ECUAȚII DIFERENȚIALE
SEMINAR 12

11.01.2023

Algoritm (Metoda caracteristicilor)

$F(\cdot,\cdot,\cdot): D \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

$$\begin{cases} F(x, z, \frac{\partial z}{\partial x}) = 0 \\ z(\cdot)|_{G_0} = \varphi_0(\cdot) \end{cases}$$

1. Se determină o parametrizare a varietății inițiale Γ_{φ_0}.

$$\begin{cases} x = \alpha(\sigma) \\ z = \beta(\sigma), \quad \sigma \in A \subset \mathbb{R}^{n-1} \end{cases}$$

2. Se det. o fct de compatibilitate $\gamma(\cdot)$.
Se rezolvă în raport cu p sist. algebric:

$$\begin{cases} F(\alpha(\sigma), \beta(\sigma), p) = 0 \\ p D\alpha(\sigma) = D\beta(\sigma) \end{cases}$$

3. Determină curentul caracteristicilor $k(\cdot,\cdot) = (x(\cdot,\cdot), z(\cdot,\cdot), p(\cdot,\cdot))$
Integrează sist. caracteristicilor.

$$\begin{cases} \frac{dx}{dt} = \frac{\partial F}{\partial p}(x,z,p) & x(0) = \alpha(\sigma) \\ \frac{dz}{dt} = \langle p, \frac{\partial F}{\partial p}(x,z,p) \rangle & z(0) = \beta(\sigma) \\ \frac{dp}{dt} = -\frac{\partial F}{\partial x}(x,z,p) - p \cdot \frac{\partial F}{\partial z}(x,z,p) & p(0) = \gamma(\sigma) \end{cases}$$

$\underline{2n+1}$ ec.

- Caută ec. independente $\longrightarrow$ metode elementare
- caută subsisteme independente $\to$ liniare cu coef. constanți
  - afine
  - integrale prime

- $F(k(t,\sigma)) \equiv 0$

4. Scrie sol. sub formă parametrizată $\begin{cases} x = X(t,\sigma) \\ z = Z(t,\sigma) \end{cases}$

5. Inversează $x = X(t,\sigma) \longrightarrow t = T(x)$
$\sigma = \Sigma(x)$

Scrie sol. explicită $\varphi(x) = Z(T(x), \Sigma(x))$

Model exercițiu examen

Ex 1 $2z+3xy-pq=0$ $\qquad x=0,\ z=y^2$

$$\left[2z(x,y)+3xy-\frac{\partial z}{\partial x}(x,y)\frac{\partial z}{\partial y}(x,y)=0 \qquad z(0,y)=y^2\right]$$

1. $x=0$ $\qquad \alpha(\sigma)=\begin{pmatrix}0\\ \sigma\end{pmatrix}$

$y=\sigma$

$z=\sigma^2$ $\qquad \beta(\sigma)=\sigma^2$

2. $\begin{cases} F(\alpha(\sigma),\beta(\sigma),(p,q))=0 \\ (p,q)\,D\alpha(\sigma)=D\beta(\sigma)\end{cases}$ $\qquad \begin{cases} 2\sigma^2+3\cdot 0\sigma-p\cdot q=0 \Rightarrow p=\sigma \\ (p,q)\begin{pmatrix}0\\1\end{pmatrix}=2\sigma \Rightarrow q=2\sigma\end{cases}$

$$\Rightarrow \gamma(\sigma)=\begin{pmatrix}\sigma\\ 2\sigma\end{pmatrix}$$

3. $$\begin{cases} \frac{dx}{dt}=\frac{\partial F}{\partial p}((x,y),z,(p,q)) \\ \frac{dy}{dt}=\frac{\partial F}{\partial q}((x,y),z,(p,q)) \\ \frac{dz}{dt}=p\cdot\frac{\partial F}{\partial p}((x,y),z,(p,q))+q\cdot\frac{\partial F}{\partial q}((x,y),z,(p,q)) \\ \frac{dp}{dt}=-\frac{\partial F}{\partial x}((x,y),z,(p,q))-p\cdot\frac{\partial F}{\partial z}((x,y),z,(p,q)) \\ \frac{dq}{dt}=-\frac{\partial F}{\partial y}((x,y),z,(p,q))-q\cdot\frac{\partial F}{\partial z}((x,y),z,(p,q)) \end{cases}$$

