TUTORIAT 3

1. Să se determine inversa matricii A folosind metoda Gauss-Jordan.

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 1 & 0 & -1 & 1 \\ 2 & 1 & 3 & -4 \\ 0 & -1 & 1 & -1 \end{bmatrix} \qquad n = 4 \Rightarrow k = \overline{1,3}$$

Folosim metoda Gauss-Jordan împreună cu MEGFP

Pentru $k=1$:

$$\bar{A} = \left[\begin{array}{cccc|cccc} \boxed{1} & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & -4 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$a_{11}^{(1)} = 1 \neq 0 \rightarrow$ putem aplica MEGFP

$i = \overline{2,4}: \quad m_i^{(1)} = a_{i1}^{(1)} / a_{11}^{(1)}$

$m_2^{(1)} = a_{21}^{(1)} / a_{11}^{(1)} = \frac{1}{1} = 1 \Rightarrow E_2 \leftarrow E_2 - m_2^{(1)} E_1$

$\Leftrightarrow E_2 \leftarrow E_2 - E_1$

$m_3^{(1)} = a_{31}^{(1)} / a_{11}^{(1)} = \frac{2}{1} = 2 \Rightarrow E_3 \leftarrow E_3 - m_3^{(1)} E_1$

$\Leftrightarrow E_3 \leftarrow E_3 - 2E_1$

$m_4^{(1)} = a_{41}^{(1)} / a_{11}^{(1)} = \frac{0}{1} = 0 \Rightarrow E_4 \leftarrow E_4 - m_4^{(1)} E_1$

$\Leftrightarrow E_4 \leftarrow E_4$

Pentru $k=2$:

$$\bar{A}^{(2)} = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & -2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$a_{22}^{(2)} = 1 \neq 0 \Rightarrow$ putem aplica MEGFP

$i = \overline{3,4}: \quad m_i^{(2)} = a_{i2}^{(2)} / a_{22}^{(2)}$

$m_3^{(2)} = a_{32}^{(2)} / a_{22}^{(2)} = \frac{3}{1} = 3 \Rightarrow E_3 \leftarrow E_3 - m_3^{(2)} E_2$

$\Leftrightarrow E_3 \leftarrow E_3 - 3E_2$

$m_4^{(2)} = a_{42}^{(2)} / a_{22}^{(2)} = \frac{-1}{1} = -1 \Rightarrow E_4 \leftarrow E_4 - m_4^{(2)} E_2$

$\Leftrightarrow E_4 \leftarrow E_4 + E_2$

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Pentru $k=3$:
$$\bar{A}^{(3)} = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{8} & -8 & 1 & -3 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$a_{33}^{(3)} = 8 \neq 0 \Rightarrow$ putem aplica MEGF

$i=\overline{4,4}$: $m_i^{(3)} = a_{i3}^{(3)} / a_{33}^{(3)}$

$$m_4^{(3)} = a_{43}^{(3)} / a_{33}^{(3)} = \frac{-2}{8} = -\frac{1}{4} \Rightarrow E_4 \leftarrow E_4 - m_4^{(3)} E_3$$
$$\Leftrightarrow E_4 \leftarrow E_4 + \frac{1}{4} E_3$$

Am obținut
$$\bar{A}^{(4)} = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 8 & -8 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3/4 & 1/4 & 1/4 & 1 \end{array}\right]$$

Obținem 4 sisteme liniare superior triunghiulare :

$$\begin{cases} x_1^1 - x_2^1 + 2x_3^1 - x_4^1 = 1 \\ x_2^1 - 3x_3^1 + 2x_4^1 = -1 \\ 8x_3^1 - 8x_4^1 = 1 \\ -x_4^1 = -\frac{3}{4} \end{cases}$$

$\leftarrow$ Sistemul 1, care va da coloana 1 a lui A^{-1}

$$-x_4^1 = -\frac{3}{4} \Rightarrow x_4^1 = \frac{3}{4}$$

$$8x_3^1 = 1 + 8x_4^1 = 1 + 8 \cdot \frac{3}{4} = 1 + 2 \cdot 3 = 1 + 6 = 7 \Rightarrow x_3^1 = \frac{7}{8}$$

$$x_2^1 = -1 - 2x_4^1 + 3x_3^1 = -1 - 2 \cdot \frac{3}{4} + 3 \cdot \frac{7}{8} = \frac{-8 - 12 + 21}{8} = \frac{1}{8}$$

$$x_1^1 = 1 + x_4^1 - 2x_3^1 + x_2^1 = 1 + \frac{3}{4} - 2 \cdot \frac{7}{8} + \frac{1}{8} = \frac{8 + 6 - 14 + 1}{8} = \frac{1}{8}$$

$$\begin{cases} x_1^2 - x_2^2 + 2x_3^2 - x_4^2 = 0 \\ x_2^2 - 3x_3^2 + 2x_4^2 = 1 \\ 8x_3^2 - 8x_4^2 = -3 \\ -x_4^2 = \frac{1}{4} \end{cases}$$

