Examen

2 Iunie 2018

Timp de lucru 2h30. Toate documentele, computerele personale, telefoanele mobile și/sau calculatoarele electronice de mână sunt autorizate. Orice modalitate de comunicare între voi este **strict interzisă**. Mult succes !

Exercițiul 1

10p

Fie $X_1, X_2, \ldots, X_n$ un eșantion de talie n dintr-o populație Poisson de parametru $\theta > 0$.

a) Determinați estimatorul de verosimilitate maximă $\hat{\theta}$ și verificați dacă acesta este deplasat, consistent și eficient.

b) Găsiți estimatorul de verosimilitate maximă pentru $\mathbb{P}_\theta(X_1 = 1 \,|\, X_1 > 0)$. Este acesta consistent ?

c) Verificați dacă estimatorul aflat la punctul b) este sau nu nedeplasat.

Exercițiul 2

10p

Fie X o variabilă aleatoare repartizată $\mathbb{P}_\theta(X = k) = A(k+1)\theta^k, \quad k \in \mathbb{N}$ unde $\theta \in (0,1)$ un parametru necunoscut și $A \in \mathbb{R}$ este o constantă.

1. Determinați constanta A și calculați $\mathbb{E}[X]$ și $Var(X)$.

Dorim să estimăm pe θ plecând de la un eșantion $X_1, X_2, \ldots, X_n$ de talie n din populația dată de repartiția lui X.

2. Determinați estimatorul $\tilde{\theta}$ a lui θ obținut prin metoda momentelor și calculați $\mathbb{P}_\theta(\tilde{\theta} = 0)$.
3. Determinați estimatorul de verosimilitate maximă $\hat{\theta}$ a lui θ și verificați dacă acesta este bine definit.
4. Studiați consistența estimatorului $\tilde{\theta}$ și determinați legea lui limită.

Exercițiul 3

10p

Calculați marginea Rao-Cramer pentru familia $\mathcal{N}(\mu, 1)$ unde μ este necunoscut. Determinați estimatorul obținut prin metoda momentelor și verificați dacă este eficient.

Exercițiul 4

10p

Considerăm următorul eșantion de talie 20 dintr-o populație Bernoulli de parametru $\theta \in (0,1)$:

`0 1 1 0 1 1 0 1 1 0 0 1 1 1 1 1 0 0 0 0`

a) Găsiți estimatorul de verosimilitate maximă $\hat{\theta}$ și determinați informația lui Fisher $I(\theta)$.

b) Determinați estimatorul de verosimilitate maximă pentru $\mathbb{V}_\theta[X_1]$. Este acesta nedeplasat? Dar consistent? Justificați răspunsul.

c) Construiți un interval de încredere pentru $\hat{\theta}$ de nivel 95%.

Ex 1

$$X_1, X_2, \dots X_n \sim P(\theta)$$

$$f_\theta(k) = \mathbb{P}(X=k) = e^{-\theta}\frac{\theta^k}{k!}, \quad k \in \mathbb{N}$$

$$L(\theta; x_1, \dots x_n) = e^{-n\theta}\frac{\theta^{x_1+\dots+x_n}}{(x_1+\dots+x_n)!}$$

$$l(\theta; x_1, \dots x_n) = (x_1+\dots+x_n)\log(\theta) - n\theta - \ln(x_1+\dots+x_n)!$$

$$\frac{dl}{d\theta}(\theta; x_1 \dots x_n) = (x_1+\dots+x_n)\cdot\frac{1}{\theta} - n$$