$$\begin{pmatrix}x(0)\\y(0)\end{pmatrix}=\alpha(\sigma) \qquad z(0)=\beta(\sigma) \qquad \begin{pmatrix}p(0)\\q(0)\end{pmatrix}=\gamma(\sigma)$$

$$\begin{cases} x'=-q & x(0)=0 \\ y'=-p & y(0)=\sigma \\ z'=-2pq & z(0)=\sigma^2 \\ p'=-3y-2p & p(0)=\sigma \\ q'=-3x-2q & q(0)=\sigma^2 \end{cases}$$

$$\begin{cases} x' = -q \\ q' = -3x - 2q \end{cases} \quad \begin{array}{l} x(0) = 0 \\ q(0) = \sigma^2 \end{array} \longrightarrow q = -x'$$

$$\Downarrow$$

$$-x'' = -3x - 2x'$$

$$x'' + 2x' - 3x = 0.$$

ec. caract: $\lambda^2 + 2\lambda - 3 = 0 \Rightarrow \lambda_1 = 1$, $\lambda_2 = -3$

$$x(t) = c_1 e^{t} + c_2 e^{-3t}$$

$$q(t) = -x'(t) = -c_1 e^{t} + 3c_2 e^{-3t}$$

$$x(0) = c_1 + c_2 = 0$$

$$q(0) = -c_1 + 3c_2 = 2\sigma$$

$$\Downarrow$$

$$c_2 = \frac{\sigma}{2} ; \quad c_1 = \frac{-\sigma}{2}$$

$$\Rightarrow \boxed{\begin{cases} x(t,\sigma) = -\frac{\sigma}{2} e^{t} + \frac{\sigma}{2} e^{-3t} \\ q(t,\sigma) = \frac{\sigma}{2} e^{t} + \frac{3\sigma}{2} e^{-3t} \end{cases}}$$

$$\begin{cases} y' = -p \\ p' = -3y - 2p \end{cases} \quad \begin{array}{l} y(0) = \sigma \\ p(0) = \sigma \end{array}$$

$$y(t) = c_1 e^{t} + c_2 e^{-3t}$$

$$p(t) = -c_1 e^{t} + 3c_2 e^{-3t}$$

$$y(0) = c_1 + c_2 = \sigma$$

$$p(0) = -c_1 + 3c_2 = \sigma$$

$$\Downarrow$$

$$c_2 = \frac{\sigma}{2}, \quad c_1 = \frac{\sigma}{2}$$

$$\boxed{\begin{array}{l} y(t,\sigma) = \frac{\sigma}{2} e^{t} + \frac{\sigma}{2} e^{-3t} \\ p(t,\sigma) = -\frac{\sigma}{2} e^{t} + \frac{3\sigma}{2} e^{-3t} \end{array}}$$

$$F(k(t,\sigma)) \equiv 0$$

$$2z(t,\sigma) + 3x(t,\sigma)y(t,\sigma) - p(t,\sigma)q(t,\sigma) \equiv 0.$$

$$z(t,\sigma) = -\frac{3}{2} x(t,\sigma)y(t,\sigma) + \frac{1}{2} p(t,\sigma)q(t,\sigma)$$

$$z(t,s) = -\frac{3}{2}\left(-\frac{s}{2}e^{t} + \frac{s}{2}e^{-3t}\right)\left(\frac{s}{2}e^{t} + \frac{s}{2}e^{-3t}\right) + \frac{1}{2}\left(-\frac{s}{2}e^{t} + \frac{3s}{2}e^{-3t}\right)\left(\frac{s}{2}e^{t} + \frac{3s}{2}e^{-3t}\right)$$

