TUTORIAT 4

1. Să se determine factorizarea LU cu pivotare a matricei

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix} \qquad A = PLU$$

$$\det(A) = \begin{vmatrix} 1 & 1 & -1 \\ \boxed{2} & 2 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 2 + 2 + 4 \cancel{+2} + 4 \cancel{-2} = 12 \neq 0 \Rightarrow$$

$\Rightarrow$ matricea A este inversabilă $\Rightarrow$ A admite factorizarea PLU

Căutăm maximul pe coloana 1 a lui A:

$\max\limits_{i=\overline{1,3}} |a_{i1}| = \max\{|1|, |2|, |1|\} = |2| = 2 = |a_{21}| \Rightarrow l = 2 \Rightarrow$ trebuie

să interschimbăm liniile 1 și 2 $\Leftrightarrow E_1 \leftrightarrow E_2$

Matricea de permutare simplă: $P^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

După interschimbarea liniilor 1 și 2, matricea A devine:

$$P^{(1)}A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{U}$$

Considerăm următoarea partiționare a lui $P^{(1)}A$:

$$P^{(1)}A = \left[\begin{array}{c|cc} 2 & 2 & 4 \\ \hline 1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right] = \left[\begin{array}{c|c} \ell_{11} & \underline{0} \\ \hline \underline{L}_{21} & L_{22} \end{array}\right]\left[\begin{array}{c|c} u_{11} & \underline{U}_{12} \\ \hline \underline{0} & U_{22} \end{array}\right] = \left[\begin{array}{c|c} \ell_{11}u_{11} & \ell_{11}\underline{U}_{12} \\ \hline u_{11}\underline{L}_{21} & \underline{L}_{21}\underline{U}_{12} + L_{22}U_{22} \end{array}\right]$$

- $\ell_{11}u_{11} = 2 \Rightarrow \ell_{11} = 1,\ u_{11} = 2$
- $\ell_{11}\underline{U}_{12} = \begin{bmatrix} 2 & 4 \end{bmatrix} \Rightarrow \underline{U}_{12} = \frac{1}{\ell_{11}}\begin{bmatrix} 2 & 4 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \end{bmatrix} \Rightarrow \begin{cases} u_{12} = 2 \\ u_{13} = 4 \end{cases}$
- $u_{11}\underline{L}_{21} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \underline{L}_{21} = \frac{1}{u_{11}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \Rightarrow \begin{cases} \ell_{21} = 1/2 \\ \ell_{31} = 1/2 \end{cases}$
- $\underline{L}_{21}\underline{U}_{12} + L_{22}U_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow L_{22}U_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \underline{L}_{21}\cdot \underline{U}_{12}$

$$L_{22}U_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}_{2,1}\underbrace{\begin{bmatrix} 2 & 4 \end{bmatrix}}_{1,2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -2 & -1 \end{bmatrix} = S$$

Problema se reduce la factorizarea PLU a matricii S

Căutăm maximul pe coloana 1 a matricii S:

$\max\limits_{j=\overline{1,2}} |a_{j1}| = \max\{|0|, |-2|\} = |-2| = 2 = |a_{21}| \Rightarrow \ell = 2 > 1 \Rightarrow$ trebuie să interschimbăm liniile 1 și 2 $\Leftrightarrow E_1 \leftrightarrow E_2$ în S, ceea ce înseamnă că, în matricea $P^{(1)}A$, are loc interschimbarea $E_2 \leftrightarrow E_3$

Matricea de permutare simplă: $P^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

După interschimbarea $E_2 \leftrightarrow E_3$, matricea $P^{(1)}A$ devine:

$$P^{(2)}P^{(1)}A = \left[\begin{array}{c|cc} 2 & 2 & 4 \\ \hline 1 & -1 & 1 \\ 1 & \underbrace{1 \quad -1}_{(P^{(2)}P^{(1)}A)_{22}} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \quad 0 \\ \hline 1/2 & \\ \underbrace{1/2}_{\underline{L}_{21}} & P^{(2)}L_{22} \end{array}\right]\left[\begin{array}{c|c} 2 & \overset{U_{12}}{2 \quad 4} \\ \hline 0 & \\ 0 & U_{22} \end{array}\right] = \begin{bmatrix} 2 & U_{12} \\ 2P^{(2)}\underline{L}_{21} & P^{(2)}\underline{L}_{21}U_{12} + P^{(2)}L_{22}U_{22} \end{bmatrix}$$

