$$= x_1^2 + (y_1^2 - 4y_1 z_1 - 10y_1) - z_1^2 + 2z_1 - 16$$

$$= x_1^2 + (\underbrace{y_1 - 2z_1 - 5}_{y_2})^2 - 4z_1^2 - 25 - 20z_1 - z_1^2 + 2z_1 - 16$$

$$= x_2^2 + y_2^2 - 5z_2^2 - 18z_2 - 41$$

$$= x_2^2 + y_2^2 - \frac{1}{5}\left(5^2 z_2^2 + 18\cdot 5 z_2\right) - 41$$

$$= x_2^2 + y_2^2 - \frac{1}{5}(\underbrace{5z_2 + 9}_{z_3})^2 + \frac{81}{5} - 41$$

$$= x_3^2 + y_3^2 - \frac{1}{5}z_3^2 - \frac{124}{5} = 0 \quad \Big| \cdot \frac{5}{124}$$

$$\Leftrightarrow \frac{5}{124}x_3^2 + \frac{5}{124}y_3^2 - \frac{1}{124}z_3^2 - 1 = 0$$

$$\begin{cases} x_4 = \sqrt{\frac{5}{124}}\, x_3 \\ y_4 = \sqrt{\frac{5}{124}}\, y_3 \\ z_4 = \sqrt{\frac{1}{124}}\, z_3 \end{cases} \qquad \Leftrightarrow x_4^2 + y_4^2 - z_4^2 - 1 = 0$$

hiperboloid cu o pânză.

b) $P = (1,1,0) \in \Gamma$ pt. că $1^2 + 2\cdot 1^2 - 0^2 + 2\cdot 1\cdot 1 - 4\cdot 1\cdot 0 + 6\cdot 1 - 4\cdot 1 + 2\cdot 0 - 7 = 1 + 2 + 2 + 6 - 4 - 7 = 0$

Calculăm $T_P\Gamma$ cu formula de mai sus:

Fie $f(x,y,z) = x^2+2y^2-z^2+2xy-4yz+6x-4y+2z-7$

$\Gamma = f^{-1}(0)$

$\Rightarrow T_P\Gamma: \frac{\partial f}{\partial x}(P)\,(x-1) + \frac{\partial f}{\partial y}(P)(y-1) + \frac{\partial f}{\partial z}(P)(z-0) = 0$

$\Leftrightarrow (2x+2y+6)\Big|_{(1,1,0)}(x-1) + (4y+2x-4z-4)\Big|_{(1,1,0)}(y-1)$

$+ (-2z-4y+2)\Big|_{(1,1,0)}(z-0) = 0$

$\Leftrightarrow 10(x-1) + 2(y-1) - 2z = 0$

<u>Altfel</u>, se poate calcula și cu formula

$T_P Q: \ {}^tX_0AX + B(X+X_0) + c = 0$ (dedublare).

c) În forma $\Gamma: x_4^2+y_4^2-z_4^2-1=0$, putem găsi o dreaptă $d\subset\Gamma$:

$x_4^2 - z_4^2 = 1-y_4^2$

$$x_4^2 - z_4^2 = 1 - y_4^2$$

$$\Leftrightarrow (x_4 - z_4)(x_4 + z_4) = (1 - y_4)(1 + y_4)$$

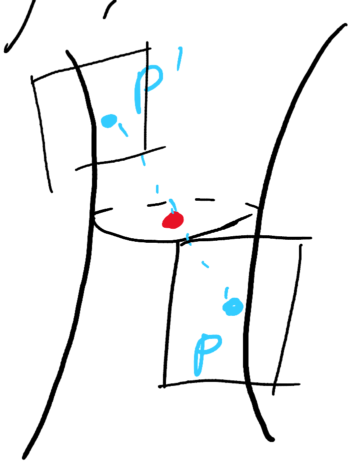
$$\Rightarrow \text{de exemplu, } d: \begin{cases} x_4 - z_4 = 1 - y_4 \\ x_4 + z_4 = 1 + y_4 \end{cases}$$

