Facultatea de Matematica-Informatica
Universitatea din Bucuresti

23 Ianuarie 2019

Examen final

Disciplina: Ecuatii cu derivate partiale
Tipul examinarii: Examen
Nume student: ______________________________
Grupa 321
Timp de lucru : 3 ore

Nu uitati sa va scrieti numele si prenumele in rubrica Nume student.

Acest examen contine 5 probleme (toate obligatorii).

Verificati foile cu subiecte fata-verso !

Examenul este individual. La sfarsitul examenului nu uitati sa aduceti foaia cu subiectele o data cu lucrarea scrisa pentru a le capsa impreuna. Astfel, corectura se va face mai usor.
Pentru elaborarea lucrarii scrise puteti folosi ca unic material ajutator o foaie format A4 care sa contina doar notiuni teoretice. Exercitiile rezolvate sunt excluse ca material ajutator.

Pentru redactare tineti cont de urmatoarele sugestii:

- Daca folositi o teorema fundamentala, rezultat cunoscut, etc **indicati** acest lucru si explicati cum se poate aplica rezultatul respectiv.
- **Organizati-va munca** intr-un mod coerent pentru a avea toti de castigat ! Incercati ca la predarea lucrarii fiecare problema sa fie redactata in ordinea aparitiei pe foaia cu subiecte. Ideal ar fi ca si subpunctele sa fie redactate in ordine. Daca nu stiti a rezolva un anumit subpunct scrieti numarul subpunctului si lasati liber.
- Raspunsurile corecte dar argumentate incomplet (din punct de vedere al calculelor/explicatiilor) vor primi punctaj partial.

Barem: P1 (2.5p) + P2 (1.5p)+ P3 (2p) +P4 (1.5p) + P5 (1.5p)+ 1 oficiu= **10p**.

Rezultatele le veti primi in cel mai scurt timp posibil pe e-mailul sefului de grupa. Pentru orice nelamuriri scrieti-mi la adresa cristian.cazacu@fmi.unibuc.ro.

Problema 1. *(2.5p). Fie functia* $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x-1|$.

1). Definiti spatiul $H^1(0,2)$.

2). Argumentati ca $f \in H^1(0,2)$ *si calculati norma lui* f *in* $H^1(0,2)$.

Consideram functia $u : B_1(0) \setminus \{0\} \to \mathbb{R}$ *data de*

$$u(x) = |x|^{-3}, \quad x = (x_1, \dots, x_5),$$

unde $B_1(0)$ *este bila unitate din* $\mathbb{R}^5$ *centrata in origine.*

3). Folosind eventual formula operatorului Laplacian Δ *pentru functii cu simetrie radiala din* $\mathbb{R}^5$, *gasiti* $\lambda \in \mathbb{R}$ *astfel incat*

$$\Delta u = \lambda \frac{u}{|x|^2}, \quad \forall x \in B_1(0) \setminus \{0\}.$$

4). Sa se determine pentru ce valori $p \geq 1$ *are loc* $u \in L^p(B_1(0))$.

5). Sa se determine pentru ce valori $p \geq 1$ *are loc* $u \in L^p(\mathbb{R}^5 \setminus \overline{B_1(0)})$.

Problema 2. *(1.5p). Se considera problema la limita*

$$(1) \qquad \begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, & (x,y) \in (0,1) \times (0,1) \\ u(x,0) = u(x,1) = 0, & x \in (0,1),\ y \in (0,1) \\ u(0,y) = \sin(2\pi y),\ u(1,y) = e^{-2\pi} \sin(2\pi y), & y \in (0,1). \end{cases}$$

1). Aratati ca (1) *are cel mult o solutie de clasa* C^2.

2). Determinati solutia problemei (1) *cautand-o in variabile separate sub forma* $u(x,y) = A(x)B(y)$.

3). Calculati $\max_\Omega u$, $\min_\Omega u$ *unde* $\Omega := [0,1] \times [0,1]$.

Problema 3. *(2p). Consideram urmatoarea problema*

$$(2) \qquad \begin{cases} u_{tt}(x,t) + 3u_{tx}(x,t) + 2u_{xx}(x,t) = t, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

unde $f, g \in C^2(\mathbb{R})$ *sunt functii date.*

1). Gasiti o functie $\phi : \mathbb{R} \to \mathbb{R}$ *astfel incat functia* $v(x,t) := u(x,t) + \phi(t)$ *sa verifice ecuatia*

$$(3) \qquad v_{tt}(x,t) + 3v_{tx}(x,t) + 2v_{xx}(x,t) = 0, \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

(Se considera faptul ca lucram cu functii de clasa C^2.*)*

2). Pentru ϕ *de mai sus scrieti conditiile initiale indeplinite de* v.

3). Rezolvati problema cu valori initiale satisfacuta de v *(scrieti forma generala a lui* v*) reducand-o la rezolvarea a doua ecuatii de transport (una omogena si alta neomogena).*

4). Folosind conditiile asupra lui v *la* $t = 0$ *deduceti solutia* u *a problemei* (2) *in cazul particular* $f(x) = \cos x$ *si* $g(x) = x^2$.

