Integrale triple. Exemple.

Mentionam urmatoarele variante ale Propozitiei 7 din cursul 12.

Propozitie 1. Fie $D \subset \mathbb{R}^2$ o multime masurabila Jordan si $\alpha, \beta : D \to \mathbb{R}$ doua functii continue si marginite pe D astfel incat $\alpha(x, z) \leq \beta(x, z)$ pentru orice $(x, z) \in D$. Atunci multimea

$$V = \{(x, y, z) \in \mathbb{R}^3 : (x, z) \in D,\ \alpha(x, z) \leq y \leq \beta(x, z)\}$$

este masurabila Jordan. Daca $f : V \to \mathbb{R}$ este continua si marginita pe V atunci f este integrabila Riemann pe V si

$$\iiint_V f(x, y, z)dxdydz = \iint_D \left(\int_{\alpha(x,z)}^{\beta(x,z)} f(x, y, z)dy \right) dxdz.$$

Propozitie 2. Fie $D \subset \mathbb{R}^2$ o multime masurabila Jordan si $\alpha, \beta : D \to \mathbb{R}$ doua functii continue si marginite pe D astfel incat $\alpha(y, z) \leq \beta(y, z)$ pentru orice $(y, z) \in D$. Atunci multimea

$$V = \{(x, y, z) \in \mathbb{R}^3 : (y, z) \in D,\ \alpha(y, z) \leq x \leq \beta(y, z)\}$$

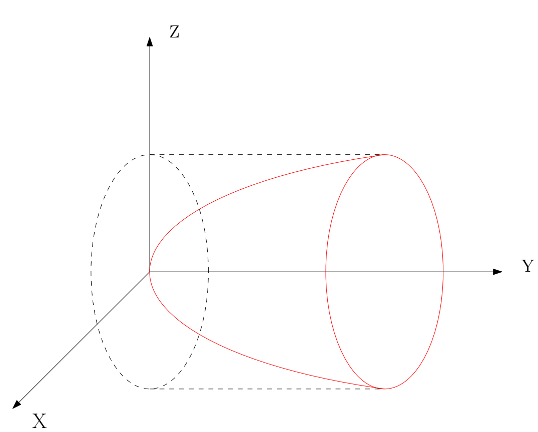
este masurabila Jordan. Daca $f : V \to \mathbb{R}$ este continua si marginita pe V atunci f este integrabila Riemann pe V si

$$\iiint_V f(x, y, z)dxdydz = \iint_D \left(\int_{\alpha(y,z)}^{\beta(y,z)} f(x, y, z)dx \right) dydz.$$

Exemplul 3. Calculati

$$\iiint_V \sqrt{x^2 + z^2}dxdydz$$

unde V este multimea marginita de planul $y = 4$ si paraboloidul $x^2 + z^2 = y$.



Asadar,

$$V = \{(x,y,z) \in \mathbb{R}^3 \ : x^2 + z^2 \leq y \leq 4, (x,y) \in D\}$$

unde

$$D = \{(x,z) \in \mathbb{R}^2 \ : x^2 + z^2 \leq 4\}$$

$$\iiint_V \sqrt{x^2+z^2} dxdydz = \iint_D \left(\int_{x^2+z^2}^4 \sqrt{x^2+z^2} dy \right) dxdz =$$

$$= \iint_D (4 - x^2 - z^2)\sqrt{x^2+z^2} \ dxdz$$

In continuare integrala se calculeaza trecand la coordonate polare (exercitiu!)

Exercitiu. Incercati sa calculati integrala de mai sus proiectand pe planul xOy.

Propozitie 4. Fie $V \subset \mathbb{R}^3$ o multime compacta masurabila Jordan cuprinsa intre planele $z = a$ si $z = b$. Notam cu D_{z_0} proiectia pe planul xOy a intersectiei lui V cu planul $z = z_0$ unde $a \leq z_0 \leq b$, adica

$$D_{z_0} = \{(x,y) \in \mathbb{R}^2 : (x,y,z_0) \in V\}.$$

Fie $f : V \to \mathbb{R}$ o functie continua. Atunci f este integrabila pe V si

$$\iiint_V f(x,y,z) dxdydz = \int_a^b \left(\iint_{D_z} f(x,y,z) dxdy \right) dz.$$

Exemplul 5. Calculati volumul piramidei a carei baza este patratul $[-1,1] \times [-1,1]$ din planul xOy si al carei varf este punctul de coordonate $(0,0,1)$.

