4

Arătați că dacă o fct. netedă $f: \mathbb{R}^4 \to \mathbb{R}$ verifică $f(\lambda x) = \lambda^6 f(x)$ pt $\forall x \in \mathbb{R}^4$ și $x \cdot \nabla f(x) = 6 f(x)$, $\forall x \in \mathbb{R}^4$

Fie x ales arbitrar

- $(f(\lambda x))'_\lambda = \sum_{i=1}^{4} \frac{\partial f}{\partial x_i} \cdot \frac{\partial (\lambda x_i)}{\partial \lambda}$

$(f(\lambda x))'_\lambda = \sum_{i=1}^{4} \frac{\partial f}{\partial x_i} \cdot x_i = x \nabla f(x)$

$(\lambda^6 f(x))'_\lambda = 6\lambda^5 f(x)$ Luăm $\lambda = 1$

$\Rightarrow x \nabla f(x) = 6 f(x)$ c.c.t.d

II

$\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$ $\partial\Omega = \text{front } \Omega$

④

$A_n = 1$.

③

Arătați că $\Delta(x_8 |x|^{-8}) = 0 \ \forall x \in \mathbb{R}^8 \setminus \{0\}$

- $\Delta(f \cdot g) = \Delta f \cdot g + f \cdot \Delta g + 2 \nabla f \cdot \nabla g$
- $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$
- $\Delta f = \lambda(\lambda + n - 2)|x|^{\lambda - 2}$
- $\Delta f = g''(r) + \frac{n-1}{r} g'(r)$

$$\Delta(f \cdot g) = \Delta(x_8) \cdot |x|^{-8} + x_8 \cdot \Delta(|x|^{-8}) +$$
$$+ 2 \nabla(x_8) \cdot \nabla(|x|^{-8})$$

$$\Delta(|x|^{-8}) = (-8|x|^{-9})' = (-9)(-8)(|x|)^{-10} +$$
$$+ \frac{7}{|x|} \cdot (-8(|x|)^{-9})$$

- $f(x) = |x|^{\lambda}$
- $\nabla f = \lambda x |x|^{\lambda - 2}$

$$\nabla(|x|^{-8}) = -8x|x|^{-10}$$

$$\nabla(x_8) = (0,0,0,0,0,0,0,1)$$

$$\nabla(|x|^{-8}) \cdot \nabla(x_8) = -8x_8 |x|^{-10}$$

③

Arătați că dacă o fct. netedă $f: \mathbb{R}^4 \to \mathbb{R}$ verifică $f(\lambda x) = \lambda^6 f(x)$ pt $\forall x \in \mathbb{R}^4$ și
$x \cdot \nabla f(x) = 6 f(x)$, $\forall x \in \mathbb{R}^4$

Fie x ales arbitrar

- $$(f(\lambda x))'_\lambda = \sum_{i=1}^{4} \frac{\partial f}{\partial x_i} \cdot \frac{\partial (\lambda x_i)}{\partial \lambda}$$

$$(f(\lambda x))'_\lambda = \sum_{i=1}^{4} \frac{\partial f}{\partial x_i} \cdot x_i = x \nabla f(x)$$

$(\lambda^6 f(x))'_\lambda = 6\lambda^5 f(x)$ Luăm $\lambda = 1$

$\Rightarrow x \nabla f(x) = 6 f(x)$ c.c.t.d

II

$\Omega := (-1,1) \times (-1,1) \subset \mathbb{R}^2$ $\partial\Omega = \text{front } \Omega$

$$\begin{cases} -\Delta u(x,y) = \dfrac{y^2}{1+x^2}, & (x,y) \in \Omega \\ u(x,y) = 0 & (x,y) \in \partial\Omega \end{cases}$$

①
$c = ?$ a.î. $v(x,y) = c(x^2+y^2)$ să verifice
$-\Delta v = 1$ în Ω

- $$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial(c(x^2+y^2))}{\partial x}\right) = \frac{\partial}{\partial x}(2cx) = 2c$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial(c(x^2+y^2))}{\partial y}\right) = \frac{\partial}{\partial y}(2cx) = 2c$$

