TEORIA MĂSURII

SEMINAR 12

Întrebare Răzvan:

$f : [0,1] \to \mathbb{R}$ integrabilă

Arătați că $\lim\limits_{n\to\infty} \int_{[0,1]} x^{2n+1} f(x)\, d\lambda(x) = 0$

Soluție:

$(\forall)\, x \in [0,1)$

$$x^{2n+1} f(x) \xrightarrow{n\to\infty} 0$$

$\forall x \in [0,1]$, $\left| x^{2n+1} f(x) \right| < |f(x)|$, f integrabilă

TLCD $\Rightarrow$ GATA

Exemplu de funcție:

$f : [0,1] \to \mathbb{R}$ integrabilă

a. î. f^2 nu e integrabilă

Fie

$$f(x) = \frac{1}{\sqrt{x}}, \quad x \in [0,1]$$

$$\int_{[0,1]} f(x)\, d\lambda(x) = \int_0^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_0^1 = 2$$

$$\int_{[0,1]} (f(x))^2\, d\lambda(x) = \int_0^1 \frac{1}{x} = \ln x\Big|_0^1 = \infty$$

Invers: $(X, \mathcal{A}, \mu)$ sp. cu măsură

$A \subseteq \mathcal{A}$, $\underline{\mu(A) < \infty}$

Atunci $f : A \to \mathbb{R}$ integrabilă

f^2 integrabilă $\Rightarrow f$ integrabilă

Dem:

Metoda I:

Inegalitatea mediilor:

$$\frac{f^2(x) + 1}{2} \geq |f(x)| \quad \Big| \int_A$$

$$\frac{1}{2}\Big(\underbrace{\int_A f^2(x)\, d\mu(x)}_{< \infty} + \underbrace{\int_A d\mu}_{\substack{\shortparallel \\ \mu(A) < \infty}}\Big) \geq \int_A |f(x)|\, d\mu(x)$$

Metoda II (Cauchy - Schwarz)

$$\int_A |f| \, d\mu = \int_A |f| \cdot 1 \, d\mu \leq$$

$$\leq \underbrace{\sqrt{\int_A |f|^2 d\mu}}_{< \infty} \cdot \underbrace{\sqrt{\int_A 1 \, d\mu}}_{\sqrt{\mu(A)} < \infty}$$

Contraexemplu

$f : X \to \mathbb{R}$

f^2 integrabilă $\quad f \in L^2(X)$

f neintegrabilă $\quad f \notin L^1(X)$

$X = ([1, \infty), \lambda)$

$f(x) = \frac{1}{x}$

$$\int_1^\infty \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^\infty = 1$$

$$\int_1^\infty \frac{1}{x}\, dx = \ln x\Big|_1^\infty = \infty$$

Demonstrație Cauchy - Schwarz

$L^2(X) = \{ f : X \to \mathbb{R} \text{ măs.} \mid f^2 \text{ integr.} \}$

Enunț: $L^1(X) = \{ f : X \to \mathbb{R} \text{ integrabile} \}$

Fie $f, g \in L^2(X)$ (i.e. f^2, g^2 integrabile)

Atunci $f \cdot g \in L^1(X)$ și

$$\int_X |fg| \leq \left(\int_X f^2 \right)^{\frac{1}{2}} \cdot \left(\int_X g^2 \right)^{\frac{1}{2}}$$

Dem:

$$|fg| \leq \frac{f^2 + g^2}{2} \quad \Big| \int_X$$

$$\int_X |fg| \, d\mu \leq \frac{1}{2} \left(\int_X f^2 d\mu + \int_X g^2 d\mu \right) < \infty,$$

deci $fg \in L^1$

$(\forall)\, \lambda \in \mathbb{R} \quad (|f| + \lambda|g|)^2 \geq 0 \quad \Big| \int_X d\mu$

$$\int_X (|f| + \lambda|g|)^2 \, d\mu \geq 0 \implies$$

$$\implies \int_X (f^2 + 2|fg|\lambda + \lambda^2 g) \, d\mu \geq 0 \implies$$

$$\implies \lambda^2 \cdot \int_X g^2 + \lambda \cdot 2\int_X |fg| + \int_X f^2 \geq 0,$$