Avem că $(P^{(2)}P^{(1)}A)_{22} = P^{(2)}\underline{L}_{21}U_{12} + P^{(2)}L_{22}U_{22}$

$$\Rightarrow (P^{(2)}L_{22})\,U_{22} = (P^{(2)}P^{(1)}A)_{22} - P^{(2)}\underline{L}_{21}U_{12}$$

$P^{(2)}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}[2 \ \ 4]$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} = S_1$$

- Factorizăm LU S_1:

$$S_1 = \left[\begin{array}{c|c} -2 & -1 \\ \hline 0 & -3 \end{array}\right] = \begin{bmatrix} l_{22} & 0 \\ l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} u_{22} & u_{23} \\ 0 & u_{33} \end{bmatrix} = \left[\begin{array}{c|c} l_{22}u_{22} & l_{22}u_{23} \\ \hline l_{32}u_{22} & l_{32}u_{23} + l_{33}u_{33} \end{array}\right]$$

- $l_{22}u_{22} = -2 \Rightarrow l_{22} = 1,\ u_{22} = -2$
- $l_{22}u_{23} = -1 \Rightarrow u_{23} = -1$
- $l_{32}u_{22} = 0 \Rightarrow l_{32} = 0$
- $l_{32}u_{23} + l_{33}u_{33} = -3 \Rightarrow l_{33}u_{33} = -3 - l_{32}u_{23} = -3 \Rightarrow l_{33} = 1$, $u_{33} = -3$

Am obținut $PA = LU$, unde

$$P = P^{(2)}P^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

2. Să se verifice dacă matricea $A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4{,}25 & 2{,}75 \\ 1 & 2{,}75 & 3{,}5 \end{bmatrix}$ admite factorizarea Cholesky și, în caz afirmativ, să se determine această factorizare.

verificăm criteriul lui Sylvester:

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4,25 & 2,75 \\ 1 & 2,75 & 3,5 \end{bmatrix}$$

$A^{(1)} = 4 \Rightarrow \det(A^{(1)}) = \det|4| = 4 > 0$ ✓

$A^{(2)} = \begin{bmatrix} 4 & -1 \\ -1 & 4,25 \end{bmatrix} \Rightarrow \begin{cases} \det(A^{(2)}) = 4 \cdot 4,25 - 1 = 16 > 0 \\ A^{(2)} = (A^{(2)})^T \end{cases}$ ✓

$A^{(3)} = A \Rightarrow \begin{cases} \det(A^{(3)}) = \det(A) = 16 > 0 \\ A^{(3)} = (A^{(3)})^T \end{cases}$ ✓

Conform criteriului lui Sylvester, A admite factorizarea Cholesky

$$A = \left[\begin{array}{c|cc} 4 & -1 & 1 \\ \hline -1 & 4,25 & 2,75 \\ 1 & 2,75 & 3,5 \end{array}\right] = \left[\begin{array}{c|c} a_{11} & \underline{A}_{21}^T \\ \hline \underline{A}_{21} & A_{22} \end{array}\right] = \begin{bmatrix} l_{11} & \underline{0} \\ \underline{L}_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & \underline{L}_{21}^T \\ \underline{0} & L_{22}^T \end{bmatrix} = LL^T$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \left[\begin{array}{c|c} l_{11}^2 & l_{11}\underline{L}_{21}^T \\ \hline \underline{L}_{21} l_{11} & \underline{L}_{21}\underline{L}_{21}^T + L_{22}L_{22}^T \end{array}\right]$$

