Extreme cu legaturi

Reamintim (vezi notatia de la diferentiala de ordinul intai) ca daca $T : \mathbb{R}^n \to \mathbb{R}$ este o aplicatie liniara definita prin

$$T(u_1, u_2, \dots, u_n) = a_1u_1 + \cdots + a_nu_n, \quad u = (u_1, \dots, u_n) \in \mathbb{R}^n.$$

unde $a_1, \dots, a_n \in \mathbb{R}$ aceasta poate fi scrisa mai simplu

$$T = a_1dx_1 + a_2dx_2 + \cdots + a_ndx_n.$$

Cu acesata notatie, daca $\varphi : \mathbb{R}^n \to \mathbb{R}$ este o forma patratica definita prin

$$\varphi(u_1, u_2, \dots, u_n) = \sum_{i,j=1}^{n} a_{ij}u_iu_j \quad u = (u_1, \dots, u_n) \in \mathbb{R}^n$$

unde $a_{ij} \in \mathbb{R}$ atunci

$$\varphi = \sum_{i,j=1}^{n} a_{ij}dx_idx_j.$$

Astfel, daca f este o functie de clasa C^2 pe o multime deschisa D din $\mathbb{R}^n$, atunci diferentialala lui f in punctul $a \in D$ se scrie

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)dx_i$$

si diferentiala de ordinul doi in punctul a se scrie

$$d^2f(a) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)dx_idx_j.$$

Fie $f : D \subset \mathbb{R}^n \to \mathbb{R}$ si $A \subset D$. Spunem ca functia f are intr-un punct $a \in A$ un extrem local relativ la multimea A daca restrictia functiei f la multimea A are in punctul a un punct de extrem local obisnuit. Astfel, functia f are in punctul $a \in A$ un punct de minim (maxim) relativ la multimea A daca exista o vecinatate V a lui a astfel incat $f(x) \geq f(a)$ (respectiv $f(x) \leq f(a)$) pentru orice $x \in A \cap V$. Extremele functiei f relativ la o submultime $A \subset D$ se numesc extreme conditionate.

Fie $f : D \subset \mathbb{R}^n \to \mathbb{R}$, $g_1, g_2, \dots, g_m : D \to \mathbb{R}$ si

$$A = \{x \in D : g_1(x) = g_2(x) = \cdots = g_m(x) = 0\}.$$

Un punct $a \in A$ se numeste punct de extrem local al functiei f cu legaturile $g_1(x) = 0$, $g_2(x) = 0, \dots,$ $g_m(x) = 0$ daca a este un punct de extrem local al functiei f relativ la multimea A.

Teorema 1. Fie $D \subset \mathbb{R}^n \times \mathbb{R}^m$ deschisa si $a \in D$. Fie $f : D \to \mathbb{R}$ de clasa C^1 si $g = (g_1, g_2, \ldots, g_m) : D \to \mathbb{R}^m$ de clasa C^1 astfel incat

$$g(a) = 0, \quad \text{rang} J_g(a) = m, \text{ unde } \quad J_g(a) = \left(\frac{\partial g_i}{\partial x_j}\right)$$

Daca a este punct de extrem local pentru restrictia functiei f la multimea

$$\{x \in D : g_1(x) = 0, g_2(x) = 0, \ldots, g_m(x) = 0\},$$

(adica a este punct de extrem local al functiei f cu legaturile

$$g_1(x) = 0, \; g_2(x) = 0 \;, \ldots, g_m(x) = 0 \tag{1}$$

atunci exista numerele reale $\lambda_1, \lambda_2, \ldots, \lambda_m$, numite multiplicatori ai lui Lagrange astfel incat, daca

$$L(x) = f(x) + \lambda_1 g_1(x) + \cdots + \lambda_m g_m(x)$$

atunci

$$\frac{\partial L}{\partial x_i}(a) = 0, \quad \text{pentru } i = 1, 2, \ldots, n + m. \tag{2}$$

