① Tema TN10_311 30.04.24

1. Descompuneți în factori primi numărul $2^{48}+1$

2. Câte cifre are în perioadă numărul $\frac{1}{2023}$

Sol 1. $N \stackrel{not}{=} 2^{48}+1 = (2^{16}+1)(2^{32}-2^{16}+1)$

• Fie p prim cu $p \mid 2^{16}+1$.

Atunci $2^{16} \equiv -1 \pmod p$ (1)

$2^{32} \equiv 1 \pmod p$ $\Big\} \Rightarrow \gamma_p(2) = 32$.

Dar $32 = \gamma_p(2) \mid p-1 \Rightarrow p \equiv 1 \pmod{32}$

$N = 65537$ $\quad 97 \nmid N$

$193 \nmid N$

$257^2 = (2^8+1)^2 = 2^{16}+2^9+1 > 2^{16}+1 = N$

Întrucât ~~...~~ pt orice nr. prim $p \in \mathbb{N}$ cu $p < \sqrt{N}$ avem $p \nmid N$, N e prim.

$N = 2^{2^4}+1$, $F_0 \stackrel{not}{=} 2^{2^0}+1 = 3$; $F_1 = 2^{2^1}+1 = 5$;

$F_2 = 2^{2^2}+1 = 17$; $F_3 = 2^{2^3}+1 = 257$; $F_4 = N$

FERMAT: „Toate $F_n = 2^{2^n}+1$ sunt prime"

<u>EULER</u>

$F_5 = 2^{32}+1 = 2^{32} + 2^{28}\cdot 5^4 - 2^{28}\cdot 5^4 + 1 =$

$= 2^{28}\underbrace{(2^4+5^4)}_{641} + (1-5\cdot 2^7)\underbrace{(1+5\cdot 2^7)}_{641}(1+5^2\cdot 2^{14})$

$\vdots\ 641$

• Fie p prim cu $p \mid 2^{32}-2^{16}+1 \overset{not}{=} T \mid 2^{48}+1$

Atunci $2^{48} \equiv -1 \pmod p$

$\Downarrow$

$2^{96} \equiv 1 \pmod p$ $\Big\} \Rightarrow o_p(2) \mid 96 \wedge o_p(2) \nmid 48$

$\Updownarrow$

$o_p(2) \in \{32, 96\}$.

<u>$o_p(2) = 32$</u> $\Rightarrow$ $2^{32} \equiv 1 \pmod p$, $2^{48} \equiv -1 \pmod p$ $\Big\} \xrightarrow[\text{deci } 2 \in U(\mathbb{Z}_p)]{(2,p)=1} 2^{16} \equiv -1 \pmod p$

$\Downarrow$

$p = N$. $\overset{N \text{ e}}{\underset{\text{prim}}{\Longleftarrow}}$ $p \mid 2^{16}+1 = N$

Atunci $2^{16}+1 = N \mid 2^{32}-2^{16}+1 = (2^{16}+1)^2 - 3\cdot 2^{16} \Rightarrow$

$2^{16}+1 \mid 3\cdot 2^{16}$, ↯ (generată de TFAritm.)

(sau de considerente elementare: $2^{16}+1 \nmid 3$)

Rămânem deci că $o_p(2) = 96$.

Ca urmare, $96 = o_p(2) \mid p-1$, deci $p \equiv 1 \pmod{97}$

$M = 2^{32} - 2^{16} + 1 = 65536 \cdot 65535 + 1 = 4.294.901.761 =$

$= 193 \cdot (22.253.377) \overset{not}{=} L$

$193 \nmid L$; $481 \nmid L$; $577 \nmid L$; $673 \nmid L$;

$769 \nmid L$; $1057 \nmid L$; $1153 \nmid L$;

$$\begin{array}{r|l} 1057 & 96 \\ \hline 96 & 11 \\ \hline 97 & \\ 96 & \\ \hline 1 & \end{array}$$

$1249 \nmid L$; $1441 \nmid L$; $1537 \nmid L$;

$1633 \nmid L$; $1729 \nmid L$; $1921 \nmid L$;

$2017 \nmid L$; $2113 \nmid L$; $2497 \nmid L$;

$2593 \nmid L$; $2689 \nmid L$; $2881 \nmid L$; $2977 \nmid L$;

$3073 \nmid L$; $3169 \nmid L$; $3361 \nmid L$; $3457 \nmid L$;

$3553 \nmid L$; $3649 \nmid L$; $3649 \nmid L$; $3841 \nmid L$;

$3937 \nmid L$; $4033 \nmid L$; $4129 \nmid L$; $4321 \nmid L$;

$4417 \nmid L$; $4513 \nmid L$; $4609 \nmid L$; $4703 \nmid L$;

$4799 > \sqrt{L}$, deci încercările se pot opri aici. Se conchide că L e prim.