$$= -\frac{3}{2}\left(-\frac{s^2}{4}e^{2t} + \frac{s^2}{4}e^{-6t}\right) + \frac{1}{2}\left(-\frac{s^2}{4}e^{2t} + \frac{9s^2}{4}e^{-6t}\right)$$

$$\boxed{z(t,s) = \frac{1}{4}s^2e^{2t} + \frac{3}{4}s^2e^{-6t}}$$

4. sol. sub formă parametrizată

$$\begin{cases} x(t,s) = -\frac{s}{2}e^{t} + \frac{s}{2}e^{-3t} \\ y(t,s) = \frac{s}{2}e^{t} + \frac{s}{2}e^{-3t} \\ z(t,s) = \frac{1}{4}s^2e^{2t} + \frac{3}{4}s^2e^{-6t} \end{cases}$$

5. $\begin{cases} x(t,s) = x \\ y(t,s) = y \end{cases} \Rightarrow \begin{cases} -\frac{s}{2}e^{t} + \frac{s}{2}e^{-3t} = x \\ \frac{s}{2}e^{t} + \frac{s}{2}e^{-3t} = y \end{cases}$

$\oplus \quad se^{-3t} = x+y$

$\ominus \quad se^{t} = y-x$

$$\Rightarrow \begin{cases} se^{-3t} = x+y \\ se^{t} = y-x \end{cases}$$

(le împărțim) $e^{-4t} = \frac{x+y}{y-x} \Rightarrow -4t = \ln\left|\frac{x+y}{y-x}\right| \Rightarrow t = -\frac{1}{4}\ln\left|\frac{x+y}{y-x}\right|$

$s = \ldots$

$$z(t,s) = \frac{1}{4}s^2e^{2t} + \frac{3}{4}s^2e^{-6t}$$

$$= \frac{1}{4}(se^{t})^2 + \frac{3}{4}(se^{-3t})^2$$

$$\Rightarrow \varphi(x,y) = \frac{1}{4}(y-x)^2 + \frac{3}{4}(x+y)^2 = x^2 + y^2 + xy$$

$\Rightarrow \boxed{z(x,y) = x^2+y^2+xy} \rightarrow$ polinom

verificare în cond inițiale $\xrightarrow{x=0} z=y^2$

Ex. ② $qy^2+p^2+xyp=0$ $\qquad y=1,\ z=x.$

1. $x=\sigma$ $\qquad \alpha(\sigma)=\begin{pmatrix}\sigma\\1\end{pmatrix}$

$y=1$

$z=\sigma$ $\qquad \beta(\sigma)=\sigma$

2) $\begin{cases} F(\alpha(\sigma),\beta(\sigma),(p,q))=0 \\ (p,q)\,D\alpha(\sigma)=D\beta(\sigma)\end{cases} \Rightarrow \begin{cases} q+p^2+\sigma p=0 \Rightarrow q=-1-\sigma \\ (p,q)\begin{pmatrix}1\\0\end{pmatrix}=1 \Rightarrow p=1\end{cases}$

$\Rightarrow \gamma(\sigma)=\begin{pmatrix}1\\-1-\sigma\end{pmatrix}$

3. $\begin{cases} x'=2p+xy \\ y'=y^2 \\ z'=2p^2+pxy+qy^2 \\ p'=-yp-p\cdot 0 \Rightarrow p'=-yp \\ q'=-2qy-xp \end{cases}$ $\qquad \begin{matrix} x(0)=\sigma \\ y(0)=1 \\ z(0)=\sigma \\ p(0)=1 \\ q(0)=-\sigma-1 \end{matrix}$

• $y'=y^2 \qquad y(0)=1$

$\frac{dy}{dt}=y^2$ – ec cu var sep.

