SEMINAR 1 9.10.2020

1+2+3

Algebră

$\boxed{\begin{array}{l} Y \subseteq X \\ \overline{Y} = X \setminus Y \end{array}}$ → complem.

(dublă incluziune) ($\subseteq$)

2) (de Morgan) $\boxed{\text{b) - temă}}$

complementara ($X-Y$)

$A_i \subseteq X, (\forall) i \in I$

$\overline{\bigcup_{i \in I} A_i} \overset{?}{=} \bigcap_{i \in I} \overline{A_i}$

(directă) „$\subseteq$" Fie $x \in \overline{\bigcup_{i \in I} A_i} \Rightarrow x \in X$ și $\underline{x \notin \bigcup_{i \in I} A_i} \Rightarrow$

$\left[\bigcup_{i \in I} A_i := \{ y \mid (\exists) i \in I \text{ a.î. } y \in A_i \}.\right]$

$\Rightarrow (\forall) i \in I \quad x \notin A_i \Rightarrow \underline{x \in X \setminus A_i, (\forall) i \in I}$

$\Rightarrow x \in \overline{A_i}, (\forall) i \in I \Rightarrow x \in \bigcap \overline{A_i}$

$\Rightarrow \overline{\bigcup_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$

(inversă) „$\supseteq$" Fie $y \in \bigcap_{i \in I} \overline{A_i} \Rightarrow y \in \overline{A_i}, \forall i \in I \Rightarrow$

$\Rightarrow y \in X \setminus A_i, (\forall) i \in I$

Vreau: $y \in \overline{\bigcup_{i \in I} A_i}$ ($\Leftrightarrow y \in X$ și $y \notin \bigcup_{i \in I} A_i$)

$y \in X$ și $y \notin \bigcup_{i \in I} A_i \Rightarrow \boxed{y \in \overline{\bigcup_{i \in I} A_i}}$

$\Rightarrow \boxed{\bigcap_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}}$

$\boxed{\text{b) - temă}}$

3) $\mathcal{P}(A) := \{Y \mid Y \subseteq A\} \ni \emptyset$

$f : A \to B$ funcție

imaginea

$X \in A \quad X \subseteq A,\ f(X) := \{ f(x) \mid x \in X \} \subseteq B$

$Y \subseteq B,\ f^{-1}(Y) := \{ a \in A \mid f(a) \in Y \} \subseteq A$

preimaginea/fibra

a) $f\left(\bigcup_{i \in I} A_i\right) \overset{?}{=} \bigcup_{i \in I} f(A_i)$

„$\subseteq$" Fie $x \in f\left(\bigcup_{i \in I} A_i\right) \Rightarrow (\exists)\ a \in \bigcup_{i \in I} A_i$ a.î. $x = f(a) \Rightarrow$

$\Rightarrow \exists\ i \in I$ a.î. $a \in A_i$ și $x = f(a) \Rightarrow (\exists)\ i \in I$ a.î. $x \in f(A_i)$

$\Rightarrow x \in \bigcup_{i \in I} f(A_i)$

„$\supseteq$" Fie $z \in \bigcup_{i \in I} f(A_i) \Rightarrow (\exists)\ i \in I$ a.î. $z \in f(A_i) \Rightarrow$

$\Rightarrow i \in I$ și $(\exists)\ a_i \in A_i$ a.î. $z = f(a_i) \Rightarrow z \in f\left(\bigcup_{j \in I} A_j\right)$

$\Downarrow$

$a_i \in \bigcup_{j \in I} A_j$

e) $f^{-1}\left(\bigcap_{j \in J} B_j\right) \overset{?}{=} \bigcap_{j \in J} f^{-1}(B_j)$

$Y \subseteq B,\ f : A \to B,\ f^{-1}(Y) := \{ a \in A \mid f(a) \in Y \}$.

„$\subseteq$". Fie $x \in f^{-1}\left(\bigcap_{j \in J} B_j\right) \Rightarrow \cancel{(\exists)}\ f(x) \in \bigcap_{j \in J} B_j \Rightarrow$

$f(x) \in B_j,\ (\forall)\ j \in J \Rightarrow x \in f^{-1}(B_j),\ \forall\ j \in J \Rightarrow$

$x \in \bigcap_{j \in J} f^{-1}(B_j) \Rightarrow \boxed{\text{OK}}$.

„2". Fie $x \in \bigcap_{j\in J} f^{-1}(B_j) \Rightarrow x \in f^{-1}(B_j), (\forall) j \in J \Rightarrow$

$f(x) \in B_j, (\forall) j \in J \Rightarrow f(x) \in \bigcap_{j\in J} B_j \Rightarrow x \in f^{-1}(\bigcap_{j\in J} B_j)$

$\boxed{b)+c)+d) = \text{TEMĂ!}}$

SEA = sunt echivalente afirmațiile

6) a) $f: A \to B$, $Im(f) := \{f(a) \mid a \in A\} = f(A)$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{5}{3+x^2}$

$\boxed{y \in Im(f)?}$

$y \in Im(f) \Leftrightarrow (\exists) a \in \mathbb{R}$ a.î. $y = f(a) = \frac{5}{3+a^2} \Leftrightarrow$ $\Delta = -4y(3y-5)$

$\Leftrightarrow ya^2 + 3y - 5 = 0 \Leftrightarrow (\exists) a \in \mathbb{R}$ a.î. $ya^2 + 3y - 5 = 0 \Leftrightarrow$

$\Leftrightarrow \Delta = -4y(3y-5) \geq 0$

$y \in Im(f) \Leftrightarrow -4y(3y-5) \geq 0 \Leftrightarrow$

$y(3y-5) \leq 0 \Leftrightarrow y \in [0, \frac{5}{3}] \Rightarrow Im(f) = [0, \frac{5}{3}]$

Temă: Ex 2) b; Ex 3) b, c, d; Ex 5); Ex 6) a, c,

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Algebră

Ex 4) $f : A \to B$

$f_* : \mathcal{P}(A) \to \mathcal{P}(B)$, $f_*(X) := f(X)$

$f^* : \mathcal{P}(B) \to \mathcal{P}(A)$, $f^*(Y) := f^{-1}(Y)$

a) f inj ; b) f_* inj ; c) $f^* \circ f_* = id$; d) f^* surj

e) $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$, $(\forall) X_1, X_2 \subseteq A$

f) $f(A \setminus X) \subseteq B - f(X)$, $(\forall) X \subseteq A$

Rez: $\boxed{a) \Leftrightarrow b)}$ banal!

$\boxed{a \Rightarrow b}$ f inj. Vreau f_* inj?? $f_*(X) = f_*(Y)$

$X, Y \subseteq A \overset{?}{\Rightarrow} X \overset{?}{=} Y$

Știu: $\{f(x) \mid x \in X\} = \{f(y) \mid y \in Y\}$

Pp. că $(\exists) x \in X, x \notin Y \Rightarrow f(x) \in f_*(X) = f_*(Y) \Rightarrow$

$\Rightarrow (\exists) y \in Y$ a.î. $f(x) = f(y) \Rightarrow x = y \in Y$, fals! $\Rightarrow X \subseteq Y$

Analog $Y \subseteq X$

$\Rightarrow X = Y$

$\boxed{b \Rightarrow a}$ Pp. f_* inj. Vreau: f inj! $\left.\begin{matrix} f(x) = f(y) \\ x, y \in A \end{matrix}\right\} \overset{?}{\Rightarrow} x = y$

$f_*(\{x\}) = \{f(x)\}$, $f_*(\{y\}) = \{f(y)\}$

$\Rightarrow (f_*$ inj$)$ $\{x\} = \{y\} \Rightarrow x = y$.

$\boxed{a) \Rightarrow c)}$ Pp. că f e inj. Vreau $f^* \circ f_* = id_{\mathcal{P}(A)}$

ie. $f^{-1}(f(X)) = X$, $(\forall) X \subseteq A$

$\boxed{X \overset{?}{\subseteq} f^{-1}(f(X))}$, Fie $\xi \in X$, Vrei $\xi \in f^{-1}(f(X))$

i.e. $f(\xi) \in f(X)$, $\boxed{\text{o.k.}}$

$f^{-1}(f(X)) \overset{??}{\subseteq} X$ Fie $\underline{a \in f^{-1}(f(X))} \Rightarrow \underline{f(a) \in f(X)} \Rightarrow$

$f(X) = \{f(x) \mid x \in X\}$

$f(a) \in f(X) \Rightarrow \underline{\exists x \in X \text{ a.î. }} f(a) = f(x) \Rightarrow \underline{a = x \in X} \Rightarrow$

$\Rightarrow a \in X \Rightarrow \boxed{f^{-1}(f(X)) \subseteq X}$

$\boxed{c) \Rightarrow d)}$ f^* surj. căci f_* e secțiune pt. f^*

$\left(f : A \to B \xrightarrow{g} A \text{ a.î. } g \circ f = Id_A \right.$

$\boxed{\text{T.C.}}$ $\Rightarrow f$ injectivă și g surjectivă ? $\left. \right)$

$\boxed{\text{inj.} + \text{surj.}}$

$\boxed{d) \Rightarrow e)}$ Știm f^* surj. Vrem: $f(X_1 \cap X_2) \overset{??}{=} f(X_1) \cap f(X_2)$, $(\forall)\, X_1, X_2 \subseteq A$.

$f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$ $\boxed{\text{o.k.}}$

$a = f(\xi) \in f(X_2)$, $\in f(X_1)$ $\Big\} \Rightarrow f(\xi) \in f(X_1) \cap f(X_2) \Rightarrow a \in f(X_1) \cap f(X_2)$

$f(X_1) \cap f(X_2) \overset{??}{\subseteq} f(X_1 \cap X_2)$, $(\forall)\, X_1, X_2 \subseteq A$

Fie $X_1, X_2 \subseteq A$ $\quad (f^* : \mathcal{P}(B) \to \mathcal{P}(A)$ e surj$)$

$\xRightarrow{f^* \text{ surj.}} \exists\, Y_1, Y_2 \subseteq B$ a.î. $X_1 = f^*(Y_1) = f^{-1}(Y_1)$

$X_2 = f^*(Y_2) = f^{-1}(Y_2)$

$f^{-1}(Y_1 \cap Y_2) \underset{\boxed{e)}}{\overset{\boxed{\text{Ex. 3)}}}{=}} f^{-1}(Y_1) \cap f^{-1}(Y_2) = \underline{X_1 \cap X_2}$

Fie $y \in f(X_1) \cap f(X_2) \Rightarrow (\exists)\, x_1 \in X_1$ și $(\exists)\, x_2 \in X_2$ a.î.

$y = f(x_1) = f(x_2) \Rightarrow \boxed{y \in Y_1 \cap Y_2}$ $\left(\begin{array}{l} x_1 \in X_1 = f^{-1}(Y_1) \Rightarrow f(x_1) \in Y_1 \Rightarrow y \in Y_1 \\ x_2 \in X_2 = f^{-1}(Y_2) \Rightarrow f(x_2) \in Y_2 \Rightarrow y \in Y_2 \end{array} \right)$

$y \in Y_1 \cap Y_2 \Rightarrow f^{-1}(y) \subseteq f^{-1}(Y_1 \cap Y_2) = X_1 \cap X_2 \Rightarrow$

$\Rightarrow (\exists)\, x \in X_1 \cap X_2$ a.î. $y = f(x) \Rightarrow y \in f(X_1 \cap X_2) \Longrightarrow$

$(z \in f^{-1}(y) \subseteq \boxed{X_1 \cap X_2} \Rightarrow f(z) = y \in X_1 \cap X_2\ )$

$y = f(z),\ z \in X_1 \cap X_2$

$\Rightarrow f(X_1) \cap f(X_2) \subseteq f(X_1 \cap X_2)$

$\boxed{e) \Rightarrow f)}$ Vreau: $f(A \setminus X) \overset{??}{\subseteq} B - f(X)$, $(\forall)\ X \subseteq A$

Fie $X \subseteq A$. Fie $X_1 := X$, $X_2 := A \setminus X$

$f(\underbrace{X \cap (A \setminus X)}) = f(X) \cap f(A \setminus X)$

$\overset{\shortparallel}{\emptyset} \Rightarrow f(X) \cap f(A \setminus X) = \emptyset$

$\Rightarrow f(A \setminus X) \subseteq B - f(X)$

$\boxed{f) \Rightarrow a)}$ Vreau: f e inj ??

$(\forall)\ a \neq b \in A \Rightarrow f(a) \neq f(b)$

Fie $a \neq b \in A$; $X = \{a\}$, $b \in A \setminus X$

$f(a) \in f(X)$ $\qquad f(b) \in f(A \setminus X) \overset{f)}{\subseteq} B \setminus f(X)$

$\Rightarrow f(b) \in B \setminus \{f(a)\} \Rightarrow f(a) \neq f(a) \Rightarrow \boxed{f \text{ e inj.}}$

$\boxed{Ex.6}$ b) $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $f(m,n) = m^2 - n^2$

$im(f) \overset{?}{=} \mathbb{Z} \setminus \{4\mathbb{Z}+2\}$

$4\mathbb{Z}+2 = \{4K+2 \mid K \in \mathbb{Z}\}$

$im(f) = \{m^2 - n^2 \mid m, n \in \mathbb{Z}\} \subseteq \mathbb{Z}$

$\mathbb{Z} \ni \boxed{x = 2K+1} = \underbrace{(K+1)^2 - K^2}_{f(K+1,K)} \in Im(f)$

$K \in \mathbb{Z}$

impare

$\mathbb{Z} \ni x = 2K$, $2 \neq m^2 - n^2 \overset{Im(f)}{\Leftarrow}$, $m, n \in \mathbb{Z}$

$4 = 2^2 - 0^2$, $2, 0 \in \mathbb{Z} \in Im(f)$

$16 = 4^2 - 0^2$

- $x = 4K \Rightarrow \underset{\overset{\cap}{Im(f)}}{x = 4K} = (K+1)^2 - (K-1)^2 = f(K+1, K-1)$

- $x = 4K+2 \Rightarrow (\nexists)\ m, n \in \mathbb{Z}$ a.î. $m^2 - n^2 = 4K+2$.

$m^2 \equiv \overset{(0,1)}{} \bmod 4 \Rightarrow m^2 - n^2 \equiv 0, 3, 1 \pmod 4 \Rightarrow$

$\Rightarrow 4K+2 \in Im(f)$

Ex: $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $f(x,y) = x^3 - y^3$ (se dat la examen)

$\searrow x^3 - y^2$

$Im(f) = ?$

6 c) $g = bij$: Vreau: $(\forall) y \in \mathbb{R} \setminus \{\frac{2}{5}\}$ $(\exists!)\ x \in \mathbb{R} - \{\frac{1}{5}\}$ a.î. $\frac{2x+1}{5x-1} = y$

Fie $y \in \mathbb{R}$, $y \neq \frac{5}{2}$. Caut (dacă există !!) un $x \in \mathbb{R}$ a.î.

$\frac{2x-1}{5x-1} = y$ ----

$\boxed{Ex\ 7}$. a) $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $f(x,y) = (2x+1, 2y+x^2)$

Rezolvare: Întreb: f e bijectivă ??

"Testez $(\forall)(a,b) \in \mathbb{R}\times\mathbb{R}$ $(\exists!)$ $(x,y)\in \mathbb{R}\times\mathbb{R}$ a.î. $f(x,y) = (a,b)$ $\boxed{!!}$

$\iff (2x+1, 2y+x^2) = (a,b)$

Fie $(a,b)\in \mathbb{R}\times\mathbb{R}$ $(\exists!)$ $(x,y)\in\mathbb{R}\times\mathbb{R}$ a.î.

$$\begin{cases} 2x+1 = a \\ 2y+x^2 = b \end{cases} \iff \begin{cases} x = \dfrac{a-1}{2} \\ y = \left(b - \left(\dfrac{a-1}{2}\right)^2\right)\cdot\dfrac{1}{2} = \dfrac{4b-a^2+2a-1}{8} \end{cases}$$

$$y = \frac{4b-a^2+2a-1}{8}$$

$x = \frac{a-1}{2}$, $y = \frac{4b-a^2+2a-1}{8}$ $\Rightarrow$ f bijectivă

$f^{-1}: \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $f^{-1}(a,b) = \left(\frac{a-1}{2}, \frac{4b-a^2+2a-1}{8}\right)$

b) $f: \mathbb{N}\to\mathbb{N}$, $f(m) = \begin{cases} 0, & m \le 5 \\ m-5, & m \ge 6 \end{cases}$

- f nu e injectivă $f(0) = f(1) = 0$, $0 \ne 1$.
- f e surjectivă

  $\boxed{0 \in \operatorname{Im}(f)}$, $f(0) = 0$, $f(m+5) = m$, $m \ge 1$

  $m \in \mathbb{Z}$, $m \ge 1 \Rightarrow \boxed{m} = f(m+5) \in \operatorname{Im}(f)$

- SECȚIUNI!

  $s: \mathbb{N}\to\mathbb{N}$ a.î. $f\circ s = id_{\mathbb{N}}$ i.e. $f(s(n)) = n$, $(\forall)\, n\in\mathbb{N}$

  $m\in\mathbb{N}$, $f^{-1}(m) = \{x\in\mathbb{N} \mid f(x) = m\}$

  $f^{-1}(0) = \{0,1,2,3,4,5\}$

  $f^{-1}(1) = \{6\}$, $f^{-1}(2) = \{7\}$, $f^{-1}(3) = \{8\}$

$f^{-1}(m) = \{m+5\}, (\forall)\ m \geq 1.$

$$s: \mathbb{N} \to \mathbb{N}, \quad s(m) = \begin{cases} m+5, & m \geq 1 \\ 0;2;3\ldots, & m = 0 \end{cases}$$

$\Rightarrow$ s poate fi ales în 6 moduri

f are 6 secțiuni

Întrebarea ce nu sunt și altele!

$f(s(n)) = n$

0 sau $s(n) \leq 5$ $\quad\to s(n) - 5$, dacă $s(n) \geq 6$

$$\Rightarrow \begin{cases} s(n) = n+5, & \text{dacă } s(n) \geq 6 \\ 0 = n, & s(n) \leq 5 \end{cases} \Rightarrow \boxed{s(0) \leq 5}$$

$s(1), s(2), \ldots \geq 6$

$f: A \to B$, $f(s(b)) = b$, $(\forall)\ b \in B$

$\Rightarrow s(b) \in f^{-1}(b)$, ... alegi $s(b) \in f^{-1}(b)$

8C) Funcția $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$g((m,n)) = 1 + 2 + \ldots + (m+n) + m$

$(\forall)\ (m,n) \in \mathbb{N} \times \mathbb{N}$ este bijectivă

Soluție: $g(m,n) = \dfrac{(m+n)(m+n+1)}{2} + m = \dfrac{(m+n)^2 + 3m + n}{2}$

- g e surjectivă? Prin inducție! $Im(g) \overset{?}{=} \mathbb{N}$

$\mathbb{N} = \{0, 1, \ldots\}$

$0 \in Im(g)$, $g(0,0) = 0$

$k \longmapsto k+1$. Pp că $k \in Im(g)$, vreau $k+1 \overset{?}{\in} Im(g)$

$(\exists)(m,n) \in \mathbb{N}^2$ a.î. $k = g(m,n) = \frac{(m+n)^2 + 3m + n}{2}$

$g(m+1, n-1)$, ~~$g(m-1, n)$~~, ~~$g(m, +n-1)$~~, $g(m-1, n+1)$

Cazul 1 $n \neq 0$

$g(m+1, n-1) = \frac{(m+n)^2 + 3(m+1) + n-1}{2} =$

$= \frac{(m+n)^2 + 3m + n}{2} + 1 = k+1 \Rightarrow k+1 \in Im(g)$ O.K.

Cazul 2 : $n = 0 \Rightarrow k = g(m,0) = \frac{m^2 + 3m}{2}$

$k+1 = g(???)$
$(0, m+1)$

$g(0, m+1) = \frac{(m+1)^2 + m+1}{2} = \frac{m^2 + 3m}{2} + 1 = k+1 \Rightarrow$

$\Rightarrow k+1 \in Im(g)$! $\Rightarrow g$ este surjectivă

• g este injectivă !

Fie $(m,n) \neq (m',n') \in \mathbb{N} \times \mathbb{N}$: Vreau $g(m,n) \neq g(m',n')$.

Pp. că $g(m,n) = g(m',n')$

$g(m,n) = 1+2+ \dots + (m+n) + m$. Trei cazuri

Cazul 1 : $m+n = m'+n' \Rightarrow$

$\cancel{1+2+\dots+(m+n)} + m = \cancel{1+2+\dots+(m+n)} + m' \Rightarrow$

$\Rightarrow m = m' \Rightarrow n = n' \Rightarrow (m,n) = (m',n')$ Fals!

Cazul 2 : $m+n < m'+n'$. Din $g(m,n) = g(m',n') \Rightarrow$

$\Rightarrow 1+2+ \ldots + \cancel{(m+n)} + m = 1+2+ \ldots + \cancel{(m+n)} + (m+n) + 1 + \ldots$
$+ (m'+n') + m'$

$\cancel{m'} = (\cancel{m}+n+1) + \ldots + (m+n') + n' \Rightarrow$

$\Rightarrow 0 =$ sumă de numere naturale <u>fals</u>!

<u>Cazul</u> 3: $m+n > m'+n'$, analog cu 2) $\Rightarrow$

<u>g inj deci f e bijectivă</u>

$\boxed{8b)}$ $f: \mathbb{N} \times \mathbb{N} \xrightarrow{\sim} \mathbb{N}$, $f(m,n) = 2^m(2n+1) - 1$
$\forall (m,n) \in \mathbb{N}^2$ e <u>bijectivă</u>

($\Rightarrow g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ e bije $\Leftrightarrow$
$(\forall)\ x \in \mathbb{N}\ (\exists!)\ m,n \in \mathbb{N}$ a.î. $2x = (m+n)^2 + 3m+n$
Temă: dem. direct pt. 8c), 8a))

Pp. $f(m,n) = f(a,b) \overset{?}{\Rightarrow} (m,n) = (a,b)$
$2^m(2n+1) - \cancel{1} = 2^a(2b+1) - \cancel{1}$

Dacă $m > a \Rightarrow \underbrace{2^{m-a}}_{2} (\underbrace{2n+1}_{\text{impar}}) = \underbrace{2b+1}_{\text{impar}}$ <u>fals</u>!

$\Rightarrow \boxed{m=a} \Rightarrow 2n+1 = 2b+1 \Rightarrow \boxed{n=b} \Rightarrow (m,n) = (a,b)$,
<u>f e inj.</u>

Fie $x \in \mathbb{N}$. Vreau: $(\exists)$ <u>$m,n \in \mathbb{N}$</u> a.î. $2^m(2n+1) = \underline{x+1}$

T.F.A: $(\forall)\ x \in \mathbb{N}\ (\exists)\ p_1, p_2, \ldots p_k =$ nr. prime
$\alpha_1, \ldots, \alpha_k \in \mathbb{N}$ a.î. $x = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$

$x+1 = \underbrace{2^{\alpha_1}} \cdot \underbrace{3^{\alpha_2} \cdot 5^{\alpha_3} \ldots p_k^{\alpha_k}}_{\overset{\shortparallel}{2n+1}}$, <u>$\alpha_1 = m$</u>

$\Rightarrow f$ e <u>surj</u>!

