Data: 26 Iunie 2023
Timp de lucru: 2h 30m
Punctaj total: 90p + 10p oficiu

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grupa 212

Examen Analiză complexă

Subiecte:

1. (a) (5 p) Scrieti seria Taylor în 0 pentru funcţia $f(z) = \sin(z^2) - ze^z$.
   (b) (5 p) Determinaţi $a, b \in \mathbb{R}$ astfel încât funcţia $f(x + iy) = x^2 + ay^2 + ibxy$ să fie olomorfă pe $\mathbb{C}$.
   (c) (5 p) Daţi exemplu de doua funcţii olomorfe $f, g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ cu pol în 0, astfel încât $\operatorname{res}(f, 0)\operatorname{res}(g, 0) = \operatorname{res}(fg, 0) = 1$.
   (d) (5 p) Determinaţi $\max_{|z|\leq 1} |z^2 - z|$ şi $\min_{|z|\leq 1} |z^2 - z|$.
   (e) (5 p) Calculaţi $\int_A (z + \bar{z}^2)dz$, unde A este semicercul de rază 1 din semiplanul superior, orientat în sens trigonometric.

2. (a) (10 p) Determinaţi numărul zerourilor funcţiei $f(z) = 3e^z - z$ din $|z| \leq 1$ (presupunem cunoscuta inegalitatea $e < 3$).
   (b) (10 p) Considerăm funcţia
   $$f(z) = \frac{z^2}{\sinh z},$$
   unde $\sinh z = \frac{e^z - e^{-z}}{2}$. Determinaţi polii, ordinul acestora şi reziduul funcţiei f în aceşti poli.

3. (20 p) Considerăm numerele $a, b > 0$, $a \neq b$. Folosind teorema reziduurilor pentru funcţia $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$, demonstraţi că
   $$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

4. (a) (10 p) Reprezentati grafic domeniul $\Omega = \{x + i \cdot y \mid x > 0, y \in \mathbb{R}\} \setminus \{x + i \cdot 0 \mid x \in [0, 1]\}$. Considerăm funcţia $h(z) = \frac{z-1}{z+1}$. Demonstraţi că
   $$h(\Omega) = \{x + i \cdot y \mid x^2 + y^2 < 1\} \setminus \{x + i \cdot 0 \mid x \in [-1, 0]\}.$$
   (b) (5 p) Determinaţi o aplicaţie biolomorfă între $h(\Omega)$ şi semidiscul
   $$\{x + i \cdot y \mid x^2 + y^2 < 1, y > 0\}.$$

5. (10 p) Demonstraţi că dacă $c > 0$, $c \neq 1$, şi $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$, atunci
   $$\left\{z \in \mathbb{C} : \left|\frac{z - z_1}{z - z_2}\right| = c\right\}$$
   reprezintă un cerc.

   Hint: Putem folosi proprietăţile transformărilor omografice.

Iunie 2023

Examen Mate-Info

1. (a) Scrieți seria Taylor în 0 pt. funcția

$$f(z) = \sin(z^2) - ze^z$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!}$$

(b) $a, b \in \mathbb{R}$ a.î. $f(x+iy) = x^2 + ay^2 + ibxy$ olomorfă

$\boxed{\text{C-R}}$ $u = x^2 + ay^2$ $\quad v = bxy$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Rightarrow \end{cases}$$

$$\begin{cases} 2x = bx \Rightarrow \boxed{b=2} \\ 2ay = -by \end{cases}$$

$2ay = -2y$

$2a = -2 \Rightarrow \boxed{a = -1}$

c) exemplu de două funcții olomorfe $f,g:\mathbb{C}$ $\to \mathbb{C}$ cu pol în 0, a.î.

$$res(f,0)\,res(g,0) = res(fg,0) = 1$$

$$f = \frac{1}{z}$$

$res(f,0) = 1$
$res(f,0)\,res(g,0) = 1$ $\Rightarrow res(g,0) = 1$

$$g = \frac{1}{z} + h$$

$res(fg,0) = 1$
$fg = \frac{1}{z}\left(\frac{1}{z}+h\right) = \frac{1}{z^2} + \frac{h}{z}$ $\Rightarrow h(0) = 1$

$$g = \frac{1}{z} + e^{z}$$

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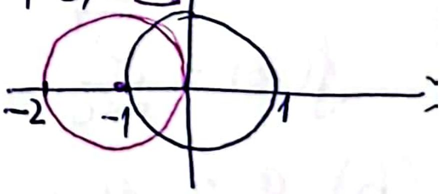
d) $\boxed{\max\limits_{|z|\le 1} |z^2 - z|}$; $\min\limits_{|z|\le 1} |z^2 - z|$

