Tema 2 (Probleme cu repartiții de v. a.)
ONUȚU RADU
Grupa 312

1. $X_t \sim Exp(\lambda)$

$\Rightarrow F(x) = 1 - e^{-\lambda x}$

a) $\mathbb{P}(X > 3) = 1 - \mathbb{P}(X \leq 3) = 1 - (1 - e^{-\frac{1}{3} \cdot 3}) = 1 - (1 - \frac{1}{e})$

$= 1 - (\frac{e-1}{e})$

$= \frac{e - e + 1}{e}$

$= \frac{1}{e}$

b) $\mathbb{P}(X = 12 \mid X > 9) = \frac{\mathbb{P}(X = 12)}{\mathbb{P}(X > 9)} = \frac{0,00\ldots1}{1 - \mathbb{P}(X \leq 9)} = \ldots =$

$= \frac{10^{-m}}{1 - 1 + e^{-\frac{1}{3} \cdot 9}} = \frac{10^{-m}}{e^{-3}} = e^3 \cdot 10^{-m}, \quad m \in \mathbb{N}$

2. A = o persoană să se îmbolnăvească de 2 ori în același an

B = vaccinul să fie eficient pt. o persoană.

$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)}$

$\mathbb{P}(B) = \frac{3}{4}$

$\mathbb{P}(A \mid B) = \mathbb{P}(X = 2) = \frac{e^{-3} \cdot 3^2}{2!} = \frac{9}{2 \cdot e^3}$

$$\mathbb{P}(A) = \mathbb{P}(X=2) = \frac{e^{-2}\cdot 5^2}{2!} = \frac{25}{2\cdot e^2}$$

$$\Rightarrow \mathbb{P}(B|A) = \frac{\frac{9}{2\cdot e^3}\cdot\frac{3}{4}}{\frac{25}{2\cdot e^2}} = \frac{27}{8\cdot e^3}\cdot\frac{2\cdot e^2}{25} = \frac{27}{100\cdot e^3}$$

3. 100 - produse: 95 - bune; 5 - defecte

a) $\mathbb{P}$(1 - produs bun [redacted]) $= \frac{95}{100} = \frac{19}{20}$

$\mathbb{P}$(5 - produse bune) $= \left(\frac{19}{20}\right)^5 = \frac{2476099}{3200000} \simeq 0,77$

b) ~~$\mathbb{P}$(1 - produs defect) $= \frac{5}{100}$~~

$\mathbb{P}$(5 - produse bune) = $\mathbb{P}$(acceptat)

$\mathbb{P}$(respins) = 1 - $\mathbb{P}$(acceptat) $\simeq 1 - 0,77 \simeq 0,23$

c) $\mathbb{P}$(1 - prd. bun) · $\mathbb{P}$(1 - prd. bun) · $\mathbb{P}$(1 - prd. bun) · $\mathbb{P}$(1 - prd. def.) $= \frac{19}{20}\cdot\frac{19}{20}\cdot\frac{19}{20}\cdot\frac{1}{20} \simeq 0,04$

4 $\mathbb{E}(X) = 173$ $\quad X \sim \mathcal{N}(\mu, \sigma^2)$

$\text{Var}(X) = 16$ $\quad X \sim \mathcal{N}(173, 16)$

$\mu = 173$

$\sigma^2 = 16 \Rightarrow \sigma = 4$

$$Z = \frac{X - \mu}{\sigma} = \frac{176 - 173}{4} = \frac{3}{4} = 0,75 \Rightarrow 77,34\%$$

5. a) $X \approx \text{Bin}\left(5, \frac{150}{250}\right) = \text{Bin}\left(5, \frac{3}{5}\right)$; $k = \overline{0,5}$

$$\mathbb{P}(X=0) = C_5^0 \cdot \left(\frac{3}{5}\right)^0 \cdot \left(\frac{2}{5}\right)^5 = \frac{32}{3125} \simeq 0,01$$

$$\mathbb{P}(X=1) = C_5^1 \cdot \left(\frac{3}{5}\right)^1 \cdot \left(\frac{2}{5}\right)^4 = \frac{48}{625} \simeq 0,07$$

$$\mathbb{P}(X=2) = C_5^2 \cdot \left(\frac{3}{5}\right)^2 \cdot \left(\frac{2}{5}\right)^3 = \frac{144}{625} \simeq 0,23$$

$$\mathbb{P}(X=3) = C_5^3 \cdot \left(\frac{3}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2 = \frac{216}{625} \simeq 0,34$$

