Examen

I a) ~~m~~ teorema de convergență - enunț + dem.
b) estimarea erorii - enunț + dem.
c) Dem că ponderile oricărei cuadraturi Gauss de ordin $n \geq 0$ sunt nr reale > 0.

II a) $\varphi_i : [0,3] \to \mathbb{R}$, $i \in \overline{0,3}$

$$\varphi_0(x) = \begin{cases} 1-x, & x \in [0,1] \\ 0, & x \in [1,3] \end{cases}$$

$$\varphi_1(x) = \begin{cases} x, & x \in [0,1] \\ 2-x, & x \in [1,2] \\ 0, & x \in [2,3] \end{cases}$$

$$\varphi_2(x) = \begin{cases} 0, & x \in [0,1] \\ x-1, & x \in [1,2] \\ 3-x, & x \in [2,3] \end{cases}$$

$$\varphi_3(x) = \begin{cases} 0, & x \in [0,2] \\ x-2, & x \in [2,3] \end{cases}$$

Arăt că $\varphi_i\big|_{[i,i+1]}$ și $\varphi_{i+1}\big|_{[i,i+1]}$ polinoame liniare (de grad I) de bază Lagrange pe int. $[i, i+1]$, $i = \overline{0,2}$

b) Arăt că mulțimea polinoamelor continue liniare pe porțiuni pe intervalul $[0,3]$, asociate partiției $\{[i,i+1]\}_{i=\overline{0,2}}$

$$X_h^1 := \{ f \in C([0,3]) \mid f|_{[i,i+1]} \in P_1([i,i+1]),\ i=\overline{0,2} \}$$

este un sp. liniar peste $\mathbb{R}$, iar mulțimea $\{ \varphi_i \mid i=\overline{0,3} \}$ este o bază pt X_h^1.

c) P.p că afirmația de la b) este adev., det. cea mai bună aprox. din X_h^1, în norma $\|\cdot\|_{L^2(0,3)}$, a fct.

$$f:[0,3] \to \mathbb{R},\quad f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \in (1,3] \end{cases}.$$

③ c) Fie $g \in X_h^1$ cea mai bună aproximare a lui f în $L^2([0,3])$.

$\Rightarrow E = \|f-g\|$ este minimă.

$$E = \|f-g\| = \int_0^3 (f-g)^2 = \int_0^3 f^2 - 2fg + g^2 =$$

$$= \underbrace{\int_0^3 f^2}_{A} - \underbrace{2\int_0^3 fg}_{B} + \underbrace{\int_0^3 g^2}_{C} = A - B + C$$

$$\bullet\ A = \int_0^3 f^2 dx = \int_0^1 1\, dx + \int_1^3 0\, dx = 1$$

$g \in X_h^1$ care are baza $\{\varphi_i\}_{i=\overline{0,3}} \Rightarrow \exists\ a, b, c, d \in \mathbb{R}$ a.î.

$$g = a\varphi_0 + b\varphi_1 + c\varphi_2 + d\varphi_3$$

$$\bullet\ B = 2\int_0^3 f(a\varphi_0 + b\varphi_1 + c\varphi_2 + d\varphi_3)dx =$$

$$= 2\left(\int_0^1 f(\ldots)dx + \int_1^3 f(\ldots)dx\right)$$

Dar $f \equiv 0$ pe $[1;3] \Rightarrow B = 2\left(a\int_0^1 f\varphi_0 + b\int_0^1 f\varphi_1 + c\underbrace{\int_0^1 f\varphi_2}_{0} + d\underbrace{\int_0^1 f\varphi_3}_{0}\right) = 2\left(a\int_0^1 f\varphi_0 + b\int_0^1 f\varphi_1\right)$$

①

II a) Pt $i=0$, $a=0, b=1$

$L_{1,1} = \frac{x-0}{1-0} = x = \varphi_1(x)/_{[0;1]}$

$L_{1,0} = \frac{x-1}{0-1} = 1-x = \varphi_0/_{[0;1]}$

Pt $i=1$, $a=1, b=2$

$L_{1,0} = \frac{x-2}{1-2} = 2-x = \varphi_1/_{[1;2]}$

$L_{1,1} = \frac{x-1}{2-1} = x-1 = \varphi_2/_{[1;2]}$

Pt $i=2$, $a=2, b=3$

$L_{1,0} = \frac{x-3}{2-3} = 3-x = \varphi_2/_{[2;3]}$

$L_{1,1} = \frac{x-2}{3-2} = x-2 = \varphi_3/_{[2;3]}$

$\Rightarrow \varphi_i/_{[i;i+1]}, \varphi_{i+1}/_{[i;i+1]}$ sunt polin de bază Lag. pe $[i;i+1]$, pe porțiuni

