Seminarul 5 111

Tuesday, November 2, 2021 1:42 PM

Translații și rotații. Exerciții

Ex 1. Fie $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y)=\left(\frac{\sqrt3 x-y+4-\sqrt3}{2}, \frac{x+\sqrt3 y+3-2\sqrt3}{2}\right)$.

a) Demonstrați că f este o rotație și aflați-i centrul și unghiul de rotație.

b) $d: x+y+3=0$. $\underbrace{f(d)}_{\text{tot o dreaptă}}=?$

Dem a) O rotație $R_{P,\alpha}$ are forma:

$$\boxed{R_{P,\alpha}(x,y)=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x-x_P\\ y-y_P\end{pmatrix}+\begin{pmatrix}x_P\\ y_P\end{pmatrix}} \quad (*)$$

$$R_{O,\alpha}(x,y)=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \text{ și } \boxed{R_{P,\alpha}=T_{\vec{OP}}\circ R_{O,\alpha}\circ T_{\vec{OP}}^{-1}}$$

$$f(x,y)=\left(\frac{\sqrt3 x-y+4-\sqrt3}{2}, \frac{x+\sqrt3 y+3-2\sqrt3}{2}\right)$$

scriere pe coloană

$$\stackrel{\downarrow}{=}\underbrace{\begin{pmatrix}\frac{\sqrt3}{2} & -\frac12\\ \frac12 & \frac{\sqrt3}{2}\end{pmatrix}}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}\frac{4-\sqrt3}{2}\\ \frac{3-2\sqrt3}{2}\end{pmatrix}$$

$$=\boxed{\begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}\frac{4-\sqrt3}{2}\\ \frac{3-2\sqrt3}{2}\end{pmatrix}} \quad (**)$$

$$\begin{pmatrix} \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} \begin{pmatrix} y \end{pmatrix} + \begin{pmatrix} \frac{3-2\sqrt{3}}{2} \end{pmatrix}$$

matrice de rotație

$\overset{\text{ucs}}{\Longrightarrow}$ deja știm că e o rotație (de unghi $\frac{\pi}{6}$)

$$(*) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \left[I_2 - \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \right] \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

$$(*) = (**) \overset{\alpha = \frac{\pi}{6}}{\iff} \left[I_2 - \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} \right] \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} \frac{4-\sqrt{3}}{2} \\ \frac{3-2\sqrt{3}}{2} \end{pmatrix}$$

$$\iff \underbrace{\begin{pmatrix} 1-\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} \end{pmatrix}}_{\det \neq 0} \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} \frac{4-\sqrt{3}}{2} \\ \frac{3-2\sqrt{3}}{2} \end{pmatrix} \leftarrow \text{sistem (valeu) compatibil determinat}$$

$$\begin{cases} (2-\sqrt{3})\, x_p + y_p = 4-\sqrt{3} \\ -x_p + (2-\sqrt{3})\, y_p = 3-2\sqrt{3} \quad \Big| \cdot (2-\sqrt{3}) \end{cases}$$

$$\left[1 + (2-\sqrt{3})^2\right] y_p = 4-\sqrt{3} + (2-\sqrt{3})(3-2\sqrt{3})$$

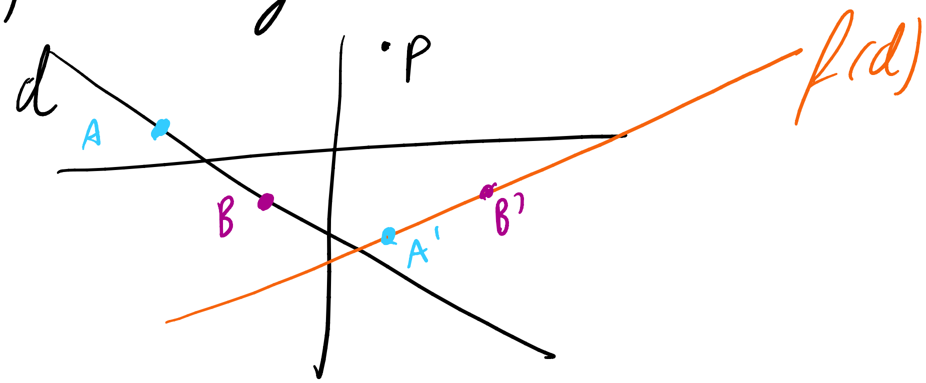
$$\iff (1+4-4\sqrt{3}+3)\, y_p = 4-\sqrt{3} + 6 - 4\sqrt{3} - 3\sqrt{3} + 6$$