Rescriem d în coordonatele (x, y, z):

$$\begin{cases} x_4 = \sqrt{\frac{5}{124}}\, x_3 = \sqrt{\frac{5}{124}}\, x_2 = \sqrt{\frac{5}{124}}\, x_1 = \sqrt{\frac{5}{124}}\,(x+y+3) \\ y_4 = \sqrt{\frac{5}{124}}\, y_3 = \sqrt{\frac{5}{124}}\, y_2 = \sqrt{\frac{5}{124}}\,(y_1 - 2z_1 - 5) = \sqrt{\frac{5}{124}}\,(y - 2z - 5) \\ z_4 = \sqrt{\frac{1}{124}}\, z_3 = \sqrt{\frac{1}{124}}\,(5z_2 + 9) = \sqrt{\frac{1}{124}}\,(5z_1 + 9) = \sqrt{\frac{1}{124}}\,(5z + 9) \end{cases}$$

$$\Rightarrow d: \begin{cases} \sqrt{\frac{1}{124}}\left(\sqrt{5}(x+y+3) - (5z+9)\right) = 1 - \sqrt{\frac{5}{124}}\,(y - 2z - 5) \\ \sqrt{\frac{1}{124}}\left(\sqrt{5}(x+y+3) + (5z+9)\right) = 1 + \sqrt{\frac{5}{124}}\,(y - 2z - 5) \end{cases}$$

d) P' = simetricul lui P față de centrul lui Γ



Calculăm centrul lui Γ:

P_0: $x_4 = y_4 = z_4$

$$\Rightarrow \begin{cases} x + y + 3 = 0 \\ y - 2z - 5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x+y+3=0 \\ y-2z-5=0 \\ 5z-9=0 \end{cases}$$

$$y = 5+2\cdot\frac{9}{5} = \frac{43}{5}$$

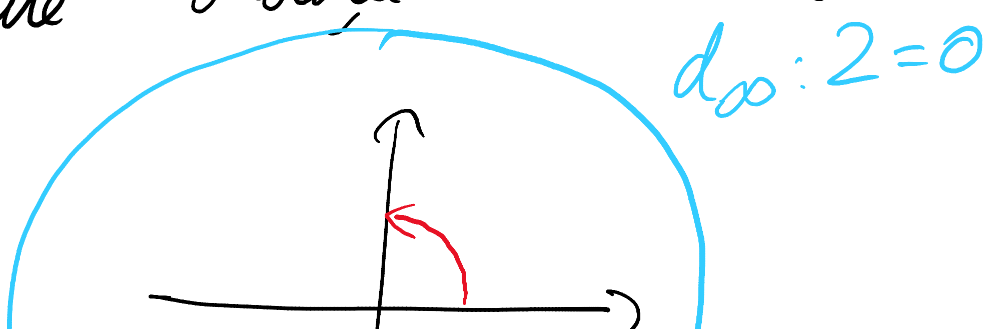
$$x = -\frac{43}{5}-3 = -\frac{58}{5}$$

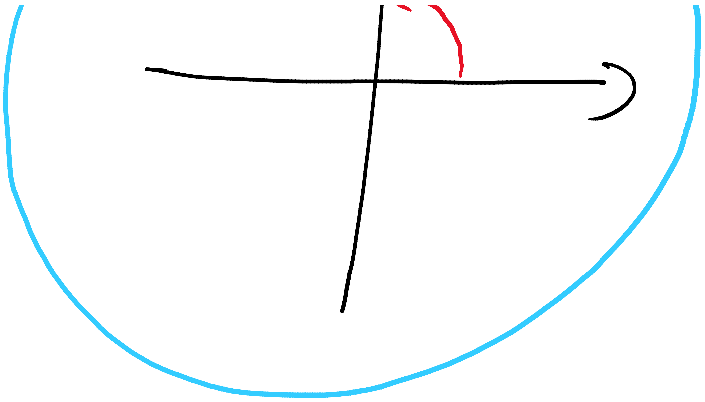
$$\Rightarrow P_0 = \left(-\frac{58}{5}, \frac{43}{5}, \frac{9}{5}\right)$$

$$\Rightarrow \frac{1}{2}P+\frac{1}{2}P' = P_0 \Rightarrow P' = 2P_0 - P$$
$$= \left(-\frac{116}{5}-1, \frac{86}{5}-1, \frac{18}{5}\right)$$
$$= \left(-\frac{121}{5}, \frac{81}{5}, \frac{18}{5}\right).$$

3. a) f este o proiectivitate, provine din $g:\mathbb{R}^3\to\mathbb{R}^3$, $g(x,y,z)=(-y,x,z)$ evident un izomorfism liniar.

b) Observăm că, în planul afin $z=1$, f este o rotație cu 90° (în sens trigonometric)





$\Rightarrow$ Evident nu are pcte fixe pe d_∞

$\Rightarrow$ singurul pct fix este $[0:0:1]$.

