Examen Probabilități

Exercițiul 1

$f(x) = ax + b$ pt. $0 \le x \le 3$, 0 în rest

$\mathbb{P}(X \le 2) = 0,64 \Rightarrow \mathbb{E}(X) = ?$

$$f(x) = \begin{cases} ax+b, & 0 \le x \le 3 \\ 0 & ; \text{ rest} \end{cases}$$

Soluție:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 f(x)\,dx = \int_0^3 (ax+b)\,dx =$$

$$= \int_0^3 ax\,dx + \int_0^3 b\,dx = a\frac{x^2}{2}\Big|_0^3 + bx\Big|_0^3 =$$

$$= a \cdot \frac{9}{2} + 3b = 1$$

$$\int_0^2 ax + b\,dx = 0,64 = \int_0^2 ax\,dx + \int_0^2 b\,dx =$$

$$= a\frac{x^2}{2}\Big|_0^2 + bx\Big|_0^2 = a \cdot 2 + 2b$$

$$\begin{cases} \frac{9a}{2} + 3b = 1 \\ 2a + 2b = 0,64 \end{cases} \Rightarrow \begin{cases} a = \frac{2}{75} \\ b = \frac{22}{75} \end{cases}$$

$$f(x) = \frac{2}{75}\cancel{x}x + \frac{22}{75}$$

$$\cancel{\mathbb{E}[f(x)] = \int_{-\infty}^{\infty} x \cdot \underbrace{f(x)}_{\text{densitatea}} = \underbrace{\mathbb{E}[X]}_{\text{variabila aleatoare}}}$$

$$X \sim pdx$$

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x) \cdot pd(x)\, dx$$

$$\mathbb{E}[g(X)] \underset{g(x)=x}{=} \int_{\mathbb{R}} x \cdot f(x)\, dx =$$

$$= \int_{\mathbb{R}} x \cdot (ax + b)\, dx$$

$$= \int_0^3 x\left(\frac{2}{75}x + \frac{22}{75}\right) dx =$$

$$= \int_0^3 \left(\frac{2}{75}x^2 + \frac{22}{75}x\right) dx = \frac{2}{75}\,\frac{x^3}{3}\Big|_0^3 + \frac{22}{75}\,\frac{x^2}{2}\Big|_0^3 =$$

$$= \frac{2}{\cancel{75}_{25}} \cdot \cancel{9}^{3} + \frac{\cancel{22}^{11}}{\cancel{75}_{25}} \cdot \frac{\cancel{9}^{3}}{\cancel{2}_{1}} = \frac{6}{25} + \frac{33}{25} = \frac{39}{25}$$

lō

2) $Var(x) = 0{,}61$

$Var(y) = 2{,}5$

$Cov(x; y) = -0{,}37$

$Var(x + y) = ?$

$$\begin{array}{r} 3{,}11 - \\ 0{,}74 \\ \hline 2{,}37 \end{array} \qquad \begin{array}{r} 2{,}50 + \\ 0{,}61 \\ \hline 3{,}11 \end{array}$$

Soluție:

Th. R.J. generalizată:

$Var(x + y) = Var\, x + Var\, y + 2\, Cov(x, y)$

$Var(x + y) = 0{,}61 + 2{,}5 + 2 \cdot (-0{,}37) =$

$= 3{,}11 - 0{,}74 = 2{,}37$

3) Prob. de a aștepta mai mult de 1,95 ms este 0,3933661765130499

Timpul mediu de așteptare = ? → distribuție exponențială

Soluție:

• ce distribuție folosim ?

$E[t] = ?$

$E[t] = \frac{1}{\lambda}$ → formula mediei pt. exponențială

$P(t > k) = e^{-\lambda k}$

$e^{-\lambda \cdot 1{,}95} = 0{,}3933661765130499$

$-\lambda \cdot 1{,}95 = \ln 0{,}3933661765130499$

$\lambda = \frac{-\ln 0{,}3933661765130499}{1{,}95}$

3

$\lambda = 2,09$

$\mathbb{E}[T] = \cancel{\frac{1}{2,09} = 0,478}$ $2,09$.