Demonstratie. Vom demonstra teorema pentru cazul $m = n = 2$. Asadar fie $f : D \subset \mathbb{R}^4 \to \mathbb{R}$ de clasa C^1, $g, h : D \to \mathbb{R}$ functii de clasa C^1 astfel incat

$$\frac{D(g,h)}{D(z,w)}(x_0, y_0, z_0, w_0) \neq 0. \tag{3}$$

Trebuie sa aratam ca daca (x_0, y_0, z_0, w_0) este un punct de extrem local al functiei f cu legaturile $g(x, y, z, w) = 0$ si $h(x, y, z, w) = 0$ atunci exista doua numere reale λ_1 si λ_2 astfel incat, daca $L(x, y, z, w) = f(x, y, x, w) + \lambda_1 g(x, y, z, w) + \lambda_2 h(x, y, z, w)$ atunci

$$\frac{\partial L}{\partial x}(x_0, y_0, z_0, w_0) = 0, \qquad \frac{\partial L}{\partial y}(x_0, y_0, z_0, w_0) = 0$$

$$\frac{\partial L}{\partial z}(x_0, y_0, z_0, w_0) = 0, \qquad \frac{\partial L}{\partial w}(x_0, y_0, z_0, w_0) = 0.$$

Sistemul

$$\begin{cases} g(x, y, z, w) = 0 \\ h(x, y, z, w) = 0 \end{cases}$$

indeplineste conditiile Teoremei Functiilor Implicite in punctul $a = (x_0, y_0, z_0, w_0)$ si atunci exista U o vecinatate deschisa a punctului (x_0, y_0), exista V o vecinatate deschisa a punctului (z_0, w_0) si o unica pereche de functii $(\varphi, \psi) : U \to V$ de clasa C^1 astfel incat $\varphi(x_0, y_0) = z_0$, $\psi(x_0, y_0) = z_0$ si

$$g(x, y, \varphi(x, y), \psi(x, y)) = 0, \; h(x, y, \varphi(x, y), \psi(x, y)) = 0, \; \forall (x, y) \in U.$$

Avem

$$\frac{\partial g}{\partial x}(a)+\frac{\partial g}{\partial z}(a)\frac{\partial \varphi}{\partial x}(x_0,y_0)+\frac{\partial g}{\partial w}(a)\frac{\partial \psi}{\partial x}(x_0,y_0)=0 \tag{4}$$
$$\frac{\partial g}{\partial y}(a)+\frac{\partial g}{\partial z}(a)\frac{\partial \varphi}{\partial y}(x_0,y_0)+\frac{\partial g}{\partial w}(a)\frac{\partial \psi}{\partial y}(x_0,y_0)=0 \tag{5}$$
$$\frac{\partial h}{\partial x}(a)+\frac{\partial h}{\partial z}(a)\frac{\partial \varphi}{\partial x}(x_0,y_0)+\frac{\partial h}{\partial w}(a)\frac{\partial \psi}{\partial x}(x_0,y_0)=0 \tag{6}$$
$$\frac{\partial h}{\partial y}(a)+\frac{\partial h}{\partial z}(a)\frac{\partial \varphi}{\partial y}(x_0,y_0)+\frac{\partial h}{\partial w}(a)\frac{\partial \psi}{\partial y}(x_0,y_0)=0 \tag{7}$$

Functia $H : U \to \mathbb{R}$

$$H(x,y)=f(x,y,\varphi(x,y),\psi(x,y))$$

are in a un punct de extrem local neconditionat, deoarece functia f are in a un punct de extrem local conditionat. Cum H este de clasa C^1 rezulta ca

$$\frac{\partial H}{\partial x}(a)=\frac{\partial H}{\partial y}(a)=0$$

Asadar

$$\frac{\partial f}{\partial x}(a)+\frac{\partial f}{\partial z}(a)\frac{\partial \varphi}{\partial x}(x_0,y_0)+\frac{\partial f}{\partial w}(a)\frac{\partial \psi}{\partial x}(x_0,y_0)=0 \tag{8}$$
$$\frac{\partial f}{\partial y}(a)+\frac{\partial f}{\partial z}(a)\frac{\partial \varphi}{\partial y}(x_0,y_0)+\frac{\partial f}{\partial w}(a)\frac{\partial \psi}{\partial y}(x_0,y_0)=0 \tag{9}$$

Din (3), rezulta ca exista si sunt unice doua numere reale λ_1, λ_2 astfel incat

$$\begin{pmatrix} \frac{\partial f}{\partial z}(a) \\ \frac{\partial f}{\partial w}(a) \end{pmatrix} = -\begin{pmatrix} \frac{\partial g}{\partial z}(a) & \frac{\partial h}{\partial z}(a) \\ \frac{\partial g}{\partial w}(a) & \frac{\partial h}{\partial w}(a) \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

