Examen la TEORIA MĂSURII ŞI INTEGRĂRII [1]
an II, sem. I, grupele 201, 202, 221, 222

2.02.2021

Numele şi prenumele ....

Grupa ....

Punctaj seminar ....

Subiectul 1. Calculaţi:

$$\lim_{n\to\infty}\int_{(1,\infty)}\frac{x}{1+x^n}d\lambda,$$

unde λ este măsura Lebesgue pe $\mathbb{R}$.

Subiectul 2. Considerăm mulţimile A, B, C definite după cum urmează:

$$A=\{(x,y)\in\ \mathbb{R}^2|\ x^2+y^2\leq -1+2x-2y,\ y+1\leq 0,\ x-1\leq 0\},$$
$$B=A\cap(\mathbb{Q}\times(\mathbb{R}\setminus\mathbb{Q})),$$
$$C=\{y\in\ \mathbb{R}|\ \text{există } x\in\mathbb{R} \text{ cu } (x,y)\in A\}.$$

Precizaţi care din mulţimile A, B, C este măsurabilă Lebesgue şi, in caz că există, calculaţi măsura Lebesgue a acestora.

Subiectul 3. Calculaţi integrala curbilinie următoare în două moduri (direct şi cu teorema lui Green):

$$I=\int_\gamma (x+1)dx+(xy+1)dy,$$

unde γ este conturul triunghiului OAB, $O(0,0)$, $A(7,-7)$ şi $B(14,14)$, parcurs în sens trigonometric.

Subiectul 4. Calculaţi volumul corpului mărginit de paraboloidul de ecuaţie $z=x^2+y^2$ şi planul de ecuaţie $z=5$.

Subiectul 5. Cosiderăm $f:\mathbb{R}\longrightarrow\mathbb{R}$ o funcţie integrabilă Lebesgue (λ măsura Lebesgue pe $\mathbb{R}$ şi $\mathcal{M}_\lambda$ σ-algebra mulţimilor măsurabile Lebesgue din $\mathbb{R}$).

a) Demonstraţi că $\mu:\mathcal{M}_\lambda\longrightarrow[0,\infty]$, definită prin $\mu(A)=\int_A|f|\,d\lambda$ pentru orice $A\in\mathcal{M}_\lambda$, este o măsură finită.

b) Demonstraţi că pentru orice şir de mulţimi $(E_n)_{n\geq 0}\subseteq\mathcal{M}_\lambda$ cu proprietatea că $E_n\subseteq E_{n+1}$ pentru orice $n\in\mathbb{N}$ şirul $\left(\int_{E_n} f d\lambda\right)_n$ este convergent.

[1]**Toate subiectele sunt obligatorii. Toate răspunsurile trebuie justificate. Timp de lucru 2h. Fiecare subiect se noteaza se la 1 la 10. Rezolvările trebuie scanate şi trimise împreună cu subiectul primit sub forma unui singur fişier pdf in formularul Google corespunzător grupei dumneavoastră. Succes!**

Morariu Medeea
Grupa 201

Subiectul 1:

$$\lim_{n\to\infty} \int_{(1,\infty)} \frac{x}{1+x^n} \, dx = ?$$

Fie $f_n:(1,\infty) \to [0,\infty]$, $f_n(x) = \dfrac{x}{1+x^n}$, $\forall n \in \mathbb{N}$

f_n - funcție continuă $\Rightarrow f_n$ - măsurabilă $\mathcal{L}$, $\forall n \in \mathbb{N}$

~~$x \in (1,\infty) \Rightarrow x^n \le x^{n+1}$ $\forall n \in \mathbb{N} \Rightarrow f_n(x) \ge f_{n+1}(x)$, $\forall n \in \mathbb{N}$ $\forall x \in (1,\infty)$~~

~~$f_0(x) = \frac{x}{2}$~~

$|f_n(x)| \le 1$, $\forall n \in \mathbb{N}$

$g(x) = 1$ - integrabilă, $g:(1,\infty) \to [0,\infty]$

$$\xrightarrow{TCD} \lim_{n\to\infty} \int_{(1,\infty)} \frac{x}{1+x^n} \, dx = \int_{(1,\infty)} \lim_{n\to\infty} \frac{x}{1+x^n} \, dx = \int_{(1,\infty)} 0 \, dx =$$

$$= \int 0 \cdot 1\!\!1_{(1,\infty)} \, dx = \int 0 \, dx = 0.$$

Subiectul 2

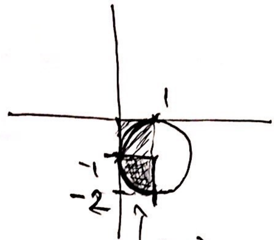
$A = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le -1 + 2x - 2y, \ y+1 \le 0, \ x-1 \le 0 \}$

