Data: 13 septembrie 2022
Timp de lucru: 2h 30m
Punctaj total: 90p + 10p oficiu **Nume:** ______________________________________

Examen **Analiză complexă**

Subiecte:

1. (a) (5 p) Determinaţi soluţiile $z \in \mathbb{C}$ ale ecuaţiei $z^2 - 2z + \mathrm{i} = 0$.

   (b) (5 p) Considerăm $f : \mathbb{C} \to \mathbb{C}$ definită prin
$$f(x + \mathrm{i}\, y) = (2x^2 - 2xy - 2y^2) + \mathrm{i}(x^2 + 4xy - y^2),$$
   pentru orice $x, y \in \mathbb{R}$. Este f olomorfă pe $\mathbb{C}$? Justificaţi răspunsul!

   (c) (5 p) Daţi exemplu de funcţie $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, olomorfă, cu pol de ordin 3 în 0, şi $\mathrm{res}(f, 0) = 1$. Justificaţi, pe scurt, de ce funcţia aleasă îndeplineşte condiţiile cerute.

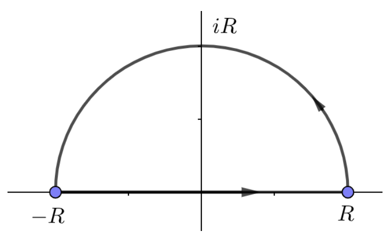
   (d) (5 p) Pentru $f(z) = \dfrac{z^2}{z - \sin z}$, calculaţi $\mathrm{res}(f, 1)$.

   (e) (5 p) Demonstraţi că pentru orice $x, y \in \mathbb{R}$, are loc egalitatea
$$\sin(x + \mathrm{i}\, y) = \sin x \cosh y + \mathrm{i} \cos x \sinh y.$$

2. (a) (15 p) Determinaţi câte dintre rădăcinile ecuaţiei $z^4 - 5z + 1 = 0$ se află în coroana circulară $\mathcal{A} = \{1 \le |z| \le 2\}$.

   (b) (10 p) Calculaţi, folosind eventual principiul argumentului,
$$\int_{|z-i|=2} \frac{z+1}{z^2 + 2z + 2} dz,$$
   unde cercul $|z - i| = 2$ este pozitiv orientat.

3. (20 p) Calculaţi
$$\int_0^\infty \frac{\cos x}{x^4 + 16} dx,$$
   folosind funcţia $f(z) = \dfrac{e^{\mathrm{i}\, z}}{z^4 + 16}$ şi conturul de integrare din desenul următor:



4. (10 p) Descrieţi cum putem obţine o aplicaţie biolomorfă între Ω_1 şi Ω_2, unde
$$\Omega_1 = \{z = x + \mathrm{i}\, y \mid x, y \in \mathbb{R} \text{ şi } 0 < x - y < 1\} \text{ şi } \Omega_2 = \{z \in \mathbb{C} \mid |z| < 1\}.$$

5. (10 p) Determinaţi toate funcţiile olomorfe $f : \mathbb{C} \to \mathbb{C}$, pentru care $f(x + \mathrm{i}\, y) = u(x) + \mathrm{i}\, v(y)$ pentru orice $x, y \in \mathbb{R}$, unde u şi v sunt funcţii cu valori reale.

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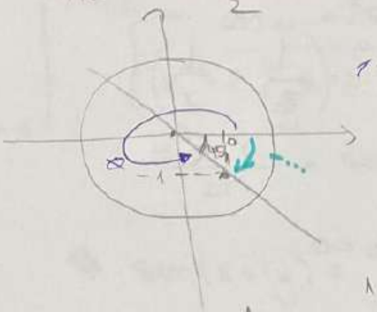
1)

a) $z^2 - 2z + i = 0$

$\Delta = 4 - 4i$

$$z_{1,2} = \frac{2 \pm \sqrt{4-4i}}{2} = \frac{\cancel{2} \pm \cancel{2}\sqrt{1-i}}{\cancel{2}} = 1 \pm \sqrt{1-i}$$



