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gr. 113

Subiectul 1.

a) i) $A=\begin{bmatrix} -4 & 2 & 4 \\ 0 & -3 & 3 \\ 2 & -2 & 4 \end{bmatrix} \in \mathcal{M}_3(\mathbb{R}) \Rightarrow A$ e pătratică (1)

$\det A = \begin{vmatrix} -4 & 2 & 4 \\ 0 & -3 & 3 \\ 2 & -2 & 4 \end{vmatrix} = 60 \neq 0 \Rightarrow A$ e inversabilă (2)

$$\left.\begin{array}{l} a_{11} = -4 \neq 0 \\ a_{22} = \begin{vmatrix} -4 & 2 \\ 0 & -3 \end{vmatrix} = 12 \neq 0 \\ a_{33} = \det A \neq 0 \end{array}\right\} \Rightarrow a_{kk} \neq 0,\ k=\overline{1,3} \quad (3)$$

(1), (2), (3) $\Rightarrow$ A admite fact LU fără pivotare

ii) $A \in \mathcal{M}_3(\mathbb{R}) \Rightarrow A$ pătratică

$$\left.\begin{array}{l} \det A = \begin{vmatrix} -4 & 2 & 4 \\ 0 & -3 & 3 \\ 2 & -2 & 4 \end{vmatrix} = 60 \neq 0 \Rightarrow A \text{ inversabilă} \end{array}\right\} \Rightarrow$$

$\Rightarrow$ A admite factorizarea LU cu pivotare

iii) $A \in \mathcal{M}_3(\mathbb{R}) \Rightarrow A$ pătratică

$$\left.\begin{array}{l} \det A = 60 \neq 0 \Rightarrow A \text{ inversabilă} \\ \left.\begin{array}{l} a_{11} = -4 \neq 0 \\ a_{22} = \begin{vmatrix} -4 & 2 \\ 0 & -3 \end{vmatrix} = 12 \neq 0 \\ a_{33} = \det A \neq 0 \end{array}\right\} \Rightarrow a_{kk} \neq 0,\ k=\overline{1,3} \\ b \in \mathbb{R}^3 \Rightarrow A \text{ și } b \text{ sunt compatibili} \end{array}\right\} \Rightarrow$$

$\Rightarrow$ A admite metoda Gauss fara pivotare

iv) $A \in \mathcal{M}_3(\mathbb{R}) \Rightarrow A$ pătratică

$\left.\begin{array}{l} \det A = 60 \neq 0 \Rightarrow A \text{ inversabilă} \\ b \in \mathbb{R}^3 \Rightarrow A \text{ și } b \text{ sunt compatibile} \end{array}\right\} \Rightarrow$ A admite MEG cu pivotare

v) $A^T = \begin{bmatrix} -4 & 0 & 2 \\ 2 & -3 & -2 \\ -4 & 3 & 4 \end{bmatrix}$

$A = \begin{bmatrix} -4 & 2 & -4 \\ 0 & -3 & 3 \\ 2 & -2 & 4 \end{bmatrix}$

$\Rightarrow A^T \neq A \Rightarrow$ A nu admite factorizare Cholesky

vi) $|a_{11}| = 4$

$\left.\begin{array}{l} |a_{11}| = 4 \\ |a_{12}| + |a_{13}| = 2+4 \end{array}\right\} \Rightarrow |a_{11}| < |a_{12}| + |a_{13}| \Rightarrow$

A nu e (strict) diagonal dominantă

b) $A = \left[\begin{array}{c|cc} -4 & 2 & -4 \\ \hline 0 & -3 & 3 \\ 2 & -2 & 4 \end{array}\right] = \underbrace{\left[\begin{array}{c|c} l_{11} & \underline{0} \\ \hline \underline{L}_{21} & L_{22} \end{array}\right]}_{L} \underbrace{\left[\begin{array}{c|c} u_{11} & \underline{U}_{12} \\ \hline \underline{0} & U_{22} \end{array}\right]}_{U} =$

$= \left[\begin{array}{c|c} l_{11} u_{11} & l_{11} \underline{U}_{12} \\ \hline \underline{L}_{21} u_{11} & \underline{L}_{21} \underline{U}_{12} + L_{22} U_{22} \end{array}\right] \Rightarrow$

$\left.\begin{array}{l} \Rightarrow l_{11} u_{11} = -4 \\ \text{Vrem ca } l_{11} = 1 \end{array}\right\} \Rightarrow \boxed{l_{11} = 1}\ \boxed{u_{11} = -4}$