Problema 4. *(1.5p). Se considera problema Dirichlet*

$$(4) \qquad \begin{cases} -((1+x^2)u'(x))' + u(x) = x, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

Definim o solutie slaba pentru (4) *ca fiind o functie* $u \in H_0^1(0,1)$ *ce satisface formularea variationala*

$$(5) \qquad \int_0^1 (1+x^2)u'v'dx + \int_0^1 uvdx = \int_0^1 xv(x)dx, \quad \forall v \in H_0^1(0,1).$$

1). Aratati ca daca $u \in C^2([0,1])$ *este solutie clasica pentru* (4) *atunci* u *este solutie slaba pentru* (4) *in sensul lui* (5).

2). Dati exemplu de o norma pe $H_0^1(0,1)$ *si aratati ca termenii din membrul stang in* (5) *sunt bine definiti pentru* $u, v \in H_0^1(0,1)$.

3). Aratati ca functionala liniara $F : H_0^1(0,1) \to \mathbb{R}$ *definita prin* $F(v) = \int_0^1 uvdx$ *este continua.*

4). Aratati ca forma biliniara $a : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$ *definita prin*

$$a(u,v) := \int_0^1 (1+x^2)u'v'dx + \int_0^1 uvdx$$

este coerciva.

Problema 5. *(1.5p) Consideram problema Cauchy*

$$(6) \qquad \begin{cases} u_t(x,t) - u_{xx}(x,t) + u(x,t) = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R}. \end{cases}$$

1). Gasiti o functie $\phi : \mathbb{R} \to \mathbb{R}$ *astfel incat functia* $v(x,t) := u(x,t) + \phi(x)$ *sa verifice ecuatia caldurii*

$$(7) \qquad v_t(x,t) - v_{xx}(x,t) = 0, \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

2). Scrieti problema Cauchy verificata de v *si calculati* $v(0,1)$.

3). $\star$ *Determinati explicit solutia problemei* (6).

1. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=|x-1|$

a) Def. $H^1(0,2)$

b) $f\in H^1(0,2)$, $\|f\|_{H^1(0,2)}=?$

$w: B_1(0)\setminus\{0\}\to\mathbb{R}$, $w(x)=|x|^{-3}$, $x=(x_1,\dots,x_5)$

c) Δ pt. funcții radiale $\Rightarrow \lambda\in\mathbb{R}$ a.î. $\Delta w=\lambda\frac{w}{|x|^2}$, $\forall x\in B_1(0)\setminus\{0\}$

d) $p\geq 1$ a.î. $w\in L^p(B_1(0))$

e) —— " —— $w\in L^p(\mathbb{R}^5\setminus\overline{B_1(0)})$

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a) $H^1(0,2)=W^{1,2}(0,2)=\{f\in L^2(0,2)\mid f'_s\in L^2(0,2)\}$

spațiu Sobolev pe $(0,2)\subset\mathbb{R}$

derivata slabă

sau $=\{f\in L^2(0,2)\mid \exists\, g\in L^2(0,2)$ a.î. $\int_{(0,2)} f\varphi'\,dx=-\int g\varphi\,dx, \forall\varphi\in C_c^\infty(0,2)\}$

$L^2(0,2)=\{f\mid \int_{(0,2)}|f(x)|^2dx<\infty\}$

b) $f:\mathbb{R}\to\mathbb{R}$

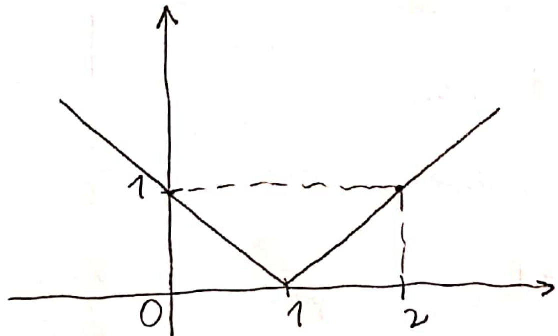
$$f(x)=|x-1|=\begin{cases}x-1, & x\geq 1\\ 1-x, & x<1\end{cases}$$

$\underline{f\in L^2(0,2)}$

$$(1)\int_{(0,2)}|f(x)|^2dx=\int_{(0,2)}\underbrace{(x-1)^2}_{\text{funcție continuă și }(0,2)\text{ mărginit}\ \Downarrow\ \int<\infty}dx$$

sau pot să o calculez efectiv

$$\overset{\text{sau}}{=}\int_0^2(x^2-2x+1)\,dx=\left(\frac{x^3}{3}-2\frac{x^2}{2}+x\right)\Big|_0^2==\frac{8}{3}-4+2=\frac{2}{3}<\infty$$



f nu e derivabilă în 1

f deriv. în sens slab și putem lua

$$f'_s(x)=\begin{cases}1, & x\geq 1\\ -1, & x<1\end{cases}$$

Dem. că f'_s este derivată slabă pt. f.

Fie $\varphi\in C_c^\infty(0,2)$, $\Rightarrow \varphi(0)=\varphi(2)=0$ ($\varphi=0$ pe frontieră)

$$\int_{(0,2)}f\varphi'\,dx=\int_0^2 f\varphi'\,dx=\int_0^1 f\varphi'\,dx+\int_1^2 f\varphi'\,dx\overset{\text{int. prin părți}}{=}$$

$$=\cancel{f(1)\varphi(1)}-\cancel{f(0)\varphi(0)}-\int_0^1 f'\varphi\,dx+\cancel{f(2)\varphi(2)}-\cancel{f(1)\varphi(1)}-\int_1^2 f'\varphi\,dx=$$

$$=-\int_0^1\underbrace{f'}_{-1}\varphi\,dx-\int_1^2\underbrace{f'}_{1}\varphi\,dx=\int_0^1\varphi\,dx-\int_1^2\varphi\,dx=-\int_0^2 f'_s\varphi\,dx,$$

unde $f'_s(x)=\begin{cases}-1, & x\in(0,1)\\ 1, & x\in[1,2)\end{cases}$

$\Rightarrow \int_{(0,2)} f\varphi' dx = -\int_{(0,2)} f_1'\varphi\, dx, \ \forall \varphi \in C_c^\infty(0,2)$

$f_1' \in L^2(0,2)$

(2) $\int_{(0,2)} |f_1'(x)|^2 dx = \int_0^2 1\, dx = x\Big|_0^2 = 2 < \infty$

Deci $f \in H^1(0,2)$.

