Schimbarea de variabila in integrala multipla

Propozitie 1. Fie $A \in \mathcal{J}(\mathbb{R}^n)$ cu $\lambda(A) = 0$. Atunci orice functie marginita $f : A \to \mathbb{R}$ este integrabila Riemann pe A si

$$\int_A f(x)dx = 0.$$

Demonstratie. Fie $M = \sup\{|f(x)| : x \in A\}$. Fie J un interval din $\mathbb{R}^n$ astfel incat $A \subset J$ si $\tilde{f} : J \to \mathbb{R}$

$$\tilde{f}(x) = \begin{cases} f(x), & x \in A \\ 0, & x \in J \setminus A \end{cases}.$$

Fie $\varepsilon > 0$. Deoarece $\lambda(A) = 0$, exista o multime elementara $E \subset J$ astfel incat $A \subset E$ si $\lambda(E) < \varepsilon$. Fie $J_1, J_2, \ldots, J_p$ intervale mutual disjuncte astfel incat

$$E = J_1 \cup J_2 \cup \cdots \cup J_p.$$

Multimea $J \setminus E$ este elementara si la randul ei poate fi scrisa ca o reuniune disjuncta de intervale $K_1, \ldots, K_q$ din $\mathbb{R}^n$. Deci

$$\mathcal{P} = \{J_i, K_s : 1 \leq i \leq p, 1 \leq s \leq q\}$$

este o descompunere a lui J. Avem

$$S_{\mathcal{P}}(\tilde{f}) = \sum_{i=1}^{p} \sup\{\tilde{f}(x), x \in J_i\}\mathrm{vol}(J_i) \leq M \sum_{i=1}^{p} \mathrm{vol}(J_i) \leq \varepsilon \cdot M$$
$$s_{\mathcal{P}}(\tilde{f}) = \sum_{i=1}^{p} \inf\{\tilde{f}(x), x \in J_i\}\mathrm{vol}(J_i) \geq -M \sum_{i=1}^{p} \mathrm{vol}(J_i) \geq -\varepsilon \cdot M$$

Cum ε a fost ales arbitrar rezulta ca $\tilde{f}$ este integrabila pe J si $\int_J \tilde{f} = 0$. Deci f este integrabila pe A si $\int_A f = 0$.

Propozitie 2. Fie $A, B \in \mathcal{J}(\mathbb{R}^n)$ si $f : A \cup B \to \mathbb{R}$ o functie marginita integrabila Riemann pe A si pe B. Atunci f este integrabila Riemann pe $A \cap B$ si pe $A \cup B$ si

$$\int_{A\cup B} f(x)dx = \int_A f(x)dx + \int_B f(x)dx - \int_{A\cap B} f(x)dx.$$

Corolar 3. Fie $A, B \in \mathcal{J}(\mathbb{R}^n)$ astfel incat $\lambda(A \cap B) = 0$ si $f : A \cup B \to \mathbb{R}$ o functie marginita integrabila Riemann pe A si pe B. Atunci f este integrabila Riemann pe $A \cup B$ si

$$\int_{A\cup B} f(x)dx = \int_A f(x)dx + \int_B f(x)dx.$$

Propozitie 4. Fie $A, B \in \mathcal{J}(\mathbb{R}^n)$ astfel incat $\lambda(B) = 0$ si $f : A \cup B \to \mathbb{R}$ o functie marginita. Atunci f este integrabila Riemann pe $A \cup B$ daca si numai daca f este integrabila Riemann pe A si

$$\int_{A \cup B} f(x)dx = \int_A f(x)dx$$

Reamintim ca o aplicatie $\phi : U \to V$ intre doua multimi deschise U, V din $\mathbb{R}^n$ se numeste difeomorfism (sau schimbare de coordonate sau schimbare de variabila) daca ϕ este bijectiva si ϕ si inversa ei ϕ^{-1} sunt de clasa C^1. Asa cum am vazut, $\phi : U \to V$ este un difeomorfism daca si numai daca ϕ este bijectiva, de clasa C^1 si jacobianul J_ϕ nu se anuleaza pe U.

