SEMINAR GEOMETRIE II – 11 oct 2017

ex: V/K sp vectorial
$f: V \to W$ bijecție
"+": $W \times W \to W$
$y_1 + y_2 = f(f^{-1}(y_1) + f^{-1}(y_2))$
"·": $K \times W \to W$
$\alpha \cdot y = f(\alpha f^{-1}(y))$
Atunci $(W, +, \cdot)$ este K-sp vectorial
• $(W,+)$ grup comutativ
(A) Fie $y_1, y_2, y_3 \in W$
$y_1 + (y_2 + y_3) = y_1 + f(f^{-1}(y_2) + f^{-1}(y_3)) =$
$= f(f^{-1}(y_1) + f^{-1}(y_2) + f^{-1}(y_3))$
$(y_1 + y_2) + y_3 = f(f^{-1}(y_1) + f^{-1}(y_2)) + y_3 =$
$= f(f^{-1}(y_1) + f^{-1}(y_2) + f^{-1}(y_3))$
(C) Fie $y_1 + y_2 \in W$
$y_1 + y_2 = f(f^{-1}(y_1) + f^{-1}(y_2)) = f(f^{-1}(y_2) + f^{-1}(y_1)) =$
$= y_2 + y_1$
(N) $(\exists)\ \theta \in W$ a.î. $y + \theta = \theta + y = y\ (\forall)\ y \in W$
Fie $y \in W$
$y + \theta = y \Leftrightarrow f(f^{-1}(y) + f^{-1}(\theta)) = y \,/\, f^{-1} \Leftrightarrow$
$\Leftrightarrow f^{-1}(y) + f^{-1}(\theta) = f^{-1}(y) \Rightarrow f^{-1}(\theta) = 0_V \Rightarrow \theta = f(0_V)$
(S) $(\forall)\ y \in W\ (\exists)\ y' \in W \quad y + y' = \theta$
$y + y' = \theta \Leftrightarrow f(f^{-1}(y) + f^{-1}(y')) = \theta = f(0_V) \xRightarrow{f \text{ injectie}}$
$f^{-1}(y) + f^{-1}(y') = 0$
$f^{-1}(y') = -f^{-1}(y)$
$y' = f(-f^{-1}(y)) \in W$
Fie $\alpha \in K$, $y_1, y_2 \in W$ $\quad \alpha(y_1 + y_2) = \alpha y_1 + \alpha y_2$
$\alpha(y_1 + y_2) = \alpha f(f^{-1}(y_1) + f^{-1}(y_2)) = f(\alpha \cdot (f^{-1}(y_1) + f^{-1}(y_2))) =$
$= f(\alpha f^{-1}(y_1) + \alpha f^{-1}(y_2))$
$\alpha y_1 + \alpha y_2 = f(\alpha f^{-1}(y_1)) + f(\alpha f^{-1}(y_2))$

Fie $\alpha, \beta \in K$, $y \in W$
$(\alpha + \beta) y = \alpha y + \beta y$
$(\alpha + \beta) y = f((\alpha + \beta) f^{-1}(y)) = f(\alpha f^{-1}(y) + \beta f^{-1}(y))$
$\alpha y + \beta y = f(\alpha f^{-1}(y)) + f(\beta f^{-1}(y)) = f(\alpha f^{-1}(y) + \beta f^{-1}(y))$
$(\alpha \cdot \beta) y = \alpha \cdot (\beta y)$
$\alpha (\beta y) = \alpha \cdot f(\beta f^{-1}(y)) = f(\alpha (\beta f^{-1}(y))) = f((\alpha\beta) f^{-1}(y)) =$
$= (\alpha\beta) y$

$1_K \cdot y = y \quad y \in W$
$1_K y = f(1_K \cdot f^{-1}(y)) = f(f^{-1}(y)) = y$
$f: V \to W \quad \theta = f(0_V) \quad y_1 + y_2 = f(f^{-1}(y_1) + f^{-1}(y_2))$

