TEORIA MĂSURII

SEMINAR 13

Integrale de suprafață

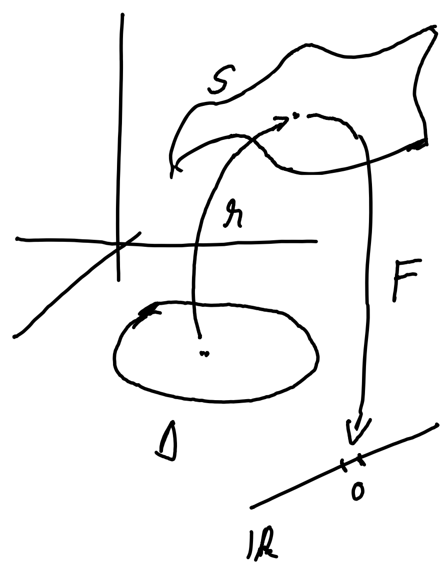
Fie $r: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ clasă C^1

$S = \operatorname{Im} r$

$r(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v)\big)$

- De prima speță

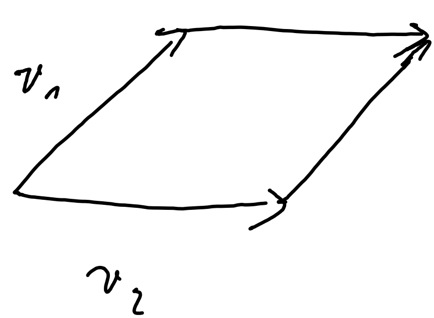
Fie $F: S \to \mathbb{R}$ continuă

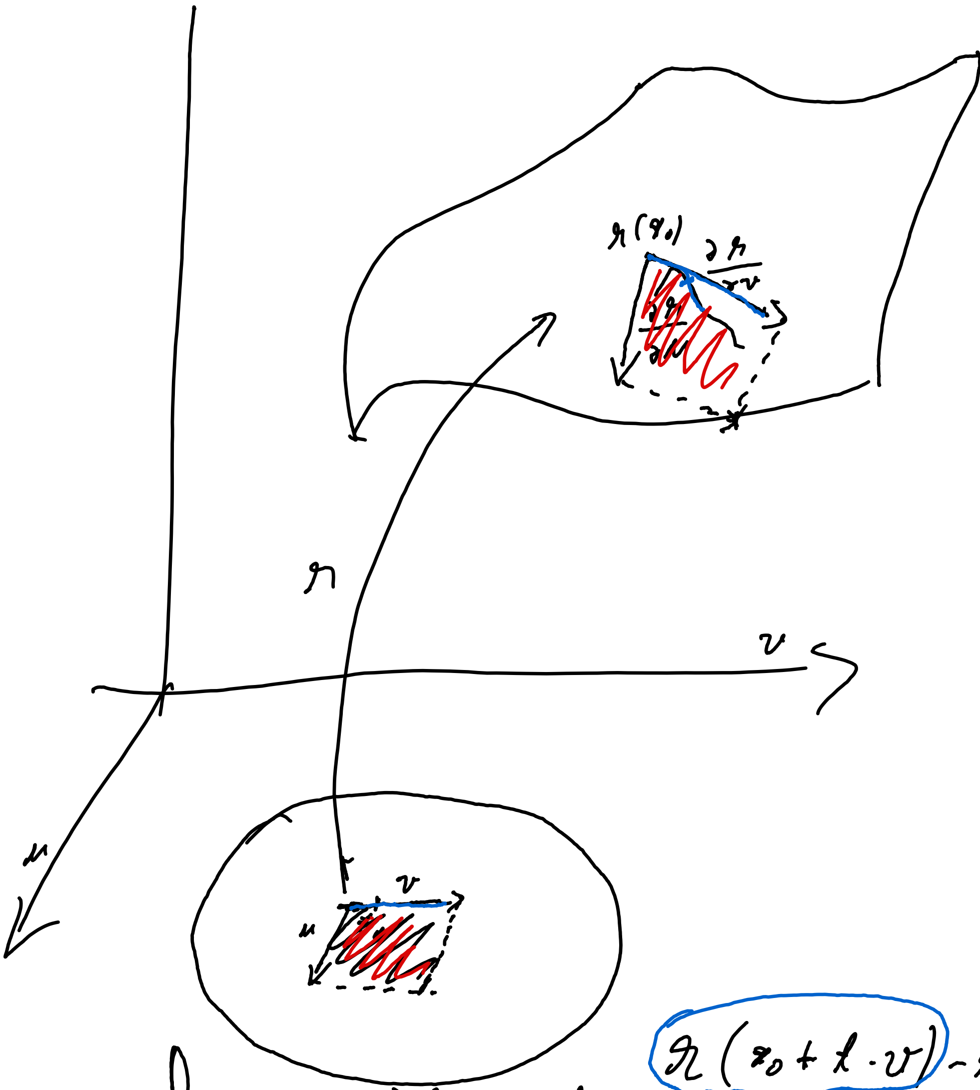


$$\iint_S F \, d\sigma = \iint_D (F \circ r)(u,v) \cdot \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du \, dv$$

$$\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| = \left\| \frac{\partial r}{\partial u} \right\| \cdot \left\| \frac{\partial r}{\partial v} \right\| \cdot \sin\left(\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v} \right)$$

$=$ aria paralelogram format de $\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}$

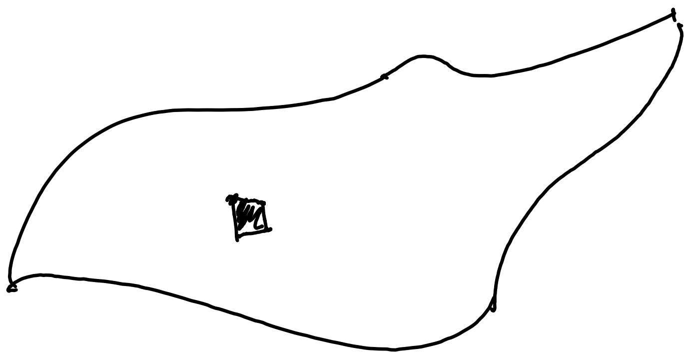




