Consultație

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(1) Fie $X_1, X_2, \dots X_n$ eșantion de vol n din pop $P_\theta X^?$

$$P(X=k) = \frac{\theta^k}{(1+\theta)^{k+1}} \quad \left(\overset{A}{=} \frac{\theta^k}{(1+\theta)^k} \right)$$

$\theta > 0$

Det. estimatorii ai. prin met momentelor și met ver. maxime și studiați prop. acestora.

Sol: Met. momentelor:

$$\overline{X}_n = E_\theta[X_1] \longrightarrow g(\theta)$$

$$\tilde{\theta}_n = h(\overline{X}_n)$$

$$(q^k)'$$

$$\sum_k k q^{k-1}$$

$$E_\theta[X_1] = \sum_{k \geq 0} k P_\theta(X_1 = k)$$

$$= \sum_{k \geq 0} k \frac{\theta^k}{(1+\theta)^{k+1}} = \frac{\theta}{(1+\theta)^2} \sum_{k \geq 1} k \left(\frac{\theta}{1+\theta}\right)^{k-1}$$

$$= \frac{\theta}{(1+\theta)^2} \underline{\sum_{k \geq 1} k q^{k-1}} \quad \text{unde } q = \frac{\theta}{1+\theta}$$

$$E_\theta[X_1] = \frac{\theta}{(1+\theta)^2}\sum_{k\geq 0}(k+1)q^k \quad \to \frac{d}{dq}q^{k+1}$$

$$= \frac{\theta}{(1+\theta)^2}\sum_{k\geq 0}\left(q^{k+1}\right)'$$

$$= \frac{\theta}{(1+\theta)^2}\frac{d}{dq}\left(\sum_{k\geq 0}q^{k+1}\right)$$

$$(f\circ g)' = f'(g)\cdot g'$$

$$= \frac{\theta}{(1+\theta)^2}\frac{d}{dq}\left(\frac{1}{1-q}-1\right)$$

$$= \frac{\theta}{(1+\theta)^2}\underbrace{\frac{d}{dq}\left(\frac{q}{1-q}\right)} = \frac{\theta}{(1+\theta)^2}\frac{1}{(1-q)^2}$$

$$= \frac{\theta}{(1+\theta)^2}\cdot\frac{1}{\left(1-\frac{\theta}{1+\theta}\right)^2} = \theta$$

$\overline{X}_n = \theta \to \boxed{\tilde{\theta}_n = \overline{X}_n}$ est. obținut prin met. mom.

<u>Met. ver. maxime:</u>

$$L_n(\theta;x) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n}\frac{\theta^{x_i}}{(1+\theta)^{x_i+1}}$$

$$= \frac{\theta^{\sum x_i}}{(1+\theta)^{n+\sum x_i}}$$

$$\ell_n(\theta;x) = \log L_n(\theta;x)$$
$$= \left(\sum_{n} x_i\right)\log(\theta) - \left(n+\sum_{i=1}^{n} x_i\right)\log(1+\theta)$$

Ec. de verosim.:

$$\frac{\partial \ell_n(\theta, x)}{\partial\theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n+\sum_{i=1}^{n} x_i}{1+\theta} = 0$$

$$\Leftrightarrow \frac{\sum x_i}{\theta} = \frac{n+\sum x_i}{1+\theta} \Leftrightarrow \theta = \frac{\sum x_i}{n}$$

$\underline{\hat{\theta}_n = \overline{X}_n}$ est. de verosimilitate maximă

<u>Prop</u>: 1) deplasarea

$\mathbb{E}_\theta[\overline{X}_n] = \mathbb{E}_\theta[X_1] = \theta \Rightarrow \hat{\theta}_n$ este un estimator nedeplasat

2) consistența

LNM: $\overline{X}_n \xrightarrow{a.s.} \mathbb{E}_\theta[X_1] = \theta \Rightarrow \hat{\theta}_n$ este consistent

