TEORIA MĂSURII

SEMINAR 5

Propoziție: $f : (X, \mathcal{A}) \longrightarrow \mathbb{R}$

Fie $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$

$\alpha_i = \alpha_j \Rightarrow$
$\Rightarrow i = j$
$A_i \cap A_j \neq \emptyset \Rightarrow$
$\Rightarrow i = j$

f este măsurabilă $\Longleftrightarrow A_i \in \mathcal{A}$
$(\forall)\, i = \overline{1, n}$

Dem: „$\Rightarrow$"

$$A_i = f^{-1}(\underbrace{\{\alpha_i\}}_{\in \mathcal{B}(\mathbb{R})}) \in \mathcal{A}$$

„$\Leftarrow$"

Fără a restrânge gen: $\alpha_1 < \alpha_2 < \ldots < \alpha_n$

Fie $t \in \mathbb{R}$

$(\exists)\, k = \overline{0, n}$ a.î. $\alpha_1, \alpha_2, \ldots, \alpha_k < t$

$$f^{-1}((-\infty, t)) = f^{-1}(\{\alpha_1, \alpha_2, \ldots \alpha_k\}) =$$

$$= \bigcup_{i=1}^{k} A_i \in \mathcal{A}$$

Fie $A \subseteq X$

$$\chi_A : X \longrightarrow \{0, 1\}$$

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Exemple:

Fie $\mathbb{N}$ cu măsura de numărare

i.e. $(\mathbb{N}, \mathcal{P}(\mathbb{N}), |\cdot|)$

$$f = \sum_{i=1}^{n} i \cdot \chi_{\{i\}}$$

adică

$$f(m) = \sum_{i=1}^{n} i \cdot \chi_{\{i\}}(m) = \begin{cases} m, & m \in \{1, 2, \ldots n\} \\ 0, & \text{în rest} \end{cases}$$

$n = 4$

$$f(m) = 1 \cdot \chi_{\{1\}}(m) + 2 \cdot \chi_{\{2\}}(m) + 3 \cdot \chi_{\{3\}}(m) + 4 \cdot \chi_{\{4\}}(m)$$

$g : \mathbb{N} \to \mathbb{N}$

$$g = \sum_{i=1}^{n} i \cdot \chi_{\{2i, 2i+1\}}$$

$$g(m) = \begin{cases} \frac{m}{2}, & \text{dacă } m \text{ par în } \{2, \dots, 2n\} \\ \frac{m-1}{2}, & \text{dacă } m \text{ impar în } \{3, 5, \dots, 2n+1\} \\ 0, & \text{în rest} \end{cases}$$

$g(m) = i$, dacă $m \in \{2i, 2i+1\}$ cu $i = \overline{1, n}$

$$\mu(A) = |A|, \quad (\forall) A \in \mathcal{P}(\mathbb{N})$$

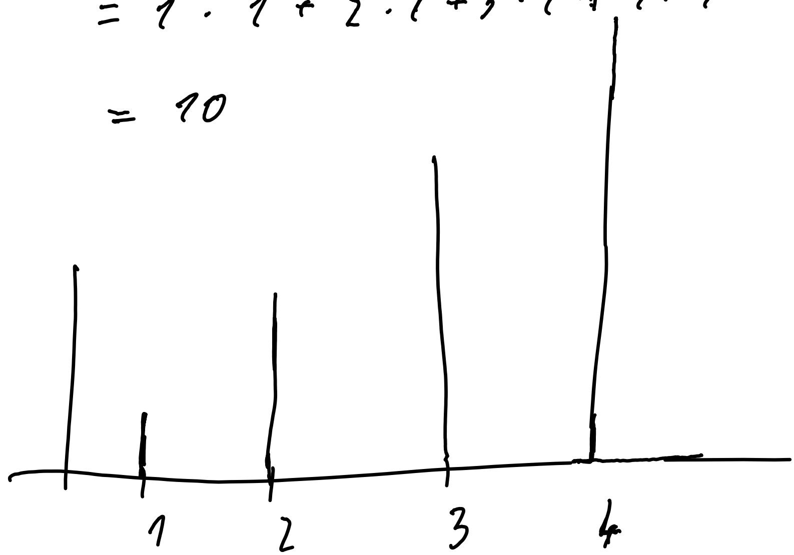
$$\int_{\mathbb{N}} f(x)\, d\mu(x)$$

$$f(x) = 1\cdot \chi_{\{1\}} + 2\cdot \chi_{\{2\}} + 3\cdot \chi_{\{3\}} + 4\cdot \chi_{\{4\}}$$