$y^2=0 \Rightarrow y=0 \Rightarrow y(t)=0$ $\qquad$ NU E BUNĂ

$\frac{1}{y^2}dy=dt \Rightarrow \int\frac{1}{y^2}dy=\int 1\,dt$

$-\frac{1}{y}=t+c \Rightarrow y(t)=\frac{-1}{t+c},\ c\in\mathbb{R}$

$y(0)=1 \Rightarrow y(0)=\frac{-1}{c}$

$\Downarrow$

$-\frac{1}{c}=1 \Rightarrow c=-1.$

$\Rightarrow \boxed{y(t,\sigma)=\frac{1}{1-t}}$

• $p'=-yp \Rightarrow p'=\frac{1}{t-1}\cdot p \quad ; \quad p(0)=1$

$\Rightarrow p(t)=c\cdot e^{\int\frac{1}{t-1}dt}=c\cdot e^{\ln(1-t)}=\frac{c}{1-t}$

$p(0)=\frac{c}{1}=1 \Rightarrow c=1 \Rightarrow \boxed{p(t,\sigma)=\frac{1}{1-t}}$

• $x' = \frac{x}{1-t} + \frac{2}{1+t}$ $\qquad x(0) = \sigma$

$$\bar{x}' = \frac{\bar{x}}{1-t} \Rightarrow \bar{x}(t) = c \cdot e^{-\ln(1-t)} = \frac{c}{1-t}$$

$$x(t) = \frac{c(t)}{1-t}$$

$$\left(\frac{c(t)}{1-t}\right)' = \frac{c(t)}{(1-t)^2} + \frac{2}{1-t}$$

$$\frac{c'(t)\cdot(1-t) + c(t)\cdot(+1)}{(1-t)^2} = \frac{c(t)}{(1-t)^2} + \frac{2}{1+t}$$

$$\frac{c'(t)}{1-t} + \cancel{\frac{c(t)}{(1-t)^2}} = \cancel{\frac{c(t)}{(1-t)^2}} + \frac{2}{1-t}$$

$$c'(t) = 2 \Rightarrow c(t) = 2t + k,\ k \in \mathbb{R}$$

$$\Rightarrow x(t) = \frac{2t+k}{1-t},\ k \in \mathbb{R}.$$

$$\left.\begin{matrix} x(0) = \sigma \\ x(0) = \frac{k}{1} \end{matrix}\right\} \Rightarrow k = \sigma \Rightarrow \boxed{x(t,\sigma) = \frac{2t+\sigma}{1-t}}$$

• $z' = 2p^2 + pxy + 2y^2$

initial: $2y^2 + p^2 + xyp = 0$

$$F(k(t,\sigma)) \equiv 0 \Rightarrow 2y^2 + p^2 + xyp = 0 \Rightarrow z' = p^2 + \underbrace{p^2 + xyp + 2y^2}_{0}$$

$$\Rightarrow z' = p^2 \Rightarrow z' = \frac{1}{(1-t)^2} \Rightarrow$$

$$z(t) = \int \frac{1}{(1-t)^2}\,dt = \frac{1}{1-t} + c,\ c \in \mathbb{R}$$

$$z(0) = \sigma \Rightarrow c = \sigma$$

$$\Rightarrow \boxed{z(t,\sigma) = \frac{1}{1-t} + \sigma - 1}$$

4. sol. sub formă parametrizată

$$\begin{cases} x(t,s) = \dfrac{2t+s}{1-t} \\ y(t,s) = \dfrac{1}{1-t} \\ z(t,s) = \dfrac{1}{1-t} + s - 1 \end{cases}$$

5. $\begin{cases} x(t,s) = x \\ y(t,s) = y \end{cases} \Rightarrow \begin{cases} \dfrac{2t+s}{1-t} = x \Rightarrow s = x\dfrac{1}{y} - 2\left(1 - \dfrac{1}{y}\right) \\ \dfrac{1}{1-t} = y \Rightarrow y = 1 - \dfrac{1}{y} \end{cases}$

$$z(x,y) = y + \frac{x}{y} - 3 + \frac{2}{y}$$

Ex. ③ $py + qx - 4xy = 0$. $\quad y = 1, \ z = x^2 + 1$.