③

$$B = 2\left(a\int_0^1 1\cdot(1-x) + b\int_0^1 1\cdot x\right) =$$

$$= 2\left(a\left(x-\frac{x^2}{2}\right)\Big|_0^1 + b\left(\frac{x^2}{2}\right)\Big|_0^1\right) =$$

$$= 2\left(a\left(1-\frac{1}{2}\right) + b\cdot\frac{1}{2}\right) = a + b$$

$$\bullet\ C = \int_0^3 (a\varphi_0 + b\varphi_1 + c\varphi_2 + d\varphi_3)^2 = \int_0^1 (a\varphi_0 + b\varphi_1 + c\varphi_2 + d\varphi_3)^2 +$$

$$+ \int_1^2 (a\varphi_0 + b\varphi_1 + c\varphi_2 + d\varphi_3)^2 + \int_2^3 (a\varphi_0 + b\varphi_1 + c\varphi_2 + d\varphi_3)^2 =$$

$$= \int_0^1 (a\varphi_0 + b\varphi_1)^2 + \int_1^2 (b\varphi_1 + c\varphi_2)^2 + \int_2^3 (c\varphi_2 + d\varphi_3)^2 =$$

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$$= \int_0^1 (a - xa + bx)^2 + \int_1^2 (x\cdot 2b - bx + cx - c)^2 +$$

$$+ \int_2^3 (3c - cx + dx - 2d)^2 =$$

$$= \int_0^1 (a + x(b-a))^2 + \int_1^2 (x(c-b) + 2b - c)^2 +$$

$$+ \int_2^3 ((3c-2d) + (d-c)x)^2 =$$

$$= \int_0^1 a^2 + 2a(b-a)x + x^2(b-a)^2 +$$

$$+ \int_1^2 x^2(c-b)^2 + 2x(c-b)(2b-c) + (2b-c)^2 +$$

$$+ \int_2^3 ((3c-2d)^2 + x(3c-2d)(d-c) + (d-c)^2 x^2 =$$

$$= \left(a^2 x + a(b-a)x^2 + \frac{x^3}{3}(b-a)^2\right)\Big|_0^1 +$$

$$+ \left(\frac{x^3}{3}(c-b)^2 + x^2(c-b)(2b-c) + x(2b-c)^2\right)\Big|_1^2 +$$

$$+ \left((3c-2d)^2 x + \frac{x^2}{2}(3c-2d)(d-c) + \frac{x^3}{3}(d-c)^2\right)\Big|_2^3 =$$

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$$= a^2 + a(b-a) + \frac{(b-a)^2}{3} + (2b-c)^2 +$$

$$+ 3(c-b)(2b-c) + \frac{7}{3}(c-b)^2 + 3(c-b)(2b-c) + \frac{19}{3}(d-c)^2 =$$

$$+ (3c-2d)^2 + \frac{5}{2}(3c-2d)(d-c) +$$

$$= \frac{1}{3}a^2 + \frac{1}{3}b^2 + \frac{1}{3}ab +$$

$$+ 6cb - 3c^2 - 6b^2 + 3bc +$$

$$+ \frac{7}{3}c^2 - \frac{14}{3}cb + \frac{7}{3}b^2 +$$

$$+ 4b^2 - 4bc + c^2 +$$

$$+ 9c^2 - 12cd + 4d^2 + \frac{15}{2}cd - \frac{15}{2}c^2 - 5d^2 + 5dc +$$

$$+ \frac{19}{3}d^2 - \frac{38}{3}dc + \frac{19}{3}c^2 =$$

~~$E = \frac{1}{3}a^2 + \frac{1}{3}$ A+B+C~~

$$E = A - B + C =$$

$$= 1 + a + \frac{1}{3}a^2 + \frac{1}{3}ab + b + \frac{8}{3}b^2 \ldots$$