$$(8-4\sqrt{3})\, y_p = 16-8\sqrt{3} \Rightarrow \boxed{y_p = 2} \Rightarrow x_p = 2(2-\sqrt{3}) + 2\sqrt{3} - 3$$

$$= 1$$

$\Rightarrow P(1,2)$ e centrul rotației

b) $d: x+y+3=0$. $f(d)=?$



<u>Știu</u> că $f(d)$ e dreaptă!

<u>Var 1</u> Aleg aleatoare $A, B \in d$, $f(A) \overset{not}{=} A'$, $f(B) \overset{not}{=} B'$

$\Rightarrow f(d)$ e dreapta $A'B'$.

Fie $A=(0,-3)$, $B=(-3,0)$ $\Rightarrow f(A) = \left(\frac{\sqrt{3}\cdot 0-(-3)+4-\sqrt{3}}{2}, \frac{0+\sqrt{3}\cdot(-3)+3-2\sqrt{3}}{2}\right)$

$$= \left(\frac{7-\sqrt{3}}{2}, \frac{3-5\sqrt{3}}{2}\right)$$

$$f(B) = \left(\frac{4-4\sqrt{3}}{2}, \frac{-2\sqrt{3}}{2}\right) = \left(2-2\sqrt{3}, -\sqrt{3}\right)$$

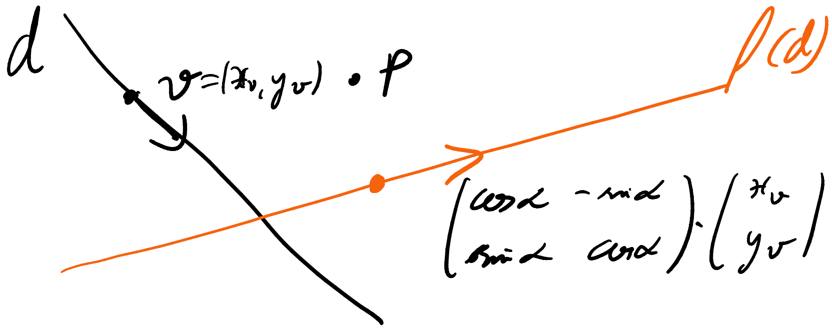
$$\Rightarrow \boxed{f(d)=A'B': \frac{x-x_{B'}}{x_{A'}-x_{B'}} = \frac{y-y_{B'}}{y_{A'}-y_{B'}} \Leftrightarrow \frac{x-(2-2\sqrt{3})}{\frac{7-\sqrt{3}}{2}-(2-2\sqrt{3})} = \frac{y+\sqrt{3}}{\frac{3-5\sqrt{3}}{2}+\sqrt{3}}}.$$

ecuație carteziană

[illegible] $f(d)$ e linia dată de un punct și

Var 2 Caracterizăm $f(d)$ ca fiind dată de un punct și un vector director:

$$B' = (2-2\sqrt{3}, -\sqrt{3}) \in f(d).$$



$d: x+y+3=0 \Rightarrow v=(1,-1)$ este vector director

$$\Rightarrow f(d) \text{ are vector director } \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{pmatrix}$$

$$\Rightarrow \boxed{f(d): \underbrace{(2-2\sqrt{3}, -\sqrt{3})}_{B'} + t\left(\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right), \ t \in \mathbb{R}.}$$

ecuație parametrică

Ex 2 $P=(1,2)$, $Q=(-2,3)$.