θ	0		$\overline{x}_n$	
$l(\theta; x_1 \dots x_n)$		$\nearrow$		$\searrow$

Deci $\operatorname{argmax}_\theta l(\theta; x_1 \dots x_n) = \overline{x}_n$

$$\hat{\theta} = \overline{X}_n$$

Deplasarea:

$$\mathbb{E}[X_1] = \sum_{k \geq 0} k \cdot e^{-\theta} \cdot \frac{\theta^k}{k!}$$

$$\sum_{k \geq 0} e^{-\theta} \cdot \frac{\theta^k}{k!} = 1$$

$$\sum_{k \geq 0} \frac{\theta^k}{k!} = e^{\theta} \;\Big|\; \frac{d}{d\theta}$$

$$\sum_{k \geq 0} k \cdot \frac{\theta^{k-1}}{k!} = e^{\theta} \;\Big|\cdot \theta$$

$$\sum_{k \geq 0} k \cdot e^{-\theta} \cdot \frac{\theta^k}{k!} = \theta$$

Deci $\mathbb{E}[X_1] = \theta = \mathbb{E}(\overline{X}_n) = \mathbb{E}(\hat{\theta})$

Deci $\hat{\theta}$ e nedeplasat.

$\hat{\theta} = \overline{X}_n \xrightarrow{\mathbb{P}} \mathbb{E}[X_1] = \theta \quad (L.N.M.) \Rightarrow$

$\Rightarrow \hat{\theta}$ e consistent

$$\mathrm{Var}(\overline{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var}(X_1)$$

$$= \frac{1}{n} \mathrm{Var}(X_1)$$

$$\mathrm{Var}(X_1) = \sum_{k=0}^{n} k^2 \cdot e^{-\theta} \cdot \frac{\theta^k}{k!} - \theta^2$$

$$\sum_{k=0}^{n} k \cdot \frac{\theta^k}{k!} = e^{\theta} \cdot \theta \quad \Big| \frac{d}{d\theta}$$

$$\sum_{k=0}^{n} k^2 \cdot \frac{\theta^{k-1}}{k!} = e^{\theta} \cdot \theta + e^{\theta} \quad \Big| \cdot \theta$$

$$\sum_{k=0}^{n} k^2 \cdot \frac{\theta^k}{k!} = e^{\theta}(1+\theta) \cdot \theta$$

$$\mathrm{Var}(X_1) = (1+\theta) \cdot \theta - \theta^2 = \theta$$

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}(\overline{X}_n) = \frac{\theta}{n}$$

$$I_1 = E_{\theta}\left[\left(\frac{d}{d\theta} \log f_{\theta}(X_1)\right)^2\right] = \sum_{k=0} \left(\frac{d}{d\theta} \log f_{\theta}(k)\right)^2 \cdot f_{\theta}(k)$$

$$= \sum_{k \geq 0} \frac{\left(f_\theta'(k)\right)^2}{\left(f_\theta(k)\right)^2} \cdot f_\theta(k)$$

(derivare în rap. cu θ)

$$f_\theta(k) = e^{-\theta} \cdot \frac{\theta^k}{k!}$$

$$f_\theta'(k) = -e^{-\theta} \cdot \frac{\theta^k}{k!} + e^{-\theta} \cdot k \cdot \frac{\theta^{k-1}}{k!}$$

$$= -e^{-\theta} \cdot \frac{\theta^k}{k!} + e^{-\theta} \cdot \frac{\theta^{k-1}}{(k-1)!}$$

$$= e^{-\theta} \cdot \frac{\theta^k}{k!} \cdot \left(-1 + \frac{k}{\theta}\right)$$

$$I_1 = \sum_{k \geq 0} \left(\frac{k}{\theta} - 1\right)^2 \cdot e^{-\theta} \cdot \frac{\theta^k}{k!}$$

$$= \frac{1}{\theta^2} \sum_{k \geq 0} k^2 \cdot e^{-\theta} \cdot \frac{\theta^k}{k!} - \frac{2}{\theta} \cdot \sum_{k \geq 0} k \cdot e^{-\theta} \cdot \frac{\theta^k}{k!} +$$

$$+ \sum_{k \geq 0} e^{-\theta} \cdot \frac{\theta^k}{k!}$$

$$= \frac{1}{\theta^2} \cdot \theta \cdot (\theta + 1) - \frac{2}{\theta} \cdot \theta + 1 = \frac{1}{\theta}$$

Ineq. Rao-Cramer:

$$Var(\hat{\theta}) \geq \frac{1}{I_n} = \frac{1}{n \cdot I_1}$$

$$\frac{\theta}{n} \geq \frac{1}{n \cdot \frac{1}{\theta}}$$ Avem egalitate $\Rightarrow$