Teorema 5. Fie U si V doua multimi deschise din $\mathbb{R}^n$ si ϕ un difeomorfism de la U la V astfel incat frontiera multimii U este neglijabila Lebesgue. Atunci pentru orice $A \subset V$, $A \in \mathcal{J}(\mathbb{R}^n)$ astfel incat ϕ este marginita pe A avem $\phi(A) \in \mathcal{J}(\mathbb{R}^n)$ si pentru orice functie $f : \phi(A) \to \mathbb{R}$ marginita si integrabila Riemann functia $f \circ \phi$ este integrabila Riemann pe A. Daca in plus Jacobianul J_ϕ este marginit pe A, atunci

$$\int_{\phi(A)} f(x)dx = \int_A f \circ \phi(u) \cdot |J_\phi(u)|du$$

Observatie. In toate situatiile uzuale urmatoarele conditii din teorema de mai sus

(1) frontiera multimii U este neglijabila Lebesgue,

(2) ϕ marginita pe A,

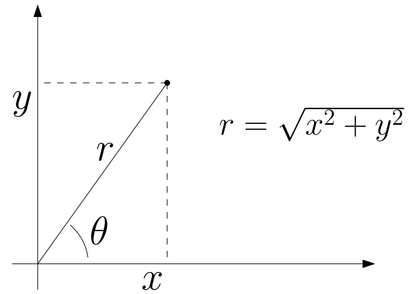
(3) J_ϕ marginita pe A,

sunt satisfacute.

Trecerea la coordonate polare

Fie $\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Phi(r, \theta) = (x, y)$ unde

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



adica

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta).$$

Atunci,

$$J_\Phi(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

pentru orice $(r, \theta) \in \mathbb{R}^2$ si evident Φ este de clasa C^1. Restrictia lui Φ la semibanda deschisa $(0, \infty) \times (0, 2\pi) \subset \mathbb{R}^2$ notata cu ϕ defineste un difeomorfism intre $(0, \infty) \times (0, 2\pi) \subset \mathbb{R}^2$ si imaginea sa, $\mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$. Asadar

$$\phi : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}, \quad \phi(r, \theta) = (r\cos\theta, r\sin\theta)$$

Observam insa ca restrictia lui Φ la orice semibanda deschisa de forma $(0, \infty) \times (a, a + 2\pi)$ este un difeomorfism. Uneori poate fi mai convenabil sa consideram difeomorfismul rezultat in urma restrictionarii la banda $(0, \infty) \times (-\pi, \pi)$, ca de exemplu in cazul in care avem de calculat o integrala pe multimea $\{(x, y) \in \mathbb{R}^2 \ : x^2 + y^2 \leq 1, x \geq 0\}$.

Exemplu. Calculati

$$\iint_D x^2 dxdy, \qquad D = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 4\}$$

Solutie. Avem

$$D = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\}.$$

Deoarece functiile $\alpha, \beta : [-2, 2] \to \mathbb{R}$

$$\alpha(x) = -\sqrt{4 - x^2}, \beta(x) = \sqrt{4 - x^2}$$

sunt continue pe $[-2, 2]$ rezulta ca D este masurabila Jordan. Functia $f : D \to \mathbb{R}$ este continua pe multimea compacta D si deci marginita. Asadar f este integrabila Riemann pe D. Vom trece la coordonate polare

