iam 2020

Problema 1 $\quad f: \mathbb{R} \to \mathbb{R},\ f(x) = |x-1|$

1) Arăt. că $f \in H^1(0,2)$ și arg că $f \notin H_0^1(0,2)$

$H =$ spațiul

$W^{1,2} = \{ u: \mathbb{R} \to \mathbb{R},$ u admite derivate slabe de ord 1 și $\in L^2(\mathbb{R})$

$$\boxed{L^p = \left\{ f: \mathbb{R} \to \mathbb{R} \ \middle|\ \int_{-\infty}^{\infty} (f(x))^p \, dx < \infty \right\}}$$

$$\int_{\mathbb{R}} |x-1| \, \varphi' \, dx = \int_0^1 (1-x) \, \varphi' \, dx + \int_1^2 (x-1) \, \varphi' \, dx$$

3) Folos. formula op Laplacian Δ pt fct cu simetr. rad din R^5, găs $\lambda \in R$ a.?.

$$\Delta(x \cdot \nabla u) = \lambda \frac{4}{|x|^2}, \quad \forall x \in B_1(0) \setminus \{0\}$$

$$\nabla u = \lambda x |x|^{\lambda - 2} \qquad \lambda = -\frac{3}{7}$$

$$= -\frac{3}{7} x |x|^{-\frac{17}{7}}$$

$$x \cdot \nabla u = -\frac{3}{7} x^2 \cdot |x|^{-\frac{17}{7}} = -\frac{3}{7} |x|^{2-\frac{17}{7}} = -\frac{3}{7} |x|^{-\frac{3}{7}}$$

Fie $f(x) = x \cdot \nabla u = g(|x|) = g(r) \qquad r := |x|$

$\boxed{u(x) = u(y) \Rightarrow |x| = |y|} \Rightarrow u$ e fct radială

f este funct radială $\Rightarrow \Delta f = g''(r) + \frac{5-1}{r} g'(r)$

$$g(r) = -\frac{3}{7} r^{-\frac{3}{7}}$$

$$g'(r) = \frac{9}{49} r^{-\frac{10}{7}} \qquad \Delta(3f) = 3 \cdot \Delta f$$

$$g''(r) = -\frac{90}{343} r^{-\frac{17}{7}}$$

$$\Delta f = \frac{-90}{343} r^{-\frac{17}{7}} + \frac{4}{r} \cdot \frac{9}{49} r^{-\frac{10}{7}}$$

$$\Delta f = \frac{9}{49} r^{-\frac{17}{7}} \left(-\frac{10}{7} + 4\right)$$

$$\Delta f = \frac{9}{49} r^{-\frac{17}{7}} \cdot \frac{18}{7}$$

$$\Delta f = \frac{162}{343} r^{-\frac{17}{7}} = \lambda \frac{r^{-\frac{3}{7}}}{r^2} \Rightarrow \lambda = \frac{162}{343}$$

4) Să se det pt ce val $p \geq 1$ are loc $u \in \mathcal{L}^1(R^5 \setminus \overline{B_1(0)})$

$$\int_{R^5 \setminus \overline{B_1(0)}} |u|^p dx < \infty$$

$$= \int_1^\infty \left(\int_{\partial B_s(0)} |u|^p(t) \, d\sigma(t) \right) ds = \int_1^\infty s^{-\frac{3}{7}p} \cdot |\partial B_s(0)| \, ds$$

$$= \omega_5 \cdot \int_1^\infty s^{4 - \frac{3}{7}p} \, ds < \infty \Rightarrow 5 - \frac{3}{7} p < 0$$

$$\Rightarrow p > \frac{35}{3}$$

5) Pt ce valori $p \geq 1$, $v: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$, $v(x) := |x|^{-\frac{3}{5}p} \underbrace{\sin(|x_3|)}_{\vee}$ apart. $L^1(B_1(0))$, unde $B_1(0)$ bila unit $\mathbb{R}^3$.

$$\int_{B_1(0)} v(x)\,dx = \int_0^1 \left(\int_{\partial B_s(0)} v(t)\,d\sigma(t) \right) ds$$

$$\int_{B_1(0)} |x|^{-\frac{3}{5}} \sin(|x_3|)\,dx \sim \int_{B_1(0)} \underbrace{|x|^{-\frac{3}{5}p}}_{w}\,dx = \int_0^1 \left(\int_{\partial B_s(0)} w(t)\,d\sigma(t) \right) ds$$

$$= \int_0^1 s^{-\frac{3}{5}p} \cdot \underbrace{|\partial B_s(0)|}_{s^{3-1}\cdot \omega_3}\,ds$$

$$= \omega_3 \cdot \int_0^1 s^{2-\frac{3}{5}p}\,ds < \infty \Rightarrow 3 - \frac{3}{5}p > 0$$

$$\Rightarrow \frac{15-3p}{5} > 0 \Rightarrow p < 5$$

6) Arăt că fct. $z: \mathbb{R}^5 \setminus \{0\} \to \mathbb{R}$, $z(x) := x_5 |x|^{-5}$ este armonică.

