~~VINERI~~ : 15:00.

Consultații statistică 1.

Variabile aleatoare: $\begin{cases} \text{discrete} \to \text{f. de masă (dă probabilitatea într-un punct)} \\ \text{continue} \to \text{f. de repartiție} \end{cases}$

I. Cazul discret:

$$X: \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \qquad P(X=x_n)=p_n.$$

Funcție de masă:

$$\boxed{f(x)=\begin{cases} p_i; & x=x_i \\ 0; & \text{în rest} \end{cases}}$$

Cum știm dacă X este discret sau continuu:

$f(x)=\{\dots;\ x\in\mathbb{N}$ (discretă, deoarece $\mathbb{N}$ numărabilă)

$f(x)=\begin{cases} \dots; & x\in[0;2] \\ 0; & \text{în rest} \end{cases}$ (continuă)

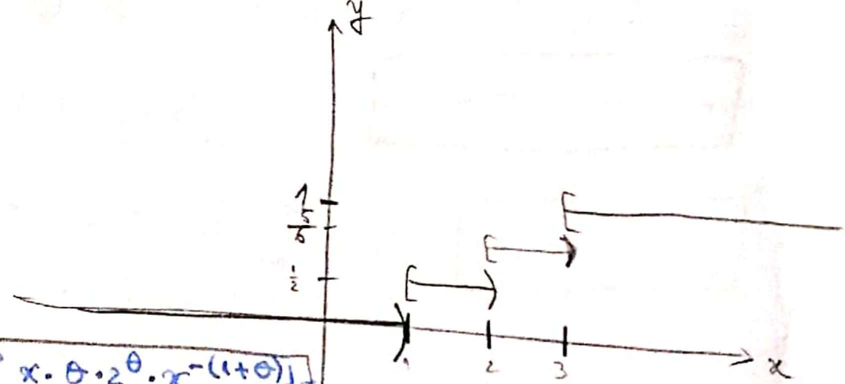
F. de repartiție: $F:\mathbb{R}\to\mathbb{R}$, $\boxed{F(x)=\mathbb{P}(X\le x)}$ $\to$ f. crescătoare. $\operatorname{Im} f=[0;1]$.

1. $X: \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$ $\qquad f(2)=\frac{1}{3}$

$F(2)=\mathbb{P}(X\le 2)=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}.$

$$\boxed{F(x)=\sum_{x_i\le x} f(x_i)}$$

$$F(x)=\begin{cases} 0; & x<1 \\ \frac{1}{2}; & x\in[1;2) \\ \frac{1}{2}+\frac{1}{3}=\frac{5}{6}; & x\in[2;3) \\ \frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1; & x\in[3;\infty) \end{cases}$$



f în trepte.

Ultimul ex: $E(X)=\int_{-\infty}^{\infty} x\, f(x)\, dx=\int_2^{\infty} x\cdot\theta\cdot 2^{\theta}\cdot x^{-(1+\theta)}dx$

$=\theta\cdot 2^{\theta}\cdot\int_2^{\infty} x^{-\theta}dx=\theta\cdot 2^{\theta}\cdot\frac{x^{1-\theta}}{1-\theta}\Big|_2^{\infty}=$

$=\theta\cdot 2^{\theta}\cdot\frac{1}{1-\theta}\left(\lim_{x\to\infty} x^{1-\theta}-2^{1-\theta}\right)$

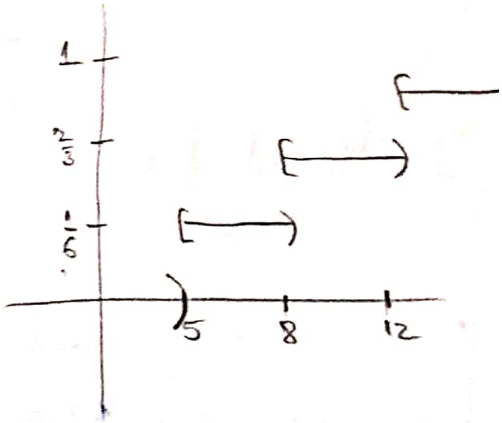
$=\theta\cdot 2^{\theta}\cdot\frac{1}{1-\theta}\cdot\frac{2^{1-\theta}}{1-\theta}=\frac{-2\theta}{1-\theta}$

MM: $E(X)=\overline{X}\Rightarrow -\frac{2\theta}{1-\theta}=\overline{X}$

Cum aflăm variabila X dacă avem gf F(x)?