$R_{P,\frac{\pi}{3}} \circ R_{Q,\frac{\pi}{6}}$ e o rotație de unghi $\frac{\pi}{2}$. Calculați centrul ei.

Dem $\left(R_{P,\frac{\pi}{3}} \circ R_{Q,\frac{\pi}{6}}\right)(x,y) = R_{P,\frac{\pi}{3}}\left(\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}\begin{pmatrix} x+2 \\ y-3 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}\right)$

$$\underline{\text{Dim}}\quad \left(R_{P,\frac{\pi}{3}} \circ R_{Q,\frac{\pi}{6}}\right)(x,y) = R_{P,\frac{\pi}{3}}\left(\underbrace{\begin{pmatrix} \cdots & \cdots \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}\begin{pmatrix} \cdots \\ y-3\end{pmatrix}} + \begin{pmatrix} \cdots \\ 3\end{pmatrix}\right)$$

$$= \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}\left[\underbrace{\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}\begin{pmatrix} x+2 \\ y-3\end{pmatrix} + \begin{pmatrix} -2 \\ 3\end{pmatrix}} - \begin{pmatrix} 1 \\ 2\end{pmatrix}\right]$$

$$+ \begin{pmatrix} 1 \\ 2\end{pmatrix} = \boxed{\underbrace{\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}}}\begin{pmatrix} x \\ y\end{pmatrix}$$

è una rotazione di angolo $\frac{\pi}{2}$

$$+ \left(\quad\right)\begin{pmatrix} 2 \\ -3\end{pmatrix} + \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}\begin{pmatrix} -3 \\ 1\end{pmatrix} + \begin{pmatrix} 1 \\ 2\end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 2 \\ -3\end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} -3 \\ 1\end{pmatrix} + \begin{pmatrix} 1 \\ 2\end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} + \begin{pmatrix} 3 - \frac{3}{2} - \frac{\sqrt{3}}{2} + 1 \\ 2 - \frac{3\sqrt{3}}{2} + \frac{1}{2} + 2\end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} + \boxed{\begin{pmatrix} \frac{5-\sqrt{3}}{2} \\ \frac{9-3\sqrt{3}}{2}\end{pmatrix}} = (*)$$

$$= \begin{pmatrix} \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} y \end{pmatrix} \quad \boxed{\begin{pmatrix} \frac{9-3\sqrt{3}}{2} \end{pmatrix}}$$

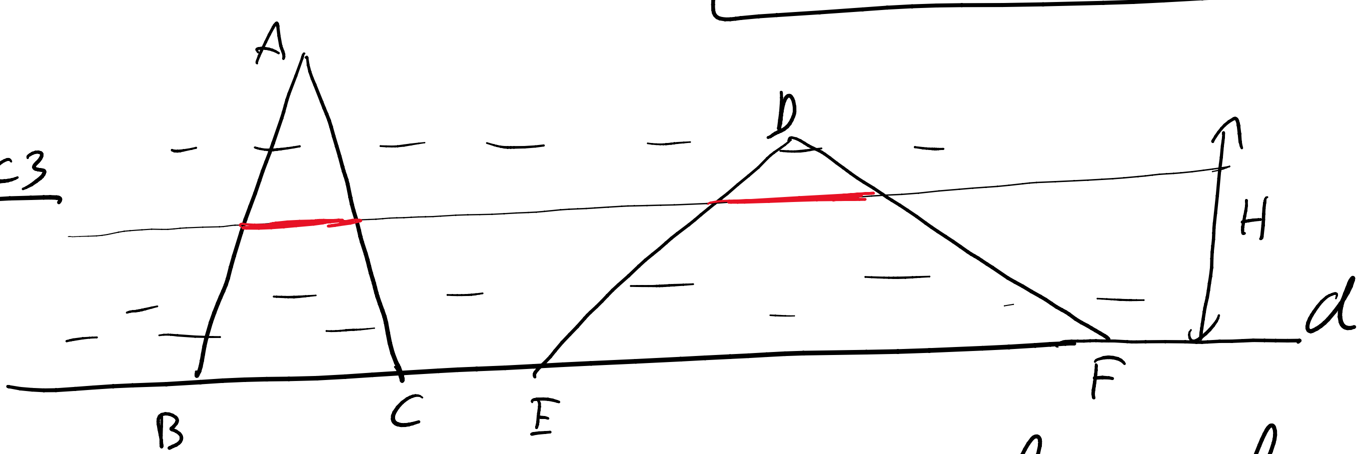
$$\text{Vrem } (*) = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \boxed{\left[I_2 - \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \right] \begin{pmatrix} x_R \\ y_R \end{pmatrix}}$$