$\Rightarrow$ Principiul slab de maxim $\Rightarrow$

$$\Rightarrow \Delta v = 2c + 2c = 4c$$

$$4c = -1 \Rightarrow c = -\frac{1}{4}$$

②

Folos „Principiile de maxim" deduceți că

$$0 < u(x,y) \le \frac{1}{2}, \quad \forall (x,y) \in \Omega$$

Considerăm $w = u - v$

$$\Delta w = \Delta u - \Delta v$$

$$\Delta w = -\frac{y^2}{1+x^2} - (-1) = 1 - \frac{y^2}{1+x^2} \ge 0$$

$\Rightarrow -\Delta w \le 0 \Rightarrow w$ subarmonică

Aplic principiul slab de maxim pt fct. subarmonice

$$\Rightarrow \max_{\overline{\Omega}} w = \max_{\partial\Omega} w$$

$$\max_{\partial\Omega} w = \max_{\partial\Omega}(\overset{0}{u} - v) = \max_{\partial\Omega}(-v) =$$

$$= \max \frac{1}{4}(x^2+y^2) = \frac{1+1}{4} = \frac{1}{2} \Rightarrow \max_{\partial\Omega} w = \frac{1}{2}$$

$$\Rightarrow u - v \le \frac{1}{2} \Rightarrow u \le \frac{1}{2} + v$$

$$v = \underbrace{-\frac{1}{4}}_{<0}\underbrace{(x^2+y^2)}_{\ge 0} \Rightarrow v \le 0 \qquad \Rightarrow u \le \frac{1}{2}$$

$$-\Delta u(x,y) = \frac{y^2}{1+x^2} \ge 0 \Rightarrow u \text{ superarmonică}$$

$\Rightarrow$

$\Rightarrow$ Principiul slab de maxim $\Rightarrow$

$\Rightarrow \min_{\bar{\Omega}} u = \min_{\partial\Omega} u = 0$

$\Rightarrow u \geq 0$

$0 \leq u \leq \frac{1}{2}$

Pp că $\exists\ (x,y) \in \Omega$ a.î. $u(x,y) = 0$, u super-armonică $\xRightarrow{PTM}$ u e constantă $\Rightarrow \Delta u = 0$ $\cancel{\text{C}}$ $\Rightarrow$

$\Rightarrow 0 < u \leq \frac{1}{2}$

III

Considerăm următoarea problemă de tip „unde"

$$\begin{cases} 3u_{tt}(x,t) + 11\,u_{tx}(x,t) - 4u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

①

Arătați că $(3\partial_t - \partial_x)(v_t(x,t) + 4v_x(x,t)) = 3v_{tt}(x,t) + 11\,v_{tx}(x,t) - 4v_{xx}(x,t)\quad \forall x, \forall t$

$(3\partial_t - \partial_x)(v_t(x,t) + 4v_x(x,t)) =$

$= 3v_{tt}(x,t) + 12v_{tx}(x,t) - v_{tx}(x,t) - 4v_{xx}(x,t) =$

$= 11\,v_{tx}(x,t) + 3v_{tt}(x,t) - 4v_{xx}(x,t)$ c.c.t.d.

$X_\tau(\tau, s) = 4$

② Rezolvați problema cu valori inițiale (2) satisfăcută de u (scrieți forma generală a lui u) reducând-o la rezolvarea a două ec. de transport (una omogenă, alta neomogenă)

Din ex (1): $(3\partial_t - \partial_x)(\partial_t + 4\partial_x)u =$

$= 3u_{tt} + 11u_{tx} - 4u_{xx}$

Fie $v(x,t) = (\partial_t + 4\partial_x)u = u_t + 4u_x$

v verifică ec. de transport: $3v_t - v_x = 0$

$$v(x,0) = \underbrace{u_t(x,0)}_{g(x)} + \underbrace{4u_x(x,0)}_{4f'(x)} = g(x) + 4f'(x)$$

$3v_t - v_x = 0 \iff (v_t, v_x)(3,-1) = 0$

$\Rightarrow v$ e constantă pe direcția $(3,-1)$

$\Rightarrow v(x,t) = v((x,t) + t(3,-1))$ mereu dă 0

$= v(x+3t, t-t) = v(x+3t, 0)$

$= g(x+3t) + 4f'(x+3t)$

u verifică ecuația neomogenă

$$\begin{cases} u_t + 4u_x = v(x,t) = g(x+3t) + 4f'(x+3t) \\ u(x,0) = f(x) \end{cases}$$

