Seminar geometrie 8

V/K $\quad B_1 = \{f_1, \dots, f_n\}$, $B_2 = \{g_1, \dots, g_n\}$ baze în V/K

$B_1^* = \{f_1^*, \dots, f_n^*\}$, $B_2^* = \{g_1^*, \dots, g_n^*\}$ sunt baze în V^*/K

$B_1 \xrightarrow{A} B_2$

$B_1^* \xrightarrow{C} B_2^*$ $\quad C = ?$

Fie $i \in \overline{1,n}$

$g_i^* = \sum_{j=1}^{n} c_{ji} \cdot f_j^*$ $\quad \} f_k$

$k \in \overline{1, n}$

$g_i^*(f_k) = \sum_{j=1}^{n} c_{ji} \cdot \delta_{kj}$

$g_i^*(f_k) = \sum_{j=1}^{n} c_{ji} \cdot \delta_{kj}$

$g_i^*(f_k) = c_{ki}$

$B_2 \xrightarrow{D} B_1$ $\quad D = A^{-1}$

$f_k = \sum_{j=1}^{n} d_{jk} \cdot g_j \Rightarrow g_i^*(f_k) = d_{ik}$

$g_i^*(f_k) = c_{ki} \Rightarrow d_{ik} = c_{ki} \Rightarrow C = {}^tD = {}^t(A^{-1})$

$B_1^* \xrightarrow{C} B_2^*$ $\quad B_2 \xrightarrow{D} B_1$

$C = {}^tD$

$\varphi_i : \mathbb{R}^3 \to \mathbb{R}$ $\quad i \in \overline{1,3}$ $\quad \varphi_1(x) = x_1 + x_2 + x_3$

$\varphi_2(x) = x_2 + x_3$

$\varphi_3(x) = 2x_3$

a) Să se arate că $\{\varphi_1, \varphi_2, \varphi_3\}$ este bază în $(\mathbb{R}^3)^*/\mathbb{R}$

b) Să se determine $B = \{u_1, u_2, u_3\}$ bază în $\mathbb{R}^3/\mathbb{R}$

a.î. $B^* = \{\varphi_1, \varphi_2, \varphi_3\}$

$B^* = \{u_1^*, u_2^*, u_3^*\}$ $\quad u_i^* = \varphi_i$ $\quad i \in \overline{1,3}$

$u_i^*(u_j) = \delta_{ij} \Rightarrow \varphi_i(u_j) = \delta_{ij}$

$B_c^* \xrightarrow{A} B^*$ $\quad B^* \to B_i$

$B \xrightarrow{{}^tA} B_i$

Fie $a_1, a_2, a_3 \in \mathbb{R}$ a.î. $a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 = 0/\varphi_1$

$a_1 + a_2 \cdot 0 + a_3 \cdot 0 = 0 \Rightarrow a_1 = 0$

$a_1 + a_2 + a_3 \cdot 0 = 0 \Rightarrow a_2 = 0$

$a_1 + a_2 + 2a_3 = 0 \Rightarrow a_3 = 0$

~~(x_1, x_2, x_3)~~

$\begin{cases} \varphi_1 = e_1^* + e_2^* + e_3^* \\ \varphi_2 = e_2^* + e_3^* \Rightarrow e_2^* = \\ \varphi_3 = 2e_3^* \Rightarrow e_3^* = \frac{1}{2}\varphi_3 \end{cases}$

$\begin{cases} e_1^* = \varphi_1 - \varphi_2 \\ e_2^* = \varphi_2 - \frac{1}{2}\varphi_3 \\ e_3^* = \frac{1}{2}\varphi_3 \end{cases}$

$u_1 = (1, 0, 0)$

$u_2 = (-1, 1, 0)$

$u_3 = (0, -\frac{1}{2}, \frac{1}{2})$

$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

Fie $B_1 = \{f_1, f_2, f_3\}$ $\quad f_1 = (0,1,1)$, $f_2 = (1,0,1)$, $f_3 = (1,1,0)$

$B_2 = \{g_1, g_2, g_3\}$ $\quad g_1 = (1,1,1)$, $g_2 = (0,1,1)$, $g_3 = (0,0,1)$

