Seminar geometrie 6

Ex. Fie $d = \{x \in \mathbb{R}^3 / \begin{cases} x_1 + 2x_2 - x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}\}$ și

$\pi = \{x \in \mathbb{R}^3 / x_1 - x_2 + 2x_3\}$

Să se găsească o aplicație liniară $f: \mathbb{R}^3 \to \mathbb{R}^3$ a.î.

a) $\mathrm{Ker} f = d$, $\mathrm{Im} f = \pi$

b) $\mathrm{Ker} f = \pi$, $\mathrm{Im} f = d$

a) $\underset{d}{\cancel{\mathrm{Ker} f}} = Sp_{\mathbb{R}}\{u\}$, $\pi = Sp_{\mathbb{R}}\{v, w\}$

$f(u) = 0_{\mathbb{R}^3}$

$\{u\}$ l.i. $\Rightarrow \exists x, y \in \mathbb{R}^3$ a.î. $\{u, x, y\}$ să fie bază în $\mathbb{R}^3/\mathbb{R}$

$f(u) = 0_{\mathbb{R}^3}$

$f(x) = v$ $\quad \underline{\mathrm{Im} f = \pi}$ „$\subset$" fie $z \in \mathrm{Im} f \Rightarrow \exists \xi \in \mathbb{R}^3$ a.î. $f(\xi) = z$ (1)

$f(y) = w$

$\xi \in \mathbb{R}^3$, $\{u, x, y\}$ bază în $\mathbb{R}^3/\mathbb{R}$ $\Big| \Rightarrow \exists a, b, c \in \mathbb{R}$ a.î.

$\xi = au + bx + cy$ (2)

din (1) și (2) $\Rightarrow a f(u) + b f(x) + c f(y) = z \Rightarrow z = bv + cw \Rightarrow$

$\Rightarrow z \in Sp_{\mathbb{R}}\{v, w\} \Rightarrow z \in \pi$

„$\supset$" Fie $z \in \pi$, $\pi = Sp_{\mathbb{R}}\{v, w\}$ $\Big\} \Rightarrow \exists a, b \in \mathbb{R}$ a.î. $z = av + bw \Rightarrow z = af(x) + bf(y) \Rightarrow$

$\Rightarrow z = f(\underset{\mathbb{R}^3}{ax + by}) \Rightarrow \underline{z \in \mathrm{Im} f}$

$\dim_{\mathbb{R}} \mathrm{Ker} f + \dim_{\mathbb{R}} \mathrm{Im} f = \dim_{\mathbb{R}} \mathbb{R}^3$

$\dim_{\mathbb{R}} \mathrm{Ker} f + 2 = 3 \Rightarrow \dim_{\mathbb{R}} \mathrm{Ker} f = 1$

$\underline{d \subset \mathrm{Ker} f}$ fie $z \in d$, $d = Sp_{\mathbb{R}}\{\ \}$ $\Big| \Rightarrow \exists a \in \mathbb{R}$ a.î. $z = au \Rightarrow f(z) = f(au) \Rightarrow$

$\Rightarrow f(z) = a f(u) \Rightarrow f(z) = a \cdot 0_{\mathbb{R}^3} \Rightarrow f(z) = 0_{\mathbb{R}^3} \Rightarrow z \in \mathrm{Ker} f$

$d \subset \mathrm{Ker} f$

$d, \mathrm{Ker} f \leq_{\mathbb{R}} \mathbb{R}^3$ $\quad \Big| \Rightarrow d = \mathrm{Ker} f$

$\dim_{\mathbb{R}} d = \dim_{\mathbb{R}} \mathrm{Ker} f = 1$

Fie $x \in d \Rightarrow \begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + 2x_2 = 3\alpha \\ x_1 + x_2 = -\alpha \ (-1) \end{cases}$

$x_3 =$ nec sec $= \alpha$ $\qquad x_2 = 4\alpha$, $x_1 = -5\alpha$

$x = (-5\alpha, 4\alpha, \alpha) = \alpha(-5, 4, 1)$

$d = \{\alpha(-5, 4, 1) \mid \alpha \in \mathbb{R}\} = \langle \{(-5, 4, 1)\} \rangle$

$u = (-5, 4, 1)$

Fie $x \in \pi \Rightarrow x_1 - x_2 + 2x_3 = 0$, $x_2 = \alpha$, $x_3 = \beta$ $\Big| \Rightarrow x_1 = \alpha - 2\beta \Rightarrow$

$\Rightarrow x = (\alpha - 2\beta, \alpha, \beta) = \alpha(1, 1, 0) + \beta(-2, 0, 1)$