$$\Delta z = 0 \Rightarrow z \text{ armonică}$$

$$\Delta z = z_{x_1x_1} + z_{x_2x_2} + \dots + z_{x_5x_5} \qquad \frac{\partial}{\partial x_i}(|x|^\lambda) = \lambda x_i |x|^{\lambda-2}$$

$$\Delta\left(\frac{x_5}{|x|^5}\right) = -\frac{x_5\left(-2x_5^2 + \left(\sum_{i=1}^{4} x_i^2\right)\right)}{|x|^{\frac{5}{2}}} + \frac{-3\left(\sum_{i=1}^{4} x_i^2\right)x_5}{|x|^{\frac{5}{2}}}$$

$$= 0 \Rightarrow z \text{ armonică}$$

Problema 2

$$\Omega := \{(x,y) \in \mathbb{R}^2,\ x^2+y^2<4\}$$

$$\begin{cases} -\Delta u(x,y) = 4|x| & (x,y) \in \Omega \\ u(x,y) = 0 & (x,y) \in \Omega \end{cases}$$

1) Arăt. că pb. are cel mult o sol $u \in C^2(\Omega) \cap C(\bar{\Omega})$

2)

Se rezolvă la fel ca ex 2 anterior ♡

3) Găs. constanta C a.î. $u(x,y) = C(x^2+y^2)$ ...

$-\Delta u = 8$ în Ω

4) Exact la fel toată problema. :) Prin urmare, vom rezolva altă pb. 2 din alt examen.

Se cons. pb la limită:

$$\begin{cases} u_{xx}(x,y) + 2u_{yy}(x,y) = 0 & (x,y) \in (0,1)\times(0,1) \\ u(x,0) = u(x,1) = 0 \\ u(0,y) = \sin(2\pi y), \quad u(1,y) = e^{-2\sqrt{2}\pi}\sin(2\pi y) & y \in (0,1) \end{cases}$$

1) Det sol pb. căut. în var sep $u(x,y) = A(x)B(y)$

$u_{xx}(x,y) = A''(x)\cdot B(y)$

$2u_{yy}(x,y) = 2A(x)\cdot B''(y)$

$\Rightarrow A''(x)\cdot B(y) + 2A(x)\cdot B''(y) = 0 \Rightarrow \dfrac{A''(x)}{2A(x)} = -\dfrac{B''(y)}{B(y)} = 4\pi^2$ ⊛

$u(0,y) = A(0)B(y) = \sin(2\pi y) \Leftrightarrow B(y) = \dfrac{\sin(2\pi y)}{A(0)}$

$\Leftrightarrow B''(y) = \dfrac{-4\pi^2\sin(2\pi y)}{A(0)} = -4\pi^2 B(y) \Leftrightarrow \dfrac{B''(y)}{B(y)} = -4\pi^2$

⊛ $\Rightarrow A''(x) - 8\pi^2 A(x) = 0$

$(\lambda^2 - 8\pi^2 = 0 \Rightarrow \lambda_{1,2} = \pm 2\sqrt{2}\pi)$

$u(x,0) = u(x,1) = A(x)B(0) = A(x)B(1) = 0$

$A(x) = C_1\cdot\cos(2\sqrt{2}\pi x) + C_2\cdot\sin(2\sqrt{2}\pi x)$

$A(0) = C_1\cdot\cos(0) + C_2(\sin(0))$. $\quad |A(x) = C_1e^{i\lambda_1} + C_2e^{i\lambda_2}$

$A(0) = C_1$

$A(1) = C_1\cdot\cos(2\sqrt{2}\pi) + C_2\cdot\sin(2\sqrt{2}\pi)$

$A(1)\cdot B(y) = e^{-2\sqrt{2}\pi}\sin(2\pi y)$

$A(0)B(y) = \sin(2\pi y)$

$B(y) = \dfrac{\sin(2\pi y)}{C_1} = \dfrac{e^{-2\sqrt{2}\pi}\sin(2\pi y)}{C_1\cos(2\sqrt{2}\pi) + C_2\sin(2\sqrt{2}\pi)}$

$$\Rightarrow c_1 \cdot e^{-2\sqrt{2}\pi} = c_1 \cos(2\sqrt{2}\pi) + c_2 \sin(2\sqrt{2}\pi)$$

$$c_1\left(e^{-2\sqrt{2}\pi} - \cos(2\sqrt{2}\pi)\right) = c_2 \sin(2\sqrt{2}\pi)$$

$$\boxed{e^{it} = \cos(t) + i\sin(t)}$$