3) normalitatea asimptotică

TLC: $\sqrt{n}(\overline{X}_n - \theta) \xrightarrow{\mathcal{L}} N(0, \underline{Var_\theta(X_1)})$

$$Var_\theta(X_1) = \underbrace{\mathbb{E}_\theta[X_1^2]}_{?} - \underbrace{\mathbb{E}_\theta[X_1]^2}_{=\theta^2}$$

$$\mathbb{E}_\theta[X_1^2] = \sum_{k\geq 0} k^2\, \mathbb{P}_\theta(X_1=k) = \sum_{k\geq 0} k^2 \frac{\theta^k}{(1+\theta)^{k+1}}$$

$$= \sum_{k\geq 1} k^2 \frac{\theta^k}{(1+\theta)^{k+1}} = \frac{\theta}{(1+\theta)^2}\sum_{k\geq 1} k^2 \Big(\underbrace{\frac{\theta}{1+\theta}}_{q}\Big)^{k-1}$$

$$= \frac{\theta}{(1+\theta)^2}\sum_{k\geq 0} (k+1)^2 q^k$$

$(k+1)(k+1) - (k+1)$

k

$$= \frac{\theta}{(1+\theta)^2}\sum_{k\geq 0} (k^2+2k+1) q^k$$

$$= \frac{\theta}{(1+\theta)^2}\Big[\sum_{k\geq 0} k^2 q^k + 2\sum_{k\geq 0} k q^k + \sum_{k\geq 0} q^k\Big]$$

$$= \frac{\theta}{(1+\theta)^2}\Big[\underbrace{\sum_{k\geq 0} k^2 q^k}_{(\theta+1)\mathbb{E}_\theta[X_1^2]} + 2\underbrace{\sum_{k\geq 0} (k+1) q^k}_{\left(\sum_{k\geq 0} q^{k+1}\right)'} - \underbrace{\sum_{k\geq 0} q^k}\Big]$$

$\frac{q}{1-q}$

$$= \frac{\theta}{(1+\theta)^2}\Big[(\theta+1)\mathbb{E}_\theta[X_1^2] + 2\Big(\frac{q}{1-q}\Big)' - \frac{1}{1-q}\Big]$$

$$= \frac{\theta}{(1+\theta)^2}\Big[(\theta+1)\mathbb{E}_\theta[X_1^2] + 2(1+\theta)^2 - (\theta+1)\Big]$$

$$E_\theta[X_1^2] = \frac{\theta}{1+\theta} E_\theta[X_1^2] + 2\theta - \frac{\theta}{1+\theta}$$

$$\frac{E_\theta[X_1^2]}{1+\theta} = \frac{(2\theta + 2\theta^2 - \theta)}{1+\theta}$$

$$\Rightarrow E_\theta[X_1^2] = 2\theta^2 + \theta \Rightarrow Var_\theta(X_1) = \theta^2 + \theta$$
$$= \theta(\theta+1)$$

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \theta(\theta+1))$$

$$MSE_\theta(\hat{\theta}_n) = Var_\theta(\hat{\theta}_n) + \underbrace{b_\theta(\hat{\theta}_n)^2}_{=0} = Var_\theta(\hat{\theta}_n)$$

$$= \underline{\frac{\theta(1+\theta)}{n}} \xrightarrow[n\to\infty]{} 0$$

Ex2 $X_1, \dots X_n \sim Exp(\theta)$, $\theta > 0$

1) Est. obținut prin met momentelor
2) EVM și aproape
3) $\boxed{T_n = \frac{n-1}{n} \hat{\theta}_n}$, unde $\hat{\theta}_n$ este est. de ver. max

Comparăm cei 3 estimatori

1) $Exp(\theta)$ $\quad f_\theta(x) = \theta e^{-\theta x}$, $\theta > 0$, $x \geq 0$

$\bar{X}_n = \mathbb{E}_\theta[X_1]$

dar $\mathbb{E}_\theta[X_1] = 1/\theta$ $\Bigg\} \Rightarrow \boxed{\tilde{\theta}_n = \frac{1}{\bar{X}_n}}$ est. obț. prin met. mom.

Este bine definit $\tilde{\theta}_n$? $\mathbb{P}_\theta(\bar{X}_n = 0) = ?$

$\mathbb{P}_\theta(\bar{X}_n = 0) = \mathbb{P}_\theta(X_1 = 0)^n = 0$ pt că $\mathbb{P}_\theta(X_1 = 0) = 0$

$\mathbb{P}_\theta(\bar{X}_n > 0) = 1$. deci este bine definit.

