Examen Algebră liniară, Seria 11
17.01.2022

La rezolvarea problemelor veţi folosi următorii parametri:
$M =$ numărul de litere din primul vostru nume de familie
$N =$ numărul de litere din primul vostru prenume.
De exemplu, pentru *Ionescu Ana-Maria* avem $M = 7$ şi $N = 3$.

Scrieţi pe prima pagină cu rezolvări valorile parametrilor voştri:
$M = \dots,\quad N = \dots.$
Timp pentru rezolvarea problemelor şi încărcarea soluţiilor în MoodleUB: 3 ore

(1) (2,25 pct.) În $\mathbb{R}^4$ considerăm vectorii

$$u_1 = \begin{pmatrix} 1 \\ M+2 \\ 2N+1 \\ N \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ N-1 \\ M+2 \\ N-1 \end{pmatrix}, u_3 = \begin{pmatrix} 7 \\ M+2N \\ 2N+2M+5 \\ 3N-2 \end{pmatrix}, u_4 = \begin{pmatrix} -1 \\ 2M-N+3 \\ 4N-M \\ N+1 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 4 \\ 3 \\ N+1 \end{pmatrix}$$

şi subspaţiul vectorial $V = \langle u_1, u_2, u_3, u_4 \rangle$.

(a) Aflaţi $\dim V$ şi o bază $\mathcal{B}$ pentru V.
(b) Completaţi $\mathcal{B}$ la o bază pentru $\mathbb{R}^4$.
(c) Aflaţi $pr_V(w)$ proiecţia ortogonală a lui w pe V şi $||pr_V(w)||$.
(d) Este w in V? Argumentaţi.

(2) (1,5 pct.) Pentru orice $n \in \mathbb{N}, n \geq 2$ notăm

$$A_n = \begin{pmatrix} 2 & N & 0 & \dots & \dots & 0 \\ N & 2 & N & 0 & \dots & 0 \\ N & N & 2 & N & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ N & N & \dots & N & 2 & N \\ N & N & \dots & N & N & 2 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}) \text{ şi } d_n = \det(A_n).$$

(a) Calculaţi d_2, d_3, d_4.
(b) Stabiliţi dacă matricea A_3 este inversabilă, iar în caz afirmativ găsiţi-i inversa.
(c) Arătaţi că are loc relaţia $d_n = 2d_{n-1} + N^2(N-2) \cdot d_{n-3}$, pentru orice $n \geq 5$.

(3) (2,75 pct.) Fie aplicaţia liniară $T : \mathbb{R}^3 \to \mathbb{R}^3$ dată de

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x-2y-4z \\ y \\ 12x-4y-7z \end{pmatrix}, \text{ pentru orice } x, y, z \in \mathbb{R}.$$

(a) Determinaţi subspaţiul $\operatorname{Ker} T \cap \operatorname{Im} T$.

Subiectele continuă pe pagina următoare...

(b) Verificaţi dacă $w = \begin{pmatrix} 3 \\ M-1 \\ N+1 \end{pmatrix}$ este vector propriu pentru T.

(c) Aflaţi valorile proprii ale lui T.

(d) Găsiţi o bază $\mathcal{B}$ in $\mathbb{R}^3$ astfel încât matricea lui T în baza $\mathcal{B}$ să fie diagonală. Argumentaţi dacă poate fi $\mathcal{B}$ aleasă să fie ortonormată.

(4) (1,5 pct.) Fie v_1, v_2, v_3 vectori din $\mathbb{R}$-spaţiul vectorial V. Arătaţi că dacă v_1, v_2, v_3 sunt liniar independenţi, atunci şi $Mv_1 + v_2$, $Mv_2 + v_3$, $Mv_3 + v_1$ sunt liniar independenţi.

Este reciproca adevărată? Argumentaţi.

(5) (1 pct.) Fie $A \in \mathcal{M}_{m,n}(\mathbb{R})$ o matrice de rang r. Arătaţi că există două matrici $B \in \mathcal{M}_{m,r}(\mathbb{R})$, $C \in \mathcal{M}_{r,n}(\mathbb{R})$ ambele de rang r astfel încât $A = BC$.

Se acordă 1 punct din oficiu.

Justificaţi toate răspunsurile date, arătând calculele efectuate.

