Factorizarea QR: metoda Gram-Schmidt modificată

Determinați factorizarea QR a matricii

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

folosind metoda Gram-Schmidt modificată.

$$A := [\underline{a}_1 \;\; A_2] = [\underline{q}_1 \;\; Q_2] \begin{bmatrix} r_{11} & \underline{R}_{12} \\ \underline{0} & R_{22} \end{bmatrix} =: QR$$

$$\underbrace{\left[\begin{array}{c|cc} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{array}\right]}_{=: [\underline{a}_1 \;\; A_2]} = [\underline{q}_1 r_{11} \;\; \underline{q}_1 \underline{R}_{12} + Q_2 R_{22}]$$

$$\Rightarrow \begin{cases} \underline{a}_1 = \underline{q}_1 r_{11} & (1) \\ A_2 = \underline{q}_1 \underline{R}_{12} + Q_2 R_{22} & (2) \end{cases}$$

(1) $a_1 = q_1 r_{11} \Rightarrow \begin{cases} r_{11} = \|a_1\| \\ q_1 = \dfrac{1}{r_{11}} a_1 \end{cases}$

$$\|a_1\| = \sqrt{1+1+1+1} = 2 \Rightarrow \boxed{r_{11} = 2}$$

$$q_1 = \frac{1}{2} a_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

(2) $A_2 = q_1 \underline{R}_{12} + Q_2 R_{22}$

$$q_1^T A_2 = q_1^T \left(q_1 \underline{R}_{12} + Q_2 R_{22} \right) = \underline{R}_{12} \Rightarrow$$

$$\underline{R}_{12} = q_1^T A_2 =$$

$$= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 5 & 15 \end{bmatrix} \Rightarrow$$

$$\boxed{\underline{R}_{12} := \begin{bmatrix} r_{12} & r_{13} \end{bmatrix} = \begin{bmatrix} 5 & 15 \end{bmatrix}}$$

$$\underline{q}_2 R_{22} = A_2 - \underline{q}_1 R_{12}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} - \begin{bmatrix} 5/2 & 15/2 \\ 5/2 & 15/2 \\ 5/2 & 15/2 \\ 5/2 & 15/2 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & -13/2 \\ -1/2 & -7/2 \\ 1/2 & 3/2 \\ 3/2 & 17/2 \end{bmatrix} =: \begin{bmatrix} \underline{a}_2 & \underline{a}_3 \end{bmatrix} = \begin{bmatrix} \underline{q}_2 & \underline{q}_3 \end{bmatrix} \begin{bmatrix} r_{22} & r_{23} \\ 0 & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} \underline{a}_2 & \underline{a}_3 \end{bmatrix} = \begin{bmatrix} \underline{q}_2 r_{22} & \underline{q}_2 r_{23} + \underline{q}_3 r_{33} \end{bmatrix} \Rightarrow$$

$$\begin{cases} \underline{a}_2 = \underline{q}_2 r_{22} & (3) \\ \underline{a}_3 = \underline{q}_2 r_{23} + \underline{q}_3 r_{33} & (4) \end{cases}$$

(3) $\underline{a}_2 = \underline{q}_2 r_{22} \Rightarrow \begin{cases} r_{22} = \|\underline{a}_2\| \\ \underline{q}_2 = \frac{1}{r_{22}} \underline{a}_2 \end{cases}$

$$r_{22} = \|a_2\| = \sqrt{\frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4}} = \sqrt{5} \Rightarrow$$

$$\boxed{r_{22} = \sqrt{5}}$$

$$q_2 = \begin{bmatrix} -3\sqrt{5}/10 \\ -\sqrt{5}/10 \\ \sqrt{5}/10 \\ 3\sqrt{5}/10 \end{bmatrix}$$

(4) $$a_3 = q_2 r_{23} + q_3 r_{33}$$

$$q_2^T a_3 = r_{23} \Rightarrow$$

$$r_{23} = -\frac{3\sqrt{5}}{10}\left(-\frac{13}{2}\right) - \frac{\sqrt{5}}{10}\left(-\frac{7}{2}\right) + \frac{\sqrt{5}}{10}\frac{3}{2} + \frac{3\sqrt{5}}{10}\frac{17}{2}$$

$$= \frac{\sqrt{5}}{20}(39 + 7 + 3 + 51) = 5\sqrt{5} \Rightarrow$$

$$\boxed{r_{23} = 5\sqrt{5}}$$

$$q_3 r_{33} = a_3 - q_2 r_{23} \Rightarrow$$

$$\tilde{q}_3 r_{33} = \begin{bmatrix} -13/2 \\ -7/2 \\ 3/2 \\ 17/2 \end{bmatrix} - \begin{bmatrix} -3\sqrt{5}/10 \\ -\sqrt{5}/10 \\ \sqrt{5}/10 \\ 3\sqrt{5}/10 \end{bmatrix} 5\sqrt{5} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$r_{33} = \sqrt{1+1+1+1} = 2 \Rightarrow \boxed{r_{33} = 2}$$

$$\tilde{q}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Am obținut:

$$Q = \begin{bmatrix} 1/2 & -3\sqrt{5}/10 & 1/2 \\ 1/2 & -\sqrt{5}/10 & -1/2 \\ 1/2 & \sqrt{5}/10 & -1/2 \\ 1/2 & 3\sqrt{5}/10 & 1/2 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}$$