Geometrie seminar 3

• $n \in \mathbb{N}^*$, K corp, iar $k \neq 2$ ($1_k + 1_k \neq 0_k$)

$S_1 = \{A \in \mathcal{M}_n(K) \mid {}^tA = A\}$

$S_2 = \{A \in \mathcal{M}_n(K) \mid {}^tA = -A\}$

a) $S_1, S_2 \leq_K \mathcal{M}_n(K)$

b) $\mathcal{M}_n(K) = S_1 \oplus S_2$

a) $S_2 = \{A \in \mathcal{M}_n(K) / A^T = -A\}$

Fie $\left.\begin{array}{l}\alpha,\beta \in K\\ A_1, A_2 \in S_2\end{array}\right\} \overset{?}{\Rightarrow} \alpha A_1 + \beta A_2 \in S_2$

$\Leftrightarrow (\alpha A_1 + \beta A_2)^T = -(\alpha A_1 + \beta A_2)$

$(\alpha A_1 + \beta A_2)^T = \alpha A_1^T + \beta A_2^T = -\alpha A_1 - \beta A_2 = -(\alpha A_1 + \beta A_2)$

b) $\begin{cases} \mathcal{M}_n(K) = S_1 + S_2 \\ S_1 \cap S_2 = \{O_n\} \end{cases}$

$\mathcal{M}_n(K) = S_1 + S_2$

„$\subset$" $S_1 + S_2 \subset \mathcal{M}_n(K)$

„$\supset$" fie $B \in \mathcal{M}_n(K)$. Căutăm o matrice C în S_1, respectiv o matr. D în S_2 a.î. $B = C + D \;/\; {}^T \Rightarrow$

$\Rightarrow B^T = C^T + D^T \Leftrightarrow B^T = C - D$

$\begin{cases} C + D = B \\ C - D = B^T \end{cases}$

$\left.\begin{array}{l} 2_k C = B + B^T \\ \text{dar } 2_k \neq 0 \end{array}\right\} \Rightarrow C = \frac{1}{2_k}(B + B^T)$

$2_k D = B - B^T \Rightarrow D = \frac{1}{2_k}(B - B^T)$

* $C^T = C$, $D^T = D$ *

$C = \frac{1}{2_k}(B + B^T)$

$C^T = \frac{1}{2_k}((B + B^T))^T = \frac{1}{2_k}(B^T + B) = C \Rightarrow C \in S_1$

$D^T = \frac{1}{2_k}((B - B^T))^T = \frac{1}{2_k}(B^T - B) = -D \Rightarrow D \in S_2$

$S_1 \cap S_2 = \{O_n\}$

$S_1 \cap S_2 \supset \{O_n\}$

Arăm că $S_1 \cap S_2 \subset \{O_n\}$

Fie $B \in S_1 \cap S_2 \Rightarrow B \in S_1 \wedge B \in S_2 \Rightarrow \begin{cases} B^T = B \\ B^T = -B \end{cases} \Rightarrow$

$\Rightarrow B = -B \Rightarrow 2_k B = O_n$, $B = O_n$

• $n \in \mathbb{N}^*$, K corp, $\underbrace{1_k + \ldots + 1_k}_{n \text{ ori}} \neq 0_k$

$S = \{A \in \mathcal{M}_n(K) / Tr(A) = 0_k\}$

a) $S \leq_K \mathcal{M}_n(K)$

b) $\mathcal{M}_n(K) = S \oplus Sp_K = \{I_n\}$

a) fie $A, B \in S$ $\quad Tr A = 0_k$

$A + B \in S$? $\quad Tr B = 0_k$

$Tr(A+B) = 0_k$?

$Tr(A+B) = Tr A + Tr B = 0_k + 0_k = 0_k \Rightarrow A + B \in S$

Fie $\alpha \in K$ și $M \in S \Rightarrow Tr M = 0_k$

$Tr(\alpha M) = \alpha Tr M = \alpha 0_k = 0_k \Rightarrow \alpha M \in S$

b) $\mathcal{M}_n(K) = S + Sp_K\{I_n\}$

fie $A \in \mathcal{M}_n(K)$. Căutăm $B \in S$ și $C \in Sp_K\{I_n\}$

$\Downarrow \; Tr B = 0_k \qquad \Downarrow \; \exists \alpha \in K$ a.î. $C = \alpha I_n$

a.î. $A = B + C$

$A = B + \alpha I_n \Rightarrow tr A = tr B + \alpha\, tr\, I_n \Leftrightarrow Tr A = 0_k + \alpha n \Rightarrow Tr A = \alpha n$

$\alpha = \frac{Tr A}{n}$

$C = \frac{1}{n}(Tr A) I_n$

$B = A - \left(\frac{1}{n}(Tr A) I_n\right)$

$Tr B = Tr A - \frac{1}{n}(Tr A) n = 0_k \Rightarrow B \in S$

$(\forall) A \in \mathcal{M}_n(K)\; (\exists)\; B \in S \wedge C \in Sp_K\{I_n\}$ a.î. $A = B + C \Rightarrow \mathcal{M}_n(K) = S + Sp_K\{I_n\}$

