Curs geometrie 8-22 noiembrie

OBS $f \in End(V)$ este diagonalizab

• f

OBS $g: V \times V \to K$ formă biliniară

• $g_1: V \to V^*$

$w \to g(\cdot, w)$

$g_1(w): V \to \mathbb{K}, g_1(w)(u) = g(u, w)$

• $g_2: V \to V^*$

$u \to g(u, \cdot)$

$g_2(u): V \to \mathbb{K}, g_2(u)(w) = g(u, w)\ \forall u, w \in V$

Spații vectoriale euclidiene

$(V, +, \cdot)/\mathbb{R}$ sp vect real, $g: V \times V \to \mathbb{R}$ formă biliniară → simetrică → pozitiv definită

• g s.n. produs scalar

• $(V, g) = (E, g) = (E, \langle \cdot, \cdot \rangle) = (E, (\cdot, \cdot))$ spațiu vectorial euclidian real

Ex $(\mathbb{R}^n, g_0)$ $g_0: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, g_0(x, y) = x_1 y_1 + \dots + x_n y_n$ produs scalar canonic, $\forall x, y \in \mathbb{R}^n$

• $R = \{e_1, \dots, e_n\}$ reper în V

a) ortogonal $\Leftrightarrow \langle e_i, e_j \rangle = 0, \forall i \neq j$

b) ortonormat $\Leftrightarrow \langle e_i, e_j \rangle = \delta_{ij}$

• $R = \{e_1, \dots, e_n\} \xrightarrow{A} R' = \{e_1', \dots, e_n'\}$ repere ortonormate

$e_i' = \sum_{j=1}^n a_{ij} e_j, \quad \forall i = \overline{1, n}$

$A \in O(n) \Leftrightarrow A \cdot A^T = I_n$

Dc R, R' sunt la fel orientate: $A \in SO(n)$

Produs vectorial

$(E, \langle \cdot, \cdot \rangle)$ sp vectorial euclidian real, $\dim E = n$

$S = \{v_1, \dots, v_{n-1}\}$ sistem de vectori din E.

Definim vectorul $w = v_1 \times \dots \times v_{n-1}$ numit produs vectorial astfel

1) Dacă S este SLD, at. $w = 0$

2) Dacă S este SLI, atunci

a) $\|w\|^2 = \begin{vmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_{n-1} \rangle \\ \vdots & & & \\ \langle v_{n-1}, v_1 \rangle & \langle v_{n-1}, v_2 \rangle & \dots & \langle v_{n-1}, v_{n-1} \rangle \end{vmatrix}$

b) $w \perp v_k, \forall k = \overline{1, n-1}$ i.e. $\langle w, v_k \rangle = 0, \forall k = \overline{1, n-1}$

c) Reperul $\{v_1, \dots, v_{n-1}, w\}$ este reper pozitiv orientat

OBS Determinant formal. Fie $R = \{e_1, \dots, e_n\}$ reper ortonormat

$$w = \begin{vmatrix} v_1^1 & v_2^1 & \dots & v_{n-1}^1 & e_1 \\ \vdots & \vdots & & \vdots & \vdots \\ v_1^n & v_2^n & \dots & v_{n-1}^n & e_n \end{vmatrix} =$$

$$= e_1 (-1)^{n+1} \begin{vmatrix} v_1^2 & \dots & v_{n-1}^2 \\ \vdots & & \\ v_1^n & \dots & v_{n-1}^n \end{vmatrix} + \dots + e_n (-1)^{2n} \begin{vmatrix} v_1^1 & \dots & v_{n-1}^1 \\ \vdots & & \vdots \\ v_1^{n-1} & & v_{n-1}^{n-1} \end{vmatrix}$$

Caz particular $(\mathbb{R}^3, g_0)$ sp vectorial euclidian, cu str canonică.