$\pi = \langle \{(1, 1, 0), (-2, 0, 1)\} \rangle$

$v = (1, 1, 0)$

$w = (-2, 0, 1)$

$\det \begin{pmatrix} -5 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \{u, e_1, e_2\}$ bază în $\mathbb{R}^3/\mathbb{R}$

$x = e_1$, $y = e_2$

$\cancel{f(e_3)}$ $f(u) = 0_{\mathbb{R}^3} \Rightarrow f(-5e_1 + 4e_2 + e_3) = 0_{\mathbb{R}^3} \Rightarrow$

$\Rightarrow -5f(e_1) + 4f(e_2) + f(e_3) = 0$, $f(e_1) = v$, $f(e_2) = w$ $\Big| \Rightarrow f(e_3) = 5v - 4w = (13, 5, -4)$

$(5, 5, 0) - (-8, 0, 4)$

Fie $x \in \mathbb{R}^3$ atunci $f(x) = f(x_1 e_1 + x_2 e_2 + x_3 e_3) =$

$= x_1 f(e_1) + x_2 f(e_2) + x_3 f(e_3) =$

$= x_1 (1, 1, 0) + x_2 (-2, 0, 1) + x_3 (13, 5, -4) =$

$f(x) = (x_1 - 2x_2 + 13x_3, x_1 + 5x_3, x_2 - 4x_3)$

* nu e unică *

b) $\begin{cases} f(v) = 0_{\mathbb{R}^3} \\ f(w) = 0_{\mathbb{R}^3} \\ f(x) = u \end{cases}$ $\{v, w\}$ l.i. $\Rightarrow$

$\exists x \in \mathbb{R}^3$ a.î. $\{v, w, x\}$ să fie bază în $\mathbb{R}^3/\mathbb{R}$

$v, w \in \mathrm{Ker} f \Rightarrow Sp_{\mathbb{R}}\{v, w\} \subset \mathrm{Ker} f \Rightarrow$

$\Rightarrow \dim_{\mathbb{R}} \mathrm{Ker} f \geq 2$

dacă $\dim_{\mathbb{R}} \mathrm{Ker} f = 3 \Rightarrow \mathrm{Ker} f = \mathbb{R}^3$, $x \in \mathbb{R}^3$ $\Big\} \Rightarrow$

$\Rightarrow f(x) = 0_{\mathbb{R}^3}$, $f(x) = u$ $\Big| \Rightarrow u = 0_{\mathbb{R}^3}$ (F)

$\dim \mathrm{Ker} f = 2$, $Sp_{\mathbb{R}}\{v, w\} \leq_{\mathbb{R}} \mathrm{Ker} f$ $\Big| \Rightarrow Sp_{\mathbb{R}}\{v, w\} = \mathrm{Ker} f \Rightarrow$

$\Rightarrow \mathrm{Ker} f = \pi \Rightarrow \dim_{\mathbb{R}} \mathrm{Im} f = 3 - \dim \mathrm{Ker} f =$

$= 3 - \dim \pi = 3 - 2 = 1$

$\underline{d \subset \mathrm{Im} f}$ fie $z \in d = Sp_{\mathbb{R}}\{u\} \Rightarrow \exists a \in \mathbb{R}$ a.î.

$z = au \Rightarrow z = a \cdot f(x) \Rightarrow z = f(\underset{\mathbb{R}^3}{ax}) \Rightarrow z \in \mathrm{Im} f$

$d \subset \mathrm{Im} f$

$\dim_{\mathbb{R}} d = 1$

$\dim_{\mathbb{R}} \mathrm{Im} f = 1$

Fie $x \in \mathbb{R}$

$\begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow \{v, w, e\}$ bază în $\mathbb{R}^3/\mathbb{R}$

alegem $x = e$

$f(v) = 0_{\mathbb{R}^3} \Rightarrow f(e_1 + e_2) = (0, 0, 0) \Rightarrow f(e_1) + f(e_2) = (0, 0, 0)$

$f(w) = 0_{\mathbb{R}^3} \Rightarrow f(-2e_1 + e_2) = 0_{\mathbb{R}^3} \Rightarrow 2f(e_1) = -(-5, 4, 1)$

$f(e_1) = (-\frac{5}{2}, 2, \frac{1}{2})$ $\qquad f(e_2) = -f(e_1) = +(\frac{5}{2}, -2, -\frac{1}{2})$

$f($

$= (-\frac{5}{2}x_1, 2x_1, \frac{1}{2}x_1) + (\frac{5}{2}x_2, -2x_2, \frac{x_2}{2}) +$

