LU, LDU, LDL^T

Să se determine factorizările LU fără pivotare, LDU și LDL^T ale matricei

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

• Factorizarea LU fără pivotare:

$$A = \left[\begin{array}{c|cc} 25 & 15 & -5 \\ \hline 15 & 18 & 0 \\ -5 & 0 & 11 \end{array}\right] = \underbrace{\left[\begin{array}{c|c} l_{11} & \underline{0} \\ \hline \underline{L}_{21} & L_{22} \end{array}\right]}_{L} \underbrace{\left[\begin{array}{c|c} u_{11} & \underline{U}_{12} \\ \hline \underline{0} & U_{22} \end{array}\right]}_{U}$$

$$= \left[\begin{array}{c|c} l_{11} u_{11} & l_{11} \underline{U}_{12} \\ \hline \underline{L}_{21} u_{11} & \underline{L}_{21} \underline{U}_{12} + L_{22} U_{22} \end{array}\right] \Rightarrow$$

• $l_{11} u_{11} = 25 \Rightarrow \boxed{l_{11} = 1}$ (luăm)

$\boxed{u_{11} = 25}$

• $l_{11} \underline{u}_{12} = [15 \ \ -5] \Rightarrow$

$\underline{u}_{12} = [15 \ \ -5] \Rightarrow \boxed{u_{12} = 15} \ \boxed{u_{13} = -5}$

• $\underline{L}_{21} u_{11} = \begin{bmatrix} 15 \\ -5 \end{bmatrix} \Rightarrow \underline{L}_{21} = \frac{1}{25}\begin{bmatrix} 15 \\ -5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix} \Rightarrow$

$\boxed{l_{21} = 3/5} \quad \boxed{l_{31} = -1/5}$

• $\underline{L}_{21} \underline{u}_{12} + L_{22} U_{22} = \begin{bmatrix} 18 & 0 \\ 0 & 11 \end{bmatrix} \Rightarrow$

$L_{22} U_{22} = \begin{bmatrix} 18 & 0 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix} [15 \ \ -5]$

$= \begin{bmatrix} 18 & 0 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 10 \end{bmatrix} = S$

Problema s-a redus la factorizare LU a matricei S, ie

$$\left[\begin{array}{c|c} 9 & 3 \\ \hline 3 & 10 \end{array}\right] = \left[\begin{array}{c|c} l_{22} & 0 \\ \hline l_{32} & l_{33} \end{array}\right] \left[\begin{array}{c|c} u_{22} & u_{23} \\ \hline 0 & u_{33} \end{array}\right] =$$

$$= \left[\begin{array}{c|c} l_{22}u_{22} & l_{22}u_{23} \\ \hline l_{32}u_{22} & l_{32}u_{23}+l_{33}u_{33} \end{array}\right] \Rightarrow$$

- $l_{22}u_{22} = 9 \Rightarrow \boxed{l_{22}=1}$ (luăm) $\boxed{u_{22}=9}$

- $l_{22}u_{23} = 3 \Rightarrow \boxed{u_{23}=3}$

- $l_{32}u_{22} = 3 \Rightarrow l_{32}\cdot 9 = 3 \Rightarrow \boxed{l_{32}=1/3}$

- $l_{32}u_{23}+l_{33}u_{33} = 10 \Rightarrow$

$$l_{33}u_{33} = 10 - l_{32}u_{23} = 10 - \frac{1}{3}3 = 9 \Rightarrow$$

Problema s-a redus la factorizarea LU a lui 9 (1-dimensională)

$$l_{33}u_{33} = 9 \Rightarrow \boxed{l_{33}=1} \; \boxed{u_{33}=9}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

• Factorizarea LDU:

Am obținut

$$A = L\tilde{U} = \begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

Vrem $\tilde{U} = DU$

$$D = \operatorname{diag}(25, 9, 9)$$

$$U = D^{-1}\tilde{U} \Rightarrow$$

$$U = \begin{bmatrix} 1/25 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix} \Rightarrow$$

$$U = \begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

• Factorizarea LDL^T coincide, în acest caz, cu fact LDU.