26-11-2020

TEST - TEORIA MASURII SI A INTEGRALEI

I. Fie $f, g : [0,1] \to \mathbb{R}$ definite astfel: $f(0) = 3$,

$$f(x) = (-1)^n(n+1) \quad \text{daca } x \in \left(\frac{1}{(2n+1)^3}, \frac{1}{(2n-1)^3}\right], \quad n \in \mathbb{N}, n \geq 1$$

si

$$g(x) = \begin{cases} (f(x))^3 & \text{daca } x \in [0,1] \setminus \{\frac{1}{n} \ : \ n \in \mathbb{N}^*\} \\ 1 & \text{daca } x = \frac{1}{n}, \ n \in \mathbb{N}^*. \end{cases}$$

1) Aratati ca f este masurabila Borel si ca f este masurabila Lebesgue.

2) Studiati integrabilitatea Lebesgue a functiei f.

3) Este g masurabila Borel? Este g masurabila Lebesgue? Este g integrabila Lebesgue? Justificati raspunsurile!

Nota: Nu se cere calcularea integalelor, ci doar justificarea faptului ca functiile sunt sau nu integrabile!

II. Fie $E \subset [0,9]$ o multime masurabila Lebesgue cu $\lambda(E) = 3$ si $f : [0,\infty) \to \mathbb{R}$, $f(x) = \lambda(E \cap [0,x])$. Aratati ca f este uniform continua. Aratati ca exista $A \subset E$ masurabila Lebesgue astfel incat $\lambda(A) = 2$.

Bonus: Ramane adevarata afirmatia in cazul in care E este nemarginita?

III. Fie $A_1, A_2, \ldots, A_{2003}$ multimi masurabile Borel ale intervalului $[0,1]$ astfel incat $\lambda(A_k) > 1 - \frac{1}{3^k}$ pentru orice $1 \leq k \leq 2003$. Aratati ca

$$\lambda\left(\bigcap_{k=1}^{2003} A_k\right) > 0$$

Nota. Timp de lucru: 1 ora. Rezolvarile trebuie scanate si trimise impreuna cu lista de subiecte sub forma unui **singur** fisier pdf la adresele radu.munteanu@unibuc.ro si radu-bogdan.munteanu@g.unibuc.ro cel tarziu la ora 13.10.

PREGĂTIRE TEST

TEORIA MĂSURII

1. $f(0) = 3$

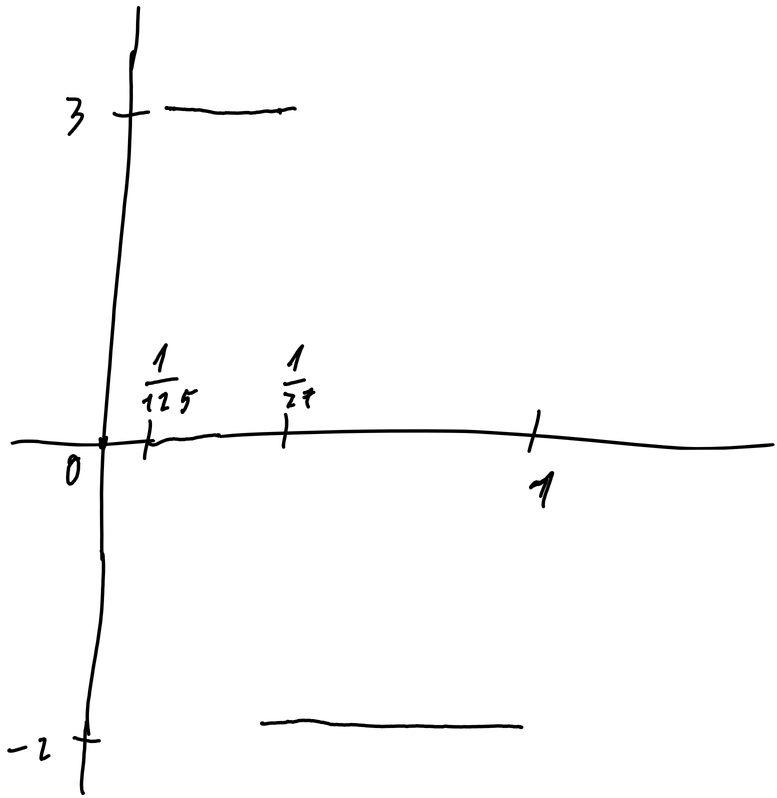
$f(x) = (-1)^n (n+1)$,

$x \in \left(\frac{1}{(2n+1)^3}, \frac{1}{(2n-1)^3} \right]$

1)