Algebră

$g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$g(m,n) = 1+2+\dots+(m+n)+m$ e bijectivă

$g^{-1}(m) = (\underline{\ ?\ }, \underline{\ ?\ })$

⑦. d) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = 3n+2$

- f e injectivă: $f(m) = f(n) \Rightarrow 3m+2 = 3n+2 \Rightarrow \underline{m=n}$
- f nu e surjectivă: $(\nexists)\ n \in \mathbb{N}$ a.î. $f(n) = 1$.

$\Rightarrow f$ are o retractă $r : \mathbb{N} \to \mathbb{N}$ i.e. $r \circ f = id_{\mathbb{N}}$

$\boxed{r = ??}$

$r(f(n)) = n, (\forall)\ n \in \mathbb{N} \Leftrightarrow$

$\underline{r : \mathbb{N} \to \mathbb{N}}$

$\Leftrightarrow r(\underbrace{3n+2}_{x}) = n, (\forall)\ n \in \mathbb{N} \Rightarrow 3n+2 = x$

$r(x) = \frac{x-2}{3}$

$$r : \mathbb{N} \to \mathbb{N},\ r(x) := \begin{cases} \frac{x-2}{3}, & x = 3k+2, k \in \mathbb{N} \\ 13, & x \notin 3\mathbb{Z}+2 \end{cases}$$

([illegible] poate fi definită [illegible] în mai multe moduri)

⑨. X, Y, Z - trei mulțimi

$$\alpha : Hom(X, Hom(Y,Z)) \underset{\beta}{\overset{\sim}{\rightleftarrows}} Hom(X \times Y, Z)$$

$\alpha(f)((x,y)) := f(x)(y)$ $\underline{\text{este bijectivă}}$.

$(\forall)\ f \in Hom(X, Hom(Y,Z))$, $x \in X$, $y \in Y$

Rezolvare:

Definim $\beta : Hom(X \times Y, Z) \longrightarrow Hom(X, Hom(Y,Z))$

$\beta(g)(x)(y) := g((x,y))$

$\forall g \in Hom(X \times Y, Z)$, $\forall x \in X$, $y \in Y$

Afirm: α și β sunt inverse una alteia

(β e inversa lui α): $\alpha\circ\beta \overset{?}{=} id$ și $\beta\circ\alpha \overset{?}{=} id$

$Hom(X\times Y, Z)$ $\qquad$ $Hom(X, Hom(Y,Z))$

$\alpha\circ\beta = id$

- Fie $g\in Hom(X\times Y, Z)$

$(\alpha\circ\beta)(g) = \alpha(\beta(g)) \overset{?}{=} g$

Fie $(x,y)\in X\times Y \Rightarrow$

$LHS(x,y) = \alpha(\beta(g))(x,y) \overset{def\,\alpha}{=} \beta(g)(x,y) = g(x,y) \Rightarrow$

$\Rightarrow \alpha(\beta(g)) = g$, $(\forall)$ $g\in Hom(X\times Y, Z)$

Analog, $(\beta\circ\alpha)(f) \overset{?}{=} f$, $(\forall)$ $f\in Hom(X, Hom(Y,Z))$

- Fie $f\in Hom(X, Hom(Y,Z))$

Fie $x\in X$: $(\beta\circ\alpha)(f) = \beta(\alpha(f)) \overset{?}{=} f(x)$

Fie $y\in Y$: $\beta(\alpha(f))(x)(y) = \alpha(f)(x)(y) = f(x)(y)$, $\forall y\in Y$

$\Rightarrow \beta(\alpha(f))(x) = f(x)$, $(\forall)$ $x\in X \Rightarrow$

$\beta(\alpha(f)) = f$, ie. $\beta\circ\alpha = id$

(10). Fie X o mulțime. Atunci funcția

$\varphi: \mathcal{P}(X) \xrightarrow{\sim} Hom(X, \{0,1\})$ definită prin:

$$\varphi(A)(x) := \begin{cases} 1, & x\in A \\ 0, & x\in X\setminus A \end{cases} = \varphi_A(x)$$

$(\forall)$ $A\subseteq X$, $x\in X$ este bijectivă.

$\varphi(A) \overset{not}{=} \varphi_A$ = funcția caracteristică a mulțimii $A\subseteq X$

Rezolvare:

Fie $\gamma : Hom(X, \{0,1\}) \longrightarrow \mathcal{P}(X)$

$$\gamma(\theta) := \theta^{-1}(1) = \{x \in X \mid \theta(x) = 1\}$$

$$(\forall)\, \theta : X \longrightarrow \{0,1\}$$

Afir: φ și γ sunt inverse una alteia?

- $\gamma \circ \varphi = id_{\mathcal{P}(X)}$, $\gamma(\varphi(A)) \overset{?}{=} A$, $(\forall) A \subseteq X$

$\gamma(\varphi_A) \overset{?}{=} A \Leftrightarrow \varphi_A^{-1}(1) = A \Leftrightarrow \{x \in X \mid \varphi_A(x) = 1\} = A$

$\Leftrightarrow A = A$, adevărat.

- $\varphi \circ \gamma \overset{?}{=} id_{Hom(X,\{0,1\})} \Leftrightarrow \varphi(\gamma(\theta)) = \theta$, $\boxed{(\forall)\theta : X \to \{0,1\}}$

Fie $\theta : X \to \{0,1\}$

$\varphi(\gamma(\theta))(x) \overset{?}{=} \theta(x)$, $(\forall)\, x \in X$.

$\varphi_{\gamma(\theta)}(x) = \theta(x) \Leftrightarrow \varphi_{\theta^{-1}(1)}(x) \overset{?}{=} \theta(x)$

$$\varphi_{\theta^{-1}(1)}(x) = \begin{cases} 1, & x \in \theta^{-1}(1) \\ 0, & x \notin \theta^{-1}(1) \end{cases} \overset{?}{=} \theta(x)$$

adevărat

$$\left.\begin{array}{l} \varphi_{\theta^{-1}(1)}(x) = 1,\ x \in \theta^{-1}(1) \Rightarrow \theta(x) = 1 \\ \varphi_{\theta^{-1}(1)}(x) = 0,\ x \notin \theta^{-1}(1) \Rightarrow \theta(x) = 0 \end{array}\right\} \Rightarrow \boxed{\{0,1\}}$$

Obs: La ce e bună funcția caracteristică?

1) $A, B \subseteq X$, $\varphi_A, \varphi_A : X \to \{0,1\}$

$\varphi_A = \varphi_B \Leftrightarrow A = B$ (transformă o egalitate de mulțimi în egalitate de funcții)

2) $\boxed{|\mathcal{P}(X)| \text{"="} 2^{|X|}}$ pt. mulțimi finite.

ex. $|X| = n \Rightarrow |\mathcal{P}(X)| = 2^n$

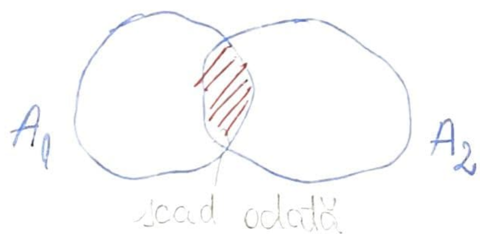
EX 11. Fie $A_1, \dots A_n$ mulțimi finite (Principiul includerii și excluderii)

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \dots +$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \dots A_n| \quad , n \in \mathbb{N}^*$$

Dem: Prin inducție după n

$n=2$ $\quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ adev



$n \xrightarrow{??} n+1$

$$|(A_1 \cup A_2 \cup \dots \cup A_n) \cup A_{n+1}| = |A_1 \cup A_2 \dots \cup A_n| + |A_{n+1}| -$$

$$- |(A_1 \cup A_2 \cup \dots \cup A_n) \cap A_{n+1}| =$$

$$= |A_1 \cup A_2 \dots \cup A_n| + |A_{n+1}| - |(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1})| =$$

$A_i \cap A_j \cap A_{n+1}$

$$= |A_1 \cup A_2 \cup \dots \cup A_n| + |A_{n+1}| - \Big(\sum_{i=1}^{n} |A_i \cap A_{n+1}| - \sum_{1 \le i < j \le n} |(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})|$$

inducția

inducție

$$+ \dots (-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cap \dots \cap A_{n+1}| \Big) = \text{(inducție)}$$

$$= \sum_{i=1}^{n+1} |A_i| - \sum_{1 \le i < j \le n+1} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n+1} |A_i \cap A_j \cap A_k| - \dots +$$

$$+ (-1)^{n+2} |A_1 \cap A_2 \cap \dots A_{n+1}|. $$ ($\Rightarrow$ formula principiului includerii și excluderii (este demonstrată)

EX 12) $A=\{1,\dots,m\}$, $B=\{1,\dots,n\}$

a) $|Hom(A,B)| \overset{??}{=} n^m$

$1 \xmapsto{f} n$ moduri; $2 \xmapsto{f} n$ moduri; ...; $m \xrightarrow{f} n$ moduri

$\Rightarrow$ avem $\underbrace{n \cdot n \cdot \dots \cdot n}_{\text{de } m \text{ ori}} = n^m$ funcții

b) $\left|\{f \in Hom(A,B) \mid f = \text{injectivă}\}\right| = \begin{cases} 0, & \underline{m>n} \\ ?, & \underline{m \le n} \end{cases}$

$1 \xmapsto{f} n$ moduri $f(1)$; $2 \xrightarrow{f} n-1$ moduri $f(2)$ .....

$m \xrightarrow{f} (n-m+1)$ moduri $f(m)$

$n \cdot (n-1) \cdot \dots \cdot (n-m+1) = \dfrac{n!}{(n-m)!} = A_n^m$

Mulțimi numărabile

A s.n. numărabilă dacă $(\exists)\ f: A \xrightarrow{\sim} \mathbb{N}$ o funcție bijectivă.

Exemple: $\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N} \Longrightarrow$

• A și B sunt numărabile $\overset{?}{\Rightarrow}$ $A \times B$ este numărabilă

$(\exists)\ f: A \xrightarrow{\sim} \mathbb{N}$ bijectivă, $g: B \xrightarrow{\sim} \mathbb{N}$ bijectivă

$\Rightarrow f \times g : A \times B \xrightarrow{\sim} \mathbb{N} \times \mathbb{N} \cong \mathbb{N}$

$(f \times g)(a,b) := (f(a), g(b)), (\forall)(a,b) \in A \times B$

e bijectivă. $\Rightarrow A \times B$ e numărabilă

($f: A \to B$, $g: C \to D \Rightarrow f \times g : A \times C \to B \times D$,

$(f \times g)(a, c) := (f(a), g(c))$, f, g bijective $\Rightarrow f \times g$ e bijectivă)

(EX14) a) $i: B \longrightarrow A$ o funcție injectivă și A numărabilă $\Rightarrow B$ este finită sau numărabilă (cel mult numărabilă)

b) Fie $p: A \to B$ surjectivă și A = numărabilă $\Rightarrow B$ este cel mult numărabilă.

Rezolvare:

a) $i: B \longrightarrow A$ injectivă

$$B \overset{bij}{\simeq} \underline{i(B)} \subseteq A \overset{g}{\simeq} \mathbb{N}$$
$$b \longmapsto i(b)$$

$$i(B) \simeq g(i(B)) \subseteq g(A) = \mathbb{N}$$

$A := \mathbb{N}$

$\left[X \subseteq \mathbb{N} \overset{?}{\Longrightarrow} X \text{ e finită sau numărabilă!} \right]$

$X \neq \phi$; $x_0 \overset{def}{:=} \min(X)$

Dacă $X \setminus \{x_0\} = \phi \Rightarrow X = \{x_0\}$ OK.

Dacă $X \setminus \{x_0\} \neq \phi$, fie $x_1 \overset{\min}{:=} (X - \{x_0\})$

Dacă $X \setminus \{x_0, x_1\} = \phi \Rightarrow X = \{x_0, x_1\}$ OK.

Dacă $X \setminus \{x_0, x_1\} \neq \phi$, $x_2 := \min(X \setminus \{x_0, x_1\})$.

..... dacă se găsește $\Rightarrow X$ = finită dacă nu $\Rightarrow X = \{x_0, x_1, x_2 \dots\} \simeq \mathbb{N}$

b) Fie $p: A \to B$ surjectivă, A numărabilă $\Rightarrow$
$\Rightarrow (\exists)\ s: B \to A$ a.î. $p \circ s = id_B \Rightarrow$
$\Rightarrow s =$ injectivă $\overset{a)}{\Rightarrow} B$ e cel mult numărabilă.

Ex 15. Fie $(X_n)_{n\in\mathbb{N}}$ e o familie numărabilă de mulțimi numărabile $\Rightarrow \bigcup\limits_{n\in\mathbb{N}} X_n$ e numărabilă.

Rezolvare: Folosim: • $B \subseteq A$ = numărabilă $\Rightarrow B$ e cel mult numărabilă

• $\mathbb{N} \times \mathbb{N}$ e numărabilă

$(X_n)_{n\in\mathbb{N}}$; $\bigcup\limits_{n\in\mathbb{N}} X_n \overset{?}{=}$ numărabilă!

Fie $(\overline{X_n})_{n\in\mathbb{N}}$ definite prin:

$\overline{X_0} := X_0$, $\overline{X_1} := X_1 \setminus X_0$, ...

$$\boxed{\overline{X_n} := X_n - \bigcup_{k=0}^{n-1} X_k}, (\forall)\ n \geq 1$$

• $\overline{X_n} \cap \overline{X_m} = \phi$, $(\forall)\ n \neq m \in \mathbb{N}$

• $\overline{X_n} \subseteq X_n$ = numărabilă $\Rightarrow \overline{X_n}$ e finită sau numărabilă, $(\forall)\ n \in \mathbb{N}$. Pentru $\boxed{n \in \mathbb{N}}$

$\Rightarrow (\exists)\ f_n : \overline{X_n} \to \mathbb{N}$ injectivă

Fie $\gamma : \bigcup_{m\in\mathbb{N}} \overline{X_m} \longrightarrow \underline{\mathbb{N}\times\mathbb{N}}$ = numărabilă

$$\gamma(\overline{x_m}) := (m, f_m(\overline{x_m})), \ (\forall)\ \overline{x_m} \in \overline{X_m}.$$

$\Rightarrow \gamma$ este <u>injectivă</u> $\Rightarrow \bigcup_{m\in\mathbb{N}} \overline{X_m}$ este <u>finită</u> sau <u>numărabilă</u>!

Dar

~~Dar~~ $\bigcup_{m\in\mathbb{N}} \overline{X_m} \overset{?}{=} \bigcup_{m\in\mathbb{N}} X_m$ "$\subseteq$" ok căci $\overline{X_m} \subseteq X_m$ $(\forall)\ m\in\mathbb{N}$.

Fie $y \in \bigcup_{m\in\mathbb{N}} X_m \Rightarrow (\exists)\ m\in\mathbb{N}$, cel mai mic număr natural a.î. $y\in X_m$ şi $y \in X_1, X_2, \dots X_{m-1} \Rightarrow$

$\Rightarrow \underline{y\in \overline{X_m}} \Rightarrow y \in \bigcup_{m\in\mathbb{N}} \overline{X_m}$.

SEMINAR 4

4+5

30.10.2020

Algebră

12) c) $f: A \to B$, $A = \{1, 2, \dots, m\}$, $B = \{1, 2, \dots, n\}$

nr. funcțiilor surjective

$s(m, n) = |\{f: A \to B \mid f \text{ surjectivă}\}| = ?$

$$s(m, n) = \begin{cases} 0, & m < n \\ ??, & m \geq n \end{cases}$$

Pp. $\boxed{m \geq n}$

$s(m, n) = n^m - |\{f: A \to B \mid f \text{ nesurjectivă}\}|$

$f: A \to B$ e nesurjectivă $\Leftrightarrow (\exists)\, i \in \{1, 2, \dots n\}$ a.î. $i \notin \operatorname{Im} f$

Pentru fiecare $i = \overline{1, n}$

$$A_i := \{f: A \to B \mid i \notin \operatorname{Im}(f)\}$$

- $f: A \to B$ este nesurjectivă $\Leftrightarrow (\exists)\, i = \overline{1, n}$ a.î. $f \in A_i \Leftrightarrow$ $f \in A_1 \cup A_2 \cup \dots \cup A_n$

$$|A_1 \cup A_2 \cup \dots \cup A_n| \overset{11)}{=} \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \dots (-1)^{n+1} |A_1 \cap \dots \cap A_n|$$

$|A_i| = (n-1)^m$

$|A_i \cap A_j| = (n-2)^m$

$|A_i \cap A_j \cap A_k| = (n-3)^m$

$s(m, n) = n^m - |A_1 \cup \dots \cup A_n| = (Ex 11)$

$= n^m - \left(C_n^1 (n-1)^m - C_n^2 (n-2)^m + C_n^3 (n-3)^m + \dots \pm (-1)^{n+1} \cdot 0\right) =$

$= n^m - \sum_{k=1}^{n-1} (-1)^{k+1} C_n^k (n-k)^m$

d) $|\{f:\{1,\dots m\}\to\{1,\dots n\} \mid f = \text{bijectivă}\}| =$

$$= \begin{cases} 0, & m \neq n \\ n!, & \text{dacă } m = n \end{cases}$$

$$= \boxed{n^n - \sum_{k=1}^{m-1} (-1)^{k+1} C_n^k (n-k)^n} = n!$$

$$\boxed{n^n = n! + \sum_{k=1}^{m-1} (-1)^{k+1} C_n^k (n-k^n)}$$

$$C_n^0 (n-0)^n$$

$$\sum_{k=0}^{n-1} (-1)^k C_n^k (n-k)^n \overset{?}{=} n!$$

13) A o mulțime. S.E.A:

a) A este finită; b) Orice funcție injectivă $f: A \to A$ este bij;

c) Orice funcție surjectivă $g: A \to A$ este bijectivă.

Rezolvare:

a) ⇒ b) și a) ⇒ c). Pp. $A = \{a_1, a_2, \dots, a_n\}$

Fie $f: A \to A$ surjectivă ⇒ $Im(f) = A$ ⇒

$\underbrace{\{f(a_1), f(a_2), \dots f(a_n)\}}_{n \text{ elemente}} = A$ → are n elemente

⇒ $f(a_i) \neq f(a_j), (\forall) i \neq j$ ⇒ $f: A \to A$ e inj.

Fie $g: A \to A$ injectivă ⇒ $\underbrace{\{g(a_1), g(a_2), \dots, g(a_n)\}}_{\text{sunt diferite}} \subseteq \underset{n \text{ elemente}}{A}$ ⇒

sunt n elemente

⇒ $\{g(a_1), g(a_2) \dots g(a_n)\} = A$ ⇒ $im(g) = A$ → g = surjecție

$\boxed{\text{b)} \Rightarrow \text{a) și c)} \Rightarrow \text{a)}}$ Pp. că A este infinită!

Vreau :) să construim $f : A \to A$ surjecție, neinjectivă

— „ — $g : A \to A$ injecție, nesurjectivă

$\Rightarrow (\exists)\, B \subseteq A$, B = numărabilă

$B = \{a_0, a_1, a_2, \dots, a_n, \dots\} \subseteq A$

Definim $f : A \to A$

$$f(x) = \begin{cases} x, & x \in A \setminus \{a_0, a_1, \dots\} \\ a_0, & x = a_0 \\ a_{n-1}, & \text{dacă } x = a_n,\ n \geq 1 \end{cases}$$

$f(a_0) = f(a_1) = a_0$. , fals! $\Rightarrow$ $\boxed{A \text{ finită}}$

f e neinjectivă

f e surjectivă

Fie $g : A \to A$, $g(x) = \begin{cases} x, & x \in A \setminus B \\ a_{n+1}, & x = a_n,\ n \geq 0 \end{cases}$

g e injectivă

g e nesurjectivă căci $\boxed{a_0 \notin \operatorname{Im}(g)}$ $\Rightarrow$ fals! $\Rightarrow$ $\boxed{A \text{ finită}}$

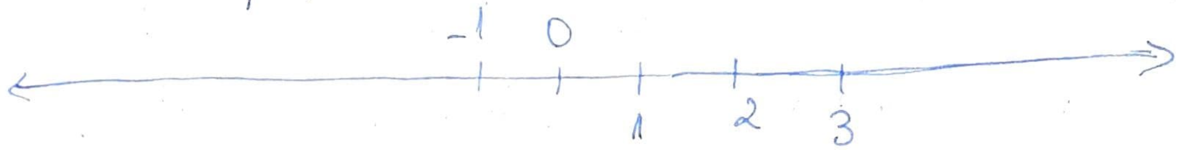
Ex 16) Arătați că $(\nexists)\, f : \mathbb{R} \to \mathbb{R}$ o funcție a.î. $|f(x) - f(y)| > 1$, $(\forall)\, x \neq y \in \mathbb{R}$.

Soluție: Pp. că $(\exists)\, f : \mathbb{R} \to \mathbb{R}$ a.î.

(*) $|f(x) - f(y)| > 1$, $(\forall)\, x \neq y \Rightarrow$ $\boxed{f \text{ e injectivă}}$

$\Rightarrow \mathbb{R} \simeq \operatorname{Im}(f) \subseteq \mathbb{R} \Rightarrow \operatorname{Im}(f)$ nu e numărabilă

$x \longmapsto f(x)$



$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1)$

(*) $\Rightarrow |\operatorname{Im}(f) \cap [n, n+1)| \leq 1$

$\Rightarrow \operatorname{Im}(f) \cap [n, n+1)$ are cel mult un element! $(\forall)\, n \in \mathbb{Z}$

$\operatorname{Im}(f) = \operatorname{Im}(f) \cap \mathbb{R} = \operatorname{Im}(f) \cap \left(\bigcup_{n \in \mathbb{Z}} [n, n+1) \right) =$

$= \bigcup_{n\in\mathbb{Z}} \underbrace{(\operatorname{im}(f)\cap[n,n+1))}_{\text{are cel mult un element}}$ = cel mult numărabilă

Fals!

Căci $\operatorname{im}(f)$ nu e numărabilă.

$\boxed{\text{Ex 26}^*}$ Dacă X, Y sunt mulțimi nevide $\Rightarrow$

$(\exists)\ f: X \to Y$ injectivă sau $g: Y \to X$ injectivă.

$|X| \le |Y|$ sau $|Y| \le |X|$.

!! $m, n \in \mathbb{N} \Rightarrow m \le n$ sau $n \le m$!!

Dem: Fie

$\mathcal{F} := \{ (A, f) \mid \emptyset \ne A \subseteq X,\ f: A \to Y \text{ injectivă} \}$

• $\boxed{\mathcal{F} \ne \emptyset}$, Fie $x_0 \in X$, $A := \{x_0\}$. orice funcție $f: \{x_0\} \to Y$ e injectivă

• Pe $\mathcal{F}$ definim relația:

$(A, f) \le (B, g) \overset{def}{\Longleftrightarrow} A \subseteq B$ și $f = g|_A$

$(\forall)\ (A, f), (B, g) \in \mathcal{F}$

$(\mathcal{F}, \le)$ este ordonată: reflexivă, antisimetrică, tranzitivă!

• $(\mathcal{F}, \le)$ este inductiv ordonată!

Fie $C = (A_\alpha, f_\alpha)_{\alpha\in\Lambda}$ o parte total ordonată a lui $\mathcal{F}$, i.e. $(\forall)\ \alpha_1, \alpha_2 \in \Lambda$ avem $(A_{\alpha_1}, f_{\alpha_1}) \le (A_{\alpha_2}, f_{\alpha_2})$ sau "pe dos"

Vreau: un majorant al lui C, i.e. $(\bar{A}, \bar{f}) \in \mathcal{F}$ a.î. $(A_\alpha, f_\alpha) \le (\bar{A}, \bar{f})$, $(\forall)\ \alpha \in \Lambda$.

