TEORIA MĂSURII

SEMINAR 11

Exercițiu lăsat data trecută:

$f: I \times I \longrightarrow \mathbb{R} \qquad I = [0,1]$

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$J_1 = \int_{[0,1]} \left(\int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2} \, d\lambda(y) \right) d\lambda(x)$$

Fix. $x \neq 0$

$$I_x = \int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2} \, d\lambda(y)$$

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Pt. $x \neq 0$

Funcția $y \longrightarrow \frac{x^2-y^2}{(x^2+y^2)^2}$ e cont. $\Rightarrow$

$\Rightarrow \begin{cases} \text{măsurabilă} \\ \text{integrabilă Riemann pe } [0,1] \\ \text{și integralele coincid} \end{cases}$

$$I_x = \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} \, dy$$

$$I_x = \int_0^1 \frac{x^2+y^2}{(x^2+y^2)^2} \, dy - \int_0^1 \frac{2y^2}{(x^2+y^2)^2} \, dy$$

$$= \int_0^1 \frac{1}{x^2+y^2} \, dy + \int_0^1 \left(\frac{1}{x^2+y^2}\right)' \cdot y \, dy$$

$$\overset{\text{prin}}{\underset{\text{părți}}{=}} \int_0^1 \frac{1}{x^2+y^2}\,dy + \left.\frac{y}{x^2+y^2}\right|_0^1 -$$

$$- \int_0^1 \frac{1}{x^2+y^2}\,dy =$$

$$= \frac{1}{x^2+1}$$

Deci $I_x = \dfrac{1}{x^2+1}$, $(\forall)\, x$.

$$J_1 = \int_{[0,1]} \frac{1}{x^2+1}\,dx$$

$=$ Blablabla Riemann

$$= \left.\operatorname{arctg} x\right|_0^1 = \operatorname{arctg} 1 = \frac{\pi}{4}$$

$$J_2 = \int_{[0,1]} \left(\int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2} \, d\lambda(x) \right) d\lambda(y)$$

$$= \int_{[0,1]} \left(\int_{[0,1]} \frac{y^2 - x^2}{(y^2+x^2)^2} \, d\lambda(y) \right) d\lambda(x)$$

$$= -J_1 = -\frac{\pi}{4}$$

De ce nu-s egale?

Este f integrabilă?

Adică: $J = \int_{[0,1]\times[0,1]} |f(x,y)| \, d\lambda(x,y)$

Din T. Tonelli,

$$J = \int_{[0,1]} \left(\int_{[0,1]} \left| \frac{x^2 - y^2}{(x^2+y^2)^2} \right| d\lambda(y) \right) d\lambda(x)$$

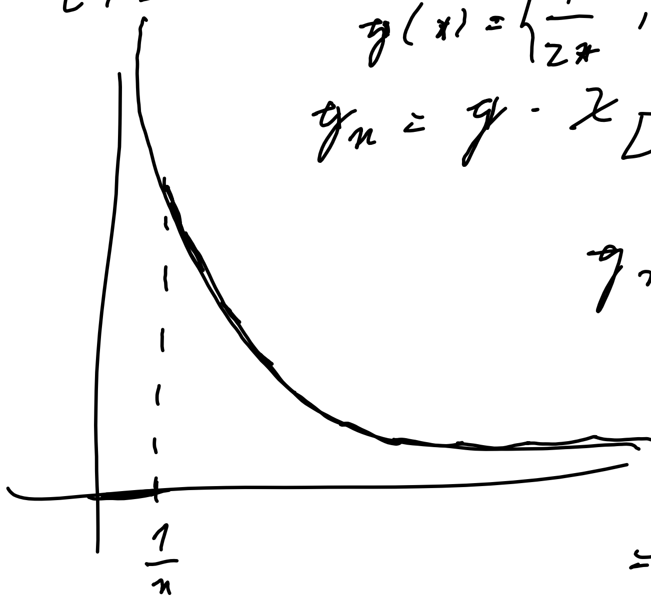
$$\mathcal{J} \geq \int_{[0,1]} \left(\int_{[0,x]} \frac{x^2-y^2}{(x^2+y^2)^2} \, d\lambda(y) \right) d\lambda(x) =$$