$\Rightarrow$ $\hat{\theta}$ e eficient

b)

$$P_\theta(X_1 = 1 \mid X_1 > 0) = \frac{P_\theta(X_1 = 1)}{P_\theta(X_1 > 0)}$$

$$= \frac{e^{-\theta} \cdot \frac{\theta}{1!}}{1 - e^{-\theta} \cdot \frac{1}{0!}}$$

$$= \frac{e^{-\theta} \cdot \theta}{1 - e^{-\theta}}$$

$g : (0, \infty) \to (0, \infty)$

$$g(\theta) = \frac{e^{-\theta} \cdot \theta}{1 - e^{-\theta}}$$

$\hat{X}_n$ e estimatorul de verosimilitate maximă pt. θ $\Rightarrow$

$\Rightarrow g(\overline{X}_n)$ e est. de ver. maximă

pt. $P_\theta(X_1 = 1 \mid X_1 > 0)$

Consistența: g continuă

L.N.M: $\overline{X}_n \xrightarrow{\mathbb{P}} \theta$ $\Bigg/ \overset{\text{T. apl. cont.}}{\Rightarrow}$

$$g(\overline{X}_n) \xrightarrow{\mathbb{P}} g(\theta) = \mathbb{P}_\theta(X_1 = 1 \mid X_1 > 0)$$

Deci g e consistent

c) $$g(\overline{X}_n) = \frac{e^{-\overline{X}_n} \cdot \overline{X}_n}{1 - e^{-\overline{X}_n}} = \frac{\overline{X}_n}{e^{\overline{X}_n} - 1}$$

$$= \frac{X_1 + \dots + X_n}{n \cdot \left(e^{\frac{X_1 + \dots + X_n}{n}} - 1\right)}$$

$$\mathbb{P}(X_1 + \dots + X_n = l) =$$

$$= \sum_{i=0}^{l} \mathbb{P}(X_1 + \dots + X_{n-1} = i) \cdot \mathbb{P}(X_n = l - i)$$

$$\mathbb{P}(X_1+X_2=l) = \sum_{i=0}^{l} e^{-2\theta} \frac{\theta^i}{i!} \cdot \frac{\theta^{l-i}}{(l-i)!}$$

$$= \sum_{i=0}^{l} e^{-2\theta} \frac{\theta^l}{i! \cdot (l-i)!}$$

$$= \frac{e^{-2\theta}}{l!} \cdot \theta^l \cdot \underbrace{\sum_{i=0}^{l} \binom{l}{i}}_{2^l}$$

$$= e^{-2\theta} \cdot \frac{2\theta}{l!}$$

$$\mathbb{P}(X_1+X_2+X_3=l) =$$

$$= \sum_{i=0}^{l} e^{-3\theta} \cdot \frac{(2\theta)^i}{i!} \cdot \frac{\theta^{l-i}}{(l-i)!}$$

$$= e^{-3\theta} \cdot \frac{\theta^l}{l!} \cdot \underbrace{\sum_{i=0}^{l} 2^i \cdot \binom{l}{i}}_{\overset{\text{Binom Newton}}{=} (1+2)^l}$$

$$= e^{-3\theta} \cdot \frac{(3\theta)^l}{l!}$$

Prin inducție:

$$\mathbb{P}(X_1 + \dots + X_n = k) = e^{-n\theta} \frac{(n\theta)^k}{k!}$$

$$E[g(\overline{X}_n)] = \sum_{k \geq 0} \frac{k}{n \cdot (e^{\frac{k}{n}} - 1)} \cdot \mathbb{P}(X_1 + \dots + X_n = k)$$

$$= \sum_{k \geq 0} \frac{k}{n \cdot (e^{\frac{k}{n}} - 1)} \cdot e^{-n\theta} \cdot \frac{(n\theta)^k}{k!}$$