$B = A \cap (\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}))$

$C = \{ y \in \mathbb{R} \mid \exists x \in \mathbb{R} \ (x,y) \in A \}$

$A = \{(x,y) \in \mathbb{R}^2 : y \leq -1,\ x \leq 1,\ x^2 - 2x + 1 + y^2 + 2y \leq 0\}$

$(x-1)^2 + y(y+2) \leq 0 \Rightarrow (x-1)^2 + (y+1)^2 \leq 1$



(x,y) se află în această porțiune din cerc (sfertul de cerc)

$\Rightarrow A$ - măsurabilă $\mathcal{L}$.

$\lambda(A)$ = aria suprafeței $\rightarrow \lambda(A) = \frac{\pi}{4}$ (un sfert din aria cercului cu raza 1)

$B = A \cap (\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}))$

$\mathbb{Q} = \bigcup_{a \in \mathbb{Q}} \{a\}$ - reuniune numărabilă de mulțimi ~~măsurabile $\mathcal{L}$~~ boreliene

$\Rightarrow \mathbb{Q}$ - ~~măsurabilă $\mathcal{L}$~~ boreliană

$\mathbb{R} \setminus \mathbb{Q} = C_{\mathbb{Q}} \in \mathcal{M}_{\lambda^*}$ (în $\mathbb{R}$) $\Big\} \Rightarrow \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) \in$

$\rightarrow \mathbb{R} \setminus \mathbb{Q}$ - ~~măsurabilă $\mathcal{L}$~~ boreliană

$\in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{M}_{\lambda^*}$

$\Rightarrow \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ - măsurabilă $\mathcal{L}$.

$B = A \cap \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ - intersecție de mulțimi măsurabile

$\Rightarrow B$ - măsurabilă Lebesgue

$\lambda^*(\mathbb{Q}) = 0$ (în $\mathbb{R}$) $\rightarrow$

$\Rightarrow \lambda(\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})) = 0$
$B \subset \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ $\Big\} \Rightarrow \lambda(B) = 0$ (λ - măsură completă pe $\mathbb{R}^2$)

~~$C = \{y \in \mathbb{R} : y$~~

Pt. $x = 1 \Rightarrow (x-1)^2 + (y+1)^2 \le 1 \Leftrightarrow -1 \le y+1 \le 1 \Leftrightarrow$ $\begin{cases} -2 \le y \le 0 \\ y \le -1 \end{cases} \Rightarrow$

$\Rightarrow y \in [-2, -1]$

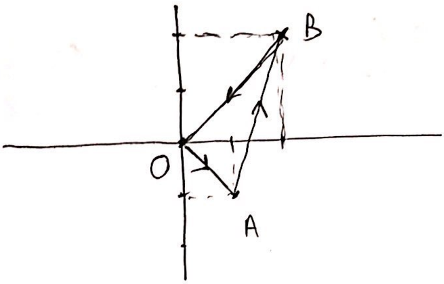
$C = [-2, -1]$ - interval $\Rightarrow$ măsurabil Lebesgue

$\lambda(C) = \lambda([-2, -1]) = 1.$

Subiectul 3:

$$\int_{\gamma} (x+1)\,dx + (xy+1)\,dy$$

ΔOAB $\quad O(0,0),\ A(7,-7)\quad B(14,14)$



Direct.

$C_1 ::= \{OA\}$

$\gamma: [0,7] \to \mathbb{R}^2,\ \gamma(t) = (\underset{\gamma_1(t)}{t}, \underset{\gamma_2(t)}{-t})$