Fie $\bar{A} := \bigcup_{\alpha\in\Lambda} A_\alpha$, $A_\alpha \subseteq \bar{A}$, $(\forall)\ \alpha \in \Lambda$

Vreau: $\bar{f}: \bar{A} \to Y$ injectivă

$f_\alpha: A_\alpha \to Y$ sunt injective $(\forall)\ \alpha \in \Lambda$.

$$\left[\begin{array}{l}\underset{\substack{\cap \\ A_\alpha}}{\bar{f}(x_\alpha)} := f_\alpha(x_\alpha)\end{array}\right], (\forall)\, \alpha \in \lambda$$

- $\bar{f}$ e definită <u>corect</u>!

$$f_{\alpha_2} = f_{\alpha_1}|_{A_{\alpha_2}}$$

$$Y \in A_{\alpha_1},\ Y \in A_{\alpha_2} \Rightarrow$$

$$(A_{\alpha_1}, f_{\alpha_1}) \leq (A_{\alpha_2}, f_{\alpha_2}) \text{ sau } (A_{\alpha_2}, f_{\alpha_2}) \leq (A_{\alpha_1}, f_{\alpha_1})$$

$$f_{\alpha_1} = f_{\alpha_2}|_{A_{\alpha_1}}$$

$$f_{\alpha_1}(Y) = f_{\alpha_2}(Y)$$

$\bar{f}$ e <u>injectivă</u> căci f_α sunt injective $\Rightarrow$

$\Rightarrow (\bar{A}, \bar{f}) \in \mathcal{F}$ și $(A_\alpha, f_\alpha) \leq (\bar{A}, \bar{f}), (\forall)\, \alpha \in \lambda$

$$\underline{f_\alpha = \bar{f}|_{A_\alpha}}$$

$\Rightarrow$ <u>$(\mathcal{F}, \leq)$ este inductiv ordonat.</u>

Lema Zorn $\Rightarrow (\mathcal{F}, \leq)$ are un <u>element</u> <u>maximal</u> notat cu $(A^*, f^*) \in \mathcal{F}$

$$f^*: A^* \longrightarrow Y \text{ e injectivă}, \quad A^* \subseteq X$$
$$A^* \neq \emptyset.$$

<u>Cazul 1</u>: Dacă <u>$A^* = X$</u> ok. ($f^*: X \to Y$ e injectivă)

<u>Cazul 2</u>: Dacă $Im(f^*) = Y \Rightarrow f^*$ <u>surj</u>

$$A^* \underset{\sim}{\xrightarrow{bij}} Im(f^*) = Y$$
$$\longleftarrow$$

$g: Y \xrightarrow[\sim]{(f^*)^{-1}} A^* \xhookrightarrow{i} X$, $g := i \circ (f^*)^{-1}$ e injectivă

Cazul 3 Pp. că $A^* \subsetneq X$ și $\operatorname{Im}(f^*) \subsetneq Y$.

Fie $x_0 \in X \setminus A^*$ și $y_0 \in Y \setminus \operatorname{Im}(f^*)$

Fie $A^* \cup \{x_0\}$ și $h: A^* \cup \{x_0\} \longrightarrow Y$

$$h(x) := \begin{cases} f^*(x), & x \in A^* \\ y_0, & x = x_0 \end{cases}$$

h = injectivă, $y_0 \neq f^*(x)$, $(\forall)\ x \in A^*$

$\Rightarrow (A^*, f^*) \subsetneq (A^* \cup \{x_0\}, h) \in \mathcal{F}$

maximal, fals!

Opusul unei relații: $\rho \subseteq A \times A$ relație

$\rho^{op} \subseteq A$ definit astfel:

$$[a \rho^{op} b \overset{def}{\Longleftrightarrow} b \rho a] \qquad (\rho^{op} = \rho^{-1})$$

SEMINAR 5 — 6.11.2020

Lucrare 27.11 - ora 14^{00}

Algebră

20) Pe $\mathbb{R}$ definim relația : $x \sim y \overset{def}{\Leftrightarrow} x - y \in \mathbb{Z}$.

Arătați că $\mathbb{R}/_{\sim} \simeq [0,1)$

SOLUȚII :

- Metoda 1 (Sisteme de reprezentanți) $\Rightarrow [0,1) \simeq \mathbb{R}/_{\sim}$

$S :\overset{def}{=} [0,1)$ e sistem de reprezentanți ??? $\quad x \longrightarrow \hat{x}$

- Vrem : $(\forall)\ x \in \mathbb{R}\ (\exists!)\ y \in [0,1)$ a.î. $x \sim y$??$(\Leftrightarrow x - y \in \mathbb{Z})$

Fie $x \in \mathbb{R}$. Fie $y = \{x\} \in [0,1)$ și $x - \{x\} = [x] \in \mathbb{Z}$

i.e. $x \sim y$.

unicitatea lui y: Fie $z \in [0,1)$ a.î. $x \sim z \Rightarrow$

$\boxed{x - z \in \mathbb{Z}} \Rightarrow \{x - z\} = 0$

Vrem : $z \overset{?}{=} \{x\}$ $\qquad \longrightarrow x - z = k,\ k \in \mathbb{Z}$

$$\left\{\begin{array}{l} x = k + z,\ k \in \mathbb{Z},\ z \in [0,1) \\ x = [x] + \{x\},\ [x] \in \mathbb{Z},\ \{x\} \in [0,1) \end{array}\right| \Rightarrow \text{unicitatea}$$

$\Rightarrow k = [x]$ și $z = \{x\}$

Altfel : $x - z \in \mathbb{Z} \Rightarrow [x] + \{x\} - z \in \mathbb{Z} \Rightarrow \underbrace{\{x\}}_{\in[0,1)} - z \in \mathbb{Z}$

$\boxed{\{x\} = z}$

- Metoda 2 (cu P.U.M.F)

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\pi"] \arrow[d, "f"'] & \mathbb{R}/_{\sim} \arrow[dl, dashed, "(\exists!)\ \bar{f}"] \\ {[0,1)} & \end{tikzcd}$$

, $\pi(r) = \hat{r}$

Vrem : un f "bun"??

(i.e. $\rho_f = \sim$ și f surj)

$f: \mathbb{R} \to [0,1),\ f(x) := \{x\},\ (\forall)\, x \in \mathbb{R}$.

- f surjectivă

Fie $r \in [0,1)$ avem că $\underset{\underset{f(r+1)}{\shortparallel}}{f(r)} = r \Rightarrow f$ e surj.

- $\mathcal{I}_f \overset{?}{=} \sim$

Fie $x, y \in \mathbb{R}$

$x\, \mathcal{I}_f\, y \Leftrightarrow f(x) = f(y) \Leftrightarrow \{x\} = \{y\} \overset{??}{\Leftrightarrow} \underline{x - y \in \mathbb{Z}} \Leftrightarrow x \sim y$

„$\Rightarrow$" $x - [x] = y - [y] \Leftrightarrow x - y = [x] - [y] \in \mathbb{Z}$

$\{x\} - \{y\} = 0$

„$\Leftarrow$" $x - y \in \mathbb{Z}$, $\underline{[x]} + \{x\} - \underline{[y]} - \{y\} \in \mathbb{Z} \Rightarrow \{x\} - \{y\} \in \mathbb{Z} \Rightarrow$

$\Rightarrow \{x\} = \{y\}$

$\Rightarrow \mathcal{I}_f = \sim \Rightarrow$ „$\sim$" e echivalență

$\Rightarrow$ (P.U.M.F.) $(\exists!)\ \boxed{\bar{f} : \mathbb{R}/_{\sim} \xrightarrow{\sim} [0,1)}$ a.î.

$\bar{f}(\hat{r}) = \{r\},\ (\forall)\, \hat{r} \in \mathbb{R}/_{\sim}$

În plus, $\bar{f}$ e bijectivă, căci f e surj ($\bar{f}$ e surj.) și $\mathcal{I}_f = \sim$ ($\bar{f}$ e inj.)

21) Pe $\mathbb{C}^*$ definim relația $z_1 \sim z_2 \overset{def}{\Leftrightarrow} \arg(z_1) = \arg(z_2)$

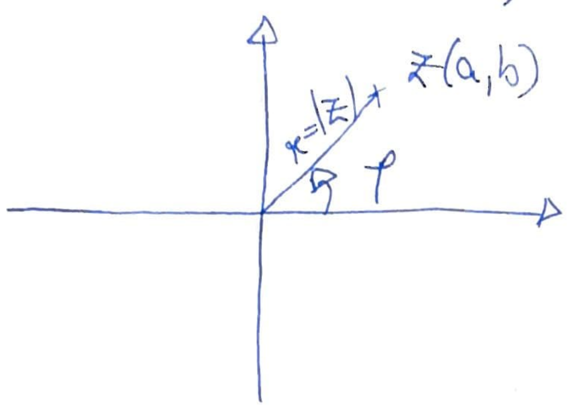
Atunci $\mathbb{C}^*/_{\sim} \simeq [0, 2\pi)$

<u>Sol:</u>

- Reprezentarea nr. complexe sub formă trigonometrică:

$(\forall)\, z \in \mathbb{C}\ (\exists!)\ \varphi \in [0, 2\pi)$ și $(\exists!)\, r \in [0, +\infty)$ a.î.

$$z = r(\cos\varphi + i \sin\varphi)$$



$z = a + bi$

$r = \sqrt{a^2+b^2} = |z|$

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "\pi"] \arrow[d, "f"'] & \mathbb{C}^*/_\sim \arrow[dl, dashed, "(\exists!)\bar{f}"] \\ {[0,2\pi)} & \end{tikzcd}$$

$f(z) := \arg(z)$

- f e surjectivă

(Fie $\varphi \in [0, 2\pi)$, $\varphi = 0$, $f(z) = 0$; $\varphi \neq 0$, $f(\cos\varphi + i\sin\varphi) = \varphi$)

- $\rho_f \overset{??}{=} \sim$

Fie $z_1, z_2 \in \mathbb{C}^*$. Atunci:

$z_1\, \rho_f\, z_2 \Leftrightarrow f(z_1) = f(z_2) \Leftrightarrow \arg(z_1) = \arg(z_2) \Leftrightarrow z_1 \sim z_2 \Rightarrow$

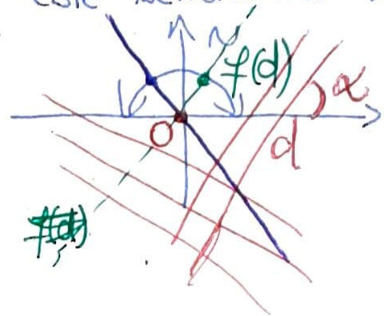
$\Rightarrow \rho_f = \sim$

$\Rightarrow (\exists!)\ \bar{f} : \mathbb{C}^*/_\sim \xrightarrow{\sim} [0, 2\pi)$, $\bar{f}(\hat{z}) := \arg(z)$, $(\forall)\, \hat{z} \in \mathbb{C}^*/_\sim$

și $\bar{f}$ e <u>bijectivă</u> căci f e surj și $\rho_f = \sim$.

22) Fie $\mathcal{D}$:= mulțimea dreptelor dintr-un plan fixat și pe $\mathcal{D}$ relația: $d_1 \sim d_2 \overset{def}{\Longleftrightarrow} \begin{cases} d_1 = d_2 & \text{sau} \\ d_1 \cap d_2 = \emptyset \end{cases}$

Arătați că $\mathcal{D}/_{\sim} \simeq B$, unde B este semicercul unitate



$$\begin{tikzcd} \mathcal{D} \arrow[r, "\overline{\pi}"] \arrow[d, "f"'] & \mathcal{D}/_{\sim} \arrow[dl, dashed, "(\exists!)\overline{f}"] \\ B & \end{tikzcd}$$

$f(d)$:= intersecția lui B cu paralela la d ce trece prin O

(Valentin: $f(d) = \cos(\operatorname{arctg} m_d) + i \sin(\operatorname{arctg} m_d) =$
$= \cos(\alpha) + i\sin(\alpha)$)

- $\boxed{f \text{ e surjectivă}}$ ok.
- $\rho_f \overset{??}{=} \sim$; $d_1, d_2 \in \mathcal{D}$ $\quad d_1 \rho_f d_2 \Leftrightarrow \underline{f(d_2) = f(d_2)} \Leftrightarrow$

$\Leftrightarrow \begin{cases} d_1 \parallel d_2 \\ \text{sau} \\ d_1 = d_2 \end{cases} \Leftrightarrow d_1 \sim d_2 \Rightarrow \boxed{\rho_f = \sim}$

DIN P.U.M.F $\Rightarrow (\exists!)\overline{f} : \mathcal{D}/_{\sim} \longrightarrow B$ a.î. $\overline{f}(\hat{d}) := f(B)$, $(\forall)\, \hat{d} \in \mathcal{D}/_{\sim}$ și $\overline{f}$ bijectivă (căci $\rho_f = \sim$ și f e surj.)

$\hat{d} := \{ f \in \mathcal{D} \mid f \parallel d \}$

dreaptă

24) (PRODUS DE RELAȚII) Fie $A_1, \dots, A_m$ mulțimi și $\rho_1, \dots, \rho_m$ relații de echivalență pe $A_1, \dots$ resp A_m.

Pe $A_1 \times \dots \times A_m$ definim relația:

$$(a_1, \dots, a_m) \sim (b_1, \dots, b_m) \overset{def}{\Leftrightarrow} a_i \, \rho_i \, b_i, (\forall) \, i = \overline{1, m}$$

Arătați că $\sim$ e relație de echivalență și

$$(A_1 \times \dots \times A_m)/_{\sim} \cong A_1/_{\rho_1} \times A_2/_{\rho_2} \times \dots \times A_m/_{\rho_m}.$$

în bijecție (izomorfe)

Sol:

$$\begin{tikzcd} A_1 \times A_2 \times \dots \times A_m \arrow[r, "\pi"] \arrow[d, "??? = f"'] & A_1 \times \dots \times A_m/_{\sim} \arrow[dl, dashed, "(\exists!)\, \overline{f}"] \\ A_1/_{\rho_1} \times A_2/_{\rho_2} \times \dots \times A_m/_{\rho_m} & \end{tikzcd}$$

Cine e f ???

$$f((a_1, \dots a_n)) := (\hat{a}_1^{\rho_1}, \hat{a}_2^{\rho_2}, \dots \hat{a}_n^{\rho_m})$$

$(\forall)(a_1, \dots, a_n) \in A_1 \times A_2 \times \dots \times A_m$

• $\boxed{\rho_f \overset{?}{=} \sim}$??? Fie $(a_1, \dots a_m), (b_1, \dots b_m) \in A_1 \times \dots \times A_n$

Atunci:

$(a_1, \dots, a_n) \rho_f (b_1, \dots, b_m) \Leftrightarrow f(a_1, \dots, a_n) = f(b_1, b_2 \dots b_m) \Leftrightarrow$

$\Leftrightarrow \hat{a}_i^{\rho_i} = \hat{b}_i^{\rho_i}, (\forall) \, i = \overline{1, m} \Leftrightarrow a_i \, \rho_i \, b_i, (\forall) \, i = \overline{1, m} \Leftrightarrow$

$\Leftrightarrow (a_1, \dots, a_m) \sim (b_1, \dots, b_m) \Rightarrow \boxed{\rho_f = \sim}$

• f e surjectivă!

$(\forall)\ z = (\hat{a}_1^{\rho_1}, \hat{a}_2^{\rho_2}, \ldots \hat{a}_m^{\rho_m}) \in A_1/\rho_1 \times \ldots \times A_m/\rho_m$ $(\exists)\ (a_1, \ldots a_n) \in A_1 \times \ldots \times A_n$ a.î.

$f(a_1, \ldots, a_n) = z$, i.e. f e surjectivă

$\Rightarrow \sim$ e relație de echivalență și

$(\exists!)\ \bar{f} : (A_1 \times \ldots \times A_n)/_\sim \longrightarrow A_1/\rho_1 \times \ldots \times A_n/\rho_m$

clasă = ~~λ-relația~~

a.î. $\bar{f}\left(\overline{(a_1, a_2 \ldots a_m)}\right) = \left(\hat{a}_1^{\rho_1}, \ldots\ldots, \hat{a}_m^{\rho_m}\right), (\forall)$

$\overline{(a_1, \ldots a_n)} \in (A_1 \times \ldots \times A_n)/_\sim$

25) Să se calculeze numărul tuturor relațiilor de echivalență care se pot defini pe o mulțime cu m elemente, $m \in \mathbb{N}^*$.

Sol. $E(m) :=$ numărul acestor relații.

$M =$ mulțime, $|M| = m$.

$\sim =$ rel. de echivalență pe M, $M \xrightarrow{\pi} M/_\sim$ surjectivă

$\pi(x) = \hat{x}, (\forall)\ x \in M$

Fie $e_j :=$ numărul tuturor relațiilor de echivalență pe M a.î. mulțimea factor are j elemente. $(\forall)\ j = \overline{1, m}$.

Atunci:

$$\boxed{E(m) = e_1 + e_2 + \ldots + e_m}$$

Vrem $e_j = ??$, pentru j fixat, $j \in \{1, \ldots, m\}$

$\{f : M \longrightarrow \{1, 2, \ldots j\} \mid f = \text{surjectivă}\}$ $\boxed{s(m, j)}$

vezi seminar 4

$\downarrow F \qquad \qquad f \mapsto \rho_f$

$R_j := \{\rho \subseteq M \times M \mid \rho$ relație de echivalență pe M a.î. $|M/_\rho| = j\}$.

$$\boxed{e_j = |R_j|}$$

$$F(f) := \rho_f \underset{??}{\in} R_j, \quad \boxed{|M/_{\rho_f}| = j}$$

$$\begin{tikzcd} M \arrow[r, "\overline{\pi}"] \arrow[d, "\text{surj } f"'] & M/_{\rho_f} \arrow[dl, dashed, "(\exists!)\overline{f}"] \\ \{1,\dots,j\} & \end{tikzcd}$$

ρ_f = rel prin care factorizez

$\overline{f}(\hat{x}) := f(x), (\forall)\hat{x} \in M/_{\rho_f}$

și $\overline{f}$ e bijectivă

$\Rightarrow \mathrm{Im}(F) \subseteq R_j$

- F e surjectivă?

Fie $\rho \in R_j$, $M \xrightarrow{\overline{\pi}} M/_\rho \overset{\theta}{\cong} \{1, 2, \dots j\}$

$\rho_f \qquad f := \theta \circ \overline{\pi}$

$F(f) \overset{?}{=} \rho \; (\Rightarrow F$ e surjectivă$)$

Fie $x, y \in M$. Atunci

$x \rho_f y \Leftrightarrow f(x) = f(y) \Leftrightarrow \theta(\overline{\pi}(x)) = \theta(\overline{\pi}(y)) \Leftrightarrow$

$\Leftrightarrow \overline{\pi}(x) = \overline{\pi}(y) \Leftrightarrow \hat{x} = \hat{y} \Leftrightarrow x \rho y \Rightarrow \rho_f = \rho \Rightarrow$

$\Rightarrow F(f) = \rho$

$\Rightarrow j!$ surjecții $M \to \{1, 2, \dots j\}$ induc prin M aceeași relație de echivalență!

$e_j = |R_j| = \dfrac{s(m,j)}{j!}, (\forall)\, j = \overline{1,m}$

$$\Rightarrow \boxed{E(m) = \sum_{k=1}^{m} e_k = \sum_{k=1}^{m} \frac{s(m,k)}{k!}} \Rightarrow$$

$$\boxed{E(m)=\sum_{k=1}^{m}\frac{1}{k!}\left(k^{m}-\sum_{i=1}^{k-1}(-1)^{i+1}C_k^{i}(k-i)^{m}\right)}$$

Problemă (Birkhoff): Numărați și $\boxed{\text{clasificați}}$ toate relațiile de ordine pe o mulțime cu m elemente?

$\boxed{O(m)}$

A, B sunt două mulțimi ordonate. O funcție $f: A \to B$ s.n. morfism dacă $x \leq y \Rightarrow f(x) \leq f(y)$, $\forall x, y \in A$.

În plus, $f: A \to B$ s.n. izomorfism dacă $(\exists)\ g: B \to A$ a.î. g = morfism de mulțimi ordonate și $f \circ g = id_B$ și $g \circ f = id_A$.

$\boxed{(A, \leq) \simeq (B, \leq)}$ $\boxed{O(m) = ??}$

Afirm: Fie $f, g: M \to \{1, 2, \dots j\}$ surjective

$\boxed{\bullet\ \rho_f = \rho_g \overset{?}{\Leftrightarrow} (\exists)\ \sigma \in \boxed{S_j}\ \text{a.î.}\ g = \sigma \circ f}$

$S_j = \{\sigma: \{1, \dots j\} \to \{1, \dots j\} \mid \sigma = \text{bijectivă}\}$

$|S_j| = j!$

"$\Leftarrow$" $g = \sigma \circ f$, $\sigma \in S_j$

$x \rho_g y \Leftrightarrow g(x) = g(y) \Leftrightarrow \sigma(f(x)) = \sigma(f(y)) \overset{\sigma}{\underset{bij.}{\Longleftrightarrow}}$

$f(x) = f(y) \Leftrightarrow x \rho_f y \Rightarrow \rho_f = \rho_g$.

"$\Rightarrow$" Pp. $\rho_f = \rho_g \overset{not.}{=} \rho$

$$\begin{tikzcd}
M \arrow[r, "\overline{\pi}"] \arrow[d, "f"'] \arrow[d, dashed, shift left=2, "g"] & M/\rho \arrow[dl, "(\exists!)\,\overline{f}"] \arrow[dl, dashed, shift right=2, "(\exists!)\,\overline{g}"'] \\
\{1,2,\dots j\}
\end{tikzcd}$$

$\rho_f = \rho$

$f = \underline{\text{surj}}.$

$\Rightarrow (\exists!)\ \overline{f} : M/\rho \xrightarrow{\sim} \{1,2,\dots j\}$ a.î. $\overline{f} \circ \overline{\pi} = f$ și $\overline{f}$ e <u>bijectivă</u> Analog

$(\exists)\ \overline{g} : M/\rho \xrightarrow{\sim} \{1,2,\dots j\}$ a.î. $\overline{g} \circ \overline{\pi} = g$ și $\overline{g}$ e <u>bijectivă</u>

$$\Rightarrow g = \overline{g} \circ \overline{\pi} = \underbrace{\overline{g} \circ (\overline{f})^{-1}}_{\text{not}} \circ f = \sigma \circ f$$

$\sigma \in S_j$

Sistem de reprezentanți (exercițiu)

$A := \mathbb{Z}$, $n \in \mathbb{N}^*$, $n \geq 2$

$x \equiv y \pmod{n} \overset{def}{\Leftrightarrow} n \mid x - y, (\forall)\, x, y \in \mathbb{Z}$

$S = \{0, 1, 2, \dots, n-1\}$ e sist. de reprezentanți pt $\equiv$

- $(\forall)\, i \neq j,\ i, j \in \{0, 1, \dots, n-1\} \overset{?}{\Rightarrow}$

  $i \not\equiv j \pmod{n}$

Pp. că $i \equiv j \pmod{n} \Rightarrow n \mid i - j$, <u>fals</u>!

$\underline{i - j} = n \cdot k,\ k \in \mathbb{Z}$, $\underline{|i-j| < n}$ <u>fals</u>!