$= \cancel{(\ldots)} - (5x_3, 4x_3, x_3) =$

$= (-\frac{5}{2}x_1 + \frac{5}{2}x_2 - 5x_3, 2x_1 - 2x_2 + 4x_3, \frac{1}{2}x_1 - \frac{1}{2}x_2 + x_3)$

Ex Să se determine o aplicație liniară $f: \mathbb{R}^2 \to \mathbb{R}^2$ a.î. $\operatorname{Ker} f = \operatorname{Im} f$

$\operatorname{Ker} f = Sp_{\mathbb{R}}\{e_1\}$

$\operatorname{Im} f = Sp_{\mathbb{R}}\{e_1\}$

$\begin{cases} f(e_1) = 0_{\mathbb{R}^2} \\ f(e_2) = e_1 \end{cases}$

$f(x) = x_1 \cdot f(e_1) + x_2 f(e_2)$

$f(x) = x_2 e_1 = (x_2, 0)$

Fie V/K, V'/K sp. vect.

$B_1 = \{f_1, \dots, f_n\}$ bază în V/K

$B_1' = \{f_1', \dots, f_m'\}$ bază în V'/K și $f: V \to V'$ liniară

Spunem că $A \in \mathcal{M}(K)$ este matricea asociată lui f la fixarea bazelor B_1 și B_1' ($B_1 \xrightarrow[A]{f} B_1'$) dacă $f(f_1) = a_{11} f_1' + a_{21} f_2' + \dots + a_{m1} f_m'$

$f(f_2) = a_{12} f_1' + a_{22} f_2' + \dots + a_{m2} f_m'$ $\iff$

$\vdots$

$f(f_n) = a_{1n} f_1' + a_{2n} f_2' + \dots + a_{mn} f_m'$

$\forall i \in \overline{1,n}$ $f(f_i) = \sum_{j=1}^{m} a_{ji} \cdot f_j'$

Fie $x \in V$

Notăm cu $x_1, \dots, x_n$ coordonatele lui x în B_1

$x = \sum_{i=1}^{n} x_i f_i$ $\quad x_1', \dots, x_m'$ — // — $f(x)$ în B_2

$f(x) = \sum_{j=1}^{m} x_j' \cdot f_j'$ (1)

$B_1 \xrightarrow[A]{f} B_1'$

Atunci $\begin{pmatrix} x_1' \\ \vdots \\ x_m' \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad V = K^n \quad B_1 = B_c$; $\quad V' = K^m \quad B_1' = B_c'$

$f(x) = A \cdot x$

$f(x) = f\left(\sum_{i=1}^{n} x_i f_i\right) = \sum_{i=1}^{n} x_i \cdot f(f_i) = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} a_{ji} f_j'$ (2)

Fie V/K, W/K sp. vect cu K corp com.

~~$V = K^n$~~ Din (1) și (2) $\Rightarrow \forall j \in \overline{1,m}$ $x_j' = \sum_{i=1}^{n} a_{ji} x_i \Rightarrow$

$\Rightarrow \begin{pmatrix} x_1' \\ \vdots \\ x_m' \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\begin{tikzcd} B_1 \arrow[r, "f", "A"'] \arrow[d, "C"'] & B_1' \arrow[d, "D"] \\ B_2 \arrow[r, "f", "A'"'] & B_2' \end{tikzcd}$$

$B_2 = \{g_1, \dots, g_n\}$

$B_2' = \{g_1', \dots, g_m'\}$

$f(g_i) = \sum_{j=1}^{m} a_{ji}' g_j' = \sum_{j=1}^{m} a_{ji}' \sum_{k=1}^{m} d_{kj} f_k' =$

$= \sum_{k=1}^{m} \left(\sum_{j=1}^{m} d_{kj} a_{ji}'\right) f_k'$

$f(g_i) = f\left(\sum_{j=1}^{n} c_{ji} f_j\right) = \sum_{j=1}^{n} c_{ji} f(f_j) = \sum_{j=1}^{n} c_{ji} \sum_{k=1}^{m} a_{kj} f_k' =$

$= \sum_{k=1}^{m} \left(\sum_{j=1}^{n} a_{kj} c_{ji}\right) f_k'$ $\Bigg\} \Rightarrow$

$\Rightarrow \forall i = \overline{1,n}$, $\forall k = \overline{1,m}$ $\sum_{j=1}^{m} d_{kj} \cdot a_{ji}' = \sum_{j=1}^{n} a_{kj} c_{ji} \Rightarrow DA' = AC \Rightarrow$

