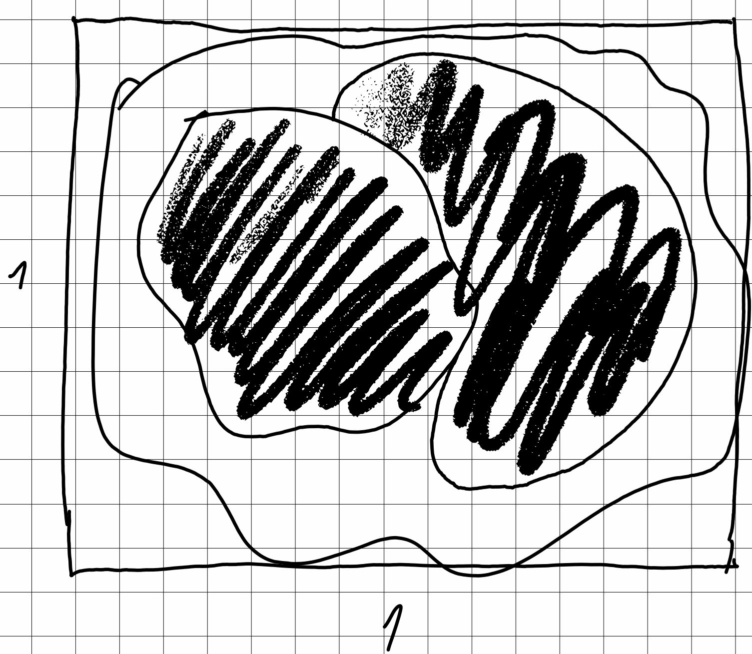
TEORIA MĂSURII
SEMINAR 1

$(X, \mathcal{A}, \mu)$

Exemple de măsuri:

- $\operatorname{card}(\cdot) : \mathcal{P}(X) \to \mathbb{N} \cup \{\infty\}$
- măsura nulă



Exemplu de măsură:

$X = \mathbb{N}$

$\mathcal{A} = \mathcal{P}(\mathbb{N})$

$$\mu(A) = \begin{cases} card(A) \in \mathbb{N}, & A \text{ finită} \\ \infty, & A \text{ infinită} \end{cases}$$

$\mu(A) = card(A)$, $(\forall)\ A \subseteq X$

- $\mu : \mathcal{A} \to [0, \infty]$
- $\mu(\emptyset) = 0$ ✓
- $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$,

dacă $A_n \cap A_m = \emptyset$, $(\forall)\ n \neq m$

$$card\left(\bigcup_{n=1}^{\infty} A_n\right) = \underbrace{\sum_{n=1}^{\infty} card(A_n)}_{\text{există?}}$$

De ce?

$card(A_n) \geq 0$

$$\boxed{\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \overbrace{\sum_{n=1}^{N} a_n}^{S_N}}$$

Propoziție

Dacă, $a_n \geq 0$, $(\forall)\, n \in \mathbb{N}$

Atunci

$\sum_{n=1}^{\infty} a_n$ există și aparține lui $[0, \infty]$

Demonstrație

Vreau să studiez

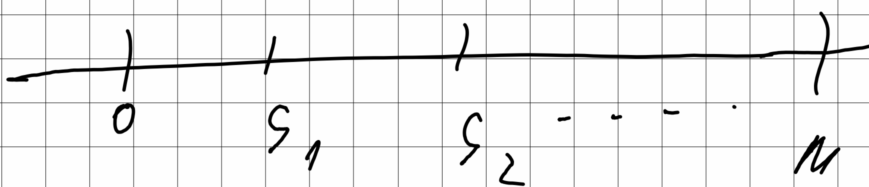
$$\lim_{N\to\infty} S_N$$

$$S_{N+1} - S_N = a_{N+1} \geq 0, \text{ deci } S_N \nearrow$$

Clar $0 \leq S_N$, $(\forall)\, N \in \mathbb{N}$

I Dacă $S_N < M$, $(\forall)\, N \in \mathbb{N}$,

pentru un $M \geq 0$



Atunci, cf. criteriului Weierstrass,

$$(\exists) \lim_{N \to \infty} S_N \in [0, \infty)$$

II Dacă $(S_N)_N$ nu e mărginit superior,

$$S_N \to \infty$$

În primul caz, $\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N < \infty$