Rezolvare. Daca $z \in [0,1]$ atunci (exercitiu)

$$D_z = [-1+z, 1-z] \times [-1+z, 1-z].$$

Cu alte cuvinte,

$$V = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq 1, (x,y) \in D_z\}.$$

Volumul piramidei este

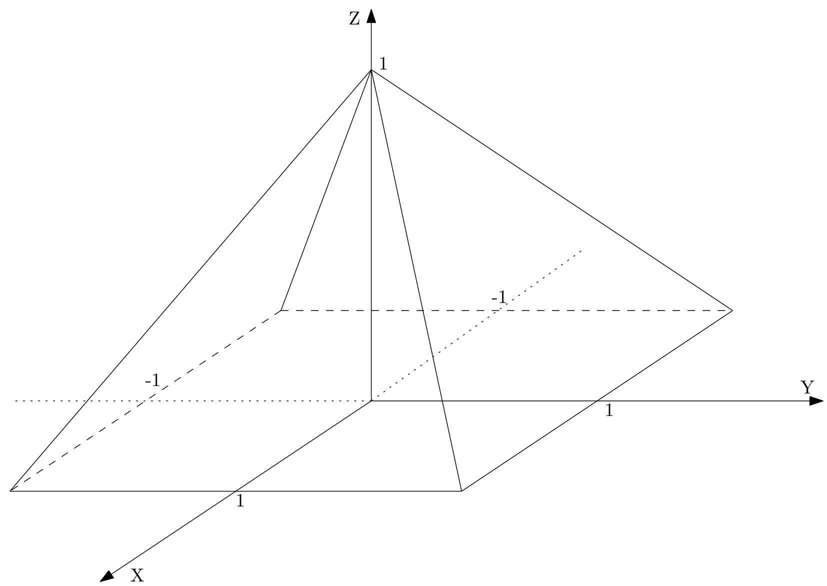
$$\iiint_V dxdydz = \int_0^1 \left(\iint_{D_z} dxdy \right) dz$$

Deoarece

$$\iint_{D_z} dxdy = \int_{-1+z}^{1-z} \left(\int_{-1+z}^{1-z} dy \right) dx = (2-2z)^2$$

rezulta ca

$$\iiint_V dxdydz = \int_0^1 (2z-2)^2 dz = 4 \cdot \left. \frac{(z-1)^3}{3} \right|_0^1 = \frac{4}{3}.$$



Exemplul 6. Calculati
$$\iiint_V \sqrt{x^2+y^2+z^2}\, dxdydz,$$
unde $V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1, x^2+y^2 \le z^2, z \ge 0\}$.

Rezolvare. Intersectia dintre dintre conul $x^2+y^2 = z^2$ si sfera unitate este

$$\begin{cases} x^2+y^2 = z^2 \\ x^2+y^2+z^2 = 1 \\ z \ge 0 \end{cases},$$

adica cercul
$$x^2+y^2 = \frac{1}{2}$$
situat in planul $z = \frac{1}{\sqrt{2}}$.