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Transformarea

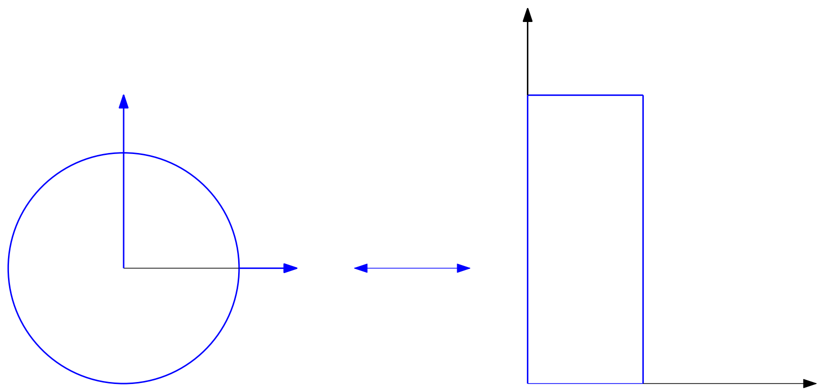
$$\phi : (0,\infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \geq 0\}$$

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

este difeomorfism si

$$J_\phi(r,\theta) = r.$$

Observam ca



$D \setminus ([0,2] \times \{0\} \cup Fr(D))$ $(0,2) \times (0,2\pi)$

$$(x,y) = \phi(r,\theta) \in D \setminus ([0,2] \times \{0\} \cup \mathrm{Fr}(D)) \iff \begin{cases} 0 < r < 2 \\ 0 < \theta < 2\pi \end{cases}$$

Fie

$$A = (0,2) \times (0,2\pi)$$

Deoarece

$$D = \phi(A) \cup ([0,2] \times \{0\} \cup \mathrm{Fr}(D)) \quad \text{si} \quad \lambda([0,2] \times \{0\} \cup \mathrm{Fr}(D)) = 0$$

si

$$dxdy = |J_\phi(r,\theta)|\, drd\theta = rdrd\theta$$

avem

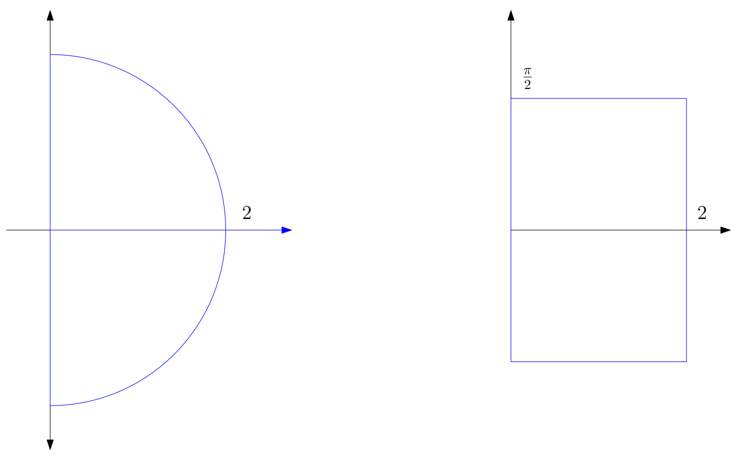
$$\begin{aligned}\iint_D x^2 dxdy &= \iint_{\phi(A)} x^2 dxdy = \iint_A r^2\cos^2\theta \cdot r dr d\theta = \int_0^2 \left(\int_0^{2\pi} r^3 \cos^2\theta d\theta\right) dr \\ &= \int_0^2 \left(r^3 \int_0^{2\pi} \frac{\cos 2\theta + 1}{2} d\theta\right) dr = \int_0^2 r^3 \left(\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right)\Big|_0^{2\pi} dr \\ &= \int_0^2 \pi r^3 dr = \frac{\pi r^4}{4}\Big|_0^2 = 4\pi.\end{aligned}$$

Exemplu. Calculati

$$\iint_D y dxdy, \qquad D = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 \le 4, x \ge 0\}$$

Solutie. Vom trece la coordonate polare

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



Aplicatia

$$\phi : (0,\infty) \times (-\pi, \pi) \to \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \le 0\}$$

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

este difeomorfism si

$$J_\phi(r,\theta) = r.$$

$$(x,y) = \phi(r,\theta) \in \mathring{D} = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 < 4, x > 0\} \Longleftrightarrow \begin{cases} 0 < r < 2 \\ -\pi/2 < \theta < \pi/2 \end{cases}.$$