$(\forall)\, \lambda \in \mathbb{R}$

Ecuație de gradul II în λ

Dacă $\int_X g^2 = 0 \implies$ gata

Dacă nu, $\Delta \leq 0 \implies$

$$\Rightarrow \quad 4\left(\int_X |fg|\right)^2 - 4\left(\int_X g^2\right)\cdot\left(\int_X f^2\right) \leq 0,$$

adică gata $\square$

Enunț inegalitatea Hölder:

Fie $p, q \in [1, \infty)$ a.î.

$$\frac{1}{p} + \frac{1}{q} = 1$$

Fie $f \in L^p(X) = \{f : X \to \mathbb{R} \text{ măs.} \mid |f|^p \text{ integr.}\}$

$g \in L^q(X)$

Atunci $f \cdot g \in L^1(X)$ și

$$\int_X |fg| \leq \left(\int_X |f|^p\right)^{\frac{1}{p}} \cdot \left(\int_X |g|^q\right)^{\frac{1}{q}}$$

② $f: X \to [0,\infty)$ integrabilă

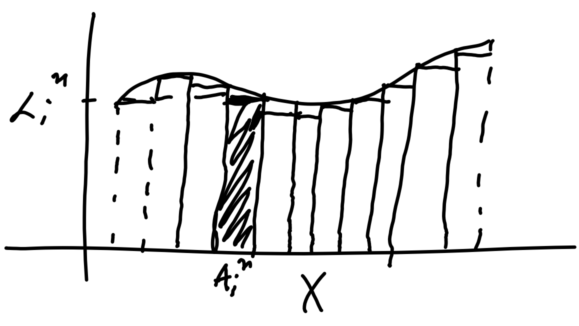
$\mu \otimes \lambda$ măsură completă

$S_f = \{(x,y) \in X \times [0,\infty) \mid y \leq f(x)\}$

Arătați S_f măsurabil în $X \times [0,\infty)$

și $(\mu \otimes \lambda)(S_f) = \int_X f \, d\mu$

Dem:



Fie $f_n : X \to [0,\infty)$ fct. simple

a.î. $f_n \nearrow f$ și $\int_X f_n \, d\mu \to \int_X f \, d\mu$

$$f_n = \sum_{i=1}^{m_n} \alpha_i^n \cdot \chi_{A_i^n} \qquad A_i^n \text{ măsurabilă}$$

$$S_{f_n} = \{(x,y) \in X \times [0,\infty) \mid y \le f_n(x)\}$$

$$= \bigcup_{i=1}^{m_n} \underbrace{A_i^n \times [0, \alpha_i^n]}_{\text{disjuncte, căci } A_i^n \cap A_j^n = \emptyset} \in \mathcal{A} \otimes \mathcal{B}([0,\infty))$$

$$(\mu \otimes \lambda)(S_{f_n}) = \sum_{i=1}^{m_n} \mu(A_i^n) \cdot \alpha_i^n = \int_X f_n \, d\mu$$

Arătăm că:

$$S_f = \bigcup_{n \ge 1} S_{f_n}$$

și

$$S_{f_n} \subseteq S_{f_{n+1}}, \ (\forall) n$$

Fie $(x, y) \in X \times [0, \infty)$

$f_n \leq f_{n+1}$ (ipoteză asupra lui $(f_n)_n$)

Deci $y \leq f_n(x) \Rightarrow y \leq f_{n+1}(x)$

Prin urmare, $S_{f_n} \subseteq S_{f_{n+1}}$

Demonstrăm $S_f \supseteq \bigcup_{n \geq 1} S_{f_n}$

Dem:

$f_n \leq f \quad , (\forall) n \Rightarrow$

$\Rightarrow$ Pt. $(x, y) \in X \times [0, \infty)$

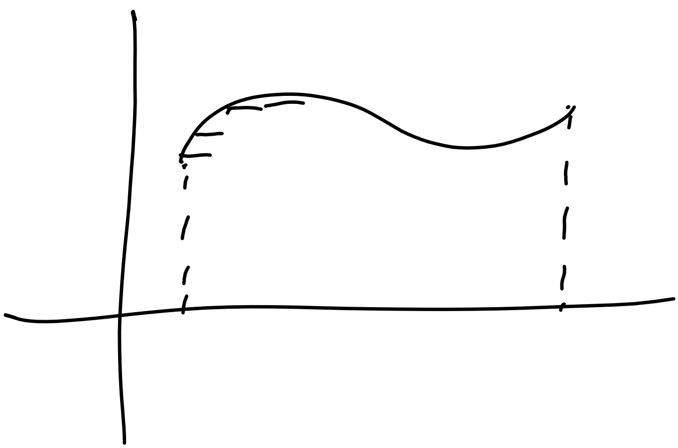
$y \leq f_n(x) \Rightarrow y \leq f(x)$

Prin urmare $S_{f_n} \subseteq S_f$, $(\forall) n$,

de unde $\bigcup_{n \geq 1} S_{f_n} \subseteq S_f$

Fie $c > 1$

Arătăm $S_f \subseteq \bigcup_{n \geq 1} S_{c \cdot f_n}$



Fie $(x, y) \in S_f$ adică $y \leq f(x)$

Presupun $(x, y) \notin S_{c \cdot f_n}$, $(\forall)\ n \geq 1$,

de unde $y > c \cdot f_n(x)$, $(\forall)\ n \geq 1$

$f_n(x) \xrightarrow{n \to \infty} f(x)$ $\Rightarrow$

$\Rightarrow y \geq c \cdot f(x)$

$$f(x) = y \geq c \cdot f(x) \Rightarrow \begin{cases} f(x) = 0 \\ y = 0 \end{cases}$$

Contradicție cu

$(x, y) \in S_{c \cdot f_1}$

Avem

$$\bigcup_{n \geq 1} S_{f_n} \subseteq S_f \subseteq \bigcup_{n \geq 1} S_{c \cdot f_n},$$

$(\forall)\ c > 1$

Ca mai sus, $(\mu \otimes \lambda)(S_{c \cdot f_n}) =$

$$= c \cdot \int_X f_n \, d\mu$$

și $S_{c \cdot f_n} \subseteq S_{c \cdot f_{n+1}}$

Din continuitatea în sus,

$$(\mu \otimes \lambda)\left(\bigcup_{n \geq 1} S_{f_n}\right) =$$

$$= \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

şi

$$(\mu \otimes \lambda)\left(\bigcup_{n \geq 1} S_{c \cdot f_n}\right) = c \cdot \int_X f \, d\mu$$

Fie $S = \bigcup_{n \geq 1} S_{f_n}$

$$S_c = \bigcup_{n \geq 1} S_{c \cdot f_n}$$

$$S \subseteq S_f \subseteq S_c \,, \ (\forall) c > 1$$

$$S \subseteq S_f \subseteq S_{1+\frac{1}{n}} \quad , (\forall)\ n \in \mathbb{N}$$

$$S_{1+\frac{1}{n}} \subseteq S_{1+\frac{1}{n-1}} \quad , (\forall)\ n$$

Deci $S' = \bigcap_{n \geq 1} S_{1+\frac{1}{n}}$ are propr.