Ex 2 $\quad \mathbb{P}_\theta(X=k) = A(k+1)\theta^k$

a)

$$\sum_{k\geq 0} A\cdot(k+1)\cdot\theta^k = A\cdot\sum_{k\geq 0}(k+1)\theta^k$$

$$\sum_{k\geq 0}\theta^{k+1} = \theta\cdot\frac{1}{1-\theta} \quad \Big| \frac{d}{d\theta}$$

$$\sum_{k\geq 0}(k+1)\theta^k = \frac{1}{(1-\theta)^2} \Rightarrow$$

$$\Rightarrow A = (1-\theta)^2$$

$$\mathbb{E}[X] = (1-\theta)^2\cdot\sum_{k\geq 0} k\cdot(k+1)\cdot\theta^k$$

$$\sum_{k\geq 0}\theta^k = \frac{1}{1-\theta} \quad \Big| \frac{d}{d\theta}$$

$$\sum_{k\geq 0} k\cdot\theta^{k-1} = \frac{1}{(1-\theta)^2} \Rightarrow \sum_{k\geq 0} k\cdot\theta^k = \frac{\theta}{(1-\theta)^2}$$

Mai derivăm o dată :

$$\sum_{k\geq 0} k^2 \theta^{k-1} = \frac{(1-\theta)^2+2\theta(1-\theta)}{(1-\theta)^4}$$

$$= \frac{1-2\theta+\theta^2+2\theta-2\theta^2}{(1-\theta)^4}$$

$$= \frac{1-\theta^2}{(1-\theta)^4} = \frac{1+\theta}{(1-\theta)^3}$$

$$\sum_{k\geq 0} k^2 \cdot \theta^k = \frac{\theta+\theta^2}{(1-\theta)^3} \quad \Big| \frac{d}{d\theta}$$

$$\sum_{k\geq 0} k^3 \cdot \theta^{k-1} = \frac{(1+2\theta)\cdot(1-\theta)^3+3(\theta+\theta^2)\cdot(1-\theta)^2}{(1-\theta)^6}$$

$$= \frac{1+2\theta-\theta-2\theta^2+3\theta+3\theta^2}{(1-\theta)^4}$$

$$= \frac{1+4\theta+\theta^2}{(1-\theta)^4}$$

$$\sum_{k\geq 0} k^3 \cdot \theta^k = \frac{\theta+4\theta^2+\theta^3}{(1-\theta)^3}$$

$$\mathbb{E}[X] = (1-\theta)^2 \cdot \left(\frac{\theta}{(1-\theta)^2} + \frac{\theta+\theta^2}{(1-\theta)^3} \right)$$

$$= \theta + \frac{\theta+\theta^2}{1-\theta}$$

$$= \frac{\theta-\theta^2+\theta+\theta^2}{1-\theta} = \frac{2\theta}{1-\theta}$$

Analog varianța

2. $\mathbb{E}[X] = \dfrac{2\theta}{1-\theta} \Rightarrow$

$$\Rightarrow 2\theta = (1-\theta)\,\mathbb{E}[X]$$

$$\theta(2+\mathbb{E}[X]) = \mathbb{E}[X]$$

$$\theta = \frac{\mathbb{E}[X]}{2+\mathbb{E}[X]} \Rightarrow$$

$$\Rightarrow \tilde{\theta} = \frac{\overline{X_n}}{2+\overline{X_n}}$$ (metoda momentelor

$P_\theta(\tilde{\theta} = 0) =$

$= \mathbb{P}(\overline{X}_n = 0)$

$X_1, \dots X_n$ iau doar valori pozitive $\Rightarrow$

$\Rightarrow \overline{X}_n = 0 \Leftrightarrow X_1 = X_2 = \dots = X_n = 0$

$$P_\theta(\tilde{\theta} = 0) = \left(\mathbb{P}(X_1 = 0)\right)^n$$
$$= \left((1-\theta)^2 \cdot (0+1)\right)^n$$
$$= (1-\theta)^{2n}$$