$\|f\|_{H^1(0,2)} = \|f\|_{L^2(0,2)} + \|f_1'\|_{L^2(0,2)} = \left(\int_{(0,2)} |f(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_{(0,2)} |f_1'(x)|^2 dx\right)^{\frac{1}{2}}$

$= \sqrt{\frac{2}{3}} + \sqrt{2}$ (din (1) și (2))

c) $u : B_1(0) \setminus \{0\} \to \mathbb{R}$

$u(x) = |x|^{-3}$

Obs. că $u(x) = g(|x|)$, unde $g : (0,\infty) \to \mathbb{R}$, $g(r) = r^{-3} \Rightarrow$

$\Rightarrow u$ este funcție radială $\Rightarrow \Delta u(x) = g''(|x|) + \frac{n-1}{|x|} g'(|x|)$

$n = 5$

$g'(r) = (r^{-3})' = (-3) r^{-4} = -3r^{-4}$

$g''(r) = (-3r^{-4})' = -3 \cdot (-4) r^{-5} = 12 r^{-5}$

$g''(r) + \frac{4}{r} g'(r) = 12 r^{-5} + \frac{4}{r}(-3r^{-4}) = 12r^{-5} - 12 r^{-5} = 0$

$\Rightarrow \Delta u(x) = 0$

$\Delta u(x) = \lambda \frac{u}{|x|^2} \Leftrightarrow 0 = \lambda |x|^{-5} \Leftrightarrow \underline{\lambda = 0}.$

d) $u \in L^p(B_1(0)) \Leftrightarrow \int_{B_1(0)} |u(x)|^p dx < \infty \Leftrightarrow \int_{B_1(0)} |x|^{-3p} dx < \infty$

[din ex. seminar $\Rightarrow -3p > -5 \Rightarrow 3p < 5 \Rightarrow p < \frac{5}{3}$ $\quad p \in [1, \frac{5}{3})$

sau calculăm:

$\int_{B_1(0)} |x|^{-3p} dx \overset{\text{formula}}{\underset{\text{co-ariei}}{=}} \int_0^1 \left(\int_{\partial B_s(0)} |x|^{-3p} d\sigma\right) ds = \int_0^1 \left(\int_{\partial B_s(0)} s^{-3p} d\sigma\right) ds =$

$= \int_0^1 s^{-3p} \cdot \left(\int_{\partial B_s(0)} d\sigma\right) ds = \int_0^1 s^{-3p} \cdot |\partial B_s(0)|\, ds = \int_0^1 s^{-3p} \cdot \omega_5 \cdot s^4 ds =$

$= \omega_5 \cdot \int_0^1 s^{-3p+4} ds \sim \begin{cases} \frac{s^{-3p+5}}{-3p+5}\Big|_0^1 & -3p+4 \neq -1 \\ \ln s \Big|_0^1, & -3p+4 = -1 \end{cases} = \begin{cases} \frac{1}{-3p+5}, & -3p+5 > 0 \\ \infty, & -3p+5 \leq 0 \end{cases}$

$\Rightarrow \int_{B_1(0)} |x|^{-3p} dx < \infty \Leftrightarrow -3p+5 > 0 \Leftrightarrow p < \frac{5}{3}$

e) $u \in L^p(\mathbb{R}^5 \setminus \overline{B_1(0)}) \Leftrightarrow \int_{\mathbb{R}^5 \setminus \overline{B_1(0)}} |u(x)|^p dx < \infty \Leftrightarrow \int_{\mathbb{R}^5 \setminus \overline{B_1(0)}} |x|^{-3p} dx < \infty$

$\int_{\mathbb{R}^5 \setminus \overline{B_1(0)}} |x|^{-3p} dx \overset{\text{formula}}{\underset{\text{co-ariei}}{=}} \int_1^\infty \left(\int_{\partial B_s(0)} |x|^{-3p} d\sigma\right) ds = \int_1^\infty s^{-3p} |\partial B_s(0)|\, ds =$

$= \omega_5 \cdot \int_1^\infty s^{-3p+4} ds < \infty \Leftrightarrow -3p+5 < 0 \Leftrightarrow p > \frac{5}{3}$

2. $(1)\begin{cases} (a)\ u_{xx}(x,y)+u_{yy}(x,y)=0, \ (x,y)\in(0,1)\times(0,1) \iff \Delta u(x,y)=0 \\ (b)\ u(x,0)=u(x,1)=0, \ x\in(0,1) \\ (c)\ u(0,y)=\sin(2\pi y) \\ (d)\ u(1,y)=e^{-2\pi}\sin(2\pi y), \ y\in(0,1) \end{cases}$

a) (1) are cel mult o sol. C^2

b) sol. variabile separate $u(x,y)=A(x)B(y)$

c) $\max_{\Omega} u$, $\min_{\Omega} u$, $\Omega=[0,1]\times[0,1]$

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a) Pp. că (1) are două soluții, u_1 și u_2.