Ex Fie V/K sp vectorial – $x_0 \in V$. Să se arate că V admite o structură de spațiu vectorial pt care elem. neutru la adunare să fie x_0.
Considerăm funcția
Fie $f: V \to V \quad f(x) = x + x_0$ – bijecție
$f^{-1}: V \to V \quad f^{-1}(x) = x - x_0$
$\oplus: V \times V \to V$
$x \oplus y = f(f^{-1}(x) + f^{-1}(y)) = f(x - x_0 + y - x_0) =$
$= f(x + y - 2x_0) = x + y - x_0$
~~$x \odot y =$~~
$\odot: K \times V \to V$
$\alpha \odot x = f(\alpha f^{-1}(x)) = f(\alpha(x - x_0)) = \alpha x - \alpha x_0 + x_0 =$
$= \alpha x + (1 - \alpha) x_0$
$(V, \oplus, \odot)$ este K-sp vectorial)

Def. Fie V/K sp vectorial și $S \subset V$, $S \neq \emptyset$. Spunem că S este subsp vectorial V/K ($S \leq_K V$) dacă
1) $(\forall)\ \alpha \in K, (\forall)\ x \in S \quad \alpha x \in S$
2) $(\forall)\ x, y \in S \quad x + y \in S$

Propoziție $(S, +, \cdot)$ este K sp vect.
Fie $x \in S$
$-x = (-1_K) x$
$\left.\begin{array}{l} -1_K \in K \\ x \in S \\ S \text{ subsp vectorial } S \leq_K V \end{array}\right| \Rightarrow (-1) x \in S \Big| \Rightarrow -x \in S$
$\left.\begin{array}{l} S \text{ este subgrup în } (V,+) \\ (V,+) \text{ gr. comutativ} \end{array}\right| \Rightarrow (S,+)$ gr. com.

Ex Să se decidă care dintre submulțimi ale lui $\mathbb{R}^3 / \mathbb{R}$ sunt subsp. vectoriale în $\mathbb{R}^3 / \mathbb{R}$
1) $S_1 = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\}$
2) $S_2 = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 1\}$
3) $S_3 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = x_3^2\}$
4) $S_1 = \{x \in \mathbb{R}^3 \mid |x_1| < 1\} \quad |x_1| < 1 \Leftrightarrow -1 < x_1 < 1$
1) Fie $\alpha \in \mathbb{R}$, $x \in S_1$
dem $\alpha x \in S_1$
$\alpha x = \alpha(x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3)$
$\alpha x \in S_1 \Leftrightarrow \alpha x_1 + 2\alpha x_2 - \alpha x_3 = 0 \Leftrightarrow$
$\Leftrightarrow \alpha \underbrace{(x_1 + 2x_2 - x_3)}_{0} = 0$ pt că $x \in S_1$

II Fie $x, y \in S_1 \Rightarrow x_1+2x_2-x_3=0$

$y_1+2y_2-y_3=0$

$x+y=(x_1+y_1, x_2+y_2, x_3+y_3)$

$x+y\in S_1 \Leftrightarrow x_1+y_1+2(x_2+y_2)-(x_3+y_3)=0 \Leftrightarrow$

$\Leftrightarrow (x_1+2x_2-x_3)+(y_1+2y_2-y_3)=0$

$=0, x\in S_1$ $\quad$ $=0, y\in S_1$

2) $x\in S_2 \Rightarrow x_1+2x_2-x_3=1$

$\alpha x\in S_2 \Leftrightarrow \alpha x_1+2\alpha x_2-\alpha x_3=1 \Leftrightarrow$

$\Leftrightarrow \alpha(x_1+2x_2-x_3)=1 \Leftrightarrow \alpha=1 \Rightarrow S_2$ nu e subsp. vectorial în $\mathbb{K}^3$

$\alpha x\notin S_2$

3) $S_3=\{x\in\mathbb{K}^3 / x_1^2+x_2^2=x_3^2\}$

I $\forall \alpha\in\mathbb{R}$ $\forall x\in S$ $\alpha x\in S_3$

$x=(x_1,x_2,x_3)$

$\alpha x=(\alpha x_1,\alpha x_2,\alpha x_3)\in S_3 \Leftrightarrow$

$\Leftrightarrow (\alpha x_1)^2+(\alpha x_2)^2=(\alpha x_3)^2 \Leftrightarrow \alpha^2x_1^2+\alpha^2x_2^2=\alpha^2x_3^2 \Rightarrow$