4) $X_1, \dots, X_n, \dots \sim \begin{pmatrix} -4 & 2 & 8 \\ 0,22 & 0,35 & 0,43 \end{pmatrix}$

distrib. discretă

$\lim_{n \to \infty} \frac{X_1 + \dots + X_n}{n} = ?$

Soluție

$$\frac{X_1 + \dots + X_n}{n} = \overline{S_n} \xrightarrow{n \to \infty} \mathbb{E}[X].$$

$-4 \cdot 0,22 + 2 \cdot 0,35 + 8 \cdot 0,43 = -0,88 + 0,7 + 8 \cdot 0,43$

$= 3,26$.

5) Dacă $X \sim N(9;7)$

$a = -2$

$b = 4$

media, variația

$f(x) = ax + b$ are distribuție:

Soluție:

$\mathbb{E}[X] = 9$

$Var(X) = 7$

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x) \cdot p(x)\, dx$$

$Var(-2X+4) = Var(-2X) = 4\,Var(X) = 28$

~~E[αX~~

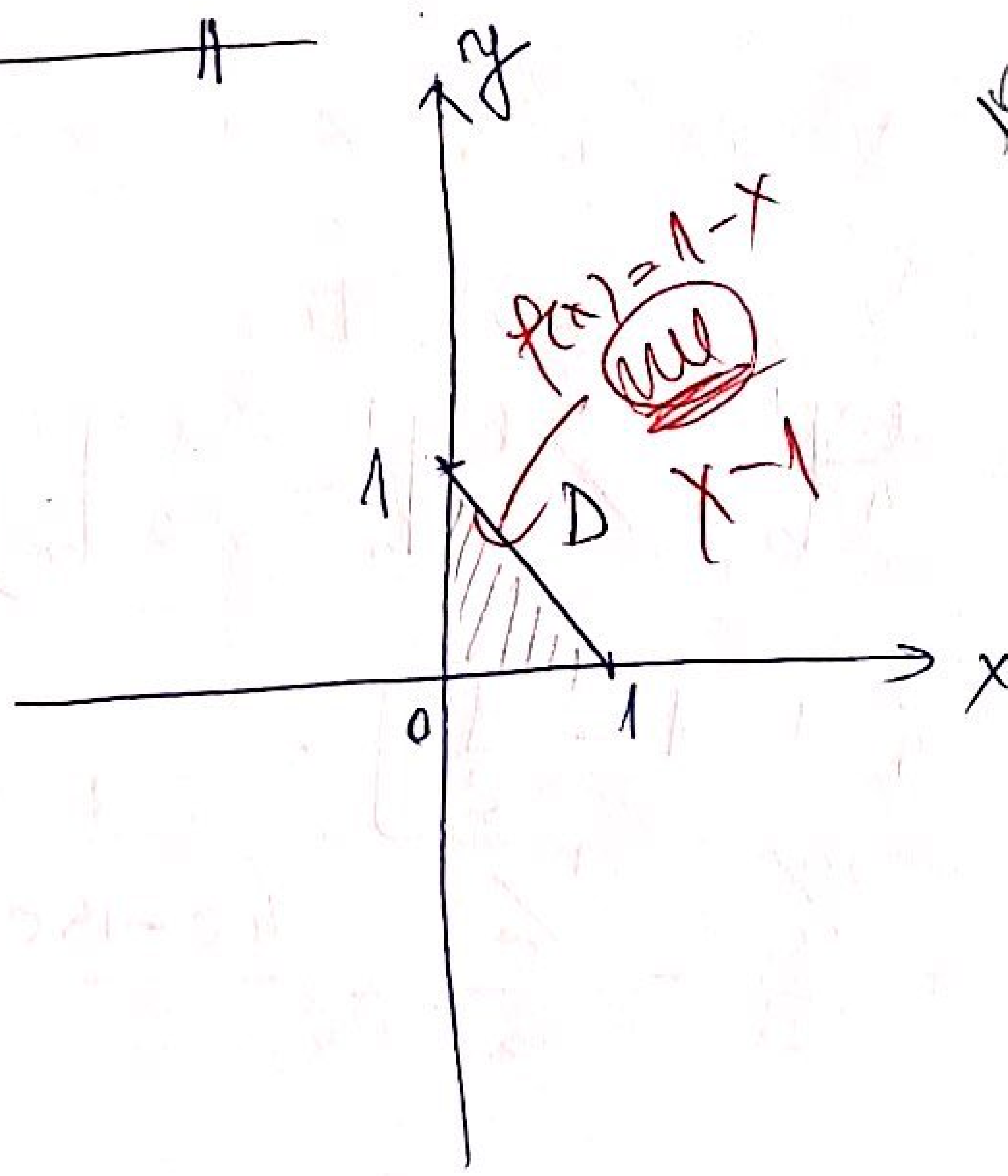
$\mathbb{E}[-2X+4] = -2\,\mathbb{E}[X] + 4 = -14$

