Spatiul $\mathbb{R}^n$

Reamintim ca

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

este un spatiu vectorial peste $\mathbb{R}$ pentru care adunarea si inmultirea cu scalari sunt definite prin

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$\alpha(x_1, x_2, \ldots x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

Daca $n = 2$ vom nota un punct curent (x_1, x_2) cu (x, y) si daca $n = 3$ vom nota un punct curent (x_1, x_2, x_3) cu (x, y, z), adica

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}, \quad \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

Functia $\|\cdot\| : \mathbb{R}^n \to [0, +\infty)$ definita prin

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_2^n}, \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

este o norma pe $\mathbb{R}^n$, adica are proprietatile

(1) $\|x + y\| \leq \|x\| + \|y\|$ pentru orice $x, y \in \mathbb{R}^n$

(2) $\|\alpha x\| = |\alpha| \cdot \|x\|$ pentru orice $x \in \mathbb{R}^n$ si orice $\alpha \in \mathbb{R}$

(3) $\|x\| \geq 0$ pentru orice $x \in \mathbb{R}^n$ si $\|x\| = 0 \Leftrightarrow x = 0_{\mathbb{R}^n}$

Aceasta norma se numeste norma euclidiana. Functia $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ definita pentru orice doua puncte $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ prin

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$$

este o distanta (metrica) numita distanta sau metrica euclidiana.

Daca $a \in \mathbb{R}^n$ si $r > 0$ multimea

$$B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

se numeste bila cu centrul a si raza r. Topologia lui $\mathbb{R}^n$ va fi topologia asociata distantei euclidiene: $D \subset \mathbb{R}^n$ este deschisa daca si numai daca pentru orice $a \in D$ exista $r > 0$ astfel incat $B(a, r) \subset D$

Exercitiu. Aratati ca $B(a, r)$ este o multime deschisa.

Definitie. Spunem ca doua norme $\|\cdot\|$ si $\|\cdot\|'$ pe $\mathbb{R}^n$ sunt echivalente daca exista $\alpha, \beta > 0$ astfel incat

$$\alpha\|x\| \leq \|x\|' \leq \beta\|x\| \text{ pentru orice } x \in \mathbb{R}^n$$

Exercitiu. Aratati ca $\|\cdot\|_\infty : \mathbb{R}^n \to [0, +\infty)$ si $\|\cdot\|_1 : \mathbb{R}^n \to [0, +\infty)$

$$\|x\|_\infty = \max\{|x_i| : i = 1, 2, \ldots, n\} \qquad \|x\|_1 = |x_1| + |x_1| + \cdots + |x_n|$$

sunt norme pe $\mathbb{R}^n$ echivalente cu norma euclidiana. Desenati bila cu centrul in origine si de raza 1 in cazul celor doua norme.

Propozitie 1. Oricare doua norme pe $\mathbb{R}^n$ sunt echivalente.

Demonstratie. Vezi de exemplu N. Boboc -Analiza Matematica II -pag 32.

Observatie. Se poate arata ca daca doua norme pe $\mathbb{R}^n$ sunt echivalente atunci ele genereaza aceeasi topologie. Daca nu se face alta precizare, norma utilizata va fi norma euclidiana.

Definitie. O aplicatie $T : \mathbb{R}^n \to \mathbb{R}^m$ se numeste liniara daca $T(x + y) = T(x) + T(y)$ si $T(\alpha x) = \alpha T(x)$, pentru orice $\alpha \in \mathbb{R}$, si orice $x, y \in \mathbb{R}^n$

Propozitie 2. Daca $T : \mathbb{R}^n \to \mathbb{R}^m$ este o aplicatie liniara atunci T este continua.

Demonstratie. Fie $x = x_1e_1 + x_2e_2 + \cdots x_ne_n \in \mathbb{R}^n$. Atunci

$$\|T(x)\| = \|x_1T(e_1) + x_2T(e_2) + \cdots x_nT(e_n)\| \leq \sum_{i=1}^{n} |x_i|\|T(e_i)\| \leq M\|x\|$$

unde

$$M = \sqrt{\sum_{i=1}^{n} \|T(e_i)\|}.$$

De aici deducem ca pentru orice $x, y \in \mathbb{R}^n$

$$\|T(x - y)\| \leq M \cdot \|x - y\|$$

si deci T este uniform continua.