2) EVM

$$L_n(\theta; x) = \prod_1 \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum x_i}$$

$$l_n(\theta; x) = n \log \theta - \theta \sum x_i \Rightarrow \frac{\partial l_n}{\partial \theta} = 0 \iff \hat{\theta}_n = \frac{1}{\bar{X}_n}$$

Prop: a) consistența

LNM: $\bar{X}_n \xrightarrow{a.s} \mathbb{E}_\theta[X_1] = \frac{1}{\theta}$

$\hat{\theta}_n = g(\bar{X}_n)$, $g(x) = \frac{1}{x}$

Th. Apl. Cont. $\quad g(\bar{X}_n) \xrightarrow{a.s} g(1/\theta) = \theta \Rightarrow \hat{\theta}_n \xrightarrow{a.s} \theta$

b) Normalitatea asimptotică

TLC: $\quad \sqrt{n}\left(\bar{X}_n - \frac{1}{\theta}\right) \xrightarrow{d} N\big(0, \overbrace{Var_\theta(X_1)}^{=\frac{1}{\theta^2}}\big)$

$$\sqrt{n}\left(\overline{X}_n - 1/\theta\right) \xrightarrow{d} N\left(0, \frac{1}{\theta^2}\right)$$

$$\hat{\theta}_n = \frac{1}{\overline{X}_n} = g(\overline{X}_n)$$

Metoda Delta (g deriv)

$$\sqrt{n}\left(g(\overline{X}_n) - g(1/\theta)\right) \xrightarrow{d} g'\left(\frac{1}{\theta}\right) N\left(0, \frac{1}{\theta^2}\right)$$

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} N\left(0, g'\left(\frac{1}{\theta}\right)^2 \frac{1}{\theta^2}\right)$$

$$g'\left(\frac{1}{\theta}\right) = -\theta^2 \Rightarrow \sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} N\left(0, \frac{\theta^4}{\theta^2}\right)$$

$$= \underline{N(0, \theta^2)}$$

3) Deplasarea

$$E_\theta[\hat{\theta}_n] = E_\theta\left[\frac{1}{\overline{X}_n}\right] = E_\theta\left[\frac{n}{\sum_{i=1}^{n} X_i}\right] = n\, E_\theta\left[\frac{1}{\sum_{i=1}^{n} X_i}\right]$$

$X_1, X_2, \ldots X_n \sim Exp(\theta)$ indep

at $X_1 + \ldots + X_n \sim \Gamma(n, \theta)$

$$h(x) = \frac{\theta^n x^{n-1} e^{-\theta x}}{(n-1)!}$$

$\underline{Obs}$: $X \perp\!\!\!\perp Y$, $Z = X + Y$

$$f_Z(z) = \int f_X(z-x) f_Y(x)\, dx$$

$X \perp\!\!\!\perp Y$ indep discrete, $Z = X+Y$

$$\mathbb{P}(Z=k) = \mathbb{P}(X+Y=k)$$

$$= \sum_x \mathbb{P}(X+Y=k \mid X=x)\, \mathbb{P}(X=x)$$

$$= \sum_x \underbrace{\mathbb{P}(Y=k-x \mid X=x)}_{\text{indep}}\, \mathbb{P}(X=x)$$

$$= \sum_x \mathbb{P}(Y=k-x)\, \mathbb{P}(X=x)$$

$$f_Z(z) = \int f_Y(z-t)\, f_X(t)\, dt$$

$$\mathbb{E}_\theta\left[\frac{1}{\sum X_i}\right] = \int \frac{1}{x} h_n(x)\, dx$$

$$= \int_0^\infty \frac{1}{\cancel{x}}\, \theta^n \frac{\overset{x^{n-2}}{\cancel{x^{n-1}}}}{(n-1)!} e^{-\theta x}\, dx$$

$$= \frac{\theta}{n-1} \underbrace{\int_0^\infty h_{n-1}(x)\, dx}_{=1} = \theta/(n-1)$$

$$\boxed{\mathbb{E}_\theta[\hat{\theta}_n] = \frac{n\theta}{n-1}}$$ deplasat

$\sum X_i \sim \Gamma(n, \theta)$

densitatea lui $\Gamma(n,\theta)$ notam cu $h_n(x)$

$$\underline{h_n(x)} = \theta^n \frac{x^{n-1}}{(n-1)!} e^{-\theta x}, \quad x>0$$