$1 - i = r \cdot e^{\theta i}$

$2\pi - \frac{\pi}{4} = \frac{7\pi}{4} = \theta$

$|1-i| = \sqrt{2}$

$1 - i = \sqrt{2}\, e^{\frac{7\pi}{4} i} \;|\sqrt{\ }$

$\Rightarrow \sqrt{1-i} = 2^{\frac{1}{4}} \cdot e^{\frac{7\pi}{8} i}$

$\Rightarrow z_1 = 1 + 2^{\frac{1}{4}} \cdot e^{\frac{7\pi}{8} i}$

$\Rightarrow z_2 = 1 - 2^{\frac{1}{4}} \cdot e^{\frac{7\pi}{8} i}$

b) $f(x+iy) = \underbrace{(2x^2 - 2xy - 2y^2)}_{u} + i\underbrace{(x^2 + 4xy - y^2)}_{v}$

f olom $\Leftrightarrow \exists$ ec. Cauchy - Riemann

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \qquad \begin{cases} 4x - 2y = 4x - 2y \\ -2x - 4y = -2x - 4y \end{cases}$$

$\Rightarrow$ olom. pe $\mathbb{C}$, $(\forall)\, x, y \in \mathbb{R}$

c) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

olom.

$$f(z) = \frac{1}{z^3} + \frac{1}{z}$$

- ord 3
- pol = 0

$$\operatorname{res}(f, 0) = \frac{1}{(3-1)!} \left[(z-0)^3 \cdot \left(\frac{1}{z^3} + \frac{1}{z} \right) \right]^{(3-1)} \Bigg|_{z=0}$$

$$= \frac{1}{2} \left(1 + z^2 \right)'' \Bigg|_{z=0}$$

$$= \frac{1}{2} \cdot 2 = 1.$$

d) $$f(z) = \frac{z^2}{z - \sin z}$$

$\operatorname{res}(f, 1) = ?$

$$f(1) = \frac{1^2}{\underbrace{1 - \sin 1}_{\neq 0}} \Rightarrow z = 1 \text{ nu e pol} \Rightarrow \operatorname{res}(f, 1) = 0$$

e) (V) $x, y \in \mathbb{R}$

$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$e^{zi} = \cos z + i \sin z$ $\qquad 2\cos z = e^{zi} + e^{-zi}$

$e^{-zi} = \cos(-z) + i\sin(-z) =$ $\qquad \Rightarrow \cos z = \frac{e^{zi} + e^{-zi}}{2}$

$= \cos z - i \sin z$ $\qquad \Rightarrow \cosh z = \frac{e^{z} + e^{-z}}{2}$

$e^{zi} - e^{-zi} = 2i \sin z$ $\ominus$

$\frac{e^{zi} - e^{-zi}}{2i} = \sin z$ $\quad \cancel{z}i \Rightarrow \sinh(z) = \frac{e^{z} - e^{-z}}{2}$

$$\sin(x+iy) = \frac{e^{(x+iy)i} - e^{-(x+iy)i}}{2i} =$$

$$= \frac{e^{(xi-y)} - e^{(y-xi)}}{2i}$$

$$= \frac{e^{xi} \cdot e^{-y} - e^{-xi} \cdot e^{y}}{2i}$$

$$= \frac{(\cos x + i \sin x) e^{-y} - (\cos \cancel{x} - i \sin x) e^{y}}{2i}$$

$$= \frac{\cos x e^{-y} + i \sin x e^{-y} - \cos x e^{y} + i \sin x e^{y}}{2i}$$

$$= \frac{\cos x (e^{-y} - e^{y}) + i \sin x (e^{-y} + e^{y})}{2i}$$

$$= \frac{\cancel{i} \sin x (e^{-y} + e^{y})}{2\cancel{i}} + \frac{\cos x (e^{-y} - e^{y})}{2i}$$

$$= \underbrace{\sin x \frac{(e^{-y} + e^{y})}{2}}_{= \cosh y} + \frac{i \cos x (e^{-y} - e^{y})}{-2}$$

$$= \sin x \cosh y + i \cos x \underbrace{\frac{(e^{y} - e^{-y})}{2}}_{\sinh y}$$

2)

a) $z^4 - 5z + 1 = 0 = F(z)$

$A = \{ 1 \leq |z| \leq 2 \}$

$|z| \leq 2 \Rightarrow$ disc de rază $2 = D_1$

$\underbrace{z^4}_{f} \mid \underbrace{-5z+1}_{g}$

T. Rouché: $|f| \big|_{Fr D_1} > |g| \big|_{Fr D_1}$

$\inf\limits_{|z|=2} |f(z)|$ și $\sup\limits_{|z|=2} |g(z)|$

$\inf\limits_{|z|=2} |f(z)| = \min\limits_{\theta \in (0,2\pi)} |f(2e^{\theta i})| = \min |2^4 e^{4\theta i}| =$