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Examen algebră liniară 113

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$M=5$

$N=4$

(1) $u_1 = \begin{pmatrix} 1 \\ 7 \\ 9 \\ 4 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ 3 \\ 7 \\ 3 \end{pmatrix}, u_3 = \begin{pmatrix} 7 \\ 13 \\ 23 \\ 10 \end{pmatrix}, u_4 = \begin{pmatrix} -1 \\ 9 \\ 11 \\ 5 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 5 \end{pmatrix}$

$V = \langle u_1, u_2, u_3, u_4 \rangle$

(a) Verific liniar independența lui u_1, u_2, u_3, u_4

$$\begin{pmatrix} 1 & 3 & 7 & -1 \\ 7 & 3 & 13 & 9 \\ 9 & 7 & 23 & 11 \\ 4 & 3 & 10 & 5 \end{pmatrix} \xrightarrow[\substack{L_3' = L_3 - 9L_1 \\ L_4' = L_4 - 4L_1}]{L_2' = L_2 - 7L_1} \begin{pmatrix} 1 & 3 & 7 & -1 \\ 0 & -18 & -36 & 16 \\ 0 & -20 & -40 & 20 \\ 0 & -9 & -18 & 9 \end{pmatrix} \xrightarrow[L_2 \leftrightarrow L_3]{L_3' = -\frac{1}{20} L_3}$$

$$\begin{pmatrix} 1 & 3 & 7 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & -18 & -36 & 16 \\ 0 & -9 & -18 & 9 \end{pmatrix} \xrightarrow[\substack{L_3' = L_3 + 18L_2 \\ L_4' = L_4 + 9L_2}]{L_1' = L_1 - 3L_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_3' = -\frac{1}{2} L_3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[L_2' = L_2 + L_3]{L_1' = L_1 - 2L_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{am 3 pivoți} \Rightarrow$$

$\Rightarrow u_1, u_2, u_4$ sunt liniar independenți $\Rightarrow u_3$ e o combinație liniară de $u_1, u_2, u_4 \Leftrightarrow u_3 \in \langle u_1, u_2, u_4 \rangle$

$\Rightarrow \mathcal{B} = \langle u_1, u_2, u_4 \rangle$ bază pt. $V \Rightarrow \dim \mathcal{B} = 3$

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(b) Verific dacă B poate fi completată cu e_1:

$$\begin{pmatrix} 1 & 3 & -1 & 1 \\ 7 & 3 & 9 & 0 \\ 9 & 7 & 11 & 0 \\ 4 & 3 & 5 & 0 \end{pmatrix} \xrightarrow[\substack{L'_3 = L_3 - 9L_1 \\ L'_4 = L_4 - 4L_1}]{L'_2 = L_2 - 7L_1} \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -18 & 16 & -7 \\ 0 & -20 & 20 & -9 \\ 0 & -9 & 9 & -4 \end{pmatrix} \xrightarrow{L'_2 = -\frac{1}{18} L_2}$$

$$\begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -8/9 & 7/18 \\ 0 & -20 & 20 & -9 \\ 0 & -9 & 9 & -4 \end{pmatrix} \xrightarrow[\substack{L'_3 = L_3 + 20L_2 \\ L'_4 = L_4 + 9L_2}]{L'_1 = L_1 - 3L_2} \begin{pmatrix} 1 & 0 & 5/3 & -1/6 \\ 0 & 1 & -8/9 & 7/18 \\ 0 & 0 & 20/9 & -11/9 \\ 0 & 0 & 1 & -1/2 \end{pmatrix} \xrightarrow{L'_4 = L_4 - \frac{9}{20} L_3}$$

$$\begin{pmatrix} 1 & 0 & 5/3 & -1/6 \\ 0 & 1 & -8/9 & 7/18 \\ 0 & 0 & 20/9 & -11/9 \\ 0 & 0 & 0 & 1/20 \end{pmatrix}$$

$\Rightarrow$ am pivot pe fiecare coloană

$\Rightarrow u_1, u_2, u_4, e_1$ sunt liniar independenți

$\Rightarrow B = \{u_1, u_2, u_4, e_1\}$ bază pt. $\mathbb{R}^4$

(d) $w \in V \Leftrightarrow \exists\, a,b,c,d \in \mathbb{R}$ a.î. $w = au_1 + bu_2 + cu_3 + du_4$

