TUTORIAT 1

1. Să se rezolve sistemul de ecuații liniare folosind MEGFP și metoda substituției descendente:

$$\begin{cases} 4x_1 + 2x_2 - x_3 = -5 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 = -1 \\ x_1 + 4x_2 + 2x_3 = 9 \end{cases} \Rightarrow A = \begin{bmatrix} 4 & 2 & -1 \\ \frac{1}{9} & \frac{1}{9} & -\frac{1}{3} \\ 1 & 4 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} -5 \\ -1 \\ 9 \end{bmatrix} \qquad n = 3, \quad k = \overline{1,2}$$

Pentru $k = 1$:

$$\begin{array}{c} \\ i=2 \\ i=3 \end{array} \bar{A}^{(1)} = \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 1/9 & 1/9 & -1/3 & -1 \\ 1 & 4 & 2 & 9 \end{array}\right] = \left[A^{(1)} \mid b^{(1)}\right]$$

$a_{11}^{(1)} = 4 \neq 0 \Rightarrow$ putem aplica MEGFP

$i = \overline{2,3}: \quad m_i^{(1)} = a_{i1}^{(1)} / a_{11}^{(1)}$

$E_2 \leftarrow E_2 - m_2^{(1)} E_1$

$$m_2^{(1)} = a_{21}^{(1)} / a_{11}^{(1)} = \frac{\frac{1}{9}}{4} = \frac{1}{36}$$

$j = \overline{2,3}: \quad a_{2j}^{(2)} = a_{2j}^{(1)} - m_2^{(1)} \cdot a_{1j}^{(1)}$

$$a_{22}^{(2)} = a_{22}^{(1)} - m_2^{(1)} \cdot a_{12}^{(1)} = \frac{1}{9} - \frac{1}{36} \cdot 2 = \frac{1}{9} - \frac{1}{18} = \frac{2-1}{18} = \frac{1}{18}$$

$$a_{23}^{(2)} = a_{23}^{(1)} - m_2^{(1)} \cdot a_{13}^{(1)} = -\frac{1}{3} - \frac{1}{36} \cdot (-1) = -\frac{1}{3} + \frac{1}{36} = \frac{-12+1}{36} = -\frac{11}{36}$$

$$b_2^{(2)} = b_2^{(1)} - m_2^{(1)} \cdot b_1^{(1)} = -1 - \frac{1}{36} \cdot (-5) = -1 + \frac{5}{36} = \frac{-36+5}{36} = -\frac{31}{36}$$

$$\left\{ a_{21}^{(2)} = a_{21}^{(1)} - m_2^{(1)} \cdot a_{11}^{(1)} = \frac{1}{9} - \frac{1}{36} \cdot 4 = \frac{1}{9} - \frac{1}{9} = 0 \right\}$$

$E_3 \leftarrow E_3 - m_3^{(1)} \cdot E_1$

$m_3^{(1)} = a_{31}^{(1)} / a_{11}^{(1)} = \frac{1}{4}$

$j = \overline{2,3}: \quad a_{3j}^{(2)} = a_{3j}^{(1)} - m_3^{(1)} \cdot a_{1j}^{(1)}$

$a_{32}^{(2)} = a_{32}^{(1)} - m_3^{(1)} \cdot a_{12}^{(1)} = 4 - \frac{1}{4} \cdot 2 = 4 - \frac{1}{2} = \frac{8-1}{2} = \frac{7}{2}$

$a_{33}^{(2)} = a_{33}^{(1)} - m_3^{(1)} \cdot a_{13}^{(1)} = 2 - \frac{1}{4} \cdot (-1) = 2 + \frac{1}{4} = \frac{8+1}{4} = \frac{9}{4}$

$b_3^{(2)} = b_3^{(1)} - m_3^{(1)} \cdot b_1 = 9 - \frac{1}{4} \cdot (-5) = 9 + \frac{5}{4} = \frac{41}{4}$

$\{ a_{31}^{(2)} = 0 \}$

Pentru $k=2$, am obținut $\bar{A}^{(2)} = \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 1/18 & -11/36 & -31/36 \\ 0 & 7/2 & 9/4 & 41/4 \end{array}\right] = [A^{(2)} \mid b^{(2)}]$