$$\mathbb{P}(X=4) = C_5^4 \cdot \left(\frac{3}{5}\right)^4 \cdot \left(\frac{2}{5}\right)^1 = \frac{162}{625} \simeq 0,25$$

$$\mathbb{P}(X=5) = C_5^5 \cdot \left(\frac{3}{5}\right)^5 \cdot \left(\frac{2}{5}\right)^0 = \frac{243}{3125} \simeq 0,07$$

$$\Rightarrow X : \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0,01 & 0,07 & 0,23 & 0,34 & 0,25 & 0,07 \end{pmatrix}$$

b) $\mathbb{P}(X=4) \simeq 0,25$

$$\mathbb{P}(X \geq \tfrac{1}{2}) = 1 - \mathbb{P}(X < \tfrac{1}{2}) = 1 - 0,01 = 0,99$$

$$\mathbb{P}(X < \tfrac{4}{3}) = \mathbb{P}(X=0) + \mathbb{P}(X=1) = 0,01 + 0,07 = 0,08$$

$$\mathbb{P}(X \leq 2 \mid X > 0,2) = \frac{\mathbb{P}(0,02 < X \leq 2)}{\mathbb{P}(X > 0,2)} =$$

$$= \frac{\mathbb{P}(X=1) + \mathbb{P}(X=2)}{1 - \mathbb{P}(X \leq 0,2)} = \frac{0,07 + 0,23}{1 - 0,01} = \frac{0,3}{0,99} \simeq$$

$$\simeq 0,3$$

c) $\mathbb{P}(X \leq \frac{7}{2}) = (5 - \frac{7}{2}$

$$F\left(\frac{7}{2}\right) = \mathbb{P}\left(X \leq \frac{7}{2}\right) = \mathbb{P}(0) + \mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) =$$

$$= 0,01 + 0,07 + 0,023 + 0,34 = 0,65$$

d) $\mathbb{E}[X] = n \cdot p = 5 \cdot \frac{3}{5} = 3$

$$\mathrm{Var}[X] = n \cdot p \cdot (1-p) = 3 \cdot \left(1 - \frac{3}{5}\right) = 3 \cdot \frac{2}{5} = 1,2$$

6. $X \sim \mathrm{NBin}\left(10, \frac{1}{2}\right)$

$$\mathbb{P}(X = k) = C^{k}_{10+k-1} \cdot \left(\frac{1}{2}\right)^{k} \cdot \left(\frac{1}{2}\right)^{10}$$

$$= \frac{(9+k)!}{k! \cdot (9+k-k)!} \cdot \left(\frac{1}{2}\right)^{k+10}$$

$$= \frac{(k+9)!}{k! \cdot 9!} \cdot \left(\frac{1}{2}\right)^{k+10}$$

7. X
E
Va
IP
Z
H
H
1

8.

b

7. $X \sim \mathcal{N}(\mu, \sigma^2)$; $X \sim \mathcal{N}(3, 49)$

$\mathbb{E}(X) = 3 \Rightarrow \mu = 3$

$\text{Var}(X) = 49 \Rightarrow \sigma = 7$

$\mathbb{P}(X > c) = 0,15$

$Z = \frac{X - \mu}{\sigma} = \frac{X-3}{7} \Rightarrow 7Z + 3 = X$

~~$\mathbb{P}(7Z+3 > c) = 1 - \mathbb{P}(7Z+3 < c) = 0,15$~~

~~$\mathbb{P}(7Z+3 < c) = 0,85$~~

$\mathbb{P}(X \geq c) = 1 - \mathbb{P}(X \leq c)$

$\mathbb{P}(X \leq c) = 0,85 \Rightarrow Z \simeq 1,04$

~~[illegible]~~

$\Rightarrow X \simeq 7 \cdot 1,04 + 3 \simeq 10,28$

8. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = k \cdot x^5 \cdot (1-x)^7$, $x \in (0,1)$, $k \in \mathbb{R}$

a) f - densitatea de probabilitate $\Rightarrow f(x) \geq 0 \Rightarrow k \geq 0$

$\int_0^1 f(x)\,dx = 1$

$\int_0^1 k \cdot x^5 \cdot (1-x)^7\,dx = k \cdot B(6,8) = k \cdot \frac{\Gamma(6)\cdot\Gamma(8)}{\Gamma(14)} =$

$= k \cdot \frac{1}{10296} = 1 \Rightarrow k = 10296$

b) $\mathbb{E}(X) = \int_0^1 x \cdot f(x)\,dx = 10296 \cdot \int_0^1 x^6 \cdot (1-x)^7\,dx =$

