Fie $X_1, X_2, \dots X_n$ i.i.d. Determinați funcția de repartiție a v.a. X în următoarele cazuri:

a) $X = \max(X_1, X_2, \dots X_n)$

b) $X = \min(X_1, X_2, \dots X_n)$

a) $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\max(X_1, X_2, \dots X_n) \le x) = \mathbb{P}(\overbrace{(X_1 \le x) \cap \dots \cap (X_n \le x)}^{\text{ÎND.}})$

$= \underbrace{\mathbb{P}(X_1 \le x)}_{= F_{X_1}(x)} \cdot \underbrace{\mathbb{P}(X_2 \le x)}_{= F_{X_2}(x)} \cdot \dots \cdot \underbrace{\mathbb{P}(X_n \le x)}_{= F_{X_n}(x)}$

! Cum $X_1, X_2 \dots X_n$ sunt identic distribuite $F_{X_1}(x) = \dots = F_{X_n}(x)$.

Așadar, $\boxed{F_X(x) = (F_{X_1}(x))^n}$

b) $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\min(X_1, X_2, \dots X_n) \le x) = 1 - \mathbb{P}(\min(X_1, \dots X_n) > x)$

$= 1 - \mathbb{P}(\underbrace{(X_1 > x) \cap (X_2 > x) \cap \dots \cap (X_n > x)}_{\text{ÎND.}})$

$= 1 - \mathbb{P}(X_1 > x) \cdot \mathbb{P}(X_2 > x) \cdot \dots \cdot \mathbb{P}(X_n > x)$

$= 1 - (1 - \underbrace{\mathbb{P}(X_1 \le x)}_{= F_{X_1}(x)}) \cdot \dots \cdot (1 - \underbrace{\mathbb{P}(X_n \le x)}_{= F_{X_1}(x)})$

$= 1 - (1 - F_{X_1}(x))^n$

Obs.: Aceeași abordare funcționează și dacă $X_1, X_2 \dots X_n$ sunt doar independente, nu și identic distribuite.

Fie $X_1, X_2, \dots X_n \sim Bern(p)$ independente. Demonstrați că $X = \sum_{i=1}^{n} X_i \sim Binom(n, p)$.

Considerăm, pentru început cazul $n = 2$.

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X_1 + X_2 \le x) = \sum_{k=0}^{1} \mathbb{P}(X_1 + X_2 \le x \mid X_2 = k) \cdot \mathbb{P}(X_2 = k)$$

$$= \sum_{k=0}^{1} \underbrace{\mathbb{P}(X_1 \le x - k)}_{F_{X_1}(x-k)} \cdot \mathbb{P}(X_2 = k)$$

$$= F_{X_1}(x - 0) \cdot \underbrace{\mathbb{P}(X_2 = 0)}_{1-p} + F_{X_1}(x - 1) \cdot \underbrace{\mathbb{P}(X_2 = 1)}_{p}$$

$$X_1 : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

$$F_{X_1}(x) = \begin{cases} 0, & x < 0 \\ 1-p, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$$\underbrace{f_{X_1}(x)}_{\mathbb{P}(X_1 = x)} = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \\ 0, & \text{în rest} \end{cases}$$

$$\mathbb{P}(X = x) = \mathbb{P}(X_1 + X_2 = x) = \sum_{k=0}^{1} \mathbb{P}(X_1 + X_2 = x \mid X_2 = k) \cdot \mathbb{P}(X_2 = k)$$

$$= \sum_{k=0}^{1} \mathbb{P}(X_1 = x - k) \cdot \mathbb{P}(X_2 = k)$$

$$= \mathbb{P}(X_1 = x) \cdot \underbrace{\mathbb{P}(X_2 = 0)}_{1-p} + \mathbb{P}(X_1 = x - 1) \cdot \underbrace{\mathbb{P}(X_2 = 1)}_{p}$$

$$\mathbb{P}(X = x) = \begin{cases} (1-p) \cdot (1-p) + 0 \cdot p, & x = 0 \\ p \cdot (1-p) + (1-p) \cdot p, & x = 1 \\ 0 \cdot (1-p) + p \cdot p, & x = 2 \end{cases}$$

$$\mathbb{P}(X = x) = \begin{cases} (1-p)^2, & x = 0 \\ p(1-p), & x = 1 \end{cases}$$

$$X : \begin{pmatrix} 0 & 1 & 2 \\ (1-p)^2 & 2p(1-p) & p^2 \end{pmatrix}$$

$$\Downarrow$$

$$X \sim Binom(2, p)$$

$$F_X(x) = \mathbb{P}(X \le x) = \begin{cases} 0, & x < 0 \\ (1-p)^2, & 0 \le x < 1 \\ (1-p)^2 + 2p(1-p), & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

Pentru $n \ge 3$ putem face demonstrația prin inducție.