$= \min(|2^4| \cdot \underbrace{|e^{4\theta i}|}_{1})$

$= 16$

$\sup\limits_{|z|=2} |g(z)| = \max\limits_{\theta (0,2\pi)} |g(2e^{\theta i})| = \max | -10e^{\theta i} + 1 | \leq$

$\max(\underbrace{|-10e^{\theta i}|}_{10} + \underbrace{|1|}_{1}) = 11$

$16 > 11 \Rightarrow |f| \big|_{Fr D_1} > |g| \big|_{Fr D_1} \Rightarrow$

$\xRightarrow{\text{T. Rouché}}$ f și $f+g$ au același nr. de zerouri pe D_1

$\Rightarrow f$ are 4 soluții $\Rightarrow F(z)$ are 4 sol pe D_1.

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$|z| \leq 1 \Rightarrow$ disc de rază 1 $\Rightarrow D_2$

$z^4 - 5z + 1$

$f = -5z$

$g = z^4 + 1$

$\inf_{|z|=1} |f(z)|$ și $\sup_{|z|=1} |g(z)|$

$\inf |f(z)| = \min_{\theta \in (0,2\pi)} |f(e^{\theta i})| = \min |-5e^{\theta i}| = 5$

$\sup |g(z)| = \max |g(e^{\theta i})| = \max |e^{4\theta i} + 1| \leq 2$

$5 > 2 \Rightarrow |f|\big|_{Fr D_2} > |g|\big|_{Fr D_2} \overset{T.R.}{\Rightarrow} f$ și $f+g$ au

aceiași nr. de zerouri pe $D_2 \Rightarrow F(z)$ are o sol pe D_2

$A = D_1 \setminus D_2 \Rightarrow 4 - 1 = 3$ rădăcini în coroana circulară

b) $\displaystyle\int_{|z-i|=2} \frac{z+1}{z^2+2z+2} dz$

$|z-i|=2$ are por orientat (sens trigonometric)

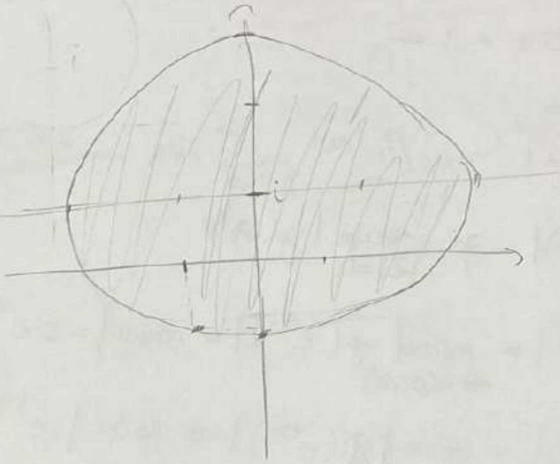
$\displaystyle\frac{1}{2}\int_{\gamma} \frac{2z+2}{z^2+2z+2} dz = \pi i$ (Nr zero $-$ Nr poli) $-$ (*)

Fie $f_1(z) = \frac{1}{}$

Nr. zerouri:

$f(z) = z^2+2z+2$

$|z-i| = 2$



$\Delta = 4-4 = 0 \Rightarrow z_{1,2} = \frac{-b \pm 0}{2a} = -1$

$|-1-i| = \sqrt{2} < 2 \in Int(\gamma) \Rightarrow$ 2 zerouri

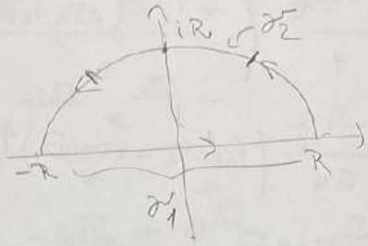
Nr poli $= 0$

$* = 2\pi i$

Fie $f_1(z) = \frac{1}{\ }$

3) $\displaystyle\int_0^\infty \frac{\cos x}{x^4+16}\,dx$

$$f(z) = \frac{e^{iz}}{z^4+16}$$



$$\int_\gamma f(z)\,dz \xlongequal{T.\ Rez.} 2\pi i \sum_{\substack{k \text{ pt } f \\ \text{pol}}} res\left(f|_{Int\,\gamma}, k\right) \cdot \underbrace{Ind_k \gamma}_{1}$$