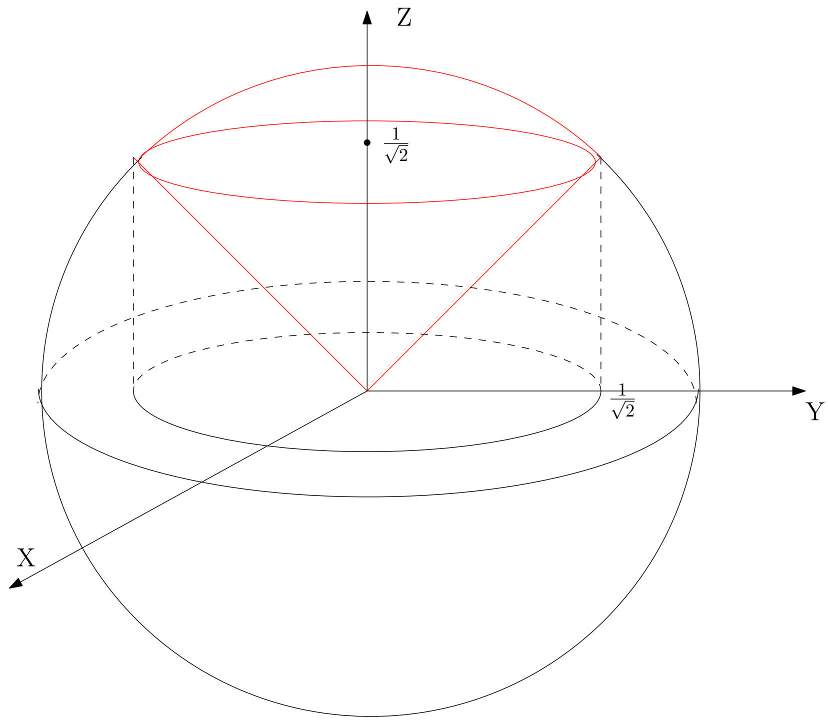
Vom trece la coordonate sferice

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}.$$

Transformarea

$$\phi : (0,\infty)\times(0,2\pi)\times(0,\pi) \to \mathbb{R}^3 \setminus \{(x,0,z) : x \ge 0, z \in \mathbb{R}\}$$

$$\phi(r,\theta,\varphi) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$$



este difeomorfism si

$$J_\phi(r,\theta,\varphi) = -r^2 \sin\varphi \qquad \text{pentru orice } (r,\theta,\varphi) \in (0,\infty)\times(0,2\pi)\times(0,\pi).$$

Atunci

$$(x,y,z) = \phi(r,\theta,\varphi) \in \overset{\circ}{V} \Longleftrightarrow \begin{cases} \sin^2\varphi < \cos^2\varphi \\ \cos\varphi > 0 \\ 0 < \theta < 2\pi \\ 0 < r < 1 \end{cases} \Longleftrightarrow \begin{cases} \sin\varphi < \cos\varphi \\ 0 < \theta < 2\pi \\ 0 < r < 1 \end{cases}$$

Asadar

$$(x,y,z) = \phi(r,\theta,\varphi) \in \overset{\circ}{V} \Longleftrightarrow \begin{cases} 0 < r < 1 \\ 0 < \theta < 2\pi \\ 0 < \varphi < \pi/4 \end{cases}$$

Daca

$$A = (0,1)\times(0,2\pi)\times(0,\pi/4)$$

atunci

$$\overset{\circ}{V} = \phi(A)$$

si cum $\mathrm{Fr}(V) = V \setminus \phi(A)$ are masura Jordan zero avem

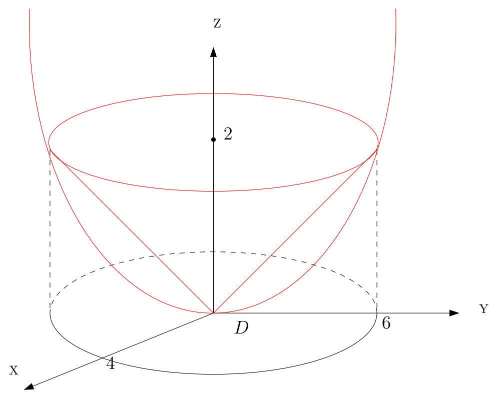
$$\iiint_V \sqrt{x^2+y^2+z^2}dxdydz = \iiint_{\overset{\circ}{V}} \sqrt{x^2+y^2+z^2}dxdydz = \iiint_{\phi(A)} \sqrt{x^2+y^2+z^2}dxdydz$$

$$= \iiint_A r\cdot r^2 \sin\varphi dr d\theta d\varphi = \int_0^1 \left(\int_0^{2\pi}\left(\int_0^{\frac{\pi}{4}} r^3 \sin\varphi d\varphi\right) d\theta\right) dr$$
$$= \left(\int_0^1 r^3 dr\right)\cdot\left(\int_0^{2\pi} d\theta\right)\cdot\left(\int_0^{\frac{\pi}{4}} \sin\varphi d\varphi\right)$$
$$= \frac{1}{4}\cdot 2\pi\cdot(-\cos\varphi)\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right).$$