$$\frac{\partial r}{\partial v}(x_0) = \lim_{t \to 0} \frac{r(x_0 + t \cdot v) - r(x_0)}{t}$$

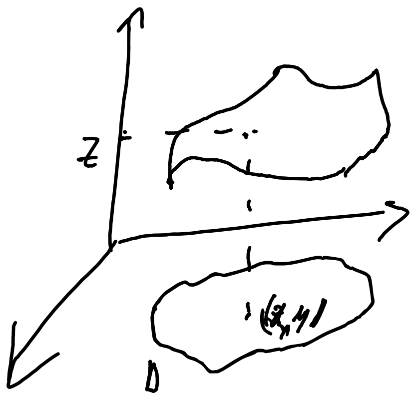
$$\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|$$

$$\iint_D \left| \frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v) \right| du\, dv$$



$$\iint_S F \, d\sigma = \iint_D F \circ r(u,v) \cdot \left| \frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v) \right| du\, dv$$

Formulă pt. suprafețe de tip grafic



$$z = f(x,y)$$

$$r(x,y) = (x, y, f(x,y))$$

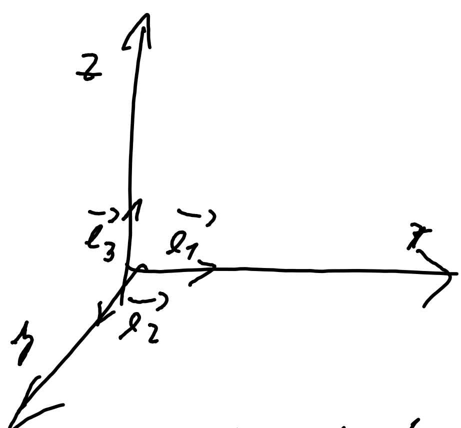
$$\left\| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} \right\| =$$

$$= \left\| \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right) \right\| =$$

$$= \sqrt{1 + \underbrace{\left(\frac{\partial f}{\partial x}\right)^2}_{p} + \underbrace{\left(\frac{\partial f}{\partial y}\right)^2}_{q}}$$

