Tema 4 (Monte Carlo)

ONUȚU RADU

Grupa 312

1) $\int_0^\infty x^6 \cdot e^{-3x}\,dx = \frac{1}{3}\int_0^\infty \left(\frac{t}{3}\right)^6 \cdot e^{-t}\,dt =$

$3x = t$

$3dx = dt$

$= \frac{1}{3^7}\cdot\int_0^\infty t^6 \cdot e^{-t}\,dt = \frac{1}{3^7}\cdot\Gamma(7) = \frac{6!}{3^7} = \frac{80}{3^5} = \frac{80}{243}$

$\simeq 0,32$

2) $\int_0^\infty x^7 \cdot e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty t^3 \cdot e^{-t}\,dt = \frac{1}{2}\Gamma(4)$

$x^2 = t$ $\qquad = \frac{3!}{2} = 3$

$2x\,dx = dt$

3) $\int_0^1 (x-x^2)^5\,dx = \int_0^1 (x(1-x))^5\,dx = \beta(6,6) =$

$= \frac{\Gamma(6)\cdot\Gamma(6)}{\Gamma(12)} = \frac{5!\cdot 5!}{11!} = \frac{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}}{\cancel{2}\cdot\cancel{6}\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11} = \frac{1}{630}$

$4 \cdot 10 \cdot 11$

$= \frac{1}{2772} \simeq$

$0,0003$

4) $\int_{-\infty}^\infty x^4 \cdot e^{-x^2}\,dx = 2\int_0^\infty x^4 \cdot e^{-x^2}\,dx = \int_0^\infty t^{\frac{3}{2}} \cdot e^{-t}\,dt$

$x^2 = t \Rightarrow$ ~~$t = 2\sqrt{x}$~~ $x = \sqrt{t}$

$2x\,dx = dt$

$= \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4} \simeq 1,32$

5) $\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \left(x(1-x)\right)^{\frac{1}{2}} dx = \beta\left(\frac{3}{2},\frac{3}{2}\right)$

$= \frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$

$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2!}$

$= \frac{1}{8}\cdot\pi = \frac{\pi}{8} \simeq 0,39$

6) $\int_0^1 \frac{1}{\sqrt[6]{x^5(1-x)}}\,dx = \int_0^1 x^{-\frac{5}{6}}\cdot(1-x)^{-\frac{1}{6}}\,dx = \beta\left(\frac{1}{6},\frac{5}{6}\right) =$

$= \frac{\pi}{\sin\left(\frac{\pi}{6}\right)} = 2\pi \simeq 6,28$

7) $\int_0^2 x^2\sqrt{4-x^2}\,dx = \int_0^2 x^2\sqrt{4\left(1-\frac{x^2}{4}\right)}\,dx = 2\cdot\int_0^1 2\sqrt{t}\cdot\sqrt{4(1-t)}\,dt$

$\frac{x^2}{4} = t$ $\quad x=0 \Rightarrow t=0 \quad x^2 = 4t$

$\quad x=2 \Rightarrow t=1 \quad x = 2\sqrt{t}$

$\frac{x}{2}\,dx = dt$

$= \int_0^1 8\cdot t^{\frac{1}{2}}\cdot(1-t)^{\frac{1}{2}}\,dt = 8\,\beta\left(\frac{3}{2},\frac{3}{2}\right) = 8\cdot\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$

$= 8\cdot\frac{\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{2!} = 4\cdot\frac{1}{4}\cdot\pi = \pi \simeq 3,14$

8) $\int_0^1 x \cdot \ln^5 x \, dx$

$\ln x = t \qquad x = 0 \Rightarrow t = -\infty;\ x = 1 \Rightarrow t = 1$

$\frac{1}{x} dx = dt \qquad \ln x = t \Rightarrow x = e^t$

$\int_{-\infty}^{0} e^{2t} \cdot t^5 \, dt$

$2t = -u \qquad \Rightarrow t = -\infty \Rightarrow u = \infty$

$2dt = -du \qquad t = 0 \Rightarrow u = 0$

$-\frac{1}{2}\int_{\infty}^{0} e^{-u} \cdot \left(-\frac{u}{2}\right)^5 du = -\frac{1}{2^6} \cdot \int_{0}^{\infty} e^{-u} \cdot u^5 \, du =$

$= -\frac{1}{2^6} \cdot \Gamma(6) = -\frac{1}{2^6} \cdot 5! = -\frac{15}{8} = -1,875$