$$z^4 + 16 = 0$$

$$(z^2+4i)(z^2-4i) =$$

$$(z+2\sqrt{-i})(z-2\sqrt{-i}) = z^2 + 4i$$

$$(z+2\sqrt{i})(z-2\sqrt{i})$$

$$z_1 = -2\sqrt{-i} = -2e^{\frac{3\pi}{4}i} \notin Int\,\gamma$$

$$z_2 = 2\sqrt{-i} = 2e^{\frac{3\pi}{4}i} \in Int\,\gamma$$

$$z_3 = -2\sqrt{i} = -2e^{\frac{\pi}{4}i} \notin Int\,\gamma$$

$$z_4 = 2\sqrt{i} = 2e^{\frac{\pi}{4}i} \in Int\,\gamma$$

$$i = e^{\frac{\pi}{2}i} \qquad -i = e^{-\frac{\pi}{2}i}$$

$$\sqrt{i} = e^{\frac{\pi}{4}i} \qquad \sqrt{-i} = e^{\frac{3\pi}{4}i}$$

$$2\pi - \frac{\pi}{2} = \frac{3\pi}{2} = \frac{3\pi}{4}$$

$$\Rightarrow f(z) = \frac{e^{iz}}{(z+2e^{\frac{3\pi}{4}i})(z-2e^{\frac{3\pi}{4}i})(z+2e^{\frac{\pi}{4}i})(z-2e^{\frac{\pi}{4}i})}$$

$$res\left(f, 2e^{\frac{3\pi}{4}i}\right) =$$

$$= \frac{e^{i2e^{\frac{3\pi}{4}i}}}{(2e^{\frac{3\pi}{4}i}+2e^{\frac{3\pi}{4}i})(2e^{\frac{3\pi}{4}i}+2e^{\frac{\pi}{4}i})(2e^{\frac{3\pi}{4}i}-2e^{\frac{\pi}{4}i})}$$

Fie $f_1(z) = \frac{1}{\quad}$

$$e^{i2e^{\frac{3\pi}{4}i}} = e^{i2\cdot\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)} = e^{i2\cdot\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} =$$

$$= e^{-i\sqrt{2}-\sqrt{2}} = e^{-i\sqrt{2}}\cdot e^{-\sqrt{2}}$$

$$4e^{\frac{3\pi}{4}i} = 4\cdot\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -2\sqrt{2} + 2i\sqrt{2}$$

$$4e^{\frac{3\pi}{2}i} - 4e^{\frac{\pi}{2}i} = 4(-i) - 4i = -8i$$

$$res\left(f, 2e^{\frac{3\pi}{4}i}\right) = \frac{e^{-i\sqrt{2}}\cdot e^{-\sqrt{2}}}{(-2\sqrt{2}+2i\sqrt{2})(-8i)}$$

$$res\left(f, \underbrace{2e^{\frac{\pi}{4}i}}_{a}\right) = res\left(f, 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right) =$$

$$= res\left(f, \sqrt{2}+i\sqrt{2}\right) =$$

$$= \frac{e^{i\sqrt{2}}\cdot e^{-\sqrt{2}}}{(a+b)(a-b)2a} = \frac{e^{i\sqrt{2}}\cdot e^{-\sqrt{2}}}{(a^2-b^2)2a}$$

$$a = 2e^{\frac{\pi}{4}i}$$

$$b = 2e^{\frac{3\pi}{4}i}$$

$$= \frac{e^{i\sqrt{2}}\cdot e^{-\sqrt{2}}}{\left(4e^{\frac{\pi}{2}i} - 4e^{\frac{3\pi}{2}i}\right)\left(2\sqrt{2}+2i\sqrt{2}\right)}$$

$$= \frac{e^{i\sqrt{2}}\cdot e^{-\sqrt{2}}}{(4i+4i)(2\sqrt{2}+2i\sqrt{2})} = \frac{e^{i\sqrt{2}}\cdot e^{-\sqrt{2}}}{8i\left(2\sqrt{2}+2i\sqrt{2}\right)}$$

$$\int_\gamma f(z)\,dz = \cancel{2}\pi\cancel{i}\cdot\frac{e^{-\sqrt{2}}}{{}_{4}\cancel{8i}}\left(\frac{e^{-i\sqrt{2}}}{2\sqrt{2}-2i\sqrt{2}} + \frac{e^{i\sqrt{2}}}{2\sqrt{2}+2i\sqrt{2}}\right) =$$