- Integrale de suprafață de speța a doua

$$\omega(x,y,z) = L(x,y,z) \cdot dx \wedge dy + M(x,y,z) \cdot dy \wedge dz + N(x,y,z) \cdot dx \wedge dy$$



$$dx \wedge dy\big((a_1, a_2, a_3), (b_1, b_2, b_3)\big) =$$

$$= dx \wedge dy\,(a_1 \cdot e_1 + a_2 \cdot e_2 + a_3 \cdot e_3,\ b_1 \cdot e_1 + b_2 \cdot e_2 + b_3 \cdot e_3)$$

$e_1 = (1, 0, 0)$

$$dx \wedge dy (a_1 e_1 + a_2 e_2 + a_3 e_3,$$
$$b_1 e_1 + b_2 e_2 + b_3 e_3) =$$

$$= \sum_{i,j} a_i \cdot b_j \, dx \wedge dy (e_i, e_j)$$

$dx \wedge dy (e_i, e_j) = 0$, dacă $i = 3$ sau $j = 3$

e_1 = versor pe Ox

e_2 = versor pe Oy

e_3 = versor pe Oz

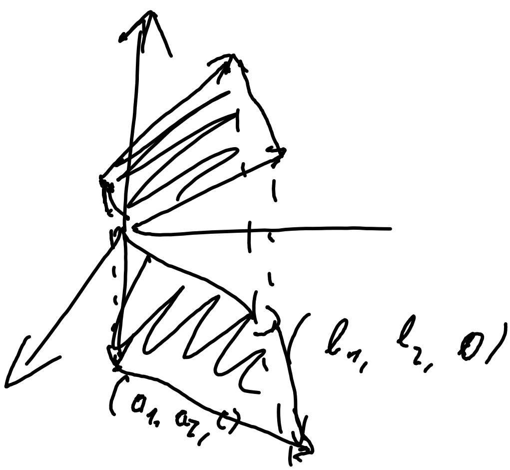
$$dx \wedge dy (e_1, e_2) = 1$$

$$dx \wedge dy (e_2, e_1) = -1$$

$$dx \wedge dy\big((a_1, a_2, a_3), (b_1, b_2, b_3)\big) =$$

$$= a_1 b_2 - a_2 b_1 =$$ aria paralelog. det. de vectori

$(a_1, a_2, 0), (b_1, b_2, 0)$



$$\iint_S \omega = \iint_S \Big(\alpha(x,y,z)\, dy \wedge dz +$$

$$+ \beta(x,y,z)\, dz \wedge dx +$$

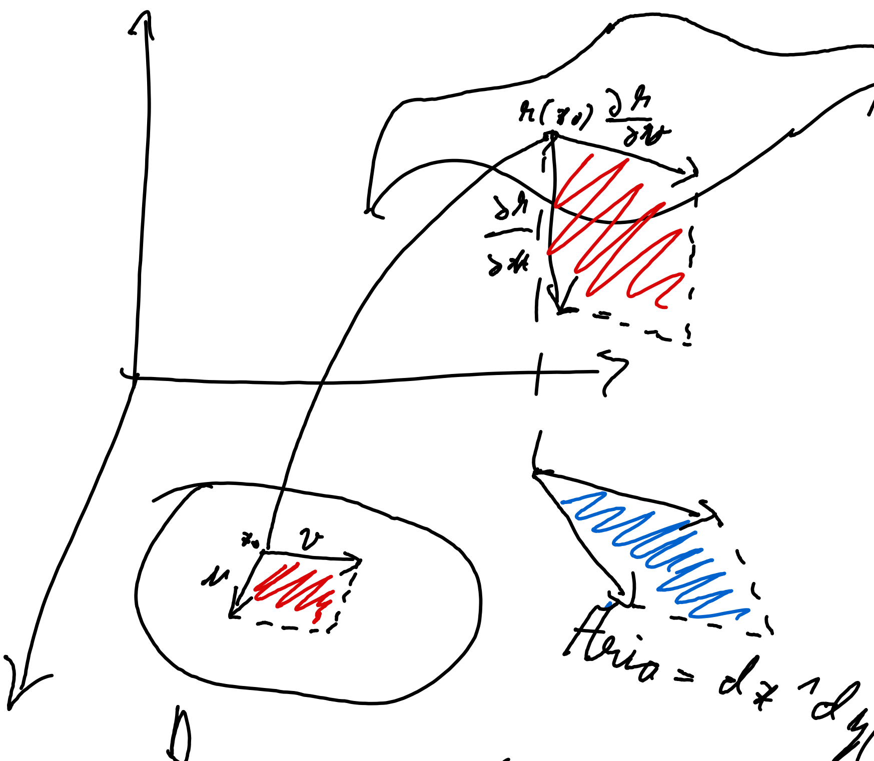
$$+ \gamma(x,y,z)\, dx \wedge dy \Big) =$$