Să se determine B_1^*, B_2^* și matricea de trecere de la B_1^* la B_2^*

I. $B_c^* \xrightarrow{A} B_1^*$

$B_1 \to B_c$

$f_1 = e_2 + e_3$

$f_2 = e_1 + e_3$

$f_3 = e_1 + e_2$

$f_1 + f_2 + f_3 = 2e_1 + 2e_2 + 2e_3$

$e_1 + e_2 + e_3 = \frac{1}{2}(f_1 + f_2 + f_3)$

$e_1 = \frac{1}{2}(f_1 + f_2 + f_3) - f_1 \Rightarrow e_1 = -\frac{1}{2}f_1 + \frac{1}{2}f_2 + \frac{1}{2}f_3$

$e_2 = \frac{1}{2}f_1 - \frac{1}{2}f_2 + \frac{1}{2}f_3$

$e_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2 - \frac{1}{2}f_3$

$A = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$f_1^* = \frac{1}{2}(-e_1^* + e_2^* + e_3^*) \Rightarrow f_1^* : \mathbb{R}^3 \to \mathbb{R}^3$, $f_1^*(x) = \frac{1}{2}(-x_1 + x_2 + x_3)$

$f_2^* = \frac{1}{2}(e_1^* - e_2^* + e_3^*) \Rightarrow f_2^* : \mathbb{R}^3 \to \mathbb{R}^3$, $f_2^*(x) = \frac{1}{2}(x_1 - x_2 + x_3)$

$f_3^* = \frac{1}{2}(e_1^* + e_2^* - e_3^*) \Rightarrow f_3^* : \mathbb{R}^3 \to \mathbb{R}^3$, $f_3^*(x) = \frac{1}{2}(x_1 + x_2 - x_3)$

II. $B_c^* \xrightarrow{C} B_2^*$

$B_2 \to B_c$

$\begin{cases} g_1 = e_1 + e_2 + e_3 \Rightarrow e_1 = g_1 - g_2 \\ g_2 = e_2 + e_3 \Rightarrow e_2 = g_2 - e_3 = g_2 - g_3 \\ g_3 = e_3 \end{cases}$

~~$g_1 + g_2 + g_3 = e_1 + 2e_2 + 3e_3$~~

$\Rightarrow e_1 =$

$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$g_1^* = e_1^*$

$g_2^* = -e_1^* + e_2^*$

$g_3^* = -e_2^* + e_3^*$

$g_1^*(x) = x_1$

$g_2^*(x) = -x_1 + x_2$

$g_3^*(x) = -x_2 + x_3$

$B_1^* \xrightarrow{D} B_c^* \xrightarrow{C} B_2^*$

$B_c \to B_1$

$f_1 = e_2 + e_3$

$f_2 = e_1 + e_3$

$f_3 = e_1 + e_2$

$D = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$M = DC = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

Def Fie V/K sp. vect
$g: V\times V \to K$

Spunem că g este formă biliniară dacă $\forall \alpha,\beta \in K$ $\forall x,y,z \in V$

1) $g(\alpha x+\beta y, z)=\alpha g(x,z)+\beta g(y,z)$
2) $g(z,\alpha x+\beta y)=\alpha g(z,x)+\beta g(z,y)$

$B=\{e_1,\dots,e_n\}$ bază în V/K $g \xrightarrow{B} A$

$\forall i,j\in\overline{1,n}$ $a_{ij}=g(e_i,e_j)$

$x=\sum_{i=1}^{n} x_i e_i$ fie K corp comutativ $x,y\in V$

$y=\sum_{j=1}^{n} y_j x_j$

$g(x,y)=g(\sum_{i=1}^{n}x_ie_i,\sum_{j=1}^{n}y_je_j)=\sum_{i,j\in\overline{1,n}} g(e_i,e_j)x_iy_j=$
$=\sum_{i,j\in\overline{1,n}} a_{ij}\cdot x_i y_j$

$g: \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ $g(x,y)=x_1y_1+x_2y_2-x_2y_3+x_3y_1$

a) Să se arate că g este formă biliniară și să se determine matricea asociată lui g în (baza canonică) B_c.

