Examen Olaru

Sunday, 19 June, 2022 4:51 PM

① $X \sim \begin{cases} f(x) = ax+b, & x \in [0,3] \\ 0, & \text{rest} \end{cases}$

$\mathbb{P}(X \le 2) = 0{,}64$, $E[X] = ?$

$$\int_0^2 f(x)dx = a\frac{x^2}{2} + bx \Big|_0^2 = 2a + 2b = 0{,}64$$

$$a + b = 0{,}32$$

$$\int_0^3 f(x)dx = a\frac{x^2}{2} + bx\Big|_0^3 = a\cdot\frac{9}{2} + 3b = 1$$

$$9a + 6b = 2$$
$$6a + 6b = 1{,}92$$
$$3a = 0{,}08$$
$$9a = 0{,}24$$

$$6b = 1{,}76$$
$$3b = 0{,}88$$
$$9b = 2{,}64$$

$$\int_0^3 ax^2 + bx\,dx = a\frac{x^3}{3} + b\frac{x^2}{2}\Big|_0^3$$

$$= 9a + \frac{9}{2}b = 0{,}24$$

$$0{,}24 + 1{,}32 = 1{,}56$$

② $Var(X) = 0{,}61$

$Var(Y) = 2{,}5$

$Cov(X,Y) = -0{,}37$

$Var(X+Y) = ?$

Rez

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$
$$= 0{,}61 + 2{,}5 - 0{,}74$$
$$= 3{,}11 - 0{,}74 = 2{,}37$$

③ $P(t > 1{,}95) = 0{,}393366176513 0499$

α

$E[t] = ?$

Rez $X \sim Exp(\lambda)$ $\lambda = ?$ $E[X] = \frac{1}{\lambda}$

$$\mathbb{P}(X > k) = e^{-\lambda k}$$

$$\alpha = e^{-\lambda\cdot 1{,}95} \qquad -\lambda\cdot 1{,}95 = \ln\alpha$$

$$\lambda = \frac{\ln\alpha}{-1{,}95} = -\frac{1{,}95}{\ln\alpha} \approx 2{,}09$$

④ $X_1 \dots X_n$ iid $\sim \begin{pmatrix} -4 & 2 & 8 \\ 0{,}22 & 0{,}35 & 0{,}43 \end{pmatrix}$ $\lim\limits_{n\to\infty} \frac{X_1 + X_2 + \dots + X_n}{n}$

$$-0{,}88 + 0{,}7 + 3{,}44 = 4{,}14 - 0{,}88 = 3{,}26 \checkmark$$

⑤ $X \sim N(9, 7)$

$(-2X + 4) \sim$

$E[X] = 9$

$Var(X) = 7$

$E[-2X+4] = -14$

$$Var(-2X+4) = Var(-2X) \cancel{+4}$$
$$= 4\cdot Var(X) \cancel{+4}$$
$$= \cancel{32}\ 28$$

$N(-14, 28)$

⑥ $X \sim Unif(0,1)$

$Z := -\ln(X)/8$

i) $Z \sim Exp(8)$

ii) Propuneți o metodă de simulare a unei v.a. Exp(8) și scrieți pseudocod

$Exp(8) = 8\cdot e^{-8x}$

iii) $X_1 \dots X_n \dots$ sunt iid $\sim Unif(0,1) \to n\cdot\min(X_1 \dots X_n)$ converge

$Exp(8) = 8 \cdot e^{-8x}$

$x \in [0,1]$

$Z \cdot [0,1] \to [0, +\infty)$

iii) $X_1 ... X_n ...$ sunt iid $\sim Unif(0,1) \to n \cdot \min(X_1 .. X_n)$ converge în distribuție la $Exp(1)$

iv) Propuneți altă metodă de simulare, alta decât la (ii)

1) $\mathbb{P}(Z > k) = \mathbb{P}\left(-\frac{\ln(x)}{8} > k\right)$, $\mathbb{P}(-\ln(x) > 8k)$, $\mathbb{P}(x < e^{-8k}) = \int_0^{e^{-8k}} dx = e^{-8k} = Exp(8)$

$k \in [0, +\infty)$

ii) Simulăm uniform $x \sim Unif(0,1)$, apoi $k = -\ln x / 8$ are distribuție $Exp(8)$ (i)

$x \leftarrow random(0,1)$

$z \leftarrow -\ln(x) / 8$

iii) $X_n \sim Unif(0,1)$; $\min(X_1 ... X_n) = Y$ $\quad n \cdot Y \sim Exp(1) \Rightarrow \mathbb{P}(Y > k) = e^{-k}$

$\mathbb{P}\left(Y < \frac{k}{n}\right) = 1 - e^{-\frac{k}{n}}$

$\mathbb{P}\left(X_i > \frac{k}{n}\right) = ?$

$\int_{\frac{k}{n}}^{1} dx = 1 - \frac{k}{n} = \frac{n-k}{n}$

$\mathbb{P}\left(\min(X_1 .. X_n) > \frac{k}{n}\right) = \left(1 - \frac{k}{n}\right)^n \xrightarrow{n \to \infty} \lim_{n \to \infty} \left(\left(1 - \frac{k}{n}\right)^{-\frac{n}{k}}\right)^{(-k)} \to e^{-k}$