- salturile graficului sunt chiar valorile funcției.

$$\Rightarrow X : \begin{pmatrix} 5 & 8 & 12 \\ \frac{1}{6} & \frac{3}{6} & \frac{2}{6} \end{pmatrix}$$
$$\left(\frac{2}{3}-\frac{1}{6}\right) \quad \left(1-\frac{2}{3}\right)$$



Media și dispersia:

$$\boxed{E(X) = \sum_{i=1}^{n} x_i \cdot p_i} \qquad \boxed{Var(X) = E(X^2) - E(X)^2}$$

Fie g funcție continuă: Vreau media $\boxed{E(g(x)) = \sum_{i=1}^{n} g(x_i) \cdot p_i}$

Ex:
$$E(x^2) = \sum_{i=1}^{n} x_i^2 \cdot p_i$$

Problema 1)

$$f(x) = \begin{cases} \frac{K}{2^{n+2}} ; & x = x_n \\ 0 ; & \text{în rest} \end{cases} \quad ; n \in \mathbb{N} ; n \geq 1.$$

Determinați K a.î. f să fie funcție de masă.

SOL: Începem cu proprietățile funcției de masă:

- $\boxed{f(x) \geq 0 ; \forall x \in \mathbb{R}}$ $\Leftrightarrow \frac{K}{2^{n+2}} \geq 0 \Rightarrow K > 0$ (nu poate fi ≥ 0 deoarece ar fi funcția identic nulă)

- $\boxed{\sum_{x_n} f(x) = 1.}$ $\Leftrightarrow \sum_{n=1}^{\infty} \frac{K}{2^{n+2}} = 1 \Leftrightarrow K \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} = 1 \Leftrightarrow$

scriem suma după n deoarece era după x_n

Seria geometrică: $\boxed{r \in (0;1) ; \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}}$

$$\Leftrightarrow K \cdot \frac{1}{2^2} \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \Leftrightarrow \frac{K}{4} \cdot \left(\sum_{n=0}^{\infty} \frac{1}{2^n} - \frac{1}{2^0} \right) = 1. \Leftrightarrow \frac{K}{4} \cdot \left(\frac{1}{1-\frac{1}{2}} - 1 \right) = 1$$

$$\Leftrightarrow \frac{K}{4} \cdot 1 = 1 \Rightarrow K = 4 \geq 0.$$

2) $\int_{-\infty}^{\infty} f(x)dx = 1 \iff \underbrace{\int_{-\infty}^{-\frac{1}{2}} f(x)dx}_{=0} + \int_{-\frac{1}{2}}^{0} f(x)dx + \int_{0}^{\frac{1}{2}} f(x)dx + \underbrace{\int_{\frac{1}{2}}^{\infty} f(x)dx}_{=0} = 1.$

$\iff \int_{-\frac{1}{2}}^{0} 1-\theta \, dx + \int_{0}^{\frac{1}{2}} 1+\theta \, dx = 1 \iff (1-\theta)x\Big|_{-\frac{1}{2}}^{0} + (1+\theta)x\Big|_{0}^{\frac{1}{2}} = 1.$

$\iff (1-\theta)\cdot\frac{1}{2} + (1+\theta)\cdot\frac{1}{2} = 1 \iff \frac{1}{2} - \frac{\theta}{2} + \frac{1}{2} + \frac{\theta}{2} = 1 \iff 1 = 1. \Rightarrow \theta \in \mathbb{R}.$

$\Rightarrow \theta \in [-1;1]$

Media pe caz continuu:

$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$ $\qquad \mathbb{E}(g(x)) = \int_{-\infty}^{\infty} g(x)\cdot f(x)dx$

Dispersia pe caz continuu: $Var(X) = E(X^2) - E(X)^2$

$F(x) = \int_{-\infty}^{x} f(t)\,dt \iff F'(x) = f(x)$

Ex examen (feb. 2018)

$f_\theta(x) = \frac{1}{\theta}\cdot e^{-\frac{x-\theta}{\theta}} \cdot \mathbb{1}_{(\theta;\infty)}(x)$ → densitate; $\theta \geq 0$ param. cunoscut.

când avem θ la f. indicator nu merge MVM ⇒ încercăm M.M.