caz în care R e centrul!

$$\Longleftrightarrow \text{Caut } R = (x_R, y_R) \text{ a.î. } \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_R \\ y_R \end{pmatrix} = \begin{pmatrix} \frac{5-\sqrt{3}}{2} \\ \frac{9-3\sqrt{3}}{2} \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} x_R + y_R = \frac{5-\sqrt{3}}{2} \\ -x_R + y_R = \frac{9-3\sqrt{3}}{2} \end{cases} \Rightarrow \boxed{\begin{aligned} y_R &= \frac{14-4\sqrt{3}}{4} = \frac{7-2\sqrt{3}}{2} \\ x_R &= \frac{-4+2\sqrt{3}}{4} = \frac{\sqrt{3}-2}{2} \end{aligned}}$$

Ex 3



ΔABC, ΔDEF isoscele, $|BC| < |EF|$ și $h_{\Delta ABC} > h_{\Delta DEF}$.

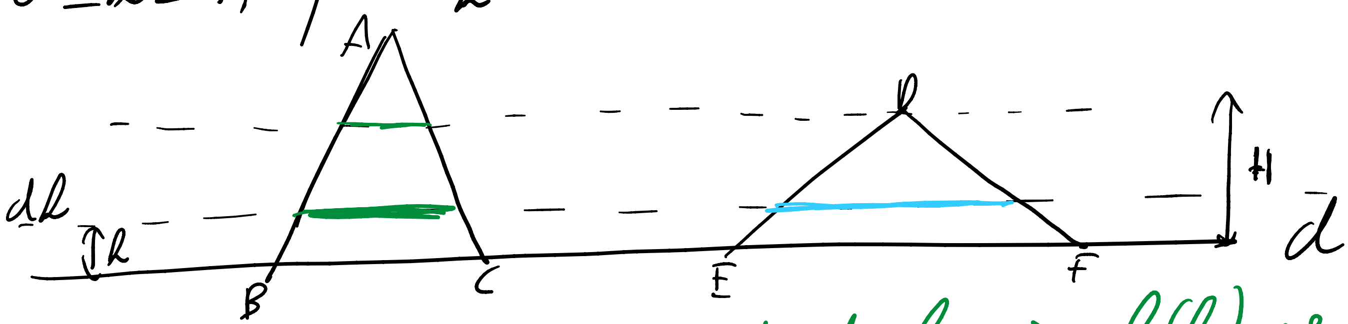
Arătați că există (construiți?) o dreaptă $\parallel d$ care determină pe cele două triunghiuri segmente congruente.

determină pe cele două triunghiuri [illegible] ?

Mai ușoară dacă Δ nu sunt isoscele?

Idee Var 1 (cu principiul lui Darboux)

Fie H = lungimea înălțimii în Δ DEF. Pentru orice h, $0 \le h \le H$, fie d_h dreapta la distanța h de d, $d_h \parallel d$.



d_h determină un segment de lungime $l(h)$ pe Δ ABC și un segment de lungime $L(h)$ pe Δ DEF.

Caut h a.î. $l(h) = L(h)$.

