Factorizarea QR: metoda Gram-Schmidt clasică - standard

Determinați factorizarea QR a matricii

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

folosind metoda Gram-Schmidt clasică / standard.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} := \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix} =$$

$$= \begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} =: QR \Rightarrow$$

Obtinem:

$$\begin{cases} \underline{a}_1 = \underline{q}_1 r_{11} & (1) \\ \underline{a}_2 = \underline{q}_1 r_{12} + \underline{q}_2 r_{22} & (2) \\ \underline{a}_3 = \underline{q}_1 r_{13} + \underline{q}_2 r_{23} + \underline{q}_3 r_{33} & (3) \end{cases}$$

(1) $\underline{a}_1 = \underline{q}_1 r_{11} \implies \begin{cases} r_{11} = \|\underline{a}_1\| \\ \underline{q}_1 = \underline{a}_1 / r_{11} \end{cases}$

$$\|\underline{a}_1\| = \sqrt{1+1+1+1} = 2 \implies \boxed{r_{11} = 2}$$

$$\underline{q}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

(2) $\underline{a}_2 = \underline{q}_1 r_{12} + \underline{q}_2 r_{22} \implies$

$$\begin{cases} r_{12} = \underline{q}_1^T \underline{a}_2 \\ r_{22} = \|\underline{a}_2 - \underline{q}_1 r_{12}\| \\ \underline{q}_2 = \frac{1}{r_{22}} (\underline{a}_2 - \underline{q}_1 r_{12}) \end{cases}$$

$$r_{12} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 5 \Rightarrow$$

$$\Rightarrow \boxed{r_{12} = 5}$$

$$\underline{a}_2 - \underline{q}_1 r_{12} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - 5 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$$

$$\| \underline{a}_2 - \underline{q}_1 r_{12} \| = \sqrt{\frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4}} = \sqrt{5} \Rightarrow$$

$$\boxed{r_{22} = \sqrt{5}}$$

$$\underline{q}_2 = \begin{bmatrix} -3\sqrt{5}/10 \\ -\sqrt{5}/10 \\ \sqrt{5}/10 \\ 3\sqrt{5}/10 \end{bmatrix}$$

(3) $\underline{a}_3 = \underline{q}_1 r_{13} + \underline{q}_2 r_{23} + \underline{q}_3 r_{33} \Rightarrow$

$r_{13} = \underline{q}_1^T \underline{a}_3$

$r_{23} = \underline{q}_1^T \underline{a}_3$

$$r_{33} = \| a_3 - q_1 r_{13} - q_2 r_{23} \|$$

$$q_3 = \frac{1}{r_{33}} \left(a_3 - q_1 r_{13} - q_2 r_{23} \right)$$

$$r_{13} = \frac{1}{2} \cdot 1 + \frac{1}{2} 4 + \frac{1}{2} 9 + \frac{1}{2} 16$$

$$= \frac{1}{2}(1+4+9+16) = 15 \implies$$

$$\boxed{r_{13} = 15}$$

$$r_{23} = -\frac{3\sqrt{5}}{10} \cdot 1 - \frac{\sqrt{5}}{10} \cdot 4 + \frac{\sqrt{5}}{10} 9 + \frac{3\sqrt{5}}{10} 16$$

$$= \frac{\sqrt{5}}{10}(-3-4+9+48) = 5\sqrt{5} \implies$$

$$\boxed{r_{23} = 5\sqrt{5}}$$

$$a_3 - q_1 r_{13} - q_2 r_{23} =$$

$$= \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix} - 15 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - 5\sqrt{5} \begin{bmatrix} -3\sqrt{5}/10 \\ -\sqrt{5}/10 \\ \sqrt{5}/10 \\ 3\sqrt{5}/10 \end{bmatrix} = \begin{bmatrix} 1 - 15/2 + 15/2 \\ 4 - 15/2 + 5/2 \\ 9 - 15/2 - 5/2 \\ 16 - 15/2 - 15/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow r_{33} = \| a_3 - q_1 r_{13} - q_2 r_{23} \| = 2$$

$$\boxed{r_{33} = 2}$$

$$\Rightarrow q_3 = \frac{1}{r_{33}} \left(a_3 - q_1 r_{13} - q_2 r_{23} \right) = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Am obtinut:

$$Q = \begin{bmatrix} 1/2 & -3\sqrt{5}/10 & 1/2 \\ 1/2 & -\sqrt{5}/10 & -1/2 \\ 1/2 & \sqrt{5}/10 & -1/2 \\ 1/2 & 3\sqrt{5}/10 & 1/2 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}$$