3. $L(\theta; x_1, \dots x_n) = (1-\theta)^{2n} \cdot \prod_{i=1}^{n} (x_i + 1) \cdot \theta^{\sum_{i=1}^{n} x_i}$

$$\ell(\theta; x_1 \dots x_n) = 2n \ln(1-\theta) + \sum_{i=1}^{n} \ln(x_i + 1) + \left(\sum_{i=1}^{n} x_i\right) \ln \theta$$

$$\frac{dl}{d\theta} = -\frac{2n}{1-\theta} + \left(\sum_{i=1}^{n} x_i\right) \cdot \frac{1}{\theta}$$

$$\frac{dl}{d\theta} = 0 \iff \frac{2n}{1-\theta} = \frac{1}{\theta} \cdot \sum_{i=1}^{n} x_i \iff$$

$$\iff 2n\theta = \sum x_i - \theta \cdot \sum x_i \iff \theta\left(2n + \sum x_i\right) = \sum x_i$$

$$\theta = \frac{\sum x_i}{2n + \sum x_i}$$

$$l \nearrow \text{ pt. } \theta < \frac{\sum x_i}{2n + \sum x_i}$$

$$l \searrow \text{ pt. } \theta > \frac{\sum x_i}{2n + \sum x_i}$$

Deci $\hat{\theta} = \dfrac{\sum X_i}{2n + \sum X_i} = \dfrac{\overline{X_n}}{2 + \overline{X_n}}$

$\overline{X_n} \geq 0 \Rightarrow 2 + \overline{X_n} \geq 2 \Rightarrow \hat{\theta}$ e

bine definit $\quad g : (0, \infty) \to (0, \infty)$

4) $\tilde{\theta} = \dfrac{\overline{X_n}}{2 + \overline{X_n}}$ $\quad g(x) = \dfrac{x}{2 + x}$ continuă

Din T. apl. continue:

$$\overline{X_n} \xrightarrow{\mathbb{P}} \mathbb{E}[X_1] = \frac{2\theta}{1-\theta} \Rightarrow$$

$$\Rightarrow \tilde{\theta}_n = g(\overline{X}_n) \xrightarrow{\mathbb{P}} g(\mathbb{E}[X]) =$$

$$= g\left(\frac{2\theta}{1-\theta}\right) = \frac{\frac{2\theta}{1-\theta}}{2+\frac{2\theta}{1-\theta}} = \theta \Rightarrow$$

$\Rightarrow \tilde{\theta}_n$ e consistent

Din T.L.C $\quad \sqrt{n}\left(\overline{X}_n - \frac{2\theta}{1-\theta}\right) \xrightarrow{d} N(0, Var(X_1))$

calculată anterior

Din T. privind metoda Delta:

$$\sqrt{n}\left(g(\overline{X}_n) - g\left(\frac{2\theta}{1-\theta}\right)\right) \xrightarrow{d} g'\left(\frac{2\theta}{1-\theta}\right) \cdot N(0, Var(X_1))$$

$$\sqrt{n}(\tilde{\theta}_n - \theta) \xrightarrow{d} N\left(0, \left(g'\left(\frac{2\theta}{1-\theta}\right)\right)^2 \cdot Var(X_1)\right)$$

$$g'(x) = \frac{2+x-x}{(2+x)^2} = \frac{2}{(2+x)^2}$$

$$g'\left(\frac{2\theta}{1-\theta}\right) = \dots$$

Ex 3 $\quad N(\mu, 1)$

$$X_1, X_2, \dots, X_n \sim N(\mu, 1)$$

$$f_\mu(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2}}$$

$$\frac{d}{d\mu} \log f_\mu(x) = \frac{d}{d\mu}\left(-\frac{(x-\mu)^2}{2}\right)$$

$$= x - \mu$$

$$I_1 = E\left[\left(\frac{d}{d\mu} \log f_\mu(X)\right)^2\right] = E\left[(X-\mu)^2\right]$$

$$= Var(X) = 1$$

$$I_n = n \Rightarrow MIRC = \frac{1}{n}$$

$E[X_1] = \mu \Rightarrow \tilde{\mu} = \overline{X}_n$ → metoda momentelor

$$Var(\tilde{\mu}) = Var\,\overline{X}_n = \frac{1}{n^2} \cdot n \cdot Var(X_1)$$