$\Rightarrow \alpha^2\underbrace{(x_1^2+x_2^2-x_3^2)}_{0}=0$

II $\forall x,y\in S_3$, $x+y\in S_3$

$(x_1+y_1)^2+(x_2+y_2)^2=(x_3+y_3)^2$

$\underbrace{x_1^2+x_2^2-x_3^2}+\underbrace{y_1^2+y_2^2-y_3^2}-2(x_1y_1+x_2y_2-x_3y_3)=0 \Rightarrow$

$\Rightarrow x_1y_1+x_2y_2-x_3y_3=0$

Considerăm $x=(0,1,1)\in S_3$

$y=(1,0,1)\in S_3$

$x+y\in S_3 \Leftrightarrow x_1y_1+x_2y_2-x_3y_3=0 \Rightarrow 0+0-1=0 \Rightarrow -1=0$ Contradicție (fals!)

Exerciții

Fie K corp, $m,n\in\mathbb{N}^*$ $A\in\mathcal{M}_{m,n}(K)$, K corp comutativ

$S=\left\{x\in K^n \middle/ \begin{matrix} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=0 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=0\end{matrix}\right\}$ Atunci $S\le_K K^n$

$S=\left\{x\in K^n / A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix}\right\}$ m ori

Fie $\alpha\in K$ $x\in S \Rightarrow A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix}$

$\alpha x\in S \Leftrightarrow A\begin{pmatrix}\alpha x_1\\ \vdots\\ \alpha x_n\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix} \Leftrightarrow \alpha\left(A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\right)=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix} \Leftrightarrow \alpha\begin{pmatrix}0\\ \vdots\\0\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\0\end{pmatrix}$ (A)

Fie $x,y\in S \Rightarrow AX=AY=0_m$

$x+y\in S \Leftrightarrow A(x+y)=0_m \Leftrightarrow \underline{Ax}=\underline{A\cdot y}=0_m$ (A)

$AX=0_m$ $A\cdot y=0_m$

Ecuațiile liniar omogene → spațiu vectorial

Proprietăți

Fie V/K - sp vectorial, $S\subset V$, $S\neq\emptyset$. Atunci

1) $S\le_K V \Leftrightarrow \forall\alpha,\beta\in K\ (\forall) x,y\in S$ $\alpha x+\beta y\in S$

2) dacă $S\le_K V$, $n\in\mathbb{N}^*$, $\alpha_1,\dots,\alpha_n\in K$, $x_1,\dots,x_n\in S$ (vectori, nu componente)

Atunci $\sum_{k=1}^n \alpha_k\cdot x_k\in S$ – se dem prin inducție matematică

Propoziție: Fie V/K sp vectorial $I\neq\emptyset$ și $(S_i)_{i\in I}$ a.î. $\forall i\in I$ $S_i\le_K V$

Atunci: $\bigcap_{i\in I} S_i\le_K V$

Fie $\alpha,\beta\in K$ $x,y\in\bigcap_{i\in I}S_i$

Fie $j\in I$ $x,y\in\bigcap_{i\in I}S_i \Rightarrow x,y\in S_j$; $\alpha,\beta\in K$; $S_j\le_K V$ $\Rightarrow \alpha x+\beta y\in S_j$

$\forall j\in I$ $\alpha x+\beta y\in S_j \Rightarrow \alpha x+\beta y\in\bigcap_{i\in I}S_i$

Propoziție Fie V/K sp vectorial $S_1,S_2\le_K V$

Atunci $S_1\cup S_2\le_K V \Leftrightarrow (S_1\subset S_2)$ sau $(S_2\subset S_1)$

"⇒" pp că $S_1\not\subset S_2$ și $S_2\not\subset S_1$

$S_1\not\subset S_2 \Rightarrow \exists x_1\in S_1\setminus S_2$

$S_2\not\subset S_1 \Rightarrow \exists x_2\in S_2\setminus S_1$

$x_1\in S_1\Rightarrow x_1\in S_1\cup S_2$; $x_2\in S_2\Rightarrow x_2\in S_1\cup S_2$; $S_1\cup S_2\le_K V$ $\Rightarrow x_1+x_2\in S_1\cup S_2 \Rightarrow x_1+x_2\in S_1$ sau $x_1+x_2\in S_2$