$S = \{x, y\} \subset \mathbb{R}^3$; $z = x \times y$ produs vectorial

1) Dc S este SLD, at $z = 0_{\mathbb{R}^3}$

2) Dc S este SLI, at

a) $\|z\|^2 = \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix}$

b) $\langle z, x \rangle = 0, \langle z, y \rangle = 0$

c) $\{x, y, z\}$ reper pozitiv orientat

~~Determinarea~~ $R = \{e_1, e_2, e_3\}$ reper canonic (ortonormat)

OBS $z = x \times y = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix} =$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \Rightarrow x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

Def (produsul mixt)

$$z \wedge x \wedge y = \langle z, x \times y \rangle = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} =$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x \wedge y \wedge z$$

Prop 1) $x \times y = -y \times x$

2) $(x \times y) \times z = \langle x, z \rangle y - \langle y, z \rangle x$

3) Id. Jacobi: $(x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$

OBS $\|z\|^2 = \|x \times y\|^2 = \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix} = \begin{vmatrix} \|x\|^2 & \|x\|\cdot\|y\|\cos\varphi \\ \|x\|\cdot\|y\|\cos\varphi & \|y\|^2 \end{vmatrix} =$

$= \|x\|^2 \cdot \|y\|^2 \cdot \sin^2\varphi$

$\|x \times y\| = \|x\| \cdot \|y\| \cdot \sin\varphi$

$\mathcal{A}$ paralelogram

Aplicatii

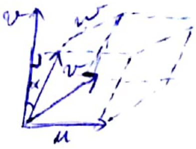
1) $\mathcal{A}_{\Delta A_1A_2A_3} = \frac{1}{2} \|\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}\| = \frac{1}{2} \left\| \begin{vmatrix} e_1 & e_2 & e_3 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} \right\| =$

$= \frac{1}{2}\sqrt{\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_3-x_1 & z_3-z_1 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}^2}$

2) $\{u, v, w\}$ sistem liniar independent (Li) $\Rightarrow$
$\Rightarrow R = \{u, v, w\}$ reper în $\mathbb{R}^3$

$R' = \{u, v, u \times v\}$ reper pozitiv orientat

a) R, R' sunt la fel orientate

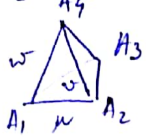


$\mathcal{V}_{paralelipiped} = \|u \times v\| \cdot \|w\| \cos\alpha =$
$= \langle w, u \times v\rangle = w \wedge u \wedge v$

b) R', R nu sunt la fel orientate

$\mathcal{V}_{paralelipiped} = |w \wedge u \wedge v| = \left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|$

$\mathcal{V}_{tetraedru} = \frac{1}{6} |\underbrace{\overrightarrow{A_1A_4}}_{w} \wedge \underbrace{\overrightarrow{A_1A_2}}_{u} \wedge \underbrace{\overrightarrow{A_1A_3}}_{v}| =$



$= \frac{1}{6} \left|\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}\right| =$

$= \frac{1}{6} \left|\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}\right|$ $\quad L_2' = L_2 - L_1$, $L_3' = L_3 - L_1$, $L_4' = L_4 - L_1$

Exemple

$(\mathbb{R}^3, g_0)$ sp. v. e., str. canonică

$u = (1, 2, -1)$
$v = (0, 1, 2)$
$w = (1, 1, 1)$

a) $u \times v$
b) $w \wedge u \wedge v$
c) $z = ?$ a.î. $\{u, v, z\}$ reper ortogonal, pozitiv orientat

Sol

a) $rg\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \end{pmatrix} = 2 \Rightarrow \{u, v\}$ sist Li

$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = (5, -2, 1)$

$= e_1 \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} - e_2 \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + e_3 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = e_1 5 - e_2 2 + e_3 1 = (5, -2, 1)$

b) $w \wedge u \wedge v = \langle w, u \times v\rangle = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