$$\left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 7 & 3 & 13 & 9 & 4 \\ 9 & 7 & 23 & 11 & 3 \\ 4 & 3 & 10 & 5 & 5 \end{array}\right) \xrightarrow[\substack{L'_3 = L_3 - 9L_1 \\ L'_4 = L_4 - 4L_1}]{L'_2 = L_2 - 7L_1} \left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 0 & -18 & -36 & 16 & -31 \\ 0 & -20 & -40 & 20 & -42 \\ 0 & -9 & -18 & 9 & -15 \end{array}\right) \xrightarrow[L_3 \leftrightarrow L_2]{L'_3 = -\frac{1}{20} L_3}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 0 & 1 & 2 & -1 & 21/10 \\ 0 & -18 & -36 & 16 & -31 \\ 0 & -9 & -18 & 9 & -15 \end{array}\right) \xrightarrow[\substack{L'_3 = L_3 + 18L_2 \\ L'_4 = L_4 + 9L_2}]{L'_1 = L_1 - 3L_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -6/5 \\ 0 & 1 & 2 & -1 & 21/10 \\ 0 & 0 & 0 & -2 & 34/5 \\ 0 & 0 & 0 & 0 & -15 + \frac{9 \cdot 21}{10} \end{array}\right)$$

ultima ecuație devine $0 \cdot a + 0 \cdot b + 0 \cdot c + 0 \cdot d = -15 + \frac{9 \cdot 21}{10}$ Fals

$\Rightarrow \nexists\, a,b,c,d \in \mathbb{R}$ a.î. $w = au_1 + bu_2 + cu_3 + du_4 \Leftrightarrow$

$\Leftrightarrow w \notin V$

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(2) $A_n = \begin{pmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & & & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{pmatrix}$

$\det(A_n) = d_n$

(a) $d_2 = \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = -12$

$d_3 = \begin{vmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{vmatrix} = 8 + 64 - 32 - 32 = 8$

$d_4 = \begin{vmatrix} 2 & 4 & 0 & 0 \\ 4 & 2 & 4 & 0 \\ 4 & 4 & 2 & 4 \\ 4 & 4 & 4 & 2 \end{vmatrix} = 2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{vmatrix} + 4 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{vmatrix}$

$= 2 \cdot d_3 - 4 \cdot (16 + 64 - 64 - 32)$

$= 16 + 4 \cdot 16$

$= 5 \cdot 16 = 80$

(b) $\left(\begin{array}{ccccc|ccccc} 2 & 4 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 4 & 4 & 4 & \dots & 2 & 0 & 0 & 0 & \dots & 1 \end{array}\right) \xrightarrow[\substack{L'_2 = L_2 - 4L_1 \\ L'_n = L_n - 4L_1}]{L'_1 = \frac{1}{2}L_1} \left(\begin{array}{ccccc|ccccc} 1 & 2 & 0 & \dots & 0 & 1/2 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ 4 & 4 & 4 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{array}\right)$

$\xrightarrow{}$ $L'_2 = L_2 - 4L_1$
$\vdots$
$L'_n = L_n - 4L_1$

$\left(\begin{array}{ccccc|ccccc} 1 & 2 & 0 & \dots & 0 & 1/2 & 0 & 0 & \dots & 0 \\ 0 & -6 & 4 & \dots & 0 & -2 & 1 & 0 & \dots & 0 \\ 0 & -4 & -14 & \dots & 0 & -2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & & & & & & \\ 0 & -4 & 4 & \dots & 4 & -2 & 0 & 0 & \dots & 2 \end{array}\right)$

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(c) $$d_n = \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{\cancel{d_n \neq 0} \text{ n coloane}} = 2\cdot(-1)^{1+1}\cdot \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} +$$

$$+ 4\cdot(-1)^{1+2} \underbrace{\begin{vmatrix} 4 & 4 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & 0 & \dots & 0 \\ 4 & 4 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} \underset{C_1' = C_1 - C_2}{=}$$

$$= 2\cdot d_{n-1} - 4\cdot \underbrace{\begin{vmatrix} 0 & 4 & 0 & \dots & 0 \\ 2 & 2 & 4 & \dots & 0 \\ 0 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} = 2d_{n-1} - 4\cdot 4\cdot(-1)^{1+2}\cdot$$

$$\cdot \underbrace{\begin{vmatrix} 2 & 4 & 0 & 0 & \dots & 0 \\ 0 & 2 & 4 & 0 & \dots & 0 \\ 0 & 4 & 2 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-2 \text{ coloane}} = 2d_{n-1} + 16\cdot 2\cdot(-1)^{1+1}\cdot \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-3 \text{ coloane}} =$$

$$= 2d_{n-1} + 32 d_{n-3} \iff d_n = 2d_{n-1} + 4^2(4-2)d_{n-3},$$

$\forall n \geq 5$

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(2)(b) $A_3 = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{pmatrix}$

$\det A_3 = d_3 = 8 \neq 0 \Rightarrow A_3$ - inversabilă

${}^tA_3 = \begin{pmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \Rightarrow A_3^* = \begin{pmatrix} -12 & -8 & 16 \\ 8 & 4 & -8 \\ 8 & 8 & -12 \end{pmatrix}$

