TUTORIAT 5

1. Determinați factorizarea Crout a matricii tridiagonale

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ 0 & l_{32} & l_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & \mu_{12} & 0 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

tridiagonale

$$= \begin{bmatrix} l_{11} & l_{11}\mu_{12} & 0 \\ l_{21} & l_{21}\mu_{12} + l_{22} & l_{22}\mu_{23} \\ 0 & l_{32} & l_{32}\mu_{23} + l_{33} \end{bmatrix}$$

$$A = \left[\begin{array}{c|cc} 2 & -1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right] = \left[\begin{array}{c|c} l_{11} & \underline{0} \\ \hline \underline{L_{21}} & L_{22} \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{U_{12}} \\ \hline \underline{0} & U_{22} \end{array}\right] = \left[\begin{array}{c|c} l_{11} & l_{11}\underline{U_{12}} \\ \hline \underline{L_{21}} & \underline{L_{21}}\,\underline{U_{12}} + L_{22}U_{22} \end{array}\right]$$

- $l_{11} = 2$
- $\underline{l_{21}} = -1$
- $l_{11}\mu_{12} = -1 \Rightarrow \mu_{12} = \dfrac{-1}{l_{11}} = -1/2$
- $\underline{L_{21}}\,\underline{U_{12}} + L_{22}U_{22} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow L_{22}U_{22} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \underline{L_{21}}\,U_{12}$

$$L_{22}U_{22} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\begin{bmatrix} -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3/2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{c|c} 3/2 & -1 \\ \hline -1 & 2 \end{array}\right] = \begin{bmatrix} l_{22} & 0 \\ l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & \mu_{23} \\ 0 & \underset{\shortparallel}{\mu_{33}} \end{bmatrix} = \left[\begin{array}{c|c} l_{22} & l_{22}\mu_{23} \\ \hline l_{32} & l_{32}\mu_{23} + l_{33} \end{array}\right]$$

- $l_{22} = 3/2$
- $l_{22} u_{23} = -1 \Rightarrow u_{23} = \frac{-1}{l_{22}} = \frac{-1}{\frac{3}{2}} = -2/3$
- $l_{32} u_{23} + l_{33} = 2 \Rightarrow l_{33} = 2 - l_{32} u_{23} = 2 - (-1)\cdot(-2/3) = 4/3$

Am obținut $L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3/2 & 0 \\ 0 & -1 & 4/3 \end{bmatrix}$ $U = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$

Subiect de EXAMEN

Fie sistemul $\begin{bmatrix} 3 & -2 & 2 \\ 2 & 4 & -3 \\ 2 & -2 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$ (1)

(a) Menționați dacă matricea asociată sistemului (1):

(i) admite factorizare LU fără pivotare

(ii) admite factorizare LU cu pivotare (PLU)

(iii) admite MEGFP

(iv) admite MEGPP, MEGPPS, MEGPT

(v) admite factorizare Cholesky

(vi) este (strict) diagonal dominantă

Justificați răspunsurile date.

(b) Determinați soluția sistemului (1), $x \in \mathbb{R}^3$, folosind MEGFP.

SOLUȚIE

Notăm matricea asociată sistemului (1) cu A.

(i) factorizare LU fără pivotare

$A \in \mathcal{M}_3(\mathbb{R}) \Rightarrow A$ pătratică

$\det A = \begin{vmatrix} 3 & -2 & 2 \\ 2 & 4 & -3 \\ 2 & -2 & 1 \end{vmatrix} = 12 - 8 + 12 - 16 - 18 + 4 = -14 \neq 0$
$\Rightarrow A$ inversabilă

$$\begin{bmatrix} \boxed{3} & -2 & 2 \\ 2 & 4 & -3 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow[E_3 \leftarrow E_3 - \frac{2}{3}E_1]{E_2 \leftarrow E_2 - \frac{2}{3}E_1} \begin{bmatrix} 3 & -2 & 2 \\ 0 & \boxed{16/3} & -13/3 \\ 0 & -2/3 & -1/3 \end{bmatrix} \xrightarrow{E_3 \leftarrow E_3 + \frac{1}{8}E_2}$$

$a_{11} = 3 \neq 0$ $\qquad a_{22} = 16/3 \neq 0$

$$\xrightarrow{E_3 \leftarrow E_3 + \frac{1}{8} E_2} \begin{bmatrix} 3 & -2 & 2 \\ 0 & 16/3 & -13/3 \\ 0 & 0 & \boxed{-7/8} \end{bmatrix} \Rightarrow \text{avem } a_{kk} \neq 0 \text{ la fiecare pas}$$

$$a_{33} = -7/8 \neq 0$$

Așadar, A admite MEGFP $\Rightarrow$ conform teoremei de caracterizare, A admite LU fără pivotare