Fie

$$A = (0,2) \times (-\pi/2, \pi/2).$$

Atunci $\phi(A) = D \setminus \mathrm{Fr}(D) = \overset{\circ}{D}$. Deoarece

$$\phi(A) \cup \mathrm{Fr}(D) = D, \quad \text{si} \quad \lambda(\mathrm{Fr}(D)) = 0,$$

$$\overline{A} = [0,2] \times [-\pi/2, \pi/2]$$

si $dxdy = rdrd\theta$ avem

$$\begin{aligned}\iint_D ydxdy &= \iint_{\phi(A)} ydxdy = \iint_A r\sin\theta \cdot rdrd\theta = \iint_{\overline{A}} r\sin\theta \cdot drd\theta \\ &= \int_0^2 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \sin\theta d\theta\right) dr = \int_0^2 (-r^2\cos\theta)\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} dr = 0.\end{aligned}$$

Trecerea la coordonate polare generalizate

Se foloseste atunci cand multimea pe care calculam integrala este delimitata de o portiune de elipsa. Fie $\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Phi(r,\theta) = (x,y)$ unde

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}, \qquad a, b > 0$$

Restrictionand Φ la $(0,\infty) \times (0, 2\pi) \subset \mathbb{R}^2$ obtinem un difeomorfism

$$\phi : (0,\infty) \times (0,2\pi) \to \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}, \quad \phi(r,\theta) = (ar\cos\theta, br\sin\theta)$$

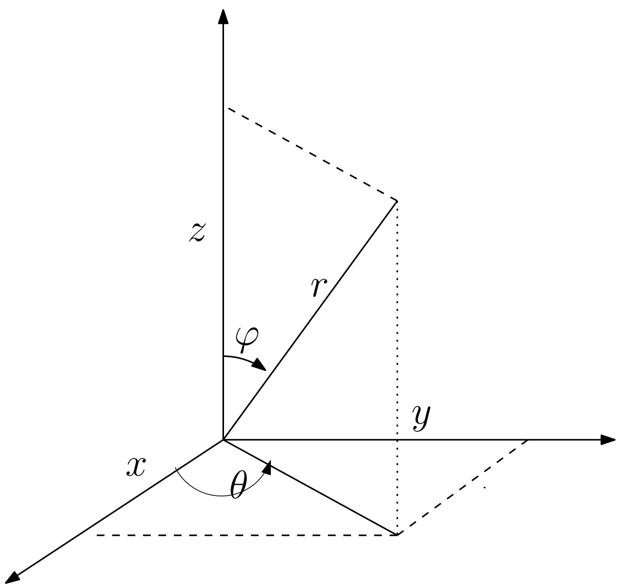
cu

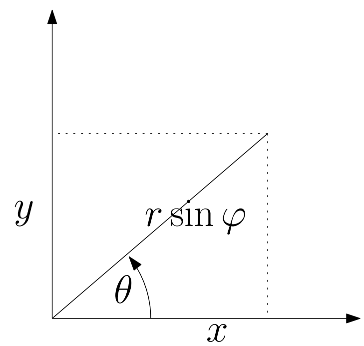
$$J_\phi(r,\theta) = abr \text{ pentru orice } (r,\theta) \in \mathbb{R}^2.$$

Trecerea la coordonate sferice

Fie $\Phi : \mathbb{R}^3 \to \mathbb{R}^3, \quad \Phi(r,\theta,\varphi) = (x,y,z)$ unde

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$





Daca restrictionam Φ la $U = (0,\infty)\times(0,2\pi)\times(0,\pi)$ obtinem un difeomorfism intre U si $\Phi(U)$ pe care il notam cu ϕ, adica

$$\phi : (0,\infty)\times(0,2\pi)\times(0,\pi) \to \mathbb{R}^3 \setminus \{(x,0,z) : x \geq 0, z \in \mathbb{R}\}.$$

$$\phi(r,\theta,\varphi) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi).$$