Aceasta inseamna ca

$$\frac{\partial L}{\partial z}(a)=0, \quad \frac{\partial L}{\partial w}(a)=0$$

Din (4), (6) si (8) rezulta ca

$$\begin{aligned}\frac{\partial L}{\partial x}(a) &= \frac{\partial f}{\partial x}(a)+\lambda_1\frac{\partial g}{\partial x}(a)+\lambda_2\frac{\partial h}{\partial x}(a)\\ &= \frac{\partial f}{\partial x}(a)-\lambda_1\left(\frac{\partial g}{\partial z}(a)\frac{\partial \varphi}{\partial x}(x_0,y_0)+\frac{\partial g}{\partial w}(a)\frac{\partial \psi}{\partial x}(x_0,y_0)\right)-\lambda_2\left(\frac{\partial h}{\partial z}(a)\frac{\partial \varphi}{\partial x}(x_0,y_0)+\frac{\partial h}{\partial w}(a)\frac{\partial \psi}{\partial x}(x_0,y_0)\right)\\ &= \frac{\partial f}{\partial x}(a)+\frac{\partial f}{\partial z}(a)\frac{\partial \varphi}{\partial x}(x_0,y_0)+\frac{\partial f}{\partial w}(a)\frac{\partial \psi}{\partial x}(x_0,y_0)=0.\end{aligned}$$

Similar se demonstreaza ca $\frac{\partial L}{\partial y}(a)=0$. Demonstratia este incheiata.

Cu aceleasi notatii ca in Teorema 1, orice punct $a \in D$ care verifica conditiile (1) pentru care matricea

$$J_g(a)=\left(\frac{\partial g_i}{\partial x_j}(a)\right) \tag{10}$$

are rangul m si care verifica si conditiile (2) pentru anumite valori $\lambda_1, \ldots, \lambda_m$ se numeste punct stationar sau critic al functiei f conditionat de (1). Valorile $\lambda_1, \ldots, \lambda_m$ se schimba in functie de punctul stationar a.

Fie $D \subset \mathbb{R}^n \times \mathbb{R}^m$ si $a \in D$. Fie $f : D \to \mathbb{R}$ de clasa C^1. Pentru a determina punctele critice conditionate de legaturile

$$g_1(x) = 0,\ g_2(x) = 0, \cdots, g_m(x) = 0 \tag{11}$$

unde $g_1, g_2, \ldots, g_m$ sunt functii de clasa C^2 pe D procedam astfel:

1) Definim functia lui Lagrange

$$L(x) = f(x) + \lambda_1 g_1(x) + \ldots + \lambda_m g_m(x)$$

cu $\lambda_1, \ldots, \lambda_m$ nedeterminati.

2) Consideram sistemul cu $n + 2m$ ecuatii

$$\begin{cases} \frac{\partial L}{\partial x_1}(x_1, x_2, \ldots, x_{n+m}) = 0 \\ \vdots \\ \frac{\partial L}{\partial x_{n+m}}(x_1, x_2, \ldots, x_{n+m}) = 0 \\ g_1(x_1, x_2, \ldots, x_{n+m}) = 0 \\ \vdots \\ g_m(x_1, x_2, \ldots, x_{n+m}) = 0 \end{cases} \tag{12}$$

cu necunoscutele $x_1, x_2, \ldots, x_{n+m}, \lambda_1, \ldots, \lambda_m$.

3) Punctele de extrem local conditionat ale functiei f se gasesc printre punctele critice conditionate ale lui f.

Vom furniza in continuare conditii suficiente pentru a decide care dintre punctele critice conditionate sunt puncte de extrem local conditionat.

