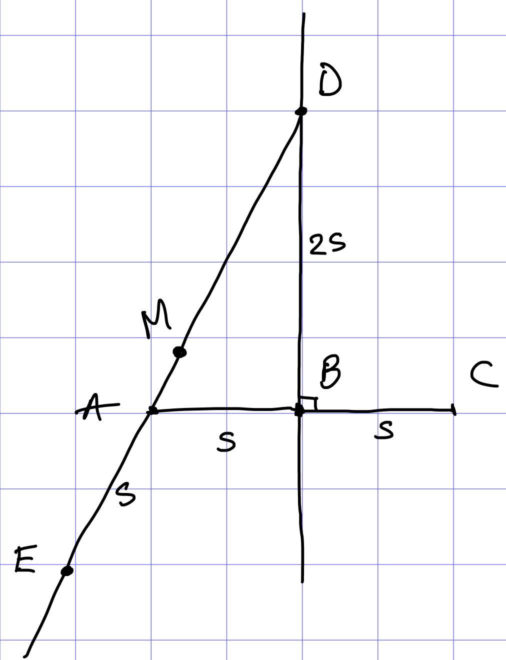
Geometrie
SEMINARUL 2

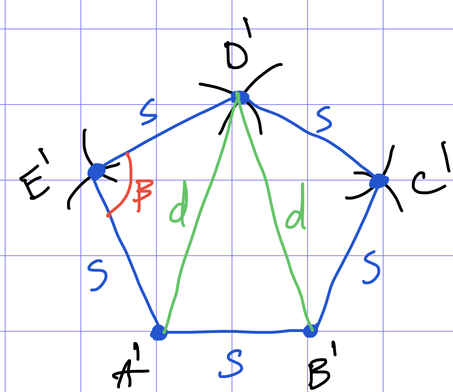


$AD = \sqrt{5}s$

$M =$ mijlocul lui ED

$$DM = \frac{1}{2}(s + \sqrt{5}s) = \frac{1+\sqrt{5}}{2}s$$

$$a = \frac{1+\sqrt{5}}{2} \qquad d = a\cdot s = \frac{1+\sqrt{5}}{2}s$$



Th. cosinusului în $\Delta E'A'D'$

$$A'D'^2 = E'A'^2 + E'D'^2 - 2E'A'\cdot E'D' \cos\beta$$

$$d^2 = s^2 + s^2 - 2s\cdot s\cdot\cos\beta$$

$$(as)^2 = 2s^2 - 2s^2\cos\beta$$

$$a^2\cdot s^2 = 2s^2(1-\cos\beta) \quad /\cdot\frac{1}{s^2}$$

$$a^2 = 2(1-\cos\beta)$$

$$a+1 = 2(1-\cos\beta)$$

$$\cos\beta = 1 - \frac{a+1}{2} = \frac{1-a}{2} = \frac{\frac{1-\sqrt{5}}{2}}{2} = \frac{1}{4}(1-\sqrt{5})$$

$$a = \frac{1+\sqrt{5}}{2}$$

$$a^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

$$= a+1$$

Care este măsura unghiului unui pentagon regulat?

$S =$ Suma măsurilor unghiurilor unui poligon cu n laturi: $180^\circ(n-2)$

$$S = 180^\circ \cdot 3 = 540^\circ$$

măsura unui unghi: $\frac{S}{5} = 108^\circ = \frac{3\pi}{5}$

Vrem să demonstrăm că $\beta = 108^\circ$. Va fi suficient să arătăm că $\cos 108^\circ = \frac{1}{4}(1-\sqrt{5})$.

$$\cos(108^\circ) = \cos(90^\circ + 18^\circ) = \cos 90^\circ \cos 18^\circ - \sin 90^\circ \sin 18^\circ = -\sin 18^\circ$$

Notăm $18^\circ = \alpha = \frac{\pi}{10}$

$$\begin{aligned}\cos(3\alpha) &= \cos(2\alpha+\alpha) \\ &= \cos(2\alpha)\cos\alpha - \sin 2\alpha \sin\alpha \\ &= (1-2\sin^2\alpha)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha \\ &= (1-2\sin^2\alpha)\cos\alpha - 2\sin^2\alpha\cos\alpha \\ &= \boxed{(1-4\sin^2\alpha)\cos\alpha}\end{aligned}$$

$$= \underbrace{\sin\frac{\pi}{2}}_{=1}\cos 3\alpha - \underbrace{\cos\frac{\pi}{2}}_{=0}\sin 3\alpha$$