În al doilea caz, $\sum_{n=1}^{\infty} a_n = \infty$

Revenim la problemă:

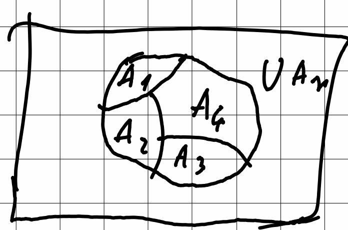
$$\operatorname{card}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \operatorname{card}(A_n)$$

$$\bigcup_{n=1}^{\infty} A_n = \{a \in X \mid (\exists)\ n \in \mathbb{N} \text{ a.î. } a \in A_n\}$$

Distingem două cazuri:

I $(\exists)\ N \in \mathbb{N}$ a.î. $A_n = \emptyset,\ (\forall)\ n > N$

$\underline{\operatorname{card}(A_n) < \infty}$



$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{N} A_n$$

$$\operatorname{card}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \operatorname{card}(A_n) =$$

$$= \sum_{n=1}^{N} card(A_n) + \underbrace{\sum_{n=N+1}^{\infty} card(A_n)}_{=0}$$

$$= \sum_{n=1}^{\infty} card(A_n)$$

I' $card(A_{n_0}) = \infty$ pentru un $n_0 \in \mathbb{N}$

$$A_{n_0} \subseteq \bigcup_{n=1}^{\infty} A_n \Rightarrow card\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$$

$$\sum_{n=1}^{\infty} card(A_n) \geq card(A_{n_0}) = \infty \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} card(A_n) = \infty$$

$\underline{II}$ $(\forall)\ N \in \mathbb{N}\ (\exists)\ n \geq N$ a.i.

$A_n \neq \emptyset$

$card(A_n) \geq 1$

Echivalent cu $(\exists)\ (n_k)_k$ șir strict crescător de nr. naturale a.î.

$card(A_{n_k}) \geq 1$

Fie $a_{n_k} \in A_{n_k}$

Pentru $k \neq l$ $\quad a_{n_k} \neq a_{n_l}$

$$\mathrm{card}\left(\{a_{n_k} \mid k \in \mathbb{N}\}\right) = \infty$$

$$\left.\{a_{n_k} \mid k \in \mathbb{N}\} \subseteq \bigcup_{n=1}^{\infty} A_n \right/ \Rightarrow$$

$$\Rightarrow \mathrm{card}\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$$

$$\sum_{n=1}^{\infty} \mathrm{card}(A_n) \geq \sum_{k=1}^{\infty} \mathrm{card}(A_{n_k}) \geq$$

$$\geq \sum_{k=1}^{\infty} 1 = \infty$$

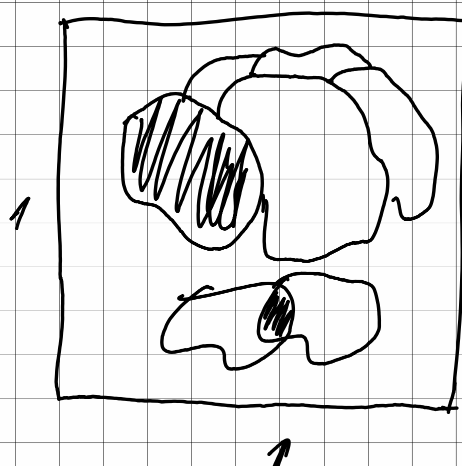
Deci, şi în cazul II,

$$\mathrm{card}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathrm{card}(A_n)$$

$$\emptyset \cap \emptyset = \{ x \in \emptyset \mid x \in \emptyset \}$$

$$= \emptyset$$

σ - algebre :