Sa presupunem in continuare ca functia f si functiile $g_1, g_2, \ldots, g_m$ sunt de clasa C^2 si ca $(a_1, \ldots, a_{n+m})$ este punct critic conditionat. Asadar, exista $\lambda_1, \ldots, \lambda_m$ astfel incat $a_1, \ldots, a_{n+m}, \lambda_1, \ldots, \lambda_m$ sa fie solutie a sistemului (12). Pentru a vedea daca a este sau nu punct de extrem conditionat trebuie sa studiem diferenta

$$f(x) - f(a) = f(x_1, x_2, \ldots, x_{n+m}) - f(a_1, a_2, \ldots, a_{n+m})$$

cu conditia ca punctele x sa verifice ecuatiile de legatura, deci pentru care $g_1(x) = 0, \ldots, g_m(x) = 0$. Este usor de vazut ca pentru asemenea puncte

$$f(x) - f(a) = L(x) - L(a)$$

Asadar studiul diferentei $f(x) - f(a)$ pentru punctele care verifica sistemul (1), revine la studiul diferentei

$$L(x) - L(a)$$

pentru asemenea puncte. Dar punctul a verifica sistemul (2) si deci a este punct stationar obisnuit pentru functia L. Functia L are derivate partiale de ordinul 2 continue pe D si atunci

$$L(x) - L(a) = \frac{1}{2} \sum_{i,j=1}^{n+m} \frac{\partial^2 L}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + \frac{1}{2}\omega(x)\|x - a\|^2$$

cu $\omega(a) = 0$ si $\lim_{x \to a} \omega(x) = 0$. Sa presupunem acum ca determinantul minorului corespunzator ultimelor m coloane ale matricii (10) este nenul. Atunci, intr-o vecinatate a lui a sistemul

$$\begin{cases} g_1(x_1, x_2, \ldots, x_{n+m}) = 0 \\ \vdots \\ g_m(x_1, x_2, \ldots, x_{n+m}) = 0 \end{cases}$$

defineste implicit $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ ca functii de $(x_1, \ldots, x_n)$ Asadar,

$$\begin{array}{l}
\frac{\partial g_1}{\partial x_1}(a)dx_1 + \cdots + \frac{\partial g_1}{\partial x_n}(a)dx_n + \frac{\partial g_1}{\partial x_{n+1}}(a)dx_{n+1} + \cdots + \frac{\partial g_1}{\partial x_{n+m}}(a)dx_{n+m} = 0 \\
\frac{\partial g_2}{\partial x_1}(a)dx_1 + \cdots + \frac{\partial g_2}{\partial x_n}(a)dx_n + \frac{\partial g_2}{\partial x_{n+1}}(a)dx_{n+1} + \cdots + \frac{\partial g_2}{\partial x_{n+m}}(a)dx_{n+m} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial g_m}{\partial x_1}(a)dx_1 + \cdots + \frac{\partial g_m}{\partial x_n}(a)dx_n + \frac{\partial g_m}{\partial x_{n+1}}(a)dx_{n+1} + \cdots + \frac{\partial g_m}{\partial x_{n+m}}(a)dx_{n+m} = 0
\end{array}$$

unde $dx_{n+1}, \ldots dx_{n+m}$ reprezinta diferentialele functiilor implicite $x_{n+1}, \ldots, x_{n+m}$ in punctul $(a_1, a_2, \ldots a_n)$. Deoarece matricea acestui sistem liniar are rangul m putem exprima m dintre acestea in functie de celelalte n si inclocuindu-le in

$$d^2 L(a) = \sum_{i,j=1}^{n+m} \frac{\partial^2 L}{\partial x_i \partial x_j}(a) dx_i dx_j$$

obtinem forma patratica

$$\sum_{i,j=1}^{n} a_{ij} dx_i dx_j.$$

De fapt

$$f(x) - f(a) = L(x) - L(a) = \sum_{i,j=1}^{n} a_{ij}(x_i - a_i)(x_j - a_j) + \frac{1}{2}\alpha(x)\|x - a\|^2$$

cu $\alpha(a) = 0$ si $\lim_{x \to a} \alpha(x) = 0$. Procedand ca in cazul punctelor de extrem local neconditionat, se demonstreaza ca daca forma patratica $\sum_{i,j=1}^{n} a_{ij} dx_i dx_j$ este pozitiv (resp. negativ) definita atunci a este punct de minim (resp. maxim) local conditionat.