Pe de altă parte, $\cos(3\alpha) = \sin\left(\frac{\pi}{2} - 3\alpha\right) = \sin(5\alpha - 3\alpha) = \sin 2\alpha$
$$= \boxed{2\sin\alpha\cos\alpha}$$

$\Downarrow$

$$(1-4\sin^2\alpha)\cos\alpha = 2\sin\alpha\cos\alpha \quad \Big/ \cdot \frac{1}{\cos\alpha}$$

$$1 - 4\sin^2\alpha = 2\sin\alpha$$

$$4\sin^2\alpha + 2\sin\alpha - 1 = 0 \qquad (\text{Știm că } \sin\alpha > 0)$$

$$\sin\alpha = \frac{-2+\sqrt{4-4\cdot 4\cdot(-1)}}{8}$$

$$= \frac{-2+\sqrt{20}}{8} = \frac{-2+2\sqrt{5}}{8} = \frac{-1+\sqrt{5}}{4}$$

$$\cos 108^\circ = -\sin 18^\circ = -\sin\alpha = -\frac{-1+\sqrt{5}}{4} = \frac{1-\sqrt{5}}{4} = \frac{1}{4}(1-\sqrt{5})$$

Deci $\beta = 108^\circ$:)

$\boxed{\text{Exc. 2.1}}$ $|\langle u, w\rangle| \leq \|u\|\cdot\|w\|$, $\forall u, w$ vectori în plan.

Demo.: $\langle u, w\rangle = \|u\|\cdot\|w\|\cdot\cos\alpha \Rightarrow$ exact ce vrem, dacă punem module și folosim $|\cos\alpha| \leq 1$.

Altfel: $\langle u+\lambda w, u+\lambda w\rangle = \|u+\lambda w\|^2 \geq 0$

$\lambda \in \mathbb{R}$

$$\| $$

$$\langle u, u+\lambda w\rangle + \langle \lambda w, u+\lambda w\rangle$$

$$\|$$

$$\langle u,u\rangle + \langle u,\lambda w\rangle + \langle \lambda w, u\rangle + \langle \lambda w, \lambda w\rangle$$

$$\|$$

$$\|u\|^2 + \|\lambda w\|^2 + 2\langle u, \lambda w\rangle$$

$$\|$$

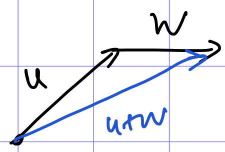
$$\|u\|^2 + \lambda^2\|w\|^2 + 2\lambda\langle u, w\rangle \geq 0 \quad \forall \lambda\in\mathbb{R}$$

Deci $\Delta = \left(2\langle u,w\rangle\right)^2 - 4\|w\|^2\|u\|^2 \leq 0$

$$4(\langle u,w\rangle)^2 \leq 4\|u\|^2\|w\|^2$$

$$\langle u,w\rangle \leq |\langle u,w\rangle| \leq \|u\|\cdot\|w\|$$

Exc. 2.2



$$\|u+w\| \le \|u\| + \|w\| \quad , \ \forall\, u, w \text{ vectori în plan}$$

de demonstrat

$$\|u+w\| = \sqrt{\langle u+w, u+w\rangle}$$

$$= \sqrt{\langle u,u\rangle + \langle w,w\rangle + 2\langle u,w\rangle}$$

$$= \sqrt{\|u\|^2 + \|w\|^2 + 2\langle u,w\rangle}$$

$$\overset{\text{Cauchy-Schwarz}}{\le} \sqrt{\|u\|^2 + \|w\|^2 + 2\|u\|\cdot\|w\|}$$

$$= \sqrt{(\|u\| + \|w\|)^2}$$

$$= \|u\| + \|w\|$$

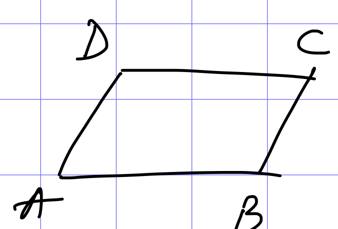
Exc 2.3

ABCD paralelogram

M, N, P, Q puncte în plan a.î. $\overrightarrow{AM} + \overrightarrow{CP} = \overrightarrow{BN} + \overrightarrow{DQ}$

Demonstrați că $MNPQ$ este paralelogram.