$$\int_{\mathbb{N}} f(x) = 1\cdot \mu(\{1\}) + 2\cdot \mu(\{2\}) + 3\cdot \mu(\{3\}) + 4\cdot \mu(\{4\})$$

$$= 1\cdot 1 + 2\cdot 1 + 3\cdot 1 + 4\cdot 1$$

$$= 10$$



$$\int_{\mathbb{N}} g(x)\, d\mu(x) = ?$$

$$g = \sum_{i=1}^{n} i \cdot \chi_{\{2i, 2i+1\}}$$

$$\int_{\mathbb{N}} g(x)\, d\mu(x) = \sum_{i=1}^{n} i \cdot \underbrace{\mu(\{2i, 2i+1\})}_{=2}$$

$$= 2 \sum_{i=1}^{n} i = n(n+1)$$

Sei $(X, \mathcal{A}, \mu)$

$A \in \mathcal{A}$

$$\int_A f\, d\mu = \int_X f \cdot \chi_A\, d\mu$$

Pentru: $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$

Cine e $f \cdot \chi_A$

$$(f \cdot \chi_A)(x) = f(x) \cdot \chi_A(x) =$$

$$= \left(\sum_{i=1}^{n} \alpha_i \chi_{A_i}(x) \right) \cdot \chi_A(x)$$

$$= \sum_{i=1}^{n} \alpha_i \underbrace{\chi_{A_i}(x) \cdot \chi_A(x)}_{\chi_{A_i \cap A}}$$

$$= \sum_{i=1}^{n} \alpha_i \chi_{A_i \cap A}(x)$$

$$f \cdot \chi_A = \sum_{i=1}^{n} \alpha_i \cdot \chi_{A_i \cap A}$$

Aplicație:

$f : X \to \mathbb{R}$ măsurabilă

Dem. că $|f|$ e măsurabilă.

Soluție:

$$\{|f| < r\} = \{f > -r\} \cap \{f < r\}$$

$\square$

$$\frac{(f+g)^2 - f^2 - g^2}{2} = f \cdot g \Rightarrow$$

$\Rightarrow f \cdot g$ măsurabilă

Aplicație:

$f: \mathbb{R} \to \mathbb{R}$ derivabilă

Folosind eventual $f_n = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$

arătați că f' e măsurabilă

Soluție:

Fie $x \in \mathbb{R}$

$$f_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} \xrightarrow{n \to \infty} f'(x)$$

Deci $f' = \lim\limits_{n \to \infty} f_n$ (punctual)

Dacă $(f_n)_n$ măsurabilă, $(\forall)\, n$,

atunci f' măsurabilă.

f continuă $\Rightarrow$ f_n e continuă $(\forall)\, n$ $\Rightarrow$

$\Rightarrow$ f_n măsurabilă $(\forall)\, n$ ✓

$$\{f^2 < t\} = \begin{cases} \{f \geq -\sqrt{t}\} \cap \{f < \sqrt{t}\}, & t \geq 0 \\ \emptyset, & t < 0 \end{cases}$$

Deci f^2 măsurabilă

Lema Borel - Cantelli

Fie $(X, \mathcal{A}, \mu)$ spațiu cu măsură

și $(A_n)_n \subseteq \mathcal{A}$ a.î.

$$\sum_{n \geq 1} \mu(A_n) < \infty$$

Atunci $\mu(\limsup_{n \to \infty} A_n) = 0$

$\limsup_{n \to \infty} A_n$

Remember:

$$\limsup_{n \to \infty} x_n = \inf_{n \geq 1} \sup_{m \geq n} x_m$$

$$\limsup_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{m \geq n} A_m$$

$$\mu\left(\bigcap_{n \geq 1} \bigcup_{m \geq n} A_m\right) \leq$$

$$\leq \mu\left(\bigcup_{m \geq n} A_m\right) \leq \sum_{m \geq n} \mu(A_m) \xrightarrow{n \to \infty} 0$$

De ce?