$$\int \frac{1}{x} h_n(x)\,dx = \int_0^\infty \frac{1}{\cancel{x}} \theta^n \frac{x^{\cancel{n-1}}}{(n-1)!} e^{-\theta x} dx$$

$$= \int_0^\infty \theta^n \frac{x^{n-2}}{(n-1)!} e^{-\theta x} dx$$

$$= \int_0^\infty \frac{\theta}{n-1} \underbrace{\theta^{n-1} \frac{x^{n-2}}{(n-2)!} e^{-\theta x}}_{h_{n-1}(x)} dx$$

$$= \frac{\theta}{n-1} \underbrace{\int h_{n-1}(x)\,dx}_{=1} = \theta/n-1$$

Putem să calculăm și $MSE_\theta(\hat{\theta}_n)$

$$MSE_\theta(\hat{\theta}_n) = Var_\theta(\hat{\theta}_n) + b_\theta(\hat{\theta}_n)^2$$

$$b_\theta(\hat{\theta}_n) = E_\theta[\hat{\theta}_n] - \theta = \frac{n}{n-1}\theta - \theta$$

$$= \frac{\theta}{n-1}$$

$$\mathrm{Var}_\theta(\hat\theta_n) = \mathbb{E}_\theta[\hat\theta_n^2] - \underbrace{\mathbb{E}_\theta[\hat\theta_n]^2}_{\left(\frac{n}{n-1}\theta\right)^2}$$

$$\mathbb{E}_\theta[\hat\theta_n^2] = n^2\,\mathbb{E}_\theta\left[\frac{1}{(\sum X_i)^2}\right]$$

$$= n^2 \underline{\int \frac{1}{x^2} f_{n}(x)\,dx} = \frac{n^2\theta^2}{(n-1)(n-2)}$$

$$\mathrm{Var}_\theta(\hat\theta_n) = \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

$T_n = \frac{n-1}{n}\hat\theta_n$ nedyplosat

$$\left.\begin{array}{l}\hat\theta_n \xrightarrow{a.s} \theta \\ \frac{n-1}{n} \to 1\end{array}\right\} \Rightarrow \frac{n-1}{n}\hat\theta_n \xrightarrow{a.s} \theta \quad \text{consist.}$$

$$\sqrt{n}(T_n-\theta) = \sqrt{n}\left(\frac{n-1}{n}\hat\theta_n - \theta\right)$$

$$= \sqrt{n}\left(\hat\theta_n - \theta - \frac{1}{n}\hat\theta_n\right)$$

$$= \underbrace{\sqrt{n}(\hat\theta_n-\theta)}_{\xrightarrow{d}\ \mathcal{N}(0,\theta^2)} - \underbrace{\frac{1}{\sqrt{n}}\hat\theta_n}_{\xrightarrow{P}\ 0}$$

$$\sqrt{n}(T_n-\theta) = \underbrace{\sqrt{n}(\hat{\theta}_n-\theta)}_{\xrightarrow{d}\ N(0,\theta^2)} - \underbrace{\frac{1}{\sqrt{n}}\hat{\theta}_n}_{\xrightarrow{P}\ 0} \quad \overset{\text{Th. Slutsky}}{\Longrightarrow}$$

$$\left.\begin{array}{l} \frac{1}{\sqrt{n}} \longrightarrow 0 \\ \hat{\theta}_n \xrightarrow{P} \theta \end{array}\right\} \Rightarrow \frac{1}{\sqrt{n}}\hat{\theta}_n \xrightarrow{P} 0 \qquad \sqrt{n}(T_n-\theta) \xrightarrow{d} N(0,\theta^2)$$

$\hat{\theta}_n, T_n$? pe care îl alegem.