Propozitie 3. Multimea $L(\mathbb{R}^n, \mathbb{R}^m)$ a aplicatiilor liniare de la $\mathbb{R}^n$ la $\mathbb{R}^m$ este un spatiu vectorial si aplicatia $T \to \|T\|$

$$\|T\| = \sup\{\|T(x)\| \mid \|x\| \leq 1\} = \inf\{M \geq 0 \mid \|Tx\| \leq M\|x\|\}$$

este o norma pe $L(\mathbb{R}^n, \mathbb{R}^m)$.

Pentru demonstratie vezi N. Boboc -Analiza Matematica II - pag 34.

Derivate partiale. Functii diferentiabile

Pe parcursul intregului curs D va fi o multime deschisa din $\mathbb{R}^n$. Fie $f : D \to \mathbb{R}^m$ si $a \in D$. Daca $u \in R^n$ este un vector nenul, multimea

$$\{a + tu \;|t \in \mathbb{R}\} \subset \mathbb{R}^n$$

este o dreapta care trece prin a. Deoarece D este deschisa rezulta ca multimea

$$\{t \in \mathbb{R} \;\, : a + tu \in D\}$$

este o multime deschisa din $\mathbb{R}$ care include un interval deschis centrat in $t = 0$.

Definitie. Se spune ca f este derivabila dupa vectorul u in punctul a daca exista limita (in $\mathbb{R}^m$)

$$\lim_{t\to 0} \frac{f(a + tu) - f(a)}{t}$$

Daca acesta limita exista, se numeste derivata partiala dupa vectorul u (sau dupa directia u daca u este versor, adica $\|u\| = 1$) in punctul a si se noteaza cu $\frac{\partial f}{\partial u}(a)$. Daca f are derivata partiala dupa vectorul u in orice punct $x \in D$ functia

$$D \ni x \mapsto \frac{\partial f}{\partial u}(a)$$

se noteaza cu $\frac{\partial f}{\partial u}$ si se numeste derivata lui f dupa vectorul u.

Daca $\{e_1, e_2, \dots, e_n\}$ este baza canonica a lui $\mathbb{R}^n$, atunci derivata lui f dupa directia e_i, adica $\frac{\partial f}{\partial e_i}(a)$ se noteaza cu $\frac{\partial f}{\partial x_i}(a)$ si se numeste derivata partiala a lui f in raport cu x_i in punctul a.

$$\frac{\partial f}{\partial x_i}(a) = \lim_{t\to 0} \frac{f(a + te_i) - f(a)}{t}$$

$$= \lim_{t\to 0} \frac{f(a_1, \dots, a_{i-1}, a_i + t, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)}{t}$$

$$= \lim_{x_i\to a_i} \frac{f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)}{x_i - a_i}$$

Deci $\frac{\partial f}{\partial x_i}(a)$ exista daca si numai daca functia

$$x \mapsto f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$$

este derivabila in a_i. Daca $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ si $(x_0, y_0, z_0) \in D$, atunci

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \lim_{x\to x_0} \frac{f(x, y_0, z_0) - f(x_0, y_0, z_0)}{x - x_0}$$

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) = \lim_{y \to y_0} \frac{f(x_0, y, z_0) - f(x_0, y_0, z_0)}{y - y_0}$$
$$\frac{\partial f}{\partial z}(x_0, y_0, z_0) = \lim_{z \to z_0} \frac{f(x_0, y_0, z) - f(x_0, y_0, z_0)}{z - z_0}$$

cu conditia ca limitele sa existe in $\mathbb{R}$.

Exemplu. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = e^{x^2+yz} - x^2yz + xy^4 + z^2$.

$$\frac{\partial f}{\partial x}(x, y, z) = 2xe^{x^2+yz} - 2xyz + y^4 \qquad \frac{\partial f}{\partial y}(x, y, z) = ze^{x^2+yz} - x^2z + 4xy^3$$
$$\frac{\partial f}{\partial z}(x, y, z) = ye^{x^2+yz} - x^2y + 2z$$

Definitie. Fie $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ $f = (f_1, f_2, \ldots, f_m)$. Spunem ca f are derivate partiale in $a \in D$ daca exista toate derivatele partiale $\frac{\partial f}{\partial x_1,}(a)$, $\frac{\partial f}{\partial x_2}(a), \ldots, \frac{\partial f}{\partial x_n}(a)$.