9) $\int_1^3 \frac{1}{\sqrt{(3-x)(x-1)}} dx = \int_1^3 \frac{1}{\sqrt{-x^2+4x-3}} dx =$

$= \int_1^3 \frac{1}{\sqrt{-(x^2-4x+3)}} dx =$

$= \int_1^3 \frac{1}{\sqrt{-(x^2-4x+4)+1}} dx = \int_1^3 \frac{1}{\sqrt{-(x-2)^2+1}} dx$

$x - 2 = t \qquad x = 1 \Rightarrow t = -1$

$dx = dt \qquad x = 3 \Rightarrow t = 1$

$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} dt = \arcsin t \Big|_{-1}^{1} = \arcsin 1 - \arcsin(-1)$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi \simeq 3,14$

10) $\int_0^1 \sqrt{\ln \frac{1}{x}}$ …

$\ln x = t$ …

$\frac{1}{x} dx =$ …

$\int_{\infty}^{0} -e^{-u}$ …

11) $\int_0^{\frac{\pi}{2}} \sin^3 x$ …

$\sin x = t$

$\cos x \, dx = dt$ …

$t^2 = u$

$2t \, dt = du$

$\Rightarrow \frac{1}{2} \int_0^1 u$ …

12) $\int_{-4}^{0} x^6 \cdot \sqrt{16-}$ …

$\frac{x}{16}$ …

$\frac{x}{8} d$ …

$= 8 \int_1^0 (-4\sqrt{}$ …

$= 4^4 \cdot 128 \int_0^1 t$ …

10) $\int_0^1 \sqrt{\ln \frac{1}{x}}\,dx = \int_0^1 \sqrt{-\ln x}\,dx = \int_{-\infty}^0 \cancel{x} e^t \cdot \sqrt{-t}\,dt =$

$\ln x = t \Rightarrow \cancel{x} = e^t$ $\quad x=0 \Rightarrow t=-\infty \quad -t=u \quad t=-\infty \Rightarrow u=\infty$

$\frac{1}{x}dx = dt \quad x=1 \Rightarrow t=0 \quad -dt=du \quad t=0 \Rightarrow u=0$

$\int_\infty^0 -e^{-u}\cdot\sqrt{u}\,du = \int_0^\infty e^{-u}\cdot u^{\frac{1}{2}}\,du = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$

$= \frac{\sqrt{\pi}}{2} \simeq 0{,}88$

11) $\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos^5 x\,dx = \int_0^1 t^3\cdot(1-t^2)^2\,dt$

$\sin x = t \quad x=0 \Rightarrow t=0$

$\cos x\,dx = dt \quad x=\frac{\pi}{2} \Rightarrow t=1$

$t^2 = u$

$2t\,dt = du$

$\Rightarrow \frac{1}{2}\int_0^1 u\cdot(1-u)^2\,du = \frac{1}{2}\beta(2,3) = \frac{1}{2}\,\frac{\Gamma(2)\cdot\Gamma(3)}{\Gamma(5)} =$

$= \cancel{\frac{1}{2}\frac{2}{2\cdot3\cdot4\cdot8}} = \cancel{\frac{1}{20}} = \frac{1}{2}\cdot\frac{1!\cdot2!}{4!} = \frac{1}{2\cdot3\cdot4} = \frac{1}{24} \simeq 0{,}04$

12) $\int_{-4}^0 x^6\cdot\sqrt{16-x^2}\,dx = \int_{-4}^0 x^6\cdot\sqrt{16\left(1-\frac{x^2}{16}\right)}\,dx =$

$\frac{x^2}{16} = t \quad x=0 \Rightarrow t=0 \quad x=-4\sqrt{t}$

$x=-4 \Rightarrow t=1$

$\frac{x}{8}dx = dt$

$= 8\int_1^0 \left(-4\sqrt{t}\right)^5\cdot 4\sqrt{1-t}\,dt$

$= 4^4\cdot128\int_0^1 t^{\frac{5}{2}}\cdot(1-t)^{\frac{1}{2}}\,dt = 4^4\cdot128\cdot\beta\left(\frac{7}{2},\frac{3}{2}\right) =$

$$= 128 \cdot \frac{\Gamma\left(\frac{7}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma(5)} = 128 \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{4!}$$

$$= \frac{2^7 \cdot 5 \cdot 3 \cdot \pi \cdot 4^4}{2^4 \cdot 2 \cdot 3 \cdot 4} = 5\pi \cdot 4^4 = 1280\pi \simeq 4021,23$$