$A \in (S \cap Sp_K\{I_n\})$

$A \in S \Rightarrow Tr A = 0_k$

$\left.\begin{array}{l} A \in S_{p_K}\{I_n\} \Rightarrow \exists \alpha \in K \text{ a.î. } A = \alpha I_n \end{array}\right\} \Rightarrow Tr(\alpha I_n) = 0_k$

$\Downarrow \begin{cases} \alpha n = 0_k \\ n \neq 0_k \end{cases}$

$\alpha = 0 \Rightarrow A = O_n$

DEFINIȚIE fie $m \in \mathbb{N}^*$, V/K - sp vectorial, $M = \{x_1, \ldots, x_m\} \subset V$

Spunem că M este:

a) sistem de vectori liniar independenți dacă
- $(\alpha_1, \ldots, \alpha_m \in K)$
- $\left(\sum_{i=1}^{m} \alpha_i x_i = 0 \Rightarrow \alpha_1 = \ldots = \alpha_m = 0\right)$

b) sistem de generatori pt V/K dacă $V = Sp_K(M)$ ($\Leftrightarrow (\forall) x \in V\; \exists\, \alpha_1, \ldots, \alpha_m \in C$ a.î. $x = \sum_{i=1}^{m} \alpha_i x_i$)

c) bază a lui V/K dacă M este sistem de vectori l.i. și M este sist. de gen. pt V/K

Ex1. K sp vect $n \in \mathbb{N}^*$ $\quad K^n/K$

Fie $i = \overline{1,n}$ $\quad e_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)$

Atunci B_C (baza canonică) $= \{e_1, \ldots, e_n\}$ este bază a lui K^n/K

fie $x \in K^n$ $\quad x = (x_1, \ldots, x_n) = (x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \ldots + (0, \ldots, x_n) = x_1 e_1 + \ldots + x_n e_n \Rightarrow B_C$ este sistem de generatori (1)

fie $\alpha_1, \ldots, \alpha_n \in K$ a.î. $\alpha_1 e_1 + \ldots + \alpha_n e_n = 0_{K^n} \Leftrightarrow$

$\Leftrightarrow (\alpha_1, 0, \ldots, 0) + (0, \alpha_2, \ldots, 0) + \ldots + (0, 0, \ldots, \alpha_n) = (0, \ldots, 0) \Leftrightarrow$

$\Leftrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, \ldots, 0) \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0 \Rightarrow B_C$ este și (2)

din (1) și (2) $\Rightarrow B_C$ bază pt K^n/K

Ex2 $m, n \in \mathbb{N}^*$, K corp $\quad \mathcal{M}_{m,n}(K)/K$

fie $i \in \overline{1,m}$, $j = \overline{1,n}$, $E_{ij} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$ $\quad B_C = \{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ e bază în $\mathcal{M}_{m,n}(K)$

Ex3 $n \in \mathbb{N}$, K corp

$K_n[X] = \{f \in K[X] / \text{grad } f \leq n\}$

$B_C = \{1, X, X^2, \ldots, X^n\}$

$\dim(V_1 \oplus V_2') = 2+1 = 3 = \dim \mathbb{R}^3$
$V_1 \oplus V_2' \subset \mathbb{R}^3$ $\Big\} \Rightarrow \mathbb{R}^3 = V_1 \oplus V_2$

$\mathbb{R}^3 = V_1 \oplus V_2 = V_1 \oplus V_2'$

OBS. $V = V_1 + V_2$

R_k reper în V_k, $k = \overline{1,2}$

dacă $R = R_1 \cup R_2$ este reper în $V \Rightarrow V = V_1 \oplus V_2$

TEOREMĂ $(V, +, \cdot)_{/\mathbb{K}}$ sp. vect., f. finit generat, $\dim_{\mathbb{K}} V = n$

$x \in V$, $R = \{e_1, \dots, e_n\}$ reper în V

$x = \sum_{i=1}^{n-1} x_i e_i$, $(x_i, \dots, x_n)$ componentele lui x în raport cu reperul R

$A \in \mathcal{M}_{m,n}(\mathbb{K})$

$S(A) = \{(x_1, \dots, x_n) \in \mathbb{K}^n / \underset{(m,n)(n,1)}{AX} = 0_{m,1}\}$

a) $S(A) \subset \mathbb{K}^n$ este subsp vect

b) $\dim_{\mathbb{K}} S(A) = n - \operatorname{rg} A$

Demonstrație

$k = \operatorname{rg} A$, $k \le \min\{m, n\}$

$n - k = p$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

Pp (fără a restrânge generalitatea) că $\Delta_p = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ a_{k1} & \dots & a_{kk} \end{vmatrix} \neq 0$,