(ii) - factorizarea LU cu pivotare

Am arătat la punctul (i) că matricea A este inversabilă $\Rightarrow$ A admite factorizarea LU cu pivotare

(iii) MEGFP

MEGFP impune condițiile verificate la punctul (i) și, cum toate sunt adevărate, $\Rightarrow$ A admite MEGFP

(iv) MEGPP, MEGPPS, MEGPT

Din (i) $\Rightarrow$ A pătratică și inversabilă $\Rightarrow$ A admite MEGPP, MEGPPS, MEGPT

(v) factorizarea Cholesky

Verificăm criteriul lui Sylvester

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 2 & 4 & -3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A^{(1)} = 3 \quad \begin{cases} A^{(1)} = (A^{(1)})^T \\ \det A^{(1)} = 3 > 0 \end{cases} \quad ✓$$

$$A^{(2)} = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{cases} \det(A^{(2)}) = 12 + 4 = 16 > 0 \\ A^{(2)} \neq (A^{(2)})^T \quad \left(\begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix} \right) \end{cases} \quad ✗$$

$\Rightarrow$ A nu admite factorizare Cholesky

(vi) A (strict) diagonal dominantă

$i=1: \quad |a_{11}| \geq |a_{12}| + |a_{13}|$

$\Leftrightarrow |3| \geq |-2| + |2|$

$\Leftrightarrow 3 \geq 2+2$ (Fals)

$\Rightarrow$ A nu e (strict) diagonal dominantă

(b) MEGFP

$$\underbrace{\begin{bmatrix} 3 & -2 & 2 \\ 2 & 4 & -3 \\ 2 & -2 & 1 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}}_{\underline{b}} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

$n = 3 \Rightarrow k = \overline{1,2}$

Pentru $k=1$: $\bar{A}^{(1)} = \left[\begin{array}{ccc|c} \boxed{3} & -2 & 2 & 5 \\ 2 & 4 & -3 & 1 \\ 2 & -2 & 1 & 1 \end{array}\right] = [A^{(1)} | \underline{b}^{(1)}]$

$a_{11} = 3 \neq 0 \Rightarrow$ putem aplica MEGFP

$i = \overline{2,3}$: $m_i^{(1)} = a_{i1}^{(1)} / a_{11}^{(1)}$

- $m_2^{(1)} = a_{21}^{(1)} / a_{11}^{(1)} = 2/3 \rightarrow E_2 \leftarrow E_2 - m_2^{(1)} E_1$

$E_2 \leftarrow E_2 - \frac{2}{3} E_1$

$j = \overline{2,3}$: $a_{2j}^{(2)} = a_{2j}^{(1)} - m_2^{(1)} a_{1j}^{(1)}$

$a_{22}^{(2)} = a_{22}^{(1)} - m_2^{(1)} a_{12}^{(1)} = 4 - \frac{2}{3} \cdot (-2) = 4 + \frac{4}{3} = \frac{16}{3}$

$a_{23}^{(2)} = a_{23}^{(1)} - m_2^{(1)} a_{13}^{(1)} = -3 - \frac{2}{3} \cdot 2 = -3 - \frac{4}{3} = -\frac{13}{3}$

$b_2^{(2)} = b_2^{(1)} - m_2^{(1)} b_1^{(1)} = 1 - \frac{2}{3} \cdot 5 = 1 - \frac{10}{3} = -\frac{7}{3}$

$\{ a_{21}^{(2)} = 0 \}$

- $m_3^{(1)} = a_{31}^{(1)} / a_{11}^{(1)} = 2/3 \Rightarrow E_3 \leftarrow E_3 - m_3^{(1)} E_1$