Calculăm curbele caracteristice asociate $(X(\tau,s), Y(\tau,s), Z(\tau,s))$

$\partial u(x+3t,t) \cdot \partial(x+3t)$

$x_\tau(\tau,s) = 4$

$y_\tau(\tau,s) = 1$

$z_\tau(\tau,s) = g(x(\tau,s) + 3y(\tau,s) +$

$+ 4f'(x(\tau,s) + 3y(\tau,s))$

$$\begin{cases} x(0,s) = s \\ y(0,s) = 0 \\ z(0,s) = f(s) \end{cases}$$

$$\begin{cases} x_\tau(\tau,s) = 4 \Rightarrow x(\tau,s) = 4\tau + c_1 \\ x(0,s) = s \end{cases}$$

$\Rightarrow c_1 = s \Rightarrow \boxed{x(\tau,s) = 4\tau + s}$

$$\begin{cases} y_\tau(\tau,s) = 1 \Rightarrow y(\tau,s) = \tau + c_2 \\ y(0,s) = 0 \end{cases}$$

$\Rightarrow c_2 = 0 \Rightarrow \boxed{y(\tau,s) = \tau}$

$$\left.\begin{array}{l} \underset{x}{\underline{x(\tau,s)}} = 4\tau + s \\ \underset{t}{\underline{y(\tau,s)}} = \tau \end{array}\right\} \Rightarrow \begin{array}{l} s = x - 4t \\ \tau = t \end{array}$$

$z_\tau(\tau,s) = g(7\tau + s) + 4f'(7\tau + s)$

$z(0,s) = f(s)$

$z(\tau,s) = z(0,s) + \int_0^\tau \left(g(s+7\xi) + 4f'(s+7\xi)\right) d\xi$

$$v_x(x,t) = \frac{\partial u(x+3t,t)}{\partial x} \cdot \frac{\partial(x+3t)}{\partial x} +$$

$$+ \partial \quad \overset{s \,\vartheta\, \alpha = s+7\tau}{\Longrightarrow} \quad f(s) + \int_s^{s+7\tau} g(\alpha) \cdot \frac{1}{7} \, \partial\alpha +$$

$$v_x \quad + \int_s^{s+7\tau} 4 f'(\alpha) \cdot \frac{1}{7} \, \partial\alpha =$$

$$= f(s) + \frac{1}{7} \left(\int_s^{s+7\tau} g(\alpha) \partial\alpha + 4 f(\alpha) \Big|_s^{s+7\tau} \right) =$$

$$\Rightarrow \quad = f(s) + \frac{1}{7} \int_s^{s+7\tau} g(\alpha) \, \partial\alpha + \frac{4}{7} \left(f(s+7\tau) - f(s) \right) =$$

$$\Rightarrow u(x,t) = f(x-4t) + \frac{1}{7} \int_{x-4t}^{x+3t} g(\alpha) \, \partial\alpha +$$

2

$$+ \frac{4}{7} \left(f(x+3t) - f(x-4t) \right) =$$

$$= \frac{4}{7} f(x+3t) + \frac{3}{7} f(x-4t) + \frac{4}{7} \int_{x-4t}^{x+3t} g(\alpha) \, \partial\alpha$$

IV

v

Considerăm problema Cauchy

$$\begin{cases} u_t(x,t) + 3u_x(x,t) - u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

1

$v : \mathbb{R} \to \mathbb{R}$

$v(x,t) = u(x+3t,t)$

v verifică $v_t(x,t) - v_{xx}(x,t) = 0 \quad \forall x \in \mathbb{R}$
$\forall t > 0$

$v(x,t) = u(x+3t,t)$

$$v_t(x,t) = \frac{u(x+3t,t)}{\partial x} \cdot \frac{\partial(x+3t)}{\partial t} + \frac{\partial u(x+3t,t)}{\partial t} \cdot \frac{\partial t}{\partial t}$$

$$= u_x(x+3t,t) \cdot 3 + u_t(x+3t,t)$$