Ca urmare, descompunerea lui $2^{16}+1$ în factori primi este

$193 \cdot 65537 \cdot 22253377$.

$$\text{sol 2} \qquad \exists t, a_1, a_2, \dots, a_t$$

$$\frac{1}{2023} = 0,(a_1 a_2 \dots a_t) = \frac{\overline{a_1 a_2 \dots a_t}}{10^t - 1} \Leftrightarrow$$

$$(\exists t, a_1, \dots, a_t) \quad 2023 \cdot \overline{a_1 a_2 \dots a_t} = 10^t - 1 \Leftrightarrow$$

$$2023 \mid 10^t - 1 \Leftrightarrow 10^t \equiv 1 \pmod{2023}$$

$$\Leftrightarrow \gamma_{2023}(10) \mid t$$

De fapt, nsă, numărul de cifre din perioada lui 2023 înseamnă CEL MAI MIC astfel de $t \in \mathbb{N}^*$, adică $\gamma_{2023}(10)$.

Mai trebuie, deci, să-l găsim pe $\gamma_{2023}(10)$

Dar $\gamma_{2023}(10) = \underset{U(\mathbb{Z}_{2023})}{\mathrm{ord}} \hat{10} =$

$$= \cancel{\underset{U(\mathbb{Z}_7 \times \mathbb{Z}_{17^2})}{\mathrm{ord}}} (\hat{3}, \overline{10}) = \underset{U(\mathbb{Z}_7) \times U(\mathbb{Z}_{17^2})}{\mathrm{ord}} \qquad (\hat{3}, \overline{10}) =$$

Căci dacă $(a, b) = 1$,

$$\mathbb{Z}_a \times \mathbb{Z}_b \xrightarrow{\sim} \mathbb{Z}_{ab}$$

$$(\hat{x}, \overline{x}) \longleftarrow \tilde{x}$$

e izomorfism

$$= [\underset{U(\mathbb{Z}_7)}{\mathrm{ord}} (\hat{3}), \underset{U(\mathbb{Z}_{17^2})}{\mathrm{ord}} (\overline{10})] = [6, \underset{U(\mathbb{Z}_{17^2})}{\mathrm{ord}} (\overline{10})]$$

Notăm $\gamma = \text{ord}_{U(\mathbb{Z}_{17^2})}(\hat{10})$.

Atunci $10^{\gamma} \equiv 1 \pmod{17^2} \Rightarrow$

$10^{\gamma} \equiv 1 \pmod{17} \Rightarrow \gamma \,\vdots\, \text{ord}_{U(\mathbb{Z}_{17})}(\hat{10}) = \gamma_{17}(10)$

Dar: $10^2 \overset{17}{\equiv} -2$; $10^4 \overset{17}{\equiv} 4$; $10^8 \overset{17}{\equiv} 16 \overset{17}{\equiv} -1$; $10^{16} \overset{17}{\equiv} 1$.

Ca urmare, $\gamma_{17}(\hat{10}) = 16$.

De mai sus, $16 \mid \gamma$.

Dar, $\gamma \mid |U(\mathbb{Z}_{17^2})| = \varphi(17^2) = 16 \cdot 17$ $\Big\} \Rightarrow$

$\gamma \in \{16, 16 \cdot 17\}$. (1)

$10^2 \overset{17^2}{\equiv} 100$; $10^4 \equiv 100^2 \overset{17^2}{\equiv} 174$; $10^8 \overset{17^2}{\equiv} 174^2 \equiv -69$;

$10^{16} \overset{17^2}{\equiv} 69^2 \overset{17^2}{\equiv} 137 \not\equiv 1$

Ca urmare, $\gamma \neq 16$ $\underset{(1)}{\Big\}} \Rightarrow \gamma = 16 \cdot 17$.

În consecință, $\frac{1}{2023}$ are o perioadă

$\gamma_{2023}(10) = [6, 16 \cdot 17] = 16 \cdot 51 = 816$ cifre.