Soluție:



ABCD paralelogram $\iff \overrightarrow{AB} = \overrightarrow{DC}$ (știm)

Vrem să demo.: MNPQ paralelogram $\iff \overrightarrow{MN} = \overrightarrow{QP}$

$$\overrightarrow{AM} + \overrightarrow{CP} = \overrightarrow{BN} + \overrightarrow{DQ}$$

$$\cancel{\overrightarrow{AB}} + \cancel{\overrightarrow{BN}} + \overrightarrow{NM} + \cancel{\overrightarrow{CP}} = \cancel{\overrightarrow{BN}} + \cancel{\overrightarrow{DC}} + \cancel{\overrightarrow{CP}} + \overrightarrow{PQ}$$

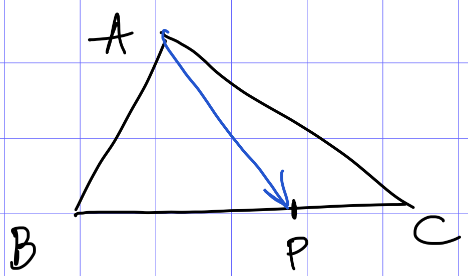
$$\overrightarrow{NM} = \overrightarrow{PQ}$$

$$\overrightarrow{MN} = \overrightarrow{QP}$$

$\boxed{\text{Exc } 2.4}$

ΔABC

$P \in BC$ a.î. $\overrightarrow{BP} = k \cdot \overrightarrow{PC}$



Obs: suma lor este 1

Vrem să demonstrăm că $\overrightarrow{AP} = \frac{1}{k+1}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{AC}$

Demo.:

$$\overrightarrow{BA} + \overrightarrow{AP} = k\left(\overrightarrow{PA} + \overrightarrow{AC}\right)$$

$$\overrightarrow{BA} + \overrightarrow{AP} + k\overrightarrow{AP} = k\overrightarrow{AC}$$

$$(k+1)\overrightarrow{AP} = k\overrightarrow{AC} + \overrightarrow{AB}$$

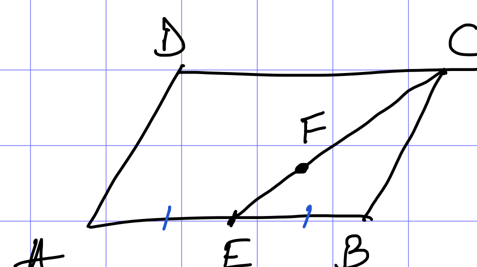
$$\overrightarrow{AP} = \frac{1}{k+1}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{AC}$$

Exc 2.5

ABCD paralelogram

$E \in AB$ a.î. $\overrightarrow{AE} = \overrightarrow{EB}$

$F \in EC$ a.î. $\overrightarrow{EC} = 3\overrightarrow{EF}$



Vrem să demonstrăm că B, F, D coliniare.

$\lambda\overrightarrow{BD} = \overrightarrow{BF}$ ← vrem

Demo.:

$$\overrightarrow{BF} = \overrightarrow{BC} + \overrightarrow{CF}$$

$$= \overrightarrow{BC} + \frac{2}{3}\overrightarrow{CE}$$

$$= \overrightarrow{BC} + \frac{2}{3}\left(\frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}\right)$$

$$= \overrightarrow{BC} + \frac{1}{3}\overrightarrow{CA} + \frac{1}{3}\overrightarrow{CB}$$

$$= \overrightarrow{BC} - \frac{1}{3}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{CA}$$

$$= \frac{2}{3}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{CA}$$

$$= \frac{2}{3}\overrightarrow{BC} + \frac{1}{3}\left(\overrightarrow{CB} + \overrightarrow{BA}\right)$$

$$= \frac{2}{3}\overrightarrow{BC} - \frac{1}{3}\overrightarrow{BC} + \frac{1}{3}\overrightarrow{BA}$$

$$= \frac{1}{3}\left(\overrightarrow{BC} + \overrightarrow{BA}\right)$$

$$= \frac{1}{3}\overrightarrow{BD}$$

Am obținut: $\overrightarrow{BF} = \frac{1}{3}\overrightarrow{BD}$

$\Downarrow$

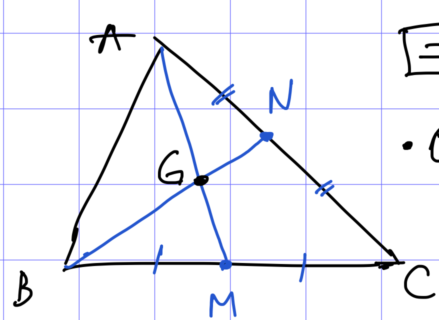
B, F, D coliniare

$\boxed{\text{Exc } 2.6}$ $\triangle ABC$

G punct în plan.