$x_1, \dots, x_k$ = variabile principale

$x_{k+1} = \lambda_1, \dots, x_n = \lambda_p$ = variabile secundare

$a_{11}x_1 + \dots + a_{1k}x_k = a_{1k+1}\lambda_1 - \dots - a_{1n}\lambda_p$

$\vdots$

$a_{k1}x_1 + \dots + a_{kk}x_k = -a_{kk}\lambda_1 - \dots - a_{kn}\lambda_p$

$$x_1 = \frac{\begin{vmatrix} -a_{1k+1}\lambda_1 - \dots - a_{1n}\lambda_p & -a_{12} \dots a_{1k} \\ \vdots & \vdots \\ -a_{kk+1}\lambda_1 - \dots - a_{kn}\lambda_p & -a_{k2} \dots a_{kk} \end{vmatrix}}{\Delta} =$$

$$= \frac{\lambda_1 \begin{vmatrix} -a_{1k+1} & \dots & a_{1k} \\ -a_{kk} & & a_{kk} \end{vmatrix} + \dots + \lambda_p \begin{vmatrix} -a_{1n} & \dots & a_{1k} \\ \vdots & & \\ -a_{kn} & \dots & a_{kk} \end{vmatrix}}{\Delta} =$$

$$= \frac{\Delta_{11}\lambda_1}{\Delta} + \dots + \frac{\Delta_{1p}\lambda_p}{\Delta}$$

$$x_k = \frac{\Delta_{k1}}{\Delta}\lambda_1 + \dots + \frac{\Delta_{kp}}{\Delta}\lambda_p$$

Soluție

$(x_1, \dots, x_k, \lambda_1, \dots, \lambda_p) =$

$(*) = \left(\frac{\Delta_{11}\lambda_1}{\Delta} + \dots + \frac{\Delta_{1p}}{\Delta}\lambda_p, \dots, \frac{\Delta_{k1}}{\Delta}\lambda_1 + \dots + \frac{\Delta_{kp}}{\Delta}\lambda_p, \lambda_1, \dots, \lambda_p\right)$

$= \lambda_1 \underbrace{\left(\frac{\Delta_{11}}{\Delta}, \dots, \frac{\Delta_{k1}}{\Delta}, 1, 0, \dots, 0\right)}_{y_1} + \dots + \lambda_p \underbrace{\left(\frac{\Delta_{1p}}{\Delta}, \dots, \frac{\Delta_{kp}}{\Delta}, 0, \dots, 0, 1\right)}_{y_p}$

$B = \{y_1, \dots, y_p\}$ sistem de generatori pt $S(A)$

Dem că B este sistem LI

Fie $\lambda_1, \dots, \lambda_p \in \mathbb{K}$ a. î. $\lambda_1 y_1 + \dots + \lambda_p y_p = 0_{\mathbb{K}^n} \overset{(*)}{\Longrightarrow}$

$\Rightarrow \lambda_1 = \dots = \lambda_p = 0_{\mathbb{K}}$

Deci B bază în $S(A) \Rightarrow \dim S(A) = p = n - k = n - \operatorname{rg} A$

a) $S(A) \subset \mathbb{K}^n$ subsp. vect.

$(\forall)\ (x_1, \dots, x_n), (y_1, \dots, y_n) \overset{S(A)}{\Longrightarrow} a(x_1, \dots, x_n) + b(y_1, \dots, y_n) \in S(A)$

$x \in \sum_1^n x_i e_i$, $y = \sum^n$

$A(ax) + by$

Exemplu

$(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$

$V_1 = \{(x, y, z) \in \mathbb{R}^3 / 2x - y - z = 0\}$ $A_1 = \begin{pmatrix} 2 & -1 & -1 \end{pmatrix}$

$V_2 = \{(x, y, z) \in \mathbb{R}^3 / \begin{smallmatrix} x + y + z = 0 \\ 2x + z = 0 \end{smallmatrix}\}$ $A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

a) $\dim V_1$

b) $\dim V_2$

c) $\dim V_1 \cap V_2$

Soluție

a) $\dim V_1 = 3 - \operatorname{rg} A_1 = 3 - 1 = 2$

b) $\dim V_2 = 3 - \operatorname{rg} A_2 = 3 - 2 = 1$

c) $V_1 \cap V_2 = \{(x, y, z) \in \mathbb{R}^3 / \begin{cases} 2x - y - z = 0 \\ x + y + z = 0 \\ 2x + z = 0 \end{cases}\}$

$A_3 = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

$\det A_3 = \begin{vmatrix} 4 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} \neq 0 \Rightarrow \operatorname{rg} A_3 = 3$

$\dim V_1 \cap V_2 = 0$
$\dim(V_1 \oplus V_2) = 3$
$V_1 \oplus V_2 \subset \mathbb{R}^3$ $\Big\} \Rightarrow \mathbb{R}^3 = V_1 \oplus V_2$