$= 10296 \cdot B(7,8) = 10296 \cdot \frac{\Gamma(7)\cdot\Gamma(8)}{\Gamma(15)} = \frac{3}{7}$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \int_0^1 x^2 \cdot f(x)\,dx = 10296 \int_0^1 x^7 \cdot (1-x)^7\,dx$$

$$= 10296 \cdot B(8,8) = 10296 \cdot \frac{\Gamma(8)\cdot\Gamma(8)}{\Gamma(16)} = \frac{1}{5}$$

$$\Rightarrow \mathrm{Var}(X) = \frac{1}{5} - \left(\frac{3}{7}\right)^2 = \frac{1}{5} - \frac{9}{49} = \frac{49-45}{245} = \frac{4}{245}$$

9. a) $X \sim P_o(\lambda) = P_o(3)$

0 pers intră în medie la fiecare 10 min $\Rightarrow$

3 pers intră în medie la fiecare 30 min.

$\Rightarrow \lambda = 3$

$$\mathbb{P}(X=0) = e^{-3} \cdot \frac{3^0}{0!} = \frac{1}{e^3}$$

b) $\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X < 4) = 1 - \mathbb{P}(0) - \mathbb{P}(1) - \mathbb{P}(2) - \mathbb{P}(3)$

$= 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) - \mathbb{P}(X=2) - \mathbb{P}(X=3)$

$$\mathbb{P}(X=1) = e^{-3} \cdot \frac{3^1}{1!} = \frac{3}{e^3}$$

$$\mathbb{P}(X=2) = e^{-3} \cdot \frac{3^2}{2!} = \frac{9}{2e^3}$$

$$\mathbb{P}(X=3) = e^{-3} \cdot \frac{3^3}{3!} = \frac{9}{2e^3}$$

$$\mathbb{P}(X \geq 4) = 1 - \frac{1}{e^3} - \frac{3}{e^3} - \frac{9}{2e^3} - \frac{9}{2e^3} = 1 - \frac{4}{e^3} - \frac{9}{e^3}$$

$$= \frac{e^3 - 13}{e^3} \simeq 0{,}35$$

10. a) $X \sim Bin\left(5, \frac{50}{250}\right) = Bin\left(5, \frac{1}{5}\right)$

$$\mathbb{P}(X=0) = C_5^0 \cdot \left(\frac{1}{5}\right)^0 \cdot \left(\frac{4}{5}\right)^5 \simeq 0{,}32$$

$X \sim Bin\left(5, \frac{49}{249}\right)$

$$\mathbb{P}(X=1) = C_5^1 \cdot \left(\frac{49}{249}\right)^1 \cdot \left(\frac{200}{249}\right)^4 \simeq 0{,}4$$

$X \sim Bin\left(5, \frac{48}{248}\right)$

$$\mathbb{P}(X=2) = C_5^2 \cdot \left(\frac{48}{248}\right)^2 \cdot \left(\frac{200}{248}\right)^3 \simeq 0{,}19$$

$X \sim Bin\left(5, \frac{47}{247}\right)$

$$\mathbb{P}(X=3) = C_5^3 \cdot \left(\frac{47}{247}\right)^3 \cdot \left(\frac{200}{247}\right)^2 \simeq 0{,}04$$

$X \sim Bin\left(5, \frac{46}{246}\right)$

$$\mathbb{P}(X=4) = C_5^4 \cdot \left(\frac{46}{246}\right)^4 \cdot \left(\frac{200}{246}\right)^1 \simeq 0{,}004 \ (0{,}00496\ldots)$$

$X \sim Bin\left(5, \frac{45}{245}\right)$

$$\mathbb{P}(X=5) = C_5^5 \cdot \left(\frac{45}{245}\right)^5 \cdot \left(\frac{200}{245}\right)^0 \simeq 0{,}0002$$

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0{,}32 & 0{,}4 & 0{,}19 & 0{,}04 & 0{,}004 & 0{,}0002 \end{pmatrix}$$

b) $\mathbb{P}(X=1)=0,4$

$\mathbb{P}(X \geq \frac{5}{2}) = 1 - \mathbb{P}(X < \frac{5}{2}) = 1 - \mathbb{P}(0) - \mathbb{P}(1) - \mathbb{P}(2) =$

$= 1 - 0,32 - 0,4 - 0,19 = 0,19$

$\mathbb{P}(X < \frac{4}{3}) = \mathbb{P}(0) + \mathbb{P}(1) = 0,32 + 0,4 = 0,72$