$A' = D^{-1} AC$

Ex. Fie $f: \mathbb{R}^2 \to \mathbb{R}^3$ $f(x) = (x_1 + x_2, x_1 - x_2, 2x_2)$

a) Să se determine matr. asociată lui f la fixarea bazelor canonice

b) Fie $B_2 = \{g_1, g_2\}$ $g_1 = (1,1), g_2 = (0,1)$

$B_2' = \{g_1', g_2', g_3'\}$ $g_1' = (0,1,1), g_2' = (1,0,1), g_3' = (1,1,0)$

Să se determine matricea asociată lui f la fixarea bazelor B_2 și B_2'.

$$\begin{tikzcd} B_1 \arrow[r, "f", "A"'] \arrow[d, "C"'] & B_1' \arrow[d, "D"] \\ B_2 \arrow[r, "f"] & B_2' \end{tikzcd}$$

$B_2 = B_1 = \{e_1, e_2\}$

$B_1' = B_2' = \{e_1', e_2', e_3'\}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$

b) $g_1 = e_1 + e_2$, $g_2 = e_2$ $\Rightarrow C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$g_1' = e_2' + e_3'$

$g_2' = e_1' + e_3'$ $\Rightarrow D = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$g_3' = e_1' + e_2'$

$D^{-1} = \frac{1}{\det D} D^*$

$\det D = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 \neq 0$

$D^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $\quad D^* = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$D^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$A' = D^{-1} AC = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 2 & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 4 & 4 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 2 \\ 0 & -1 \end{pmatrix}$

$AC = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 2 & 2 \end{pmatrix}$

$\begin{cases} f(g_1) = 2g_2' \\ f(g_2) = 2g_2' - g_3' \end{cases} \Rightarrow \operatorname{Im} f = Sp_{\mathbb{R}}\{g_2', g_3'\}$

$f(x) = 0 \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ 2x_2 = 0 \Rightarrow x_2 = 0 \end{cases} \Rightarrow x_1 = 0$

$\operatorname{Ker} f = \{0_{\mathbb{R}^2}\}$

$\dim_{\mathbb{R}} \operatorname{Ker} f = 0$

$\dim_{\mathbb{R}} \operatorname{Im} f = 0$

Def Fie V/K sp. vect, ~~K corp~~

$End_K(V) = \{f : V \to V \mid f \text{ liniară}\}$

Def Fie $f \in End_K(V)$ și $\lambda \in K$

Spunem că λ este valoare proprie pt f dacă $\exists\, x \in V \setminus \{0\}$ a.î. $f(x) = \lambda x$

$Spec(f) = \{\lambda \in K / \lambda \text{ valoare proprie pt } f\}$

Ex: $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (5x_1 - 2x_2, -2x_1 + 2x_2)$

Să se determine $Spec(f)$

Fie $\lambda \in Spec(f)$ atunci $\exists\, x \in \mathbb{R}^2 \setminus \{0\}$ pt care $f(x) = \lambda x$

$(5x_1 - 2x_2, -2x_1 + 2x_2) = (\lambda x_1, \lambda x_2)$

$$\begin{cases} 5x_1 - 2x_2 = \lambda x_1 \\ -2x_1 + 2x_2 = \lambda x_2 \end{cases} \Leftrightarrow \begin{cases} (5-\lambda)x_1 - 2x_2 = 0 \\ -2x_1 + (2-\lambda)x_2 = 0 \end{cases}$$

Sistemul S admite soluții nebanale $\Leftrightarrow$ det matr S $= 0$

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$10 + \lambda^2 - 7\lambda - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda - 1)(\lambda - 6) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 6 \end{cases} \Rightarrow Spec f = \{1, 6\}$

pt $\lambda_1 = 1$ S devine

$$\begin{cases} 4x_1 - 2x_2 = 0 \Leftrightarrow 2x_1 - x_2 = 0 \\ -2x_1 + x_2 = 0 \end{cases}$$

$x_2 = 2x_1$

$x = (x_1, 2x_1) = x_1(1, 2) \quad x \neq 0_V \Leftrightarrow x_1 \neq 0$

$u_1 = (1, 2)$

pt $\lambda = 6$ S devine

$$\begin{cases} -x_1 - 2x_2 = 0 \\ -2x_1 - 4x_2 = 0 \end{cases}$$

$x_1 = (-2x_2) \Rightarrow x = (-2x_2, x_2) = x_2(-2, 1)$

$u_2 = (-2, 1)$

$B_1 = \{u_1, u_2\}$ bază în $\mathbb{R}^2_{\mathbb{R}}$

$B_1 \xrightarrow[C]{f} B_2 \quad f(u_1) = u_1 \quad f(u_2) = 6u_2 \quad x = (x_1, 2x_1) = x_1(1, 2)$

$x \neq 0_V \Leftrightarrow x_1 \neq 0 \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$