$\leftarrow$ Sistemul 2, care dă coloana 2 a lui A^{-1}

$$-x_4^2 = \frac{1}{4} \Rightarrow x_4^2 = -\frac{1}{4}$$

$$8x_3^2 = -3 + 8x_4^2 = -3 + \overset{2}{\cancel{8}} \cdot \left(-\frac{1}{\cancel{4}}\right) = -3 - 2 = -5 \Rightarrow x_3^2 = -\frac{5}{8}$$

$$x_2^2 = 1 - 2x_4^2 + 3x_3^2 = 1 - 2\cdot\left(-\frac{1}{4}\right) + 3\cdot\left(-\frac{5}{8}\right) = {}^{8/}1 + {}^{4/}\frac{1}{2} - \frac{15}{8} = -\frac{3}{8}$$

$$x_1^2 = 0 + x_4^2 - 2x_3^2 + x_2^2 = 0 + \left(-\frac{1}{4}\right) - 2\cdot\left(-\frac{5}{8}\right) + \left(-\frac{3}{8}\right) =$$

$$= -{}^{2/}\frac{1}{4} + {}^{2/}\frac{5}{4} - \frac{3}{8} = \frac{-2 + 10 - 3}{8} = \frac{5}{8}$$

$$\begin{cases} x_1^3 - x_2^3 + 2x_3^3 - x_4^3 = 0 \\ x_2^3 - 3x_3^3 + 2x_4^3 = 0 \\ 8x_3^3 - 8x_4^3 = 1 \\ -x_4^3 = \frac{1}{4} \end{cases}$$

Sistemul 3, care va da coloana 3 a lui A^{-1}

$$-x_4^3 = \frac{1}{4} \Rightarrow x_4^3 = -\frac{1}{4}$$

$$8x_3^3 = 1 + 8x_4^3 = 1 + 8\cdot\left(-\frac{1}{4}\right) = 1 - 2 = -1 \Rightarrow x_3^3 = -\frac{1}{8}$$

$$x_2^3 = 0 - 2x_4^3 + 3x_3^3 = 0 - 2\cdot\left(-\frac{1}{4}\right) + 3\cdot\left(-\frac{1}{8}\right) = 0 + {}^{4/}\frac{1}{2} - \frac{3}{8} = \frac{1}{8}$$

$$x_1^3 = 0 + x_4^3 - 2x_3^3 + x_2^3 = -\frac{1}{4} - 2\cdot\left(-\frac{1}{8}\right) + \frac{1}{8} = -\cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} + \frac{1}{8} = \frac{1}{8}$$

$$\begin{cases} x_1^4 - x_2^4 + 2x_3^4 - x_4^4 = 0 \\ x_2^4 - 3x_3^4 + 2x_4^4 = 0 \\ 8x_3^4 - 8x_4^4 = 0 \\ -x_4^4 = 1 \end{cases}$$

Sistemul 4, care va da coloana 4 a lui A^{-1}

$$-x_4^4 = 1 \Rightarrow x_4^4 = -1$$

$$8x_3^4 = 0 + 8x_4^4 = 8\cdot(-1) = -8 \Rightarrow x_3^4 = \frac{-8}{8} = -1$$

$$x_2^4 = 0 - 2x_4^4 + 3x_3^4 = -2\cdot(-1) + 3\cdot(-1) = 2 - 3 = -1$$

$$x_1^4 = 0 + x_4^4 - 2x_3^4 + x_2^4 = -1 - 2\cdot(-1) + (-1) = -1 + 2 - 1 = 0$$

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Prin urmare, $A^{-1} = \begin{bmatrix} 1/8 & 5/8 & 1/8 & 0 \\ 1/8 & -3/8 & 1/8 & -1 \\ 7/8 & -5/8 & -1/8 & -1 \\ 3/4 & -1/4 & -1/4 & -1 \end{bmatrix}$