① $x = s$ $\qquad \alpha(s) = \begin{pmatrix} s \\ 1 \end{pmatrix}$

$y = 1$

$z = s^2 + 1$ $\qquad \beta(s) = s^2 + 1$

② $\begin{cases} F(\alpha(s), \beta(s), (p,q)) = 0 \\ (p,q) \cdot D\alpha(s) = D\beta(s) \end{cases} \Rightarrow \begin{matrix} p + qs - 4s = 0 \Rightarrow q = \dfrac{4s - p}{s} = 2 \\ (p,q)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2s \Rightarrow p = 2s \end{matrix}$

$\Rightarrow \gamma(s) = \begin{pmatrix} 2s \\ 2 \end{pmatrix}$

③ $\begin{cases} x' = y \\ y' = x \\ z' = py + qx \\ p' = -q + 4y \\ q' = -p + 4x \end{cases}$ $\qquad \begin{matrix} x(0) = s \\ y(0) = 1 \\ z(0) = s^2 + 1 \\ p(0) = 2s \\ q(0) = 2 \end{matrix}$

$\begin{cases} x' = y \Rightarrow y = x' \\ y' = x \Rightarrow x'' = x \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1. \end{cases}$

$x(t) = c_1 e^t + c_2 e^{-t} \Rightarrow y(t) = c_1 e^t - c_2 e^{t}, c_1, c_2 \in \mathbb{R}$

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$$\begin{cases} c_1 + c_2 = \sigma \\ c_1 - c_2 = 1 \end{cases} \oplus$$

$$\begin{cases} c_1 = \frac{\sigma+1}{2} \\ c_2 = \frac{\sigma-1}{2} \end{cases}$$

$$\boxed{\begin{cases} x(t,\sigma) = \frac{\sigma+1}{2} \cdot e^t + \frac{\sigma-1}{2} e^{-t} \\ y(t,\sigma) = \frac{\sigma+1}{2} \cdot e^t - \frac{\sigma-1}{2} e^{-t} \end{cases}} \quad *$$

$$z' = py + qx = 4xy \Rightarrow z' = (\sigma+1)^2 e^{2t} - (\sigma-1)^2 e^{-2t}$$

$$z(t) = \frac{1}{2}(\sigma+1)^2 e^{2t} + \frac{1}{2}(\sigma-1)^2 e^{-2t} + k, \quad k \in \mathbb{R}$$

$$z(0) = \sigma^2 + 1 \Rightarrow \quad k = 0$$

$$\Rightarrow \boxed{z(t,\sigma) = \frac{1}{2}(\sigma+1)^2 e^{2t} + \frac{1}{2}(\sigma-1)^2 e^{-2t}} \quad **$$

④ sol sub formă param: * și **

⑤ $\begin{cases} x(t,\sigma) = x \\ y(t,\sigma) = y \end{cases} \Rightarrow \begin{cases} \frac{\sigma+1}{2} e^t + \frac{\sigma-1}{2} e^{-t} = x \\ \frac{\sigma+1}{2} e^t - \frac{\sigma-1}{2} e^{-t} = y \end{cases}$

$$\begin{cases} x+y = (\sigma+1)\, e^t \\ x-y = (\sigma-1)\, e^{-t} \end{cases}$$ am putea să le înmulțim și să explicităm σ.

Dar $z(t,\sigma) = \frac{1}{2}((\sigma+1)e^t)^2 + \frac{1}{2}((\sigma-1)e^{-t})^2$

$$= \frac{1}{2}(x+y)^2 + \frac{1}{2}(x-y^2)$$

$$\Rightarrow \underline{\underline{z(x,y) = x^2 + y^2}}$$

ECUAȚII DIFERENȚIALE

SEMINAR 13

18.01.2023

Algoritm (Ecuații cu diferențiale totale)

$$dx = F(t,x)\,dt \qquad F(t,x) = \left(f_j^i(t,x)\right)_{\substack{i=\overline{1,n}\\ j=\overline{1,k}}}$$

$$x(t_0) = x_0$$

Pas I. Verifică condiția de integrabilitate completă.