Int. Gamma
$\Gamma(a) = \int_0^\infty x^{a-1}\cdot e^{-x}dx$

$E(X) = \int_\theta^\infty x\cdot\frac{1}{\theta}\cdot e^{-\frac{x-\theta}{\theta}} dx$

$\Rightarrow x = \theta + \theta t$

S.V.: $\left.\begin{array}{l} \frac{x-\theta}{\theta} = t \Rightarrow dx = \theta dt \\ x = \theta \Rightarrow t = 0 \\ x\to\infty \Rightarrow t\to\infty. \end{array}\right\} \Rightarrow$

$\Rightarrow \int_\theta^\infty x\cdot\frac{1}{\theta}\cdot e^{-\frac{x-\theta}{\theta}}dx = \int_0^\infty (\theta+\theta t)\cdot e^{-t}dt = \int_0^\infty \theta e^{-t}dt + \int_0^\infty \theta t\cdot e^{-t}dt$

$= \theta\int_0^\infty t^0 e^{-t}dt + \theta\int_0^\infty t\cdot e^{-t}dt = \theta\cdot\underbrace{\Gamma(1)}_{=1} + \theta\,\underbrace{\Gamma(2)}_{=(n-1)!} = \theta+\theta = 2\theta$

MM: $E(X) = \overline{X}$

$\iff 2\theta = \overline{X} \Rightarrow \theta = \frac{\overline{X}}{2}$

a) det. a; b) θ prin M.M.

SOL.:

a) f. densitate de probabilitate ⇒ $f(x) \geq 0;\ \forall x$; $\int_{-\infty}^{\infty} f(x) = 1.$

$f(x) \geq 0 \iff a\cdot x^{-(1+\theta)} \geq 0,\ \forall x>2 \iff \boxed{a>0}$

Ex: 2016/2017.

$f_\theta(x) = \begin{cases} a\cdot x^{-(1+\theta)}; & x>2 \\ 0; & \text{în rest} \end{cases}$

$\theta > 1$

$\int_{-\infty}^{\infty} f(x) = 0 \Rightarrow \int_2^\infty a\cdot x^{-(1+\theta)}dx = a\int_2^\infty x^{-(1+\theta)}dx =$

$= a\cdot\frac{x^{-(1+\theta)+1}}{-(1+\theta)+1}\Big|_2^\infty = a\cdot\frac{x^{-\theta}}{-\theta}\Big|_2^\infty = \frac{a}{\theta}\cdot(\underbrace{\lim_{x\to\infty} x^{-\theta}}_{0} - 2^{-\theta})$

$= \frac{a}{\theta}\left(0 - \frac{1}{2^\theta}\right) = \frac{a}{\theta}\cdot\frac{1}{2^\theta} = \frac{a}{\theta\cdot 2^\theta} = 1 \Rightarrow \boxed{a = \theta\cdot 2^\theta}$

2) Calculați media și dispersia variabilei aleatoare: $X : \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$.

Sol.:

$\mathbb{E}(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{6} = \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = \frac{5}{3}$.

$\mathbb{E}(X^2) = 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} + 9 \cdot \frac{1}{6} = \frac{1}{2} + \frac{4}{3} + \frac{3}{2} = \frac{10}{3}$.

$Var(X) = \frac{10}{3} - \frac{25}{9} = \frac{5}{9}$

Proprietățile mediei și ale dispersiei:

$\mathbb{E}(15X - 9) = 15 \cdot \mathbb{E}(X) - \underbrace{\mathbb{E}(9)}_{=9} = 15 \cdot \frac{5}{3} - 9 = 25 - 9 = 16$.

$Var(8X - 3) = 8^2 Var(X) - \underbrace{Var(3)}_{=0} = 64\, Var(X) - 0 = 64 \cdot \frac{5}{9} = \frac{320}{9}$

Variabile aleatoare continue:
- f. de densitate de probabilitate
- f. de repartiție
- repartiție.

$f : \mathbb{R} \to \mathbb{R}$ funcție densitate de probabilitate dacă:
1) $f(x) \geq 0 \;;\; \forall x$
2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\boxed{\int_a^b f(x)\,dx = P(a < X \leq b)}$$

Examen 2023 - feb.:

2) $f : \mathbb{R} \to \mathbb{R};\; f_\theta(x) = \begin{cases} 1 - \theta; & -\frac{1}{2} < x \leq 0 \\ 1 + \theta; & 0 < x \leq \frac{1}{2} \\ 0; & \text{altfel.} \end{cases}$

Sol: 1) $f(x) \geq 0 \iff \begin{cases} 1 - \theta \geq 0; & \forall x \in (-\frac{1}{2}; 0] \\ 1 + \theta \geq 0; & \forall x \in (0; \frac{1}{2}] \end{cases} \iff \begin{cases} \theta \leq 1; & \forall x \in (-\frac{1}{2}; 0] \\ \theta \geq -1; & \forall x \in (0; \frac{1}{2}] \end{cases} \iff$

$\iff \boxed{\theta \in [-1; 1]}$