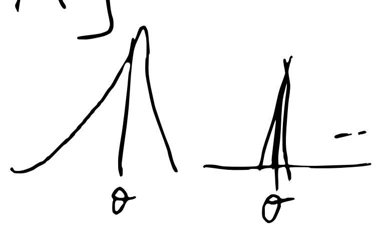
$$MSE_\theta(T_n) = Var_\theta(T_n) = \left(\frac{n-1}{n}\right)^2 Var_\theta(\hat{\theta}_n) \le MSE_\theta(\hat{\theta}_n)$$

$$= \frac{\cancel{(n-1)^2}}{\cancel{n^2}} \cdot \frac{\cancel{n^2}\theta^2}{\cancel{(n-1)^2}(n-2)} = \frac{\theta^2}{n-2}$$

$$MSE_\theta(\hat{\theta}_n) = \frac{n^2\theta^2}{(n-1)^2(n-2)} + \frac{\theta^2}{(n-1)^2}$$

$$= \frac{\theta^2}{(n-1)^2}\left[\frac{n^2}{n-2}+1\right]$$

$$T_n \simeq N\Big(\theta, \underbrace{\theta^2/n}_{\to\, 0}\Big)$$



Cât este Inf. lui Fisher?

$$I_1(\theta) = \mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log f_\theta(X_1)\right)^2\right]$$
$$= -\mathbb{E}_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X_1)\right]$$

$$\log f_\theta(x) = \log\left(\theta e^{-\theta x}\right) = \log\theta - \theta x$$

$$\frac{\partial}{\partial\theta}\log f_\theta(x) = \frac{1}{\theta} - x$$

$$\frac{\partial^2}{\partial\theta^2}\log f_\theta(x) = -\frac{1}{\theta^2}$$

$$I_1(\theta) = \frac{1}{\theta^2} \Rightarrow I_n(\theta) = n I_1(\theta) = \frac{n}{\theta^2}$$

$$\text{MIRC} = \frac{1}{I_n(\theta)} = \theta^2/n$$

$$\mathrm{Var}_\theta(T_n) = \theta^2/_{n-2} > \theta^2/n = \text{MIRC}$$

deci T_n nu este eficient

$$\mathrm{eff}_\theta(\hat{\theta}_n) = \frac{\text{MIRC}}{\mathrm{Var}_\theta(\hat{\theta}_n)} = \frac{n-2}{n} \xrightarrow[n\to\infty]{} 1$$

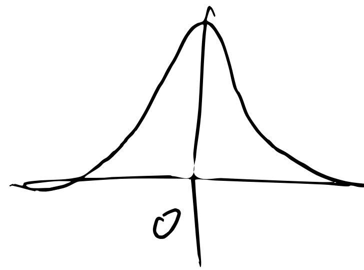
Ex $X \sim N(0,\sigma^2)$

$X_1, \dots X_n$ din pop $X+\theta$

$\boxed{F_\theta(x) = F(x-\theta)}$, F fct de rep $N(0,\sigma^2)$

Cat este mediana?

$$F_\theta^{-1}(1/2) = ?$$



$$F_\theta(\theta) = F(0) = 1/2$$

$$F_\theta^{-1}\left(\tfrac{1}{2}\right) = \theta \Rightarrow x_{1/2} = \theta$$

Med. empirică:

$$\hat{x}_n(1/2) = \hat{F}_n^{-1}(1/2)$$

$$f_\theta(\theta) = f(0) = \frac{1}{\sqrt{2\pi}\,\sigma}$$

$$\sqrt{n}\left(\hat{x}_n(1/2) - x_{1/2}\right) = \sqrt{n}\left(\hat{x}(1/2) - \theta\right)$$

$$\xrightarrow{d} N\left(0, \frac{\frac{1}{2}(1-\frac{1}{2})}{f_\theta(x_{1/2})^2}\right)$$

$$f_\theta(x_{1/2}) = f_\theta(\theta)$$

$$\sqrt{n}\left(\hat{x}_n(p) - x_p\right) \xrightarrow{d} N\left(0, \frac{p(1-p)}{f(x_p)^2}\right)$$

$$\sqrt{n}\left(\underline{\underline{\hat{x}_n(1/2)}} - \theta\right) \xrightarrow{d} N\left(0, \frac{\frac{1}{4}}{\frac{1}{2\pi\sigma^2}}\right)$$

$$= N\left(0, \frac{\pi\sigma^2}{2}\right)$$

$\underline{\bar{X}_n} \leftarrow$ EVM

$$\sqrt{n}\left(\bar{X}_n - \theta\right) \xrightarrow{d} N(0, \sigma^2)$$

Cun $\frac{\pi\sigma^2}{2} > \sigma^2$ dùi alg $\bar{X}_n$