γ de clasă C^1 $\begin{cases} \gamma_1'(t) = 1 \\ \gamma_2'(t) = -1 \end{cases}$

$C_2 ::= \{AB\}$

$\sigma: [0,1] \to \mathbb{R}^2,\ \sigma(t) = (1-t)A + tB = (7+7t,\ 21t-7)$

σ de clasă C^1 $\begin{cases} \sigma_1'(t) = 7 \\ \sigma_2'(t) = 21 \end{cases}$

$C_3 ::= \{BO\}$

$d: [0,14] \to \mathbb{R}^2,\ d(t) = (14-t, 14-t)$ $\quad (14-t,\ 14-t)$

d de cls C^1 $\begin{cases} d_1'(t) = -1 \\ d_2'(t) = -1 \end{cases}$

Fie $F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y)=(x+1,\ xy+1)$ - continuă

$$\int_{\gamma}(x+1)\,dx+(xy+1)\,dy=\int_{C_1}(x+1)\,dx+(xy+1)\,dy+$$

$$+\int_{C_2}(x+1)\,dx+(xy+1)\,dy+\int_{C_3}(x+1)\,dx+(xy+1)\overset{(-1)}{}\,dy=$$

$$=\int_0^7 (t+1)+(-t^2+1)(-1)\,dt+\int_0^1 (21t+8)\cdot 7+(147t^2+98t-48)\cdot 21\,dt$$

$$=\int_0^{14}(t+1+t^2+1)\,dt=\int_0^7(t^2+t)\,dt+\int_0^1(3087t^2+2107t-952)\,dt$$

$$=\int_0^{14}(t^2+t+2)\,dt=\frac{343}{3}+\frac{49}{2}+1029+\frac{2107}{2}-952-\frac{2744}{3}-98-28$$

$$-49-\frac{2401}{3}+1098=\frac{686}{3}$$

$$=\cancel{203+1078+1029=2310}$$

Cu Green:

$F:\mathbb{R}^2\to\mathbb{R}^2$, $F(x,y)=(x+1,\ xy+1)$

$$\cancel{F_1\,dx=\frac{\partial P}{\partial x}\ \frac{\partial P}{\partial y}}$$

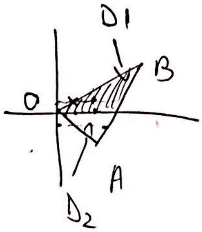
$$F_1\,dx=P\Rightarrow\frac{\partial P}{\partial y}=0$$

$$F_2\,dx=Q\Rightarrow\frac{\partial Q}{\partial x}=y$$

$$\int_\gamma P\,dx + Q\,dy = \int_\gamma P(x,y)\,dx + Q(x,y)\,dy \overset{T.\ Green}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$= \iint_D \cancel{x}\, y \,dx\,dy$, unde D = triunghiul de vârfuri $(0,0)$, $(7,-7)$, $(14,14)$

(int. triunghi + fr.)



$$\frac{x-7}{7} = \frac{y+7}{21}$$

$$21x - 49\cdot 3 = 7y + 49$$

$$\cancel{21x} \quad x = 7\cdot\frac{4}{3} + \frac{y}{3}$$

$D = D_1 \cup D_2$, $D_1 = \left\{(x,y) \mid y \in [0,14],\ y \le x \le \frac{y+28}{3}\right\}$

$D_2 = \left\{(x,y) \mid y \in [-7,0],\ -y \le x \le \frac{y+28}{3}\right\}$

$$\iint_{D_1} y\,dx\,dy = \int_{[0,14]} \left(\int_y^{\frac{y+28}{3}} y\,dx\right) dy = \int_0^{14} y\left(\frac{y+28}{3} - y\right) dy =$$

$$\cancel{= \int_0^{14} \frac{-2y^2+28y}{3}\,dy = \frac{-2}{3}\cdot 14^3 + \frac{28\cdot 14^2}{6} = 2.}$$

$$= \int_0^{14} \frac{-2y^2+28y}{3}\,dy = \frac{-2}{3}\cdot\frac{14^3}{3} + \frac{28}{3}\cdot\frac{14^2}{2} = \frac{{}^{3)}14^3\cdot 2}{6} - \frac{{}^{2)}14^3\cdot 2}{9} =$$

$$= \frac{14^3\cdot 2}{18} = \frac{14^3}{9}$$

$$\iint_{D_2} y\,dx\,dy = \int_{[-7,0]} \left(\int_{-y}^{\frac{y+28}{3}} y\,dx\right) dy = \int_{-7}^{0} y\left(\frac{y+28}{3} + y\right) dy = \ldots$$