Fie $S = \sum_{n=1}^{\infty} \mu(A_n) < \infty$

$$\sum_{m \geq n} \mu(A_m) = \lim_{m \to \infty} \sum_{i=n}^{m} \mu(A_i) =$$

$$= \lim_{m \to \infty} \sum_{i=1}^{m} \mu(A_i) - \sum_{i=1}^{n-1} \mu(A_i) =$$

$$= S - \sum_{i=1}^{n-1} \mu(A_i) \xrightarrow{n\to\infty} S - S = 0$$

$$\mu\Big(\bigcap_{n\geq 1} \bigcup_{m\geq n} \mu(A_m)\Big) \leq \sum_{n\geq m} \mu(A_m)$$

$$\text{și } \sum_{n\geq m} \mu(A_m) \longrightarrow 0$$

$$\text{deci } \mu\Big(\bigcap_{n\geq 1} \bigcup_{m\geq n} \mu(A_m)\Big) = 0$$

□

Fie $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1, & x \in [0, 2] \\ 3, & x \in [4, 5] \\ 2, & x \in [7, 9) \\ 0, & \text{în rest} \end{cases}$$

$$\int_{\mathbb{R}} f(x)\,dx = ?$$

$$f = 1 \cdot \chi_{[0,2]} + 3 \cdot \chi_{[4,5]} +$$

$$+ 2 \cdot \chi_{[7,9)}$$

$$\int_{\mathbb{R}} f(x)\,dx = 1 \cdot \lambda([0,2]) + 3 \cdot \lambda([4,5]) +$$

$$+ 2 \cdot \lambda([7,9))$$

$$= 2 + 3 \cdot 1 + 2 \cdot 2$$

$$= 7$$

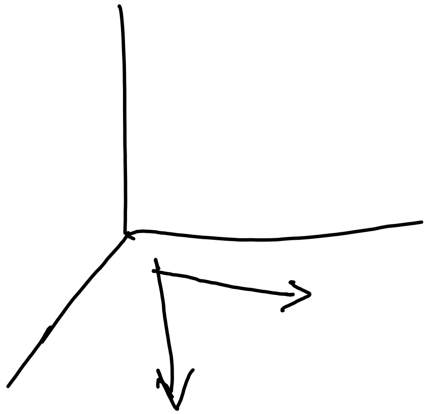
$$g(x) = \begin{cases} 1, & x \in [0,2] \\ 3, & x \in [4,8] \\ 2, & x \in [7,9] \end{cases}$$

$$g = 1 \cdot \chi_{[0,2]} + 3 \cdot \chi_{[4,8]} + 2 \cdot \chi_{[7,9]}$$

$$= 1 \cdot \chi_{[0,2]} + 3 \cdot \chi_{[4,7)} + 5 \cdot \chi_{[7,8]} + 2 \cdot \chi_{(8,9]}$$

$$g(7) = 1 \cdot \chi_{[0,2]}(7) + 3 \cdot \chi_{[4,8]}(7) + 2 \cdot \chi_{[7,9]}(7)$$

$$= 1 \cdot 0 + 3 \cdot 1 + 2 = 5$$



$\mathbb{R}^2 \subset \mathbb{R}^3$

$(x_1, x_2, x_3) \in \mathbb{R}^3$ este în $\mathbb{R}^2$ dacă

$(\exists)\ \alpha, \beta$ a.î.

$(x_1, x_2, x_3) = \alpha \cdot (1,0,0) + \beta \cdot (0,1,0)$

$(7, 8, 0) = 7 \cdot (1,1,0) + 1 \cdot (0,1,0)$

$= 7 \cdot (1,1,0) + 8 \cdot (0,1,0)$