$(\sigma 1)$ • $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$

$(\sigma 2)$ • $A_n \in \mathcal{A}$, $\forall n \Rightarrow (A_n)_n \in \mathcal{A}$

Proprietate:

$\mathcal{A}$ σ-algebră

$(A_n)_n$ şir de mulţimi din $\mathcal{A}$

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$$

Dem: Fie $B = \bigcap_{n=1}^{\infty} A_n$

$(\forall) n \in \mathbb{N}$ $A_n \in \mathcal{A} \overset{\sigma 1}{\Rightarrow} X \setminus A_n \in \mathcal{A} \overset{\sigma 2}{\Rightarrow}$

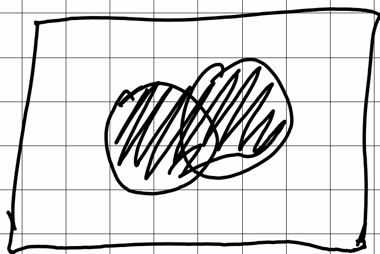
$$\overset{\sigma 2}{\Rightarrow} \underbrace{\bigcup_{n=1}^{\infty} (X \setminus A_n)} \in \mathcal{A}$$

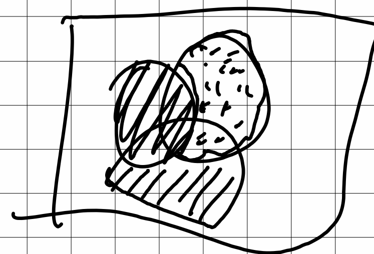
$$= X \setminus \left(\bigcap_{n=1}^{\infty} A_n \right) \in \mathcal{A} \overset{\sigma 1}{\Rightarrow}$$

$$\underbrace{X \setminus \left(X \setminus \bigcap_{n=1}^{\infty} A_n \right)} \in \mathcal{A}$$

$$\bigcap_{n=1}^{\infty} A_n = B \in \mathcal{A}$$

Proprietatea de subaditivitate a unei măsuri





μ măsură pe $\mathcal{A}$

$(A_n)_n$ şir de mulţimi din $\mathcal{A}$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n)$$

Considerăm şirul de mulţimi

$B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus (A_1 \cup A_2)$

$\vdots$

$B_n = A_n \setminus (A_1 \cup A_2 \cup \dots \cup A_{n-1})$

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

"$\subseteq$" $B_n \subseteq A_n \ (\forall)\ n \in \mathbb{N} \Rightarrow$

$$\Rightarrow \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} A_n$$

"$\supseteq$" Fie $x \in \bigcup_{n=1}^{\infty} A_n$ şi fie

$$k = \min \{ n \in \mathbb{N} \mid x \in A_n \}$$

Claim: $x \in B_k$

Proof: $x \in A_k$

Din minimalitate, $x \notin (A_1 \cup A_2 \cup \dots \cup A_{k-1})$ $\Big\} \Rightarrow$

$$\Rightarrow x \in \underbrace{A_k \setminus (A_1 \cup A_2 \cup \dots \cup A_{k-1})}_{= B_k}$$

Deci $x \in \bigcup_{n=1}^{\infty} B_n$,

de unde $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$

În final, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$B_n \cap B_m = \emptyset$, $(\forall)\ n \neq m$,

deci

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

$$B_n \subseteq A_n \Rightarrow \mu(B_n) \leq \mu(A_n)$$

$\Rightarrow$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Comentariu

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$$

$$\left.\begin{array}{l} B = A \cup (B \setminus A) \\ A \cap (B \setminus A) = \emptyset \end{array}\right| \Rightarrow$$

definiție măsură

$$\Rightarrow \mu(B) = \mu\left(A \cup (B \setminus A)\right) = \mu(A) + \underbrace{\mu(B \setminus A)}_{\geq 0} \geq$$