$A_3^{-1} = \frac{1}{\det A_3} \cdot A_3^* = \begin{pmatrix} -3/2 & -1 & 2 \\ 1 & 1/2 & -1 \\ 1 & 1 & -3/2 \end{pmatrix}$

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(3) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}7x-2y-4z\\ y\\ 12x-4y-7z\end{pmatrix} = \underbrace{\begin{pmatrix}7 & -2 & -4\\ 0 & 1 & 0\\ 12 & -4 & -7\end{pmatrix}}_{A}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

(a) $\operatorname{Ker} T = \left\{ \begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; T\begin{pmatrix}a\\b\\c\end{pmatrix} = 0_{\mathbb{R}^3} \right\}$

$$\Leftrightarrow \begin{pmatrix}7 & -2 & -4\\ 0 & 1 & 0\\ 12 & -4 & -7\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\cancel{7 \; -2 \; -4}$ $\cancel{12}$

$$\begin{pmatrix}7 & -2 & -4\\ 0 & 1 & 0\\ 12 & -4 & -7\end{pmatrix} \xrightarrow{L_1' = \frac{1}{7}L_1} \begin{pmatrix}1 & -2/7 & -4/7\\ 0 & 1 & 0\\ 12 & -4 & -7\end{pmatrix}$$

$$\xrightarrow{L_3' = L_3 - 12L_1} \begin{pmatrix}1 & -2/7 & -4/7\\ 0 & 1 & 0\\ 0 & -4/7 & -1/7\end{pmatrix} \xrightarrow[L_3' = L_3 + \frac{4}{7}L_2]{L_1' = L_1 + \frac{2}{7}L_2}$$

$$\begin{pmatrix}1 & 0 & -4/7\\ 0 & 1 & 0\\ 0 & 0 & -1/7\end{pmatrix} \xrightarrow{L_3' = -7L_3} \begin{pmatrix}1 & 0 & -4/7\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \xrightarrow{L_1' = L_1 + \frac{4}{7}L_3}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \Rightarrow a = b = c = 0 \Rightarrow \operatorname{Ker} T = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\} \Rightarrow \dim(\operatorname{Ker}(T)) = 0$$

$\Rightarrow T$ - injectivă

dim Th. rang-defect $\Rightarrow \dim \mathbb{R}^3 = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(\cancel{T}))$

$\Rightarrow \dim(\operatorname{Im}(T)) = 3 \Rightarrow T$ surjectivă $\Rightarrow \operatorname{Im} T = \mathbb{R}^3$

$\operatorname{Im} T = \left\{ \begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \exists \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \text{ a.î. } \cancel{T}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} \right\}$

$T\begin{pmatrix}x\\y\\z\end{pmatrix} = A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} \Leftrightarrow x \cdot C_1(A) + y \cdot C_2(A) + z \cdot C_3(A) = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$\Rightarrow \begin{pmatrix}a\\b\\c\end{pmatrix} \in \langle C_1(A), C_2(A), C_3(A) \rangle$

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din Ker T am arătat că $C_1(A), C_2(A), C_3(A)$ sunt liniar independenți $\Rightarrow \langle C_1(A), C_2(A), C_3(A)\rangle$ bază pt. Im T
$\Rightarrow \dim(\mathrm{Im}\, T) = 3$; $\mathrm{Ker}\, T = \{0_{\mathbb{R}^3}\}$; $\mathrm{Im}\, T = \mathbb{R}^3$

$\mathrm{Ker}\, T \cap \mathrm{Im}\, T = \{x \in \mathrm{Ker}\, T$ și $x \in \mathrm{Im}\, T\}$

$$\left.\begin{array}{l} x = \begin{pmatrix}0\\0\\0\end{pmatrix} \\ x = a\cdot C_1(A) + b\cdot C_2(A) + c\cdot C_3(A)\end{array}\right| \Rightarrow a = b = c = 0$$

$\Rightarrow \mathrm{Ker}\, T \cap \mathrm{Im}\, T = \mathrm{Ker}\, T = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

~~(b) $w = \begin{pmatrix}3\\4\\5\end{pmatrix}$~~

(c) $P_A(x) = \det(x\cdot I_3 - A)$

$$\det\left(\begin{pmatrix}x&0&0\\0&x&0\\0&0&x\end{pmatrix} - \begin{pmatrix}7&-2&-4\\0&1&0\\12&-4&-7\end{pmatrix}\right) = \det\begin{pmatrix}x-7&2&4\\0&x-1&0\\-12&4&x+7\end{pmatrix} =$$

$$= \begin{vmatrix}x-7&2&4\\0&x-1&0\\-12&4&x+7\end{vmatrix} = (x-1)(x^2-49) + 48(x-1) =$$
$$= (x-1)(x^2-49+48)$$
$$= (x-1)(x^2-1)$$
$$= (x-1)^2(x+1)$$