$$= \int_{[0,1]} \frac{x}{x^2+x^2} \, d\lambda(x)$$

$$= \int_{[0,1]} \frac{1}{2x} \, d\lambda(x) = \infty$$

$$g(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{2x}, & x \in (0,1] \end{cases}$$

$$g_n = g \cdot \chi_{[\frac{1}{n}, 1]}$$



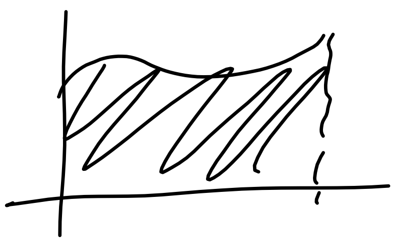
$$g_n \nearrow g$$

$$\int_{[0,1]} g_n = \int_{\frac{1}{n}}^{1} \frac{1}{2x} = \frac{1}{2}\left(\ln 1 - \ln \frac{1}{n}\right)$$

$$= \frac{1}{2} \ln n$$

② $f : X \to [0, \infty)$ integrabilă

$S_f = \{(x, y) \in X \times [0, \infty) \mid y \le f(x)\}$



Atunci S_f măsurabilă în $X \times [0, \infty)$

şi $\mu \otimes \lambda(S_f) = \int_X f \, d\mu$

25 min

③ $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$

$(\mathbb{N}, |\cdot|)$

$$f(n, m) = \begin{cases} 1, & n = m \\ -1, & n = m+1 \end{cases}$$

$$\int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(n, m) \, d|m| \right) d|n| = ?$$

$$\int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(n, m) \, d|n| \right) d|m| = ?$$

De ce nu-s egale

15 min

(4) Exemple concrete de int. curbilinii

25 min

(5) Calcul $\int_{-\infty}^{\infty} e^{-x^2}\, dx$

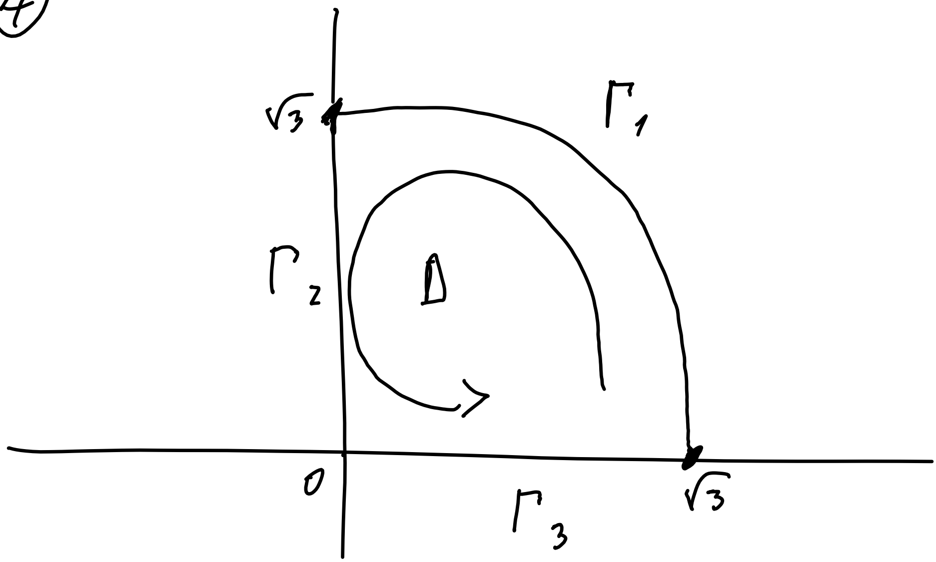
15 min

(6) Intuiții pt. integrale pe curbe și suprafețe

Intuiție demonstrație Green

40 min

4



$$D = \{(x, y) \mid x, y \geq 0, \ x^2 + y^2 \leq 3\}$$

$$\int_{\partial D} (x^2 y + x^2) \, ds$$

Parametrizăm ∂D:

$\Gamma_1 = \operatorname{Im}(\gamma_1)$

$\gamma_1 : [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}^2$

$\gamma_1(t) = (\sqrt{3} \cos t, \sqrt{3} \sin t)$

$\Gamma_2 = \mathrm{Im}(\gamma_2)$

$\gamma_2 : [0, \sqrt{3}] \longrightarrow \mathbb{R}^2$

$\gamma_2(t) = (0, \sqrt{3} - t)$

$\Gamma_3 = \mathrm{Im}(\gamma_3)$

$\gamma_3 : [0, \sqrt{3}] \longrightarrow \mathbb{R}^2$

$\gamma_3(t) = (t, 0)$

$\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$f : \partial D \longrightarrow \mathbb{R}$

$f(x, y) = x^2 y + x^2$

$$\int_{\partial D} f \, ds = \int_{\Gamma_1} f \, ds + \int_{\Gamma_2} f \, ds + \int_{\Gamma_3} f \, ds$$

$$\int_{\Gamma_1} f\, ds = \int_0^{\frac{\pi}{2}} \left[\left(\gamma_1^1(t)\right)^2 \cdot \gamma_1^2(t) + \left(\gamma_1^2(t)\right)^2 \right] \cdot$$

$$\cdot \sqrt{\left((\gamma_1^2)'(t)\right)^2 + \left((\gamma_2^2)'(t)\right)^2}\, dt =$$

$$= \int_0^{\frac{\pi}{2}} \left(3\sqrt{3}\cos^2 t \sin t + 3\cos^2 t\right) \cdot$$

$$\cdot \sqrt{3(-\sin t)^2 + 3(\cos t)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \left(3\sqrt{3}\cos^2 t \sin t + 3\cos^2 t\right)\sqrt{3}\, dt$$

$$\gamma_1(t) = \left(\gamma_1^1(t), \gamma_1^2(t)\right)$$

$$\int_{\Gamma_2} f\,ds = \int_0^{\sqrt{3}} \Big[\big(\gamma_2^1(t)\big)^2 \cdot \gamma_2^2(t) +$$

$$+ \big(\gamma_2^1(t)\big)^2\Big] \cdot$$

$$\cdot \sqrt{\big((\gamma_2^1)'(t)\big)^2 + \big((\gamma_2^2)'(t)\big)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \big(0 \cdot (\sqrt{3} - t) + 0\big) \cdot$$

$$\cdot \sqrt{0 + (-1)^2}\, dt$$

$$= 0$$

$$\int_{\Gamma_3} f\,ds = \int_0^{\sqrt{3}} \big(t^2 \cdot 0 + t^2\big) \cdot \sqrt{1^2 + 0^2}\, dt$$

$$= \int_0^{\sqrt{3}} t^2\, dt$$

∂D orientat în sens trigonometric

și

$$\omega = x^2 y\, dx + x^2 dy$$

$$\int_{\partial D} \omega = \int_{\Gamma_1} \omega + \int_{\Gamma_2} \omega + \int_{\Gamma_3} \omega :$$

$\alpha = \gamma_1$ $\qquad \alpha(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t)$

$\alpha^1(t) = \sqrt{3}\cos t$

$$\int_{\Gamma_1} \omega = \int_0^{\frac{\pi}{2}} (\alpha^1(t))^2 \cdot \alpha^2(t) \cdot (\alpha^1)'(t)\, dt$$

$$+ \int_0^{\frac{\pi}{2}} (\alpha^1(t))^2 \cdot (\alpha^2)'(t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} (\sqrt{3}\cos t)^2 \cdot \sqrt{3}\sin t \cdot \sqrt{3}\sin t\, dt$$

$$+ \int_0^{\pi/2} (\sqrt{3}\cos t)^2 \cdot (-\sqrt{3}\cos t)\, dt$$

$$\int_{\Gamma_2} \omega =$$

$$= \int_0^{\sqrt{3}} (x(t))^2 \cdot y(t) \cdot x'(t)\, dt +$$

$$+ \int_0^{\sqrt{3}} (x(t))^2 \, y'(t)\, dt$$

$$= \int_0^{\sqrt{3}} 0^2 \cdot (\sqrt{3} - t) \cdot 0 \, dt +$$

$$+ \int_0^{\sqrt{3}} 0 \cdot (\sqrt{3} - t)' \, dt = 0$$

$$\int_{\Gamma_3} \omega = \int_0^{\sqrt{3}} t^2 \cdot 0 \cdot 1 \, dt + \int_0^{\sqrt{3}} t^2 \cdot 0 \, dt = 0$$