13) $$\int_0^\infty \frac{x^2}{1+x^4}\,dx = \cancel{x} \cdot = \frac{\pi}{\sqrt{8}} \simeq 1,11$$

$$\cancel{\int x \cdot \frac{1}{1+x^4}\,dx = \frac{1}{2}\int \frac{1}{1+t^2}\,dt = \frac{1}{2} \cdot \operatorname{arctg} t}$$

$\cancel{x^2 = t}$

$\cancel{2x\,dx = dt}$

14) $$\int_1^\infty e^{-x^2+2x-4}\,dx$$

$x^2 - 2x + 4 = t \iff x^2 - 2x + 4 - t = 0$

$2x - 2\,dx = dt \qquad \Delta = 4 - 16 + 4t$

$x = 1 \Rightarrow t = 3 \qquad \Delta = -12 + 4t$

$x = \infty \Rightarrow t = \infty \qquad x_{1,2} = \frac{2+\sqrt{4t-12}}{2} = 1 + \sqrt{t-3}$

$$= \int_3^\infty \frac{1}{2+2\sqrt{t-3}-2} \cdot e^{-t}\,dt = \int_3^\infty \frac{1}{2\sqrt{t-3}} \cdot e^{-t}\,dt$$

$t - 3 = u$

$dt = du$

$t = 3 \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

$$= \int_0^\infty \frac{1}{2} \cdot \frac{1}{\sqrt{u}} \cdot e^{-(u+3)}\,du$$

$\frac{1}{2}\cdot\sqrt{4}\cdot\frac{1}{2}\cdot\sqrt{4}$

!

$4021,23$

$+\sqrt{t-3}$

dt

$$= \frac{1}{2}\int_0^\infty u^{-\frac{1}{2}}\cdot e^{-u}\cdot e^{-3}\,du = \frac{1}{2e^3}\int_0^\infty u^{-\frac{1}{2}}\cdot e^{-u}\,du =$$

$$= \frac{1}{2e^3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2e^3} \simeq 0,04$$

15) $$\int_1^e \frac{1}{x}\ln^3 x\,(1-\ln x)^4\,dx$$

Let $\ln x = t \qquad x = 1 \Rightarrow t = 0$

$\frac{1}{x}dx = dt \qquad x = e \Rightarrow t = 1$

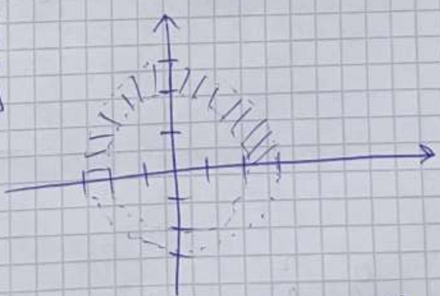
$$\int_0^1 t^3\cdot(1-t)^4\,dt = \beta(4,5) = \frac{\Gamma(4)\cdot\Gamma(5)}{\Gamma(9)} =$$

$$= \frac{3!\cdot 4!}{8!} = \frac{2\cdot 3}{5\cdot 6\cdot 7\cdot 8} = \frac{1}{280} \simeq 0,003$$

16) $$\iint_D \sqrt{x^2+y^2}\,dx\,dy$$

$D = \{(x,y)\in\mathbb{R}^2 \mid 4 \le x^2+y^2 \le 9, y \ge 0\}$

$\begin{cases} x = r\cos\theta & r\in[2,3] \\ y = r\sin\theta & \theta\in[0,\pi] \end{cases}$

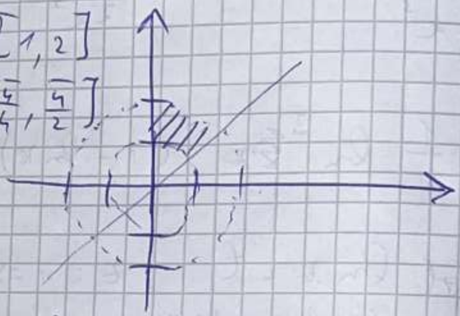


$$= \iint_{[2,3]\times[0,\pi]} \sqrt{r^2}\cdot r\,dr\,d\theta = \int_2^3 \pi r^2\,dr = \frac{\pi r^3}{3}\Big|_2^3 = \frac{19\pi}{3} \simeq 19,89$$