Matricea de transformare $(M^{(1)})$, care transformă $\bar{A}^{(1)}$ în $\bar{A}^{(2)}$ este:

$$M^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -m_2^{(1)} & 1 & 0 \\ -m_3^{(1)} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/36 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}$$

Are loc relația: $M^{(1)} [A^{(1)} \;\; b^{(1)}] = [A^{(2)} \;\; b^{(2)}] \qquad (1)$

$a_{22}^{(2)} = 1/18 \neq 0 \Rightarrow$ putem aplica MEGFP

$i = \overline{3,3}: \quad m_i^{(2)} = a_{i2}^{(2)} / a_{22}^{(2)}$

$m_3^{(2)} = a_{32}^{(2)} / a_{22}^{(2)} = \frac{7}{2} \cdot \frac{18}{1} = 63$

$E_3 \leftarrow E_3 - m_3^{(2)} E_2$

$j = \overline{3,3}: \quad a_{3j}^{(3)} = a_{3j}^{(2)} - m_3^{(2)} \cdot a_{2j}^{(2)}$

$a_{33}^{(3)} = a_{33}^{(2)} - m_3^{(2)} \cdot a_{23}^{(2)} = \frac{9}{4} - \cancel{63} \cdot \left(-\frac{11}{\cancel{36}}\right) = \frac{9}{4} + \frac{77}{4} = \frac{86}{4} = \frac{43}{2}$

$b_3^{(3)} = b_3^{(2)} - m_3^{(2)} \cdot b_2^{(2)} = \frac{41}{4} - \cancel{63} \cdot \left(-\frac{31}{\cancel{36}}\right) = \frac{41}{4} + \frac{217}{4} = \frac{258}{4} = \frac{129}{2}$

$\{ a_{32}^{(2)} = 0 \}$

Am obținut: $\bar{A}^{(3)} = \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 1/18 & -11/36 & -31/36 \\ 0 & 0 & 43/2 & 129/2 \end{array}\right]$

Matricea de transformare $(M^{(2)})$, care transformă $\bar{A}^{(2)}$ în $\bar{A}^{(3)}$ este:

$$M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_3^{(2)} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -63 & 1 \end{bmatrix}$$

Are loc relația: $M^{(2)} [A^{(2)}\ \underline{b}^{(2)}] = [A^{(3)}\ \underline{b}^{(3)}] = [U\ \underline{\tilde{b}}]$ (2)

Din (1) și (2) $\Rightarrow M^{(2)} \cdot M^{(1)} \cdot [A | \underline{b}] = [U | \underline{\tilde{b}}]$

Sistemul $A\underline{x} = \underline{b}$ devine $U\underline{x} = \underline{\tilde{b}}$

$$\left\{\begin{array}{l} 4x_1 + 2x_2 - x_3 = -5 \\ \frac{1}{18}x_2 - \frac{11}{36}x_3 = -\frac{31}{36} \\ \frac{43}{2}x_3 = \frac{129}{2} \end{array}\right.$$

aplicăm metoda substituției descendente

$\frac{43}{2}x_3 = \frac{129}{2} \Rightarrow 43x_3 = 129 \Rightarrow x_3 = 3$

$\frac{1}{18}x_2 = \frac{11}{36}x_3 - \frac{31}{36} = \frac{11}{36}\cdot 3 - \frac{31}{36} = \frac{2}{36} = \frac{1}{18} \Rightarrow x_2 = 1$

$4x_1 = -5 + x_3 - 2x_2 = -5 + 3 - 2\cdot 1 = -4 \Rightarrow x_1 = -1$

$(x_1, x_2, x_3) = (-1, 1, 3)$.

2. Să se rezolve sistemul de ecuații folosind MEGPP și metoda subst. desc.

$$\left\{\begin{array}{l} 2x_1 - 3x_2 + 2x_3 = 5 \\ -4x_1 + 2x_2 - 6x_3 = 14 \\ 2x_1 + 2x_2 + 4x_3 = 8 \end{array}\right. \Rightarrow A = \begin{bmatrix} 2 & -3 & 2 \\ -4 & 2 & -6 \\ 2 & 2 & 4 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 5 \\ 14 \\ 8 \end{bmatrix}$$

$n = 3 \Rightarrow k = \overline{1,2}$

Pentru $k=1$: $\bar{A}^{(1)} = [A^{(1)} \mid \underline{b}^{(1)}] = \left[\begin{array}{ccc|c} 2 & -3 & 2 & 5 \\ -4 & 2 & -6 & 14 \\ 2 & 2 & 4 & 8 \end{array}\right]$