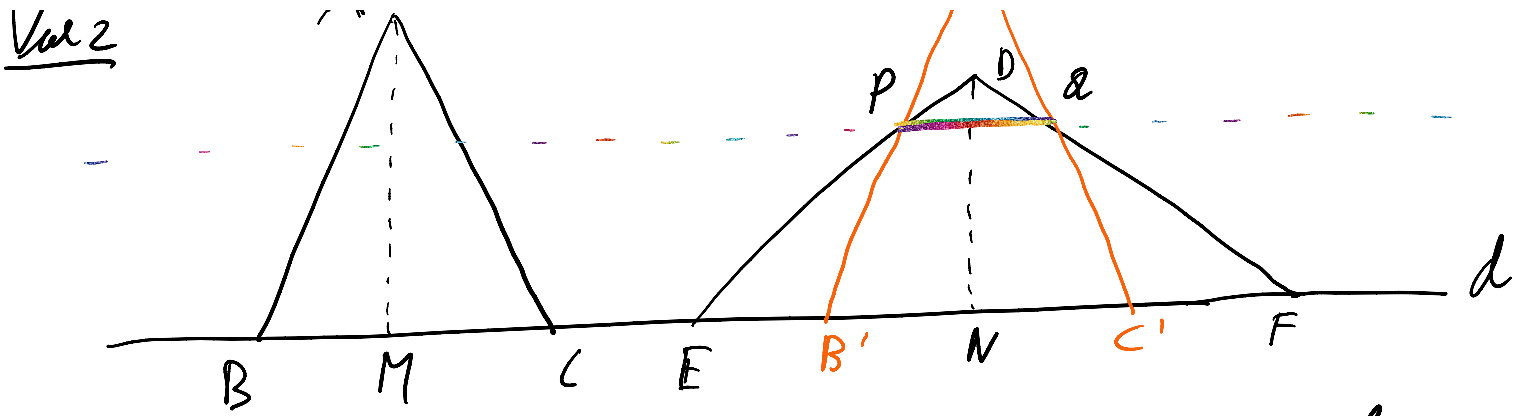
Fie $f:[0,H] \to \mathbb{R}$, $f(h) = L(h) - l(h)$.

- $f(0) = |EF| - |BC| > 0$ prin ipoteză
- $f(H) = 0 - \underbrace{l(h)}_{>0 \text{ pt că } h_{\Delta ABC} > h_{\Delta DEF} = H} < 0$
- f continuă

$\xRightarrow{\text{Darboux}} \exists\, h_0$ a.î. $f(h_0) = 0 \iff L(h) = l(h)$.

Var 2

Var 2



Translatez ΔABC cu $\overrightarrow{MN}$, unde M, N piciorele înălțimilor

$$T_{\overrightarrow{MN}}(\Delta ABC) = \Delta A'B'C' \equiv \Delta ABC.$$

$\Rightarrow$ Dacă $\{P\} = A'B' \cap DE$ și $\{Q\} = A'C' \cap DF$, PQ e dreapta căutată!

($\Rightarrow PQ \parallel EF$) pt că sunt isoscele!

Q: E adevărat rezultatul dacă triunghiurile nu sunt isoscele?

Da! Var 1 funcționează și dacă nu sunt isoscele!

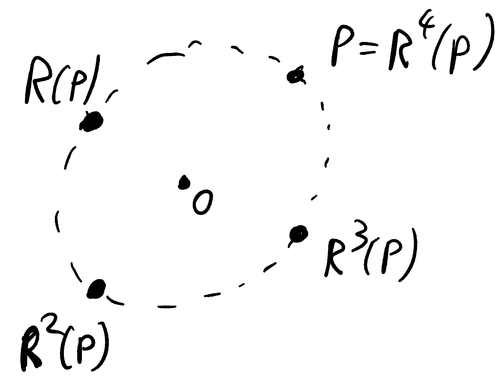
Ex 4 $\quad O, P$ pcte distincte, $\alpha \in (0, 2\pi)$ și

$$\mathcal{P} = \{ R_{O,k\alpha}(P) \mid k \in \mathbb{Z} \}$$

Cee este $\mathcal{P}$?