Propozitie 4. Functia f are derivata partiala dupa directia u in punctul a, daca si numai daca toate functiile $f_1, f_2, \ldots, f_m$ au derivata dupa directia u in a. In aceasta situatie

$$\frac{\partial f}{\partial u}(a) = \left(\frac{\partial f_1}{\partial u}(a), \frac{\partial f_2}{\partial u}(a), \ldots, \frac{\partial f_m}{\partial u}(a) \right).$$

Daca $f : D \to \mathbb{R}^m$, atunci f are derivate partiale in a, daca si numai daca functiile $f_1, f_2, \ldots, f_n$ au derivate partiale in a. Matricea cu m linii si n coloane

$$[df(a)] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & . & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}$$

se numeste matricea Jacobi asociata functiei f in punctul a. Cand $m = n$, numarul

$$J_f(a) = \det [df(a)] = \frac{D(f_1, f_2, \ldots, f_n)}{D(x_1, x_2, \ldots, x_n)}$$

se numeste Jacobianul lui f in punctul a.

Propozitie 5. Fie $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ si $a \in D$. Daca exista doua aplicatii liniare $T, T_1 : \mathbb{R}^n \to \mathbb{R}^m$ astfel incat

$$\lim_{x \to a} \frac{f(x) - f(a) - T(x - a)}{\|x - a\|} = 0 \quad \text{si} \quad \lim_{x \to a} \frac{f(x) - f(a) - T_1(x - a)}{\|x - a\|} = 0$$

atunci $T(x) = T_1(x)$ pentru orice $x \in \mathbb{R}^n$.

Demonstratie. Din ipoteza rezulta ca

$$\lim_{x \to a} \frac{T(x-a) - T_1(x-a)}{\|x-a\|} = 0. \tag{1}$$

Fie $r > 0$ astfel incat $B(a, r) \subset D$ si fie $x \in \mathbb{R}^n$, $x \neq 0$. Pentru orice $t > 0$ astfel incat $t\|x\| < r$, avem $a + tx \in D$ si $t\|x\| = \|(a + tx) - a\|$ atunci din (1) rezulta ca

$$T(x) - T_1(x) = \frac{T((a+tx) - a) - T_1((a+tx) - a)}{\|(a+tx) - a\|} \|x\|.$$

si deci

$$T(x) = T_1(x).$$

Definitie. Spunem ca $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ este diferentiabila (sau derivabila) in a daca exista o aplicatie liniara $T : \mathbb{R}^n \to \mathbb{R}^m$ astfel incat

$$\lim_{x \to a} \frac{f(x) - f(a) - T(x-a)}{\|x-a\|} = 0$$

Daca T exista atunci, din Propozitia 5, este unica, se noteaza cu $df(a)$ si se numeste diferentiala lui f in a.

Observam ca daca

$$\varepsilon_f(x) = \begin{cases} \frac{f(x) - f(a) - T(x-a)}{\|x-a\|} & \text{daca } x \neq a \\ 0 & \text{daca } x = a \end{cases}$$

atunci

$$f(x) = f(a) + T(x-a) + \varepsilon_f(x)\|x-a\|$$

unde functia ε_f este continua in 0 si $\varepsilon_f(a) = 0$.
Deducem urmatoarea caracterizare echivalenta a diferentiabilitatii.

Propozitie 6. Fie $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ si $a \in D$. Atunci f este diferentiabila in a daca si numai daca exista o aplicatie liniara $T : \mathbb{R}^n \to \mathbb{R}^m$ si o functie $\varepsilon_f : D \to \mathbb{R}^m$ cu proprietatea ca $\lim_{x \to 0} \varepsilon_f(x) = \varepsilon_f(a) = 0$ astfel incat

$$f(x) = f(a) + T(x-a) + \varepsilon_f(x)\|x-a\|.$$

Propozitie 7. Daca $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ este diferentiabila in $a \in D$ atunci f este continua in a.

Demonstratie. Deoarece $f(x) = f(a) + T(x-a) + \varepsilon_f(x)\|x-a\|$ unde T si ε_f sunt ca in Propozitia 6, avem

$$\|f(x) - f(a)\| \leq \|T\| \cdot \|x-a\| + \|\varepsilon_f(x)\| \cdot \|x-a\|,$$

de unde rezulta ca f este continua in a.

Propozitie. Fie $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ si $x_0 \in (a, b)$ Atunci f este diferentiabila in x_0 daca si numai daca f este derivabila in x_0. In acest caz

$$df(x_0)(u) = uf'(x_0), \quad u \in \mathbb{R}.$$

Demonstratie. Exercitiu!