Căutăm maxim pe coloana 1: $\max\limits_{i=\overline{1,3}} |a_{i1}| = \max\{|2|, |-4|, |2|\} = |-4|$

$= 4 = |a_{21}^{(1)}| = |a_{\ell 1}^{(1)}|,\ \ell = 2 > 1 \Rightarrow$ interschimbăm, prin intermediul matricii de permutare, $E_1 \longleftrightarrow E_2$

Matricea de permutare simplă: $P^{(1)} = P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (1a)

$P^{(1)} \cdot \bar{A}^{(1)} = P^{(1)} \cdot [A^{(1)} \mid \underline{b}^{(1)}] = \left[\begin{array}{ccc|c} -4 & 2 & -6 & 14 \\ 2 & -3 & 2 & 5 \\ 2 & 2 & 4 & 8 \end{array}\right] = [\tilde{A}^{(1)} \mid \underline{\tilde{b}}^{(1)}] = \overline{\tilde{A}}^{(2)}$

$\tilde{a}_{11}^{(1)} = -4 \neq 0 \Rightarrow$ putem aplica MEGPP

$i = \overline{2,3}$: $m_i^{(1)} = \tilde{a}_{i1}^{(1)} / \tilde{a}_{11}^{(1)}$

$m_2^{(1)} = \tilde{a}_{21}^{(1)} / \tilde{a}_{11}^{(1)} = \frac{2}{-4} = -\frac{1}{2}$

$E_2 \longleftarrow E_2 - m_2^{(1)} \cdot E_1$

$j = \overline{2,3}$: $a_{2j}^{(2)} = \tilde{a}_{2j}^{(1)} - m_2^{(1)} \tilde{a}_{1j}^{(1)}$

$a_{22}^{(2)} = \tilde{a}_{22}^{(1)} - m_2^{(1)} \cdot \tilde{a}_{12}^{(1)} = -3 - \left(-\frac{1}{2}\right) \cdot 2 = -3 + 1 = -2$

$a_{23}^{(2)} = \tilde{a}_{23}^{(1)} - m_2^{(1)} \cdot \tilde{a}_{13}^{(1)} = 2 - \left(-\frac{1}{2}\right) \cdot (-6) = 2 - 3 = -1$

$b_2^{(2)} = \tilde{b}_2^{(1)} - m_2^{(1)} \cdot \tilde{b}_1^{(1)} = 5 - \left(-\frac{1}{2}\right) \cdot 14 = 5 + 7 = 12$

$m_3^{(1)} = \tilde{a}_{31}^{(1)} / \tilde{a}_{11}^{(1)} = 2 / -4 = -\frac{1}{2}$

$E_3 \longleftarrow E_3 - m_3^{(1)} \cdot E_1$

$j = \overline{2,3}$: $a_{3j}^{(2)} = \tilde{a}_{3j}^{(1)} - m_3^{(1)} \cdot \tilde{a}_{1j}^{(1)}$

$a_{32}^{(2)} = \tilde{a}_{32}^{(1)} - m_3^{(1)} \cdot \tilde{a}_{12}^{(1)} = 2 - \left(-\frac{1}{2}\right) \cdot 2 = 2 + 1 = 3$

$a_{33}^{(2)} = \tilde{a}_{33}^{(1)} - m_3^{(1)} \cdot \tilde{a}_{13}^{(1)} = 4 - \left(-\frac{1}{2}\right) \cdot (-6) = 4 - 3 = 1$

$b_3^{(2)} = \tilde{b}_3^{(1)} - m_3^{(1)} \cdot \tilde{b}_1^{(1)} = 8 - \left(-\frac{1}{2}\right) \cdot 14 = 8 + 7 = 15$

Pentru $k=2$, am obținut: $\bar{A}^{(2)} = [A^{(2)} \mid \underline{b}^{(2)}] = \left[\begin{array}{ccc|c} -4 & 2 & -6 & 14 \\ 0 & -2 & -1 & 12 \\ 0 & 3 & 1 & 15 \end{array}\right]$

Matricea care transformă $\overline{\tilde{A}^{(1)}}$ în $\bar{A}^{(2)}$ este: $M^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$ (1b)

