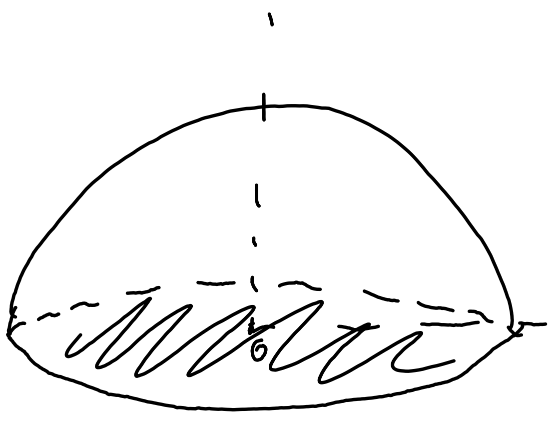
TEORIA MĂSURII

SEMINAR 14

Σ = fața exterioară a semielipsoidului

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

limitat de planul $z=0$



$$I=\iint_{\Sigma} 2x^2yz\,dy\,dz+z^2\,dz\,dx+xyz^2\,dx\,dy$$

Formula Gauss - Ostrogradski

$\Sigma = \partial D \quad , \quad D \subseteq \mathbb{R}^3$ deschis

$$\iint_{\Sigma} F \cdot n \, d\sigma = \iiint_{D} \operatorname{div} F \, dx\,dy\,dz$$

$$\iint_{\Sigma} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy =$$

$$= \iint_{\Sigma} F \cdot n \, d\sigma \quad ,$$

unde $F = (P, Q, R)$

$$\iint_{\Sigma} P\,dy\,dz = \qquad r : E \to \Sigma \text{ parametrizare pt. } \Sigma$$

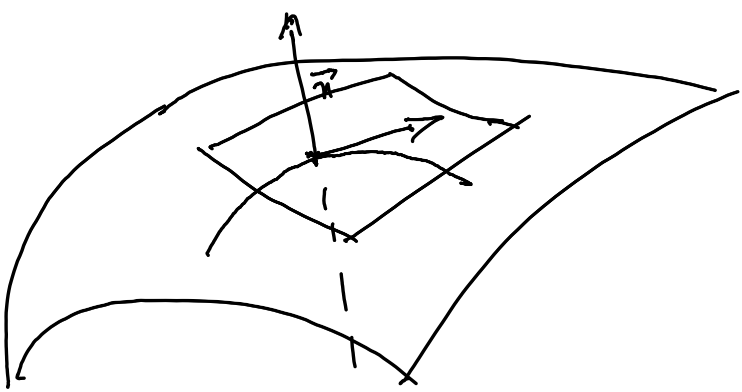
$$= \iint_{E} P \circ r(u,v) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right)^1 du\,dv$$

$$= \iint_{E} P \circ r(u,v) \cdot \underbrace{\boxed{\frac{\left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right)^1}{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}}}_{n^1} \cdot \underbrace{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}_{\sqrt{EG-F^2}} du\,dv$$

$$= \iint_{E} P \circ r(u,v) \cdot n^1(r(u,v)) \cdot \sqrt{EG-F^2}\, du\,dv =$$

$$= \iint_{\Sigma} F \cdot n^{1} \, d\sigma$$

$$n = \frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}$$



Teoremă:

$$\iint_{\Sigma} f \, d\sigma = \iint_{E} (f \circ r) \cdot \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\, dv$$

Revenind la problemă:

$$F(x,y,z) = (\underset{P}{2x^2yz}, \underset{Q}{z^2}, \underset{R}{xyz^2})$$

$$\iint_{\partial D} F \cdot n \, d\sigma = \iint_{\partial D} (P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy) =$$

$$\overset{\text{Gauss}}{\underset{\text{Ostrogradski}}{=}} \iiint_D \operatorname{div} F \, dx\,dy\,dz$$

$$\operatorname{div} F = 4xyz + 0 + 2xyz$$

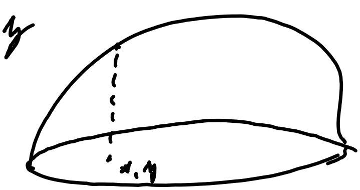
$$= 6xyz$$

$$I_1 = \iiint_D \operatorname{div} F \, dx \, dy \, dz =$$

$$I_1 = \iiint_D 6xyz \, dx \, dy \, dz$$

[E = elipsa de jos : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$]