$$\frac{\partial f_j^i}{\partial t_k}(t,x) + \sum_{m=1}^{n} \frac{\partial f_j^i}{\partial x_m}(t,x) f_k^m(t,x) \equiv \frac{\partial f_k^i}{\partial t_j}(t,x) + \sum_{m=1}^{n} \frac{\partial f_k^i}{\partial x_m}(t,x) f_j^m(t,x)$$

$$\forall\, i=\overline{1,n}$$
$$\forall\, 1 \le j < k \le k$$

Dacă nu se verifică $\rightarrow$ STOP

Dacă se verifică $\rightarrow$ Pasul II.

Pas II. Integrează ec. dif. parametrizată

$$\frac{dy}{ds} = F(t_0 + s\lambda, y)\lambda \qquad y(0) = x_0 \longrightarrow \text{sol. gen. } \varphi(s,\lambda)$$

Scrie sol. $\varphi(t) = \varphi(1, t - t_0)$

① $dz = 2xe^y dx + z\,dy$ $\qquad z(x_0, y_0) = z_0$

$n = 1$ $\qquad t \rightsquigarrow (x,y)$

$k = 2$ $\qquad x \rightsquigarrow z$

$$dz = P(x,y,z)\,dx + Q(x,y,z)\,dy$$

$$F(x,y,z) = \left(P(x,y,z), Q(x,y,z)\right)$$

$P(x,y,z) = 2xe^y$

$Q(x,y,z) = z$

Pas 1: $D_2 P(x,y,z) + D_3 P(x,y,z)\cdot Q(x,y,z) \equiv D_1 Q(x,y,z) + D_3 Q(x,y,z)\cdot P(x,y,z)$

$$2xe^y + 0\cdot Q(x,y,z) \equiv 0 + 1\cdot 2xe^y$$

$$2xe^y \equiv 2xe^y \quad (A)$$

Pas 2:

- $\frac{du}{ds} = P(x_0 + s\lambda_1, y_0 + s\lambda_2, u)\cdot\lambda_1 + Q(x_0 + s\lambda_1, y_0 + s\lambda_2, u)\cdot\lambda_2$

$u(0) = z_0$

$$\frac{du}{ds} = 2\cdot(x_0+s\lambda_1)\cdot e^{y_0+s\lambda_2}\cdot \lambda_1 + u\lambda_2 \quad ; \quad u(0)=z_0.$$

afină scalară

$\bar{u}' = \lambda_2 \cdot \bar{u}$ - ec. liniară

$$\bar{u}(s) = c\cdot e^{\int \lambda_2\, ds} = ce^{\lambda_2 s}$$

căutăm sol. de forma $u(s) = c(s)\cdot e^{\lambda_2 s}$

$$\left(c(s)\cdot e^{\lambda_2 s}\right)' = 2(x_0+s\lambda_1)e^{y_0+s\lambda_2}\cdot\lambda_1 + c(s)\cdot e^{\lambda_2 s}\cdot\lambda_2$$

$$c'(s)\cdot e^{\lambda_2 s} + \cancel{c(s)\cdot e^{\lambda_2 s}\cdot\lambda_2} = 2(x_0+s\lambda_1)e^{y_0+s\lambda_2}\cdot\lambda_1 + \cancel{c(s)e^{\lambda_2 s}\cdot\lambda_2}$$

e^{y_0}

$$c'(s) = 2(x_0+s\lambda_1)\cdot\lambda_1\cdot e^{y_0}$$

$$\Rightarrow c(s) = e^{y_0}(x_0+s\lambda_1)^2 + k,\ k\in\mathbb{R}$$

"ct."

$$\Rightarrow u(s) = \left[e^{y_0}(x_0+s\lambda_1)^2 + k\right]\cdot e^{\lambda_2 s} \Rightarrow u(0) = x_0^2 e^{y_0} + k$$