Propozitie 8. Daca f este diferentiabila in a si $u \in \mathbb{R}^n$ este un vector nenul atunci exista $\frac{\partial f}{\partial u}(a)$ si

$$df(a)(u) = \frac{\partial f}{\partial u}(a)$$

Demonstratie. Intrucat f este diferentiabila rezulta ca

$$\lim_{t\to 0} \frac{f(a+tu) - f(a) - df(a)(tu)}{|t|} = 0$$

sau echivalent

$$\lim_{t\to 0} \frac{f(a+tu) - f(a) - df(a)(tu)}{t} = 0,$$

ceea ce arata ca f are derivata partiala si

$$\frac{\partial f}{\partial u}(a) = \lim_{t\to 0} \frac{f(a+tu) - f(a)}{t} = df(a)(u)$$

Exemplu. Fie $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{x^2 y}{x^6+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

Functia f nu este continua (deci nici diferentiabila) in $(0, 0)$ deoarece daca $x_n = (\frac{1}{n}, \frac{1}{n^3})$ atunci $f(x_n) = \frac{n}{2} \to \infty$. Cu toate acestea f admite derivata partiala dupa orice vector nenul. Daca $u = (u_1, u_2)$ este un vector nenul, atunci

$$\frac{\partial f}{\partial u}(0,0) = \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{u_1^2 u_2}{t^4 u_1^6 + u_2^2} = \begin{cases} \frac{u_1^2}{u_2}, & u_2 \neq 0 \\ 0, & u_2 = 0 \end{cases}$$

Propozitie 9. Daca f este diferentiabila in a atunci exista $\frac{\partial f}{\partial x_k}(a)$ pentru orice $k \in \{1, 2, \dots, n\}$ si

$$df(a)(u_1, u_2, \dots, u_n) = \frac{\partial f}{\partial x_1}(a)u_1 + \frac{\partial f}{\partial x_2}(a)u_2 + \dots + \frac{\partial f}{\partial x_n}(a)u_n.$$

Demonstratie. Din propozitia anterioara rezulta ca exista

$$\frac{\partial f}{\partial x_k}(a) = \lim_{t\to 0} \frac{f(a+te_k) - f(a)}{t} = df(a)(e_k)$$

Daca $u = (u_1, \dots, u_n)$, $u = u_1 e_1 + u_2 e_2 + \cdots + u_n e_n$ atunci

$$\begin{aligned} df(a)(u_1, u_2, \dots, u_n) &= u_1 df(a)(e_1) + u_2 df(a)(e_2) + \cdots + u_n df(a)(e_n) \\ &= \frac{\partial f}{\partial x_1}(a)u_1 + \frac{\partial f}{\partial x_2}(a)u_2 + \cdots + \frac{\partial f}{\partial x_n}(a)u_n. \end{aligned}$$

Observatie. Matricea Jacobi $[df(a)]$ asociata functiei f in punctul a este matricea aplicatiei liniare $df(a)$.

Exemplu. Fie $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x^2 y, 2x + xy + \sin y, x + 2y)$

$$\frac{\partial f}{\partial x}(x, y) = (2xy, 2 + y, 1), \quad \frac{\partial f}{\partial y}(x, y) = (x^2, x + \cos y, 2)$$

$$[d_f(x, y)] = \begin{bmatrix} 2xy & x^2 \\ 2 + y & x + \cos y \\ 1 & 2 \end{bmatrix}$$

$$df(1, 0)(u, v) = \frac{\partial f}{\partial x}(1, 0)u + \frac{\partial f}{\partial y}(1, 0)v = (v, 2u + 2v, u + 2v)$$

$$[df(1, 0)] \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ 2u + 2v \\ u + 2v \end{bmatrix}$$

Remark 10. Orice functie liniara $T : \mathbb{R}^n \to \mathbb{R}$ este diferentiabila in orice a si

$$dT(a) = T.$$

In particular, aplicatiile $\mathrm{pr}_i : \mathbb{R}^n \to \mathbb{R}$ definite prin

$$\mathrm{pr}_i(u_1, u_2, \dots, u_n) = u_i, \text{ pentru orice } (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$$

sunt liniare, si deci

$$d\mathrm{pr}_i(a) = \mathrm{pr}_i, \quad \text{pentru orice } i = 1, 2, \dots, n.$$

