# Operatii cu functii diferentiabile

**Propozitie 1.** Fie $D$ si $G$ doua multimi deschise in $\mathbb{R}^n$ respectiv $\mathbb{R}^m$ si

$$f : D \to G, \quad g : G \to \mathbb{R}^k$$

functii diferentiabile in $a \in D$ respectiv $b = f(a) \in G$. Atunci $g \circ f : G \to \mathbb{R}^k$ este diferentiabila in $a$ si

$$d(g \circ f)(a) = dg(f(a)) \circ df(a).$$

*Demonstratie.* Avem

$$f(x) = f(a) + df(a)(x-a) + \varepsilon_f(x) \cdot \|x-a\|.$$

$$g(y) = g(b) + dg(b)(y-b) + \varepsilon_g(y) \cdot \|y-b\|.$$

unde $\varepsilon_f$ este continua in $a$, $\varepsilon_g$ este continua in $b$ si

$$\varepsilon_f(a) = 0, \varepsilon_g(b) = 0.$$

Avem

$$g(f(x)) = g(f(a)) + dg(b)(f(x) - f(a) + \varepsilon_g(f(x)) \cdot \|f(x) - f(a)\|.$$

si deci

$$\begin{aligned} g(f(x)) = g(f(a)) + dg(f(a)) \circ df(a)(x-a) + dg(f(a))(\varepsilon_f(x))\|x-a\| \\ + \varepsilon_g(f(x)) \cdot \|df(a)(x-a) + \varepsilon_f(x) \cdot \|x-a\|\|. \end{aligned}$$

Deoarece

$$\lim_{x \to a} dg(f(a))((\varepsilon_f(x)) = 0,$$

$$\lim_{x \to a} \varepsilon_g(f(x)) = 0$$

si

$$\frac{\|df(a)(x-a) + \varepsilon_f(x) \cdot \|x-a\|\|}{\|x-a\|} \leq \|df(a)\| + \|\varepsilon_f(x)\|, \text{ daca } \ x \neq a,$$

rezulta

$$\lim_{x \to a} \frac{g \circ f(x) - g \circ f(a) - d(g(f(a))) \circ df(a)(x-a)}{\|x-a\|} = 0$$

Deci $g \circ f$ este diferentiabila in $a$ si

$$d(g \circ f)(a) = dg(f(a)) \circ df(a).$$

**Corolar 2.** In conditiile propozitiei de mai sus

$$[d(g \circ f)(a)] = [dg(f(a))] \cdot [df(a)].$$

**Propozitie 3.** Fie $D$ multime deschisa in $\mathbb{R}^n$ si functiile

$$f : D \to \mathbb{R}^m, \quad g : D \to \mathbb{R}^m \text{ si } \alpha \in \mathbb{R}$$

diferentiabile in $a \in D$. Atunci $f + g$ si $\alpha f$ sunt diferentiabile in $a$ si

$$d(f + g)(a) = df(a) + dg(a)$$

$$d(\alpha f)(a) = \alpha d(f)(a)$$

*Demonstratie.* Exercitiu!

**Teorema 4.** Fie $E \subset \mathbb{R}^n$, $F \subset \mathbb{R}^m$ doua multimi deschise. Daca $u_1, \ldots, u_m : E \to \mathbb{R}$ sunt functii diferentiabile (respectiv cu derivate partiale continue) pe multimea $E$, astfel incat

$$(u_1(x_1, x_2, \ldots, x_n), \ldots, u_m(x_1, x_2, \ldots, x_n)) \in F$$

si pentru orice $(x_1, x_2, \ldots, x_n) \in E$ si daca $\varphi : F \to \mathbb{R}$ este diferentiabila (repectiv admite derivate partiale continue) pe $D$ atunci functia $f : E \to \mathbb{R}$ definita prin

$$f(x_1, x_2, \ldots, x_n) = \varphi(u_1(x_1, x_2, \ldots, x_n), \ldots, u_m(x_1, x_2, \ldots, x_n))$$

este diferentiabila (respectiv admite derivate partiale continue) si

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{m} \frac{\partial \varphi}{\partial u_j}\left(u_1(x_1, x_2, \ldots, x_n), \ldots, u_m(x_1, x_2, \ldots, x_n)\right) \frac{\partial u_j}{\partial x_i}(x_1, x_2, \ldots, x_n).$$