$u(0) = z_0$ $\Bigg/ \Rightarrow$

$$k = z_0 - x_0^2 e^{y_0}$$

$$\Rightarrow u(s,\lambda) = \left[(x_0+s\lambda_1)^2 e^{y_0} + z_0 - x_0^2 e^{y_0}\right]e^{\lambda_2\cdot s}$$

$$\Rightarrow \underline{z(x,y)} = u(1, (\underbrace{x-x_0}_{\lambda_1}, \underbrace{y-y_0}_{\lambda_2})) =$$

$$\underline{= \left[x^2\cdot e^{y_0} + z_0 - x_0^2 e^{y_0}\right]e^{y-y_0}}$$

② $dz = \frac{z-y^2}{x}\,dx + (2y-x)\,dy, \quad z(1,0)=0.$

TEMĂ

Exerciții recapitulative

③ Fie ec. $\begin{cases} x' = \frac{x^2}{y} \\ y' = -\frac{y^2}{x} \end{cases}$

a) Să se arate că funcția $F(t,(x,y)) = \operatorname{arctg}\frac{x}{y} - t$ este integrală primă.

b) Să se determine soluția generală.

Rezolvare:

a) Aplicăm criteriul:

$$\frac{\partial F}{\partial t}(t,(x,y)) + \frac{\partial F}{\partial x}(t,(x,y))\cdot\frac{x^2}{y} + \frac{\partial F}{\partial y}(t,(x,y))\cdot\left(-\frac{y^2}{x}\right) \overset{?}{=} 0.$$

$$-1 + \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}\cdot\frac{x^2}{y} + \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{\cancel{x}}{y^2}\right)\cdot\left(-\frac{y^2}{\cancel{x}}\right) \overset{?}{=} 0$$

$$-1 + \underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{1} = 0$$

$$-1+1=0$$

$$0=0 \ (A)$$

b) Soluția generală

$$F(t,(x,y))=c \Rightarrow \operatorname{arctg}\frac{x}{y} - t = c$$

$$\operatorname{arctg}\frac{x}{y} = t+c$$

$$\frac{x}{y} = \operatorname{tg}(t+c) \Rightarrow y = \frac{x}{\operatorname{tg}(t+c)} \ *$$

$$x' = \frac{x^{\cancel{2}}}{1}\cdot\frac{\operatorname{tg}(t+c)}{\cancel{x}} = x\operatorname{tg}(t+c) - \text{ec. liniară}$$

$$x(t) = c_2\cdot e^{\int \operatorname{tg}(t+c)\,dt} = c_2\cdot e^{\int\frac{\sin(t+c)}{\cos(t+c)}dt} = c_2\cdot e^{-\int\frac{\cos'(t+c)}{\cos(t+c)}dt}$$

$$= c_2\cdot e^{-\ln|\cos(t+c)|} = \frac{c_2}{|\cos(t+c)|}$$

$$* \ y(t) = \frac{c_2}{|\cos(t+c)|}\cdot\frac{1}{\operatorname{tg}(t+c)}, \quad c, c_2\in\mathbb{R}.$$

4. $p^2x^2 - 4q^2y^2 = 0$ $\quad y=1,\ z=x^2$

$$\begin{cases} x=\sigma \\ y=1 \\ z=\sigma^2 \end{cases} \qquad \alpha(\sigma)=\begin{pmatrix}\sigma\\1\end{pmatrix} \qquad \beta(\sigma)=\sigma^2$$

$$\begin{cases} F(\alpha(\sigma),\beta(\sigma),(p,q))=0 \\ (p,q)\,D\alpha(\sigma) = D\beta(\sigma) \end{cases} \Rightarrow (p,q)\cdot\begin{pmatrix}1\\0\end{pmatrix} = 2\sigma \Rightarrow p=2\sigma$$

$$4\sigma^2\sigma^2 - 4\cdot q^2\cdot 1^2 = 0 \Rightarrow q^2=\sigma^4 \Rightarrow q=\pm\sigma^2 \xrightarrow{\text{alegem}} q=\sigma^2$$

$$\begin{cases} x' = 2px^2 \\ y' = -8qy^2 \\ z' = 2p^2x^2 - 8q^2y^2 \\ p' = -2p^2x - p\cdot 0 = -2p^2x \\ q' = +8q^2y \end{cases}$$