$\langle (1, 1, 1), (5, -2, 1)\rangle = 5 - 2 + 1 = 4 = \mathcal{V}_{paralelip\ u, v, w}$

c) $\langle u, v\rangle = \langle (1, 2, -1), (0, 1, 2)\rangle$
$= 2 - 2 = 0 \Rightarrow u \perp v$

$\{u, v, u \times v\}$ reper ortogonal pozitiv orientat

$u \times v \perp u$
$u \times v \perp v$

Teoremă (Cauchy-Buniakowski-Schwarz)

$(E, \langle\cdot,\cdot\rangle)$ sp. vect. euclidian real, $x, y \in E$

$\Rightarrow |\langle x, y\rangle| \le \|x\| \cdot \|y\|$

"$=$" $\Leftrightarrow \{x, y\}$ un sist. LD

Dem

1. Dacă $x = 0$ sau $y = 0$

$|\langle x, y\rangle| = 0$
$\|x\| \cdot \|y\| = 0$ $\Big\} \Rightarrow |\langle x, y\rangle| = \|x\| \cdot \|y\|$

2. Dacă $x \ne 0$ și $y \ne 0$

Fie $\lambda \in \mathbb{R}$

$\langle x + \lambda y, x + \lambda y\rangle \ge 0, \ \forall \lambda \in \mathbb{R}$

$\lambda^2 \|y\|^2 + 2\lambda \langle x, y\rangle + \|x\|^2 \ge 0, \ \forall \lambda \in \mathbb{R} \Leftrightarrow$

$\Delta_\lambda \le 0 \Rightarrow 4\langle x, y\rangle^2 - 4\|x\|^2\|y\|^2 \le 0 \Rightarrow$

$\Rightarrow |\langle x, y\rangle| \le \|x\| \cdot \|y\|$

"$\Rightarrow$" $|\langle x, y\rangle| = \|x\| \cdot \|y\|$. Dem $\{x, y\}$ este SLD.

$\exists \lambda_0 \in \mathbb{R}$ a.î. $\langle x + \lambda_0 y, x + \lambda_0 y\rangle = 0 \xrightarrow[def]{poz}$

$x + \lambda_0 y = 0 \Rightarrow \{x, y\}$ SLD

"$\Leftarrow$" $\{x, y\}$ este SLD. Dem $|\langle x, y\rangle| = \|x\| \cdot \|y\|$

$\exists a \in \mathbb{R}^*$ a.î. $y = ax$ $\quad |\langle x, y\rangle| = |\langle x, ax\rangle| = |a| \, \|x\|^2$

$\|x\| \cdot \|y\| = \|x\| \, \|ax\| = |a| \, \|x\|^2$

$\|ax\| = \sqrt{\langle ax, ax\rangle} = \sqrt{a^2 \langle x, x\rangle} = |a| \cdot \|x\|$

OBS A da un produs scalar $\Leftrightarrow$ a declara un reper ortonormat

OBS $(V, \langle\cdot,\cdot\rangle_V) \to (V^*, \langle\cdot,\cdot\rangle_{V^*})$

$R = \{e_1, \dots, e_n\}$ reper ortonormat în V

$R^* = \{e_1^*, \dots, e_n^*\}$ reper ortonormat în V^*

$e_i^* : V \to \mathbb{R}$ apl. liniară

$e_i^*(e_j) = \delta_{ij}$

$\langle f, g\rangle_{V^*} = \langle \sum_{i=1}^n f_i e_i^*, \sum_{j=1}^n g_j e_j^*\rangle =$
$= \sum_{i=1}^n \sum_{j=1}^n f_i g_j \underbrace{\langle e_i^*, e_j^*\rangle}_{\delta_{ij}} = \sum_{i=1}^n f_i g_i$

$\langle\cdot,\cdot\rangle_{V^*}$ formă biliniară sim., poz. def

a) $\langle f, g\rangle_{V^*} = \langle g, f\rangle_{V^*}$

b) $\langle f, f\rangle = \sum_{i=1}^n f_i^2$

$\langle f, f\rangle > 0, \forall f \ne 0$ și $\langle f, f\rangle = 0 \Leftrightarrow f_i = 0 \ \forall i = \overline{1,n} \Leftrightarrow f = 0$