Introducem notatia

$$\mathrm{pr}_i = dx_i$$

Cu aceasta notatie, daca f este diferentiabila in a, avem

$$df(a)(u_1, u_2, \dots, u_n) = \frac{\partial f}{\partial x_1}(a)u_1 + \frac{\partial f}{\partial x_2}(a)u_2 + \cdots + \frac{\partial f}{\partial x_n}(a)u_n$$

$$= \frac{\partial f}{\partial x_1}(a)\mathrm{pr}_1(u_1, u_2, \dots, u_n) + \dots + \frac{\partial f}{\partial x_n}(a)\mathrm{pr}_n(u_1, u_2, \dots, u_n)$$
$$= \frac{\partial f}{\partial x_1}(a)dx_1(u_1, u_2, \dots, u_n) + \dots + \frac{\partial f}{\partial x_n}(a)dx_n(u_1, u_2, \dots, u_n)$$

pentru orice $(u_1, u_2, \dots, u_n)$ si deci

$$df(a) = \frac{\partial f}{\partial x_1}(a)dx_1 + \frac{\partial f}{\partial x_2}(a)dx_2 + \dots + \frac{\partial f}{\partial x_n}(a)dx_n.$$

Daca $f : \mathbb{R}^3 \to \mathbb{R}$, este diferentiabila in (x_0, y_0, z_0),

$$df(x_0, y_0, z_0)(u, v, w) = \frac{\partial f}{\partial x}(x_0, y_0, z_0)u + \frac{\partial f}{\partial y}(x_0, y_0, z_0)v + \frac{\partial f}{\partial z}(x_0, y_0, z_0)w$$

adica

$$df(x_0, y_0, z_0) = \frac{\partial f}{\partial x}(x_0, y_0, z_0)dx + \frac{\partial f}{\partial y}(x_0, y_0, z_0)dy + \frac{\partial f}{\partial z}(x_0, y_0, z_0)dz$$

Teorema 11. Fie $D \subset \mathbb{R}^n$ o multime deschisa si $f : D \to \mathbb{R}^m$ cu $f = (f_1, f_2, \dots, f_m)$ unde $f_i : D \to \mathbb{R}$. Functia f este diferentiabila in $a \in D$ daca si numai daca functiile f_i sunt diferentiabile in a si in acest caz

$$df(a) = (df_1(a), df_1(a), \dots, df_m(a)).$$

Demonstratie. Exercitiu!

Teorema 12 (Conditie suficienta de diferentiabilitate)**.** Fie D o multime deschisa din $\mathbb{R}^n$, fie $f : D \to \mathbb{R}^m$ si $a = (a_1, a_2, \dots, a_n) \in D$. Daca exista o vecinatate V a lui a cu proprietatea ca exista toate derivatele partiale in orice punct din V si acestea sunt continue in a, atunci f este diferentiabila in a si

$$df(a) = \frac{\partial f}{\partial x_1}(a)dx_1 + \frac{\partial f}{\partial x_2}(a)dx + \dots + \frac{\partial f}{\partial x_n}(a)dx_n.$$

Demonstratie. Avand in vedere rezultatul anterior, este suficient sa demonstram teorema pentru cazul $m = 1$. Fara a restrange generalitatea, putem presupune ca vecinatatea V lui a este $B(a, r) = \{x \in D : \|x - a\| < r\}$ bila deschisa cu centrul in a si raza $r > 0$, pe care avand in vedere ca D este multime deschisa, o putem considera inclusa in D. Fie $x = (x_1, x_2, \dots, x_n) \in D$. Definim functiile $g_1, g_2, \dots, g_n$ astfel

$$g_1 : [a_1, x_1] \to \mathbb{R}, \; g_1(t) = f(t, x_2, x_3 \dots, x_n)$$
$$g_2 : [a_2, x_2] \to \mathbb{R}, \; g_2(t) = f(a_1, t, x_3, \dots, x_n)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$g_n : [a_n, x_n] \to \mathbb{R}, \; g_n(t) = f(a_1, a_2, a_3 \dots, t)$$