- $(\forall)\, m \in \mathbb{Z}, (\exists)\, i \in S$ a.î. $m \equiv i \pmod{n}$

$$\boxed{\begin{array}{l}\text{Teorema de împărțire cu rest a numerelor} \\ (\forall)\, m, n \in \mathbb{Z},\ n \neq 0\ (\exists!)\, q, r \in \mathbb{Z} \text{ a.î. } m = n \cdot q + r \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq r < n\end{array}}$$

$\exists!\, q, r \in \mathbb{Z}$ a.î. $m = n \cdot q + r$, $r \in S$

$\Downarrow$

$\boxed{m \equiv r \pmod{n}} \Rightarrow \underline{m - r = nq}$

$\{0, 1, \dots, n-1\}$ = sistem de reprezentați pt. $\equiv$

SEMINAR 6 13.11.2020

Algebră

Setul 3 de probleme

1) $A = \{a_1, a_2, \dots, a_n\}$

Vrem: numărul legilor cu element neutru

Câte legi îl au pe a_1 element neutru?

φ	a_1	a_2	a_3	$\dots$	a_n
a_1	a_1	a_2	a_2	$\dots$	a_n
a_2	a_2	$\dots$			$\dots$
$\vdots$	$\vdots$				
a_n	a_n	$\dots$			

$n^{(n-1)^2}$ legi cu a_1 ca element neutru

$n \cdot n^{(n-1)^2} = n^{(n-1)^2+1}$ legi care au element neutru

3) M monoid, $x \in M \Rightarrow (\exists!)$ morfism de monoizi $f_x : (\mathbb{N}, +) \to M$, a.î. $f_x(1) = x$. În plus, $f_x(n) = x^n, \forall n \in \mathbb{N}$.

<u>SOLUȚIE</u>: Fie $f_x : (\mathbb{N}, +) \to M$, $f_x(n) = x^n, \forall n \in \mathbb{N}$.

- f_x e morfism de monoizi

$f_x(0) = x^0 \overset{def}{=} 1 = 1_M$

Fie $m, n \in \mathbb{N}$. $f_x(m+n) = x^{m+n} = x^m \cdot x^n = f_x(m) \cdot f_x(n) \Rightarrow f_x$ e morfism de monoizi

- <u>unicitatea lui f_x</u>

Fie $h:(\mathbb{N},+) \longrightarrow M$ morfism de monoizi a.î. $h(1)=x$.

$h(0)=1_M$, $h(1)=x$, $h(2)=h(1+1) \overset{h \text{ morf.}}{=\!=\!=} h(1)\cdot h(1)=x^2$

Prin inducție arătăm că $h(n) \overset{?}{=} x^n$, $\forall\, n\in\mathbb{N}$.

$n=1$, $n=2$ ok.

$n \longmapsto n+1$

$h(n+1) \overset{h\text{ morf.}}{=\!=\!=} h(n)\cdot h(1)=x^n\cdot x = x^{n+1}$

$\Rightarrow h=f_x \Rightarrow f_x$ unic

Consecință: $\exists$ o bijecție:

$\varphi: \underbrace{\mathrm{Hom}_{\text{monoizi}}}_{\text{mulțimea tuturor morfismelor de monoizi de la } (\mathbb{N},+) \text{ la } M}((\mathbb{N},+),M) \xrightarrow{\sim} M$, $f \longrightarrow f(1)$

$\boxed{(\varphi(f)=f(1))}$

($\forall\, x\in M\ \exists!\ f_x$ morfism de monoizi a.î. $\varphi(f_x)=x$)

(arată că f e bijectivă)

4) Fie $a,b,c\in\mathbb{Z}$, $b\neq 0$ și legea de compoziție pe $\mathbb{Z}$

$[x*y:=axy+b(x+y)+c]$, $(\forall)\ x,y\in\mathbb{Z}$

Arătați că $M_{a,b,c}:=(\mathbb{Z},*)$ este monoid $\Leftrightarrow b=b^2-ac$ și $b|c$. În acest caz, arătați că $(\exists)$ un izomorfism de monoizi $M_{a,b,c}\simeq M_{a,1,0}$.

$\boxed{\text{SOL}}$: Când $*$ e asociativă?

Fie $x,y,z\in\mathbb{Z}$

$(x*y)*z - x*(y*z)=0.$

ii) $\underline{(x*y)*z - x*(y*z)} =$

$= (axy+b(x+y)+c)*z - x*(ayz+b(y+z)+c) =$

$= a(axy+b(x+y)+c)z + b(axy+b(x+y)^{+c}+z) + \cancel{c} -$

$- ax(ayz+b(y+z)+c) - b(x+ayz+b(y+z)+c) - \cancel{c} =$

$= a(\cancel{axyz} + \cancel{bxz} + \cancel{byz} + cz - \cancel{ayzx} - \cancel{xby} - \cancel{xbz} - xc) +$

$+ b(\cancel{axy} + bx + \cancel{by} + \cancel{z}^{+c} - x - \cancel{ayz} - \cancel{by} - bz - \cancel{c}) =$

$= b^2x + acz - axc - b^2z + bz - bx =$

$= ac(z-x) + b^2(x-z) + b(z-x) =$

$= \underline{(z-x)(ac-b^2+b)} \Rightarrow *$ e asociativă $\Leftrightarrow (z-x)(ac-b^2+b)=0$
$(\forall) x,z \in \mathbb{Z}$

$\Leftrightarrow ac-b^2+b=0$

„$\Leftarrow$" ok.
„$\Rightarrow$" $z=1, x=0 \Rightarrow$ ok

În concluzie, $*$ este asociativă $\Leftrightarrow \underline{\underline{b=b^2-ac}}$

Pp. că $\boxed{b=b^2-ac}$ Când are element neutru?

Caut $e \in \mathbb{Z}$ a.î. ??

$x*e = e*x = x, (\forall) x \in \mathbb{Z}$??

$axe + b(x+e) + c = x, (\forall) x \in \mathbb{Z}$

$\Leftrightarrow \underline{(ae+b-1)x + c + be = 0}, (\forall) x \in \mathbb{Z}$

$\overset{??}{\Leftrightarrow} b | c$ $\qquad \Updownarrow \begin{cases} c+be=0 \\ ae+b-1=0 \end{cases} \Leftrightarrow \boxed{b|c}$

„$\Rightarrow$" Facem $\underline{x=0} \Rightarrow c=-be \Rightarrow b|c$

$\boxed{e := \frac{c}{-b} = -\frac{c}{b}}$, $\underline{b \neq 0}$

4 $\Longleftarrow^{4}$ Pp. că $b \mid c \Rightarrow (\exists)\, d \in \mathbb{Z}$ a.î. $\boxed{c = bd} \Rightarrow$

$$b = b^2 - abd \quad (b \neq 0) \Rightarrow 1 = b - ad \Rightarrow \boxed{b = 1 + ad}$$

Fie $x \in \mathbb{Z}$. Atunci

$$(ae + \cancel{1} + ad - \cancel{1})x + (1+ad)d + (1+ad)e \overset{??}{=} 0$$

$$a(e+d)x + (1+ad)(d+e) = 0$$

$$\Leftrightarrow (e+d)(ax + 1 + ad) = 0, \ (\forall)\, x \in \mathbb{Z}$$

Alegem $\boxed{e = -d} = \boxed{-\frac{c}{b}}$

$\Rightarrow$ OK prima parte $\left[M_{a,b,c} := (\mathbb{Z}, *) \Rightarrow \text{monoid} \right]$

Caut $f : \underset{\mathbb{Z}}{\overset{\|}{M_{a,b,c}}} \xrightarrow{\sim} \underset{\mathbb{Z}}{\overset{\|}{M_{a,1,0}}}$ izomorfism de monoizi

$$x *_1 y = axy + b(x+y) + c =$$
$$= axy + (1+ad)(x+y) + (1+ad)d$$

$$x *_2 y = axy + x + y$$

TEMĂ

[Căutați $f : \mathbb{Z} \to \mathbb{Z}$ bijectivă a.î. $f(-d) = 0$, $f(x *_1 y) = f(x) *_2 f(y)$, $\forall\, x, y \in \mathbb{Z}$]

$f(x) := x + \alpha,\ \alpha \in \mathbb{Z}$? $\to$ ce fel de α $\qquad -d + \alpha = 0$, $\alpha = d$

$\boxed{f(x) := x + d}$ e bijectivă, $f(-d) = 0$

TEMĂ $\boxed{f(x *_1 y) = f(x) *_2 f(y),\ (\forall)\, x, y \in \mathbb{Z}\ !!!}$

$\boxed{\text{Ex 5}}$. Fie A o mulțime nevidă și $\mathcal{P}(A)$ mulțimea părților lui A. Arătați că $(\mathcal{P}(A), \Delta)$ e grup abelian, unde

$$X \Delta Y \overset{def}{:=} (X \setminus Y) \cup (Y \setminus X), (\forall) X, Y \in \mathcal{P}(A)$$

EXAMEN/LUCRARE

$\boxed{\text{SOL}}$:

Vrem : $(\mathcal{P}(A), \Delta)$ este grup abelian?

1. asociativitatea : $(X \Delta Y) \Delta Z = X \Delta (Y \Delta Z), \forall X, Y, Z \subseteq A$

$$((X \setminus Y) \cup (Y \setminus X)) \Delta Z = \underbrace{(\underbrace{((X \setminus Y) \cup (Y \setminus X)) \setminus Z}_{\overset{not.}{=} M_1}) \cup (Z \setminus \underbrace{((X \setminus Y) \cup (Y \setminus X))}_{\overset{not.}{=} M_2}))}_{\overset{not.}{=} M}$$

$$X \Delta (Y \Delta Z) = X \Delta ((Y \setminus Z) \cup (Z \setminus Y)) = \underbrace{(X \setminus ((Y \setminus Z) \cup (Z \setminus Y))) \cup (((Y \setminus Z) \cup (Z \setminus Y)) \setminus X)}_{\overset{not.}{=} N}$$

$M \overset{?}{=} N$

Fie $m \in M \Rightarrow m \in M_1$ sau $m \in M_2$

Caz I.

Temă: Arătați egalitatea de mulțimi cu funcția caracteristică

$$f_M = f_N$$

$$f_{A \Delta B} \overset{(*)}{=} f_{(A \setminus B) \cup (B \setminus A)}(X) = f_{(A \setminus B)}(X) + f_{(B \setminus A)} \overset{(*)}{-} f_{(A \setminus B) \cap (A \setminus B)}^{(*)}$$

(Alexandra → cu funcția caracteristică)

2. Elementul neutru este $\emptyset$

$$1_{\mathcal{P}(A)} = \emptyset$$

$$X \Delta X = \emptyset = 1_{\mathcal{P}(A)} \Rightarrow X \text{ e inversabil} \Rightarrow X^{-1} = X$$

Scufundări ale unui semigrup într-un grup

$\underset{\text{monoid}}{\underline{(\mathbb{N},+)}} \overset{i}{\subseteq} (\mathbb{Z},+) = \text{grup}$, i.e. $i(x)=x, (\forall)\, x\in\mathbb{N}$

Când un semigrup S se poate "scufunda" într-un grup G: i.e. când $(\exists)\, G=$ grup și $f: S \longrightarrow G$ morfism injectiv de semigrupuri $S \simeq \mathrm{Im}(f) \subset G$

Ex 6) a) $S=$ semigrup $\Rightarrow$ S se scufundă într-un monoid

b) $(\mathbb{Z}_4, \cdot)$ e monoid care $\boxed{\text{nu}}$ se scufundă într-un grup

Sol:

a) $S=$ semigrup. • Dacă S e chiar monoid OK

$M=$ monoid $:= S$; $i: S \longrightarrow M = S$, $i(x)=x$ e morf. bijectiv de semigrupuri/monoizi.

•• Dacă S $\boxed{\text{nu}}$ e monoid. Fie $\infty \notin S$ și $M:=S\cup\{\infty\}$

și pe M definim:

$$\underset{M}{\overset{\cap}{x}} * \underset{M}{\overset{\cap}{y}} := \begin{cases} xy, & \text{dacă } x,y\in S \\ \infty, & \text{dacă } x=y=\infty \\ x*\infty=\infty*x=x, & \text{dacă } x\in S \end{cases}$$

$(M,*,\infty)$ e monoid, $1_M=\infty$ și $*$ e asociativă

$x*(y*z)=(x*y)*z$, $(\forall)\, x,y,z\in M$.

$x=\infty$, $y*z = y*z$ ------

$i: S\longrightarrow M:=S\cup\{\infty\}$, $i(s):=s$, $(\forall)\, s\in S$ e morfism injectiv de semigrupuri:

$s_1, s_2 \in S \Rightarrow i(s_1 s_2) \overset{?}{=} i(s_1) * i(s_2)$

$$\underset{s_1 s_2}{\parallel} = \underset{\underline{\underline{s_1 * s_2}}}{\parallel}$$

b) $(\mathbb{Z}_4, \cdot)$ = monoid

$\hat{x} \cdot \hat{y} := \widehat{xy}$, $(\forall)\ \hat{x}, \hat{y} \in \mathbb{Z}_4$, $1_{\mathbb{Z}} = \hat{1}$.

Presupunem că $(\exists)$ G = grup și $f : (\mathbb{Z}_4, \cdot) \hookrightarrow G$ morfism $\boxed{\text{injectiv}}$ de monoizi

$\hat{3} \cdot \hat{2} = \hat{6} = \hat{2}$ și $\hat{1} \cdot \hat{2} = \hat{2} \Rightarrow \hat{3} \cdot \hat{2} = \hat{1} \cdot \hat{2} \Longrightarrow$

$f(\hat{3}) \cdot f(\hat{2}) = f(\hat{1}) f(\hat{2})$ în G = grup $\Longrightarrow$

$| f(\hat{2})^{-1}$ $| f(\hat{2})^{-1}$

$f(\hat{3}) = f(\hat{1}) \Rightarrow \hat{3} = \hat{1}$, fals!

$p, m, n \in \mathbb{N}$ $m + n = p + n \Rightarrow \underline{\underline{m = p}}$

$\boxed{\text{Ex 7}}$. Fie S = semigrup $\boxed{\text{comutativ}}$. Atunci S se poate scufunda într-un grup $\Leftrightarrow$ S este "semigrup cu simplificare" (i.e. $ax = ay \Rightarrow x = y$)

Sol: "$\Rightarrow$" Fie G = grup și $f : S \to G$ e morfism injectiv de semigrupuri. Fie $a, x, y \in S$ a.î. $ax = ay \xrightarrow[f]{\text{Aplic}}$

$f(a) f(x) = f(a) f(y)$ în G = grup $\Longrightarrow$

$$\underbrace{f(a)^{-1} f(a)}_{1} f(x) = \underbrace{f(a)^{-1} f(a)}_{1} f(y) \Rightarrow f(x) = f(y) \Longrightarrow \boxed{x = y}$$

Deci

$ax = ay \Rightarrow x = y \Rightarrow S$ este "semigrup cu simplificare".

„$\Leftarrow$" $S = \boxed{\text{semigrup}}$ cu simplificare

$G := S \cup \{\infty\} \cup \bar{S}$, $\bar{S}$ e copie dijunctă a lui S

$\bar{S} = \{\bar{s} \mid s \in S\}$. Fie $g, h \in G$

$$g * h := \begin{cases} gh, & \text{dacă } g, h \in S \\ g * \infty = \infty * g = g, & (\forall)\, g \in S \\ s \cdot \bar{s} = \infty = \bar{s} \cdot s \\ s \cdot \bar{t} = ??? & \boxed{s \neq t} \end{cases}$$

//

..

Vreau: un grup, îl voi $\boxed{\text{nota}}$ $Q(S) = \text{grup}$

$f : S \longrightarrow Q(S)$, morfism injectiv de semigrupuri.

Fie $S \times S$; pe ea definim relația de echivalență.

$(a,b) \sim (c,d) \Leftrightarrow ad = bc, (\forall)\ (a,b), (c,d) \in S \times S$.

„$\sim$" e relație de echivalență pe $S \times S$ (Ex!) = banal

$\widehat{(a,b)} = \{(c,d) \in S \times S \mid ad = bc\} \overset{\text{not.}}{=} \frac{a}{b}$ $\frac{\text{numărător}}{\text{numitor}}$

Fie $\boxed{G \overset{\text{not}}{=} Q(S) :\overset{\text{def}}{=} S \times S/_{\sim}}$. Pe ea definim legea de compoziție:

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd}} \quad (\forall)\ \frac{a}{b}, \frac{c}{d} \in S \times S/_{\sim}$$

e comutativă

• legea e corect definită.

$$\frac{a}{b} = \frac{a'}{b'}, \quad \frac{c}{d} = \frac{c'}{d'} \overset{?}{\Rightarrow} \frac{a}{b} \cdot \frac{c}{d} = \frac{a'}{b'} \cdot \frac{c'}{d'}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \text{„}$$

$$\boxed{ab' = ba'} \qquad \boxed{cd' = dc'} \qquad \underbrace{\frac{ac}{bd}}_{\widehat{(ac,bd)}} \overset{?}{=} \underbrace{\frac{a'c'}{b'd'}}_{\widehat{(a'c', b'd')}}$$

Vreau: $ac'b'd' \overset{?}{=} a'c'bd$

- lege e asociativă (Ex!) $\Rightarrow$ $Q(S)$ e semigrup

Fie $s_0 \in S$ fixat. $\boxed{1_{Q(S)} := \frac{s_0}{s_0}}$ e element neutru?

$$\frac{a}{b} \cdot \frac{s_0}{s_0} = \frac{as_0}{bs_0} \overset{?}{=} \frac{a}{b} = \frac{s_0}{s_0} \cdot \frac{a}{b}$$

$$\overbrace{(as_0, bs_0)}^{\shortparallel} \overset{?}{=} \overbrace{(a,b)}^{\shortparallel}$$

$(as_0, bs_0) \sim (a,b) \Leftrightarrow \underline{as_0b} = \underline{bs_0a}$

$\Rightarrow \boxed{\frac{s_0}{s_0} = \frac{t}{t}}, (\forall)\, t \in S$

Fie $\frac{a}{b} \in Q(S) \overset{?}{\Longrightarrow} \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

$$\left(\frac{a}{b}\right)\cdot\left(\frac{b}{a}\right) = \frac{ab}{ba} = \frac{ab}{ab} = \frac{s_0}{s_0} = 1_{Q(S)} \Rightarrow$$

$\Rightarrow \left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \Rightarrow (Q(S), \cdot) = \boxed{\text{grup}}$

- Definim $f: S \to Q(S)$, $f(a) := \frac{as_0}{s_0}$, $(\forall)\, a \in S$

f e morfism de semigrupuri (Ex!) și $f \overset{?}{=}$ injectiv

Fie $a, b \in S$ a.î. $f(a) = f(b) \Rightarrow$

$$\frac{as_0}{s_0} = \frac{bs_0}{s_0} \Rightarrow a(\underbrace{s_0^2}_{\alpha}) = b(\underbrace{s_0^2}_{\alpha}) \xrightarrow[\text{simp.}]{S \text{ e cu}} a = b \text{ i.e. } f \text{ inj}$$

Întrebare: Mai rămâne adevărată concluzia dacă S nu e comutativ?

$f: A \times A \longrightarrow A, \ A \neq \emptyset$

A = finită, $A = \{a_1, a_2, \ldots, a_n\}$

$f = *$	a_1	a_2	$\ldots$	a_n
a_1	$a_1 * a_1$	$a_1 * a_2$		
a_2				
$\vdots$				
a_n				

tabla de compoziție (tabla lui Cayley)

$f(a_i, a_j) \overset{not.}{=} a_i * a_j$

$|A| = n \Rightarrow$ sunt $\boxed{n^{n^2} \text{ legi de compoziție}}$

/ $\boxed{\text{SEMINAR 7}}$ 20.11.2020

Algebră

Ex. $(\mathbb{Z},+)$; $\boxed{H \leq (\mathbb{Z},+)} \iff (\exists)\ n \in \mathbb{N}$ a.î. $\boxed{H = n\mathbb{Z}} = \{nx \mid x \in \mathbb{Z}\}$.

SOL:

„$\Leftarrow$" Pp. că $\underline{H = n\cdot\mathbb{Z} \overset{?}{\leq} \mathbb{Z}}$

Vrem: $(\forall)\, x, y \in H \overset{?}{\Rightarrow} x - y \in H$

$x = n\cdot t$; $y = n\cdot q \Rightarrow x - y = n(t-q) \in H \Rightarrow n\mathbb{Z} \leq \mathbb{Z}$.

„$\Rightarrow$" Fie $H \leq (\mathbb{Z},+)$. Dacă $H = \{0\}$ ok.

(alegem $n = 0$; $\{0\} = 0\cdot\mathbb{Z}$) Pp. $\underline{H \neq \{0\}}$

$(\exists)\ h \in H,\ h > 0$ ($x \in H$; $x < 0$) $\Rightarrow -x \in H$; $-x > 0$

Fie $n \in \mathbb{N}^*$ cel mai mic număr pozitiv din H.

$n \in H \underset{(H \leq \mathbb{Z})}{\Rightarrow} \underline{nx \in H, (\forall)\ x \in \mathbb{Z}} \Rightarrow (*) \Rightarrow \boxed{n\cdot\mathbb{Z} \subseteq H}$

A rămas: $H \overset{?}{\subseteq} n\cdot\mathbb{Z}$

Fie $\underline{h \in H} \subseteq \mathbb{Z} \xrightarrow{T.Î.R.} (\exists)\ q, r \in \mathbb{Z}$ a.î. $h = n\cdot q + r$; $0 \leq r < n$.

Dacă $r \neq 0 \Rightarrow r = \underbrace{\underbrace{h}_{\in H} - \underbrace{nq}_{\in H}}_{\in H} > 0$

$r < n$, fals! căci n era minim.

$\Rightarrow \boxed{r = 0} \Rightarrow h = nq$ i.e. $\boxed{H \subseteq n\cdot\mathbb{Z}} \overset{(*)}{\Rightarrow} \boxed{H = n\cdot\mathbb{Z}}$

$n, m \in \mathbb{N}$, $n\mathbb{Z} = m\mathbb{Z} \Rightarrow n = m$ (EX!)

Ex. (Fac): Găsiți toate mulțimile lui $\mathbb{Z} \times \mathbb{Z}$

$$(a,b) + (c,d) := (a+c, b+d), (\forall) \ldots$$

8) a) Arătați că $f: M_1 \to M_2$ este un <u>morfism</u> <u>injectiv</u> de monoizi $\Leftrightarrow$ f este <u>monomorfism</u> de monoizi.

b) Arătați că $i: (\mathbb{Z}, \cdot) \to (\mathbb{Q}, \cdot)$, $i(x) = x, (\forall) x \in \mathbb{Z}$ este <u>epimorfism de monoizi</u> care <u>nu</u> e surjectiv.

<u>SOL</u>:

a) „$\Rightarrow$" Pp. că $f: M_1 \to M_2$ e morfism injectiv de monoizi.
Vreau: f e monomorfism??

$$M: \underset{v}{\overset{u}{\rightrightarrows}} M_1 \xrightarrow{f} M$$

Fie M = monoid; u, v morfism de monoizi a.î. $f \circ u = f \circ v$.
Vreau: $\underline{u = v}$??

$f \circ u = f \circ v \Rightarrow f(u(m)) = f(v(m)); \forall m \in M \Rightarrow u(m) = v(m)$; $(\forall) m \in M$; i.e. $\boxed{u = v}$.