$N(-14; 28)$.

9) Fie (X, Y) - vector aleator cu densitatea
$$f(x;y) = \begin{cases} c\,x^1 y^2 ; & 0 \le x, y \;;\; x+y \le 1 \\ 0 ; & \text{în rest} \end{cases}$$

i) $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $Var(X)$, $Var(Y) = ?$

ii) $Cor(X;Y)$. Sunt X și Y - independente?



100!!

$c(xy^2)(x+y) =$
$= c(x^2y^2 + xy^3)$

$$\iint_D f(x;y)\,dx\,dy = 1$$

$$\iint_D f(x;y)\,dx\,dy = \cancel{\int_0^1\int_0^1 c x y^2\,dx\,dy =}$$

$$= \int_0^1\int_0^{x-1} cxy^2\,dy\,dx = \int_0^1 cx\left(\int_0^{x-1} y^2\,dy\right)dx =$$

$$= \int_0^1 cx\,\frac{y^3}{3}\Big|_0^{x-1}\,dx = \int_0^1 cx\,\frac{(x-1)^3}{3}\,dx =$$

$$= c\int_0^1 x\,\frac{(x-1)^3}{3}\,dx = \frac{c}{3}\int_0^1 x(x-1)^3\,dx =$$

$$= \frac{c}{3}\int_0^1 (x^4 - 3x^3 + 3x^2 - x)\,dx =$$

$$(x-1)(x^2-2x+1) = x^3 - 2x^2 + x - x^2 + 2x - 1 =$$

$$= x^3 - 3x^2 + 3x - 1$$

$$= \frac{c}{3}\left[\frac{x^5}{5}\Big|_0^1 - 3\,\frac{x^4}{4}\Big|_0^1 + \cancel{3}\,\frac{x^3}{\cancel{3}}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1\right] =$$

$$= \frac{c}{3}\left[\frac{1}{5} - 3\cdot\frac{1}{4} + 1 - \frac{1}{2}\right] =$$

$$= {}^{4/}\frac{c}{15} - {}^{15/}\frac{c}{4} + {}^{20/}\frac{c}{3} - {}^{10/}\frac{c}{6} = \frac{4c - 15c + 20c - 10c}{60} =$$

$$= \frac{-c}{60} \Rightarrow \quad c = +60$$

$$\mathbb{E}[X] \overset{g(x;y)=x}{=} \mathbb{E}[g(x;y)] = \int_D g(x;y)\cdot f(x;y)\,dx\,dy =$$

$$\iint_D 60x^2y^2\,dx\,dy = -60\iint_D x^2y^2\,dx\,dy =$$

$$= -60\int_0^1\int_0^{x-1} x^2y^2\,dy\,dx = -60\int_0^1 x^2\left(\int_0^{x-1} y^2\,dy\right)dx =$$

$$= -60\int_0^1 x^2\cdot \frac{y^3}{3}\Big|_0^{x-1}\,dx = -\frac{60}{3}\int_0^1 x^2(x-1)^3\,dx =$$