$\Rightarrow$ valorile proprii sunt $\lambda_1 = 1$, $\lambda_2 = -1$

(b) $\lambda_1 = 1 \Rightarrow (A - 1\cdot I_3)v_1 = 0_{\mathbb{R}^3}$

$$\begin{pmatrix}6&-2&-4\\0&0&0\\12&-4&-8\end{pmatrix} v_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}; \quad v_1 = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$$

$$\begin{pmatrix}6&-2&-4\\12&-4&-8\\0&0&0\end{pmatrix} \xrightarrow{L_1' = \frac{1}{6}L_1} \begin{pmatrix}1&-\frac{1}{3}&-\frac{2}{3}\\12&-4&-8\\0&0&0\end{pmatrix} \xrightarrow{L_2' = L_2 - 12L_1}$$

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$$\begin{pmatrix} 1 & -1/3 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 - \frac{1}{3}y_1 - \frac{2}{3}z_1 = 0$$

x_1 – nec. principală

y_1, z_1 – nec. secundare; $y_1 = s$, $z_1 = t$, $s, t \in \mathbb{R}$

$$\Rightarrow x_1 = \frac{1}{3}y_1 + \frac{2}{3}z_1$$

$$x_1 = \frac{1}{3}s + \frac{2}{3}t$$

$$V_1 = \left\{ s\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

$$\lambda_2 = -1 \Rightarrow (A + I_3)v_2 = 0_{\mathbb{R}^3};\ v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -2 & -4 \\ 0 & 2 & 0 \\ 12 & -4 & -6 \end{pmatrix} \xrightarrow{L_1' = \frac{1}{8}L_1} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 2 & 0 \\ 12 & -4 & -6 \end{pmatrix} \xrightarrow{L_3' = L_3 - 12L_1}$$

$$\begin{pmatrix} 1 & -1/4 & -1/2 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{L_2' = \frac{1}{2}L_2} \begin{pmatrix} 1 & -1/4 & -1/2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow[L_3' = L_3 + L_2]{L_1' = L_1 + \frac{1}{4}L_2}$$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow y_2 = 0 \Rightarrow x_2 - \frac{1}{2}z_2 = 0$$

$z_2 = \alpha$, $\alpha \in \mathbb{R} \Rightarrow x_2 = \frac{1}{2}\alpha$

$$V_2 = \left\{ \alpha\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

$$w = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \notin V_2$$

pt. $s = 4$ și $t = 5 \Rightarrow v_1 = \begin{pmatrix} 14/3 \\ 4 \\ 5 \end{pmatrix} \Rightarrow w \notin V_1 \Rightarrow$

w nu este vector propriu pt. T

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(5) v_1, v_2, v_3 - liniar independenți <=>

<=> $\exists a, b, c \in \mathbb{R}$ a.î. $av_1 + bv_2 + cv_3 = 0_V$

$5v_1 + v_2, 5v_2 + v_3, 5v_3 + v_1$ - lin. indep. <=>

<=> $\exists x, y, z \in \mathbb{R}$ a.î.

$5xv_1 + xv_2 + 5y v_2 + y v_3 + 5z v_3 + z v_1 = 0_V$

$v_1(5x + z) + v_2(x + 5y) + v_3(y + 5z) = 0_V$

$$\begin{cases} 5x + 0 + z = 0 \\ x + 5y + 0 = 0 \\ 0 + y + 5z = 0 \end{cases}$$

$$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{L_1 <-> L_2} \begin{pmatrix} 1 & 5 & 0 \\ 5 & 0 & 1 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{L_2' = L_2 - 5L_1} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -25 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\xrightarrow{L_2 <-> L_3} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & -25 & 1 \end{pmatrix} \xrightarrow[L_3' = L_3 + 25L_2]{L_1' = L_1 - 5L_2} \begin{pmatrix} 1 & 0 & -25 \\ 0 & 1 & 5 \\ 0 & 0 & 126 \end{pmatrix} \xrightarrow{L_3' = \frac{1}{126}L_3}$$

$$\begin{pmatrix} 1 & 0 & -25 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[L_2' = L_2 - 5L_3]{L_1' = L_1 + 25L_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} => x = y = z = 0$$

=> $\exists x, y, z \in \mathbb{R}$ a.î. $v_1(5x + z) + v_2(x + 5y) + v_3(y + 5z) = 0_V$

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