„$\Leftarrow$" Pp. că $f: M_1 \to M_2$ e monomorfism.
<u>Vreau</u>: $f = \underline{inj}$? Pp. că f $\boxed{\text{nu}}$ e injectiv.
$\Rightarrow (\exists)$ $\underline{a \neq b \in M_1}$ și $\underline{f(a) = f(b)}$

Fie $M = (\mathbb{N}, +)$

$$\mathbb{N} \underset{v}{\overset{u}{\rightrightarrows}} M_1 \xrightarrow{f} M, \quad u(n) := a^n$$
$$v(n) := b^n$$

$(\forall)\ n \in \mathbb{N}$; u și v sunt morf. de monoizi (Seminarul trecut)

$u \neq v$, $\underline{u(1) = a} \neq b = \underline{v(1)}$

$\underline{f \circ u \overset{?}{=} f \circ v}$ $(\Rightarrow f$ $\boxed{nu}$ e mono, fals!)

$\underline{(f \circ u)}(n) = f(u(n)) = f(a^n) = \underline{f(a)}^n = \underline{\underline{f(b)}}^n = f(b^n) =$

$= f(v(n)) = \underline{\underline{(f \circ v)}}(n)$, $(\forall)\ n \in \mathbb{N} \Rightarrow f \circ u = f \circ v$.

b) $i : (\mathbb{Z}, \cdot) \hookrightarrow (\mathbb{Q}, \cdot)$, $i(m) := m$, $(\forall)\ m \in \mathbb{Z}$

i = morfism de monoizi, i = $\underline{\underline{nu}}$ e surjectiv

$\left(\frac{1}{2} \notin \operatorname{im}(i)\right)$

$\underline{\text{Afirm}}$: i e epimorfism de monoizi?

$$\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q} \underset{v}{\overset{u}{\rightrightarrows}} M, \quad i(m) := m, (\forall)\ m \in \mathbb{Z}$$

$\underline{\text{Vreau}}$: $(\forall)\ M = \underline{\text{monoid}}$, $(\forall)\ u, v : \mathbb{Q} \to M$ morfisme de monoizi a.î. $\underline{u \circ i = v \circ i} \Rightarrow u = v$??

$\Downarrow$

$\underline{u(m) = v(m), (\forall)\ m \in \mathbb{Z}}$

$\underline{\text{Vreau}}$: $u\left(\frac{p}{q}\right) \overset{??}{=} v\left(\frac{p}{q}\right)$, $\forall\ p, q \in \mathbb{Z}$, $q \neq 0$.

$u\left(\frac{1}{q}\right) \overset{??}{=} v\left(\frac{1}{q}\right)$, $(\forall)\ q \in \mathbb{Z}^*$

$\underline{\underline{1_M}} = u(1) = u\left(q \cdot \frac{1}{q}\right) = \underline{u(q)} \cdot u\left(\frac{1}{q}\right)$

$u(1) = u\left(\frac{1}{q} \cdot q\right) = u\left(\frac{1}{q}\right) \cdot \underline{u(q)}$

$\Rightarrow u(q)$ e inversabil și $u\left(\frac{1}{q}\right) = u(q)^{-1}$.

Analog, $v\left(\frac{1}{q}\right) = v(q)^{-1} = u(q)^{-1} = u\left(\frac{1}{q}\right), (\forall) q \in \mathbb{Z}^*$

$\Rightarrow u\left(\frac{1}{q}\right) = v\left(\frac{1}{q}\right), \forall q \in \mathbb{Z}^*$

$$u\left(\frac{p}{q}\right) = u\left(p \cdot \frac{1}{q}\right) = u(p) \cdot u\left(\frac{1}{q}\right) = v(p) \cdot v\left(\frac{1}{q}\right) \overset{(v\,\text{morf})}{=} v\left(\frac{p}{q}\right) \Rightarrow$$

$\Rightarrow u(r) = v(r), (\forall) r \in \mathbb{Q}$ i.e. $u = v$

Întrebare: Ce se întâmplă cu semigrupurile / mono / epi?

Ex 9) Fie $n \in \mathbb{N}^*$. Dați exemple de un monoid, care nu e grup, finit (resp. infinit) care are exact n elemente inversabile.

SOL: M_1 și M_2 = monoizi

$M_1 \times M_2$ e monoid cu $(m_1, m_2) \cdot (m_1', m_2') := (m_1 m_1', m_2 m_2')$

$(1_{M_1}, 1_{M_2}) = 1_{M_1 \times M_2}$

$$\boxed{U(M_1 \times M_2) \overset{?}{=} U(M_1) \times U(M_2)}$$

$x = (m_1, m_2) \in U(M_1 \times M_2) \overset{?}{\Leftrightarrow} m_1 \in U(M_1)$ și $m_2 \in U(M_2)$

$(m_1, m_2) \in U(M_1 \times M_2) \overset{def}{\Leftrightarrow} (\exists)(n_1, n_2) \in M_1 \times M_2$ a.î.

$(m_1, m_2)(n_1, n_2) = (1_{M_1}, 1_{M_2}) = (n_1, n_2)(m_1, m_2)$

$\Leftrightarrow m_1 n_1 = n_1 m_1 = 1_{M_1}$ și $m_2 n_2 = n_2 m_2 = 1_{M_2}$

$\Leftrightarrow m_1 \in U(M_1)$ și $m_2 \in U(M_2)$

$\boxed{U((\mathbb{N},+)) = \{0\}}$, $(\mathbb{Z}_2, \cdot)$, $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$

$\hat{0}\cdot\hat{1} = \hat{0}$, $\hat{1}\cdot\hat{1} = \hat{1}$

$\boxed{U(\mathbb{Z}_2) = \{\hat{1}\}}$

$(\mathbb{Z}_n, +)$, $\hat{x} + \hat{y} := \widehat{x+y}$, e $\boxed{\text{grup}}$ cu n elemente

$\boxed{U(\mathbb{Z}_n) = n}$

Fie $M := (\mathbb{N}, +) \times (\mathbb{Z}_n, +)$ produs direct de monoizi,

$\boxed{|U(M)| = n}$

$U(M) = \{(0, \hat{x}) \mid \hat{x} \in \mathbb{Z}_n\}$.

$\boxed{(m\mathbb{N}_2 + 1, \cdot) \not\cong (n\mathbb{N} + 1, \cdot)}$

Fie $M' := (\mathbb{Z}_2, \cdot) \times (\mathbb{Z}_n, +)$

$U(M') = \{(\hat{1}, \hat{\hat{x}}) \mid \hat{\hat{x}} \in \mathbb{Z}_n\} \Rightarrow |U(M')| = \underline{n}$

10. a) Arătați că monoizii $(M_2\mathbb{Z}, \cdot)$ și $(M_3\mathbb{Z}, \cdot)$ sunt neizomorfi.

Generalizare

b) Arătați că monoizii $(\mathbb{N}^*, \cdot)$ și $(3\mathbb{N}+1, \cdot)$ sunt neizomorfi.

SOL: a) MOODLE UB

b) Monoizii $\boxed{(\mathbb{N}^*, \cdot) \not\cong (3\mathbb{N}+1, \cdot)}$ (neizomorfi!)

$3\mathbb{N}+1 = \{3k+1 \mid k \in \mathbb{N}\}$

Pp. că $(\exists)$ $f : \underline{\mathbb{N}^*} \longrightarrow \underline{3\mathbb{N}+1}$ $\boxed{\text{izo}}$ de monoizi

$\Rightarrow \boxed{f(1) = 1}$ f izo $\Rightarrow$ f e surjectiv

$\underline{4, \boxed{10}, 25 \in 3\mathbb{N}+1}$ $(4 \cdot 25 = 10^2)$

$\Rightarrow (\exists)\ x, y, z \in \underline{\mathbb{N}}^*$ a.î. $\underline{f(x)=4,\ f(y)=25},\ \underline{f(z)=10}$

[Afirm: x, y, z sunt nr. prime

Pp. că x nu nr. prim $\Rightarrow \boxed{x = ab,\ a, b \geq 2}$

$\Rightarrow 4 = f(x) = \underbrace{f(a)}\,\underbrace{f(b)}_{\in 3\mathbb{N}+1}$ $\Rightarrow$

$(2 \notin 3\mathbb{N}+1)$

$\underline{f(a)=1 \text{ și } f(b)=4}$ sau $\underline{f(a)=4 \text{ și } f(b)=1}$

$f(a) = 1 = f(1) \xRightarrow{f \text{ inj.}} a = 1$ fals!

$f(b) = 1 = f(1) \longrightarrow b = 1$ fals!

$\Rightarrow x = \underline{\text{prim}}$. Analog y și z $\underline{\text{sunt prime}}$ (Ex!)

$\underline{\underline{4}} \cdot \underline{\underline{25}} = 10^2 \Rightarrow f(x)f(y) = f(z)^2 \xRightarrow{f \text{ surj.}}$

$f(xy) = f(z^2) \xRightarrow[f \text{ inj.}]{} \boxed{xy = z^2}$, x, y, z = nr. prime

$\Rightarrow \underline{x = y = z} \Rightarrow f(x) = f(y) \Rightarrow 4 = 25$ fals!

$\Rightarrow$ monoizii NU sunt izomorfi.

13) Fie $a, b \in \mathbb{Q}$ și $\mu_{a,b} : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q} \times \mathbb{Q}$

$\mu_{a,b}((x,y)) := (x+a, by)$, $(\forall)\ (x,y) \in \mathbb{Q} \times \mathbb{Q}$

Arătați că $H = \{\mu_{a,b} \mid a \in \mathbb{Q},\ b \in \mathbb{Q} \setminus \{0\}\}$ este un subgrup în grupul de permutări $\Sigma_{\mathbb{Q}\times\mathbb{Q}} = S_{\mathbb{Q}\times\mathbb{Q}}$.

SOL: Vreau: $H \leq \Sigma_{\mathbb{Q}\times\mathbb{Q}}$

• parte stabilă?

Fie $(\mu_{a,b} \circ \mu_{c,d})(x,y) = \mu_{a,b}(x+c, dy) =$

$= (x+c+a, dby) = \mu_{c+a,db}(x,y) \Rightarrow \mu_{a,b} \circ \mu_{c,d} = \mu_{c+a,db} \Rightarrow$

$\Rightarrow H$ parte stabilă.

• $\mathbb{1} = id_{\mathbb{Q}\times\mathbb{Q}} = \mu_{0,1} \Rightarrow 1 \in H$

• $\mu_{a,b} \in H \Rightarrow \mu_{a,b^{-1}} \in H$?

$\mu_{(a,b)}^{-1} = \mu_{(-a,\frac{1}{b})} \Rightarrow H \leq \Sigma_{\mathbb{Q}\times\mathbb{Q}}$

H- comutativ

14) Fie $(G,\cdot)$ un semigrup a.î.

• $(\exists)\ e \in G$ a.î. $ex = x, (\forall)\ x \in G$

• $(\forall)\ x \in G\ (\exists)\ x' \in G$ a.î. $x'x = e$

Atunci $(G,\cdot)$ este grup.

SOL: Fie $x \in G \Rightarrow \exists\ x' \in G$ a.î. $x' \cdot x = e \Rightarrow \exists\ x \in G$ a.î.

$x'' \cdot x' = e$

$x \cdot x' = e(x \cdot x') = x'' \cdot \underbrace{x' \cdot x}_{e} \cdot x' = x'' \cdot \underbrace{\cancel{e} \cdot x'}_{x'} = e \Rightarrow$

$\Rightarrow \boxed{x \cdot x' = x' x = e}$

$xe = x \cdot (x' \cdot x) = (x \cdot x') \cdot x = e \cdot x = e \Rightarrow$

$\Rightarrow \boxed{xe = ex = x\ ;\ \forall\ x \in G}$

$\Rightarrow (G,\cdot)$ - grup.

$\boxed{\text{SEMINAR 8}}$ 4. 12. 2020

8+9

Algebră

Grupul $(\mathbb{Q},+)$

$\boxed{\text{Def}}$: Un grup $(G,+)$ s.n. divizibil dacă $(\forall)\, n\in\mathbb{N}$, $nG \overset{\text{"}\subset\text{"}}{=} G$, i.e. $\forall n\in\mathbb{N}^*, \forall x\in G\ (\exists)\, y\in G$ a.î. $x = ny = \underbrace{y+\dots+y}_{\text{de } n \text{ ori}}$

$\boxed{\text{"} y = \frac{x}{n} \text{"}}$

Notat multiplicativ: $(G,\cdot)$ e divizibil dacă $(\forall)\, n\in\mathbb{N}$ și $(\forall)\, x\in G$, $(\exists)\, y\in G$ a.î. $x = y^n$.

Exemple: 1) $(\mathbb{Q},+)$ e grup divizibil

2) $(\mathbb{Z},+)$ nu e divizibil, $\boxed{2y \neq 3}$, $y\in\mathbb{Z}$.

OBS: Dacă $f: G \xrightarrow{\sim} H$ e izo. de grupuri G e divizibil $\Leftrightarrow$ H e divizibil.

(Ex!)

$\boxed{\text{Ex 25}}$: Dați un exemplu de două grupuri neizomorfe fiecare izomorf cu un subgrup în celălalt.

[C.B: $A \overset{i}{\hookrightarrow} B$, $B \overset{f}{\hookrightarrow} A \Rightarrow A \simeq B$]

Sol: Vreau $G_1 \not\simeq G_2$ (neizomorfe) ----

Fie $\boxed{(G_1,+) := \mathbb{Q}^{\mathbb{N}}} = \{(q_0, q_1, \dots q_n \dots) \mid q_i \in \mathbb{Q}\} =$

$= \prod_{i\in\mathbb{N}} G_i$, $G_i = (\mathbb{Q},+)$

$(q_0, q_1 \dots) + (q_0', q_1', \dots) = (q_0+q_0', q_1+q_1' + \dots)$

$G_1 = \mathbb{Q}^{\mathbb{N}}$ e divizibil (Ex!)

$\boxed{G_2 := \mathbb{Z} \times \mathbb{Q}^{\mathbb{N}}} = \{(m, q_0, q_1, \dots q_n, \dots) \mid m \in \mathbb{Z}, q_i \in \mathbb{Q}\}.$

$(G_2, +) \leq (G_1, +)$

$G_1 \cong H := \underline{\{0\} \times \mathbb{Q}^{\mathbb{N}}} = \{(0, q_0, q_1, \dots q_n \dots) \mid q_i \in \mathbb{Q}\}$

$H \leq (G_2, +)$

morfism bijectiv

$(q_0, q_1, \dots q_n, \dots) \xrightarrow[\sim]{f} (0, q_0, q_1, \dots q_n \dots)$

$(q_0, q_1, \dots q_n \dots) \xleftarrow[f^{-1}]{\sim} (0, q_0, q_1, \dots q_n \dots)$

$\Rightarrow \boxed{G_1 \cong H \leq G_2}$

$G_1 \not\cong G_2$ (nu sunt izomorfe!)

$G_2 = \mathbb{Z} \times \mathbb{Q}^{\mathbb{N}}$ nu e divizibil!

$n := 3, \quad (2, 1, \frac{1}{7}, \dots) = x$

$\cancel{(\exists)}\ y \in \mathbb{Z} \times \mathbb{Q}^{\mathbb{N}}$ a.î. $(2, 1, \frac{1}{7}, \dots) = 3(\overset{\in \mathbb{Z}}{m}, \underbrace{q_0, q_1 \dots}_{\in \mathbb{Q}})$

$\Rightarrow 2 = 3m, \quad m \in \mathbb{Z}$

$\boxed{\text{Ex 30}}$. $(\mathbb{Q}, +)$ - grup abelian!

a) $H \leq (\mathbb{Q}, +)$, $\underline{H + \mathbb{Z} = \mathbb{Q}} \overset{?}{\Rightarrow} \underline{H = \mathbb{Q}}$

Sol: $H + \mathbb{Z} := \{h + m \mid h \in H, m \in \mathbb{Z}\} = \mathbb{Q} \Rightarrow \boxed{H \neq \{0\}} \Rightarrow$

$\Rightarrow (\exists)\ 0 \neq x = \frac{m}{n} \in H, \ m, n \in \mathbb{Z}, n \neq 0. \Rightarrow$

$\Rightarrow 0 \neq \underline{nx} \in \underline{H \cap \mathbb{Z}} \leq \underline{(\mathbb{Z},+)} \Rightarrow (\exists!)\ t \in \mathbb{N}^*$ a.î.

$H \cap \mathbb{Z} = \underline{t\mathbb{Z}} = \{tx \mid x \in \mathbb{Z}\}$

$\underline{H+\mathbb{Z}} = \mathbb{Q} \ni \frac{1}{t} = y + a,\ \underline{y \in H},\ a \in \mathbb{Z} \Rightarrow$

$\Rightarrow 1 = \underbrace{ty}_{\in H} + ta \in H + t\mathbb{Z} \subseteq H + H = H$

$\underline{1 \in H} \leq \mathbb{Q} \Rightarrow p \in H, (\forall) p \in \mathbb{Z} \Rightarrow \mathbb{Z} \subseteq H \Rightarrow \mathbb{Q} = \mathbb{Z} + H \subseteq \underbrace{H+H}_{= H}$

$\Rightarrow \boxed{\mathbb{Q} = H}$

$(H \lneq (\mathbb{Q},+) \Rightarrow H + \mathbb{Z} \lneq (\mathbb{Q},+))$

$\boxed{\text{b) } H \leq (\mathbb{Q},+) \text{ e finit generat} \Rightarrow H = \text{ciclic.}}$

Sol: Pp. $H = \langle \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_n}{q_n} \rangle \leq (\mathbb{Q},+)$ $\quad q_1, \dots q_n \in \mathbb{N}^*$

$= \left\{ m_1 \frac{p_1}{q_1} + m_2 \cdot \frac{p_2}{q_2} + \dots + m_n \cdot \frac{p_n}{q_n} \;\middle|\; m_1, \dots m_n \in \mathbb{Z} \right\}$

Fie $q := q_1 q_2 \cdots q_n \in \mathbb{N}^* \Rightarrow \underline{qH \leq \mathbb{Z}}$

$\Rightarrow (\exists)\ \alpha \in \mathbb{N}^*$ a.î. $qH = \underline{\alpha\mathbb{Z}} \Rightarrow H = \frac{\alpha}{q}\mathbb{Z} = \underline{\langle \frac{\alpha}{q} \rangle}, \Rightarrow$

$\Rightarrow$ i.e. $\underline{H \text{ ciclic}}$!

$\boxed{\begin{array}{l} \text{c) } (\mathbb{Q},+) \ \boxed{\text{nu}} \text{ e finit generat.} \\ \text{d) } (\mathbb{Q},+) \not\cong (\mathbb{Q},+) \times (\mathbb{Q},+) \end{array}}$

Sol: Pp că $\mathbb{Q}$ e f.g., $\mathbb{Q} = \langle \frac{p_1}{q_1}, \dots \frac{p_n}{q_n} \rangle \leq \mathbb{Q} \xrightarrow{b)}$

$(\mathbb{Q},+)$ e ciclic, fals! (temă!)

$\mathbb{Q} = \langle \frac{m}{n} \rangle = \langle \frac{1}{n} \rangle \neq \mathbb{Q}$

d) În $(\mathbb{Q},+)$ orice sub. f.g. e ciclic

$\mathbb{Z}\times\mathbb{Z} \leq \mathbb{Q}\times\mathbb{Q}$

$\langle (1,0),(0,1) \rangle$ este finit generat și $\boxed{\text{nu e ciclic!}}$

e) $(\mathbb{Q},+)$ nu are un sistem minimal de generatori: dacă S e un sistem de generatori pt. $(\mathbb{Q},+)$ și $0 \neq s \in S \Rightarrow$
$\Rightarrow S\setminus\{s\}$ e tot un sistem de generatori.

Sol: $\mathbb{Q} = \langle S \rangle \Rightarrow \mathbb{Q} \overset{??}{=} \langle S\setminus\{s\} \rangle$, $(\forall)\, 0 \neq s \in S$.

Fie $H := \langle S\setminus\{s\} \rangle \leq \mathbb{Q}$

$\mathbb{Q} = \langle S \rangle = \langle S\setminus\{s\} \rangle + \langle s \rangle = H + s\mathbb{Z}$

$\frac{1}{s}H + \mathbb{Z} = \frac{1}{s}\cdot\mathbb{Q} = \mathbb{Q} \overset{a)}{\Longrightarrow} \frac{1}{s}H = \mathbb{Q} \Rightarrow H = s\mathbb{Q} = \mathbb{Q} \Rightarrow$

$\frac{1}{s}H \leq \mathbb{Q}$

$\Rightarrow \langle S\setminus\{s\} \rangle = \mathbb{Q}$

G = grup $\Rightarrow \boxed{G = \langle G \rangle}$

$\mathbb{Q} = \langle \boxed{\mathbb{Q}} \rangle$.

Întrebare: $\mathbb{Q} \overset{?}{=} \langle \mathbb{Q} \times (0,1] \rangle$

$\mathbb{Q} \ni r = \frac{\overset{\in\mathbb{Z}}{m}}{\underset{\underset{\mathbb{N}^*}{\cap}}{n}} = m \cdot \underbrace{\frac{1}{n}}_{\in(0,1]} \in \langle \mathbb{Q} \times (0,1] \rangle$

$\boxed{(\mathbb{Z}[X],+) = \langle 1, X, X^2, \ldots \rangle}$

$\boxed{\times 31}$ a) $G =$ divizibil, $\boxed{|G| \neq 1}$ $\Rightarrow$ G este infinit!

Sol: Pp că $|G| = n \geq 2$. Fie $\boxed{a \in G \setminus \{1\}}$

G divizibil $\Rightarrow (\exists)\, x \in G$ a.î. $x^n = a$, fals!! căci

$\boxed{x^n = 1}, (\forall)\, x \in G. \Rightarrow G$ e infinit.

$$t = \theta(x) = \begin{cases} \min\{m \in \mathbb{N}^* \mid x^m = 1\} \\ +\infty, \;\; \ldots \end{cases}$$

$\theta(x) \mid n$, $x^t = 1$, $t \mid n$, $n = tq \Rightarrow x^n = x^{tq} = (x^t)^q = 1$.

$|\langle x \rangle| = \underline{\underline{K}}$, $\langle x \rangle = \{1, x, \ldots x^{K-1}\} \overset{\text{Lagrange}}{\Rightarrow}$

$\Rightarrow K \mid n$ $\qquad \boxed{\theta(x) = |\langle x \rangle|}$

b) $f : G \to H$ surjectiv, $G =$ divizibil $\Rightarrow H$ e divizibil.

Sol: Fie $\boxed{n \in \mathbb{N}}$, $\underline{\underline{x \in H}}$. Vrem $(\exists)\, \underset{G\text{ div.}}{y_0} \in H$ a.î. $x = y_0^n$.

$x \in H \Rightarrow (\exists)\, g \in G$ a.î. $f(g) = x \Rightarrow (\exists)\, y \in G$ a.î. $g = y^n \Rightarrow$

$\Rightarrow \underline{\underline{x}} = f(g) = f(y^n) = \underbrace{f(y)}_{y_0 \in H}{}^n = \underline{y_0^n}$.

grup factor.

Dacă $K \trianglelefteq G =$ grup divizibil $\Rightarrow G/K$ e divizibil

$(\pi : G \longrightarrow G/K,\ \pi(g) = \hat{g})$

$\Rightarrow (\mathbb{Q}/\mathbb{Z}, +)$ e grup divizibil.

c) $G =$ divizibil, $\{1\} \neq H \leq G \not\Rightarrow H$ e divizibil

$\mathbb{Z} \leq \mathbb{Q} =$ div.