Fie $\alpha,\beta\in\mathbb{R}$. Fie $x,y,z\in\mathbb{R}^3$

$\alpha x+\beta y=(\alpha x_1+\beta y_1,\alpha x_2+\beta y_2,\alpha x_3+\beta y_3)$
$g(\alpha x+\beta y,z)=(\alpha x_1+\beta y_1)z_1+(\alpha x_2+\beta y_2)z_2-$
$-(\alpha x_2+\beta y_2)z_3+(\alpha x_3+\beta y_3)z_1=$
$=\alpha(x_1z_1+x_2z_2-x_2z_3+x_3z_1)+\beta(y_1z_1+y_2z_2-y_2z_3+y_3z_1)=$
$=\alpha\cdot g(x,z)+\beta g(y,z)$
$g(z,\alpha x+\beta y)=z_1(\alpha x_1+\beta y_1)+z_2(\alpha x_2+\beta y_2)-z_2(\alpha x_3+\beta y_3)+$
$+z_3(\alpha x_1+\beta y_1)=\alpha(z_1x_1+z_2x_2-z_2x_3+z_3x_1)+\beta(z_1y_1+$
$z_2y_2-z_2y_3+z_3y_1)$
$=\alpha g(z,x)+\beta g(z,y)$

$$A=\begin{pmatrix}1&0&0\\0&1&-1\\1&0&0\end{pmatrix}$$

$\mathrm{Ker}^s(g)=\{x\in V/\ g(x,y)=0\ \forall y\in V\}$ — sunt subsp. vectoriale în V
$\mathrm{Ker}^d(g)=\{y\in V/\ g(x,y)=0\ \forall x\in V\}$ — mats. sistemelor au aceeași dim

$$\mathrm{Ker}^s(g)=\{x\in\mathbb{R}^3/\ g(x,e_1)=0,\ g(x,e_2)=0,\ g(x,e_3)=0\}$$

$$\begin{cases}x_1+0-0+x_3=0\\x_2=0\\-x_2=0\end{cases}\Leftrightarrow\begin{cases}x_1+x_3=0\Rightarrow x_1=-x_3\\x_2=0\end{cases}\qquad\begin{pmatrix}1&0&1\\0&1&0\\0&-1&0\end{pmatrix}$$

$<\{1,0,-1\}>$

$\mathrm{Ker}^s(g)=\{(x_1,0,-x_1)/x_1\in\mathbb{R}\}$

$$\mathrm{Ker}^d(g)=\{y\in\mathbb{R}^3/\ g(e_1,y)=0,\ g(e_2,y)=0,\ g(e_3,y)=0\}$$

$$\begin{cases}y_1=0\\y_2-y_3=0\\y_1=0\end{cases}\Rightarrow\ y_1=0,\ y_2=y_3$$

$$\begin{pmatrix}1&0&0\\0&1&-1\\1&0&0\end{pmatrix}$$

Obs: matr. Ker^s și Ker^d sunt transpuse una alteia $<\{0,1,1\}>$

$\mathrm{Ker}^d(g)=\{(0,y_2,y_2)\mid y_2\in\mathbb{R}\}$

$g(x,y)=\sum_{i,j} a_{ij}x_iy_j$

$g(x,e_1)=0\Leftrightarrow\sum a_{i1}\cdot x_i=0$

Def Fie $g: V\times V\to K$ corp comutativ biliniară. Spunem că g este nedegenerată dacă $\mathrm{Ker}^s g=\{0_V\}$
$g(x,y)=0\ \forall y\in V\Rightarrow x=0$

Prop $\{e_1,\dots,e_n\}$ bază în V/K $g\xrightarrow{B} A$
g este nedegenerată $\Leftrightarrow \det A\neq 0$

$B_1=\{e_1,\dots,e_n\}$, $B_2=\{f_1,\dots,f_n\}$ baze în V/K $g\xrightarrow{B_1}A$, $g\xrightarrow{B_2}C$ $B_1\xrightarrow{D}B_2$

Fie $i,j\in\overline{1,n}$
$c_{ij}=g(f_i,f_j)=g(\sum_{k=1}^{n}d_{ki}\cdot e_k,\sum_{s=1}^{n}d_{sj}\cdot e_s)=$
$=\sum_{k,s\in\overline{1,n}}d_{ki}d_{sj}\cdot a_{ks}=\sum {}^tD_{ik}\cdot a_{ks}\cdot d_{sj}=({}^tD\,A\,D)$