Teoremă (procedeul de ortogonalizare Gram-Schmidt)

$(E, <\cdot,\cdot>)$ sp euclidian real

$R = \{f_1, \dots, f_n\}$ reper în $E \Rightarrow \exists R' = \{e_1, \dots, e_n\}$ reper ortogonal în E a.î. $Sp\{f_1, \dots, f_i\} = Sp\{e_1, \dots, e_i\}, \forall i = \overline{1,n}$

Dem: Aplicăm o metodă inductivă

$e_1 = f_1$

$e_2 = f_2 + \alpha e_1$

$<e_1, e_2> = 0 \Rightarrow <f_2 + \alpha e_1, e_1> = 0$

$<f_2, e_1> + \alpha <e_1, e_1> = 0$

$\alpha = -\frac{<f_2, e_1>}{<e_1, e_1>}$

$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1, e_1>} \cdot e_1$

$f_1 = e_1$

$f_2 = \frac{<f_2, e_1>}{<e_1, e_1>} \cdot e_1 + e_2$

$Sp\{f_1, f_2\} = Sp\{e_1, e_2\}$

Pp adevăr P_k: $\{e_1, \dots, e_k\}$ sistem ortogonal a.î. $Sp\{e_1, \dots, e_i\} = Sp\{f_1, \dots, f_i\}, \forall i = \overline{1,k}$

Construim $e_{k+1} = f_{k+1} + \sum_{i=1}^{k} \alpha_{k+1\, i}\, e_i$

$<e_{k+1}, e_j> = 0, \forall j = \overline{1,k}$

$<f_{k+1}, e_j> + \underbrace{\sum_{i=1}^{k} \alpha_{k+1\, i} <e_i, e_j>}_{\alpha_{k+1\, j} <e_j, e_j>} = 0$

$\alpha_{k+1\, j} = -\frac{<f_{k+1}, e_j>}{<e_j, e_j>}, \forall j = \overline{1,k}$

$e_{k+1} = f_{k+1} - \sum \frac{<f_{k+1}, e_j>}{<e_j, e_j>} e_j$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{<f_2, e_1>}{<e_1, e_1>} \cdot e_1 + e_2 \\ \vdots \\ f_i = \frac{<f_i, e_1>}{<e_1, e_1>} e_1 + \frac{<f_i, e_2>}{<e_2, e_2>} \cdot e_2 + \dots + \frac{<f_i, e_{i-1}>}{<e_{i-1}, e_{i-1}>} e_{i-1} + e_i \\ \vdots \\ f_{k+1} = \frac{<f_{k+1}, e_1>}{<e_1, e_1>} e_1 + \dots + \frac{<f_{k+1}, e_k>}{<e_k, e_k>} e_k + e_{k+1} \end{cases}$$

$Sp\{f_1, \dots, f_i\} = Sp\{e_1, \dots, e_i\}, \forall i = \overline{1, k+1}$

Construim recurrent

$f_n = \frac{<f_n, e_1>}{<e_1, e_1>} e_1 + \dots + \frac{<f_n, f_{n-1}>}{<e_{n-1}, e_{n-1}>} e_{n-1} + e_n$ și avem

$Sp\{e_1, \dots, e_i\} = Sp\{f_1, \dots, f_i\}, i = \overline{1,n}$

$R = \{f_1, \dots, f_n\} \xrightarrow[\det A > 0]{A} R' = \{e_1, \dots, e_n\} \xrightarrow[\det B > 0]{B} R'' = \{\frac{e_1}{\|e_1\|}, \dots, \frac{e_n}{\|e_n\|}\}$

reper $\forall$ — reper ortogonal — reper ortonormat

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{<f_2, e_1>}{<e_1, e_1>} e_1 + e_2 \\ \vdots \\ f_n = \frac{<f_n, e_1>}{<e_1, e_1>} + \dots + \frac{<f_n, e_{n-1}>}{<e_{n-1}, e_{n-1}>} e_{n-1} + e_n \end{cases}$$