Algoritm 1. Sa considera functiile $f, g : D \subset \mathbb{R}^3 \to \mathbb{R}$ de clasa C^2. Pentru a determina punctele de extrem local ale functiei f cu legatura $g(x, y, z) = 0$ procedam astfel:

Pasul 1. Se considera functia $L : D \to \mathbb{R}$,

$$L(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$$

Pasul 2. Se rezolva sistemul

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,z) = 0 \\ \frac{\partial L}{\partial y}(x,y,z) = 0 \\ \frac{\partial L}{\partial z}(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$$

Pasul 3. Sa consideram $(\lambda_0, x_0, y_0, z_0)$ o solutie a sistemului de mai sus cu proprietatea ca

$$\text{rang}\left(\frac{\partial g}{\partial x}(x_0, y_0, z_0), \frac{\partial g}{\partial y}(x_0, y_0, z_0), \frac{\partial g}{\partial z}(x_0, y_0, z_0)\right) = 1$$

Pentru functia

$$L(x, y, z) = f(x, y, z) + \lambda_0 g(x, y, z)$$

determinam diferentiala de ordinul doi in $a = (x_0, y_0, z_0)$

$$\begin{aligned} d^2L(a) = {} & \frac{\partial^2 L}{\partial x^2}(a)dx^2 + \frac{\partial^2 L}{\partial y^2}(a)dy^2 + \frac{\partial^2 L}{\partial z^2}(a)dz^2 + 2\frac{\partial^2 L}{\partial x \partial y}(a)dxdy \\ & + 2\frac{\partial^2 L}{\partial x \partial z}(a)dxdz + 2\frac{\partial^2 L}{\partial y \partial z}(a)dydz \end{aligned} \tag{13}$$

Cazul 1. Daca $d^2L(a)$ este pozitiv (resp. negativ definita) atunci a este punct de minim (resp. maxim) al functiei f cu legatura $g(x, y, z) = 0$.

Cazul 2. Daca $d^2L(a)$ nu este nici pozitiv si nici negativ definita, diferentiem legatura, adica

$$\frac{\partial g}{\partial x}(a)dx + \frac{\partial g}{\partial y}(a)dy + \frac{\partial g}{\partial z}(a)dz = 0$$

Exprimam una dintre dx, dy, dz in functie de celelate doua, inlocuim in (13) si studiem daca forma patratica astfel obtinuta este pozitiv definita, negativ definita sau nedefinta pentru a decide natura punctului a.

Exercitiu. Sa se gaseasca punctele de extrem local ale functiei

$$f(x, y, z) = xy + xz + yz$$

cu legatura $xyz = 1$, in domeniul $x > 0$, $y > 0$, $z > 0$.

Sokutie. Fie $D = \{(x, y, z) | x, y, z > 0\}$ si $g : D \to \mathbb{R}$, $g(x, y, z) = xyz - 1$. Functiile f si g sunt de clasa C^2 pe multimea deschisa D si

$$\text{rang}\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \text{rang}\,(yz, xz, yx) = 1 \quad \forall (x, y, z) \in D.$$

Consideram functia lui Lagrange

$$L(x, y, z) = xy + xz + yz + \lambda(xyz - 1)$$

si sistemul

$$\begin{cases} \frac{\partial L}{\partial x}(x, y, z) = y + z + \lambda yz = 0 \\ \frac{\partial L}{\partial y}(x, y, z) = x + z + \lambda xz = 0 \\ \frac{\partial L}{\partial z}(x, y, z) = x + y + \lambda xy = 0 \\ xyz = 1 \end{cases}$$

Acest sistem are solutia $x = 1$, $y = 1$, $z = 1$, $\lambda = -2$. (Vezi seminarul!) Functia devine

$$L(x, y, z) = xy + xz + yz - 2(xyz - 1)$$

Trebuie sa calculam $d^2L(1, 1, 1)$.

$$\frac{\partial^2 L}{\partial x^2} = \frac{\partial^2 L}{\partial y^2} = \frac{\partial^2 L}{\partial z^2} = 0,\ \frac{\partial^2 L}{\partial x \partial y} = 1 - 2z,\ \frac{\partial^2 L}{\partial x \partial z} = 1 - 2y,\ \frac{\partial^2 L}{\partial y \partial z} = 1 - 2x$$

si deci

$$\frac{\partial^2 L}{\partial x^2}(1, 1, 1) = \frac{\partial^2 L}{\partial y^2}(1, 1, 1) = \frac{\partial^2 L}{\partial z^2}(1, 1, 1) = 0,$$
$$\frac{\partial^2 L}{\partial x \partial y}(1, 1, 1) = \frac{\partial^2 L}{\partial x \partial z}(1, 1, 1) = \frac{\partial^2 L}{\partial y \partial z}(1, 1, 1) = -1$$

