SEMINAR 3_ELTN_311

Recapitulare de la cursul de partea ? ...

Pt $x \in \mathbb{R}$,

$[x]$ = cel mai mare nr. întreg ce nu-l întrece pe x.

Prop. (i) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{Z}\ [x+n] = [x]+n$.

(ii) $\forall x, y \in \mathbb{R}\ \ [x+y] - ([x]+[y]) \in \{0,1\}$

(iii) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}^*\ \left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$

Obs (a-n):

$\forall a, x \in \mathbb{R}$

$a = [x] \Longleftrightarrow (a \in \mathbb{Z} \wedge a \le x < a+1)$

$\Longleftrightarrow (a \in \mathbb{Z} \wedge x-1 < a \le x)$

$$\lim_{x \to \infty} \frac{[3x-2\pi]}{2x+11}$$

Fie $x > 1$.

Atunci

$$\frac{3x-2\pi-1}{2x-1} < \frac{[3x-2\pi]}{2x-1} \le \frac{3x-2\pi}{2x-1}$$

$\xrightarrow{x \to \infty} \frac{3}{2}$ $\quad \downarrow \quad \swarrow_{x \to \infty}$

$\frac{3}{2}$

• Rezolvați ecuația

$$\left[\frac{7x+4}{3}\right] = \frac{5x+1}{2}, \quad (ec)$$

$$\begin{cases} x = \frac{2k-1}{5}, k \in \mathbb{Z} \\ \frac{5x+1}{2} \le \frac{7x+4}{3} < \frac{5x+3}{2} \Leftrightarrow \\ 15x+3 \le 14x+8 < 15x+9 \Leftrightarrow \\ 5 \ge x > -1 \\ 5 \ge \frac{2k-1}{5} > -1 \Leftrightarrow \\ 25 \ge 2k-1 > -5 \Leftrightarrow \\ 13 \ge k > -2 \end{cases}$$

Sol: $(ec) \Leftrightarrow \frac{5x+1}{2} \in \mathbb{Z} \wedge \frac{5x+1}{2} \le \frac{7x+4}{3} < \frac{5x+3}{2}$

$\Leftrightarrow \left(\exists k \in \mathbb{Z}\left(\frac{5x+1}{2} = k \wedge 15x+3 \le 14x+8 < 15x+9\right)\right)$

$\Leftrightarrow \left(\exists k \in \mathbb{Z}\left(x = \frac{2k-1}{5} \wedge -1 < x \le 5\right)\right) \Leftrightarrow$

$\Leftrightarrow \left(\exists k \in \mathbb{Z}\left(x = \frac{2k-1}{5} \wedge -1 < \frac{2k-1}{5} \le 5\right)\right) \Leftrightarrow$

$\Leftrightarrow \left(\exists k \in \mathbb{Z}\left(\cancel{-2} < k \le 13 \wedge x = \frac{2k-1}{5}\right)\right) \Leftrightarrow$

$\Leftrightarrow \exists k \in \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}\ x = \frac{2k-1}{5}$

$\Leftrightarrow x \in \left\{\frac{2k-1}{5} : k \in \mathbb{Z} \wedge -1 \le k \le 13\right\}$. □

1) $\forall x \in \mathbb{R}$ $\left[x+\frac{1}{2}\right] = [2x] - [x]$

1') $\forall x \in \mathbb{R}$ $\forall n \in \mathbb{N}^*$ $[x] + \left[x+\frac{1}{n}\right] + \left[x+\frac{2}{n}\right] + \dots + \left[x+\frac{n-1}{n}\right] = [nx]$

2. Determinați numărul soluțiilor naturale ale ecuației

$$\left[\frac{x}{a}\right] = \left[\frac{x}{a-1}\right], \text{ unde } a \in \mathbb{N}^* \setminus \{1\} \text{ e fixat.}$$

3. Rezolvați ecuațiile: a) $\frac{\{x\}}{x} + \frac{x}{[x]} = \frac{3}{2}$

b) $[x]^3 + \{x\} = \frac{3}{2}$

4. Fie $n \in \mathbb{N}^* \setminus \{1,2\}$ și fie $x \in \mathbb{R}$. Atunci

$$\{x\} = \{x^2\} = \{x^n\} \Longrightarrow x \in \mathbb{Z}$$

Sol 1) Fie $x \in \mathbb{R}$

Dacă $\{x\} < \frac{1}{2}$,

$$\left[x+\frac{1}{2}\right] = \Big[[x] + \underbrace{\{x\}+\frac{1}{2}}_{<1}\Big] = [x]$$

$$[2x] - [x] = \Big[2[x] + \underbrace{2\{x\}}_{<1}\Big] - \big[[x] + \{x\}\big] = 2[x] - [x] = [x]$$