Vom scrie

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial \varphi}{\partial u_j} \frac{\partial u_j}{\partial x_i}, \ i = 1, 2, \ldots, n.$$

Pentru simplitate, vom demonstra teorema in cazul particular exemplu $m = 2$ $n = 3$. Fie $u, v : E \to \mathbb{R}$ astfel incat

$$(u(x, y, z), v(x, y, z)) \in F,$$

$\varphi : F \to \mathbb{R}$ si $f : E \to \mathbb{R}$

$$f(x, y, z) = \varphi(u(x, y, z), v(x, y, z)),$$

Fie $g : E \to F$,

$$g(x, y, z) = (u(x, y, z), v(x, y, z))$$

Evident $f = \varphi \circ g$ si avem

$$[df(x,y,z)] = \left[\frac{\partial f}{\partial x}(x,y,z) \;\; \frac{\partial f}{\partial y}(x,y,z) \;\; \frac{\partial f}{\partial z}(x,y,z)\right]$$

$$[dg(x,y,z)] = \begin{bmatrix} \frac{\partial u}{\partial x}(x,y,z) & \frac{\partial u}{\partial y}(x,y,z) & \frac{\partial u}{\partial z}(x,y,z) \\ \\ \frac{\partial v}{\partial x}(x,y,z) & \frac{\partial v}{\partial y}(x,y,z) & \frac{\partial v}{\partial z}(x,y,z) \end{bmatrix}$$

$$[d\varphi(u,v)] = \left[\frac{\partial \varphi}{\partial u}(u,v), \frac{\partial \varphi}{\partial u}(u,v)\right]$$

Din Corolarul 2,

$$[df(x,y,z)] = [d\varphi(g(x,y,z))] \cdot [dg(x,y,z)]$$

si atunci

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{\partial \varphi}{\partial u}(u(x,y,z),v(x,y,z))\frac{\partial u}{\partial x}(x,y,z) + \frac{\partial \varphi}{\partial v}(u(x,y,z),v(x,y,z))\frac{\partial v}{\partial x}(x,y,z)$$
$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial \varphi}{\partial u}(u(x,y,z),v(x,y,z))\frac{\partial u}{\partial y}(x,y,z) + \frac{\partial \varphi}{\partial v}(u(x,y,z),v(x,y,z))\frac{\partial v}{\partial y}(x,y,z)$$
$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial \varphi}{\partial u}(u(x,y,z),v(x,y,z))\frac{\partial u}{\partial z}(x,y,z) + \frac{\partial \varphi}{\partial v}(u(x,y,z),v(x,y,z))\frac{\partial v}{\partial z}(x,y,z)$$

Pe scurt se va scrie

$$\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial y}$$
$$\frac{\partial f}{\partial z} = \frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial z}$$

**Exemplu.** Daca $f(x,y,z) = \varphi(xyz, x^2+y^2+z^2, x+yz)$, unde $f$ si $\varphi$ au derivate partiale continue, atunci

$$\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u}yz + \frac{\partial \varphi}{\partial v}2x + \frac{\partial \varphi}{\partial w}$$
$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial u}xz + \frac{\partial \varphi}{\partial v}2y + \frac{\partial \varphi}{\partial w}z$$
$$\frac{\partial f}{\partial z} = \frac{\partial \varphi}{\partial u}xy + \frac{\partial \varphi}{\partial v}2z + \frac{\partial \varphi}{\partial w}y$$

# Derivate partiale si diferentiale de ordin superior

Fie $D \subset \mathbb{R}^n$ o multime deschisa si $f : D \to \mathbb{R}$ astfel incat derivata partiala $\frac{\partial f}{\partial x_j}$ exista intr-o vecinatate deschisa a lui $a$ si functia $\frac{\partial f}{\partial x_j}$ are derivata partiala in raport cu $x_i$ in punctul $a$. Derivata

$$\frac{\partial f}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(a)$$

se numeste derivata partiala de ordin doi a functiei $f$ in raport cu variabilele $x_i$, $x_j$ si se noteaza cu