2. Să se determine factorizarea LU fără pivotare a matricii $A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$

Considerăm următoarea partiționare a lui A:

$$A = \left[\begin{array}{c|cc} 25 & 15 & -5 \\ \hline 15 & 18 & 0 \\ -5 & 0 & 11 \end{array}\right] = \underbrace{\left[\begin{array}{c|c} l_{11} & \underline{0} \\ \hline \underline{L}_{21} & L_{22} \end{array}\right]}_{L} \underbrace{\left[\begin{array}{c|c} u_{11} & \underline{U}_{12} \\ \hline \underline{0} & U_{22} \end{array}\right]}_{U} = \left[\begin{array}{c|c} l_{11}u_{11} & l_{11}\underline{U}_{12} \\ \hline \underline{L}_{21}u_{11} & \underline{L}_{21}\underline{U}_{12} + L_{22}U_{22} \end{array}\right]$$

$$L = \left[\begin{array}{c|cc} l_{11} & 0 & 0 \\ \hline l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{array}\right] \qquad U = \left[\begin{array}{c|cc} u_{11} & u_{12} & u_{13} \\ \hline 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{array}\right]$$

- $l_{11} u_{11} = 25 \Rightarrow l_{11} = 1$ și $u_{11} = 25$
- $l_{11} \underline{U}_{12} = [15 \ \ -5] \Rightarrow \underline{U}_{12} = [15 \ \ -5] \Rightarrow \begin{cases} u_{12} = 15 \\ u_{13} = -5 \end{cases}$
- $\underline{L}_{21} \cdot u_{11} = \begin{bmatrix} 15 \\ -5 \end{bmatrix} \Rightarrow \underline{L}_{21} = \begin{bmatrix} 15 \\ -5 \end{bmatrix} \cdot \frac{1}{25} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix} \Rightarrow \begin{cases} l_{21} = 3/5 \\ l_{31} = -1/5 \end{cases}$
- $\underline{L}_{21}\underline{U}_{12} + L_{22}U_{22} = \begin{bmatrix} 18 & 0 \\ 0 & 11 \end{bmatrix} \Rightarrow L_{22}U_{22} = \begin{bmatrix} 18 & 0 \\ 0 & 11 \end{bmatrix} - \underline{L}_{21}\underline{U}_{12} \Rightarrow$

$$L_{22}U_{22} = \begin{bmatrix} 18 & 0 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix} [15 \ \ -5] = \begin{bmatrix} 18 & 0 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 10 \end{bmatrix} = S$$

Problema s-a redus la factorizarea LU a matricei S

$$S = \left[\begin{array}{c|c} 9 & 3 \\ \hline 3 & 10 \end{array}\right] = \left[\begin{array}{c|c} l_{22} & 0 \\ \hline l_{32} & l_{33} \end{array}\right] \left[\begin{array}{c|c} u_{22} & u_{23} \\ \hline 0 & u_{33} \end{array}\right] = \left[\begin{array}{c|c} l_{22}u_{22} & l_{22}u_{23} \\ \hline l_{32}u_{22} & l_{32}u_{23} + l_{33}u_{33} \end{array}\right]$$

- $l_{22} u_{22} = 9 \Rightarrow l_{22} = 1$ și $u_{22} = 9$
- $l_{22} u_{23} = 3 \Rightarrow 1 \cdot u_{23} = 3 \Rightarrow u_{23} = 3$
- $u_{22} l_{32} = 3 \Rightarrow 9 \cdot l_{32} = 3 \Rightarrow l_{32} = 1/3$
- $l_{32} u_{23} + l_{33} u_{33} = 10 \Rightarrow l_{33} u_{33} = 10 - l_{32} u_{23} = 10 - \frac{1}{3} \cdot 3 = 10 - 1 = 9$

Problema s-a redus la factorizarea LU a lui 9

$l_{33} u_{33} = 9 \Rightarrow l_{33} = 1$ și $u_{33} = 9$

Am obținut: $L = \begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix}$ și $U = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}$

3. Să se determine factorizarea LDU a matricii $A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$

După factorizarea LU fără pivotare, am obținut

$$A = L\tilde{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}}_{\tilde{U} = DU}$$

Vrem $\tilde{U} = DU$

$$D = \operatorname{diag}(25, 9, 9) = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$D^{-1} = \operatorname{diag}(1/25, 1/9, 1/9) = \begin{bmatrix} 1/25 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/9 \end{bmatrix}$$

$\tilde{U} = DU \Rightarrow U = D^{-1}\tilde{U}$

$$U = \begin{bmatrix} 1/25 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} \cdot \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

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Am obținut $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$

4. Să se determine factorizarea LDL^T a matricii $A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$

După factorizarea LU fără pivotare am obținut

$$A = L\tilde{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}}_{\tilde{U} = DL^T}$$

Vrem $\tilde{U} = DL^T$

$$D = \mathrm{diag}(25, 9, 9) = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{U} = D \cdot L^T = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

Am obținut $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}$

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