$$= -\frac{60}{3}\int_0^1 (x^5 - 3x^4 + 3x^3 - x^2)\,dx =$$

$$= -20\left[\frac{x^6}{6}\Big|_0^1 - 3\frac{x^5}{5}\Big|_0^1 + 3\frac{x^4}{4}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1\right] =$$

$$= -20\left[\frac{1}{6} - 3\cdot\frac{1}{5} + 3\cdot\frac{1}{4} - \cancel{\#}\cdot\frac{1}{\cancel{3}}\right] =$$

$$= \frac{-10}{3} + \overset{3/}{12} - \overset{3/}{15} + \frac{20}{3} = \frac{-10+36-45+\overset{20}{\cancel{60}}}{3} =$$

$$\cancel{= \frac{-10+36+15}{3} = \frac{5+36}{3} = \frac{41}{3}}$$

$$= \frac{-10+20-45+36}{3} = \frac{-35+36}{3} = \frac{1}{3}$$

corect.

Scanned with CamScanner

$$\mathbb{E}[y] \overset{g(x;y)=y}{=} \mathbb{E}[g(x;y)] = \iint_D -60x\cdot y^3\,dx\,dy =$$

$$= -60\int_0^1 x\int_0^{x-1} y^3\,dy\,dx = -60\int_0^1 x\left(\frac{y^4}{4}\Big|_0^{x-1} dx\right) =$$

$$= -60\int_0^1 x\cdot\frac{(x-1)^4}{4}\,dx = -\frac{60}{4}\int_0^1 x\,\frac{(x-1)^4}{1}\,dx =$$

$$= -15\int_0^1 (x^5 - 4x^4 + 6x^3 - 4x^2 + x)\,dx =$$

$$(x-1)^2(x-1)^2 = (x^2-2x+1)(x^2-2x+1) =$$

$$= x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x + x^2 - 2x + 1 =$$

$$= x^4 - 4x^3 + 6x^2 - 4x + 1$$

$$= -15\left[\frac{x^6}{6}\Big|_0^1 - 4\frac{x^5}{5}\Big|_0^1 + 6\frac{x^4}{4}\Big|_0^1 - 4\frac{x^3}{3}\Big|_0^1 + \frac{x^2}{2}\Big|_0^1\right] =$$

$$= -15\left[\frac{1}{6} - \frac{4}{5} + \frac{6}{4} - \frac{4}{3} + \frac{1}{2}\right] =$$

$$= \frac{-15}{6} + 12 - \frac{15\cdot \cancel{6}^{3}}{\cancel{4}_{2}} + \frac{\cancel{15}^{5}\cdot 4}{\cancel{3}_{1}} - \frac{15}{2} = \frac{-5}{2} + 12 - \frac{45}{2} + 20 - \frac{15}{2} =$$

$$= \frac{-5-45-15}{2} + 32 = \frac{-65}{2} + \overset{2/}{32} = \frac{-65+64}{2} = -\frac{1}{2}$$

$\frac{1}{2} \to$ corect.

Scanned with CamScanner

$$\mathbb{E}[X^2] \overset{f(x;y)=x^2}{=} \mathbb{E}[f(x;y)] = \iint -60x^3y\,dx\,dy =$$

$$= \iint_D -60x^3y\,dx\,dy = -60\int_0^1\int_0^{x-1} x^3y\,dx\,dy =$$

$$= -60\int_0^1\int_0^{x-1} x^3y\,dy\,dx = -60\int_0^1 x^3\left(\int_0^{x-1} y\,dy\right)dx =$$

$$= -60\int_0^1 x^3 \cdot \frac{y^2}{2}\Big|_0^{x-1} dx = -\frac{60}{2}\int_0^1 x^3(x-1)^2\,dx =$$

$$= -30\int_0^1 x^3(x^2-2x+1)\,dx = -30\int_0^1 (x^5-2x^4+x^3)\,dx$$

$$= -30\left[\frac{x^6}{6}\Big|_0^1 - 2\,\frac{x^5}{5}\Big|_0^1 + \frac{x^4}{4}\Big|_0^1\right] =$$

$$= -30\left[\frac{1}{6} - \frac{2}{5} + \frac{1}{4}\right] = -5 + 6 - \frac{30}{4} =$$

$$= \frac{2}{1} - \frac{15}{2} = \frac{2-15}{2} = \frac{-13}{2} = \frac{1}{7}.$$

corect.