Fie $U=u_1-u_2$. Atunci:

$\Delta U=\Delta u_1-\Delta u_2=0$

$U(x,0)=u_1(x,0)-u_2(x,0)=0$

$U(x,1)=u_1(x,1)-u_2(x,1)=0$

$U(0,y)=u_1(0,y)-u_2(0,y)=\sin(2\pi y)-\sin(2\pi y)=0$

$U(1,y)=u_1(1,y)-u_2(1,y)=e^{-2\pi}\sin(2\pi y)-e^{-2\pi}\sin(2\pi y)=0$

Deci U verifică problema:

$(2)\begin{cases} \Delta U(x,y)=0, \ (x,y)\in(0,1)\times(0,1) \\ U(x,0)=U(x,1)=0, \ x\in(0,1) \\ U(0,y)=U(1,y)=0, \ y\in(0,1) \end{cases}$

$\Delta U(x,y)=0, \ \forall (x,y)\in\underbrace{(0,1)\times(0,1)}_{\Omega} \Rightarrow U$ armonică $\overset{\text{Principiul de maxim}}{\Longrightarrow}$

$\Rightarrow \begin{cases} \max_{\bar\Omega} U=\max_{\partial\Omega} U=0 \\ \min_{\bar\Omega} U=\min_{\partial\Omega} U=0 \end{cases}$

$\partial\Omega=\{(x,0),(x,1)\mid x\in(0,1)\}\cup\{(0,y),(1,y)\mid y\in(0,1)\}$

$\Rightarrow U\equiv 0$ în $\bar\Omega \Rightarrow u_1=u_2$

b) Căutăm soluții netriviale pt. $\underset{(a)+(b)}{(1)}$ în variabile separate:

$u(x,y)=A(x)B(y) \quad (A(x),B(y)\neq 0)$

$(1a)\iff A''(x)B(y)+A(x)B''(y)=0 \ \Big|\cdot\frac{1}{A(x)B(y)}$

$$\frac{A''(x)}{A(x)}=-\frac{B''(y)}{B(y)}=\lambda, \ \forall x,y\in(0,1)$$

$(1b)\iff A(x)B(0)=A(x)B(1)=0 \iff B(0)=B(1)=0$

Obținem:

$\begin{cases} B''(y)+\lambda B(y)=0, \ y\in(0,1) \\ B(0)=B(1)=0 \end{cases}$

Ec. caracteristică: $k^2+\lambda=0$

$\lambda=0 \Rightarrow B''(y)=0 \Rightarrow B(y)=c_1y+c_2$

$B(0)=0 \Rightarrow c_2=0$

$B(1)=0 \Rightarrow c_1+c_2=0$

$c_1=0$

$c_1=c_2=0 \Rightarrow B\equiv 0$ nu convine

$\lambda<0 \Rightarrow k^2=-\lambda>0$

$k_{1,2}=\pm\sqrt{-\lambda}$

$B(y)=c_1e^{\sqrt{-\lambda}y}+c_2e^{-\sqrt{-\lambda}y}$

$B(0)=B(1)=0 \Leftrightarrow \begin{cases} c_1+c_2=0 \\ c_1e^{\sqrt{-\lambda}}+c_2e^{-\sqrt{-\lambda}}=0 \end{cases}$

$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}} & e^{-\sqrt{-\lambda}} \end{vmatrix}=e^{-\sqrt{-\lambda}}-e^{\sqrt{-\lambda}}\neq 0 \Rightarrow c_1=c_2=0$

$\Rightarrow B\equiv 0$ nu convine

$\lambda>0 \Rightarrow k^2=-\lambda<0$

$k_{1,2}=\pm i\sqrt{\lambda}$

$B(y)=c_1\cos(\sqrt{\lambda}y)+c_2\sin(\sqrt{\lambda}y)$

$B(0)=B(1)=0 \Leftrightarrow \begin{cases} c_1=0 \\ \cancel{c_1\cos(\sqrt{\lambda})}+c_2\sin(\sqrt{\lambda})=0 \end{cases}$

$c_2\sin(\sqrt{\lambda})=0$

dacă $c_2=0 \Rightarrow B\equiv 0$ nu convine

$\Rightarrow \sin(\sqrt{\lambda})=0 \Rightarrow \sqrt{\lambda}=n\pi,\ n\in\mathbb{N}^*$

Rezultă că $\lambda_n=n^2\pi^2,\ n\in\mathbb{N}^*$.