Fie $t \in \mathbb{R}$

$f^{-1}((-\infty, t)) = ?$



$(\forall)\ t \in \mathbb{R}$

$$f^{-1}\big((-\infty, t)\big) =$$

$$= \bigcup_{\substack{n \geq 1 \\ (-1)^n (n+1) < t}} \left(\frac{1}{(2n+1)^3}, \frac{1}{(2n-1)^3} \right] \cup A_t$$

unde $A_t = \{0\}$, dacă $t \geq 3$

și $A_t = \emptyset$ altfel

$(\forall)\ t \in \mathbb{R}$

$f^{-1}((-\infty, t))$ e o reuniune de

borеliene, deci boreliană

Rezultă f măs. Borel

Orice boreliană e măsurabilă Lebesgue,

deci $f^{-1}((-\infty, t))$ e măsurabilă

Lebesgue, $(\forall)\ t \in \mathbb{R}$.

Rezultă f măsurabilă Lebesgue.

2) Integrabilitate:

Vreau $\int_{[0,1]} |f| \, dx < \infty$

$$[0,1] = \{0\} \cup \bigcup_{n \geq 1} \left(\frac{1}{(2n+1)^3}, \frac{1}{(2n-1)^3} \right]$$

$f_n(0) = 3$

$$f_n(x) = 3 \cdot \chi_{\{0\}}(x) + |f(x)| \cdot \chi_{\left(\frac{1}{(2n+1)^3}, 1 \right]},$$

$x \geq 0$

$$f_n(x) = 3 \cdot \chi_{\{0\}}(x) + \sum_{m=1}^{n} (m+1) \cdot \chi_{\left(\frac{1}{(2m+1)^3}, \frac{1}{(2m-1)^3} \right]}(x)$$

f_n e fct. simplă $\Rightarrow$

$$\Rightarrow \int_{[0,1]} f_n \, dx = \sum_{m=1}^{n} (m+1) \cdot \left(\frac{1}{(2m-1)^3} - \frac{1}{(2m+1)^3} \right)$$

Claim:

$f_n \nearrow f$

Proof. $\square$

Dei $\int_{[0,1]} |f| \, dx = \lim_{n \to \infty} \int_{[0,1]} f_n \, dx =$

$= \sum_{n \geq 1} (n+1) \cdot \left(\frac{1}{(2n-1)^3} - \frac{1}{(2n+1)^3} \right) < \infty$

37) f măs. Borel $\Rightarrow$ f^3 măs. Borel

Fie $B \in \mathcal{B}(\mathbb{R})$

Vreau $g^{-1}(B) \in \mathcal{B}([0,1])$

$$g^{-1}(B) = \begin{cases} \left((f^3)^{-1}(B)\right) \setminus \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}, & 1 \notin B \\ (f^3)^{-1}(B) \cup \left\{\frac{1}{n}, n \in \mathbb{N}^*\right\}, & 1 \in B \end{cases}$$

$$\left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\} = \bigcup_{n \geq 1} \underbrace{\left\{\frac{1}{n}\right\}}_{\text{boreliană}} \in \mathcal{B}([0,1])$$

Arăt g integrabilă $\Leftrightarrow$ f^3 integrabilă.

Proof: $\lambda\left(\left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}\right) = 0$

$$f^3 \equiv g \text{ pe } [0,1] \setminus \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}$$

Deci $|f^3| \equiv |g|$ pe — // —,

de unde

$$\int_{[0,1]} |g| \, d\lambda = \int_{[0,1]} |f^3| \, d\lambda$$

$$f^3(x) = (-1)^n (n+3)^3 \text{ pt. }$$

$$x \in \left(\frac{1}{(2n+1)^3}, \frac{1}{(2n-1)^3}\right]$$

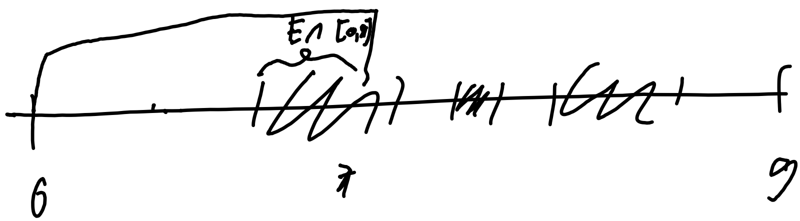
$$f^3(0) = 27$$

II $E \subset [0, 9]$

$\lambda(E) = 3$

$f : [0, \infty) \to \mathbb{R}$

$f(x) = \lambda(E \cap [0, x])$



Rem: $f : \mathbb{R} \to \mathbb{R}$ uniform continuă:

$(\forall)\ \varepsilon > 0\ (\exists)\ \delta_\varepsilon > 0$ a.î.