$x(0) = \sigma$

$y(0) = 1$

$z(0) = \sigma^2$

$p(0) = 2\sigma$

$q(0) = \sigma^2$

I $z' = 2(\underbrace{p^2x^2 - 4q^2y^2}_{0}) = 0 \Rightarrow z = k,\ k \in \mathbb{R}$

$\boxed{z(t,\sigma) = \sigma^2}$

II $\begin{cases} x' = 2px^2 \quad | \cdot p & x(0) = \sigma \\ p' = -2p^2x \quad | \cdot x & p(0) = 2\sigma \end{cases}$

$\Rightarrow \begin{cases} x'p = 2p^2x^2 \\ xp' = -2p^2x^2 \end{cases}$

$(x'p + xp') = 0$

$(xp)' = 0 \Rightarrow \exists\, c \in \mathbb{R}$ a.î. $x(t)\,p(t) = c$

pt $t = 0 \Rightarrow c = x(0)p(0) = \sigma \cdot 2\sigma = 2\sigma^2 \Rightarrow xp = 2\sigma^2 \Rightarrow$

$p = \frac{2\sigma^2}{x}$

$\Rightarrow x' = \frac{4\sigma^2}{\cancel{x}} \cdot x^{\cancel{2}} = 4\sigma^2 x$, $x(0) = \sigma$

(liniară)

$\Rightarrow x(t) = c\, e^{\int 4\sigma^2 dt} = c\, e^{4\sigma^2 t}$

$x(0) = c \cdot e^0 = c$

$x(0) = \sigma$ $\Big/ \Rightarrow c = \sigma$

$\Rightarrow \boxed{x(t,\sigma) = \sigma \cdot e^{4\sigma^2 t}}$

III $\begin{cases} y' = -8qy^2 \quad | \cdot q \\ q' = 8q^2y \quad | \cdot y \end{cases}$

$y(0) = 1$

$q(0) = \sigma^2$

$\Rightarrow \begin{cases} qy' = -8q^2y^2 \\ q'y = 8q^2y^2 \end{cases} \Rightarrow (qy)' = 0 \Rightarrow \exists\, c_2 \in \mathbb{R}$ a.î. $y(t)\,q(t) = c_2$

pt $t = 0 \Rightarrow c_2 = 1 \cdot \sigma^2 = \sigma^2 \Rightarrow yq = \sigma^2$

$q = \frac{\sigma^2}{y}$

$\Rightarrow y' = -8 \cdot \frac{\sigma^2}{\cancel{y}} \cdot y^{\cancel{2}} = -8\sigma^2 y \quad , \quad y(0) = 1$

$\Rightarrow y(t) = c \cdot e^{-8\sigma^2 t}$

$\left.\begin{aligned} y(0) &= 1 \\ y(0) &= c \end{aligned}\right\} \Rightarrow c = 1$

$\Rightarrow \boxed{y(t,\sigma) = e^{-8\sigma^2 t}}$

$\begin{cases} x(t,\sigma) = \sigma \cdot e^{4\sigma^2 t} \\ y(t,\sigma) = e^{-8\sigma^2 t} \\ z(t,\sigma) = \sigma^2 \end{cases}$ sol. param.

$z(x,y) = x^2 \cdot y$.

⑤ $py + qx - 4xy = 0 \qquad y = 1, z = x^2 + 1.$

$\begin{cases} x = \sigma \\ y = 1 \\ z = \sigma^2 + 1 \end{cases} \qquad \alpha(\sigma) = \begin{pmatrix} \sigma \\ 1 \end{pmatrix}$

$\beta(\sigma) = \sigma^2 + 1$

$F($