Scanned with CamScanner

$E[y^2] \xrightarrow{f(x,y)=y^2} \iint_D c x y^4 \, dx \, dy =$

$= -60 \int_0^1 \int_0^{x-1} x y^4 \, dy \, dx = -60 \int_0^1 x \left(\int_0^{x-1} y^4 dy \right) dx =$

$= -60 \int_0^1 x \cdot \frac{y^5}{5} \Big|_0^{x-1} dx = -12 \int_0^1 x (x-1)^5 dx =$

$= -12 \int_0^1 (x^6 - 5x^5 + 10x^4 - 10x^3 + 5x^2 - x) \, dx =$

$(x-1)(x^4 - 4x^3 + 6x^2 - 4x + 1) =$

$= x^5 - 4x^4 + 6x^3 - 4x^2 + x - x^4 + 4x^3 - 6x^2 + 4x - 1 =$

$= x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

$= -12 \left[\frac{x^7}{7} \Big|_0^1 - 5 \cdot \frac{x^6}{6} \Big|_0^1 + 10 \frac{x^5}{5} \Big|_0^1 - 10 \frac{x^4}{4} \Big|_0^1 + 5 \frac{x^3}{3} \Big|_0^1 - \frac{x^2}{2} \Big|_0^1 \right] =$

$= -12 \left[\frac{1}{7} - \frac{5}{6} + \frac{10}{5} - \frac{10}{4} + \frac{5}{3} - \frac{1}{2} \right] =$

$= \frac{-12}{7} + \frac{\cancel{12}^{2} \cdot 5}{\cancel{6}_1} - \frac{12 \cdot \cancel{10}^{2}}{\cancel{5}_1} + \frac{10 \cdot \cancel{12}^{3}}{\cancel{4}_1} - \frac{5 \cdot \cancel{12}^{4}}{\cancel{3}_1} + 6 =$

$= -\frac{12}{7} + 10 - 24 + 30 - 20 + 6 = -\frac{12}{7} + 20 + 6 - 24 =$

$= -\frac{12}{7} + 26 - 24 = -\frac{12}{7} + \frac{7}{2} = -\frac{12}{7} + \frac{14}{7} = + \frac{2}{7}$

Corect.

$$\mathrm{Var}(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2 =$$

$$= {}^{9/}\frac{1}{7} - {}^{7/}\frac{1}{9} = \frac{2}{63}$$

$$\mathrm{Var}(y) = \mathbb{E}[y^2] - \mathbb{E}[y]^2 =$$

$$= {}^{4/}\frac{2}{7} - {}^{7/}\frac{1}{4} = \frac{8-7}{28} = \frac{1}{28}$$

iii) $\mathrm{Cor}(x;y) = ?$

$$\mathrm{Cor}(x;y) = \frac{\mathrm{Cov}(x;y)}{D(x)\cdot D(y)}.$$

$$\mathrm{Var}(x+y) = \mathrm{Var}(x) + \mathrm{Var}(y) + 2\,\mathrm{Cov}(x;y)$$

$$\cancel{\mathbb{E}[x+y] \overset{f(x;y)=(x+y)}{=} \iint_D}$$

$$\mathrm{Cov}(x;y) = \mathbb{E}[xy] - \mathbb{E}[x]\cdot\mathbb{E}[y].$$

$f(x,y)=xy$

$$E[xy] = 60 \iint_D x^2 y^3 \, dy \, dx =$$

$$= +60 \int_0^1 \int_0^{-x+1} x^2 y^3 \, dy \, dx = +60 \int_0^1 x^2 \left(\int_0^{1-x} y^3 dy \right) dx =$$