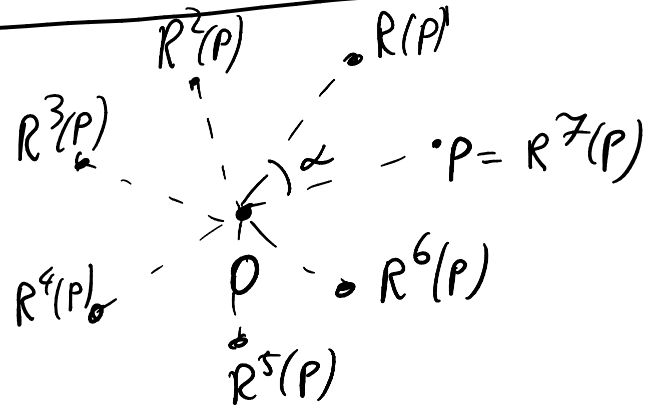
$R = R_{O,\alpha}$

Exemple - $\alpha = \frac{\pi}{2} \Rightarrow \mathcal{P}$: vârfurile unui pătrat



- $\alpha = \frac{\pi}{3} \Rightarrow \mathcal{P} =$ vârfurile unui hexagon

$\boxed{\text{Caz 0,5} \quad \alpha = \frac{2\pi}{n} \Rightarrow \mathcal{P} = \underline{\hspace{3cm}} \text{ poligon regulat cu } n \text{ vârfuri}}$



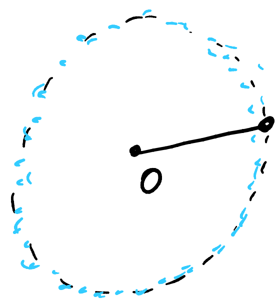
- $\alpha = \frac{49}{107} \cdot 2\pi \Rightarrow \mathcal{P} =$ vârfurile unui poligon regulat cu 107 vârfuri (ciclul se închide după 107 rotații)

Caz 1 Dacă $\frac{\alpha}{2\pi} \in \mathbb{Q}$, atunci ciclul se închide și (oleacă)

$\mathcal{P} =$ vârfurile unui poligon regulat

Caz 2 $\frac{\alpha}{2\pi} \notin \mathbb{Q} \Leftrightarrow$ ciclul $\boxed{\text{nu}}$ se închide, adică nu ajungem (într-un număr finit de pași) la repetarea punctelor.

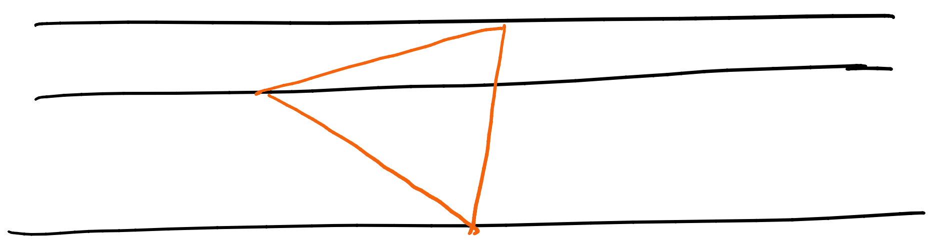
nu ajungem (într-un număr finit de [illegible] punctelor.



numărabil $\quad$ nenumărabil

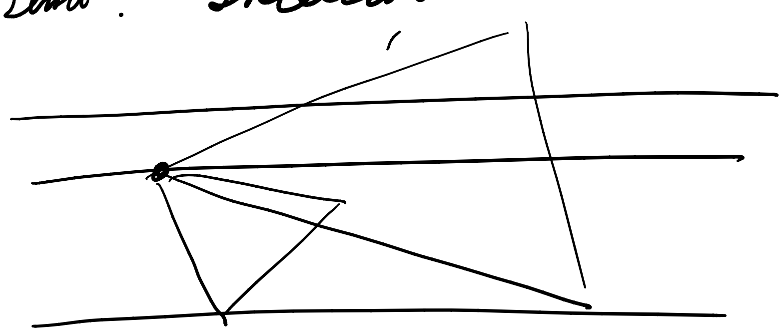
Temă Demonstrați că, în acest caz, $\mathcal{P} \neq \mathcal{C}$ dar $\forall x \in \mathcal{C}$, $\exists (x_n)_n \subset \mathcal{P}$, $x_n \to x$.

