+x^2)^2} \xrightarrow[x\to\infty]{} 1 \Rightarrow \int_1^\infty \frac{x^{2\alpha}}{(1+x^2)^2} dx \sim \int_1^\infty 1 \, dx = \infty$. $\cancel{\cdot}$

Deci, $\alpha < 2$ $\left(\alpha = 2 \Rightarrow \frac{x^4}{x^4+2x^2+1} \to 1 \right)$