Dacă $x_1+x_2\in S_1$; $x_2=(x_1+x_2)-x_1$; $x_1\in S_1$; $S_1\le_K V$ $\Rightarrow x_2\in S_1$ (F) $x_2\in S_2/S_1$

Definiție Fie V/K sp vectorial, $M\subset V$

$Sp_K(M)$ – subsp generat de mulț. M $=\bigcap_{\substack{S\le_K V\\ M\subset S}} S$

OBS. 0: $M\subset Sp_K(M)$

1: $Sp_K(\emptyset)=\bigcap_{S\le_K V}S$ = subspațiu nul $=\{0_V\}$

2: dacă $M\neq\emptyset$ atunci $Sp_K(M)=\{\sum_{k=1}^m\alpha_kx_k / m\in\mathbb{N}^*, \alpha_1,\dots,\alpha_m\in K, x_1,\dots,x_m\in M\}$

3: Fie $M=\{x_1,\dots,x_p\}\subset V$, $p\in\mathbb{N}^*$, $Sp_K(M)=\{\sum_{k=1}^p\alpha_k\cdot x_k / \alpha_1,\dots,\alpha_k\in K\}$

4: dacă $M\le_K V$ atunci $Sp_K(M)=M$

Definiție Fie V/K sp vectorial, $S_1,S_2\le_K V$

$S_1+S_2=Sp_K(S_1\cup S_2)$

Propoziție $S_1+S_2=\{x_1+x_2/x_1\in S_1,x_2\in S_2\}$

Definiție V/K sp vect $S_1,S_2\le_K V$. Sp. că $V=S_1\oplus S_2$ dacă 1) $S_1+S_2=V$ 2) $S_1\cap S_2=\{\}$

Ex: $S_1=\{(x,0)/x\in\mathbb{K}\}$, $S_2=\{(0,x)/x\in\mathbb{K}\}$

a) Să se arate că S_1 și $S_2\le_K \mathbb{K}^2/\mathbb{K}$ (sunt subsp)

b) $\mathbb{K}^2=S_1\oplus S_2$

1) $\mathbb{K}^2=S_1+S_2$ Fie $(x_1,x_2)\in\mathbb{K}^2$

$(x_1,x_2)=\underset{\in S_2}{(0,x_2)}+\underset{\in S_1}{(x_1,0)}$ $\quad x_1,x_2\in S_1+S_2 \Rightarrow \mathbb{K}^2=S_1+S_2$

2) $S_1\cap S_2=\{0_{\mathbb{K}^2}\}$ Fie $(x_1,x_2)\in S_1\cap S_2$

$(x_1,x_2)\in S_1\Rightarrow x_2=0$; $(x_1,x_2)\in S_2\Rightarrow x_1=0$ $\Rightarrow x_1,x_2=0 \Rightarrow (x_1,x_2)=(0,0)\Rightarrow$

$\Rightarrow S_1\cap S_2\subset\{0_{\mathbb{K}^2}\} \xrightarrow{(3)} \mathbb{K}^2=S_1\oplus S_2$

Ex 1. Fie K corp, iar $K\neq 2$ ($1_K+1_K\neq 0_K$) și $n\in\mathbb{N}$

$S_1=\{A\in\mathcal{M}_n(K)/ {}^tA=A\}$, $S_2=\{A\in\mathcal{M}_n(K)/{}^tA=-A\}$

a) $S_1,S_2\le_K \mathcal{M}_n(K)$

b) $\mathcal{M}_n(K)=S_1\oplus S_2$

Ex 2. Fie K corp, $n\in\mathbb{N}^*$ a.î. $1_K+1_K+\dots+1_K\neq 0$ (n ori)

$S=\{A\in\mathcal{M}_n(K)/ \mathrm{tr}\, A=0_K\}$ Atunci

a) $S\le_K M_n(K)$

b) $\mathcal{M}_n(K)=S\oplus Sp_K(\{I_n\})$