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \text{ daca } i \neq j \text{ sau cu } \frac{\partial^2 f}{\partial x_i^2}(a) \text{ daca } i = j,$$

adica

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \text{ daca } i \neq j \text{ si } \quad \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right)(a) = \frac{\partial^2 f}{\partial x_i^2}(a)$$

Derivatele

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a), \quad i \neq j$$

se numesc derivate partiale mixte de ordin doi in punctul $a$. Derivatele partiale de ordin superior se definesc in acelasi mod. Astfel derivatele partiale de ordinul 3, daca exista, se definesc ca derivate partiale de ordinul 1 ale derivatelor partiale de ordinul 2. De exemplu

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j \partial x_k}\right)(a) = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(a).$$

Functia $f$ se numeste de clasa $C^k$, daca toate derivatele partiale de ordinul $k$ exista si sunt continue pe $D$. Observam ca daca $f$ este de clasa $C^k$, $k \geq 2$ atunci $f$ este de clasa $C^{k-1}$.

**Exemplu.** Fie functia $f(x, y, z) = \sin(2xy + y^2)$

$$\frac{\partial f}{\partial x} = 2y\cos(2xy + y^2) \qquad \frac{\partial f}{\partial y} = 2(x + y)\cos(2xy + y^2)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -4y^2\sin(2xy + y^2)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2\cos(y^2 + 2xy) - 4(x + y)^2\sin(2xy + y^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2\cos(y^2 + 2xy) - 4y(x + y)\sin(2xy + y^2)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right) = 2\cos(y^2 + 2xy) - 4y(x + y)\sin(2xy + y^2)$$

**Exemplu.** Fie functia

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Observam ca

$$\frac{\partial f}{\partial x}(x,y) = y\frac{x^4 - y^4 + 4x^2y^2}{(x^2+y^2)^2} \quad (x,y) \neq (0,0)$$

si

$$\frac{\partial f}{\partial y}(x,y) = x\frac{x^4 - y^4 - 4x^2y^2}{(x^2+y^2)^2} \quad (x,y) \neq (0,0)$$

de unde

$$\frac{\partial f}{\partial x}(0,y) = -y \qquad \frac{\partial f}{\partial x}(x,0) = x, x, y \neq 0.$$

De asemenea,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$$

si

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} \frac{0}{y} = 0$$

Derivatele mixte de ordinul 2 in origine sunt

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

si

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{x} = \lim_{y\to 0} \frac{-y-0}{x} = -1$$

Asadar,

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

**Propozitie 5.** Fie $f$ o functie cu valori reale definita pe o multime deschisa $D \subset \mathbb{R}^2$ cu proprietatea ca $\frac{\partial f}{\partial x}$ si $\frac{\partial^2 f}{\partial y \partial x}$ exista in orice punct din $D$. Presupunem ca $(a,b) \in D$ si dreptunghiul determinat de $(a,b)$, $(a+h,b)$, $(a+h,b+k)$ si $(a,b+k)$ este inclus in $D$, unde $h,k \neq 0$. Fie

$$E(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) - f(a,b).$$

Atunci exista un punct $(\xi,\eta)$ in interiorul acestui dreptunghi astfel incat

$$E(h,k) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(\xi,\eta).$$

*Demonstratie.* Fie $g : [a, a + h] \to \mathbb{R}$, $g(t) = f(t, b + k) - f(t, b)$. Aplicand Teorema cresterilor finite a lui Lagrange pentru functia $g$ obtinem un punct $\xi$ intre $a$ si $a + h$ astfel incat

$$E(h, k) = g(a + h) - g(a) = g'(\xi)h = \left[\frac{\partial f}{\partial x}(\xi, b + k) - \frac{\partial f}{\partial x}(\xi, b)\right] h$$

Aplicand din nou teorema lui Lagrange pentru functia

$$t \mapsto \frac{\partial f}{\partial x}(\xi, t), \quad t \in [b, b + k]$$

obtinem $\eta$ intre $b$ si $b + k$ astfel incat

$$\frac{\partial f}{\partial x}(\xi, b + k) - \frac{\partial f}{\partial x}(\xi, b) = \frac{\partial^2 f}{\partial y \partial x}(\xi, \eta) \cdot k$$