$E_3 \leftarrow E_3 - \frac{2}{3} E_1$

$j = \overline{2,3}$: $a_{3j}^{(2)} = a_{3j}^{(1)} - m_3^{(1)} a_{1j}^{(1)}$

$a_{32}^{(2)} = a_{32}^{(1)} - m_3^{(1)} a_{12}^{(1)} = -2 - \frac{2}{3} \cdot (-2) = -2 + \frac{4}{3} = -\frac{2}{3}$

$$a_{33}^{(2)} = a_{33}^{(1)} - m_3^{(1)} a_{13}^{(1)} = 1 - \frac{2}{3}\cdot 2 = 1 - \frac{4}{3} = -\frac{1}{3}$$

$$b_3^{(2)} = b_3^{(1)} - m_3^{(1)} b_1^{(1)} = 1 - \frac{2}{3}\cdot 5 = 1 - \frac{10}{3} = -\frac{7}{3}$$

$$\{ a_{31}^{(2)} = 0 \}$$

Pentru $K=2$, am obținut $\bar{A}^{(2)} = \left[\begin{array}{ccc|c} 3 & -2 & 2 & 5 \\ 0 & 16/3 & -13/3 & -7/3 \\ 0 & -2/3 & -1/3 & -7/3 \end{array}\right] = [A^{(2)} | \underline{b}^{(2)}]$

Matricea de transformare $M^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{bmatrix}$

Are loc relația $M^{(1)}[A^{(1)} | \underline{b}^{(1)}] = [A^{(2)} | \underline{b}^{(2)}]$ (1)

$$\bar{A}^{(2)} = \left[\begin{array}{ccc|c} 3 & -2 & 2 & 5 \\ 0 & \boxed{16/3} & -13/3 & -7/3 \\ 0 & -2/3 & -1/3 & -7/3 \end{array}\right]$$

$a_{22} = 16/3 \neq 0 \Rightarrow$ putem aplica MEGFP

$i = \overline{3,3}$: $m_i^{(2)} = a_{i2}^{(2)} / a_{22}^{(2)}$

$$m_3^{(2)} = a_{32}^{(2)} / a_{22}^{(2)} = \frac{-\frac{2}{\cancel{3}}}{\frac{16}{\cancel{3}}} = -\frac{1}{8}$$

$$E_3 \leftarrow E_3 - m_3^{(2)} E_2$$

$$E_3 \leftarrow E_3 + \frac{1}{8} E_2$$

$j = \overline{3,3}$: $a_{3j}^{(3)} = a_{3j}^{(2)} - m_3^{(2)} a_{2j}^{(2)}$

$$a_{33}^{(3)} = a_{33}^{(2)} - m_3^{(2)} a_{23}^{(2)} = -\frac{1}{3} + \frac{1}{8}\cdot\left(-\frac{13}{3}\right) = -\frac{1}{3} - \frac{13}{24} = -\frac{7}{8}$$

$$b_3^{(3)} = b_3^{(2)} - m_3^{(2)} b_2^{(2)} = -\frac{7}{3} + \frac{1}{8}\left(-\frac{7}{3}\right) = -\frac{7}{3} - \frac{7}{24} = -\frac{63}{24} = -\frac{21}{8}$$

$$\{ a_{32}^{(3)} = 0 \}$$

Am obținut $\bar{A}^{(3)} = \left[\begin{array}{ccc|c} 3 & -2 & 2 & 5 \\ 0 & 16/3 & -13/3 & -7/3 \\ 0 & 0 & -7/8 & -21/8 \end{array}\right] = [A^{(3)} | \underline{b}^{(3)}]$

Matricea de transformare $M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/8 & 1 \end{bmatrix}$

Are loc relația $M^{(2)} \left[A^{(2)} \mid \underline{b}^{(2)} \right] = \left[A^{(3)} \mid \underline{b}^{(3)} \right] = \left[U \ \tilde{\underline{b}} \right]$ (2)

Din relațiile (1) și (2) $\Rightarrow M^{(2)} M^{(1)} \left[A^{(1)} / \underline{b}^{(1)} \right] = \left[U \ \tilde{\underline{b}} \right]$

Sistemul $A\underline{x} = \underline{b}$ devine de forma $U\underline{x} = \tilde{\underline{b}}$

$$\begin{cases} 3x_1 - 2x_2 + 2x_3 = 5 \\ \frac{16}{3} x_2 - \frac{13}{3} x_3 = -\frac{7}{3} \\ -\frac{7}{8} x_3 = -\frac{21}{8} \end{cases}$$

$$-\frac{7}{8} x_3 = -\frac{21}{8} \Rightarrow x_3 = 3$$

$$\frac{16}{3} x_2 = -\frac{7}{3} + \frac{13}{3} x_3 = -\frac{7}{3} + \frac{13}{3} \cdot 3 = \frac{32}{3} \Rightarrow x_2 = 2$$

$$3x_1 = 5 - 2x_3 + 2x_2 = 5 - 2 \cdot 3 + 2 \cdot 2 = 5 - 6 + 4 = 3 \Rightarrow x_1 = 1$$

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$