$$= \iint_D \Big[\alpha(r(u,v)) \cdot \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$+ \beta(r(u,v)) \cdot \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} + \gamma(\ldots) \cdot \begin{vmatrix} \ldots \end{vmatrix} \Big] du\, dv$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$$

$$\iint_D \gamma(r(u,v)) \cdot \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix} \right|$$



$$\text{Aria} = dx \wedge dy\left(\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}\right)$$

$$\left| \begin{matrix} t_1 & t_2 \\ s_1 & s_2 \end{matrix} \right| = \text{Aria}\Bigg($$  $$\Bigg)$$

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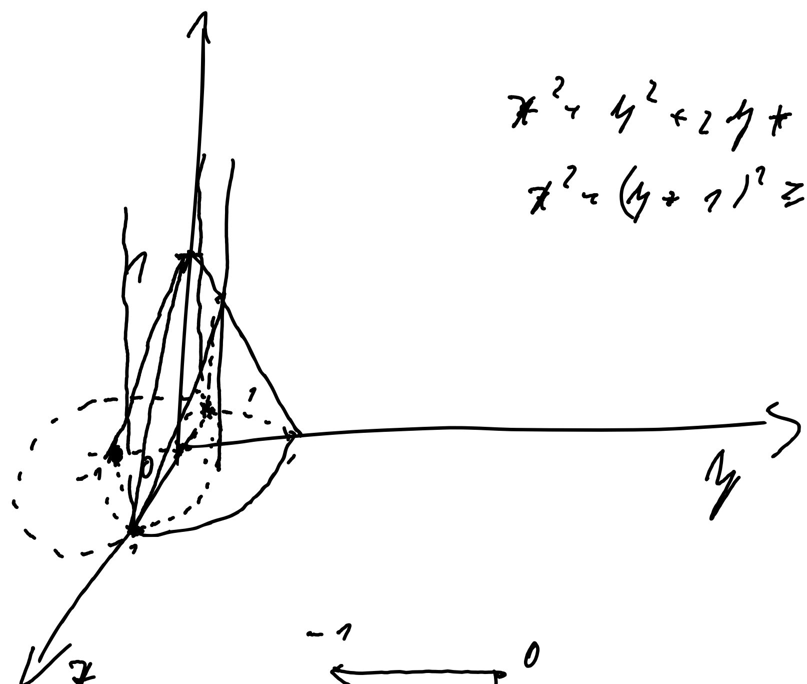
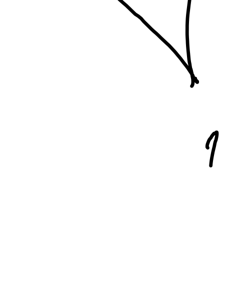
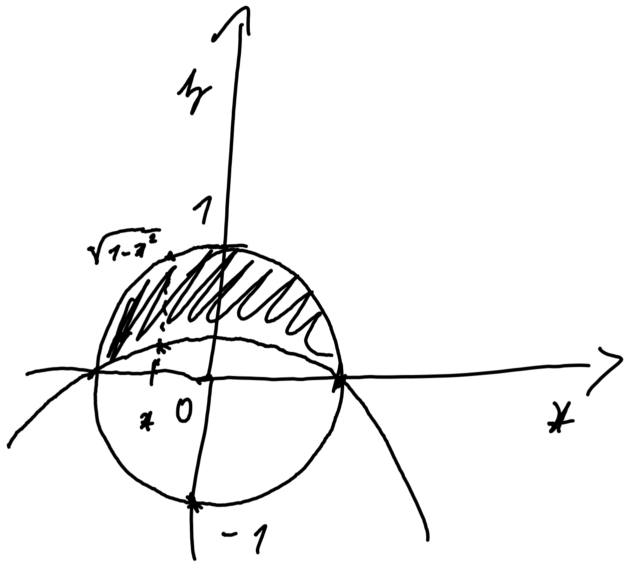
$$\Sigma = \{ (x,y,z) \mid x^2 + y^2 = (1-z)^2,\ x^2 + y^2 + 2y \geq 1,\ z \in [0,1] \}$$



$$x^2 + y^2 + 2y + 1 \geq 2$$

$$x^2 + (y+1)^2 \geq 2$$





$$x^2+(y+1)^2=2$$

$$y+1=\sqrt{2-x^2}$$

$$y=\sqrt{2-x^2}-1$$

$$1-z=\sqrt{x^2+y^2}$$

$$z=1-\sqrt{x^2+y^2}$$

$$D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; \begin{array}{l} x^2+y^2\leq 1 \\ x^2+(y+1)^2\geq 2 \end{array}\right\}$$