$B_n(y)=c_n\sin(n\pi y)$

$\frac{A''(x)}{A(x)}=\lambda_n=n^2\pi^2$

$A''(x)-A(x)\,n^2\pi^2=0$

ec. caracteristică: $k^2-n^2\pi^2=0$

$k^2=n^2\pi^2>0$

$k=\pm n\pi$

$A_n(x)=d_ne^{n\pi x}+e_ne^{-n\pi x}$

$\Rightarrow u_n(x,y)=A_n(x)B_n(y)=(C_ne^{n\pi x}+D_ne^{-n\pi x})\sin(n\pi y)$

Căutăm sol. pt. (1) de forma:

$u(x,y)=\sum_{n\geq 1}u_n(x,y)=\sum_{n\geq 1}(C_ne^{n\pi x}+D_ne^{-n\pi x})\sin(n\pi y)$

constantele C_n, D_n le determinăm din cond. (c)+(d)

$$(1c) \Longleftrightarrow \sum_{n\geq 1} (C_n+D_n)\sin(n\pi y) = \sin(2\pi y)$$

$$(1d) \Longleftrightarrow \sum_{n\geq 1} (C_n e^{n\pi}+D_n e^{-n\pi})\sin(n\pi y) = e^{-2\pi}\sin(2\pi y)$$

Fixăm $m\geq 1$. $\cdot \sin(m\pi y) + \int$:

$$\int_0^1 (C_m+D_m)\sin^2(m\pi y) + \sum_{\substack{n\geq 1\\ n\neq m}} (C_n+D_n)\underbrace{\int_0^1 \sin(n\pi y)\sin(m\pi y)\,dy}_{=0} =$$

$$= \int_0^1 \sin(2\pi y)\sin(m\pi y)\,dy$$

$$\int_0^1 \sin^2(m\pi y)\,dy = \int_0^1 \frac{1-\cos(2m\pi y)}{2}\,dy = \frac{1}{2}$$

$$\Rightarrow \boxed{C_m+D_m = 2\int_0^1 \sin(2\pi y)\sin(m\pi y)\,dy}$$

$$\int_0^1 (C_m e^{m\pi}+D_m e^{-m\pi})\sin^2(m\pi y) + \sum_{\substack{n\geq 1\\ n\neq m}} (C_n e^{n\pi}+D_n e^{-n\pi}) \cdot$$

$$\cdot \underbrace{\int_0^1 \sin(n\pi y)\sin(m\pi y)\,dy}_{=0} = \int_0^1 e^{-2\pi}\sin(2\pi y)\sin(m\pi y)\,dy$$

$$\boxed{C_m e^{m\pi}+D_m e^{-m\pi} = 2e^{-2\pi}\underbrace{\int_0^1 \sin(2\pi y)\sin(m\pi y)\,dy}_{\overset{\text{not.}}{=} I_m}}$$

$$\begin{cases} C_n+D_n = 2I_n & | \cdot e^{-n\pi} \\ C_n e^{n\pi}+D_n e^{-n\pi} = 2e^{-2\pi} I_n & - \end{cases}$$

$$C_n(e^{n\pi}-e^{-n\pi}) = 2I_n(e^{-2\pi}-e^{-n\pi})$$

$$\boxed{C_n = 2I_n \cdot \frac{e^{-2\pi}-e^{-n\pi}}{e^{n\pi}-e^{-n\pi}}}$$

$$\boxed{D_n = 2I_n \cdot \frac{e^{n\pi}-e^{-2\pi}}{e^{n\pi}-e^{-n\pi}}}$$

c) $\Delta u(x,y)=0,\ \forall (x,y)\in\Omega \Rightarrow u$ armonică $\xrightarrow{\text{Principiul de maxim}}$

$$\Rightarrow \begin{cases} \max\limits_{\bar\Omega} u = \max\limits_{\partial\Omega} u \\ \min\limits_{\bar\Omega} u = \min\limits_{\partial\Omega} u \end{cases}$$

$\max\limits_{y\in(0,1)} \sin(2\pi y) = 1 \quad (\sin(2\pi\cdot\frac{1}{4}) = \sin\frac{\pi}{2} = 1)$

$\min\limits_{y\in(0,1)} \sin(2\pi y) = -1 \quad (\sin(2\pi\cdot\frac{3}{4}) = \sin\frac{3\pi}{2} = -1)$

$\max\limits_{y\in(0,1)} e^{-2\pi}\sin(2\pi y) = e^{-2\pi}$

$\min\limits_{y\in(0,1)} e^{-2\pi}\sin(2\pi y) = -e^{-2\pi}$

$\max\{0, 1, e^{-2\pi}\} = 1 \qquad e^{-2\pi} < e^0 = 1$

$\min\{0, -1, -e^{-2\pi}\} = -1$

Deci $\begin{cases} \max\limits_{\bar\Omega} u = \max\limits_{\partial\Omega} u = 1 \\ \min\limits_{\bar\Omega} u = \min\limits_{\partial\Omega} u = -1 \end{cases}$

3. (2) $\begin{cases} u_{tt}(x,t)+3u_{tx}(x,t)+2u_{xx}(x,t)=t, & x\in\mathbb{R}, t>0 \\ u(x,0)=f(x), & x\in\mathbb{R} \\ u_t(x,0)=g(x), & x\in\mathbb{R} \end{cases}$

$f,g\in C^2(\mathbb{R})$

a) $\phi:\mathbb{R}\to\mathbb{R}$ a.î. $v(x,t)=u(x,t)+\phi(t)$ verifică
$$v_{tt}(x,t)+3v_{tx}(x,t)+2v_{xx}(x,t)=0$$

b) ϕ - condiția inițială

c) rez. pb. cu $v\to 2$ ec. de transport

d) $v|_{t=0}\Rightarrow$ sol. u a pb. (2) în cazul part. $\begin{cases} f(x)=\cos x \\ g(x)=x^2 \end{cases}$