$\to$ nu e divizibil.

d) $\boxed{\text{temă !}}$ – $\boxed{\text{Temă}}$

$\boxed{\text{Ex 32}}$: G = gr. $\boxed{\text{divizibil}}$, H = finit.
$f: G \longrightarrow H$ e morfism de grupuri $\Big\} \Rightarrow f(g) = 1_H$, $(\forall) g \in G$.

$$\boxed{\left|\mathrm{Hom}_{gr}(\mathbb{Q}, \mathbb{Z}_7 \times S_9)\right| = 1.}$$

Sol. $n = |H|$. Fie $g \in G \Rightarrow (\exists) y \in (G, +)$ a.î. $g = ny \Rightarrow$

$f(g) = f(ny) = \underbrace{f(y)}_{\in H}{}^{n} = 1_H \Rightarrow f(g) = 1_H$, $(\forall) g \in G$.

$$\left|\mathrm{Hom}_{gr}(\mathbb{R}_+^* \times \mathbb{R}, \times), S_{19} \times \mathbb{Z}_7 \times \mathbb{Z}_2)\right| = 1.$$

$\boxed{\text{– sisteme "bune" de generatori pentru } (\mathbb{Q}, +), (\mathbb{Q}^*, \cdot)}$

Temă de reflexie.

Problemă curs

$f: G \longrightarrow \mathrm{Aut}(G)$, $f(g) := \tau_g$, $(\forall) g \in G$

$\tau_g : G \longrightarrow G$, $\tau_g(x) := \underline{gxg^{-1}}$, $\forall x \in G$.

• $\tau_g \in \mathrm{Aut}(G)$??

τ_g e bijectivă

$\underline{\tau_g(x) \cdot \tau_g(y)} = gxg^{-1}gyg^{-1} = g(xy)g^{-1} = \underline{\tau_g(xy)} \Rightarrow$

$\Rightarrow \tau_g$ e morfism

$(\tau_g(\underline{g^{-1}yg}) = y)$

- f e <u>morfism</u> !

$f(\underline{g_1 g_2}) = \underline{f(g_1) \circ f(g_2)}$. Fie $x \in G$.

$\underset{\text{RHS}}{}$

$\underline{\text{RHS}}(x) \overset{i_{g_1 g_2}}{=} f(g_1)\left(\underline{f(g_2)(x)}\right) = f(g_1)\left(g_2 x g_2^{-1}\right) =$

$= \tau_{g_1}\left(g_2 x g_2^{-1}\right) = g_1 g_2 x g_2^{-1} g_1^{-1} = g_1 g_2\, x\, (g_1 g_2)^{-1} =$

$= i_{g_1 g_2}(x) = f(\underline{g_1 g_2})(x) \Rightarrow f$ e <u>morfism</u> !!

- $\underline{\text{Ker}(f) \overset{?}{=} Z(G)}$ Fie $x \in G$.

$\underline{x \in \text{Ker}(f)} \Leftrightarrow \tau_x = id_G \Leftrightarrow \underline{\tau_x(g)} = id_G(g), (\forall) g \in G$

$\Leftrightarrow x g x^{-1} = g, (\forall) g \in G \Leftrightarrow \underline{xg = gx}, (\forall) g \in G. \Leftrightarrow$

$\Leftrightarrow \boxed{x \in Z(G)} \trianglelefteq G.$

- d) $\text{Im}(f) \overset{\text{not.}}{=} \boxed{\underline{\text{Inn}}(G)} \trianglelefteq \text{Aut}(G)$

$\tau \in \text{Aut}(G), \tau_g \Rightarrow \underline{\tau \circ \tau_g \circ \tau^{-1}} \overset{?}{=} \tau_{\tau(g)} \in \text{Inn}(G)$

Temă de finalizat!

Algebră

Ex 33)

a) $Aut(\mathbb{Z}) = \{id_{\mathbb{Z}}, -id_{\mathbb{Z}}\} \simeq \mathbb{Z}_2$

$id_{\mathbb{Z}} \to \hat{0}$

$-id_{\mathbb{Z}} \to \hat{1}$

b) $\boxed{Aut(\mathbb{Z}_n) \simeq (U(\mathbb{Z}_n), \cdot)}$

$\mathbb{Z}_n = \{\hat{0}, \hat{1}, \dots, \widehat{n-1}\}$, $\hat{x} \cdot \hat{y} := \widehat{xy}$

$\hat{1}$, $U(\mathbb{Z}_n) = \{\hat{x} \in \mathbb{Z}_n \mid (x, n) = 1\}$

$|U(\mathbb{Z}_n)| = \varphi(n)$.

SOL: $f : \mathbb{Z}_n \to \mathbb{Z}_n$ e morfism $\Rightarrow f(\hat{0}) = \hat{0}$

Notez $f(\hat{1}) = \hat{a} \in \mathbb{Z}_n \Rightarrow \boxed{f(\hat{x})} = f(\underbrace{\hat{1} + \dots + \hat{1}}_{\text{de } x \text{ ori}}) =$

$= \underbrace{f(\hat{1}) + \dots + f(\hat{1})}_{\text{de } x \text{ ori}} = x\hat{a}$ $\qquad f_a : \mathbb{Z}_n \to \mathbb{Z}_n$, $f_{\hat{a}}(\hat{x}) = x \cdot \hat{a}$

$f_{\hat{a}}$ este izo $\Leftrightarrow f_{\hat{a}}$ e surjectiv $\overset{?}{\Leftrightarrow} \hat{a} \in (U(\mathbb{Z}_n), \cdot)$

"$\Rightarrow$" f surj $\Rightarrow (\exists)\, \hat{x} \in \mathbb{Z}_n$ a.î. $f_{\hat{a}}(\hat{x}) = \hat{1} \Rightarrow x\hat{a} = \hat{1} \Rightarrow$

$\Rightarrow xa \equiv 1 \pmod n \Rightarrow \hat{x} \cdot \hat{a} = \hat{1} \Rightarrow a \in U(\mathbb{Z}_n)$

"$\Leftarrow$" $a \in (U(\mathbb{Z}_n), \cdot)$, $\hat{y} \in \mathbb{Z}_n$, $f_{\hat{a}}(\hat{y}\hat{a}^{-1}) \overset{?}{=} \hat{y} \Rightarrow f_{\hat{a}}$ e surj.

$\Rightarrow f \in Aut(\mathbb{Z}_n) \Leftrightarrow (\exists!)\, \hat{a} \in (U(\mathbb{Z}_n), \cdot)$ a.î. $f(\hat{x}) = x\hat{a}$.

$G : (U(\mathbb{Z}_n), \cdot) \xrightarrow{\sim} Aut(\mathbb{Z}_n)$, $G(\hat{a})(\hat{x}) := x\hat{a} = \widehat{xa}$ e izo de grupuri (Ex!)

$\Downarrow$

$\Rightarrow |Aut(\mathbb{Z}_n)| = \varphi(n)$, indicatorul lui Euler.

d) $\boxed{Aut(S_3) \simeq S_3}$

$\varphi : G \longrightarrow Aut(G)$, $\varphi(g) := \varphi_g$, $\varphi_g(x) := gxg^{-1}$

$Ker(\varphi) = Z(G)$; $Im(\varphi) = Inn(G) \trianglelefteq Aut(G)$

? ← (temă)

$\Rightarrow G/_{Z(G)} \simeq Im(\varphi) = Inn(G) \trianglelefteq Aut(G)$

T.F.I. $\boxed{G/_{Z(G)} \simeq Inn(G)} \trianglelefteq Aut(G)$

• $S_3 = \langle (1\,2), (1\,2\,3) \rangle = \langle \sigma, \tau \rangle$, $(1\,2) = \tau$, $(1\,2\,3) = \sigma$

$\sigma^2 = (1\,2\,3)(1\,2\,3) = (1\,3\,2)$

$S_3 = \{\underbrace{e}_{ord.1}, \underbrace{\sigma, \sigma^2}_{ord.3}, \underbrace{\tau, \sigma\tau, \sigma^2\tau}_{ord.2}\} = \Delta_3 \Rightarrow \begin{cases} o(\sigma) = 3 \\ o(\tau) = 2 \end{cases}$

$f \in Aut(S_3)$, $f : S_3 \longrightarrow S_3$

$f(\sigma) \in S_3$ de ordin $\boxed{3} \Rightarrow f(\sigma) \in \{\sigma, \sigma^2\}$

$f(\tau) \in \{\tau, \sigma\tau, \sigma^2\tau\}$

f e complet determinat de $f(\sigma)$ și $f(\tau)$!

$f : S_3 \longrightarrow S_3$, $S_3 = \langle \sigma, \tau \rangle$

$\Rightarrow |Aut(S_3)| \leq 6$

$\boxed{Z(S_n) = \{e\}, (\forall)\, n \geq 3}$

$\left.\begin{array}{l} \Rightarrow \underbrace{S_3/_{Z(S_3)}}_{= S_3} \simeq Inn(S_3) \trianglelefteq Aut(S_3) \\ Z(S_3) = 1 \end{array}\right\} \Rightarrow Inn(S_3) \simeq S_3 \Rightarrow$

$\left.\begin{array}{l} \Rightarrow |Inn(S_3)| = 6 \\ Inn(S_3) \trianglelefteq Aut(S_3) \end{array}\right\} \Rightarrow |Aut(S_3)| \geq 6.$

$\Rightarrow |Aut(S_3)| = 6 = |Inn(S_3)| \Rightarrow Aut(S_3) = Inn(S_3) \Rightarrow$

$\Rightarrow Aut(S_3) \simeq S_3$ $\left(\underset{\underset{Aut(S_3)}{\trianglelefteq}}{Inn(S_3)} \simeq S_3 \right)$

$\boxed{33c)\ \text{Temă!}}$

$\boxed{\text{Ref}}$: Hölder: $Aut(S_n) \simeq S_n, (\forall) n \neq 6, n \geq 3$.

$\boxed{Ex\ 34)}$ $N \leq G$, $N \subseteq Z(G)$.

a) $N \trianglelefteq G$

$\boxed{Sol}$: Fie $x \in G$ și $n \in N$. Vreau $xnx^{-1} \overset{?}{\in} N$

$x\underbrace{n}_{\in Z(G)}x^{-1} = n\underbrace{xx^{-1}}_{1} = n \in N \Rightarrow N \trianglelefteq G$

b) G/N ciclic $\overset{??}{\Rightarrow}$ G abelian

$\boxed{Sol}$: Fie $x \in G$ a.î. $G/N = \langle \hat{x} \rangle = \langle xN \rangle$

Fie $g, g' \in G$. Vreau: $gg' \overset{?}{=} g'g$ ($\Rightarrow G$ abelian)

$\hat{g} \in G/N \Rightarrow (\exists) n \in \mathbb{N}$ a.î. $\hat{g} = \hat{x}^n \Rightarrow (\exists) y \in N$ a.î. $g = x^n y$

Analog, $(\exists) m \in \mathbb{N}$ și $y' \in N$ a.î. $g' = x^m y'$

$gg' = x^n \underbrace{y}_{\in N \subseteq Z(G)} x^m y' = x^{n+m} yy'$

$g'g = x^m \underbrace{y'}_{\in N \subseteq Z(G)} \cdot x^n y = x^m x^n y' y = x^{n+m} \cdot yy'$

$(y'y = yy')$

$\Rightarrow gg' = g'g$

c) $Aut(G)$ e ciclic $\overset{?}{\Rightarrow}$ G e abelian

$\boxed{Sol}$: $G/Z(G) \simeq Inn(G) \leq \underset{\text{ciclic}}{Aut(G)} \Rightarrow G/Z(G)$ e ciclic $\overset{b)}{\Rightarrow}$

ciclic $\Leftarrow$ ciclic

$\boxed{G \text{ e abelian}}$

$\boxed{\text{Ex. 35) Temă!}}$

$\boxed{\text{Ex 36)}}$ $Z(G)=1 \overset{?}{\Longrightarrow} \boxed{Z(\mathrm{Aut}(G))=1}$

$\boxed{\text{Sol}}$: Fie $\boxed{\sigma \in Z(\mathrm{Aut}(G))} \Rightarrow \sigma \circ \varphi_g = \varphi_g \circ \sigma, \forall g \in G$,

$\boxed{\varphi_g(x)=gxg^{-1}, (\forall)\, x\in G}$ $(\mathrm{inn}\, G)$

$\Rightarrow \sigma(\varphi_g(x)) = \varphi_g(\sigma(x)), (\forall)\, x\in G \Rightarrow$

$\Rightarrow \sigma(g)\sigma(x)\sigma(g)^{-1} = g\,\sigma(x)\,g^{-1}, (\forall)\, x\in G, g\in G$

$\Rightarrow (g^{-1}\sigma(g))\sigma(x) = \sigma(x)(g^{-1}\sigma(g)), (\forall)\, x\in G, g\in G.$

$\Rightarrow (g^{-1}\sigma(g))y = y(g^{-1}\sigma(g)), (\forall)\, y\in G, g\in G.$

$\Rightarrow g^{-1}\sigma(g) \in Z(G) = 1, (\forall)\, g\in G \Rightarrow$

$\Rightarrow \sigma(g)=g, (\forall)\, g\in G$ i.e. $\boxed{\sigma = \mathrm{id}_G} \Rightarrow Z(\mathrm{Aut}(G))=1$

$\boxed{\text{Ex 37)}}$ Fie $m,n\in\mathbb{N}$, $m,n\geq 2$. Determinați toate morfismele între grupurile $(\mathbb{Z}_m,+)$ și $(\mathbb{Z}_n,+)$.

$\boxed{\text{Sol}}$: $f:(\mathbb{Z}_m,+)\longrightarrow(\mathbb{Z}_n,+)$??

Fie, $f:\mathbb{Z}_m\longrightarrow\mathbb{Z}_n$ morfism de grupuri $\Rightarrow$

$f(\hat{0})=\bar{0}$ și notăm $f(\hat{1}) \overset{\text{not.}}{=} \bar{a}\in\mathbb{Z}_n$

$f(\hat{x}) = f(\underbrace{\hat{1}+\dots+\hat{1}}_{\text{de } x \text{ ori}}) = \underbrace{\bar{a}+\dots+\bar{a}}_{\text{de } x \text{ ori}} = x\bar{a} = \overline{xa}, (\forall)\,\hat{x}\in\mathbb{Z}_m.$

$\bar{0} = f(\hat{0}) = f(\hat{m}) = \overline{ma} \Rightarrow n \mid ma$

Fie $d=(m,n)$, $m=dm'$, $n=dn'$, $(m',n')=1.$

$\Rightarrow d n' | d m' a \Rightarrow \left.\begin{matrix} n' | m' a \\ (n', m') = 1 \end{matrix}\right\} \Rightarrow n' | a \Rightarrow$

$\Rightarrow \boxed{\frac{n}{(m,n)} \mid a}$

Rezumat. $f: \mathbb{Z}_m \to \mathbb{Z}_n$ e morfism de grupuri $\Rightarrow (\exists)$ $a \in \mathbb{Z}$ a.î. $\frac{n}{(m,n)} | a$ și $f(\hat{x}) = \overline{xa}$, $(\forall)$ $x \in \mathbb{Z}_n$

Reciproc, $a \in \mathbb{Z}$, $\frac{n}{(m,n)} | a \Rightarrow f: \mathbb{Z}_m \to \mathbb{Z}_n$,

$f(\hat{x}) := \overline{xa}$, $(\forall)$ $x \in \mathbb{Z}_m$ e morfism de grupuri (Ex!)

$\Rightarrow |\mathrm{Hom}_{gr}(\mathbb{Z}_m, \mathbb{Z}_n)| = |\{a \in \{0, 1, \dots, n-1\} \mid \frac{n}{(m,n)} | a\}| = $ (Ex!)

câte nr.

Câte sunt exact?? $\left[\frac{n}{(m,n)} | a\right]$

$\boxed{\text{Ex } 40)}$ a) Subgrupurile și grupurile factor ale lui $\mathbb{Z}_{12}$

b) Temă! $\boxed{\mathbb{Z}_{18}}$.

$\boxed{12\mathbb{Z} \leq d\mathbb{Z} \Leftrightarrow 12 \in d\mathbb{Z} \Leftrightarrow d | 12}$

$\boxed{\text{Sol}}$: $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/_{12\mathbb{Z}}$ $\qquad d\mathbb{Z}$

$\qquad\qquad\qquad\qquad\qquad\qquad ||$

$K \leq \mathbb{Z}/_{12\mathbb{Z}} \Leftrightarrow (\exists!)\ H \leq \mathbb{Z},\ H \supseteq 12\mathbb{Z}$ a.î. $K = \pi(H) \overset{not.}{=} H/_{12\mathbb{Z}}$

$\mathcal{L}(\mathbb{Z}_n) = \{d\mathbb{Z}/n\mathbb{Z} \mid d \in \mathbb{N}, d | n\}$

$\mathcal{L}(\mathbb{Z}_{12}) = ??$ $\qquad \boxed{d = 12}$ $\quad \hat{d} = \widehat{12} = \hat{0}$

- $\{\hat{0}\} \leq \mathbb{Z}_{12}$, $\quad \mathbb{Z}_{12}/\{\hat{0}\} \simeq \mathbb{Z}_{12}$
- $\cancel{\{\hat{1}\} \leq \mathbb{Z}_{12}}$ $\underline{d=1} \Rightarrow \boxed{\mathbb{Z}_{12}} \leq \mathbb{Z}_{12}$, $\quad \mathbb{Z}_{12}/\mathbb{Z}_{12} \simeq \boxed{0}$
- $d = 2$; $\langle \hat{2} \rangle = \{\hat{0}, \hat{2}, \hat{4}, \hat{6}, \hat{8}, \widehat{10}\} \leq \mathbb{Z}_{12}$

$$\mathbb{Z}_{12}/\langle\hat{2}\rangle \simeq \boxed{\mathbb{Z}_2}$$

$$\frac{\mathbb{Z}_{12}}{\langle\hat{2}\rangle} = \frac{\mathbb{Z}/12\mathbb{Z}}{2\mathbb{Z}/12\mathbb{Z}} \overset{T_1 \text{ izo}}{\simeq} \mathbb{Z}/2\mathbb{Z}$$

- $d=3$; $\langle\hat{3}\rangle = \{\hat{0},\hat{3},\hat{6},\hat{9}\} \leq \mathbb{Z}_{12}$; $\cancel{\mathbb{Z}_{12}}$

$$\frac{\mathbb{Z}_{12}}{\langle\hat{3}\rangle} = \frac{\mathbb{Z}/12\mathbb{Z}}{3\mathbb{Z}/12\mathbb{Z}} \overset{T.i}{\simeq} \mathbb{Z}/3\mathbb{Z} = \boxed{\mathbb{Z}_3}.$$

- $d=4$; $\langle\hat{4}\rangle = \{\hat{0},\hat{4},\hat{8}\}$, $\frac{\mathbb{Z}_{12}}{\langle\hat{4}\rangle} = \frac{\mathbb{Z}/12\mathbb{Z}}{3\mathbb{Z}/12\mathbb{Z}} \simeq \boxed{\mathbb{Z}_4}$

- $d=6$; $\langle\hat{6}\rangle = \{\hat{0},\hat{6}\} \leq \mathbb{Z}_{12} \Rightarrow \frac{\mathbb{Z}_{12}}{\langle\hat{6}\rangle} = \frac{\mathbb{Z}/12\mathbb{Z}}{6\mathbb{Z}/12\mathbb{Z}} \overset{T.1.}{\simeq} \boxed{\mathbb{Z}_6}$

Ex 41) $f: \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}\times\mathbb{Z}/\langle(2,3)\rangle$, $f(m) := \widehat{(m,m)}$ e izo de grupuri.

SOL: f e morfism ok (temă)

$$f(m+n) = f(m) + f(n)$$

$$\mathbb{Z} \xrightarrow{d} \mathbb{Z}\times\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}\times\mathbb{Z}/\langle(2,3)\rangle$$

$d(m) := (m,m)$ e morfism

$$\boxed{f = \pi \circ d}.$$

- f e injectiv: Fie $m \in \operatorname{Ker}(f) \Rightarrow f(m) = \widehat{(0,0)} \Rightarrow$?

$\Rightarrow \widehat{(m,m)} = \widehat{(0,0)} \Rightarrow (m,m)-(0,0) \in \langle(2,3)\rangle =$

$= \{t(2,3) \mid t\in\mathbb{Z}\} = \{(2t,3t) \mid t \in \mathbb{Z}\} \Rightarrow$

$\Rightarrow (\exists)\ t \in \mathbb{Z}$ a.î. $\begin{cases} m = 2t \\ m = 3t \end{cases} \Rightarrow \boxed{t=0}$ și $\boxed{m=0} \Rightarrow$

$\Rightarrow \operatorname{Ker}(f) = 0 \Rightarrow f$ e injectiv.

- f este surjectiv

Fie $\widehat{(a,b)} \in \mathbb{Z}\times\mathbb{Z}/_{<(2,3)>}$

Vreau: $(\exists)\, m \in \mathbb{Z}$ a.î. $\widehat{(a,b)} = f(m) = \widehat{(m,m)} \Leftrightarrow$

$\Leftrightarrow (a,b) - (m,m) \in <(2,3)> = \{(2t, 3t) \mid t \in \mathbb{Z}\} \Rightarrow$

$\Rightarrow \exists\, t \in \mathbb{Z}$ a.î. $(a,b) - (m,m) = (2t, 3t) \Rightarrow$

Rezumat: f e surj $\Leftrightarrow (\forall)\, a,b \in \mathbb{Z}, (\exists)\, m, t \in \mathbb{Z}$ a.î.

$(a,b) = (m+2t, m+3t)$

$$\begin{cases} a = m+2t \\ b = m+3t \end{cases} \Rightarrow \begin{cases} \boxed{t = b-a} \\ \boxed{m = 3a-2b} \end{cases}$$

$\widehat{(a,b)} = f(3a-2b)$, f e surj $\Rightarrow f$ e izo.

$\boxed{\text{Ex 42)}}$ $(\mathbb{R},+)/_{<\{\sqrt{2},\sqrt{3}\}>}$ elementele de ordin 2 din

$(\mathbb{R}/_{\mathbb{Z}}, +)$ $\overset{\text{not.}}{=} A$

Sol: $<\{\sqrt{2},\sqrt{3}\}> = \{m\sqrt{2} + n\sqrt{3} \mid m,n \in \mathbb{Z}\}$

- elementele de $\boxed{\text{ordin 2}}$ din $(\mathbb{R}/_{\mathbb{Z}}, +)$!!!

$\boxed{\hat{r} \in \mathbb{R}/_{\mathbb{Z}} \text{ are ordinul } 2}$ $\Leftrightarrow \hat{r} + \hat{r} = \hat{0}$ și $\hat{r} \neq \hat{0} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} r \notin \mathbb{Z} \\ 2r \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} r = \frac{t}{2}, \ t \in \mathbb{Z} \\ t = \text{impar} \end{cases} \Leftrightarrow (\exists)\, k \in \mathbb{Z} \text{ a.î.}$$

$r = \frac{2k+1}{2} = k + \frac{1}{2} \Leftrightarrow \boxed{\hat{r} = \widehat{\tfrac{1}{2}}}$

$\Rightarrow \mathbb{R}/\mathbb{Z}$ are un singur element de ordin 2 și anume $\widehat{\frac{1}{2}}$.