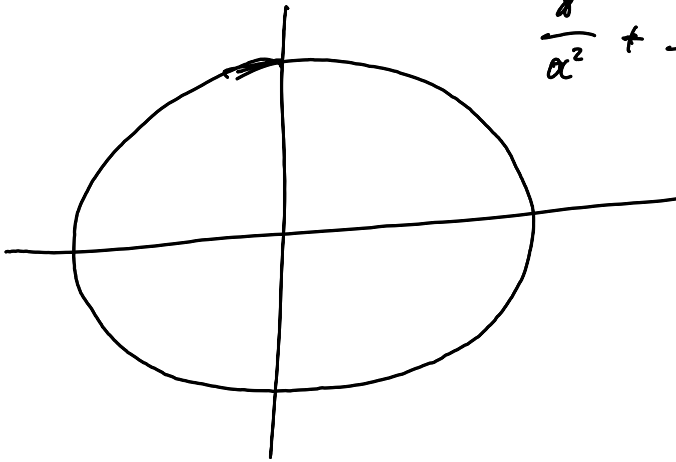
$$I_1 = \iint_E \left(\int_0^{c \cdot \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} 6xyz \, dz \right) dx \, dy$$



$$z = c \cdot \sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)}$$

$$I_1 = \iint_E 3xy \cdot c \cdot \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) dx \, dy$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$$



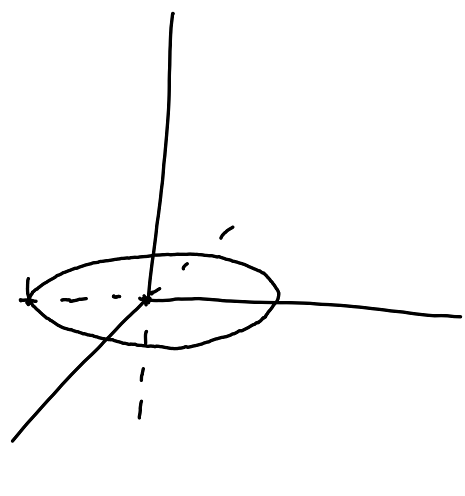
$$x = a r \cos\theta \qquad \theta \in [0, 2\pi]$$

$$y = b r \sin\theta \qquad r \in [0, 1]$$

$$|J| = a \cdot b \cdot r$$

$$I_1 = \int_0^1 \int_0^{2\pi} 3abc\, r^2 \cos\theta \sin\theta (1-r^2)\, d\theta\, dr$$

$$= 3abc \cdot \int_0^1 r^2(1-r^2)\, dr \cdot \underbrace{\int_0^{2\pi} \cos\theta \sin\theta\, d\theta}_{=0}$$



Din I_1 trebuie să scădem

$$I_2 = \iint_{\tilde{E}} F \cdot n \, d\sigma$$

$\tilde{E}$ ← elipsa cu susu-n-jos

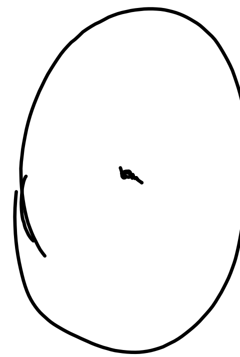
Care-i parametrizarea?

$$r(x, y) = (-x, y, 0)$$

Păstrează orientarea

$$\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$



$$= (0, 0, -1)$$

$$\left| \overset{\partial \frac{id}{\partial x}}{\frac{\partial (x, y, 0)}{\partial x}} \times \overset{\partial \frac{id}{\partial y}}{\frac{\partial (x, y, 0)}{\partial y}} \right| = (0, 0, 1)$$

Metoda 1 (cu parametrizarea r care păstrează orientarea)

$$\iint_{\tilde{E}} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy =$$

$$= \iint_{E} \left(P(r(x,y)), Q(r(x,y)), R(r(x,y))\right) \cdot \underbrace{\left(\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y}\right)}_{(0,0,-1)} dx\,dy$$

$$= \iint_{E} \underbrace{-R(-x,y,0)\,dx\,dy}_{}$$

$= 0$ SCHIMBARE DE VARIABILĂ $x \to -x$

$$= \iint_{E} -R(x,y,0)\,dx\,dy$$

Metoda 2 (cu normala exterioară intuită)

$n = (0, 0, -1)$

$$\iint_{\widehat{E}} F \cdot n \, d\sigma =$$

$$r(x, y, z) = (x, y, 0)$$

$$= \iint_E \big(P(r(x,y)), Q(r(x,y)), R(r(x,y))\big) \cdot$$

$$\cdot (0, 0, -1) \cdot \underbrace{\left\| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right\|}_{=1} dx\,dy$$

$$= \iint_E -R(r(x,y))\, dx\,dy = \iint_E -R(x,y,0)\, dx\,dy$$

$$= \iint_E -xy \cdot 0 \cdot dx\,dy = 0 .$$

În final, integrala cerută

$$\iint_{\Sigma} F \cdot n \, d\sigma = 0$$

$\square$

Răspuns pt. Răzvan:

E = elipsa din $\mathbb{R}^2$ de ecuație

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$f : E \to \mathbb{R}$ funcție continuă