$|z| = 1$
$z = 1\cdot e^{i\theta}$
$\theta \in [0, 2\pi]$

$\max\limits_{|z|\le 1} |z^2 - z| = \max\limits_{|z|=1} |z(z-1)| =$

translatat cu -1

$= \sup\limits_{\theta\in[0,2\pi]} |e^{i\theta}(e^{i\theta}-1)| = \sup\limits_{\theta\in[0,2\pi]} |e^{i\theta}|\cdot\boxed{|e^{i\theta}-1|} = |-2| = \boxed{2}$



$\min\limits_{|z|\le 1} |z^2 - z| = \min\limits_{|z|\le 1} |z(z-1)| = 0$

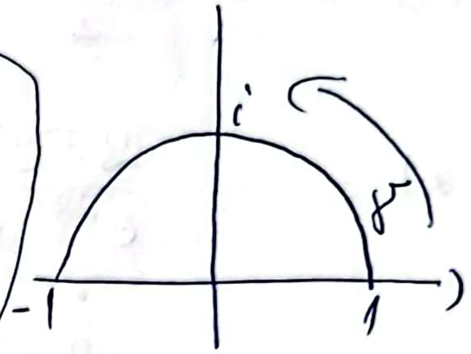
$|z(z-1)| \ge 0 \quad \forall z$

$z = 0$ sau $z = 1 \Rightarrow |z(z-1)| = 0$

1) $\int_A (z+\bar{z}^2)\,dz$, semicerc de rază 1 din semiplanul superior

$\gamma : [0,\pi] \to \mathbb{C}$

$\gamma(t) = 1\cdot e^{it} = e^{it}$



$\int_A (z+\bar{z}^2)\,dz = \int_0^{\pi} (e^{it}+\overline{e^{it}}^{\,2})\,d(e^{it})$

$= \int_0^{\pi} (e^{it}+e^{-2it})\cdot i\cdot e^{it}\,dt = i\int_0^{\pi} e^{2it}+e^{-it}\,dt$

$= i\int_0^{\pi} e^{2it}\,dt + i\int_0^{\pi} e^{-it}\,dt =$

$= i\cdot \frac{e^{2it}}{2i}\Big|_0^{\pi} + i\cdot \frac{e^{-it}}{-i}\Big|_0^{\pi} = 2.$

2. a) Nr. zerouri al fcției $f(z) = 3e^{z} - z$ din $|z| \le 1$

$e < 3.$

$f = 3e^{z}$ $\quad |f|\big|_{\partial D_1(0)} \overset{?}{>} |g|\big|_{\partial D_1(0)}$

$g = -z$

$\inf_{|z|=1} |f| = \inf_{|z|=1} |3e^{z}| = \inf_{|z|=1} |3e^{a+bi}| = \inf_{|z|=1} |3e^{a}\cdot \underbrace{e^{bi}}_{1}| =$

$= \inf_{|z|=1} |3e^{a}| = 3\inf_{|z|=1} |e^{a}| = 3\inf_{-1\le a\le 1} |e^{a}| = 3\cdot\frac{1}{e} = \frac{3}{e} > 1$

$\sup_{|z|=1} |g| = \sup_{|z|=1} |-z| = 1 < \frac{3}{e} = \inf |f|$

$\Rightarrow$ f și $f+g$ au ac. nr. de zerouri

~~au~~ (nu au zerouri)

a) $f(z) = \dfrac{z^2}{\sinh z}$ polii, ordinul, reziduu

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$f(z) = \ldots = \frac{2z^2 e^z}{e^{2z}-1} \Rightarrow \text{pol } \overset{0}{\text{de}} \text{ ordin } 1$$

$$\frac{2z^2 e^z}{(e^z-1)(e^z+1)}$$

$$\operatorname{res}(f,0) = \frac{2z^2 e^z}{e^z+1}\Big|_{z=0} = 0$$

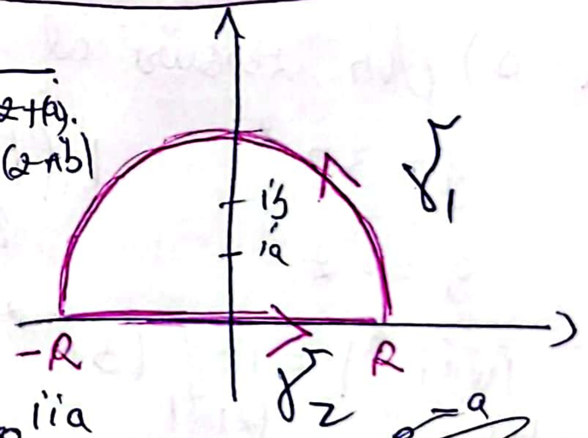
3. $$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = \frac{e^{iz}}{(z-ia)(z+ia)(z-ib)(z+ib)}$$