Fie $f_1(z) = \frac{1}{\dots}$

$$= \frac{e^{-\sqrt{2}}\pi}{4}\left(\frac{e^{-i\sqrt{2}}(2\sqrt{2}+2i\sqrt{2}) + e^{i\sqrt{2}}(2\sqrt{2}-2i\sqrt{2})}{16}\right)$$

$$= \frac{e^{-\sqrt{2}}\pi}{64}\left((\underbrace{\cos\sqrt{2}}_{c} - i\underbrace{\sin\sqrt{2}}_{s})(2\sqrt{2}+2i\sqrt{2}) + (\cos\sqrt{2}+i\sin\sqrt{2})(\dots)\right)$$

$$= \frac{e^{-\sqrt{2}}\pi\cdot 2\sqrt{2}}{\cancel{64}\ 32}\left((c-is)(1+i)+(c+is)(1-i)\right)$$

$$= \frac{e^{-\sqrt{2}}\pi\sqrt{2}}{32}\left(c+\cancel{ic}-\cancel{is}+s+c-\cancel{ci}+\cancel{is}+s\right)$$

$$= \frac{e^{-\sqrt{2}}\pi\sqrt{2}}{32}(2c+2s)$$

$$= \frac{e^{-\sqrt{2}}\pi\sqrt{2}}{\cancel{32}}\cdot\cancel{2}(c+s)$$

$$= \frac{e^{-\sqrt{2}}\pi\sqrt{2}}{16}\left(\cos\sqrt{2}+\sin\sqrt{2}\right)$$

$\gamma_1 : [-R, R] \to \mathbb{C}, \ \gamma_1(t) = t$

$\gamma_2 : [0, \pi] \to \mathbb{C}, \ \gamma_2(\alpha) = Re^{i\alpha}$

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$$

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} f(t)dt = \int_{-R}^{R}\frac{e^{-it}}{t^4+16}dt =$$

$$= \int_{-R}^{R}\frac{\cos t - i\sin t}{t^4+16}dt = \underbrace{\int_{-R}^{R}\frac{\cos t}{t^4+16}dt}_{I_1} - i\underbrace{\int_{-R}^{R}\frac{\sin t}{t^4+16}dt}_{I_2}$$

J_1:

Fie $f_1(x) = \frac{\cos x}{x^4+16}$

$f_1(-x) = \frac{\cos(-x)}{(-x)^4+16} = \frac{\cos x}{x^4+16} = f_1(x) \Rightarrow f_1$ pară $\Rightarrow$

$\Rightarrow J_1 = 2\int_0^R f_1(x)\,dx$

J_2

Fie $f_2(x) = \frac{\sin x}{x^4+16}$

$f_2(-x) = \frac{\sin(-x)}{(-x)^4+16} = \frac{-\sin x}{x^4+16} = -f_2(x) \Rightarrow f_2$ impară

$\Rightarrow J_2 = 0$

$\Rightarrow \int_{\gamma_1} = 2 \cdot J$

$$\int_{\gamma_2} f(z)\,dz = \int_0^\pi f(Re^{i\alpha})\,d(Re^{i\alpha}) =$$

$$= \int_0^\pi \frac{e^{iRe^{i\alpha}}}{Re^{4i\alpha}+16}\, iRe^{i\alpha}\,d\alpha =$$

$$\left| \frac{e^{iR(\cos\alpha + i\sin\alpha)} \cdot iRe^{i\alpha}}{Re^{4i\alpha}+16} \right| =$$

$$= \left| \frac{\overset{=1}{e^{iR\cos\alpha}} \cdot e^{-R\sin\alpha} \cdot \overset{1}{i} R \overset{1}{e^{i\alpha}}}{Re^{4i\alpha}+16} \right| = \left| \frac{e^{-R\sin\alpha} \cdot R}{Re^{4i\alpha}+16} \right|$$

$$= \left| \frac{R}{e^{R\sin\alpha}(Re^{4i\alpha}+16)} \right| \xrightarrow{R\to\infty} 0$$

$$\Rightarrow \int_{\gamma_2} f(z)\,dz \xrightarrow{R\to\infty} 0$$

$$\left.\begin{array}{l} \int_{\gamma} = 2 J_R + \underbrace{\int_{\gamma_2}}_{0} \xrightarrow{R\to\infty} 2J \\ \parallel \\ \frac{e^{-\sqrt{2}}\pi\sqrt{2}}{16}(\cos\sqrt{2}+\sin\sqrt{2}) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{J = \frac{e^{-\sqrt{2}}\pi\sqrt{2}}{32}(\cos\sqrt{2}+\sin\sqrt{2})}$$