Calcul direct:

Pct fix: $[X:Y:Z] = [-Y:X:Z]$

$\Rightarrow \exists\, \lambda \in \mathbb{R}$ a.î. $(X,Y,Z) = \lambda(-Y,X,Z)$

$\nearrow$ $Z \neq 0 \Rightarrow \lambda = 1 \Rightarrow X = -Y = -X = Y \Rightarrow$
$\Rightarrow X = Y = 0$

$\searrow$ $\underline{Z=0} \Rightarrow \underline{X} = -\lambda Y = \underline{-\lambda^2 X} \Rightarrow \dfrac{X=0 \Rightarrow Y=0}{\text{abs}}$

$\Rightarrow [0:0:1]$ punctul fix

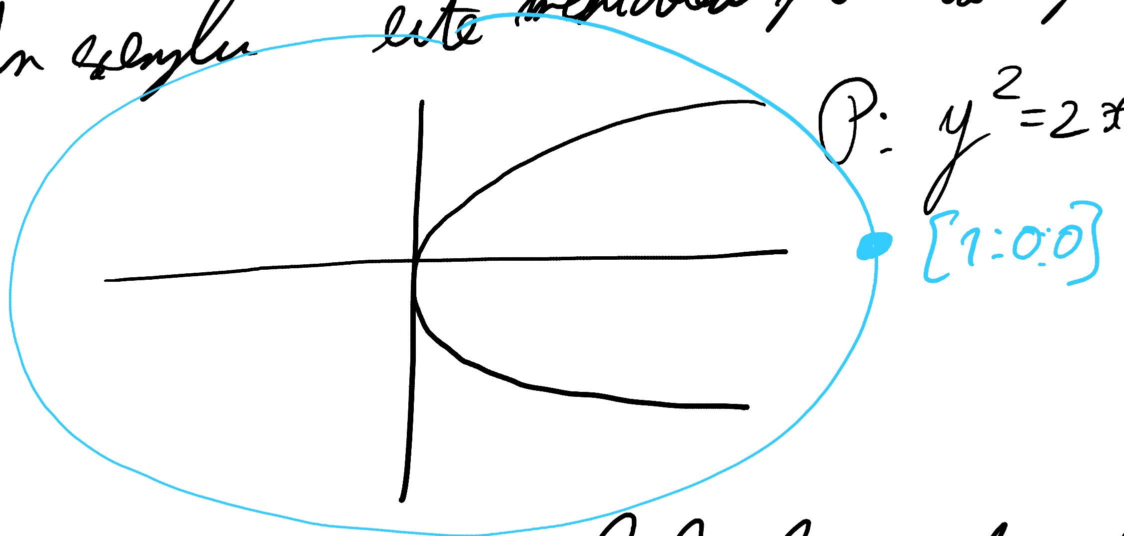
c) Evident $f(\underset{Z=0}{\underset{\uparrow}{d_\infty}}) = d_\infty$ și f (o rotație)

$$\dot{Z}=0$$

nu de drepte afine incidente

$\Rightarrow d_\infty$ singura dreaptă incidentă

d) Un exemplu este închiderea proiectivă a parabolei:



$\Rightarrow \overline{P}: Y^2 = 2XZ$. Calculăm dreapta tangentă în $P=[1:0:0]$

$$T_P\overline{P}: \frac{\partial(Y^2-2XZ)}{\partial X}(P)\cdot X + \frac{\partial(Y^2-2XZ)}{\partial Y}(P)\cdot Y + \frac{\partial(Y^2-2XZ)}{\partial Z}(P)\cdot Z = 0$$

$$\Leftrightarrow (-2Z)\Big|_{[1:0:0]}\cdot X + (2Y)\Big|_{[1:0:0]}\cdot Y + (-2X)\Big|_{[1:0:0]}\cdot Z = 0$$

ecuația lui d_∞

$(=)\ -2Z=0 \ (=)\ Z=0$ ecuația lui d_∞

4. Este echivalentul exercițiului de la examenul din iunie:

O variantă: alegem Γ reuniune de drepte proiective,

$\Gamma = PP_2 \cup P_1P_3.$ $\quad P_1=[1:1:1],\ P_2=[1:-1:1],\ P_3=[2:3:-1]$

$P_1P_2:\ Y+Z=0$

$P_1P_3:\ -4X+3Y+Z=0$

$\Rightarrow \Gamma:\ (Y+Z)\cdot(-4X+3Y+Z)=0.$