$$(\mu \otimes \lambda)(S') = \lim_{n \to \infty} (\mu \otimes \lambda)\left(S_{1+\frac{1}{n}}\right)$$

$$= \lim_{n \to \infty} \left(1+\frac{1}{n}\right) \int_X f \, d\mu$$

$$= \int_X f \, d\mu$$

$$S \subseteq S_f \subseteq S'$$

$$(\mu \otimes \lambda)(S) = (\mu \otimes \lambda)(S') = \int_X f \, d\mu$$

$\mu \otimes \lambda$ completă $\Rightarrow S_f$ măsurabilă

$$\mu \otimes \lambda (S_f) = \int_X S_f$$

Nu cumva

$$S_f = \bigcap_{n \geq 1} S_{1+\frac{1}{n}} \quad ?$$

(5) Calcul $\displaystyle\int_{\mathbb{R}} e^{-x^2} dx$

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} d\lambda(x,y) \overset{TCM}{=\!=}$$

$$= \lim_{n \to \infty} \int_{[-n,n]\times[-n,n]} e^{-x^2} \cdot e^{-y^2} d\lambda(x,y) =$$

Tonelli

$$= \lim_{n\to\infty} \int_{[-n,n]} e^{-x^2} \cdot \int_{[-n,n]} e^{-y^2} \, d\lambda(y) \, d\lambda(x)$$

$$= \lim_{n\to\infty} \int_{[-n,n]} e^{-x^2} d\lambda(x) \cdot \int_{[-n,n]} e^{-y^2} d\lambda(y)$$

$$= \left(\lim_{n\to\infty} \int_{[-n,n]} e^{-x^2} d\lambda(x) \right)^2 =$$

$$= \left(\int_{\mathbb{R}} e^{-x^2} d\lambda(x) \right)^2 < \infty$$

Am obținut

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} \, d\lambda(x,y) =$$

$$= \left(\int_{\mathbb{R}} e^{-x^2} \, d\lambda(x) \right)^2 \quad (*)$$

Pe de altă parte

$$\int_{\mathbb{R}^2} e^{-x^2-y^2} \, d\lambda(x,y) =$$

$$\overset{TCM}{=} \lim_{n \to \infty} \int_{B_n(0)} e^{-x^2-y^2} \, d\lambda(x,y)$$

$$\int_{B_r(0)} e^{-x^2-y^2}\, d\lambda(x,y) \underset{\text{Riemann} \Rightarrow \text{Lebesgue}}{=\!=\!=}$$

$$= \iint_{B_r(0)} e^{-x^2-y^2}\, dx\, dy$$

$$= \int_0^r \int_0^{2\pi} e^{-r^2} \cdot r \, d\theta\, dr$$

$$= 2\pi \int_0^r e^{-r^2} \cdot r \, dr$$

$$= \pi \int_0^r -\left(e^{-r^2}\right)' dr = \pi \cdot \left(1 - e^{-r^2}\right)$$

$$\int_{B_n(0)} e^{-x^2-y^2}\, d\lambda(x,y) =$$

$$= \pi\left(1-e^{-n^2}\right)$$

Deci,

$$\int_{\mathbb{R}^2} e^{-x^2-y^2}\, d\lambda(x,y) =$$

$$= \lim_{n\to\infty} \pi\left(1-e^{-n^2}\right) = \pi .$$

Din $(*)$ $\Rightarrow$ $\int_{\mathbb{R}} e^{-x^2}\, d\lambda(x) = \sqrt{\pi}$

Demonstrație că

$$S_f = \bigcap_{m \geq 1} S_{1+\frac{1}{m}}$$

De mai sus, $S_f \subseteq S_{1+\frac{1}{m}}$, $(\forall)\, m$ ✓

Demonstrăm $\bigcap_{m \geq 1} S_{1+\frac{1}{m}} \subseteq S_f$

Fie $(x, y) \in \bigcap_{m \geq 1} S_{1+\frac{1}{m}}$

$$S_{1+\frac{1}{m}} = \bigcup_{n \geq 1} S_{\left(1+\frac{1}{m}\right) f_n}$$