Exemplul 7. Calculati

$$\iiint_V x^2 dxdydz, \quad V = \{(x,y,z)\in\mathbb{R}^3 : z^2 \le \frac{x^2}{4}+\frac{y^2}{9} \le 2z\}$$

unde V este multimea marginita de suprafetele $\frac{x^2}{4}+\frac{y^2}{9}=2z$ si $\frac{x^2}{4}+\frac{y^2}{9}=z^2$.



Rezolvare. Paraboloidul $\frac{x^2}{4}+\frac{y^2}{9}=2z$ intersecteaza conul $\frac{x^2}{4}+\frac{y^2}{9}=z^2$ dupa elipsa $\frac{x^2}{16}+\frac{y^2}{36}=1$ situata in planul $z=2$ (deoarece $z^2=2z$). Proiectia lui V pe planul xOy este multimea masurabila Jordan

$$D = \{(x,y)\in\mathbb{R}^2 : \frac{x^2}{16}+\frac{y^2}{36}\le 1\}$$

Orice paralela la axa Oz dusa prin punctele lui D intersecteaza paraboloidul $\frac{x^2}{4}+\frac{y^2}{9}=2z$ si conul $\frac{x^2}{4}+\frac{y^2}{9}=z^2$. Deci

$$V = \{(x,y,z)\in\mathbb{R}^3 : \frac{x^2}{8}+\frac{y^2}{18}\le z\le\sqrt{\frac{x^2}{4}+\frac{y^2}{9}},\ (x,y)\in D\}.$$

Deoarece functiile $\alpha(x,y)=\frac{x^2}{8}+\frac{y^2}{18}$ si $\beta(x,y)=\sqrt{\frac{x^2}{4}+\frac{y^2}{9}}$ sunt continue pe D si $f:V\to\mathbb{R}^3$, $f(x,y,z)=x^2$ este continua pe V, conform Propozitiei 7, Curs 12 avem

$$\iiint_V x^2 dxdydz = \iint_D\left(\int_{\frac{x^2}{8}+\frac{y^2}{18}}^{\sqrt{\frac{x^2}{4}+\frac{y^2}{9}}} x^2 dz\right) dxdy =$$

$$= \iint_D x^2 \left(\sqrt{\frac{x^2}{4} + \frac{y^2}{9}} - \frac{x^2}{8} - \frac{y^2}{18} \right) dxdy$$

Pentru calcularea integralei duble vom trece la coordonate polare generalizate. Transformarea

$$\phi : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}, \quad \phi(r, \theta) = (4r \cos\theta, 6r \sin\theta)$$

este un difeomorfism cu Jacobianul

$$J_\Phi(r, \theta) = 24r \text{ pentru orice } (r, \theta)$$

Asadar, $\Phi(r, \theta) = (x, y)$

$$\begin{cases} x = 4rcos\theta \\ y = 6r\sin\theta \end{cases} \qquad \frac{x^2}{16} + \frac{y^2}{36} = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

Deci,

$$\frac{x^2}{16} + \frac{y^2}{36} < 1 \Longleftrightarrow r^2 < 1$$

si atunci

$$(x, y) = \phi(r, \theta) \in D \setminus (\mathrm{Fr}(D) \cup [0, 4] \times \{0\}) \Longleftrightarrow \begin{cases} 0 < r < 1 \\ 0 < \theta < 2\pi \end{cases}$$