Ex 6

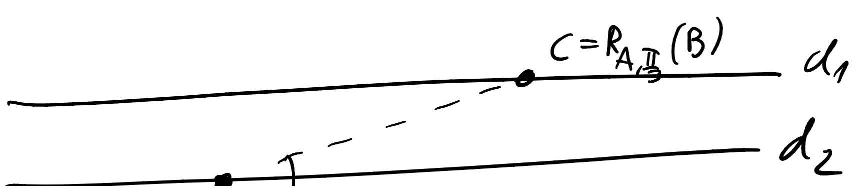


$d_1 \parallel d_2 \parallel d_3$. Construiți un triunghi echilateral cu vârfurile pe cele trei drepte.

Idee Temă: Încercați cu Darboux, ca la ex 3.



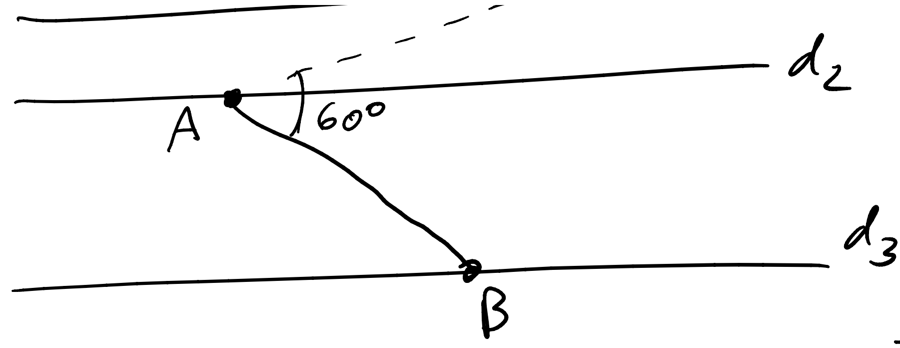
Idee



Aleg $A \in d_2$

Dacă găsesc $B \in d_3$ a.î.

Idee

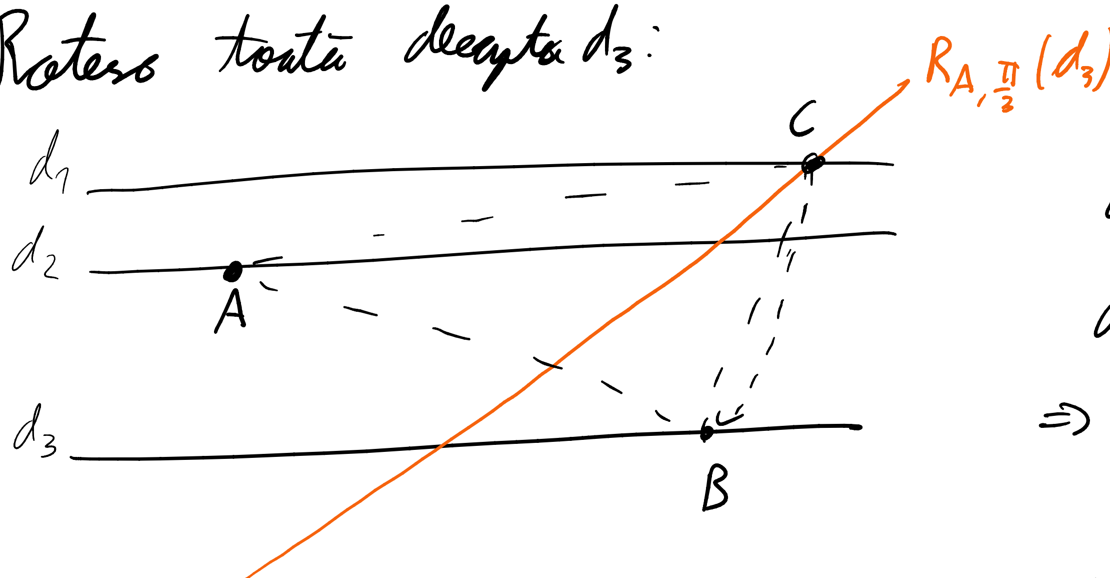


$B \in d_3$ a.î.