Asadar,

$$d^2L(1, 1, 1) = -2dxdy - 2dxdz - 2dydz$$

Diferentiem legatura $xyz = 1$ si obtinem

$$yzdx + xzdy + xydz = 0$$

In punctul $(1, 1, 1)$ aceasta relatie devine

$$dx + dy + dz = 0$$

Atunci $dz = -dx - dy$; inlocuind in d^2L obtinem forma patratica

$$2dx^2 + 2dxdy + 2dy^2 = 2\left(dx + \frac{dy}{2}\right)^2 + \frac{3}{2}dy^2$$

care este pozitiv definita si prin urmare $(1, 1, 1)$ este punct de minim local cu legatura $xyz = 1$.

Exercitiu. Sa se gaseasca punctele de extrem local ale functiei $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 6 - 4x - 3y$ cu legatura $x^2 + y^2 = 1$.

Sokutie. Fie $g : \mathbb{R}^2 \to \mathbb{R}$ $g(x,y) = x^2 + y^2 - 1$. Evident $\mathbb{R}^2$ este deschisa si functiile f si g sunt de clasa C^2.

$$\operatorname{rang}\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = \operatorname{rang}(2x, 2y) = 1 \quad \text{oricare ar fi } (x,y) \in \mathbb{R}^2 \text{ pentru care } x^2 + y^2 = 1.$$

Consideram functia lui Lagrange

$$L(x,y) = 6 - 4x - 3y + \lambda(x^2 + y^2 - 1)$$

si sistemul

$$\begin{cases} \frac{\partial L}{\partial x}(x,y) = -4 + 2x\lambda = 0 \\ \frac{\partial L}{\partial y}(x,y) = -3 + 2y\lambda = 0 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

Atunci $x = \frac{2}{\lambda}$, $y = \frac{3}{2\lambda}$ si deci $\frac{4}{\lambda^2} + \frac{9}{4\lambda^2} - 1 = 0$, adica $\frac{25}{4\lambda^2} = 1$. Obtinem solutiile $\lambda_1 = -\frac{5}{2}$, $\quad \lambda_2 = \frac{5}{2}$. Daca $\lambda_1 = -\frac{5}{2}$, obtinem $x_1 = -\frac{4}{5}$, $y_1 = -\frac{3}{5}$. Daca $\lambda_2 = \frac{5}{2}$, obtinem $x_2 = \frac{4}{5}$, $y_2 = \frac{3}{5}$.

Pentru $\lambda_1 = -\frac{5}{2}$ avem punctul critic conditionat $(-\frac{4}{5}, -\frac{3}{5})$ si in acest caz

$$L(x,y) = 6 - 4x - 3y - \frac{5}{2}(x^2 + y^2 - 1)$$

$$\frac{\partial^2 L}{\partial x^2}(x,y) = \frac{\partial^2 L}{\partial y^2}(x,y) = -5, \quad \frac{\partial^2 L}{\partial x \partial y}(x,y) = 0$$

Prin urmare $d^2L(-\frac{4}{5}, -\frac{3}{5}) = -5dx^2 - 5dy^2$ si in consecinta $(-\frac{4}{5}, -\frac{3}{5})$ este punct de maxim local al lui f cu legatura $x^2 + y^2 - 1 = 0$.

Pentru $\lambda_2 = \frac{5}{2}$ avem punctul critic conditionat $(\frac{4}{5}, \frac{3}{5})$ si in acest caz

$$L(x,y) = 6 - 4x - 3y + \frac{5}{2}(x^2 + y^2 - 1)$$

$$\frac{\partial^2 L}{\partial x^2}(x,y) = \frac{\partial^2 L}{\partial y^2}(x,y) = 5, \quad \frac{\partial^2 L}{\partial x \partial y}(x,y) = 0$$

Prin urmare $d^2L(\frac{4}{5}, \frac{3}{5}) = 5dx^2 + 5dy^2$ si in consecinta $(\frac{4}{5}, \frac{3}{5})$ este punct de minim local al lui f cu legatura $x^2 + y^2 - 1 = 0$.