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a) $v(x,t)=u(x,t)+\phi(t)$

$v_t(x,t)=u_t(x,t)+\phi'(t)$

$v_{tt}(x,t)=u_{tt}(x,t)+\phi''(t)$

$v_x(x,t)=u_x(x,t)$

$v_{tx}(x,t)=u_{tx}(x,t)$

$v_{xx}(x,t)=u_{xx}(x,t)$

$v_{tt}(x,t)+3v_{tx}(x,t)+2v_{xx}(x,t)=0\Leftrightarrow$

$\Leftrightarrow \underline{u_{tt}(x,t)}+\phi''(t)+\underline{3u_{tx}(x,t)}+\underline{2u_{xx}(x,t)}=0$

$\phi''(t)+t=0$

$\phi''(t)=-t\Rightarrow \phi'(t)=\int -t\,dt=-\frac{t^2}{2}+a$

$\phi(t)=\int\left(-\frac{t^2}{2}+a\right)dt=-\frac{t^3}{6}+at+b$

b) $\begin{cases} u(x,0)=f(x)\Rightarrow v(x,0)=u(x,0)+\phi(0)=f(x)+b \\ u_t(x,0)=g(x)\Rightarrow v_t(x,0)=u_t(x,0)+\phi'(0)=g(x)+a \end{cases}$

La a), luăm $\phi(t)=-\frac{t^3}{6}+t$

cond. inițială: $\phi(0)=0$

c) $\begin{cases} v_{tt}(x,t)+3v_{tx}(x,t)+2u_{xx}(x,t)=0. \\ v(x,0)=f(x) \\ v_t(x,0)=g(x) \end{cases}$

Ec. caracteristică: $\lambda^2+3\lambda+2=0$

$\Delta=9-8=1$

$\lambda_{1,2}=\frac{-3\pm1}{2}$ $\begin{cases}\lambda_1=-2\\ \lambda_2=-1\end{cases}$

$(\lambda+2)(\lambda+1)=0$

$(\partial_t+2\partial_x)(\partial_t+\partial_x)v=0$

Notăm $\tilde{v} = (\partial_t + \partial_x) v$.

$(\partial_t + 2\partial_x)\tilde{v} = 0$

$$\begin{cases} \tilde{v}_t + 2\tilde{v}_x = 0 \\ \tilde{v}(x,0) = v_t(x,0) + v_x(x,0) = g(x) + f'(x) \end{cases}$$

$0 = 2\tilde{v}_x + \tilde{v}_t = (\tilde{v}_x, \tilde{v}_t)\cdot(2,1) = (\nabla \tilde{v})\cdot \bar{a} = \frac{\partial \tilde{v}}{\partial \bar{a}}$, unde $a = (2,1) \Rightarrow$

$\Rightarrow \tilde{v}$ este constantă pe direcția $a = (2,1)$

$\tilde{v}(x,t) = \tilde{v}(t(2,1) + (x-2t,0)) = \tilde{v}(x-2t,0) = g(x-2t) + f'(x-2t)$

Revenim la v:

$$\begin{cases} v_t(x,t) + v_x(x,t) = g(x-2t) + f'(x-2t) \\ v(x,0) = f(x) \end{cases}$$

Fie $w(s) = v(x+s, t+s)$, $s \in \mathbb{R}$,

$w'(s) = v_x(x+s,t+s)\cdot \frac{\partial(x+s)}{\partial s} + v_t(x+s,t+s)\cdot\frac{\partial(t+s)}{\partial s} =$

$= v_x(x+s,t+s) + v_t(x+s,t+s) =$

$= g((x+s) - 2(t+s)) + f'((x+s) - 2(t+s)) =$

$= g(x-2t-s) + f'(x-2t-s) \quad \Big| \int_0^{s'} ds$

$w(s') - w(0) = \int_0^{s'} g(x-2t-s)\,ds + \int_0^{s'} \underbrace{f'(x-2t-s)}_{-(f(x-2t-s))'_s}\,ds$

$v(x+s', t+s') - v(x,t) = \int_0^{s'} g(x-2t-s)\,ds - (f(x-2t-s') - f(x-2t))$, $\forall s' \in \mathbb{R}$

Luăm $s' = -t$.

$v(x-t,0) - v(x,t) = \underbrace{\int_0^{-t} g(x-2t-s)\,ds}_{s.v.\ x-2t-s = u} - f(x-t) + f(x-2t)$

$-ds = du$

$s = 0 \Rightarrow u = x-2t$

$s = -t \Rightarrow u = x-t$

$f(x-t) - v(x,t) = \int_{x-t}^{x-2t} -g(u)\,du - f(x-t) + f(x-2t)$

$v(x,t) = 2f(x-t) - f(x-2t) + \int_{x-t}^{x-2t} g(s)\,ds$

d) $u(x,t) = v(x,t) - \phi(t)$

$v(x,t) = 2\cos(x-t) - \cos(x-2t) + \underbrace{\int_{x-t}^{x-2t} s^2\,ds}_{} =$

$\frac{s^3}{3}\Big|_{x-t}^{x-2t} = \frac{(x-2t)^3 - (x-t)^3}{3} =$

$= \frac{\cancel{x^3} - 8t^3 - 3x^2 2t + 3x\cdot 4t^2 - (\cancel{x^3} - t^3 - 3x^2 t + 3xt^2)}{3}$

$= \frac{-7t^3 - 3x^2 t + 9xt^2}{3} = -\frac{7}{3}t^3 - x^2 t + 3xt^2$

$$u(x,t)=2\cos(x-t)-\cos(x-2t)-\frac{7}{3}t^3-x^2t+3xt^2+\frac{t^3}{6}-t$$
$$=2\cos(x-t)-\cos(x-2t)-\frac{13}{6}t^3-x^2t+3xt^2-t$$