In concluzie, exista $(\xi, \eta)$ in interiorul dreptunghiului astfel incat

$$E(h, k) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(\xi, \eta).$$

**Teorema 6.** (Criteriul lui Schwarz) Fie $D \subset \mathbb{R}^2$ o multime deschisa si $f : D \to \mathbb{R}$ o functie cu proprietatea ca $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ si $\frac{\partial^2 f}{\partial y \partial x}$ exista in orice punct din $D$ si $\frac{\partial^2 f}{\partial y \partial x}$ este continua in $(a, b) \in D$. Atunci exista $\frac{\partial^2 f}{\partial x \partial y}(a, b)$ si

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

*Demonstratie.* Fie $\varepsilon > 0$. Exista $\delta_\varepsilon > 0$ astfel incat $R_\varepsilon = (a - \delta_\varepsilon, a + \delta_\varepsilon) \times (b - \delta_\varepsilon, b + \delta_\varepsilon) \subset D$ si astfel incat pentru orice $(x, y) \in R_\varepsilon$ sa avem

$$\left| \frac{\partial^2 f}{\partial y \partial x}(x, y) - \frac{\partial^2 f}{\partial y \partial x}(a, b) \right| < \varepsilon.$$

Din propozitia anterioara rezulta ca pentru orice $h$ si orice $k$ cu $|h|, |k| < \delta_\varepsilon$ exista $\xi \in (a, a + h)$ si $\eta \in (b, b + k)$ astfel incat

$$\frac{E(h, k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(\xi, \eta).$$

si deci

$$\left| \frac{E(h, k)}{hk} - \frac{\partial^2 f}{\partial y \partial x}(a, b) \right| < \varepsilon.$$

Fie $h$ fixat. Trecand la limita cu $k \to 0$ in ultima inegalitate, obtinem

$$\left| \frac{\frac{\partial f}{\partial y}(a + h, b) - \frac{\partial f}{\partial y}(a, b)}{h} - \frac{\partial^2 f}{\partial y \partial x}(a, b) \right| \le \varepsilon. \tag{1}$$

Deoarece $\varepsilon$ a fost ales arbitrar si (1) are loc pentru orice $h$ cu $|h| < \delta_\varepsilon$ rezulta ca exista

$$\lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(a+h,b) - \frac{\partial f}{\partial y}(a,b)}{h} = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

Asadar, exista $\frac{\partial^2 f}{\partial x \partial y}(a,b)$ si

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$

Teorema este valabila pentru functii si in $\mathbb{R}^n$ dupa cum urmeaza.

**Teorema 7.** (Criteriul lui Schwarz) Fie $D \subset \mathbb{R}^n$ o multime deschisa, $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$ si $f : D \to \mathbb{R}$ o functie cu proprietatea ca $\frac{\partial f}{\partial x_i}$, $\frac{\partial f}{\partial x_j}$ si $\frac{\partial^2 f}{\partial x_j \partial x_i}$ exista in orice punct din $D$ si $\frac{\partial^2 f}{\partial x_j \partial x_i}$ este continua in $a \in D$. Atunci exista $\frac{\partial^2 f}{\partial x_i \partial x_j}(a)$ si

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a).$$

Pentru aplicatii se va folosi in mod frecvent urmatorul corolar.

**Corolar 8.** Fie $D \subset \mathbb{R}^n$ o multime deschisa si $f : D \to \mathbb{R}$ o functie de clasa $C^2$. Atunci pentru orice $i \neq j$ si pentru orice $a \in D$ avem

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a).$$

Fie $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ cu proprietatea ca toate derivatele partiale de ordinul $k$ exista intr-o vecinatate a lui $(a, b)$ si sunt continue in $a$. Din Criteriul lui Schwarz rezulta ca ordinea de derivare in $(a, b)$ pana la ordinul $k$ nu are importanta.