$C = R_{A,\frac{\pi}{3}}(B) \in d_1$,

$\Rightarrow \Delta ABC$ e echilateral

Rotesc toată dreapta d_3:



$d = R_{A,\frac{\pi}{3}}(d_3)$

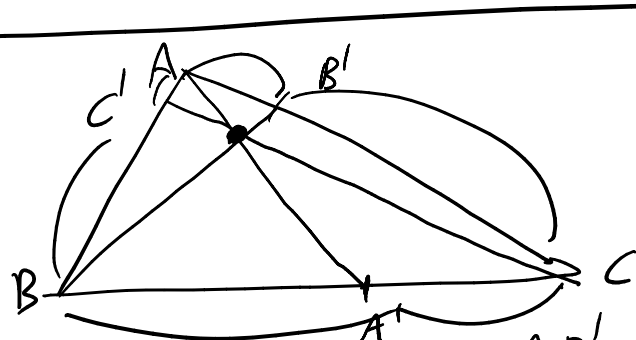
$d \nparallel d_1$ $(\sphericalangle d_1 d_3 = 60°)$

$\Rightarrow \exists \{C\} \in d \cap d_1$

$C \in d = R_{A,\frac{\pi}{3}}(d_3) \Rightarrow \exists! B$ a.î. $R_{A,\frac{\pi}{3}}(B) = C$

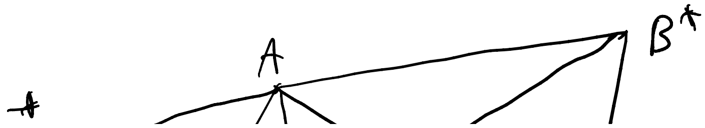
Ca mai sus, ΔABC echilateral!

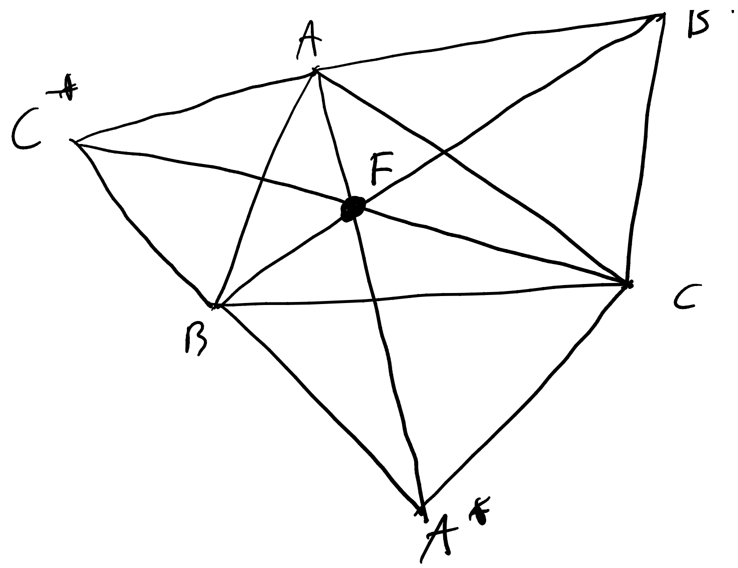
Th Ceva



AA', BB', CC' concurente $\Longleftrightarrow \dfrac{AB'}{B'C} \cdot \dfrac{CA'}{A'B} \cdot \dfrac{BC'}{C'A} = 1$

Rapoarte cu semn





A
B*
C*
F
B
C
A*