$$= 60 \int_0^1 x^2 \left(\frac{y^4}{4} \Big|_0^{1-x} \right) dx = \frac{60}{4} \int_0^1 x^2 (1-x)^4 \, dx =$$

$$= 15 \int_0^1 x^2 (x^4 - 4x^3 + 6x^2 - 4x + 1) \, dx =$$

$$= 15 \int_0^1 (x^6 - 4x^5 + 6x^4 - 4x^3 + x^2) \, dx =$$

$$= 15 \left[\frac{x^7}{7} \Big|_0^1 - 4 \frac{x^6}{6} \Big|_0^1 + 6 \frac{x^5}{5} \Big|_0^1 - 4 \frac{x^4}{4} \Big|_0^1 + \frac{x^3}{3} \Big|_0^1 \right] =$$

$$= 15 \left[\frac{1}{7} - \frac{\cancel{4}^{2} \cdot 1}{\cancel{6}_{3}} + \frac{6 \cdot 1}{5} - \frac{4 \cdot}{4} + \frac{1}{3} \right] =$$

$$\begin{array}{r} 160 - \\ 18 \\ \hline 142 \end{array}$$

$$= 15 \left[\frac{1}{7} - \frac{2}{3} + \frac{6}{5} - 1 + \frac{1}{3} \right] =$$

$$= \frac{15}{7} - \frac{2 \cdot \cancel{15}^{5}}{\cancel{3}_1} + 18 - 15 + 5 = \frac{15}{7} - 10 - 10 + 18 =$$

$$= \frac{15}{7} - 20 + 18 =$$

$$7/2$$

$$= \frac{15}{7} - 2 = \frac{15-14}{7} = \frac{1}{7}.$$

$$Cov(x;y) = \mathbb{E}[xy] - \mathbb{E}[x] \cdot \mathbb{E}[y] =$$

$$= \frac{1}{7} - \frac{1}{3} \cdot \frac{1}{2} = {}^{6/}\frac{1}{7} - {}^{7/}\frac{1}{6} = \frac{6-7}{42} =$$

$$Cor(x;y) = \frac{Cov(x;y)}{D(x) \cdot D(y)}$$

$$D(x) = \sqrt{Var(x)} \qquad D(y) = \sqrt{Var(y)}$$

$$Cor(x;y) = \frac{-\frac{1}{42}}{\sqrt{\frac{2}{63}} \cdot \sqrt{\frac{1}{28}}} = \quad =$$

$$= -\frac{1}{42\sqrt{\frac{2}{63}} \cdot \frac{1}{\sqrt{28}}} = -\frac{\sqrt{2}}{2}.$$

x și y nu sunt independente pt. că $Cor(x;y) \neq 0$.

7) O mașină de curierat poate transporta maxim 760 kg.
Un colet, în medie 8 kg, cu deviație standard de 5 kg. Care e prob. să se poată transp. 117 colete?

$\sigma = 5$ kg.

$X_1, \dots, X_{117}$

$$\mathbb{P}(\underbrace{X_1 + X_2 + \dots + X_{117}}_{S_n} \ ? \ 760)$$

Știm:
$$\frac{\overline{X_n} - m}{\frac{\sigma}{\sqrt{n}}} \sim N(0;1)$$

$$\downarrow$$

$$\frac{\frac{S_n}{n} - \mathbb{E}[X]}{\frac{\sigma}{\sqrt{n}}} \sim N(0;1)$$

8) $D = B(0;4)$

R - v.a. distr. unif. pe $(0;4)$

θ - " " " $[0;2\pi]$

Constr. v.a. 2 dim. $(x;y) = (R\cos\theta; R\sin\theta)$

Este $(x;y)$ distr. unif. pe D?

~~$(x;y)$ → unif. pe D dc. densitatea ei e 1 pe D și 0 în rest.~~

$R = 4$.

$V = 16\pi \cdot k$.

$k = ?$ a.î. $V = 1$. $\Rightarrow k = \frac{1}{16\pi}$