am polii $\pm ia, \pm ib$

$ia, ib \in \operatorname{Int}(\gamma)$

$\gamma = \gamma_1 \cup \gamma_2$



$$\operatorname{Res}(f, ia) = \frac{e^{iz}}{(z+ia)(z^2+b^2)}\Big|_{z=ia} = \frac{e^{iia}}{2ia(-a^2+b^2)} = \cancel{\frac{e^{-a}}{a\,2i}}$$

$$= \frac{e^{-a}}{-(a^2-b^2)2ia}$$

$$\operatorname{Res}(f, ib) = \frac{e^{iz}}{(z+ib)(z^2+a^2)}\Big|_{z=ib} = \frac{e^{-b}}{2ib(-b^2+a^2)} =$$

$$= \frac{e^{-b}}{-(a^2-b^2)2ib}$$

$$\int_\gamma f(z)\,dz = 2\pi i\left(\frac{e^{-a}}{-(a^2-b^2)2ia} + \frac{e^{-b}}{+(a^2-b^2)2ib}\right)$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-a}}{-a} + \frac{e^{-b}}{+b}\right) =$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Am încheiat pasul 1.

$\gamma_1 : [0,\pi] \to \mathbb{C}$

$\gamma_1(t) = Re^{it}$

$$\int_{\gamma_1} f(z)\,dz = \int_0^\pi f(Re^{it})\,d(Re^{it}) = \int_0^\pi \frac{e^{iRe^{it}} \cdot Rie^{it}}{(R^2e^{2it}+a^2)(R^2e^{2it}+b^2)}\,dt$$

$$\left|\frac{\overset{1}{e^{iR\cos t}} \cdot e^{iRi\sin t} \cdot \overset{R}{Rie^{it}}}{(R^2e^{2it}+a^2)(R^2e^{2it}+b^2)}\right| = \left|\frac{R}{(R^2e^{2it}+a^2)(R^2e^{2it}+b^2)e^{R\sin t}}\right|$$

$\xrightarrow{R\to\infty} 0$

$\int_{\gamma_1} f(z)\,dz \xrightarrow{R\to\infty} 0$

$\gamma_2 : [-R,R] \to \mathbb{R}$, $\gamma_2(t) = t$

$$\int_{\gamma_2} f(z)\,dz = \int_{-R}^{R} f(t)\,dt = \int_{-R}^{R} \frac{e^{it}}{(t^2+a^2)(t^2+b^2)}\,dt = \int_{-R}^{R}\frac{\cos t}{(\ldots)}\,dt$$

$$+ \underbrace{\int_{-R}^{R} \frac{i\sin t}{(\ldots)}\,dt}_{0}$$ impară

$$h(z) = \frac{i\sin t}{(t^2+a^2)(t^2+b^2)}$$

$$h(-z) = \frac{-i\sin t}{(t^2+a^2)(t^2+b^2)} = -h(z)$$

$$\int_{\gamma_2} f(z)\,dz = \underbrace{\int_{-R}^{R} \frac{\cos t}{(t^2+a^2)(t^2+b^2)}\,dt}_{I_R} \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{\cos t}{(t^2+a^2)(t^2+b^2)}\,dt = I_\infty$$

Am încheiat pasul 2.