$\mathbb{P}(X \leq 2 \mid X > 0,8) = \dfrac{\mathbb{P}(0,8 < X \leq 2)}{\mathbb{P}(X > 0,8)} =$

$= \dfrac{\mathbb{P}(1) + \mathbb{P}(2)}{1 - \mathbb{P}(X \leq 0,8)} = \dfrac{0,4 + 0,19}{1 - \mathbb{P}(0)} = \dfrac{0,59}{1 - 0,32} = \dfrac{0,59}{0,68} \simeq 0,86$

c) $F(\frac{11}{5}) = \mathbb{P}(X \leq \frac{11}{5}) = \mathbb{P}(X=0) + \mathbb{P}(X=1) =$

$= 0,32 + 0,4 = 0,72$

d) $\mathbb{E}(X) = 5 \cdot \frac{1}{5} = 1$

$Var(X) = 5 \cdot \frac{1}{5} \cdot (1 - \frac{1}{5}) = \frac{4}{5}$

11. $\mathbb{P}(A)$ = prob. ca o pers. să aibă sindromul

$\mathbb{P}(B)$ = — " — nu — " —

$\mathbb{P}(C)$ = prob. ca testul să fie pozitiv

$\mathbb{P}(D)$ = — " — negativ

a) $\mathbb{P}(A \mid C) = \dfrac{\mathbb{P}(C \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(C)} = \dfrac{0,9 \cdot 0,01}{\mathbb{P}(C \mid A) \cdot \mathbb{P}(A) + \mathbb{P}(C \mid B) \cdot \mathbb{P}(B)}$

$= \frac{0,009}{0,9 \cdot 0,01 + 0,1 \cdot 0,99} = \frac{0,009}{0,108} \simeq 0,08$

b) $\mathbb{P}(B \mid \Delta) = \frac{\mathbb{P}(\Delta \mid B) \cdot \mathbb{P}(B)}{\mathbb{P}(\Delta)} = \frac{0,1 \cdot 0,99}{\mathbb{P}(\Delta \mid A) \cdot \mathbb{P}(A) + \mathbb{P}(\Delta \mid B) \cdot \mathbb{P}(B)}$

$= \frac{0,099}{0,1 \cdot 0,01 + 0,1 \cdot 0,99} = \frac{0,099}{0,1} = 0,99$

12. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 30k \cdot x^{10} \cdot e^{-\frac{x}{30}}$, $x \geq 0$, $k \in \mathbb{R}$

a) f - densitatea de probabilitate $\Rightarrow f(x) \geq 0 \Rightarrow k \geq 0$

$\int_0^\infty f(x)\,dx = 1$

$30k \int_0^\infty x^{10} \cdot e^{\frac{-x}{30}}\,dx = 30k \cdot 30^{11} \int_0^\infty \frac{x}{30}^{10} \cdot t^{10} \cdot e^{-t}\,dt =$

$t = \frac{x}{30}$ $\quad x = 0 \Rightarrow t = 0$

$dt = \frac{1}{30}\,dx$ $\quad x = \infty \Rightarrow t = \infty$

$= 30^{12} \cdot k \cdot \Gamma(11) = 1 \Rightarrow k = \frac{1}{30^{12} \cdot \Gamma(11)}$

b) $\mathbb{E}(X) = \int_0^\infty x \cdot f(x)\,dx = 30 \cdot k \int_0^\infty x^{11} \cdot e^{-\frac{x}{30}}\,dx =$

$t = \frac{x}{30}$ $\quad x = 0 \Rightarrow t = 0$

$dt = \frac{1}{30}\,dx$ $\quad x = \infty \Rightarrow t = \infty$

$$30^2 \cdot k \int_0^\infty 30^{11} \cdot t^{11} \cdot e^{-t}\,dt = 30^{13} \cdot \frac{1}{30^{12} \cdot \Gamma(11)} \cdot \Gamma(12)$$

$$= 30 \cdot \frac{11!}{10!} = 30 \cdot 11 = 330$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \int_0^\infty x^2 \cdot f(x)\,dx = 30k \cdot \int_0^\infty x^{12} \cdot e^{-\frac{x}{30}}\,dx$$

$$= 30k \cdot 30^{13} \cdot \Gamma(13) = 30^{14} \cdot \frac{1}{30^{12} \cdot \Gamma(14)} \cdot \Gamma(13)$$

$$= 30^2 \cdot \frac{12!}{10!} = 900 \cdot 11 \cdot 12 = 118800$$

$$Var(X) = 118800 - 330 = 118470$$

13. $\mathbb{P}(A)$ = prob. ca o pers să aibă boala

$\mathbb{P}(A) = 0,00001$

$\mathbb{P}(B)$ = ——— " ——— nu ——— "