$(\forall)\ x, y \in \mathbb{R}\quad |x - y| < \delta_\varepsilon \Rightarrow$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

f - funcție Lipschitz: $(\exists) L$ constantă

$$|f(x) - f(y)| < L \cdot |x - y|, \ (\forall) x, y$$

Lipschitz $\Rightarrow$ uniform continuă.

Dem: Iau $\delta_\varepsilon = \dfrac{\varepsilon}{L}$

Arătăm că f din problemă e Lipschitz.

Fie $x \geq y$

$$f(x) - f(y) = \lambda(E \cap [0, x]) - \lambda(E \cap [0, y])$$

$$E \subseteq [0, 9] \Rightarrow \lambda(E) < \infty$$

$$E \cap [0, x] \supseteq E \cap [0, y]$$

$$\Rightarrow$$

$$\Rightarrow f(x) - f(y) = \lambda(E \cap [y, x])$$

Deci

$$0 \le f(x) - f(y) \le \lambda([y, x]) \Rightarrow$$

$$\Rightarrow |f(x) - f(y)| \le |x - y|$$

Deci f e Lipschitz.

Lemă: $(X, \mathcal{A}, \mu)$ sp. cu măsură

$E \in \mathcal{A}$ cu $\mu(E) < \infty$

$A, B \in \mathcal{A}$

$A \subseteq B \subseteq E$

Atunci $\mu(B) - \mu(A) = \mu(B \setminus A)$

Dem: $B = A \cup (B \setminus A)$

$A \cap (B \setminus A) = \emptyset$

μ măsură

$\square$

$$\left.\begin{array}{l} f(0) = 0 \\ f(9) = \lambda(E) = 3 \\ f \text{ continuă} \end{array}\right| \Rightarrow (\exists)\, x_0 \in [0,9] \text{ a.î. } f(x_0) = 2,$$

adică

$$\lambda(E \cap [0, x_0]) = 2 \quad \square$$

III

$$B_k = [0,1] \setminus A_k$$

$$\lambda(B_k) = 1 - \lambda(A_k) < \frac{1}{3^k}$$

$$\lambda\left(\bigcap_{k=1}^{2003} A_k\right) = \lambda\left([0,1] \setminus \bigcup_{k=1}^{2003} B_k\right) =$$

$$= 1 - \lambda\left(\bigcup_{k=1}^{2003} B_k\right)$$

$$\lambda\left(\bigcup_{k=1}^{2003} B_k\right) \le \sum_{k=1}^{\infty} \lambda(B_k) \le$$

$$\le \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{2}{3} < 1,$$

deci $\lambda\left(\bigcap_{k=1}^{2003} B_k\right) \ge \frac{1}{3} > 0$

Contraexemplu la lemă:

$B = \mathbb{R}$

$A = \mathbb{R} \setminus \{0\}$

$A \subseteq B \subseteq \mathbb{R}$

$\mu(A) = \mu(B) = \infty$

$\mu(B \setminus A) = 0$

Lema ar fi:

$\infty - \infty = 0$ nedeterminare

Bonus la II :

E nemărginită

$x > y \geq 0$

$f(x) - f(y) = \lambda(E \cap [0, x]) - \lambda(E \cap [0, y])$

$E \cap [0, y] \subseteq E \cap [0, x] \subseteq$

$[0, x]$

$\lambda([0, x]) = x < \infty$

Deci $f(x) - f(y) = \lambda(E \cap [y, x])$

$|f(x) - f(y)| < |x - y|$,

deci f Lipschitz

$f(0) = 0$

$$\lim_{x \to \infty} f(x) = \lambda(E \cap [0, \infty))$$

Atâta timp cât $\lambda(E \cap [0,\infty)) > 2$,

$(\exists)\ x_0 > 0$ a.î. $f(x_0) > 2$.

Prin urmare $(\exists)\ x_1 \in [0, x_0]$ a.î.

$f(x_1) = 2$.