$R' \xrightarrow{A^{-1}} R$

$$A^{-1} = \begin{pmatrix} 1 & \frac{<f_2, e_1>}{<e_1, e_1>} & \dots & \frac{<f_n, e_1>}{<e_1, e_1>} \\ \vdots & 1 & & \vdots \\ \vdots & \vdots & & \frac{<f_n, e_{n-1}>}{<e_{n-1}, e_{n-1}>} \\ 0 & 0 & & 1 \end{pmatrix}$$

$\det(A^{-1}) = 1 \Rightarrow \det A = \frac{1}{\det A^{-1}} = 1$

$B = \begin{pmatrix} \frac{1}{\|e_1\|} & & 0 \\ & \frac{1}{\|e_2\|} \ddots & \\ 0 & & \frac{1}{\|e_n\|} \end{pmatrix}$ $\det B = \frac{1}{\|e_1\| \cdots \|e_n\|} > 0$

R, R', R'' sunt repere la fel orientate

Def $(E, <\cdot,\cdot>)$ sp vectorial euclidian real.

a) $x \in E$

$x^\perp = \{y \in E \,/\, <x, y> = 0\}$

b) $U \subset E$ subsp v

$U^\perp = \{x \in E \,/\, <x, y> = 0, \forall y \in U\}$

Prop a) $x \in E \Rightarrow x^\perp \subset E$ subspațiu vectorial

b) $U \subset E$ subsp vect $\Rightarrow U^\perp \subset E$ subspațiu vectorial

Demonstrație

a) Fie $u, v \in x^\perp \overset{?}{\Rightarrow} \alpha u + \beta v \in x^\perp$

$<\alpha u + \beta v, x> = \alpha \underset{0}{<u, x>} + \beta \underset{0}{(v, x)} = 0 \Rightarrow$

$\Rightarrow \alpha u + \beta v \in x^\perp$

b) Fie $u, v \in U^\perp \Rightarrow \alpha u + \beta v \in U^\perp$

$x \in U$ arbitrar

$<\alpha u + \beta v, x> = \alpha <u, x> + \beta (v, x) = 0 + 0 = 0$

Prop

$U, W \subset E$ subsp vect

$U \subseteq W \Rightarrow W^\perp \subseteq U^\perp$

Dem

Fie $x \in W^\perp \Rightarrow <x, y> = 0, \forall y \in W$

$U \subseteq W \Rightarrow <x, z> = 0, \forall z \in U \Rightarrow x \in U^\perp \Rightarrow$

$\Rightarrow W^\perp \subseteq U^\perp$

Teoremă $(E, <\cdot,\cdot>)$ sp vect euclidian real

$U \subseteq E$ subsp vect

$\Rightarrow E = U \oplus U^\perp$

($U^\perp$ = complementul ortogonal al lui U)

Dem

Fie $x \in U \cap U^\perp \Rightarrow x \in U$, $x \in U^\perp \Rightarrow <x, x> = 0 \xRightarrow[\text{dy}]{\text{poz}} x = 0$

$U, U^\perp \subseteq E$ subsp vect $\Rightarrow U \oplus U^\perp \subseteq E$ subsp vect

Dem că $E \subseteq U \oplus U^\perp$

Fie $R = \{e_1, \dots, e_k\}$ reper ortonormat în U

Fie $v \in E$

$v' = v - \sum_{i=1}^{k} <v, e_i> e_i$

Dem că $v' \in U^\perp$

$<v', e_1> = <v, e_1> - \sum_{i=1}^{k} <v, e_i> \cdot \underset{\delta_{i1}}{<e_i, e_1>} = <v, e_1> - <v, e_1> = 0$

$$\langle v', e_k\rangle = \underbrace{\langle v, e_k\rangle - \sum_{i=1}^{k}\langle v, e_i\rangle \underbrace{\langle e_i, e_k\rangle}_{\delta_{ik}}}_{\langle v, e_k\rangle} = 0$$