- $l_{11}^2 = 4 \Rightarrow l_{11} = \pm 2$, dar vrem $l_{11} > 0 \Rightarrow l_{11} = 2$
- $\underline{L}_{21} l_{11} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \underline{L}_{21} = \frac{1}{l_{11}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \Rightarrow \begin{cases} l_{21} = -1/2 \\ l_{31} = 1/2 \end{cases}$
- $\underline{L}_{21}\underline{L}_{21}^T + L_{22}L_{22}^T = A_{22} \Rightarrow L_{22}L_{22}^T = A_{22} - \underline{L}_{21}\underline{L}_{21}^T$

$$L_{22}L_{22}^T = \begin{bmatrix} 4,25 & 2,75 \\ 2,75 & 3,5 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 4,25 & 2,75 \\ 2,75 & 3,5 \end{bmatrix} - \begin{bmatrix} 0,25 & -0,25 \\ -0,25 & 0,25 \end{bmatrix}$$

$$L_{22}L_{22}^T = \begin{bmatrix} 4 & 3 \\ 3 & 3,25 \end{bmatrix}$$

Problema se reduce la factorizarea Cholesky a lui $L_{22}L_{22}^T$

$$L_{22}L_{22}^T = \left[\begin{array}{c|c} 4 & 3 \\ \hline 3 & 3{,}25 \end{array}\right] = \begin{bmatrix} l_{22} & 0 \\ l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} l_{22} & l_{32} \\ 0 & l_{33} \end{bmatrix} = \left[\begin{array}{c|c} l_{22}^2 & l_{22}l_{32} \\ \hline l_{32}l_{22} & l_{32}^2 + l_{33}^2 \end{array}\right]$$

- $l_{22}^2 = 4 \Rightarrow l_{22} = \pm 2$, dar vrem $l_{22} > 0 \Rightarrow l_{22} = 2$
- $l_{22}l_{32} = 3 \Rightarrow l_{32} = \frac{3}{l_{22}} = 3/2$
- $l_{32}^2 + l_{33}^2 = 3{,}25 \Rightarrow l_{33}^2 = 3{,}25 - l_{32}^2 = \frac{325}{100} - {}^{25)}\frac{9}{4} = \frac{325-225}{100} = \frac{100}{100} = 1$

$\Rightarrow l_{33}^2 = 1 \Rightarrow l_{33} = \pm 1$, dar vrem $l_{33} > 0 \Rightarrow l_{33} = 1$

Am obținut $L = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & 2 & 0 \\ 1/2 & 3/2 & 1 \end{bmatrix}$

$$LL^T = \begin{bmatrix} 2 & 0 & 0 \\ -1/2 & 2 & 0 \\ 1/2 & 3/2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1/2 & 1/2 \\ 0 & 2 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4{,}25 & 2{,}75 \\ 1 & 2{,}75 & 3{,}5 \end{bmatrix}$$

3. Să se determine factorizarea Doolittle a matricii tridiagonale

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A = \left[\begin{array}{c|cc} 2 & -1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} u_{11} & u_{12} & 0 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{U}$$

tridiagonale

$$= \begin{bmatrix} u_{11} & u_{12} & 0 \\ l_{21}u_{11} & u_{12}l_{21} + u_{22} & u_{23} \\ 0 & u_{22}l_{32} & l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \underline{0} \\ \underline{L}_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & \underline{U}_{12} \\ \underline{0} & U_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & \underline{U}_{12} \\ \underline{0} & \underline{L}_{21}\underline{U}_{12} + L_{22}U_{22} \end{bmatrix}$$

- $u_{11} = 2$
- $u_{12} = -1$
- $u_{11}l_{21} = -1 \Rightarrow l_{21} = -1/2$
- $A_{22} = \underline{L}_{21}\underline{U}_{12} + L_{22}U_{22} \Rightarrow L_{22}U_{22} = A_{22} - \underline{L}_{21}\underline{U}_{12}$

$$L_{22}U_{22} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3/2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3/2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{22} & u_{23} \\ 0 & u_{33} \end{bmatrix} = \begin{bmatrix} u_{22} & u_{23} \\ l_{32}u_{22} & l_{32}u_{23} + u_{33} \end{bmatrix}$$

- $u_{22} = 3/2$
- $u_{23} = -1$
- $l_{32}u_{22} = -1 \Rightarrow l_{32} = \frac{-1}{u_{22}} = -2/3$
- $l_{32}u_{23} + u_{33} = 2 \Rightarrow u_{33} = 2 - l_{32}u_{23} = 2 - \left(-\frac{2}{3}\right)(-1) = 4/3$

Am obţinut

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}}_{U}$$