Avem

$$J_\phi(r,\theta,\varphi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix}$$

$$= -r^2\cos^2\theta\sin^3\varphi - r^2\sin^2\theta\cos^2\varphi\sin\varphi - r^2\cos^2\theta\cos^2\varphi\sin\varphi - r^2\sin^2\theta\sin^3\varphi$$

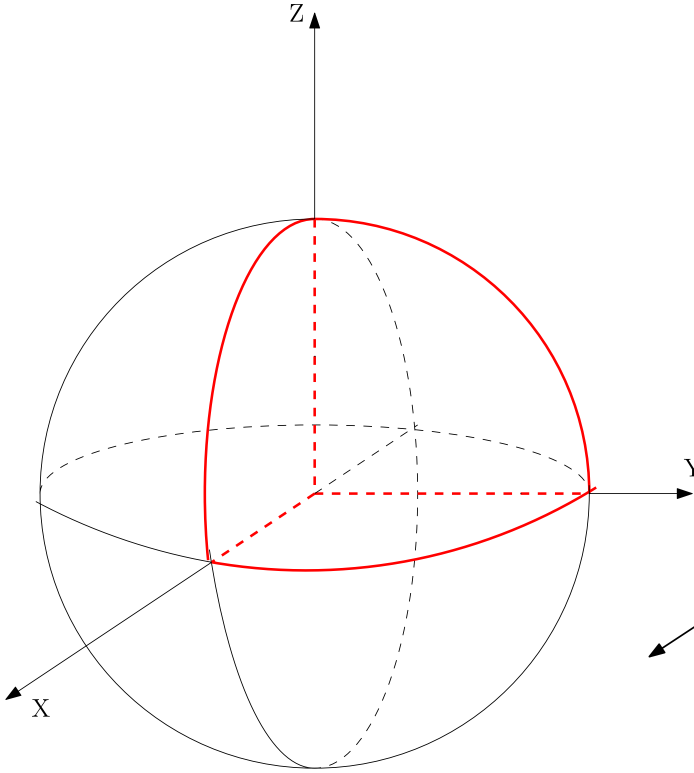
$$= -r^2\sin^3\varphi - r^2\cos^2\varphi\sin\varphi = -r^2\sin\varphi$$

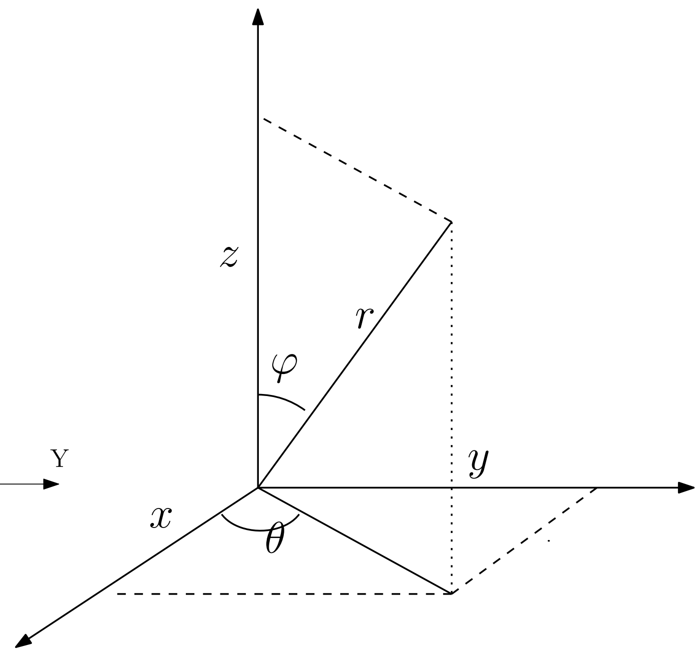
pentru orice $(r,\theta,\varphi) \in U$.