$C={}^tDAD\Rightarrow \mathrm{rang}(C)=\mathrm{rang}(A)$
$\mathrm{rang}(g)=n$

Propoziție Fie V/K sp. vect, K corp comutativ
$g: V\times V\to K$ biliniară și simetrică, $\dim_K V=n$, $n\in\mathbb{N}^*$
Atunci există o bază $B=\{e_1,\dots,e_n\}$ în V/K în care matr. asociată lui g să aibă forma diagonală

1. $g\neq 0\Rightarrow\exists x\in V/\ g(x,x)\neq 0$ $e_1=x$
$x^\perp=\{y\in V/\ g(x,y)=0\}$
$x\notin x^\perp\Rightarrow \dim x^\perp<n$
$g|_{x^\perp\times x^\perp}=0$
$V=<\{x\}>\oplus x^\perp$, $y\in x^\perp\cap<\{x\}>\Rightarrow$
$\Rightarrow\exists\lambda\in K$ $y=\lambda x$, $g(x,y)=0$ $\Big|\Rightarrow g(x,\lambda x)=0\Rightarrow\lambda g(x,x)=0$

Fie $u\in V$ $u=v+w$ $v\in<\{x\}>\Rightarrow\exists\lambda\in K$ $v=\lambda x$
$w=x^\perp\Rightarrow g(x,w)=0$ $\Rightarrow\lambda=0\Rightarrow y=0$

$u=\lambda x+w\Rightarrow g(u,x)=g(\lambda x+w,x)=\lambda g(x,x)+$
$+g(w,x)=\lambda g(x,x)$ $\lambda=\frac{g(u,x)}{g(x,x)}$ $v=\frac{g(u,x)}{g(x,x)}\cdot x\Rightarrow$
$\Rightarrow w=u-\frac{g(u,x)}{g(x,x)}x$

$g(w,x) = g(u,x) - \frac{g(u,x)}{g(x,x)} \cdot g(x,x)$

$V = \langle \{x\} \rangle \oplus x^{\perp} \Rightarrow \dim_K x^{\perp} = n-1$

1) $g|_{x^{\perp} \times x^{\perp}} = 0$

2) $g|_{x^{\perp} \times x^{\perp}} \neq 0$

$y \in x^{\perp} \quad g(y,y) \neq 0$

$y^{\perp} = \{z \in x^{\perp} / g(z,y) = 0\}$

$e_2 = y$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$g(x,y) = x_1 y_1 + x_2 y_2 + (-x_1 y_2) - x_2 y_1$

a) Să se arate că g este formă biliniară simetrică degenerată (det 0)

b) Să se determine o bază în $\mathbb{R}^3/_{\mathbb{R}}$ ortogonală în raport cu g

b) fie $x \in \mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$ $\quad x = e_1 = (1,0,0)$

$g(x,x) = 1$

$x^{\perp} = \{y \in \mathbb{R}^3 / g(x,y) = 0\}$

$g(x,y) = 0 \Leftrightarrow g(e_1,y) = 0 \Leftrightarrow y_1 - y_2 = 0 \Leftrightarrow y_1 = y_2$

$x^{\perp} = \{(y_1, y_1, y_3) / y_1, y_3 \in \mathbb{R}\} \Leftrightarrow \langle \{(1,1,0), (0,0,1)\} \rangle$

$g((1,1,0)(1,1,0)) = 1+1-1-1 = 0$

$g((1,1,1)(1,1,1)) = 0$

$\left.\begin{array}{l} g(u,u) = 0 \\ g(e_3,e_3) = 0 \\ g(u,e_3) = 0 \end{array}\right| \Rightarrow g/_{x^{\perp} \times x^{\perp}} = 0$

$B = \{e_1, u, e_3\}$ $\quad A = \begin{pmatrix} g(e_1,e_1) & g(e_1,u) & g(e_1,e_3) \\ g(u,e_1) & g(u,u) & g(u,e_2) \\ g(e_3,e_1) & g(e_3,u) & g(e_3,e_3) \end{pmatrix}$

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