4. $(4)\begin{cases}-((1+x^2)u'(x))'+u(x)=x, & x\in(0,1)\overset{\text{not.}}{=}I\\ u(0)=u(1)=0\end{cases}$

sol. slabă: $u\in H_0^1(0,1)$ a.î. $\int_0^1(1+x^2)u'v'\,dx+\int_0^1 uv\,dx=\int_0^1 xv(x)\,dx$, $\forall v\in H_0^1(I)$ (I)

a) $u\in C^2([0,1])$ sol. clasică $\Rightarrow u$ sol. slabă

b) normă pe $H_0^1(0,1)$; termenii din m.s. în (I) sunt bine def. pt. $u,v\in H_0^1(0,1)$

c) $F:H_0^1(0,1)\to\mathbb{R}$, $F(v)=\int_0^1 xv\,dx$ este continuă

d) $a:H_0^1(0,1)\times H_0^1(0,1)\to\mathbb{R}$, $a(u,v)=\int_0^1(1+x^2)u'v'\,dx+\int_0^1 uv\,dx$ este coercivă

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a) Înmulțim cu funcția test v și integrăm prin părți:

$(v\in C^1(\bar{I}))$ $v\in H_0^1(I)\Rightarrow v(0)=v(1)=0$

$$\int_0^1 -((1+x^2)u')'v+uv)\,dx=\int_0^1 xv\,dx$$
$$-(1+x^2)u'(x)v(x)\Big|_0^1+\int_0^1(1+x^2)u'v'+\int_0^1 uv\,dx=\int_0^1 xv\,dx$$
$$\left.\begin{array}{l}-2u'(1)v(1)+u'(0)v(0)+\int_0^1(1+x^2)u'v'\,dx+\int_0^1 uv\,dx=\int_0^1 xv\,dx\\ \cancel{u \text{ verifică } (4)\Rightarrow u(0)=u(1)=0}\quad v(0)=v(1)=0\end{array}\right\}\Rightarrow$$

$\Rightarrow \int_0^1(1+x^2)u'v'\,dx+\int_0^1 uv\,dx=\int_0^1 xv\,dx$, $\forall v\in C_c^1(I)$ (I)

suport compact ($v=0$ pe $\partial I=\{0,1\}$)

Fie $v\in H_0^1(I)$.

Cum $C_c^1(I)\underset{\text{dens}}{\subseteq}H_0^1(I)\Rightarrow\exists(v_n)_n\subset C_c^1(I)$ a.î. $v_n\to v$ în $H_0^1(I)$

din (I) pt. $v_n\in C_c^1(I)\Rightarrow \int_0^1(1+x^2)u'v_n'\,dx+\int_0^1 uv_n\,dx=\int_0^1 xv_n\,dx$

$$\downarrow\qquad\qquad\downarrow\qquad\qquad\downarrow$$

$$\int_0^1(1+x^2)u'v'\,dx+\int_0^1 uv\,dx=\int_0^1 xv\,dx$$

$$\left.\begin{array}{l}\|v_n-v\|_{H_0^1}\to 0\\ \quad\| \\ \|v_n-v\|_{L^2}+\|v_n'-v'\|_{L^2}\end{array}\right\}\Rightarrow \|v_n-v\|_{L^2},\ \|v_n'-v'\|_{L^2}\to 0$$

$$\left|\int_0^1 uv_n\,dx-\int_0^1 uv\,dx\right|\le\int_0^1|u||v_n-v|\,dx\overset{C-S}{\le}\|u\|_{L^2}\underbrace{\|v_n-v\|_{L^2}}_{\to 0}$$

Analog pt. ceilalți termeni.

$\Rightarrow$ (I) se verifică pt. $\forall v\in H_0^1(I)$

$u\in H_0^1(0,1)$

$\left.\begin{array}{l}u\in C^2(I)\Rightarrow u'\in C^1(I)\subset C(I)\Rightarrow u'^2\in C(I)\\ I \text{ mărginit}\end{array}\right\}\Rightarrow\int_I u'^2\,dx<\infty\Rightarrow u'\in L^2(I)$

$\left.\begin{array}{l}u\in C^2(I)\Rightarrow u\in C(I)\Rightarrow u^2\in C(I)\\ I \text{ mărginit}\end{array}\right\}\Rightarrow\int_I u^2\,dx<\infty\Rightarrow u\in L^2(I)$

$\left.\begin{array}{l} u\in C^1(I) \Rightarrow u' = u'_s \\ u(0)=u(1)=0 \\ u, u' \in L^2(I) \end{array}\right\} \Rightarrow u \in H^1_0(I)$

Deci u sol. slabă.

b) $\|u\|_{H^1_0(0,1)} = \|u\|_{H^1(0,1)} = \|u\|_{L^2(I)} + \|u'\|_{L^2(I)}$

Fie $u, v \in H^1_0(I) \Rightarrow u, v \in L^2(I) \Rightarrow \exists \langle u, v\rangle_{L^2(I)} = \int_0^1 uv\,dx < \infty$

$\Downarrow$ bine definit

$\int_0^1 |uv|\,dx \leq \|u\|_{L^2} \cdot \|v\|_{L^2} < \infty$

$u, v \in H^1(I) \Rightarrow u', v' \in L^2(I) \Rightarrow \exists \langle u', v'\rangle_{L^2(I)}$