Aria cerută devine:

$$A = \iint_D \sqrt{1+p^2+q^2}\; dx\, dy$$

$$p = \frac{\partial z}{\partial x} = -\frac{x}{\sqrt{x^2+y^2}}$$

$$q = \frac{\partial y}{\partial x} = -\frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1+p^2+q^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}$$

$$= \sqrt{2}$$

$$A = \sqrt{2} \cdot \text{Aria}(D)$$

$$\iint_D 1\,dx\,dy = \int_{-1}^{1}\left(\int_{\sqrt{2-x^2}-1}^{\sqrt{1-x^2}} 1\,dy\right)dx =$$

$$= \int_{-1}^{1}\left(\sqrt{1-x^2}-\sqrt{2-x^2}+1\right)dx$$

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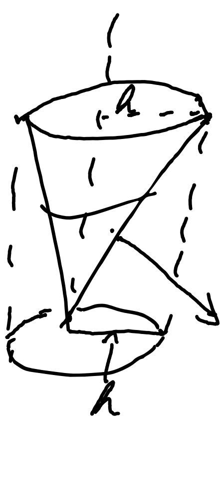
$$I = \iint_{\Sigma} (y-z)\, dy \wedge dz + \underbrace{(z-x)\, dx \wedge dz}_{= (x-z)\, dz \wedge dx} + (x-y)\, dx \wedge dy$$

Σ = frontiera conului (închis)

$$z = \sqrt{x^2+y^2}\ ,\ 0 \le z \le h$$



Integrala pe con:

$$\vec{n} = \pm(-p\ ,\ -q\ ,\ 1)$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le h^2\}$$

$$p = \frac{x}{\sqrt{x^2+y^2}}$$

$$q = \frac{y}{\sqrt{x^2+y^2}}$$

$$\vec{n} = (x, y, -1)$$

$$\iint\limits_D (y-z)\cdot\frac{x}{\sqrt{x^2+y^2}} + (x-z)\cdot\frac{y}{\sqrt{x^2+y^2}} -$$

$$-(x-y)\cdot 1 =$$

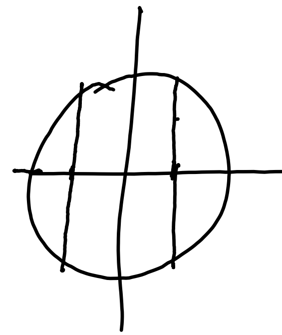
$$= \iint\limits_D \left(\frac{xy - xz + xy - yz}{\sqrt{x^2+y^2}} - (x-y) \right) dx\, dy$$

$$= \iint\limits_D \left[\frac{-(x+y)\cdot\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} + 2\frac{xy}{\sqrt{x^2+y^2}} - (x-y) \right] dx\, dy$$

$$= -\iint\limits_D (x+y)\, dx\, dy + 2\iint\limits_D \frac{xy}{\sqrt{x^2+y^2}} - \iint\limits_D (x-y)\, dx\, dy$$

$$\iint_D x\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dx\,dy =$$

$$= \underbrace{\int_{-1}^{1} x\cdot 2\cdot\sqrt{1-x^2}\,dx}_{\text{impară}}$$



$$= 0$$

$$\iint_D y\,dx\,dy = 0 \quad (\text{analog})$$

$$\iint_D \frac{xy}{\sqrt{x^2+y^2}} = \int_0^1\int_0^{2\pi} \frac{r^2\cos\theta\sin\theta}{r}\cdot r\,d\theta\,dr$$

$$= \int_0^1 r^2\cdot \underbrace{\int_0^{2\pi}\cos\theta\sin\theta\,d\theta}_{=0}\,dr = 0$$

Integrala pe capac → temă