Are loc relația: $M^{(1)} \cdot P^{(1)} \cdot \bar{A}^{(1)} = \bar{A}^{(2)}$ (1)

cu $P^{(1)}$ și $M^{(1)}$ date (1a) și (1b)

Căutăm maxim pe coloana 2 în $\bar{A}^{(2)}$: $\max\limits_{i=\overline{2,3}} |a_{i2}^{(2)}| = \max\{|-2|, |3|\} = |3| = 3$

$= |a_{32}^{(2)}| = |a_{l2}^{(2)}|$, $l = 3 > 2$ $\Rightarrow$ interschimbăm, prin intermediul matricei de permutare, liniile $E_2 \leftrightarrow E_3$

Matricea de permutare simplă: $P^{(2)} = P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (2a)

$P^{(2)} \cdot \bar{A}^{(2)} = P^{(2)} [A^{(2)} \mid \underline{b}^{(2)}] = [\tilde{A}^{(2)} \mid \tilde{\underline{b}}^{(2)}] = \overline{\tilde{A}^{(2)}} = \left[\begin{array}{ccc|c} -4 & 2 & -6 & 14 \\ 0 & 3 & 1 & 15 \\ 0 & -2 & -1 & 12 \end{array}\right]$

$\tilde{a}_{22}^{(2)} = 3 \neq 0 \Rightarrow$ putem aplica MEGFP

$i=\overline{3,3}$: $m_i^{(2)} = \tilde{a}_{i2}^{(2)} / \tilde{a}_{22}^{(2)} \Rightarrow m_3^{(2)} = \tilde{a}_{32}^{(2)} / \tilde{a}_{22}^{(2)} = -\frac{2}{3}$

$E_3 \leftarrow E_3 - m_3^{(2)} \cdot E_2$

$j=\overline{3,3}$: $a_{3j}^{(3)} = \tilde{a}_{3j}^{(2)} - m_3^{(2)} \cdot \tilde{a}_{2j}^{(2)}$

$a_{33}^{(3)} = \tilde{a}_{33}^{(2)} - m_3^{(2)} \cdot \tilde{a}_{23}^{(2)} = -1 - \left(-\frac{2}{3}\right) \cdot 1 = -1 + \frac{2}{3} = -\frac{1}{3}$

$b_3^{(3)} = \tilde{b}_3^{(2)} - m_3^{(2)} \cdot \tilde{b}_2^{(2)} = 12 - \left(-\frac{2}{3}\right) \cdot 15 = 12 + 10 = 22$

Am obținut: $\bar{A}^{(3)} = [A^{(3)} \mid b^{(3)}] = \left[\begin{array}{ccc|c} -4 & 2 & -6 & 14 \end{array}\right. = [U \mid \tilde{b}]$

$$\begin{bmatrix} 0 & 3 & 1 & \vert & 15 \\ 0 & 0 & -1/3 & \vert & 22 \end{bmatrix}$$

Matricea care transformă $\overline{\tilde{A}^{(2)}}$ în $\overline{A^{(3)}}$ este $M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}$ (2b)

Are loc relația: $\underbrace{M^{(2)} \cdot P^{(2)} \cdot \overline{\tilde{A}^{(2)}} = [U \vert \underline{\tilde{b}}]}_{(2)}$, cu $M^{(2)}$ și $P^{(2)}$ date de (2a) și (2b)

Din (1) și (2) $\Rightarrow M^{(2)} P^{(2)} M^{(1)} P^{(1)} [A \vert \underline{b}] = [U \vert \underline{\tilde{b}}]$

Sistemul $A\underline{x} = \underline{b}$ a devenit $U\underline{x} = \underline{\tilde{b}}$

$$\begin{cases} -4x_1 + 2x_2 - 6x_3 = 14 \\ 3x_2 + x_3 = 15 \\ -\frac{1}{3}x_3 = 22 \end{cases}$$

$-\frac{1}{3}x_3 = 22 \Rightarrow x_3 = -66$

$3x_2 = 15 - x_3 = 15 + 66 = 81 \Rightarrow x_2 = 27$

$-4x_1 = 14 + 6x_3 - 2x_2 = 14 + 6 \cdot (-66) - 2 \cdot 27 = 14 - 396 - 54 = -436 \Rightarrow x_1 = 109$

$(x_1, x_2, x_3) = (109, 27, -66)$.

miro