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$D_1 \cap D_2 = \{(0,0), (\frac{28}{3}, 0)\} \Rightarrow \mathcal{N}(D_1 \cap D_2) = 0$

$$\Rightarrow \int_D y\,dx\,dy = \int_{D_1} y\,dx\,dy + \int_{D_2} y\,dx\,dy$$

Subiectul 4

$z = x^2 + y^2 \qquad z = 5$

$(u\cos\theta, u\sin\theta, u^2)$, $u \in [0, 5]$, $\theta \in [0, 2\pi)$

$$V = \iint_E \left\| \frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial\theta} \right\| du\,d\theta \qquad E = [0,5] \times [0, 2\pi)$$

$$= \iint_E \| N\varphi(u,\theta) \| \, du\,d\theta$$

$$\cancel{\| N\varphi(u,\theta) \| = \sqrt{\left(\frac{\partial\varphi}{\partial u}\right)^2 + \left(\frac{\partial\varphi}{\partial\theta}\right)^2 + 1}}$$

$$N\varphi(u,\theta) = \left(\frac{\partial(\varphi_2, \varphi_3)}{\partial(u,\theta)}(u,\theta), \frac{\partial(\varphi_3, \varphi_1)}{\partial(u,\theta)}(u,\theta), \frac{\partial(\varphi_1, \varphi_2)}{\partial(u,\theta)}(u,\theta) \right)$$

$$= \left(2u^2\cos\theta, 0, u\cos^2\theta \right)$$

$$\| N\varphi(u,\theta) \| = \sqrt{4u^4\cos^2\theta + u^2\cos^4\theta}$$

$$\Rightarrow V = \iint_E \sqrt{4u^4\cos^2\theta + u^2\cos^4\theta}\, du\,d\theta$$

Subiectul 5

$f: \mathbb{R} \to \mathbb{R}$ - integrabilă $\mathcal{L}$

a) $\mu: \mathcal{M}_\lambda \to [0,\infty]$, $\mu(A) = \int_A |f| \, d\lambda$ - măsură finită

$$\left.\begin{array}{l} \mu(\Phi) = \int_\Phi |f| \, d\lambda = \int |f| \cdot 1\!\!1_\Phi \, d\lambda = \int 0 \, d\lambda = 0 \\ \text{• Fie } (A_n)_n \subseteq \mathcal{M}_\lambda, \ A_n \cap A_m = \Phi, \ \forall n \neq m. \text{ Atunci} \\ \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \int |f| \cdot 1\!\!1_{\bigcup_{n=1}^{\infty} A_n} \, d\lambda = \int |f| \left(\sum_{n=1}^{\infty} 1\!\!1_{A_n}\right) d\lambda = \\ = \int \left(\sum_{n=1}^{\infty} |f| 1\!\!1_{A_n}\right) d\lambda = \sum_{n=1}^{\infty} \int |f| 1\!\!1_{A_n} = \sum_{n=1}^{\infty} \mu(A_n) \end{array}\right\} \Rightarrow$$

$\Rightarrow \mu$ este măsură pe $(\mathbb{R}, \mathcal{M}_\lambda)$

μ - finită $\Leftrightarrow \mu(\mathbb{R}) < \infty$

$$\left.\begin{array}{l} \mu(\mathbb{R}) = \int |f| \, d\lambda \\ f \text{ - integrabilă Lebesgue} \Rightarrow \int |f| \, d\lambda < \infty \end{array}\right\} \Rightarrow \mu(\mathbb{R}) < \infty$$

$\Rightarrow \mu$ este măsură finită.

b) $\forall (E_n)_{n \geq 0} \subseteq \mathcal{M}_\lambda$, $E_n \subseteq E_{n+1}$, $\forall n \in \mathbb{N}$

$\Rightarrow \left(\int_{E_n} f \, d\lambda\right)_n$ convergent

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4. Continuare

$$\iint_{\Sigma} = \int_0^{2\pi}\left(\int_0^5 \sqrt{4u^4\cos^2 v + u^2\cos^4 v}\, du\right) dv =$$

$$= \cancel{\int_0^{2\pi}\left(\int_0^5 u\sqrt{4u^2\cos}\right.}$$

$$= \int_0^{2\pi}\left(\int_0^5 u|\cos v|\sqrt{4u^2+\cos^2 v}\, du\right) dv =$$

$$= \int_0^{2\pi}\left(|\cos v|\int_0^5 \sqrt{4u^2+\cos^2 v}\cdot\frac{1}{8}\, 8u\, du\right) dv - \int_0^{2\pi}\left(|\cos v|\cdot\frac{2\sqrt{(4u^2+\cos^2 v)^3}}{3}\Bigg|_0^5\right) dv$$

$$= 4\int_0^{\pi/2}\cos v\cdot\frac{2}{3}\left(\sqrt{(100+\cos^2 v)^3} - (\cos v)^3\right) dv$$