Factorizarea Cholesky

Să se verifice dacă matricea

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4,25 & 2,75 \\ 1 & 2,75 & 3,5 \end{bmatrix}$$

admite factorizarea Cholesky și, în caz afirmativ, să se determine această factorizare.

- Verificăm criteriul lui Sylvester:

$A^{(1)} = 4 \Rightarrow \det A^{(1)} = 4 > 0$

$$A^{(2)} = \begin{bmatrix} 4 & -1 \\ -1 & 4,25 \end{bmatrix} \Rightarrow \begin{cases} A^{(2)} = (A^{(2)})^T \\ \det A^{(2)} = 16 > 0 \end{cases}$$

$$A^{(3)} = A \Rightarrow \begin{cases} A^{(3)} = (A^{(3)})^T \\ \det A^{(3)} > 0 \end{cases}$$

Cf. criteriului lui Sylvester, A admite factorizarea Cholesky.

- Factorizarea Cholesky:

$$A = \left[\begin{array}{c|cc} 4 & -1 & 1 \\ \hline -1 & 4{,}25 & 2{,}75 \\ 1 & 2{,}75 & 3{,}50 \end{array}\right] = \begin{bmatrix} a_{11} & \underline{A}_{21}^T \\ \underline{A}_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & \underline{0} \\ \underline{L}_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & \underline{L}_{21}^T \\ \underline{0} & L_{22}^T \end{bmatrix} = LL^T \Rightarrow$$

$$\begin{cases} l_{11}^2 = a_{11} \\ l_{11}\underline{L}_{21}^T = \underline{A}_{21}^T \\ \underline{L}_{21} l_{11} = \underline{A}_{21} \\ \underline{L}_{21}\underline{L}_{21}^T + L_{22}L_{22}^T = A_{22} \end{cases} \Rightarrow$$

- $\left.\begin{array}{l} l_{11}^2 = 4 \Rightarrow l_{11} = \pm 2 \\ l_{11} > 0 \end{array}\right\} \Rightarrow \boxed{l_{11} = 2}$

- $2\underline{L}_{21} = \underline{A}_{21} \Rightarrow 2\underline{L}_{21} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\begin{array}{l} l_{21} = -1/2 \\ l_{31} = 1/2 \end{array}}$

$$L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T$$

$$= \begin{bmatrix} 4,25 & 2,75 \\ 2,75 & 3,50 \end{bmatrix} - \begin{bmatrix} -0,5 \\ 0,5 \end{bmatrix} \begin{bmatrix} -0,5 & 0,5 \end{bmatrix}$$

$$= \begin{bmatrix} 4,25 & 2,75 \\ 2,75 & 3,50 \end{bmatrix} - \begin{bmatrix} 0,25 & -0,25 \\ -0,25 & 0,25 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 3,25 \end{bmatrix}$$

Problema diviene:

$$L_{22} L_{22}^T = \begin{bmatrix} l_{22} & 0 \\ l_{23} & l_{33} \end{bmatrix} \begin{bmatrix} l_{22} & l_{23} \\ 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 3,25 \end{bmatrix}$$

$$\begin{cases} l_{22}^2 = 4 \\ l_{22} l_{23} = 3 \\ l_{23} l_{22} = 3 \\ l_{23}^2 + l_{33}^2 = 3,25 \end{cases} \qquad \left. \begin{array}{l} \Rightarrow l_{22} = \pm 2 \\ l_{22} > 0 \end{array} \right\} \Rightarrow \boxed{l_{22} = 2}$$

- $l_{22} l_{23} = 3 \Rightarrow 2 l_{23} = 3 \Rightarrow \boxed{l_{23} = 1,5}$
- $l_{23}^2 + l_{33}^2 = 3,25 \Rightarrow l_{33}^2 = 3,25 - (1,5)^2$

$$= 3{,}25 - 2{,}25 = 1 \Rightarrow \left.\begin{array}{l} l_{33} = \pm 1 \\ l_{33} > 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{l_{33} = 1}$$

Am obținut:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -0{,}5 & 2 & 0 \\ 0{,}5 & 1{,}5 & 1 \end{bmatrix}$$

<u>Verificare</u>:

$$LL^T = \begin{bmatrix} 2 & 0 & 0 \\ -0{,}5 & 2 & 0 \\ 0{,}5 & 1{,}5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -0{,}5 & 0{,}5 \\ 0 & 2 & 1{,}5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4{,}25 & 2{,}75 \\ 1 & 2{,}75 & 3{,}50 \end{bmatrix} = A \ \checkmark$$