$\int_0^1 (1+x^2)\, u'v'\,dx \leq 2\int_0^1 u'v'\,dx < \infty$

c) $F(v) = \int_0^1 xv\,dx$

$|F(v)| = \left|\int_0^1 xv\,dx\right| \leq \int_0^1 \underbrace{|x|}_{\leq 1} |v|\,dx \leq \int_0^1 |v|\,dx \overset{C-S}{\leq} \|v\|_{L^2}\|1\|_{L^2} =$

$= \|v\|_{L^2} \leq \underset{\|v\|_{H^1_0}}{\|v\|_{H^1}} \Rightarrow c = 1$ constanta de continuitate

$\Rightarrow F$ este continuă

d) $a(u,v) = \int_0^1 (1+x^2)u'v'\,dx + \int_0^1 uv\,dx$

$a(u,u) = \int_0^1 \underbrace{(1+x^2)}_{\geq 1} u'^2\,dx + \int_0^1 u^2\,dx \geq \int_0^1 u'^2\,dx + \int_0^1 u^2\,dx =$

$= \|u'\|^2_{L^2} + \|u\|^2_{L^2} \geq \frac{1}{2}\left(\|u'\|_{L^2} + \|u\|_{L^2}\right)^2 = \frac{1}{2}\|u\|^2_{H^1} \Rightarrow c_2 = \frac{1}{2}$ constanta de coercivitate

$\Rightarrow a$ este coercivă

5. (6) $\begin{cases} u_t(x,t) - u_{xx}(x,t) + u(x,t) = 0, & x\in\mathbb{R}, t>0 \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R} \end{cases}$

a) $\phi: \mathbb{R} \to \mathbb{R}$ a.î. $v(x,t) = u(x,t)\phi(t)$ verifică ec. căldurii

(7) $v_t(x,t) - v_{xx}(x,t) = 0, \ \forall x\in\mathbb{R}, t>0$

b) pb. Cauchy pt. v, $v(0,1) = ?$

c) sol. pb. (6) – explicit

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a) $v(x,t) = u(x,t)\phi(t)$

$v_t(x,t) = u_t(x,t)\,\phi(t) + u(x,t)\,\phi'(t)$

$v_x(x,t) = u_x(x,t)\,\phi(t)$

$v_{xx}(x,t) = u_{xx}(x,t)\,\phi(t)$

$v_t(x,t) - v_{xx}(x,t) = 0 \Leftrightarrow u_t(x,t)\phi(t) + u(x,t)\phi'(t) - u_{xx}(x,t)\phi(t) = 0 \Leftrightarrow$

$\Leftrightarrow \underbrace{(u_t(x,t) - u_{xx}(x,t))}_{-u(x,t)}\phi(t) + u(x,t)\phi'(t) = 0$

$u(x,t)(\phi'(t) - \phi(t)) = 0$

$\phi'(t)=\phi(t) \Rightarrow \phi(t)=c\,e^{\int 1\,dt}=c\,e^{t}$

Luăm $c=1 \Rightarrow \phi(t)=e^{t}$.

b) $\begin{cases} v_t(x,t)-v_{xx}(x,t)=0, & x\in\mathbb{R}, t>0 \\ v(x,0)=e^{-x^2} & x\in\mathbb{R} \end{cases}$

$(=u(x,0)\,\phi(0)=\underbrace{e^{-x^2}}_{u_0(x)}\cdot 1)$

$\Rightarrow v(x,t)=(K(\cdot,t)*u_0)(x)=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}}u_0(y)\,dy=$

$=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}}e^{-y^2}\,dy$

$v(0,1)=\frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} e^{-\frac{y^2}{4}}e^{-y^2}\,dy=\frac{1}{\sqrt{4\pi}}\underbrace{\int_{\mathbb{R}} e^{-\frac{5y^2}{4}}\,dy}_{\text{not. } I}$

s.v. $\frac{y\sqrt{5}}{2} \overset{\text{not.}}{=} z \Rightarrow y=\frac{2z}{\sqrt{5}}$

$dy=\frac{2}{\sqrt{5}}\,dz$

$I=\int_{\mathbb{R}} e^{-z^2}\cdot\frac{2}{\sqrt{5}}\,dz=\frac{2}{\sqrt{5}}\underbrace{\int_{\mathbb{R}} e^{-z^2}\,dz}_{=\sqrt{\pi}}=\frac{2\sqrt{\pi}}{\sqrt{5}}$

$\Rightarrow v(0,1)=\frac{1}{\cancel{\sqrt{4\pi}}}\cdot\frac{\cancel{2\sqrt{\pi}}}{\sqrt{5}}=\frac{1}{\sqrt{5}}$

---

c) $v(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}}e^{-y^2}\,dy=\frac{1}{\sqrt{4\pi t}}\int_{\infty}^{-\infty} e^{-\frac{z^2}{4t}}e^{-(x-z)^2}(-dz)=$

s.v. $x-y=z$

$-dy=dz$

$=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{z^2}{4t}}e^{-(x-z)^2}\,dz$

s.v. $\frac{z}{2\sqrt{t}}=w \Rightarrow dz=2\sqrt{t}\,dw$

$=\frac{1}{\cancel{\sqrt{4\pi t}}}\int_{\mathbb{R}} e^{-w^2}e^{-(x-2w\sqrt{t})^2}\cdot\cancel{2\sqrt{t}}\,dw$

$=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-w^2-x^2-4w^2t+4xw\sqrt{t}}\,dw$