Green:

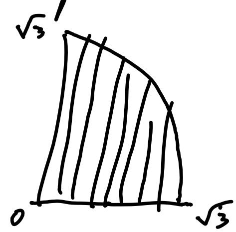
$$\int_{\partial D} \overbrace{P(x,y)\,dx + Q(x,y)\,dy}^{\omega} =$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\omega = x^2 y\,dx + x^2\,dy$$

$$P(x,y) = x^2 y \;;\; Q(x,y) = x^2$$

$$\int_{\partial D} \omega = \iint_D (2x - x^2)\,dx\,dy$$



$$= \int_0^{\sqrt{3}} \left(\int_0^{\sqrt{3-x^2}} (2x - x^2)\,dy \right) dx =$$

$$= \int_0^{\sqrt{3}} (2x - x^2) \cdot \sqrt{3 - x^2} \, dx$$

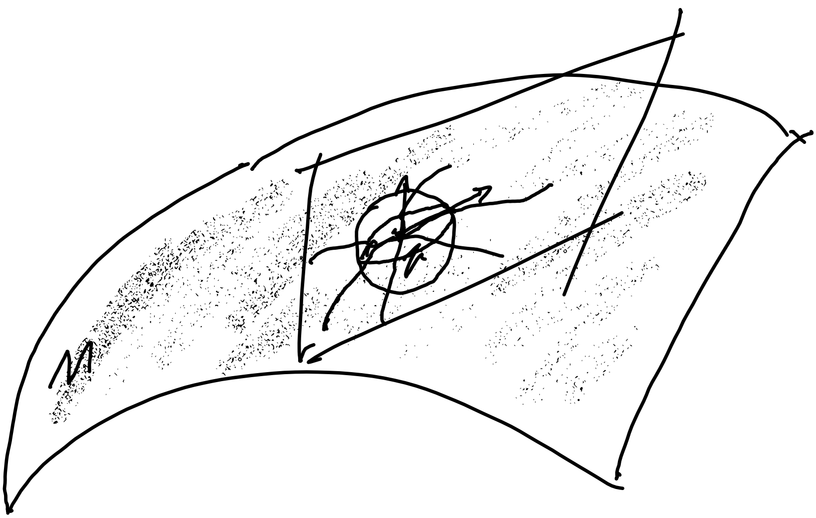
$$\iint_D (2x - x^2) \, dx \, dy =$$

$$= \int_0^{\sqrt{3}} \int_0^{\frac{\pi}{2}} \left(2r\cos\theta - (r\cos\theta)^2\right) \cdot r \, d\theta \, dr$$

$$= \int_0^{\sqrt{3}} 2r^2 dr \cdot \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta \; -$$

$$- \int_0^{\sqrt{3}} r^3 dr \cdot \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

⑥ Intuiții:



Fie M o submulțime în $\mathbb{R}^n$ cu proprietatea:

$(\forall)\, p \in M \quad (\exists)\, r > 0$

$(\exists)\, F : B_r(p) \longrightarrow \mathbb{R} \quad C^\infty$

$\nabla F(y) \neq 0 \,,\ (\forall)\, y \in B_r(p)$

$B_r(p) \cap M = F^{-1}(0)$

Fie $\gamma : (-\varepsilon, \varepsilon) \to M$ curbă

a.î. $\gamma(0) = p$

Mă uit la $\gamma'(0)$

$$T_p M = \{ \gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to M \text{ curbă } C^\infty,\ \gamma(0) = p \}$$

Proprietate:

$(\forall)\ v \in T_p(M)$,

$$\langle \nabla F(p), v \rangle = 0$$

Dem: Fie $\gamma : (-\varepsilon, \varepsilon) \to M$

$\gamma(0) = p$

$v = \gamma'(0)$

$F(\gamma(t)) = 0,\ (\forall)\ t \Rightarrow (F(\gamma(t)))'(0) = 0$

Dar $\frac{d}{dt} F(\gamma(t))\Big|_{t=0} =$

$$= \sum_{i=1}^{n} \frac{\partial F}{\partial x^i}(\gamma(0)) \cdot \frac{d}{dt}\gamma^i(0)$$

$$= \langle \nabla F(p), \gamma'(0) \rangle$$

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots \frac{\partial F}{\partial x_n} \right)$$

$$\gamma'(0) = \left(\frac{d}{dt}\gamma^1(0), \dots, \frac{d}{dt}\gamma^n(0) \right)$$

Deci $\langle \nabla F(p), v \rangle = 0$

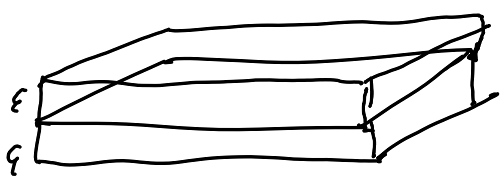
$\nabla F(p)$ normal la T_pM

Prin urmare,

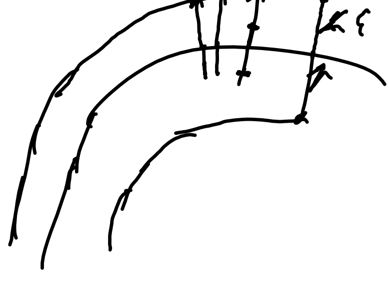
$$n(p) = \pm \frac{\nabla F(p)}{|\nabla F(p)|}$$ normala unitară (exterioară)

Fie $\Gamma \subseteq M$

$$\Gamma(\varepsilon) = \{ x \in \mathbb{R}^n \mid (\exists)\, p \in M \text{ a.î. } (x-p) \perp T_pM,\ |x-p| < \varepsilon \}$$

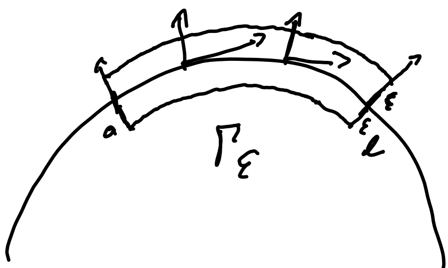


$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \cdot V_\varepsilon = A_{\square}$$



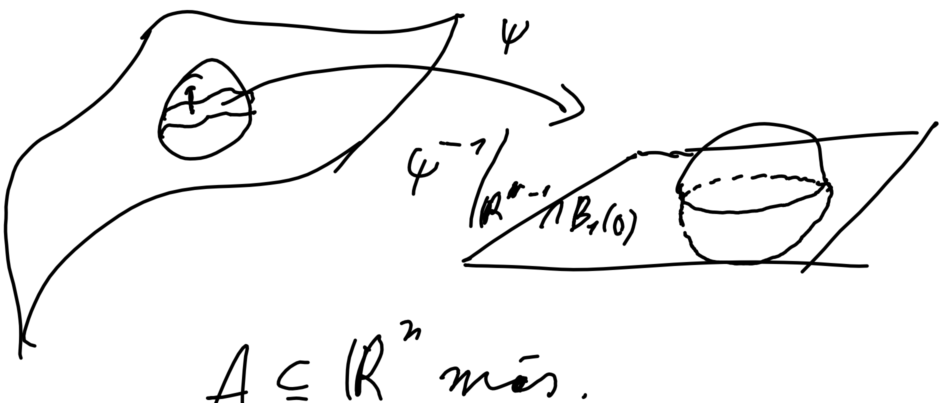
$$\lambda_n(\Gamma) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \lambda_{\mathbb{R}^n}(\Gamma(\varepsilon))$$

Ex: pe curbă:



$x \in B_r(y)$

$x = y(x) + \ell(x) \cdot n(x)$



$A \subseteq \mathbb{R}^n$ măs.

$T : \mathbb{R}^n \to \mathbb{R}^n$ liniară

$\lambda(T(A)) = |\det T| \cdot \lambda(A)$