$$\geq \mu(A)$$

Recapitulare topologie

Spațiu metric

-> Mulțime deschisă $\cup$ arbitrare, $\cap$ finite

-> Mulțime închisă $\cap$ arbitrare, $\cup$ finite

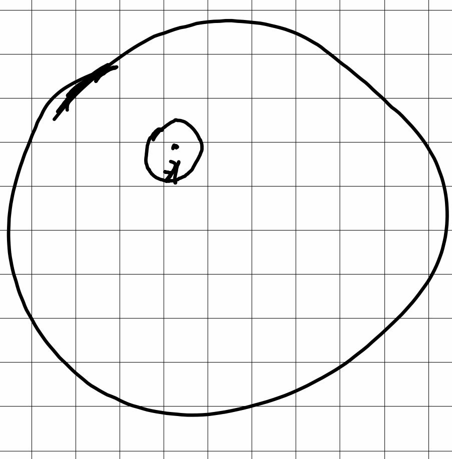
-> Închidere

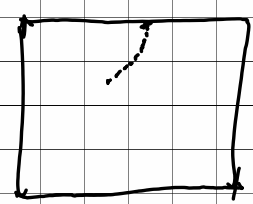
-> interior

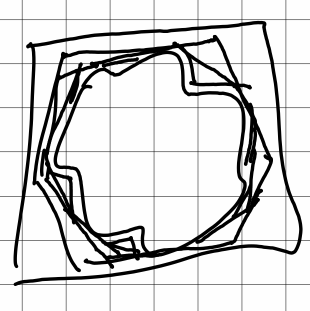
-> frontieră

-> limite de șiruri și de funcții

-> șiruri de funcții, tipuri de convergență

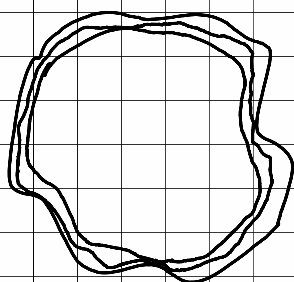






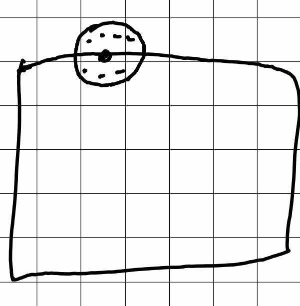
Ex: $A = [0, 1) \cup (2, 3]$

$\bar{A} = [0, 1] \cup [2, 3]$



$$\mathring{A} = \{ x \in A \mid (\exists)\, r > 0 \ \ B(x, r) \subseteq A \}$$

Frontiera: $\bar{A} \setminus \mathring{A} = \{ x \in X \mid (\forall)\, r > 0 \ \ B(x, r) \cap A \neq \emptyset, \ \emptyset \neq B(x, r) \cap (X \setminus A) \}$



Șiruri de funcții:

$$(f_n)_n \qquad f_n : X \to \mathbb{R}$$

$f_n \to f$ (simplu/punctual)

$$(\forall) x \in X \quad \lim_{n\to\infty} f_n(x) = f(x)$$

$f_n \xrightarrow{u} f$ (uniform)

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$$

σ-algebra generată de o mulțime de mulțimi.

Fie $\mathcal{B} \subseteq \mathcal{P}(X)$

σ-algebra generată de $\mathcal{B}$ este

$$\sigma(\mathcal{B}) = \bigcap_{\substack{\mathcal{A}\ \sigma\text{-algebră în } \mathcal{P}(X) \\ \mathcal{B} \subseteq \mathcal{A}}} \mathcal{A}$$

<u>Borelienele pe $\mathbb{R}^N$</u>

$$\mathcal{B}(\mathbb{R}^N) = \sigma(\mathcal{D}),$$

unde $\mathcal{D} = \{A \in \mathcal{P}(\mathbb{R}^N) \mid A \text{ \underline{deschisă}}\}$

Temă:
- $\sigma(\mathcal{B})$ definită mai sus este σ-algebră
- $\mathcal{B}(\mathbb{R}^N) = \sigma(\{A \in \mathcal{P}(\mathbb{R}^N) \mid A \text{ închisă}\})$