Asadar, daca

$$A = (0, 1) \times (0, 2\pi)$$

atunci $\phi(A) = D \setminus (\mathrm{Fr}(D) \cup [0, 4] \times \{0\})$ si cum $\lambda(\mathrm{Fr}(D) \cup [0, 4] \times \{0\}) = 0$ avem

$$= \iint_D x^2 \left(\sqrt{\frac{x^2}{4} + \frac{y^2}{9}} - \frac{x^2}{8} - \frac{y^2}{18} \right) dxdy = \iint_{\phi(A)} x^2 \left(\sqrt{\frac{x^2}{4} + \frac{y^2}{9}} - \frac{x^2}{8} - \frac{y^2}{18} \right) dxdy$$

$$\iint_A 16r^2\cos^2\theta(2r - 2r^2) \cdot 24r \; drd\theta = \int_0^1 \left(\int_0^{2\pi} 32 \cdot 24\cos^2\theta(r^4 - r^5)d\theta \right) dr$$

$$= 32 \cdot 24 \int_0^1 (r^4 - r^5)dr \cdot \int_0^{2\pi} \cos^2\theta d\theta = 32 \cdot 24 \left(\frac{1}{5} - \frac{1}{6} \right) \cdot \pi = \frac{128\pi}{5}.$$

Exemplul 8. Calculati

$$\iiint_V zdxdydz$$

unde

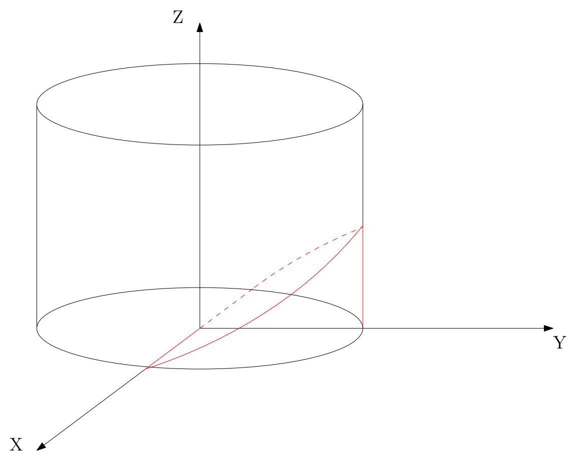
$$V = \{(x, y, z) \in \mathbb{R}^3 \; : x^2 + y^2 \leq 1, 0 \leq z \leq y\}$$

Rezolvare. Asadar,

$$V = \{(x, y, z) \in \mathbb{R}^3 \ : 0 \leq z \leq y, (x, y) \in D\}$$

unde

$$D = \{(x, y) \in \mathbb{R}^2 \ : x^2 + y^2 \leq 1, y \geq 0\}.$$



$$\iiint_V z dx dy dz = \iint_D \left(\int_0^y z dz \right) dx dy = \iint_D \frac{z^2}{2} \bigg|_{z=0}^{z=y} dx dy = \iint_D \frac{y^2}{2} dx dy$$

Integrala dubla se calculeza prin trecere la coordonate polare.

Integrala de mai sus se mai poate calcula utilizand **coordonate cilindrice**, care sunt date de difeomorfismul

$$\phi : (0, \infty) \times (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(x, 0, z) \in \mathbb{R}^3 \ : x \geq 0, z \in \mathbb{R}\}$$

$$\phi(r, \theta, z) = (r \cos \theta, r \sin \theta, z).$$

Avem $J_\phi(r, \theta, z) = r$.

$$(x, y, z) = \phi(r, \theta, z) = (r \cos \theta, r \sin \theta, z) \in \overset{\circ}{V} \Longleftrightarrow \begin{cases} r^2 \cos^2 \theta + r^2 \sin^2 \theta < 1 \\ r \sin \theta > 0 \\ 0 < z < r \sin \theta \end{cases} \Leftrightarrow \begin{cases} 0 < r < 1 \\ 0 < \theta < \pi \\ 0 < z < r \sin \theta \end{cases}$$

Fie

$$A = \{(r, \theta, \varphi) : (r, \theta) \in (0, 1) \times (0, \pi), \ 0 < z < r \sin \theta\}.$$