In aceste conditii functia $d^n f(a,b) : \mathbb{R}^2 \to \mathbb{R}$,

$$d^n f(a,b)(x,y) = \left[\frac{\partial f}{\partial x}(a,b)\cdot x + \frac{\partial f}{\partial y}(a,b)\cdot y\right]^{(n)} \text{ pentru } \quad (x,y) \in \mathbb{R}^2$$

se numeste diferentiala de ordinul $k$ a functiei $f$ in punctul $(a, b)$. Exponentul $n$ inseamna ca se dezvolta formal paranteza dupa regula binomului lui Newton unde

$$\left(\frac{\partial f}{\partial x}(a,b)\cdot x\right)^{(n)} = \frac{\partial^n f}{\partial x^k}(a,b)\cdot x^n$$

$$\left(\frac{\partial f}{\partial x}(a,b)\cdot x\right)^{(n-k)} \left(\frac{\partial f}{\partial y}(a,b)\cdot y\right)^{(k)} = \frac{\partial^n f}{\partial x^{n-k} \partial y^k}(a,b)\cdot x^{n-k} y^k$$

De exemplu pentru $n = 2$

$$d^2(f)(a,b)(x,y) = \frac{\partial^2 f}{\partial x^2}(a,b)\cdot x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)\cdot xy + \frac{\partial^2 f}{\partial y^2}(a,b)\cdot y^2$$

si pentru $n$ arbitar

$$d^n f(a,b)(x,y) = \sum_{k=0}^{n} C_n^k \left(\frac{\partial f}{\partial x}(a,b)\cdot x\right)^{(n-k)} \left(\frac{\partial f}{\partial y}(a,b)\cdot\right)^{(k)}$$

$$= \sum_{k=0}^{n} C_n^k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}(a,b)\cdot x^{n-k}y^k$$

Similar se definesc diferentialele de ordin superior pentru functii reale definite pe $\mathbb{R}^m$. Astfel, daca $f : D \subset \mathbb{R}^m \to \mathbb{R}$ are derivate partiale de ordinul doi intr-o vecinatate a lui $a$ si acestea sunt continue in $a$, diferentiala de ordinul doi a lui $f$ in punctul $a$ este functia $d^2f(a) : \mathbb{R}^m \to \mathbb{R}$ definita prin

$$d^2 f(a)(u) = \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) u_i u_j, \quad u = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m.$$

Daca $f : D \subset \mathbb{R}^m \to \mathbb{R}$ are derivate partiale de ordinul 3 intr-o vecinatate a lui $a$ si acestea sunt continue in $a$, diferentiala de ordinul 3 a lui $f$ in punctul $a$ este functia $d^3f(a) : \mathbb{R}^m \to \mathbb{R}$ definita prin

$$d^3 f(a)(u) = \sum_{i,j,k=1}^{m} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(a) u_i u_j u_k, \quad u = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m.$$

Daca $f : D \subset \mathbb{R}^m \to \mathbb{R}$ are proprietatea ca toate derivatele partiale de ordinul $n$ exista intr-o vecinatate a lui $a \in D$ si acestea sunt continue in $a$, polinomul

$$T_n(x) = f(a) + \frac{1}{1!}d(f)(a)(x-a) + \frac{1}{2!}d^2(f)(a)(x-a) + \cdots + \frac{1}{n!}d^n(f)(a)(x-a)$$

se numeste polinomul Taylor de grad $n$ asociat functiei $f$ in punctul $a$. Este un polinom in $m$ variabile. Daca avem o functie $f$ de doua variabile, atunci polinomul Taylor de gradul $n$ asociat functiei $f$ in punctul $(a, b)$ este

$$T_n(x,y) = f(a,b) + \frac{1}{1!}df(a,b)(x-a, y-b) + \frac{1}{2!}d^2 f(a,b)(x-a,y-b) + \cdots +$$

$$+\frac{1}{n!}d^n f(a,b)(x-a,y-b)$$

adica,

$$T_n(x) = f(a,b) + \frac{1}{1!}\left[\frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)\right] +$$

$$\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(a,b)\cdot(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)\cdot(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)\cdot(y-b)^2\right] + \cdots$$

$$\cdots + \frac{1}{n!}\sum_{k=0}^{n} C_n^k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}(a,b)\cdot(x-a)^n(y-b)^{n-k}$$

sau, folosind notatia introdusa mai sus

$$T_n(x) = f(a,b) + \frac{1}{1!}\left[\frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)\right]^{(1)} + \cdots$$

$$\cdots + \frac{1}{n!}\left[\frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)\right]^{(n)}$$