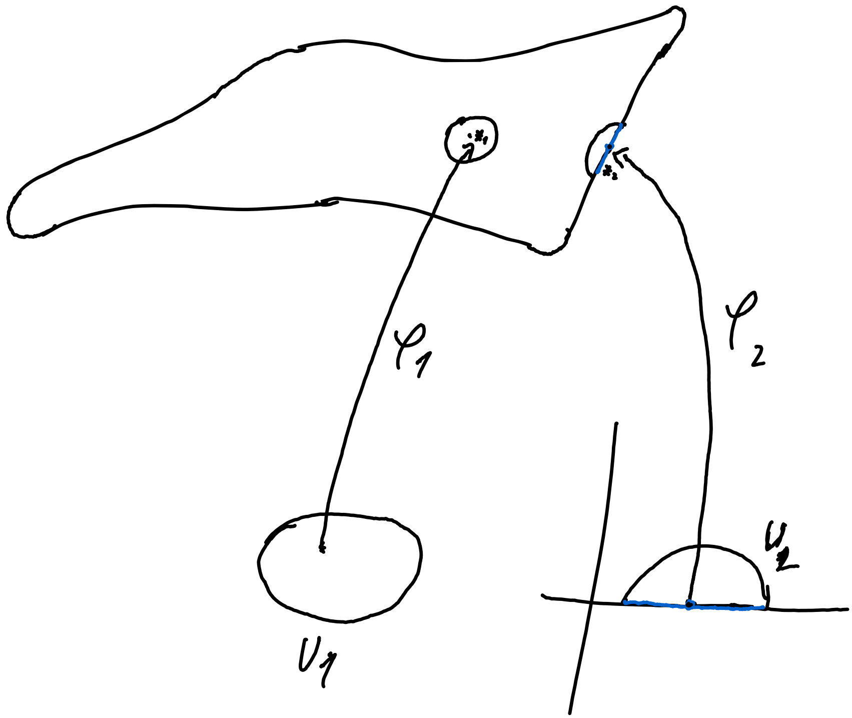
Atunci

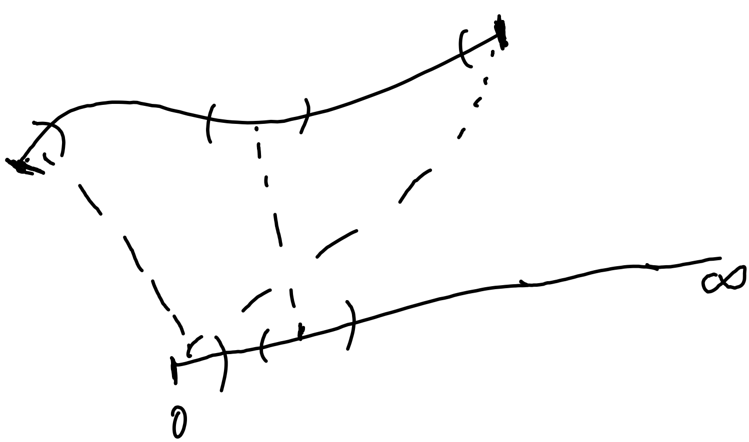
$$\iint_{E} f(x, y) \, dx \, dy = \iint_{E} f(-x, y) \, dx \, dy$$

$T : E \to E \qquad T(x, y) = (-x, y)$

$|JT| = 1$

Care e bordul unei suprafețe?





Aplicație la T. Stokes

Fie Σ suprafață compactă

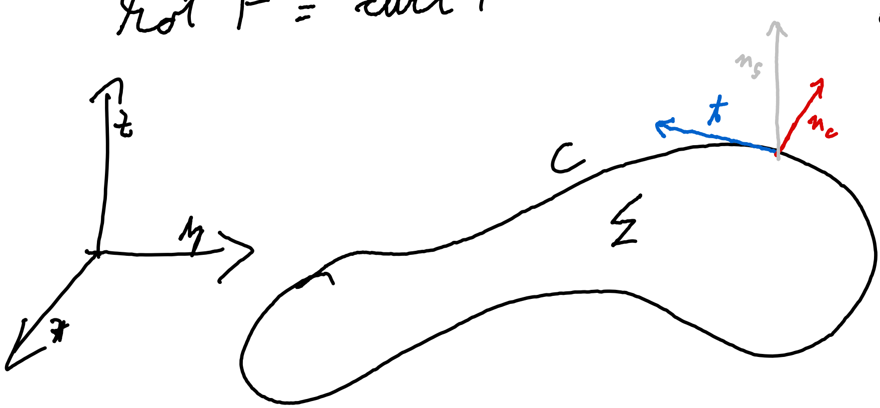
$C = \partial \Sigma$ cu orientarea indusă