Atunci, G este centrul de greutate al lui ABC $\Longleftrightarrow$ $\forall O$ în plan, $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB} + \frac{1}{3}\overrightarrow{OC}$

Demo.:



$\boxed{\Longrightarrow}$

$\cdot O$

$$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG}$$

$$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AM}$$

$$= \overrightarrow{OA} + \frac{2}{3}\left(\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}\right)$$

$$= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$$

$$= \frac{1}{3}\left(3\overrightarrow{OA} + \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{AO} + \overrightarrow{OC}\right)$$

$$= \frac{1}{3}\left(3\overrightarrow{OA} + 2\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OC}\right)$$

$$= \frac{1}{3}\left(\vec{OA}+\vec{OB}+\vec{OC}\right)$$ ✓

$\boxed{\Leftarrow}$ **Știm** că $\forall\ O \in$ plan, $\frac{1}{3}\left(\vec{OA}+\vec{OB}+\vec{OC}\right) = \vec{OG}$

Vrem: G = centrul de greutate al lui ABC.

Considerăm G' = centrul de greutate al lui ABC.

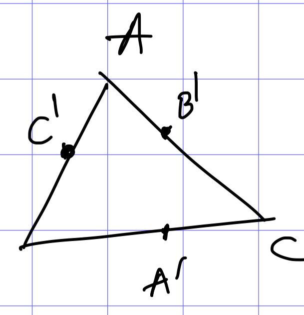
Din $\boxed{\Rightarrow}$ avem: $\vec{OG'} = \frac{1}{3}\left(\vec{OA}+\vec{OB}+\vec{OC}\right)$

Dar știm și $\vec{OG} = \frac{1}{3}\left(\vec{OA}+\vec{OB}+\vec{OC}\right)$

$$\Rightarrow \vec{OG'} = \vec{OG} \Downarrow G' = G$$

(2.7) $\rightarrow$ temă

Exc 2.8 ΔABC, G = centrul de greutate



$A' \in [BC]$, $B' \in [CA]$, $C' \in [AB]$

a.î. $\frac{BA'}{BC} = \alpha$; $\frac{CB'}{CA} = \beta$, $\frac{AC'}{AB} = \gamma$

(a) Arătați $\vec{AG} = \frac{1}{3}\left(\vec{AB}+\vec{AC}\right)$ și $\vec{AG}+\vec{BG}+\vec{CG} = \vec{0}$

(b) centrul de greutate G' al lui $A'B'C'$ $\in$ medianei din A a lui ABC

$\Leftrightarrow\ 2\alpha = \beta + \gamma$

(c) $G' = G \Leftrightarrow \alpha = \beta = \gamma$

Demo.: (a) $\vec{OG} = \frac{1}{3}\left(\vec{OA}+\vec{OB}+\vec{OC}\right)$

$\vec{AO}+\vec{OG} = \vec{AO} + \frac{1}{3}\vec{OA} + \frac{1}{3}\vec{OB} + \frac{1}{3}\vec{OC}$

$\vec{AG} = \frac{2}{3}\vec{AO} + \frac{1}{3}\vec{OB} + \frac{1}{3}\vec{OC}$

Știm: $\vec{OG} = \frac{1}{3}\left(\vec{OA}+\vec{OB}+\vec{OC}\right)$, $\forall O$ în plan

Alegem $O = G$

$\vec{0} = \vec{GA}+\vec{GB}+\vec{GC}$

$$\overrightarrow{AG} = \frac{1}{3}\left(\overrightarrow{AO}+\overrightarrow{OB}\right) + \frac{1}{3}\left(\overrightarrow{AO}+\overrightarrow{OC}\right)$$

$$\overrightarrow{AG} = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC} \quad \checkmark$$

$$\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \overrightarrow{0}$$

$\boxed{(b)+(c) \longrightarrow \text{temă}}$