$\left(1 - \frac{k}{n}\right)^{-\frac{n}{k}} \to e$

$\mathbb{P}\left(Y > \frac{k}{n}\right) = e^{-k}$

iv) Generăm un număr mare (n) de $x \sim Unif(0,1)$ și alegem min. $\to \min \cdot n \sim Exp(1)$

$\to 8 \cdot (\min \cdot n)^8 \sim Exp(8)$

$\mathbb{P}(nY > k) = e^{-k}$ ✓

$\Rightarrow nY \sim Exp(1)$

7) max 760kg

$E[colet] = 8kg$

$D(colet) = 5kg$

$P(117 \text{ colete}) \leq 760$

$Var(colet) = 25$

$N(8,5) = \frac{1}{\sqrt{2\pi 25}} \cdot e^{\frac{(x-8)^2}{225}} \to$ greutatea unui colet.

G. greutatea a 117 colete. $P(N_1 + N_2 + ... + N_{117}) \leq 760$

Fie $\tilde{S}_n = \frac{\sqrt{n}}{\sigma}(S_n - n \cdot m)$ $\quad$ T.L.C

$\mathbb{P}(\tilde{S}_n \geq t) \xrightarrow[n \to \infty]{} \int_t^{+\infty} p(x) dx$, $p \sim N(0,1)$

$\mathbb{P}(\tilde{S}_{117} \leq 760) =$

8) $D = B(0,4)$, $R \sim Unif(0,4)$

$\theta \sim Unif(0, 2\pi)$

$(X, Y) = (R \cos(\theta), R \sin \theta)$

$X = R \cos \theta$

$Y = R \sin \theta$

$\cos \theta, \sin \theta \sim Unif(-1, 1)$

$\mathbb{P}(X,Y) = (0,0) = \mathbb{P}(R = 0, \theta \in [0, 2\pi])$ $\neq$

$\mathbb{P}(X,Y) = (0,4) = \mathbb{P}(R = 4, \theta = \frac{\pi}{2})$

$\Theta \sim Unif(0, 2\pi)$

$(X,Y) = (R\cos(\Theta), R\sin\Theta)$

$(x,y) \sim Unif(D)$?

$P(x,y) = (0,0) = P(R = \dots)$

$P(x,y) = (0,4) = P(R=4, \Theta = \frac{\pi}{2})$

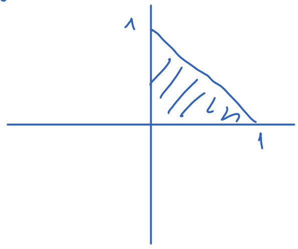
$\Rightarrow (x,y) \nsim Unif(D)$

(cluster în (0,0))

⑨ $f(x,y) = \begin{cases} cxy^2, & 0 \le x,y \text{ şi } x+y \le 1 \\ 0, & \text{altfel} \end{cases}$

(i) $E[x], E[Y], Var(x), Var(Y) = ?$

(ii) $Cov(x,y) = ?$ x, y independente?



$$\int_0^1\int_0^{1-y} cxy^2\,dx\,dy = 1$$

$$= \int_0^1 \frac{cx^2y^2}{2}\Big|_0^{1-y} dy = \int_0^1 \frac{c(1-y)^2y^2}{2} dy$$

$$= \frac{c}{2}\int_0^1 (1-2y+y^2)y^2\,dy = \frac{c}{2}\int_0^1 y^2 - 2y^3 + y^4\,dy$$

$$= \frac{c}{2}\left(\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right)\Big|_0^1$$

$$= \frac{c}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{c}{2}\left(\frac{1}{30}\right)$$

$$= \frac{c}{60} = 1 \Rightarrow c = 60$$

$f(x,y) = \begin{cases} 60x\cdot y^2, & x+y \le 1 \\ 0, & 0 \end{cases}$

$$P(x) = 60\int_0^{1-x} x\cdot y^2\,dy = 60\cdot x\frac{y^3}{3}\Big|_0^{1-x}$$

$$= 60\,\frac{x(1-3x+3x^2-x^3)}{3} = 20(x - 3x^2 + 3x^3 - x^4)$$

$$E[x] = 20\int_0^1 x^2 - 3x^3 + 3x^4 - x^5\,dx = 20\left(\frac{x^3}{3} - \frac{3}{4}x^4 + \frac{3}{5}x^5 - \frac{x^6}{6}\right)\Big|_0^1 = 20\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right)$$

$$= 20\left(\frac{20 - 45 + 36 - 10}{60}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

$$P(Y) = 60\int_0^{1-y} x\cdot y^2\,dx = \frac{60}{2}x^2\cdot y^2\Big|_0^{1-y} = 30(1-2y+y^2)y^2 = 30(y^2 - 2y^3 + y^4)$$

$$E[y] = 30\int_0^1 y^3 - 2y^4 + y^5\,dy = 30\left(\frac{y^4}{4} - \frac{2}{5}\cdot y^5 + \frac{y^6}{6}\right)\Big|_0^1 = 30\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = \frac{30}{60}(15 - 24 + 10)$$

$$= \frac{1}{2}$$

$$E[x^2] = 20\int$$