$\Rightarrow \langle v', e_j\rangle = 0, \forall j = \overline{1,k} \Rightarrow \langle v', x\rangle = 0, \forall x \in U^\perp$

$$v = \underbrace{\sum_{i=1}^{k}\langle v, e_i\rangle e_i}_{\substack{v'' \\ \cap \\ U}} + \underbrace{v'}_{\substack{\cap \\ U^\perp}} \in U \oplus U^\perp$$

$E = U + U^\perp$

Exemple

1) $(\mathbb{R}^3, g_0)$ sp vect e real cu str canonică, $x = (1, 2, -1)$

a) $x^\perp$

b) Reper ortonormat în $x^\perp$

SOL

a) $x^\perp = \{y \in \mathbb{R}^3 / \langle x, y\rangle = 0\} = \{y \in \mathbb{R}^3 / y_1 + 2y_2 - y_3 = 0\}$

$y = (y_1, y_2, y_3)$ $\quad y_3 = y_1 + 2y_2$

$(y_1, y_2, y_3) = (y_1, y_2, y_1 + 2y_2) = y_1\underbrace{(1,0,1)}_{f_1} + y_2\underbrace{(0,1,2)}_{f_2}$

din $x^\perp = 2$

$R = \{f_1, f_2\}$ reper în $x^\perp$

Aplicăm procedeul de ortogonalizare Gram-Schmidt! ← Examen

$e_1 = f_1 = (1,0,1)$

$e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1 = (0,1,2) - \frac{2}{2}$

$\langle f_2, e_1\rangle = \langle f_2, f_1\rangle = \langle (0,1,2), (1,0,1)\rangle = 2$

$\langle e_1, e_1\rangle = \langle (1,0,1), (1,0,1)\rangle = 2$

$\langle e_1, e_1\rangle$

$e_2 = (0,1,2) - (1,0,1) = (-1,1,1)$

$R' = \{e_1 = (1,0,1), e_2 = (-1,1,1)\}$ reper ortogonal

$R'' = \{e_1' = \frac{1}{\sqrt{2}}(1,0,1), e_2' = \frac{1}{\sqrt{3}}(-1,1,1)\}$ reper ortonormat în $x^\perp$

$\mathbb{R}^3 = \langle x\rangle \oplus x^\perp$

$$\tilde{R} = \{\underbrace{\tfrac{1}{\sqrt{6}}(1,2,-1)}_{\substack{R_1 \\ \downarrow \\ \langle x\rangle}}, \underbrace{\tfrac{1}{\sqrt{2}}(1,0,1), \tfrac{1}{\sqrt{3}}(-1,1,1)}_{\substack{R_2 \\ \downarrow \\ x^\perp}}\}$$

R_2 repere ortonormate

2) $(\mathbb{R}^3, g_0)$

$U = \{x \in \mathbb{R}^3 / x_1 + x_2 + x_3 = 0\}$

a) $U^\perp$

b) reper $R = R_1 \cup R_2$ ortonormat în $\mathbb{R}^3$ a.î. R_1 reper orton în U

R_2 — || — $U^\perp$

Sol

a) $U = \{x \in \mathbb{R}^3 / \langle x, (1,1,1)\rangle = 0\}$ dim $U = 2$

$U^\perp = \langle \{1,1,1\}\rangle$ dim $U^\perp = 1$

b) R_1 reper în U

$x_3 = -x_1 - x_2$

$(x_1, x_2, x_3) = (x_1, x_2, -x_1 - x_2) = x_1\underbrace{(1,0,-1)}_{f_1} + x_2\underbrace{(0,1,-1)}_{f_2}$