4) $\Omega_1 = \{ z = x+iy \mid x, y \in \mathbb{R}$ și $0 < x-y < 1 \}$

$\Omega_2 = \{ z \in \mathbb{C} \mid |z| < 1 \}$

Ω_1:

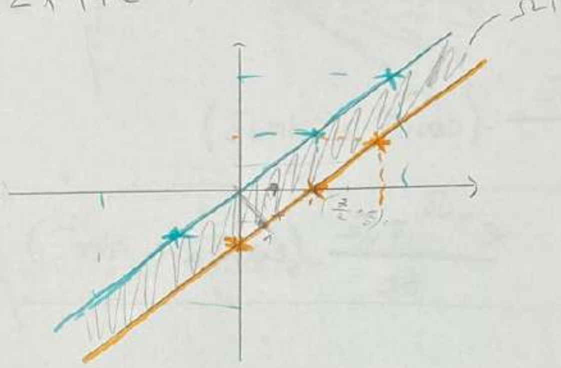
$0 < x-y < 1$

$\Rightarrow x = t \Rightarrow 45^\circ$

$x = t+1 \Rightarrow t = x-1$

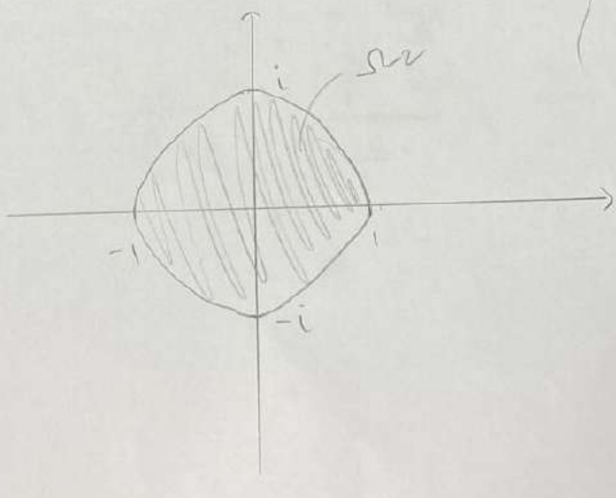
$y = t, \ t \in \mathbb{R}$

$0 < x - t < 1 \mid +t \Rightarrow t < x < t+1$



Ω_2

$|z| < 1$



Fie $f_1(z) = \frac{1}{z}$

$$f_1(-1-i) = \frac{1}{-1-i} = \frac{i-1}{2}$$

$$f(1+i) = \frac{1}{1+i} = \frac{1-i}{2}$$

$$f(2+2i) = \frac{1}{2+2i} = \frac{2-2i}{8} = \frac{1-i}{4}$$

$$f_1(1) = \frac{1}{1} = 1$$

$$f_1(-i) = \frac{1}{-i} = -\frac{1}{i} = \frac{-i}{-1} = i$$

$$f_1(2+i) = \frac{1}{2+i} = \frac{2-i}{4+1} = \frac{2-i}{5}$$

$$f_1\left(\frac{3}{2}+\frac{i}{2}\right) = \frac{1}{\frac{3}{2}+\frac{i}{2}} = \frac{\frac{3}{2}-\frac{i}{2}}{\frac{10}{4}} = \frac{3-i}{\cancel{2}}\cdot\frac{\cancel{4}^{\,2\,1}}{\cancel{10}_{5}} = \frac{3-i}{5}$$

5) $f: \mathbb{C} \to \mathbb{C}$

$f(x+iy) = u(x) + iv(y)$

f olom $\Leftrightarrow$ ec. Cauchy Riemann

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial x} = u'(x) \\ \frac{\partial v}{\partial y} = v'(y) \end{cases} \Rightarrow u'(x) = v'(y) \,\Big|\, \int dy \Leftrightarrow$$

$$\frac{du(x)}{dx} = \frac{dv(y)}{dy}$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \int u'(x)\, dy = v(y) + c_1$$

$$yu'(x) = v(y) + c_1$$

$$u'(x) = \frac{v(y) + c_1}{y}$$

$$u'(x) = \frac{v(y)}{y} + \frac{c_1(x)}{y} \,\Big|\, \int dx$$

$$u(x) + c_2(y) = \frac{x v(y)}{y} + \frac{1}{y}\int c_1(x)\, dx$$