Exemplu. Sa se calculeze

$$\iiint_V x^2 dxdydz \quad V = \{(x,y,z) : x^2+y^2+z^2 \leq 4, x,y,z \geq 0\}$$

Vom trece la coordonate sferice





$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

Transformarea

$$\phi : (0,\infty) \times (0,2\pi) \times (0,\pi) \to \mathbb{R}^3 \setminus \{(x,0,z) : x \geq 0, z \in \mathbb{R}\}$$

$$\phi(r,\theta,\varphi) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi).$$

este difeomorfism si

$$J_\phi(r,\theta,\varphi) = -r^2\sin\varphi \qquad \text{pentru orice } (r,\theta,\varphi) \in (0,\infty)\times(0,2\pi)\times(0,\pi).$$

$$(x,y,z) = \phi(r,\theta,\varphi) \in \mathring{V} = \{(x,y,z) \in \mathbb{R}^2, x^2+y^2+z^2 < 9, x,y,z > 0\} \Longleftrightarrow \begin{cases} 0 < r < 2 \\ 0 < \theta < \pi/2 \\ 0 < \varphi < \pi/2 \end{cases}.$$

Fie $A = (0,2) \times (0, \pi/2) \times (0, \pi/2)$. Atunci $\phi(A) = \overset{\circ}{V}$ si $V = \phi(A) \cup \mathrm{Fr}(V)$. Deoarece

$$\lambda(\mathrm{Fr}(V)) = 0$$

$$dxdydz = |J_\phi(r,\theta,\varphi)|\ drd\theta d\varphi = r^2 \sin\varphi drd\theta d\varphi$$

rezulta ca

$$\iiint_V x^2 dxdydz = \iiint_{\overset{\circ}{V}} x^2 dxdydz = \iiint_{\varphi(A)} x^2 dxdydz = \iiint_A r^2 \cos^2\theta \sin^2\varphi \cdot r^2 \sin\varphi drd\theta d\varphi$$

$$\int_0^2 \left(\int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} r^4 \cos^2\varphi \sin^3\varphi d\varphi \right) d\theta \right) dr = \left(\int_0^2 r^4 dr \right) \cdot \left(\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \right) \cdot \left(\int_0^{\frac{\pi}{2}} \sin^3\varphi d\varphi \right)$$

$$\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} d\theta = \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^3\varphi d\varphi = \int_0^{\frac{\pi}{2}} sin\varphi(1 - \cos^2\varphi) d\varphi = -\cos\varphi\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (\cos\varphi)' \cos^2\theta d\varphi$$

$$= 1 + \frac{\cos^3\varphi}{3}\Bigg|_0^{\frac{\pi}{2}} = 1 - 1/3 = \frac{2}{3}.$$

$$\int_0^2 r^4 dr = \frac{3^5}{5} = \frac{32}{5}$$

In concluzie

$$\iiint_V x^2 dxdydz = \frac{32}{5} \cdot \frac{2}{3} \cdot \frac{\pi}{4} = \frac{16\pi}{15}.$$

Trecerea la coordonate sferice generalizate

Fie $\Phi : \mathbb{R}^3 \to \mathbb{R}^3, \quad \Phi(r, \theta, \varphi) = (x, y, z)$ unde

$$\begin{cases} x = ar\cos\theta\sin\varphi \\ y = br\sin\theta\sin\varphi \\ z = cr\cos\varphi \end{cases} \quad , \quad a, b, c > 0$$

Daca restrictionam Φ la multimea deschisa $U = (0, \infty) \times (0, 2\pi) \times (0, \pi)$ obtinem un difeomorfism intre U si $\Phi(U)$ pe care il notam cu ϕ, adica

$$\phi : (0, \infty) \times (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3 \setminus \{(x, 0, z) : x \geq 0, z \in \mathbb{R}\}.$$

$$\phi(r, \theta, \varphi) = (ar\cos\theta\sin\varphi, br\sin\theta\sin\varphi, cr\cos\varphi).$$

Avem

$$J_\phi(r, \theta, \varphi) = -abcr^2\sin\varphi \text{ pentru orice } (r, \theta, \varphi) \in U$$

Exemplu. Calculati

$$\iiint_V (x^2+y^2)dxdydz, \quad V = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2 \leq z^2,\ 0 \leq z \leq 3\} \quad .$$