$$\int_C P\,dx + Q\,dy + R\,dz =$$

$$= \iint_\Sigma \operatorname{rot} F \cdot n \, d\sigma$$

(n_c, t, n_s) pozitiv orientat

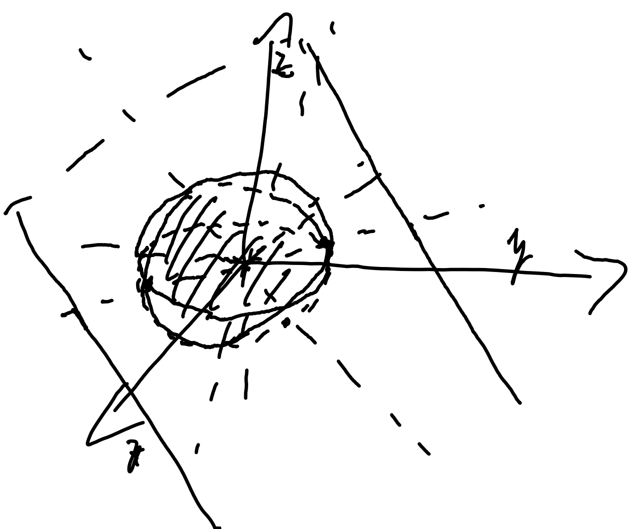
$\operatorname{rot} F = \operatorname{curl} F$



17/241 Luminița

$C = \{x^2+y^2+z^2=a^2\} \cap \{x+y+z=0\}$

C parcurs trigonometric dacă privim din direcția pozitivă a axei Ox



Discul $\Sigma = \{x^2+y^2+z^2 \leq a^2\} \cap$

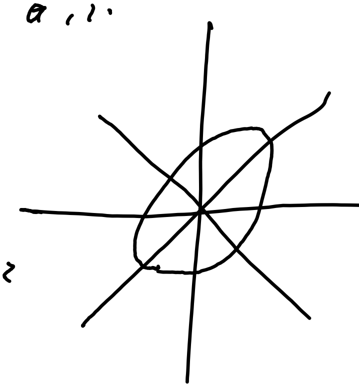
$\cap \{x+y+z=0\}$

$z = -x-y$

Domeniul de proiecție:

$D = \{(x,y) \in \mathbb{R}^2 \mid (\exists)\, z \in \mathbb{R}$ a.î.

$(x,y,z) \in \Sigma\}$



$$x^2+y^2+(x+y)^2 \leq a^2$$

$$2x^2+2y^2+2xy \leq a^2$$

$$(x-y)^2+3(x+y)^2 \leq 2a^2$$

$$x^2-2xy+y^2+3x^2+6xy+3y^2 \leq 2a^2$$

$$I = \int_C y\,dx + z\,dy + x\,dz$$

$$F = (y, z, x)$$

$$\operatorname{rot} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z},\ \frac{\partial y}{\partial z} - \frac{\partial x}{\partial x},\ \frac{\partial z}{\partial x} - \frac{\partial y}{\partial y} \right)$$

$$= (-1, -1, -1)$$

$$1\cdot x + 1\cdot y + 1\cdot z = 0$$

$$n = \frac{1}{\sqrt{3}}\cdot(1, 1, 1)$$

$$\iint_D (-1, -1, -1)\cdot\frac{1}{\sqrt{3}}(1, 1, 1)\, d\sigma =$$

$$z = -x - y$$

$$d\sigma = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$$

$$= 3 \cdot \text{Aria}(D)$$

$$\frac{(x-y)^2}{3} + (x+y)^2 \leq \frac{2a^2}{3}$$

$$(x-y) = \sqrt{3} \cdot \sqrt{\frac{2}{3}}\, a\, r \cos\theta$$

$$(x+y) = \sqrt{\frac{2}{3}}\, a\, r \sin\theta$$

$$x = \frac{\sqrt{2}\, a\, r \cos\theta + \sqrt{\frac{2}{3}}\, a\, r \sin\theta}{2}$$

$$y = \frac{-\sqrt{2}\, a\, r \cos\theta + \sqrt{\frac{2}{3}}\, a\, r \sin\theta}{2}$$

$$x, y = T(r, \theta)$$

$$\int_0^1 \int_0^{2\pi} |JT|\, dr\, d\theta$$

Rezolvați în $\mathbb{R}^3$ sistemul

$$\begin{cases} z = -x - y \\ x^2 + y^2 + z^2 \le a^2 \end{cases}$$

Aflați tripletele $(x, y, z) \in \mathbb{R}^3$ a.î.

$$\Delta = \{ (x, y) \in \mathbb{R}^2 \mid (\exists) z \in \mathbb{R} \text{ a.i. } (x, y, z) \in \Sigma \}$$