$\{f_1, f_2\}$ reper în U

Aplicăm G-S

$e_1 = f_1 = (1,0,-1)$

$e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = (0,1,-1) - \frac{1}{2}(1,0,-1) =$

$= (-\frac{1}{2}, 1, -\frac{1}{2}) = \frac{1}{2}(-1,2,-1)$

$\{e_1, e_2\}$ reper ortogonal în U

$\{\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{6}}(-1,2,-1)\}$ reper ortonormat în U

* OBS: $u = \alpha v, \alpha > 0$, $\frac{u}{\|u\|} = \frac{\alpha v}{\alpha\|v\|} = \frac{v}{\|v\|}$

$$\{\underbrace{\tfrac{1}{\sqrt{2}}(1,0,-1), \tfrac{1}{\sqrt{6}}(-1,2,-1)}_{\text{orton în } \mathbb{R}^3}, \tfrac{1}{\sqrt{3}}\underbrace{(1,1,1)}_{U^\perp}\}$$ reper în

Temă

ex1. $(\mathbb{R}^3, +, \cdot)/_\mathbb{R}$ sp vect

$g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $G = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 6 & 3 \\ 0 & 3 & 3 \end{pmatrix}$

formă biliniară

matricea asociată lui g în raport cu reperul canonic

a) $(\mathbb{R}^3, g)$ sp vectorial euclidian

b) $u = (1,0,1)$

$u^\perp = \{x \in \mathbb{R}^3 / g(x, u) = 0\} = ?$

c) Reper ortonormat în $U^\perp$

ex2. $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 2x_1x_2 - 8x_1x_3 - 8x_2x_3$

formă pătratică

a) $G = ?$ mat asociată lui Q în raport cu reperul canonic

b) Să se det forma polară $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ asociată lui Q

1) $\mathrm{Ker}\, g = ?$

2) este g nedegenerată?

c) Să se reducă Q la forma canonică

Def $(E, \langle\cdot,\cdot\rangle)$ sp v real

$f \in End(E)$

f sn ortogonală $\Leftrightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle, \forall x, y \in E$

Obs Fie $R = \{e_1, \dots, e_n\}$ reper ortonormat în E

$A =$ matricea asociată lui f în raport cu R

$f(e_i) = \sum_{j=1}^{n} a_{ji} e_j$

$\langle f(e_i), f(e_k)\rangle = \langle \sum_{j=1}^{n} a_{ji} e_j, \sum_{r=1}^{n} a_{rk} e_r\rangle =$

$= \sum_{j=1}^{n} \sum_{r=1}^{n} a_{ji} a_{rk} \underbrace{\langle e_j, e_r\rangle}_{\delta_{jr}} = \sum_{j=1}^{n} a_{ji} a_{jk}$

$\langle f(e_i), f(e_k)\rangle = \langle e_i, e_k\rangle = \delta_{ik}$

$\sum_{j=1}^{n} a_{ji} a_{jk} = \delta_{ik}, \forall i, k = \overline{1,n}$

$A^T A = I_n \Rightarrow A \in O(n) \leftarrow$ este matrice ortogonală

OBS f ortogonală $\Leftrightarrow$ schimbare de reper ortonormat

$f \in O(E) \Rightarrow A =$ matricea asoc în rap cu $R = \{e_1, \dots, e_n\}$

$A \in O(n)$

$R = \{e_1, \dots, e_n\} \xrightarrow{A} R' = \{e_1', \dots, e_n'\}$ reper orton.

$R \xrightarrow{A} R'$

$\{e_i\}_{i=\overline{1,n}}$ $\{e_i'\}_{i=\overline{1,n}}$

$f(e_i) = e_i' = \sum_{j=1}^{n} a_{ji} e_j$

$f \in O(E)$

$\langle f(e_i), f(e_k)\rangle = \langle e_i', e_k'\rangle = \delta_{ik} = \langle e_i, e_k\rangle$

$\Rightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle, \forall x, y \in E$