**Teorema 9.** (Formula lui Taylor cu restul lui Lagrange) Fie $D$ o multime deschisa convexa din $\mathbb{R}^2$ si $(a,b)$ un punct din $D$. Daca $f : D \to \mathbb{R}$ este o functie de clasa $C^{n+1}$, atunci pentru orice punct $(x,y)$ din $D$ exista $(\xi,\eta) \in D$ un punct situat pe segmentul care uneste punctele $(a,b)$ si $(x,y)$ astfel incat

$$f(x,y) = T_n(x,y) + \frac{1}{(n+1)!}d^{n+1}f(\xi,\eta)(x-a,y-b).$$

*Demonstratie.* Fie

$$x(t) = a + (x-a)t \qquad y(t) = b + (y-b)t,\ t \in [0,1]$$

Fie $\varphi : [0,1] \to \mathbb{R}$,

$$\varphi(t) = f(a+(x-a)t, b+(y-b)t) = f(x(t),y(t)).$$

Atunci

$$\begin{aligned}\varphi'(t) &= \frac{\partial f}{\partial x}(x(t),y(t))\cdot(x-a) + \frac{\partial f}{\partial y}(x(t),y(t))\cdot(y-b)\\ &= df(x(t),y(t))(x-a,y-b).\end{aligned}$$

$$\begin{aligned}\varphi''(t) = \frac{\partial^2 f}{\partial x^2}(x(t),y(t))\cdot(x-a)^2 + 2\frac{\partial^2 f}{\partial y\partial x}(x(t),y(t))\cdot(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(x(t),y(t))\cdot(y-b)^2\\ = d^2 f(x(t),y(t))(x-a,y-b).\end{aligned}$$

In general,

$$\varphi^{(n)}(t) = d^n f(x(t),y(t))(x-a,y-b).$$

Aplicand Formula lui Taylor cu restul lui Lagrange pentru o functie de o variabila reala, rezulta ca exista $\theta \in (0,1)$ astfel incat

$$\varphi(1) = \varphi(0) + \frac{1}{1!}\varphi'(0) + \frac{1}{2!}\varphi''(0) + \cdots + \frac{1}{n!}\varphi^{(n)}(0) + \frac{1}{(n+1)!}\varphi^{(n+1)}(\theta).$$

Cum $\varphi(1) = f(x,y),\ \varphi(0) = f(a,b)$ si

$$\varphi^{(k)}(0) = d^k f(a,b)(x-a, y-b),$$

punand $\xi = x(\theta)\ \eta = y(\theta)$, avem

$$f(x,y) = f(a,b) + \frac{1}{1!}df(a,b)(x-a,y-b) + \frac{1}{2!}d^2 f(a,b)(x-a,y-b) + \cdots +$$

$$+\frac{1}{n!}d^n f(a,b)(x-a,y-b) + \frac{1}{(n+1)!}d^{n+1}f(\xi,\eta)(x-a,y-b)$$

$$= T_n(x,y) + \frac{1}{(n+1)!}d^{n+1}f(\xi,\eta)(x-a,y-b)$$

Teorema este adevarata pentru functii de mai multe variabile, demonstratia decurgand intr-o maniera similara.

**Teorema 10.** (Formula lui Taylor cu restul lui Lagrange) Fie $D$ o multime deschisa si convexa din $\mathbb{R}^m$, $a \in D$ si $f : D \to \mathbb{R}$ o functie de clasa $C^{n+1}$ pe $D$. Atunci pentru orice punct $x \in D$ exista $c \in D$ un punct situat pe segmentul $[a, x]$ astfel incat

$$f(x,y) = T_n(x,y) + \frac{1}{(n+1)!}d^{n+1}f(c)(x-a,y-b).$$