Solutie. Observam ca

$$V = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} \leq z \leq 3, \quad (x, y) \in D\}$$

unde D, proiectia lui V pe planul xOy este

$$D = \{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 \leq 9\}.$$

Deoarece D este masurabila Jordan si functiile $\alpha, \beta : D \to \mathbb{R}$

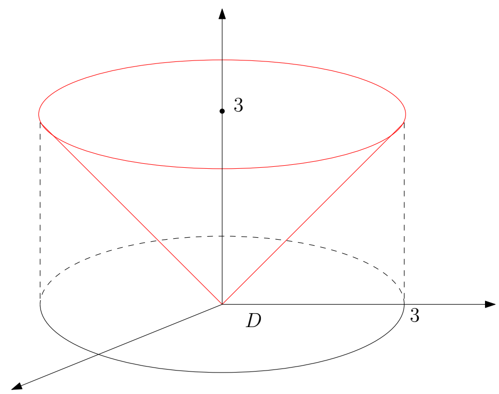
$$\alpha(x, y) = \sqrt{x^2 + y^2}, \quad \beta(x, y) = 3.$$

sunt continue si marginite pe D, rezulta ca V este masurabila Jordan.

Functia $f : V \to \mathbb{R},\ f(x, y, z) = x^2 + y^2$ este continua si deoarece V este multime compacta rezulta ca f este si marginita. Deci f integrabila Riemann.

Deci,

$$\iiint_V zdxdydz = \iint_D \left(\int_{\sqrt{x^2+y^2}}^3 (x^2 + y^2)dz \right) dxdy = \iint_D (x^2 + y^2)(3 - \sqrt{x^2 + y^2})dxdy$$



Vom trece la coordonate polare

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Transformarea

$$\phi : (0,\infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \geq 0\}$$

este un difeomorfism si pentru orice (r,θ) avem

$$J_\phi(r,\theta) = r.$$

$$(x,y) = \phi(r,\theta) \in D \setminus ([0,3] \times \{0\} \cup \mathrm{Fr}(D)) \Longleftrightarrow \begin{cases} 0 < r < 3 \\ 0 < \theta < 2\pi \end{cases} .$$

Fie

$$A = (0,3) \times (0, 2\pi).$$

Avem

$$\phi(A) = D \setminus ([0,3] \times \{0\} \cup \mathrm{Fr}(D))$$

Cum $\lambda\left([0,3] \times \{0\} \cup \mathrm{Fr}(D)\right) = 0$ avem

$$\iint_D (x^2+y^2)(3-\sqrt{x^2+y^2})dxdy = \iint_{\phi(A)} r^3(3-r)drd\theta = \int_0^3 \left(\int_0^{2\pi} (3r^3 - r^4)d\theta\right) dr$$

$$= 2\pi \int_0^3 (3r^3 - r^4)dr = \frac{243}{10}\pi.$$

Exemplu. Calculati

$$\iiint_V (x^2 + y^2) dxdydz$$

unde V este multimea marginita de cilindrul $x^2 + y^2 = 9$, conul $x^2 + y^2 = z^2$ si planul $z = 0$.

Solutie. Folosim desenul de la exeritiul precedent. Intersectia dintre cilindrul $x^2 + y^2 = 9$, conul $x^2 + y^2 = z^2$ este

$$\begin{cases} x^2 + y^2 = 9 \\ z = 3 \end{cases}$$

Asadar,

$$V = \{(x, y, z) \in \mathbb{R}^3 : \ 0 \le z \le \sqrt{x^2 + y^2}, (x, y) \in D\}$$

unde

$$D = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \le 9\}$$

Prin urmare,

$$\iiint_V (x^2+y^2) dxdydz == \iint_D \left(\int_0^{\sqrt{x^2+y^2}} (x^2 + y^2) dz \right) dxdy = \iint_D (x^2+y^2) \sqrt{x^2 + y^2} dxdy.$$

Aceasta integrala se rezolva trecand la coordonate polare.