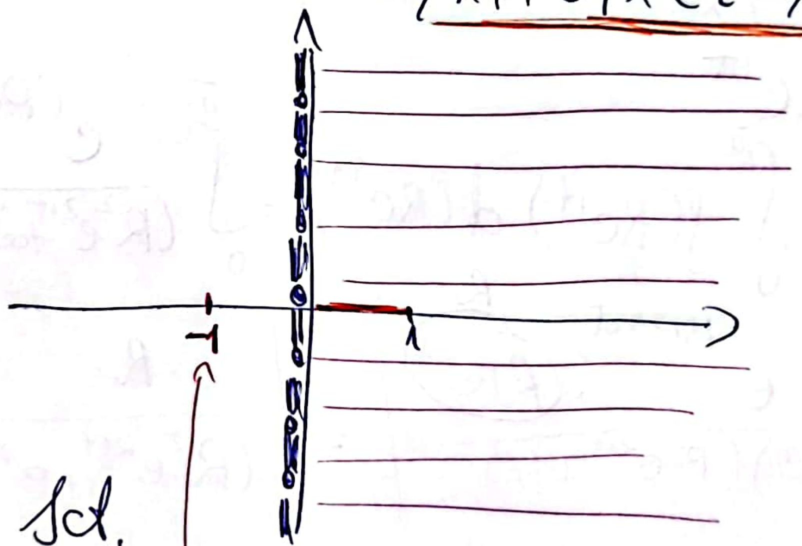
Concluzie:

$$(*) = \int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz =$$

$$= \int_{\gamma_1} f(z)dz + J_R \xrightarrow{R\to\infty} 0 + J_\infty = (**)$$

Am încheiat tot :).

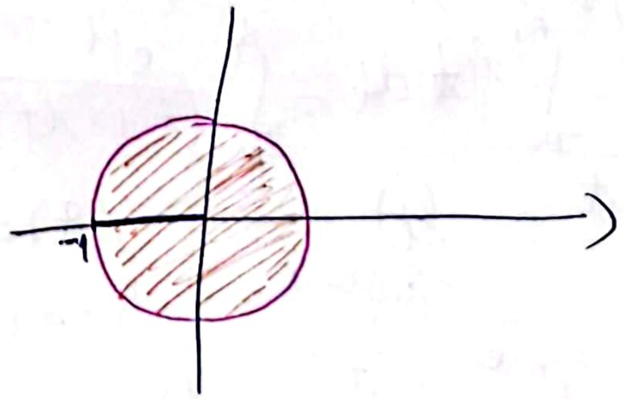
4. (a) Repr. grafic dom. $\Omega = \{x+i\cdot y \mid x>0, y\in\mathbb{R}\} \setminus \{x+i\cdot 0 \mid x\in[0,1]\}$.



Considerăm fct.

$h(z) = \dfrac{z-1}{z+1}$. Dem că

$h(\Omega) = \Lambda = \{x+iy : x^2+y^2 < 1\} \setminus \{x+i\cdot 0 : x\in[-1,0]\}$



$h(-\circ-\circ) = ?$

$-\circ-\circ = \begin{cases} x=0 \\ y=t \end{cases} \Big| \Rightarrow h(0+it) = \dfrac{it-1}{it+1} = \dfrac{-(1-it)^2}{t^2+1} =$

$= \dfrac{-(1-2it-t^2)}{1+t^2} =$

$= -\dfrac{1-t^2}{1+t^2} + \dfrac{2t}{1+t^2} i \in$ cercului roz (pe Fr) a.î. $\exists$

$\begin{cases} \cos\alpha = -\dfrac{1-t^2}{1+t^2} \\ \sin\alpha = \dfrac{2t}{1+t^2} \end{cases}$

$tg\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{-2t}{1-t^2}$

Deci $h(-\circ-\circ) =$ Fr. cercului roz $\setminus\{-1\}$

$\alpha = arctg\left(\dfrac{-2t}{1-t^2}\right)$, deci există

$[0,1] \in Ox$
$-1 \in Ox$ $\Big| \Rightarrow h(Ox) =$ dreaptă.

Este suf. să alegem două pct. din $[0,1]$.

$h(0) = -1$
$h(1) = 0$ $\Big| \Rightarrow h([0,1]) = [-1, 0]$

Verificăm $h(Int(\Omega)) \overset{?}{\in} Int(\Lambda)$

$2 \in Int(\Omega)$

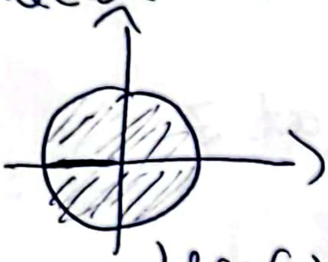
$h(2) = \frac{1}{3} < 1$ deci $h(2) \in Int(\Lambda)$.