17) $\iint_D e^{\sqrt{x^2+y^2}}\,dx\,dy$

$D=\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 4,\ 0\le x\le y\}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad r\in[1,2]$, $\quad \theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$



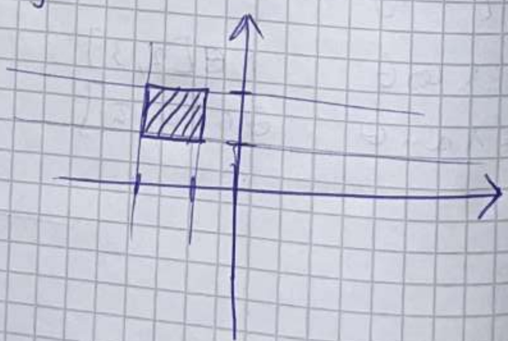
$\int_1^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^r\cdot r\,d\theta\,dr = \int_1^2 \frac{\pi}{4}\, r\cdot e^r\,dr =$

$= \frac{\pi}{4}\left(r\cdot e^r\Big|_1^2 - \int_1^2 e^r\,dr\right) = \frac{\pi}{4}\left(2e^2 - e - e^r\Big|_1^2\right)$

$= \frac{\pi}{4}\left(2e^2 - e - e^2 + e\right) = \frac{e^2\cdot\pi}{4} \simeq 5{,}80$

18) $\iint_D (5x^3y - 2xy + 7)\,dx\,dy$

$D = [-2,0]\times[1,2]$



$= \int_{-2}^{0}\left(\int_1^2 y(5x^3-2x)+7\,dy\right)dx =$

$$= \int_{-2}^{0} \frac{y^2}{2}(5x^3-2x)+7y\Big|_1^2 \, dx$$

$$= \int_{-2}^{0} \frac{3}{2}(5x^3-2x)+7 \, dx$$

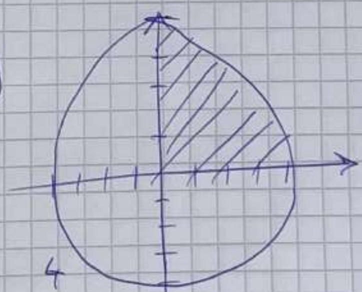
$$= \int_{-2}^{0} \frac{15}{2}x^3 - 3x + 7 \, dx$$

$$= \frac{15}{8}x^4 - \frac{3}{2}x^2 + 7x\Big|_{-2}^{0} = -30+6+14 = -10$$

19) $\iint_D e^{-(x^2+y^2)} \, dx \, dy$

$D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 16, x \ge 0, y \ge 0\}$

$$\begin{cases} x = r\cos\theta & r\in[0,4] \\ y = r\sin\theta & \theta\in[0,\frac{\pi}{2}] \end{cases}$$



$$\int_0^4 \int_0^{\frac{\pi}{2}} e^{-r^2} \cdot r \, d\theta \, dr = \frac{\pi}{2}\int_0^4 e^{-r^2} \cdot r \, dr =$$

$r^2 = t \qquad r=0 \Rightarrow t=0$

$r = \sqrt{t} \qquad r=4 \Rightarrow t=16$

$2r \, dr = dt$

$$= \frac{\pi}{4}\int_0^{16} e^{-t} \, dt = -\frac{\pi}{4} e^{-t}\Big|_0^{16} = -\frac{\pi}{4}\left(e^{-16}-1\right)$$

$$= \frac{-\pi e^{-16}+1}{4} \simeq 0,78$$

20) $\int_0^{\infty}\int_0^{x} e^{-(x+y)}\,dy\,dx = \int_0^{\infty}\int_0^{x} e^{-x}\cdot e^{-y}\,dy\,dx =$

$= \int_0^{\infty} -e^{-x}\cdot e^{-y}\Big|_0^{x}\,dx = \int_0^{\infty} -e^{-x}\cdot\left(e^{-x}-1\right)dx =$

$= \int_0^{\infty} -e^{-2x}+e^{-x}\,dx = \frac{1}{2}\cdot e^{-2x}\Big|_0^{\infty} - e^{-x}\Big|_0^{\infty}$

$= \frac{1}{2} - 1 = -\frac{1}{2}$