Atunci
$$f(x_1, x_2, \ldots, x_n) - f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} (g_i(x_i) - g_i(a_i))$$
Fiecare din functiile g_i satisface ipotezele teoremei lui Lagrange referitoare la o functie reala de variabila reala continua pe un compact si derivabila pe interiorul acelui interval. Prin urmare exista $\xi_i \in (x_i, a_i)$ astfel incat

$$g_i(x_i) - g_i(a_i) = (x_i - a_i)g_i'(\xi_i)$$

Atunci

$$\begin{aligned}
g_1(x_1) - g_1(a_1) &= (x_1 - a_1)\frac{\partial f}{\partial x_1}(\xi_1, x_2, x_3, \ldots, x_n) \\
g_2(x_2) - g_2(a_2) &= (x_2 - a_2)\frac{\partial f}{\partial x_2}(a_1, \xi_2, x_3, \ldots, x_n) \\
\ldots\ldots&\ldots\ldots\ldots\ldots\ldots\ldots \\
g_n(x_n) - g_n(a_n) &= (x_n - a_n)\frac{\partial f}{\partial x_n}(a_1, a_2, a_3, \ldots, \xi_n)
\end{aligned}$$

Definim aplicatia liniara $T : \mathbb{R}^n \to \mathbb{R}$ prin

$$T(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)u_i.$$

Obtinem

$$\begin{aligned}
&\frac{f(x) - f(a) - T(x-a)}{\|x - a\|} = \\
&\quad \frac{x_1 - a_1}{\|x - a\|}\left(\frac{\partial f}{\partial x_1}(\xi_1, x_2, x_3, \ldots, x_n) - \frac{\partial f}{\partial x_1}(a_1, x_2, x_3, \ldots, x_n)\right) + \\
&\quad \frac{x_2 - a_2}{\|x - a\|}\left(\frac{\partial f}{\partial x_2}(a_1, \xi_2, x_3, \ldots, x_n) - \frac{\partial f}{\partial x_1}(x_1, a_2, x_3, \ldots, x_n)\right) + \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad \frac{x_n - a_n}{\|x - a\|}\left(\frac{\partial f}{\partial x_n}(a_1, a_2, a_3, \ldots, \xi_n) - \frac{\partial f}{\partial x_n}(a_1, a_2, a_3, \ldots, a_n)\right)
\end{aligned}$$

Deoarece
$$\frac{|x_i - a_i|}{\|x - a\|} \leq 1, \quad i = 1, 2, \ldots, n$$

deducem ca

$$\frac{|f(x)-f(a)-T(x-a)|}{\|x-a\|} \leq \qquad (1)$$

$$\left|\frac{\partial f}{\partial x_1}(\xi_1, x_2, x_3, \ldots, x_n) - \frac{\partial f}{\partial x_1}(a_1, a_2, a_3, \ldots, a_n)\right| +$$

$$\left|\frac{\partial f}{\partial x_2}(a_1, \xi_2, x_3, \ldots, x_n) - \frac{\partial f}{\partial x_1}(a_1, a_2, a_3, \ldots, a_n)\right| +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left|\frac{\partial f}{\partial x_n}(a_1, a_2, a_3, \ldots, \xi_n) - \frac{\partial f}{\partial x_n}(a_1, a_2, a_3, \ldots, a_n)\right|$$

Deoarece derivatele partiale ale functiei f sunt continue in a, exista limita termenilor din partea dreapta a inegalitatii (1) pentru $x \to a$ si aceasta este egala cu zero. Prin urmare

$$\lim_{x\to a} \frac{f(x)-f(a)-T(x-a)}{\|x-a\|} = 0$$

Asadar f este diferentiabila in a si

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) dx_i$$

Exemplu. Fie $f(x,y,z) = xe^y + xyz + z^2$. Functia f are derivate partiale continue in raport cu toate variabilele, si deci este diferentiabila in orice punct.

$$\frac{\partial f}{\partial x} = e^y + yz, \ \frac{\partial f}{\partial y} = xe^y + xz, \quad \frac{\partial f}{\partial z} = xy + 2z.$$

Deci

$$\frac{\partial f}{\partial x}(1,0,2) = 1, \ \frac{\partial f}{\partial y}(1,0,2) = 3, \quad \frac{\partial f}{\partial z}(1,0,2) = 4$$

si atunci

$$df(1,0,2) = \frac{\partial f}{\partial x}(1,0,2)dx + \frac{\partial f}{\partial y}(1,0,2)dy + \frac{\partial f}{\partial z}(1,0,2)dz = dx + 3dy + 4dz$$

Pentru $(x,y,z) \in \mathbb{R}^3$ avem $df(1,0,2)(x,y,z) = x + 3y + 4z$.