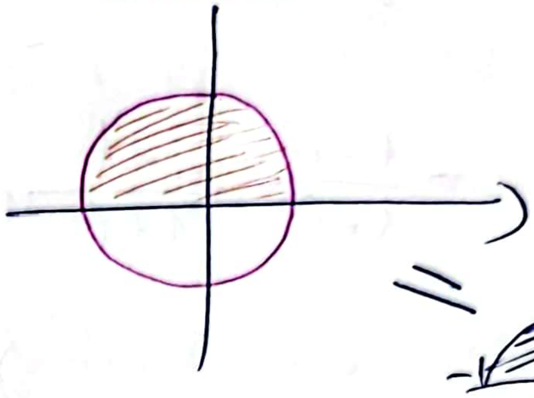
Am demonstrat, astfel, că $h(\Omega) = \Lambda$

b) Det. o aplicație biolomorfă între $h(\Omega)$ și semidiscul.

$$\{x+i\cdot y \mid x^2+y^2<1,\ y>0\}$$

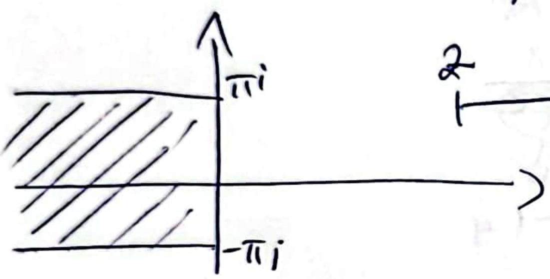
Plec de aici:

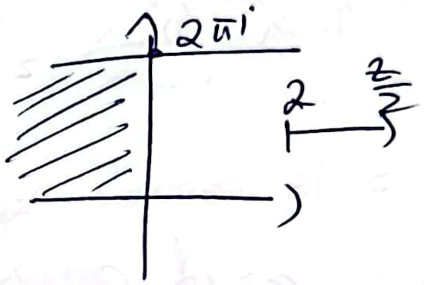


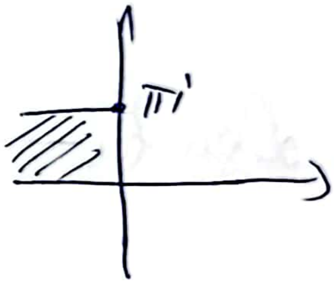


$\wr \log(z) \begin{bmatrix} \theta \in(-\pi,\pi) \\ r\in(0,1) \end{bmatrix}$

$z \longmapsto e^z$







Scriu compunerile făcute.

$$re^{i\theta} \xmapsto{\log} \log(re^{i\theta}) = \ln r + i\theta$$

$$z \longmapsto \log(z)$$

$$z \longmapsto z+\pi i$$

$$z \longrightarrow \frac{z}{2}$$

$$z \longmapsto e^z$$

$$z \longmapsto e^{\left(\frac{\log(z)+\pi i}{2}\right)} = e^{\frac{\log(z)+\pi i}{2}} =$$

$$= e^{\frac{1}{2}\log(z)+\frac{\pi i}{2}} = e^{\log(z)^{\frac{1}{2}}} \cdot e^{\frac{\pi}{2} i}$$

$$= \boxed{\sqrt{z}\cdot i}$$

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Dem că dacă $c>0$, $c\neq 1$, $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$, atunci

$$\omega = \left\{ z \in \mathbb{C} : \left| \frac{z - z_1}{z - z_2} \right| = c \right\}$$ rep. un cerc.

Hint: Putem fol. prop. transf. omografice (:-:)

$z_1 = a+bi$; $z_2 = u+iv$; $z = x+iy$

$$\frac{|z-z_1|}{|z-z_2|} = c \Rightarrow \sqrt{(x-a)^2+(y-b)^2} = c\sqrt{(x-u)^2+(y-v)^2} \Big|^2$$

$$|z-z_1| = |x+iy-(a+bi)| = \sqrt{(x-a)^2+(y-b)^2}$$

$$(x-a)^2+(y-b)^2 = c^2((x-u)^2+(y-v)^2)$$

Vreau să obțin ceva de forma:

$$(x-k_1)^2+(y-k_2)^2 = k_3, \quad k_1, k_2, k_3 \in \mathbb{R} \text{ - ct}$$
$$k_3 > 0$$

$$\Rightarrow x^2-2xa+a^2+y^2-2yb+b^2 = c^2(x^2-2xu+u^2+y^2-2yv+v^2)$$

$$(1-c^2)x^2-2xa+2c^2xu+a^2-c^2u^2 +$$
$$+ (1-c^2)y^2-2yb+2c^2yv+b^2-c^2v^2 = 0$$

$\Rightarrow$ Putem aduce acea sumă la forma $(x-k_1)^2+t_1$.
Din simetrie, analog și a doua parte.

$\Rightarrow \omega$ e un cerc.