$$= \frac{1}{n}$$

$Var(\tilde{\mu}) = MIRC$, deci $\tilde{\mu}$ e eficient

Ex 4 $\quad B(\theta) \quad \theta \in (0,1)$

$$f_\theta(x) = P(X = x) = \theta^{x} \cdot (1-\theta)^{1-x}$$

$$L(\theta; x_1, \dots x_n) = \theta^{\sum x_i} \cdot (1-\theta)^{n - \sum x_i}$$

$$l(\theta; x_1, \dots x_n) = \left(\sum x_i\right) \ln \theta + \left(n - \sum x_i\right) \ln(1-\theta)$$

$$\frac{dl}{d\theta} = \left(\sum x_i\right) \cdot \frac{1}{\theta} - \left(n - \sum x_i\right) \frac{1}{1-\theta}$$

$$\frac{dl}{d\theta} = 0 \Leftrightarrow \frac{1-\theta}{\theta} = \frac{n - \sum x_i}{\sum x_i} \Leftrightarrow$$

$$\Leftrightarrow \sum x_i - \theta \cdot \sum x_i = n\theta - \theta \sum x_i \Leftrightarrow \theta = \frac{\sum x_i}{n}$$

Pt. $\theta < \overline{x}_n \Rightarrow l \nearrow$

$\theta > \overline{x}_n \Rightarrow l \searrow$

$\Rightarrow \hat{\theta} = \overline{X}_n$ (verosim. maximă)

$$\log f_\theta = x \cdot \ln \theta + (1-x) \ln(1-\theta)$$

$$\frac{d}{d\theta} \log f_\theta = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{d^2}{d\theta^2} \log f_\theta = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$$

$$I_1 = -E_\theta\left[\frac{d^2}{d\theta^2}\log f_\theta(X)\right]$$

$$= -E_\theta\left[-\frac{X}{\theta^2} - \frac{1-X}{(1-\theta)^2}\right]$$

$$= \frac{1}{\theta^2} E_\theta[X] + \frac{1}{(1-\theta)^2}\cdot E[1-X]$$

$$= \frac{1}{\theta^2}\cdot\theta + \frac{1}{(1-\theta)^2}\cdot(1-\theta)$$

$$= \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)} \Rightarrow$$

$$\Rightarrow I_n = \frac{n}{\theta(1-\theta)}$$

b) $V_\theta[X_1] = \theta(1-\theta)$

$\hat{\theta} = \overline{X_n} \Rightarrow T_n$ est. de veros. max. pt. $V_\theta[X_1]$

$$T_n = \overline{X}_n(1-\overline{X}_n)$$

Nedeplasarea: $E[T_n] = E[\overline{X}_n] - E[(\overline{X}_n)^2]$

$$= \theta - \left(Var(\overline{X}_n) + \left(E[\overline{X}_n]\right)^2\right)$$

$$Var(\overline{X}_n) = \frac{1}{n^2} Var(X_1 + \dots + X_n)$$

$$= \frac{n}{n^2} Var(X_1) = \frac{1}{n}\theta(1-\theta)$$

$$E[T_n] = \theta - \left(\frac{1}{n}\theta(1-\theta) + \theta^2\right)$$

$$= \theta(1-\theta) - \frac{1}{n}\theta(1-\theta) = \frac{n-1}{n}\theta(1-\theta)$$

Deci $E[T_n] \neq \theta(1-\theta) \Rightarrow T_n$ nu e nedeplasat.

Consistența:

$$\overline{X}_n \xrightarrow{\mathbb{P}} \theta$$

Fie $g : [0,1] \to \mathbb{R}$

$g(\theta) = \theta(1-\theta)$ continuă

Din T. apl. continue $\Rightarrow g(\overline{X}_n) \xrightarrow{\mathbb{P}} g(\theta)$

$T_n \xrightarrow{\mathbb{P}} \theta(1-\theta) \Rightarrow$

$\Rightarrow T_n$ e consistent