Atunci $\phi(A) = \overset{\circ}{V}$ si avem

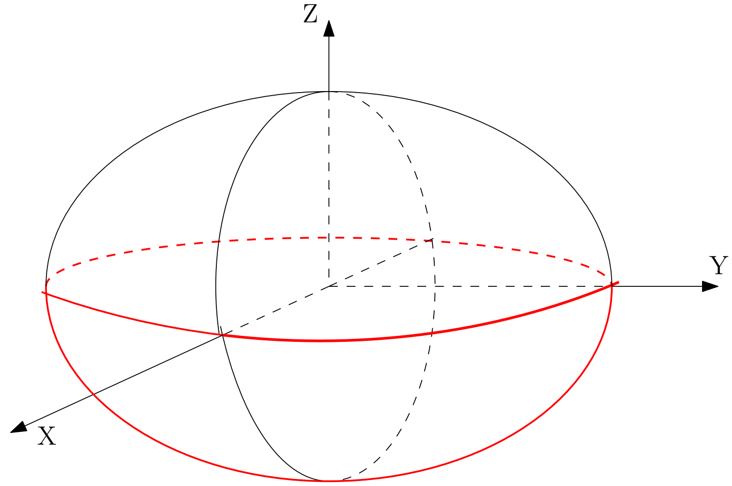
$$\iiint_V z dx dy dz = \iiint_{\overset{\circ}{V}} z dx dy dz = \iiint_{\phi(A)} z dx dy dz = \iiint_A z \cdot |J_\phi(r, \theta, \phi)| dr d\theta dz$$

$$= \iiint_A rz dr d\theta dz = \iint_{(0,1)\times(0,\pi)} \left(\int_0^{r\sin\theta} rz dz \right) dr d\theta = \int_{(0,1)\times(0,\pi)} \left(r \frac{z^2}{2} \Big|_0^{r\sin\theta} \right) dr d\theta =$$

$$= \int_0^1 \left(\int_0^\pi \frac{r^3 \sin^2\theta}{2} d\theta \right) d = \int_0^1 \left(\int_0^\pi r^3 \frac{1-\cos 2\theta}{4} d\theta \right) dr = \int_0^1 \frac{\pi r^3}{4} dr = \frac{\pi}{16}.$$

Exemplul 9. Calculati integrala

$$\iiint_V \left(\frac{x^2}{4} + \frac{y^2}{9} + z^2 \right) dx dy dz, \quad V = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{9} + z^2 \le 1, z \le 0\}.$$



Rezolvare. Consideram difeomorfismul

$$\phi : (0,\infty) \times (0,2\pi) \times (0,\pi) \to \mathbb{R}^3 \setminus \{(x,0,z) : x \ge 0, z \in \mathbb{R}\}.$$

$$\phi(r,\theta,\varphi) = (2r\cos\theta\sin\varphi, 3r\sin\theta\sin\varphi, r\cos\varphi)$$

pentru care

$$J_\phi(r,\theta,\varphi) = -6r^2 \sin\varphi \qquad (r,\theta,\varphi) \in (0,\infty) \times (0,2\pi) \times (0,\pi)$$

Daca $B = \{(x,0,z) \in \mathbb{R}^3 \ : x \ge 0, \frac{x^2}{4} + z^2 \le 1, z \le 0\}$ atunci

$$(x,y,z) = \phi(r,\theta,\varphi) \in V \setminus (\mathrm{Fr}(V) \cup B) \Longleftrightarrow \begin{cases} 0 < r < 1 \\ 0 < \theta < 2\pi \\ \frac{\pi}{2} < \varphi < \pi \end{cases}.$$

Fie $A = (0,1) \times (0,2\pi) \times (\frac{\pi}{2}, \pi)$. Avem

$$\phi(A) = V \setminus (\mathrm{Fr}(V) \cup B), \quad \lambda(\mathrm{Fr}(V) \cup B) = 0$$

si prin urmare

$$\iiint_V \left(\frac{x^2}{4}+\frac{y^2}{9}+z^2\right) dxdydz = \iiint_{\phi(A)} \left(\frac{x^2}{4}+\frac{y^2}{9}+z^2\right) dxdydz$$

$$= \iiint_A r^2 \cdot |J_\phi(r,\theta,\varphi)| drd\theta d\varphi = \iiint_A 6r^4 \sin\varphi \; drd\theta d\varphi$$

$$= \int_0^1 \left(\int_0^{2\pi} \left(\int_{\pi/2}^{\pi} 6r^4 \sin\varphi d\varphi\right) d\theta\right) dr = \int_0^1 2\pi \cdot 6r^4 dr = \int_0^1 12\pi r^4 dr = 12\pi/5$$