Calculez elem. de ordinul 2 din $(\mathbb{R},+)/\langle\{\sqrt{2},\sqrt{3}\}\rangle$

$\hat{s} \in \mathbb{R}/H$ are ordin 2 $\Leftrightarrow$ $\hat{s} \neq \hat{0}$ și $\hat{s}+\hat{s}=\hat{0}$ $\Leftrightarrow$

$\Leftrightarrow s \notin H$ și $2s \in H = \{m\sqrt{2}+n\sqrt{3} \mid m,n \in \mathbb{Z}\}$

$\boxed{\text{TEMĂ}}$

$\Leftrightarrow \hat{s} = \widehat{\frac{\sqrt{2}}{2}}, \widehat{\frac{\sqrt{3}}{2}}, \widehat{\frac{\sqrt{2}+\sqrt{3}}{2}}$

$\Rightarrow \mathbb{R}/H$ are 3 elemente de ordin 2 și anume $\widehat{\frac{\sqrt{2}}{2}}$, $\widehat{\frac{\sqrt{3}}{2}}$ și $\widehat{\frac{\sqrt{2}+\sqrt{3}}{2}}$

SEMINAR 10 $\qquad$ 18.12.2020

10+11+12

Algebră

3. probleme despre grupul de permutări

59) a) $(i_1 i_2 \dots i_{2K})^2 = (i_1 i_3 \dots i_{2K-1})(i_2 i_4, \dots i_{2K})$

b) $(i_1 i_2 \dots i_{2K+1})^2 = (i_1 i_3 i_5 \dots i_{2K+1} i_2 \dots i_{2K})$

SOL. a) $(i_1 i_2 \dots i_{2K})(i_1 i_2 \dots i_{2K}) = (i_1 i_3 i_5 \dots i_{2K-1}) \cdot$

$\cdot (i_2 i_4 \dots i_{2K})$

b) Analog.

$(i_1 i_2 \dots i_{2K+1})(i_1 \dots i_{2K+1}) = (i_1 i_3 i_5 \dots i_{2K+1} i_2 i_4 \dots i_{2K})$

58) $\boxed{\text{Examen}}$ Fie $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 6 & 5 & 7 & 1 & 2 \end{pmatrix} \in S_7$.

a) Să se descompună σ în produs de cicli disjuncți și produs de transpoziții.

b) Calculați $o(\sigma), \varepsilon(\sigma), \sigma^{-1}$ și σ^{2020}.

Sol. a) $\sigma = (1\ 3\ 6)(2\ 4\ 5\ 7)$

$\sigma = (1\ 3)(3\ 6)(2\ 4)(4\ 5)(5\ 7)$

$(i_1 i_2 \dots i_K) = (i_1 i_2)(i_2 i_3)(\dots) \dots (i_{K-1} i_K)$ → ciclu => prod de transpoziții

b) $\varepsilon(\sigma) = (-1)^5 = -1 \Rightarrow \sigma$ e impară.

$o(\sigma) = [3, 4] = 12$ $\qquad$ ordin permutare = [lg. cicli]

cmmmc

$\sigma^{-1} = (2457)^{-1}(136)^{-1} = (7\ 5\ 4\ 2)(6\ 3\ 1) = (1\ 6\ 3)(2\ 7\ 5\ 4)$

$$\sigma^{2020} = \sigma^{12\cdot 168+4} = (\underset{\overset{\parallel}{e}}{\sigma^{12}})^{168}\cdot\sigma^{4} = e = \sigma^4 =$$

$$= \left((1\,3\,6)(2\,4\,5\,7)\right)^4 = (1\,3\,6)^4\,(2\,4\,5\,7)^4 =$$

$$= \underset{\overset{\parallel}{e}}{(1\,3\,6)^3}\cdot(1\,3\,6)\cdot e = (1\,3\,6)$$

$$\sigma^{2020} = (1\,3\,6)$$

$\boxed{60)}$ $n \geq 6 \Rightarrow (\not\exists)\ \sigma \in S_n$ a.î. $\sigma^2 = (1\,2)(3\,4\,5\,6)$

$\boxed{\text{SOL}}$: $\varepsilon(\sigma^2) = \varepsilon(\sigma)\cdot\varepsilon(\sigma) = 1$, $\varepsilon((3\,4\,5\,6)) = (-1)^3 = -1$

Presupunem că $(\exists)\ \sigma \in S_n$ a.î. $\underline{\sigma^2} = (1\,2)(3\,4\,5\,6)$

$\sigma = \underline{\pi_1\pi_2\ldots\pi_k}\cdot\theta$, unde $\pi_1,\ldots\pi_k$ sunt cicli disjuncți de lungime $\underline{\geq 3}$

$\theta :=$ produsul tuturor ciclilor de lungime 2 din descompunerea lui σ.

$\theta^2 = e$ $\left((i\,j)^2 = e\right) \Rightarrow \boxed{\sigma^2 = \pi_1^2\pi_2^2\ldots\pi_k^2}$, $\pi_1, \pi_2, \ldots \pi_k$ sunt cicli de lungime ≥ 3.

<u>Cazul 1</u>: $(\exists)\ i = \overline{1,k}$ a.î. π_i are lungime impară $\Rightarrow$

$\Rightarrow l(\pi_i) = 2l_i + 1 \overset{59}{\Rightarrow} \pi_i^2$ este tot ciclu de lungime <u>impară</u>

$$\left.\begin{aligned}&\pi_1^2\pi_2^2\ldots\pi_k^2 = (1\,2)(3\,4\,5\,6)\ (=\tau)\\ &(\exists)\text{ un } \pi_i^2 \text{ de lungime impară}\end{aligned}\right\}$$

<u>Cazul 2</u>: $\pi_1, \pi_2, \ldots \pi_k$ au lungime <u>pară</u>

$l(\pi_1) = 2l_1$, $l(\pi_2) = 2l_2$, ..., $l(\pi_k) = 2l_k$.

$$\underbrace{\pi_1^2\,\pi_2^2\,\cdots\,\pi_K^2}_{} = (1\ 2)(3\ 4\ 5\ 6)(=\sigma)$$

produsul de cicli <u>disjuncți</u>

de lungimi $l_1, l_1, l_2, l_2, \dots l_K, l_K$, <u>fals</u> !! folosind unicitatea descompunerii în cicli disjuncți!!

$\boxed{44)}$ a) $f : G_1 \xrightarrow{\sim} G_2$ ~~izo~~, $H \trianglelefteq G_1 \Rightarrow \underline{f(H) \trianglelefteq G_2}$ și $G_1/_H \simeq G_2/_{f(H)}$

(Ex!)

$\boxed{SOL}$:

$$g : G_1 \xrightarrow[\sim]{f} G_2 \xrightarrow{\pi} G_2/_{f(H)}, \ \pi(x) = \hat{x}$$

$g := \pi \circ f$ e morfism <u>surjectiv</u> de grupuri.

$$g : G_1 \longrightarrow G_2/_{f(H)}, \ g(x) = \widehat{f(x)}, (\forall) x \in G_1.$$

$\boxed{\text{Vreau : T.F.i}}$ $\boxed{G/_{\ker(h)} \overset{??}{\simeq} \operatorname{im}(h)}$!!!

$\operatorname{Ker}(g) = ??$ $\boxed{\operatorname{Ker}(g) \overset{??}{=} H}$

Fie $x \in \underline{\operatorname{Ker}(g)} \Leftrightarrow g(x) = \hat{1} \Leftrightarrow \underline{\widehat{f(x)} = \hat{1}} \Leftrightarrow f(x) \in f(H) \Leftrightarrow$

$$\left(\begin{array}{l} G/_K, \ \hat{x} = \hat{y} \Leftrightarrow y^{-1}x \in K \\ \quad \hat{x} = \hat{1} \Leftrightarrow x \in K \end{array} \right) \rightarrow \text{definiții}$$

$\Leftrightarrow$ ~~xx~~ $(\exists!)\, h \in H$ a.î. $f(x) = f(h) \underset{f\text{ inj.}}{\Longleftrightarrow} \underline{x = h \in H} \Rightarrow \boxed{\operatorname{Ker}(g) = H}$.

T.F.i $\Rightarrow \tilde{g} : G_1/_H \xrightarrow{\sim} G_2/_{f(H)}, \ \tilde{g}(\bar{x}) := \widehat{f(x)}$, e <u>izo</u> de grupuri.

b) $H_1 \trianglelefteq G_1, H_2 \trianglelefteq G_2 \Rightarrow H_1 \times H_2 \trianglelefteq G_1 \times G_2$

$(\exists)$ $G_1 \times G_2/_{H_1 \times H_2} \simeq G_1/_{H_1} \times G_2/_{H_2}$.

$\boxed{\text{Sol}}$: Vreau T.F.i.

Caut $\varphi : G_1 \times G_2 \longrightarrow G_1/H_1 \times G_2/H_2$

morf + surj + $(\mathrm{Ker}(\varphi) = H_1 \times H_2)$

$\varphi((g_1, g_2)) := (\hat{g_1}, \overline{g_2})$, $(\forall)(g_1, g_2) \in G_1 \times G_2$

- φ e morfism surj. de grupuri (Ex!)

$G_1 \times G_2 \xrightarrow{\bar{\pi}_1 \times \bar{\pi}_2} G_1/H_1 \times G_2/H_2$

$\varphi := \bar{\pi}_1 \times \bar{\pi}_2$ $\quad \begin{cases} \bar{\pi}_1(g_1) := \hat{g_1} \\ \bar{\pi}_2(g_2) := \overline{g_2} \end{cases}$

- $\mathrm{Ker}(\varphi) \overset{?}{=} H_1 \times H_2$ Fie $(x, y) \in G_1 \times G_2$. Atunci

Fie $(x, y) \in \mathrm{Ker}(\varphi) \Leftrightarrow \varphi(x, y) = (\hat{1}, \bar{1}) \Leftrightarrow (\hat{x}, \bar{y}) = (\hat{1}, \bar{1}) \Leftrightarrow$

$\Leftrightarrow \begin{cases} \hat{x} = \hat{1} \\ \bar{y} = \bar{1} \end{cases} \Leftrightarrow \begin{cases} x \in H_1 \\ y \in H_2 \end{cases} \Leftrightarrow (x, y) \in H_1 \times H_2 \Rightarrow$

$\Rightarrow \boxed{\mathrm{Ker}(\varphi) = H_1 \times H_2}$

T.F.i $\Rightarrow \tilde{\varphi} : G_1 \times G_2 / H_1 \times H_2 \xrightarrow{\sim} G_1/H_1 \times G_2/H_2$,

$\tilde{\varphi}(\overline{(g_1, g_2)}) := (\hat{g_1}, \overline{g_2})$ e izo de grupuri.

43) a) $(\mathbb{R}/\mathbb{Z}, +) \simeq (U, \cdot) = \{z \in \mathbb{C} \mid |z| = 1\} \leq (\mathbb{C}^*, \cdot)$

curs: $\tilde{f}(\hat{r}) := \cos(2\pi r) + i \sin(2\pi r)$, $\tilde{f} : \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} U$

b) $(\mathbb{Q}/\mathbb{Z}, +) \simeq (U_\infty, \cdot) := \{z \in \mathbb{C} \mid (\exists)\, n \in \mathbb{N}, z^n = 1\}$
$\leq (U, \cdot)$

Sol: $f : \mathbb{Q} \longrightarrow U$, $f(r) := \cos(2\pi r) + i\sin(2\pi r)$

- f e morfism de grupuri $(f(r+r') = f(r)f(r'))$
- $\mathrm{Ker}(f) \overset{?}{=} \mathbb{Z}$, $r = \frac{m}{n} \in \mathbb{Q}$. Atunci $r \in \mathrm{Ker}(f) \Leftrightarrow$

$\Leftrightarrow f(r) = 1 \Leftrightarrow \begin{cases} \cos(2\pi r) = 1 \\ \sin(2\pi r) = 0 \end{cases} \Leftrightarrow r \in \mathbb{Z}.$

- $\mathrm{Im}(f) \overset{?}{=} U_\infty$ $\quad \mathbb{Q}$, $m, n \in \mathbb{Z}$

$\boxed{\subseteq}$ $\xi \in \mathrm{Im}(f) \Rightarrow \xi = f\left(\frac{m}{n}\right) = \cos\left(2\pi \frac{m}{n}\right) + i\sin\left(2\pi \frac{m}{n}\right) \Rightarrow$

$\Rightarrow \xi^n = \cos(2\pi m) + i\sin(2\pi m) = 1 \Rightarrow \xi \in U_\infty \Rightarrow$

$\Rightarrow \mathrm{Im} f \subseteq U_\infty$

$\boxed{\supseteq}$ Fie $z \in U_\infty \Rightarrow (\exists) n \in \mathbb{N}$ a.î. $\boxed{z^n = 1} \Rightarrow$

$\Rightarrow z \in \left\{ \cos \frac{2\pi k}{n} + i\sin\frac{2k\pi}{n} \,\middle|\, k \in \{0, \dots, n-1\} \right\}$

$\quad\quad\quad \shortparallel$

$\quad\quad f\left(\frac{k}{n}\right) \Rightarrow z \in \mathrm{Im} f \Rightarrow U_\infty = \mathrm{Im}(f)$

$\Rightarrow$ T.F.i. $\Rightarrow \tilde{f} : \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} (U_\infty, \cdot)$,

$\tilde{f}(\hat{r}) := \cos(2\pi r) + i\sin(2\pi r)$ este izo de grupuri.

c) $(\mathbb{R}/\mathbb{Q}, +) \simeq (\mathbb{R}, +)$

Sol: $\mathbb{Q} \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ e $\mathbb{Q}$-spațiu vectorial

$\dim_{\mathbb{Q}}(\mathbb{R}) = \underset{\text{Bază}}{|B|} = \underset{\text{puterea cont.}}{\mathcal{E}} = \aleph_1$

$(\mathbb{R}/\mathbb{Q}, +)$ este $\mathbb{Q}$-spațiu vectorial.

$$\dim_{\mathbb{Q}}(\mathbb{R}/\mathbb{Q}) = \underset{\substack{\downarrow \\ \text{put. continuă.}}}{\mathfrak{c}} = \aleph_1$$

$\Rightarrow$ $\mathbb{R}$ și $\mathbb{R}/\mathbb{Q}$ sunt izomorfe ca și $\mathbb{Q}$-spații vectoriale $\Rightarrow$

$$\Rightarrow (\mathbb{R}/\mathbb{Q}, +) \underset{\substack{\uparrow \\ \text{izo. de grupuri.}}}{\simeq} (\mathbb{R}, +)$$

d) $(U/U_\infty, \cdot) \simeq (\mathbb{R}, +)$

$f : \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} U$, $f(\hat{r}) = \cos(2\pi r) + i\sin(2\pi r)$

$\mathbb{Q}/\mathbb{Z} \leq \mathbb{R}/\mathbb{Z}$, $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$

$\mathbb{Q}/\mathbb{Z} \overset{def}{:=} \pi(\mathbb{Q})$

$f(\mathbb{Q}/\mathbb{Z}) = U_\infty \leq U$

Ex 44) a) $U/U_\infty \simeq \dfrac{\mathbb{R}/\mathbb{Z}}{\mathbb{Q}/\mathbb{Z}} \overset{T.1.}{\simeq} (\mathbb{R}/\mathbb{Q}, +) \simeq (\mathbb{R}, +) \Rightarrow$

$\Rightarrow (U/U_\infty, \cdot) \simeq (\mathbb{R}, +)$.

Algebră

Ex 47 a) G = grup finit și $H, K \leq G \Rightarrow |HK| = \dfrac{|H| \cdot |K|}{|H \cap K|}$

SOL: Mulțimea $H \times K$: pe ea definim relația

$$(h, k) \sim (h', k') \overset{def}{\Longleftrightarrow} hk = h'k'$$

• „$\sim$" e relație de echivalență pe $H \times K$ (Ex!)

$$\begin{tikzcd} H \times K \arrow[r, "\pi"] \arrow[d, "f"'] & H \times K/_{\sim} \arrow[dl, dashed, "(\exists!)\bar{f}"] \\ HK & \end{tikzcd}$$

$f(h,k) := hk$

f e surjectivă

$HK = \{hk \mid h \in H, k \in K\}$

$\rho_f = \sim$ (relație de echivalență)

$(h,k)\, \rho_f\, (h',k') \Leftrightarrow f(h,k) = f(h',k') \Leftrightarrow hk = h'k' \Leftrightarrow$

$\Leftrightarrow (h,k) \sim (h',k')$

P.U.M.F. $\Rightarrow (\exists!)\ \bar{f} : H \times K/_{\sim} \longrightarrow HK$ a.î.

$\bar{f}(\widehat{(h,k)}) = hk$. În plus, $\bar{f}$ e bijectivă căci f e surjectivă ($\Rightarrow \bar{f}$ e surj) și $\rho_f = \sim$ ($\Rightarrow \bar{f}$ e injectivă) $\Rightarrow \boxed{|HK| = |H \times K/_{\sim}|}$

• $\widehat{(h,k)} = \{(h',k') \in H \times K \mid hk = h'k'\} =$

$\overset{?}{=} \{(hx^{-1}, xk) \mid x \in H \cap K\}$

„$\supseteq$" $(hx^{-1}, xk) \sim (h,k)$ căci $h\underbrace{x^{-1}x}_{1}k = hk$

$x \in H \cap K$

„$\subseteq$" Fie $(h',k') \in \widehat{(h,k)} \Rightarrow h'k' = hk \Rightarrow$

$$\Rightarrow \underbrace{k^{-1}h'}_{\in H} = \underbrace{k(k')^{-1}}_{\in K} = y \in H \cap K \Rightarrow$$

$$\Rightarrow h' = hy, \quad k' = y^{-1}k, \quad y \in H \cap K$$

$$x := y^{-1} \in H \cap K, \quad \begin{cases} h' = h \cdot x^{-1} \\ k' = x \cdot k \end{cases}, \quad x \in H \cap K$$

$$\Rightarrow (h',k') = (hx^{-1}, xk), \quad x \in H \cap K$$

$$\Rightarrow \boxed{\widehat{(h,k)} = \{(hx^{-1}, xk) \mid x \in H \cap K\}} \Rightarrow$$

$$\Rightarrow |\widehat{(h,k)}| = |H \cap K|$$

Rezumat

- $\sim$ e rel. de echivalență pe $H \times K$ în care fiecare clasă de echivalență are $|H \cap K|$ elemente.

$H \times K$

$\Rightarrow$ Cum clasele de echivalență formează o partiție a lui $H \times K \Rightarrow$

$$\Rightarrow |H \times K| = |H \cap K| \cdot (\underbrace{\text{nr. claselor de echivalență}}_{\alpha})$$

$$\alpha = |H \times K/_{\sim}| = |HK| \Rightarrow$$

$$\Rightarrow |H| \cdot |K| = |H \cap K| \cdot |HK| \Rightarrow |HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$

$\boxed{Ex\ 56}$ Să se arate că:

$$S_n \overset{a)}{=} \langle (1\,2), (1\,3), \dots (1\,n)\rangle =$$

$$\overset{b)}{=} \langle (1\,2), (2,3), \dots (n-1, n)\rangle =$$

$$\overset{c)}{=} \boxed{\langle (1\,2), (1\,2\,\dots\,n)\rangle}$$

$\boxed{SOL}$:

a) $\sigma \in S_n \Rightarrow \sigma$ e un produs de transpoziții

(i, j), $i<j$

Fie $i<j$, $(i\ j) \overset{?}{=} (1\ i)(1\ j)(1\ i)$

$$\begin{cases} 1 \to 1 \\ i \to j \\ j \to i \\ K \to K \end{cases} \qquad \begin{cases} 1 \to 1, \\ i \to 1 \to j \\ j \to j \to 1 \to i, \text{ pt. } K \neq 1, i, j \\ K \to K \end{cases}$$

$\Rightarrow S_n = \langle (1\,2), (1\,3), \dots (1\,n)\rangle$

b) $i<j$

(i, j)

$(1\ 3) = (1\,2)(2\ 3)(1\ 2)$

$1 \to 3$, $2 \to 2$, $3 \to 1$

$(1\ 4) = (1\,3)(3\,4)(1\ 3) =$

$= (1\,2)(2\,3)(1\,2)(3\,4)(1\,2)(2\,3)(1\,2)$

Pp. $(1\ i) \in \langle (1\,2), (2\,3), \dots (n-1, n)\rangle \Rightarrow$

$\Rightarrow \boxed{(1, i+1) = (1\ i)(i, i+1)(1\ i)} \Rightarrow$

$\Rightarrow (1\,2), (1\ 3), \dots (1\ n) \in \langle (1\,2), (2\,3), \dots (n-1, n)\rangle$

$\Rightarrow \langle (1\ 2), (1\,3), \dots (1\,n)\rangle \subseteq \langle (1\,2), (2\,3), \dots (n-1, n)\rangle \subseteq S_n$

$\Rightarrow S_n = \langle (1\,2), (2\,3), \dots (n-1, n)\rangle$

Ex. facultativ: Se poate genera S_n cu $n-2$ transpoziții? $\boxed{\text{nu!}}$ · Pro

c) $S_n = \langle (1\,2), (1\,2\,3\,\dots\,n) \rangle$

$(K \;\; K+1) \in \underbrace{\langle (1\,2), (1\,2\,\dots\,n) \rangle}_{H}$? $\forall K = \overline{1, n-2}$

$\underline{K=1}$ $\quad (1,2) \in \langle (1\,2), (1\,2\,\dots\,n) \rangle$

$K \longmapsto K+1$ $\quad$ Pp. că $(K, K+1) \in H$

$(K+1, K+2) \overset{?}{=} (1\,2\,3\,\dots\,n)(K, K+1)(1\,2\,\dots\,n)^{-1} \in H$

$= \underline{(1\,2\,\dots\,n)(K, K+1)(n, n-1, \dots\, 2, 1)}$

$\begin{cases} K+1 \to K+2 \\ K+2 \to K+1 \end{cases}$ $\qquad \begin{cases} \underline{K+1} \to K \to K+1 \to \underline{K+2} \\ \underline{K+2} \to K+1 \to K \to \underline{K+1} \end{cases}$

Fie $t \neq K+1, K+2$

$\underline{\underline{t}} \to \boxed{t-1} \to t-1 \to \underline{t}$

$\Rightarrow (1\,2), (2\,3), \dots (n-2, n-1) \in H$

$(n-1, n) = (1\,2\,\dots\,n)(n-2, n-1)(1\,2\,\dots\,n)^{-1}$

Deci, $S = \langle (1\,2)(1\,2\,\dots\,n) \rangle$

Ex. facultativ: $n \geq 3$ arătați că A_n se poate genera cu două elemente.

$\boxed{\text{Ex 57}}$ a) A_n se poate genera cu cicli de lungime trei

b) $A_n = \langle (1\,2\,3), (1\,2\,4), \dots (1\,2\,n) \rangle$

$\boxed{\text{SOL}}$: a) $\sigma \in A_n \Rightarrow \varepsilon(\sigma) = 1 \Rightarrow$

$\sigma = (\tau_1 \tau_2) \dots (\tau_{2i})$, $\tau_1, \dots \tau_{2i}$ sunt transpoziții

• Prod. a 2 transpoziții ⇒ ciclu de lungime 3 sau un produs de lungime 3.

$* \ (a\ b)(a\ c) = (a\ c\ b)$

$a \neq b \neq c$

$* \ (a\ b)(c\ d) \overset{?}{=} (a\ b\ c)(b\ c\ d)$

$a \neq b \neq c \neq d$ $\qquad a \to b \quad c \to d$

$\qquad b \to a \quad d \to c$

Deci, A_n e generat de cicli de lungime 3.