$\mathbb{P}(B) = 0,99999$

$\mathbb{P}(C)$ = prob. ca testul să fie bun

$\mathbb{P}(C) = 0,95$

$\mathbb{P}(D)$ = ——— " ——— pozit

$\mathbb{P}(D) = 0,05$

$\mathbb{P}(E)$ = prob. ca testul să fie pentru fals

$\mathbb{P}(E) = 0,005$

$$\mathbb{P}(C|A) = \frac{\mathbb{P}(A|C)\cdot\mathbb{P}(C)}{\mathbb{P}(A)} = \frac{0,001\cdot 0,95}{\mathbb{P}(A)}$$

$$\mathbb{P}(A|C) = \frac{\mathbb{P}(C|A)\cdot\mathbb{P}(A)}{\mathbb{P}(C)} = \frac{0,95\cdot 0,00001}{\mathbb{P}(C|A)\cdot\mathbb{P}(A) + \mathbb{P}(C|B)\cdot\mathbb{P}(B)} =$$

$$= \frac{0,0000095}{0,95\cdot 0,00001 + 0,005\cdot 0,99999} \simeq 0,001$$

14. ~~$X \sim Bin(2, \frac{c}{b+c})$~~

~~$\mathbb{P}(X$~~

a) ~~$\mathbb{P}(X$~~ $\mathbb{P}(A)$ = prob. ca a doua bilă să fie albastră =
= prob. ca prima bilă să fie albastră = $\frac{c}{b+c}$

b) $\left(\frac{c}{b+c}\right)^2$

15. $X \sim Exp\left(\frac{1}{20}\right)$

a) $\mathbb{P}(X > 20) = 1 - \mathbb{P}(X \leq 20) = 1 - 1 + e^{-\frac{1}{20}\cdot 20} = \frac{1}{e}$

b) $\mathbb{P}(20 \leq X \leq 40) = 1 - e^{-\frac{1}{20}\cdot 40} - 1 + e^{-\frac{1}{20}\cdot 20} = e^{-2} + e^{-1} =$

$= \frac{1}{e^2} + \frac{1}{e} = \frac{e+1}{e^2}$

16. $X \sim \mathcal{N}(\mu, \sigma^2)$

~~$\mu = 270$~~

$\mathbb{E}(X) = 270 \Rightarrow \mu = 270$

$Var(X) = 100 \Rightarrow \sigma = 10$

$$Z_1 = \frac{290 - 270}{10} = 2 \Rightarrow 0,9772$$

$$Z_2 = \frac{240 - 270}{10} = -3 \Rightarrow 0,0013$$

$0,9772 \cdot 0,0013 \simeq 0,0012 \Rightarrow 0,12\%$

(șanse mici să fie total copulați)

17. 10 min. $\simeq$ 0,16 ore

30 min. = 0,5 ore

$X \sim Unif(9, 12) = Unif(0, 3) = Unif(0, 180)$

~~$F(X) = \begin{cases} 0, & x < 9 \\ \frac{x-9}{3}, & 9 \le x < 12 \\ 1, & x > 12 \end{cases}$~~

~~$\mathbb{P}(X > 0,16) =$~~

a) Pt. $X \sim Unif(0, 30)$

$$\mathbb{P}(X > 10) = 20 \cdot \frac{1}{30-0} = \frac{2}{3} \simeq 0,66$$

b) $\mathbb{P}(10 <$ …

c) $\mathbb{P}(X >$ …

18. $X \sim Ge$…

a) $\mathbb{P}(X =$ …

b) …

c) $(0,95)$ …

b) $\mathbb{P}(10 < X < 20) = 10 \cdot \frac{1}{30} = \frac{1}{3} \approx 0,33$

c) $\mathbb{P}(X > 25) = 5 \cdot \frac{1}{30} = \frac{1}{6} \approx 0,16$

18. $X \sim Geo(0,05)$

a) $\mathbb{P}(X = 10) = (1-0,05)^{9} \cdot 0,05 \approx 0,03$

b) $1 - (0,05)^{5} = 1 - 0,000\ldots = 0,999\ldots$

c) $(0,95)^{5} \approx 0,81$