$\Longrightarrow$

$$\left[x+\frac{1}{2}\right] = [2x] - [x]$$

Dacă $\frac{1}{2} \le \{x\} < 1$,

$$\left[x+\frac{1}{2}\right] = \Big[x + \underbrace{\{x\}+\frac{1}{2}}_{1 \le \,\cdot\, < \frac{3}{2}}\Big] = [x+1] = [x]+1$$

$$[2x] - [x] = \Big[2[x] + \underbrace{2\{x\}}_{1 \le \,\cdot\, < 2}\Big] - [x] = 2[x]+1 - [x] = [x]+1$$

$\Longrightarrow$

$$\left[x+\frac{1}{2}\right] = [2x] - [x]$$

a

Sol 1) Fie $k\in\{0,1,\dots,n-1\}$ și fie $x\in\mathbb{R}$ astfel încât $\{x\}\in[\frac{k}{n},\frac{k+1}{n})$.

Atunci $\sum_{i=0}^{n-1}[x+\frac{i}{n}] = \sum_{\substack{i=0\\ i\le n-k-1}}^{n-1}\overbrace{[x+\frac{i}{n}]}^{[x]} + \sum_{\substack{i=0\\ i\ge n-k}}^{n-1}\overbrace{[x+\frac{i}{n}]}^{[x]+1} =$

$= (n-k)[x]+k([x]+1) = n[x]+k.$ (1)

Pe de altă parte,

$\{x\}\in[\frac{k}{n},\frac{k+1}{n}) \Rightarrow x\in[[x]+\frac{k}{n},[x]+\frac{k+1}{n}) \Rightarrow$

$\Rightarrow nx\in[n[x]+k, n[x]+k+1) \Rightarrow$

$[nx]=n[x]+k$

(1) $\Rightarrow \sum_{i=0}^{n-1}[x+\frac{i}{n}]=[nx].$

3b) (3b) $\Rightarrow \{x\}=\{\frac{3}{2}\} \Leftrightarrow \{x\}=\frac{1}{2}.$

Atunci, (3b) $\Rightarrow [x]^3=1 \Leftrightarrow [x]=1.$

Ca urmare, $x=[x]+\{x\}=\frac{3}{2}$

Evident, în fine,

$[\frac{3}{2}]^3+\{\frac{3}{2}\}=1^3+\frac{1}{2}=\frac{3}{2},$

deci $x=\frac{3}{2}$ verifică (3b).

Ca urmare, (3b) are soluția unică $x=\frac{3}{2}$.

Cond: $x \in \mathbb{R} \setminus [0,1)$. Rezolvăm în această mulțime. 5

(3a) $\Leftrightarrow \frac{x-[x]}{x} + \frac{x}{[x]} = \frac{3}{2}$ $\Leftrightarrow$

$1 - \frac{[x]}{x} + \frac{x}{[x]} = \frac{3}{2} \Leftrightarrow \frac{x}{[x]} - \frac{[x]}{x} = \frac{1}{2}$ (1)

~~Dacă~~ $\cancel{\frac{x}{[x]} \geq 0} \Leftrightarrow$ Observăm că $\frac{x}{[x]} > 0 \ \forall x \in \mathbb{R} \setminus [0,1)$

~~Dacă~~ $\cancel{\frac{x}{[x]} \geq 0}$,

$\cancel{(1) \Leftrightarrow \left(\sqrt{\frac{x}{[x]}} - \sqrt{\frac{[x]}{x}}\right)^2}$

Punem $t = \frac{x}{[x]}$

$(1) \Leftrightarrow t - \frac{1}{t} = \frac{1}{2} \Leftrightarrow$

$2t^2 - t - 2 = 0 \Leftrightarrow$

$t \in \left\{ \frac{1-\sqrt{17}}{4}, \frac{1+\sqrt{17}}{4} \right\}$

Observăm că $\forall x \in \mathbb{R} \setminus [0,1)$ $\frac{x}{[x]} > 0$,

deci $t = \frac{1-\sqrt{17}}{4}$ nu dă soluții pt (3a).

Rămâne varianta $\frac{x}{[x]} = \frac{1+\sqrt{17}}{4} \Leftrightarrow$

$\frac{\{x\}}{[x]} = \frac{\sqrt{17}-3}{4}$ $\qquad \frac{[x]}{x} = \frac{4}{1+\sqrt{17}} \Rightarrow$

$\frac{x-1}{x} < \frac{4}{1+\sqrt{17}} < 1$. $\Rightarrow$

$\Rightarrow 1 - \frac{1}{x} < \frac{4}{1+\sqrt{17}} \Leftrightarrow 1 - \frac{4}{1+\sqrt{17}} < \frac{1}{x} \Leftrightarrow$

$x > \frac{\sqrt{17}+1}{\sqrt{17}-3}$