$\boxed{56\ a)}$

b) $\sigma \in A_n \Rightarrow \sigma = \underset{\boxed{\text{nr. par}}}{\prod} (1\ i)$

$= ((1\ i_1)(1\ i_2))((1\ i_3)\ \ldots)$

$(1\ i)(1\ j) = (\underline{\underline{1\ j\ i}}) \overset{??}{=} (1\ 2\ i)(1\ 2\ i)(1\ 2\ j)(1\ 2\ i)$

$i \neq j$

$1 \to 2 \to j \to j$

$2 \to i \to i \to 1 \to 2$

$i \to 1 \to 1 \to 1$

$j \to j \to 1 \to 2 \to i$

$\Rightarrow \sigma \in A_n \Rightarrow \sigma \in \langle (1\ 2\ 3), (1\ 2\ 4), \ldots (1\ 2\ n) \rangle \Rightarrow$

$\Rightarrow A_n \subseteq \langle (1\ 2\ 3), (1\ 2\ 4), \ldots, (1\ 2\ n) \rangle \subseteq A_n \Rightarrow$

$\Rightarrow A_n = \langle (1\ 2\ 3), (1\ 2\ 4), \ldots, (1\ 2\ n) \rangle \Rightarrow$

$\Rightarrow A_n$ e generat de $(n-2)$ cicli de lungime 3.

INELE → vezi pe verso.

$\boxed{Ex\ 1}$ b) $(\mathbb{Q}/\mathbb{Z}, +)$ nu are o structură de inel.

$\boxed{SOL}$: Fie $(\mathbb{Q}/\mathbb{Z}, +, *)$ este inel! Fie $e = \widehat{\frac{p}{q}}$ elem. unitate.

$$\widehat{\frac{p}{q}} * (\underbrace{\hat{x} + \hat{x} \dots + \hat{x}}_{\text{de } q \text{ ori}}) = \underbrace{\hat{x} + \dots + \hat{x}}_{\text{de } q \text{ ori}} \overset{\text{not.}}{=} q\hat{x}, (\forall)\hat{x} \in \mathbb{Q}/\mathbb{Z}$$

$$\underbrace{\left(\widehat{\frac{p}{q}} + \widehat{\frac{p}{q}} + \dots + \widehat{\frac{p}{q}}\right)}_{\text{de } q \text{ ori}} * \hat{x} = q\hat{x}, \forall \hat{x} \in \mathbb{Q}/\mathbb{Z}$$

$$\| $$

$$q \cdot \widehat{\frac{p}{q}} = \hat{p} = \hat{0} \Rightarrow \hat{0} * \hat{x} = \boxed{\hat{0} = q\hat{x}}, (\forall)\hat{x} \in \mathbb{Q}/\mathbb{Z}$$

$$x = \frac{1}{q+1} \Rightarrow \hat{0} = \widehat{\frac{q}{q+1}} \Rightarrow \frac{q}{q+1} \in \mathbb{Z} \text{ fals!}$$

Deci, nu există element unitate $\Rightarrow (\mathbb{Q}/\mathbb{Z}, +)$ nu are o structură de inel.

$\boxed{Ex\ 8}$ Fie $a \in R$ = inel a.î. $(\exists!)\ a' \in R$ a.î. $a'a = 1 \Rightarrow$
$\Rightarrow \boxed{a \cdot a' = 1}$

Dem: $a \cdot a' \overset{??}{=} 1$

Calculez $(a' + aa' - 1)a = a'a + aa'a - a =$

$= 1 + a - a = 1 \Rightarrow \cancel{a'} + aa' - 1 = \cancel{a'}$

$$\boxed{aa' = 1}$$

$\boxed{Ex\ 10}$ A = inel comutativ finit, $a \in A \setminus \{0\} \Rightarrow a \in U(A)$ sau a e divizor al lui zero.

($\Rightarrow A$ = domeniu de integ. finit $\Rightarrow A$ e corp)

$\boxed{SOL}$:

Fie $a \in A - \{0\}$ și pp. că a nu e divizor al lui zero

$f_a : A \longrightarrow A$, $f_a(x) := ax$.

f_a e injectivă $\left(f_a(x) = f_a(y) \Rightarrow ax = ay \Rightarrow \underset{\neq 0}{a}(x-y) = 0 \Rightarrow \right.$

$\left. \Rightarrow x - y = 0 \Rightarrow \underline{x = y} \right)$

$\left.\begin{array}{l} A = \underline{\text{finită}} \\ f_a \ \underline{\text{injectivă}} \end{array}\right\} \Rightarrow f_a$ e $\underline{\text{surjectivă}} \Rightarrow (\exists)\, x \in A$ a.î. $f_a(x) = 1$.

$\Rightarrow ax = 1 = xa \Rightarrow \boxed{a \in U(A)}$

$\boxed{A = \mathbb{Z}_n} \Rightarrow$ aflăm divizorii lui zero.

$\boxed{Ex\ 11}$ A = inel $\underline{\text{comutativ finit}}$, $|A| = n$

$N := \{ r \in A \mid r \neq 0,\ r = \underline{\text{neinversabil}} \} \neq \emptyset$

$\Rightarrow |N| \geq [\sqrt{n}] - 1$.

$\boxed{SOL}$: Fie $x \in N$

$f : A \longrightarrow Ax$, $f(a) := ax$

- f morfism de grupuri abeliene, f = surjectiv

$Ker(f) = \{ a \in A \mid ax = 0 \}$

$\Rightarrow$ (T.F.I de la grupuri) $\Rightarrow A/_{Ker(f)} \simeq Ax$

(izo de $\underline{\text{grupuri}}$) $\Rightarrow$

$n = |A| = |Ax| \cdot |Ker(f)|$

$\left.\begin{array}{l} Ax,\ Ker(f) \subseteq N \cup \{0\} \Rightarrow |Ax| \leq |N| + 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad |Ker(f)| \leq |N| + 1 \end{array}\right\} \Rightarrow$

$\Rightarrow (|N|+1)^2 \geq n \Rightarrow |N|+1 \geq \sqrt{n} \geq [\sqrt{n}] \Rightarrow$

$\Rightarrow |N| \geq [\sqrt{n}] - 1.$

Dacă A = inel comutativ cu $|A| = 100 \Rightarrow$

$\Rightarrow |N| \geq 9$

Algebră

Ex 40) $\operatorname{idem}\left(\mathbb{Z}[X]/_{(X^2-1)}\right) = ?$

$\boxed{\text{Sol}}$: $\hat{f}$, $f \in \mathbb{Z}[X]$, $\underline{T.Î.R.} \Rightarrow (\exists!)\ g, r \in \mathbb{Z}[X]$ a.î.

$f = (X^2-1)\cdot \underline{g} + \underline{\underline{r}}$, $r = 0$ sau $\underline{\text{grad}(r) < 2}$

$\Rightarrow (\exists!)\ g \in \mathbb{Z}[X]$, $a, b \in \mathbb{Z}$ a.î. $f = (X^2-1)g + a + bX \Rightarrow$

$$\left.\begin{array}{l} \hat{f} = \underline{\underline{\widehat{(X^2-1)}}}\cdot \hat{g} + \widehat{(a+bX)} \\ X^2-1 \in (X^2-1) \Rightarrow \boxed{\widehat{(X^2-1)} = \hat{0}} \end{array}\right\} \Rightarrow \hat{f} = \widehat{a+bX} = a\hat{1} + b\hat{X},\quad \hat{X} = x.$$

$\Rightarrow \hat{X}^2 = 1 \Rightarrow x^2 = 1$

$\left(\begin{array}{l} (a),\ R = \text{inel},\ a \in R,\ (a) = Re = \{re \mid r \in R\} \\ R/_{(a)},\ \text{ideal} \end{array} \right)$

$\Rightarrow \hat{f} = a + bx$, $a, b \in \mathbb{Z}$.

$$\mathbb{Z}[X]/_{(X^2-1)} = \{a + bx \mid a, b \in \mathbb{Z},\ x = \hat{X}\}$$
$$= \boxed{\{a + bx \mid a, b \in \mathbb{Z},\ x^2 = 1\}}$$

idempotentă

$\boxed{e = e^2}$, $e = a + bx$; $e = e^2 \Leftrightarrow$

$$a + bx = a^2 + b^2 + 2abx \Leftrightarrow \begin{cases} a^2 + b^2 = a \\ 2ab = b \Rightarrow \boxed{b = 0} \end{cases}$$

$\left(b \neq 0,\ a = \frac{1}{2} \in \mathbb{Z},\ \text{fals!}\right)$ $\quad \Rightarrow a = a^2 \Rightarrow a \in \{0, 1\}$

$\operatorname{idem}\left(\mathbb{Z}[X]/_{(X^2+1)}\right) = \{0, 1\}$

~~Ex 41)~~ (Curs $\mathbb{Z}_2[X]/(X^2+X-\hat{1})$, ... $\hat{f}=\widehat{a+bx}$ ... din T.Î.R. $a,b\in\mathbb{Z}_2$)

$\boxed{\text{Ex 41)}}$ $\mathbb{Z}[X]/(X^2-1) \not\simeq \mathbb{Z}\times\mathbb{Z}$, $\mathbb{Q}[X]/(X^2-1)\simeq\mathbb{Q}\times\mathbb{Q}$

$\boxed{\text{Sol}}$: $|\text{idem}\ \mathbb{Z}[X]/(X^2-1)|=2$ (1)

$\text{idem}(\mathbb{Z}\times\mathbb{Z})=\{(0,0),(0,1),(1,0),(1,1)\}$

$e=(a,b)$, $e^2=e \Leftrightarrow (a^2,b^2)=(a,b)$, $\begin{cases} a\in\{0,1\} \\ b\in\{0,1\}\end{cases}$

$|\text{idem}(\mathbb{Z}\times\mathbb{Z})|=4$ (2)

(1)+(2) $\Rightarrow \mathbb{Z}[X]/(X^2-1)\not\simeq\mathbb{Z}\times\mathbb{Z}$.

($R\simeq S$ (izo de inele) $\Rightarrow |\text{idem}(R)|=|\text{idem}(S)|$
$f:R\xrightarrow{\sim}S$, $r^2=r \Leftrightarrow f(r)^2=f(r)$)

• $\mathbb{Q}[X]/(X^2-1)\overset{?}{\simeq}\mathbb{Q}\times\mathbb{Q}$ (izo. de inele)

$\varphi: \underline{\mathbb{Q}[X]} \longrightarrow \underline{\mathbb{Q}\times\mathbb{Q}}$

$\varphi(f):=(f(1),f(-1))$

• φ e <u>morfism</u> de inele (Ex!) și e <u>surjectiv</u>!

$(a,b)\in\mathbb{Q}\times\mathbb{Q}$, $\exists ?\ f\in\mathbb{Q}[X]$ a.î. $\varphi(f)\overset{?}{=}(a,b)$.

$\varphi\left(\frac{a+b}{2}+\frac{a-b}{2}X\right)=(a,b)$

$\text{Ker}(\varphi)\overset{?}{=}(X^2-1)$

"$\supseteq$" $X^2-1\overset{??}{\in}\text{Ker}(\varphi)\leq\mathbb{Q}[X]$

$\varphi(X^2-1)=(0,0)\in\text{Ker}(\varphi)$.

"$\subseteq$" $f \in Ker(\varphi) \Rightarrow \varphi(f) = (0,0) \Rightarrow f(1) = 0, \ f(-1) = 0$

Teorema Bezout: a e răd pt. $f \Leftrightarrow X - a \mid f$

$\Rightarrow X-1 \mid f, \ X+1 \mid f \Rightarrow X^2-1 \mid f \Rightarrow (\exists)\ g \in \mathbb{Q}[X]$ a.î. $f = (X^2-1)g$

$(X-1, X+1) = 1$

$\Rightarrow f \in (X^2-1)$

$\boxed{Ker(\varphi) = (X^2-1)}$

T.F.i.i. $\Rightarrow \overline{\varphi} : \mathbb{Q}[X]/_{(X^2-1)} \xrightarrow{\sim} \mathbb{Q} \times \mathbb{Q}$

$\overline{\varphi}(\hat{f}) := (f(1), f(-1))$ e izo. de inele.

Ex 42) a) $\mathbb{Z}[i]/_{(1+i)} \simeq \mathbb{Z}_2$

$\boxed{\text{SOL}}$: Vrem $\varphi : \mathbb{Z}[i] \longrightarrow \mathbb{Z}_2$ morfism surj. de inele

$\varphi(\underbrace{a+bi}_{\mathbb{Z}[i]}) := \underbrace{\widehat{a+b}}_{\text{în } \mathbb{Z}_2}$ $\quad Ker\ \varphi = \{(1+i)\}$

- φ e morfism de inele (Ex!) și surjectiv

$\left(\varphi(a+bi) = \widehat{a^2+b^2} \overset{\widehat{a+b}}{\shortparallel} \quad Ker(\varphi) = ? \right)$

$Ker(\varphi) \overset{??}{=} (1+i)$

"$\supseteq$" $\varphi(1+i) = \widehat{1+1} = \hat{0} \Rightarrow (1+i) \in Ker(\varphi)$

"$\subseteq$" $Ker(\varphi) \subseteq (1+i)$

$z \in Ker(\varphi), \ z = a+bi, \ \varphi(a+bi) = \widehat{a+b} = \hat{0} \Rightarrow$

$\Rightarrow 2 \mid a+b \Rightarrow a$ și b au aceeași paritate

Vreau: $\exists\ x, y \in \mathbb{Z}$ a.î. $a+bi \overset{!!}{=} (1+i)(x+iy)$

$\Leftrightarrow a+bi = x-y+i(x+y) \Rightarrow \begin{cases} x-y = a \\ x+y = b \end{cases} \Rightarrow$

$\Rightarrow x = \frac{a+b}{2}, \ y = \frac{b-a}{2} \in \mathbb{Z} \Rightarrow a+bi = (1+i)\underbrace{\left(\frac{a+b}{2} + i\frac{b-a}{2}\right)}_{\in \mathbb{Z}[i]}$

$\cap$

$(1+i)$

T.F.i.i $\Rightarrow \tilde{\varphi} : \mathbb{Z}[i]/_{(1+i)} \simeq \mathbb{Z}_2$

$\tilde{\varphi}(\widehat{\widehat{a+bi}}) := \widehat{a+b}$

e izo. de inele.

(Ex 43)

b) $\mathbb{Z}[i]/_{(2+i)} \simeq \mathbb{Z}_5$

$\boxed{SOL}$: Vrem $\varphi : \mathbb{Z}[i] \longrightarrow \mathbb{Z}_5$ $\boxed{\text{morf. surj. și } Ker(\varphi) = (2+i)}$

$\widehat{a+3b}$

$\varphi(a+bi) := \widehat{a+3b}$

φ e morfism, φ = surj $(\varphi(a) = \hat{a})$

$Ker(\varphi) \stackrel{?}{=} (2+i)$

"$\supseteq$" $\varphi(2+i) = \widehat{2+3} = \hat{0} \Rightarrow (2+i) \subseteq Ker\,\varphi$

"$\subseteq$" Fie $a+bi \in Ker(\varphi) \Rightarrow \widehat{a+3b} = \hat{0} \Rightarrow \boxed{5 \mid a+3b}$

Caut $x, y \in \mathbb{Z}$ a.î. $a+bi = (2+i)(x+iy) \Leftrightarrow \begin{cases} a = 2x - y \\ b = x + 2y \end{cases}$

$y = 2x - a$, $b = x + 4x - 2a \Rightarrow \boxed{x = \frac{b+2a}{5}} \stackrel{?}{\in} \mathbb{Z}$

$5 \mid a+3b$, $5 \mid 5a+5b \Rightarrow 5 \mid 4a+2b \Rightarrow 5 \mid 2a+b$

$y = 2 \cdot \frac{b+2a}{5} - a \in \mathbb{Z}$

$a+bi = (2+i)\left(\frac{b+2a}{5} + i\,\frac{2b-a}{5}\right) \in (2+i)$

T.F.i.i. $\Rightarrow \tilde{\varphi} : \mathbb{Z}[i]/_{(2+i)} \xrightarrow{\sim} \mathbb{Z}_5$, $\tilde{\varphi}(\widehat{a+bi}) := \widehat{a+3b}$

e izo de inele.

Ex 43) $a_1, \dots, a_n \in \mathbb{R}$, $a_i \neq a_j$, $(\forall)\ i \neq j$

$\Rightarrow \mathbb{R}[X] \big/ (X-a_1)(X-a_2)\dots(X-a_n) \simeq \mathbb{R}^n$.

(cum aplicați lema chineză a resturilor) ???

SOL:

Fie $\varphi : \mathbb{R}[X] \longrightarrow \mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$

$\varphi(f) := (f(a_1), \dots, f(a_n))$

- φ e morfism de inele (Ex!) și e surjectiv??

$(\forall)\ (c_1, c_2, \dots c_n) \in \mathbb{R}^n$ $(\exists)\ f \in \mathbb{R}[X]$ a.î. $f(a_1) = c_1$, $f(a_2) = c_2$, $\dots$, $f(a_n) = c_n$.

f s.n. polinom de interpolare Lagrange!

$$f := \sum_{i=1}^{n} c_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{X - a_j}{a_i - a_j} \right) =$$

$$= c_1 \cdot \frac{(X-a_2)\dots(X-a_n)}{(a_1-a_2)\dots(a_1-a_n)} + \dots + c_n \cdot \frac{(X-a_1)\dots(X-a_{n-1})}{(a_n-a_1)\dots(a_n-a_{n-1})}.$$

$f(a_1) = c_1$, $f(a_2) = c_2$, $\dots$, $f(a_n) = c_n$.

- $\mathrm{Ker}(\varphi) = ((X-a_1)(X-a_2)\dots(X-a_n))$,

"$\supseteq$" $\varphi((X-a_1)\dots(X-a_n)) = (0, 0, \dots 0)$

"$\subseteq$" Fie $f \in \mathrm{Ker}(\varphi) \Rightarrow f(a_1) = 0$, $f(a_2) = 0$, $\dots$ $f(a_n) = 0$

$\Rightarrow$ (Bezout) $X - a_i \mid f$, $\forall\ i = \overline{1,n}$

$(X-a_i, X-a_j) = 1$, $\forall\ i \neq j$.

$\Rightarrow (X-a_1)(X-a_2)\dots(X-a_n) \mid f \Rightarrow f \in ((X-a_1)(X-a_2)\dots(X-a_n))$

T.7.1.i $\Rightarrow \tilde{\varphi} : \mathbb{R}[X] / ((X-a_1)\dots(X-a_n)) \xrightarrow{\sim} \mathbb{R}^n$

$\tilde{\varphi}(\hat{f}) := (f(a_1), \dots, f(a_n))$ e izo de inele.

Ex 28) $R_1, \dots R_n$ inele, $R = R_1 \times \dots \times R_n$.

$I \leq_\Delta R_1 \times R_2 \times \dots \times R_n \Leftrightarrow (\exists)\ I_k \leq_\Delta R_k\ (\forall)\ k = \overline{1,n}$ a.î.

$I = I_1 \times I_2 \times \dots \times I_n$. $((\exists)\ H \leq G \times G$ grup a.î. $H \neq K_1 \times K_2)$

$\boxed{\text{SOL}}$: "$\Leftarrow$" $I_k \leq_\Delta R_k\ (\forall)\ k = \overline{1,n} \Rightarrow$

$I := I_1 \times I_2 \times \dots \times I_n \leq_\Delta R_1 \times R_2 \times \dots \times R_n$ (Ex!)

"$\Rightarrow$" Fie $I \leq_\Delta R_1 \times R_2 \times \dots \times R_n$

Fie $\pi_k : \underline{R_1 \times R_2 \times \dots \times R_n} \longrightarrow R_k$ $\quad k = \overline{1,n}$ morfism surj. de inele

$\pi_k(r_1, r_2, \dots r_n) := r_k$

$\Rightarrow \pi_k(I) \leq_\Delta R_k, (\forall)\ k = \overline{1,n}$

Afirm: $I = \pi_1(I) \times \pi_2(I) \times \dots \times \pi_k(I)$

$\pi_k(I) = \{ a_k \in R_k \mid (\exists)\ x = (x_1, \dots, x_n) \in I$ a.î. $a_k = x_k \}$

"$\subseteq$" $x = (a_1, a_2, \dots a_n) \in I \Rightarrow a_1 \in \pi_1(I), a_2 \in \pi_2(I), \dots a_n \in \pi_n(I)$.

"$\supseteq$" Fie $x = (x_1, x_2, \dots x_n) \in \pi_1(I) \times \dots \times \pi_n(I) \Rightarrow$

$(\forall)\ i = \overline{1,n}, (\exists)\ y^i \in I, y^i = (y_1^i, y_2^i, \dots y_n^i) \in I$ a.î.

$y_i^i = x_i$

Fie $z^i := (z_1^i, z_2^i, \dots, z_n^i) \in \underline{R_1 \times \dots \times R_n}$

unde $z_j^i := \delta_{ij}$

$$\underset{\substack{\cap \\ I}}{z^1 \cdot y^1} + \underset{\substack{\cap \\ I}}{z^2 \cdot y^2} + \dots + \underset{\substack{\cap \\ I}}{z^n \cdot y^n} = (x_1, x_2, \dots, x_n) = x \in I \Rightarrow$$

$$\Rightarrow \pi_1(I) \times \pi_2(I) \times \cancel{\pi_3 I} \dots \times \pi_n(I) \subseteq I.$$

b) $(R_1 \times \dots \times R_n)/_{I_1 \times I_2 \times \dots \times I_n} \simeq R_1/_{I_1} \times \dots \times R_n/_{I_n}$

$\boxed{\text{SOL}}$: $\varphi : R_1 \times \dots \times R_n \longrightarrow R_1/_{I_1} \times \dots \times R_n/_{I_n}$

$\varphi(x_1, \dots x_n) := (\widehat{x_1}^1, \widehat{x_2}^2, \dots, \widehat{x_n}^n)$

φ e morfism surj. de inele și $\mathrm{Ker}(\varphi) = I_1 \times \dots \times I_n$.

T.F.i.i. $\Rightarrow$ izomorfismul de mai sus (cu 2 rând)

Ex 30) Ideale lui $\mathbb{Z}_6$: ← Ideale lui $\mathbb{Z}_n$! $\hat{d}\mathbb{Z}_n$, $d \mid n$.

$\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/_{6\mathbb{Z}} = \mathbb{Z}_6$ T.C. pt. ideale

$\Rightarrow \mathbb{Z}_6$ sunt: $\mathcal{L}(\mathbb{Z}_6) = \{\hat{0}, \mathbb{Z}_6, \hat{2} \cdot \mathbb{Z}_6, \hat{3} \cdot \mathbb{Z}_6\}$

$\hat{2}\mathbb{Z}_6 = \{\hat{0}, \hat{2}, \hat{4}\}$, $\hat{3}\mathbb{Z}_6 = \{\hat{0}, \hat{3}\}$

idealele lui $\mathbb{Z} \times \mathbb{Z}_6 \times \mathbb{Q}$?? $\boxed{\text{Ex 28}}$

→ Idealele lui $\mathbb{Z}$: $n\mathbb{Z}$, $n \in \mathbb{N}$

→ Idealele lui $\mathbb{Z}_6$: ok.

→ Idealele lui $\mathbb{Q}^6 = \{0, \mathbb{Q}\}$