$\left.\begin{array}{l} l_{11} \cdot \underline{U}_{12} = [2 \ \ -4] \\ l_{11} = 1 \end{array}\right\} \Rightarrow \underline{U}_{12} = [2 \ \ -4] \Rightarrow \boxed{u_{12} = 2}\ \boxed{u_{13} = -4}$

$$\underline{L}_{21} \cdot u_{11} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \underline{L}_{21} = \frac{1}{u_{11}} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \underline{L}_{21} = -\frac{1}{4} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \Rightarrow$$

$$\Rightarrow \boxed{l_{21} = 0}, \boxed{l_{31} = -\frac{1}{2}}$$

$$L_{22} U_{22} = \begin{bmatrix} -3 & 3 \\ -2 & 4 \end{bmatrix} - \underline{L}_{21} \underline{U}_{12} = \begin{bmatrix} -3 & 3 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -4 \end{bmatrix} =$$

$$= \begin{bmatrix} -3 & 3 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -1 & 2 \end{bmatrix} = S$$

$$S = \left[\begin{array}{c|c} -3 & 3 \\ \hline -1 & 2 \end{array}\right] = \left[\begin{array}{c|c} l_{22} & 0 \\ \hline l_{32} & l_{33} \end{array}\right] \left[\begin{array}{c|c} u_{22} & u_{23} \\ \hline 0 & u_{33} \end{array}\right] = \left[\begin{array}{c|c} l_{22} u_{22} & l_{22} u_{23} \\ \hline l_{32} u_{22} & l_{32} u_{23} + l_{33} u_{33} \end{array}\right] \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} l_{22} \cdot u_{22} = -3 \\ \text{Vrem ca } l_{22} = 1 \end{array}\right\} \Rightarrow \boxed{l_{22} = 1}, \boxed{u_{22} = -3}$$

$$l_{22} \cdot u_{23} = 3 \Rightarrow \boxed{u_{23} = 3}$$

$$l_{32} \cdot u_{22} = -1 \Rightarrow \boxed{l_{32} = \frac{1}{3}}$$

$$l_{32} u_{23} + l_{33} \cdot u_{33} = 2 \Rightarrow l_{33} u_{33} = 2 - \frac{1}{\cancel{3}} \cdot \cancel{3} = 1$$

$$\Rightarrow \boxed{l_{33} = 1}, \boxed{u_{33} = 1} \Rightarrow$$

$\Rightarrow$ Factorizarea LU a lui A este:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -4 & 2 & -4 \\ 0 & -3 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Am obținut sistemul: $L U \underline{x} = \underline{b}$

Rezolvăm sistemul: $L(U\underline{x}) = \underline{b}$ folosind metoda subst. asc.

Fie $y \in \mathbb{R}^3 = U\underline{x}$. $\Rightarrow$

$$\Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 3 \\ -\frac{1}{2} y_1 + \frac{1}{3} y_2 + y_3 = 2 \end{cases} \Rightarrow \boxed{y_1 = 0}, \quad \boxed{y_2 = 3}$$

$y_3 = 2 + \frac{1}{2} y_1 - \frac{1}{3} y_2 = 2 + 0 - 1 = 1 \Rightarrow$

$\Rightarrow \boxed{y_3 = 1}$

Acum rezolvam sistemul $U\underline{x} = y$, folosind metoda subst. descendente.

$$\begin{cases} -4x_1 + 2x_2 - 4x_3 = 0 \\ -3x_2 + 3x_3 = 3 \\ x_3 = 1 \end{cases} \Rightarrow \boxed{x_3 = 1}$$

$x_2 = (3 - 3x_3)/3 = 0 \Rightarrow \boxed{x_2 = 0}$

$x_1 = \dfrac{(4x_3 - 2x_2)}{-4} = -\dfrac{4}{4} = -1 \Rightarrow$

$\Rightarrow \boxed{x_1 = -1} \Rightarrow$

$\Rightarrow$ Soluția sistemului $A\underline{x} = \underline{b}$ este $\underline{x} = (-1 \ \ 0 \ \ 1)$

Subiectul 2

$P_1(-2, 1)$
$P_2(-1, 1)$
$P_3(1, -3)$
$P_4(2, 3)$.

Sistemul augumenta asociat este:

$$\underbrace{\begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -3 \\ 3 \end{bmatrix}}_{\underline{b}}$$

$$A\underline{x} = \underline{b} \ \Big| \cdot A^T \iff A^T A \underline{x} = A^T \underline{b} := \underline{\tilde{b}}$$

$$A^T A = \begin{bmatrix} 4 & 1 & 1 & 4 \\ -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 4 \end{bmatrix}$$

$$\underline{\tilde{b}} = \begin{bmatrix} 4 & 1 & 1 & 4 \\ -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 2 \end{bmatrix}$$

Vom rezolva sistemul obținut folosind MEG cu pivotare parțială scalată

Fie $A \overset{not}{=} A^T A$

PASUL $k=1$

$$\overline{A} = \overline{A}^{(1)} = \left[\begin{array}{ccc|c} 34 & 0 & 10 & 14 \\ 0 & 10 & 0 & 0 \\ 10 & 0 & 4 & 2 \end{array}\right]$$

$i=\overline{1,3}$; $S_i = \max\limits_{j=\overline{1,3}} |a_{ij}^{(1)}|$

$$S_1 = \max_{j=\overline{1,3}} |a_{1j}^{(1)}| = 34$$

$$S_2 = \max_{j=\overline{1,3}} |a_{2j}^{(1)}| = 10$$

$$S_3 = \max_{j=\overline{1,3}} |a_{3j}^{(1)}| = 10$$

$i=\overline{1,3}$, $\tilde{a}_{i1}^{(1)} = a_{i1}^{(1)} / S_i$

$$\tilde{a}_{11}^{(1)} = 34/34 = 1$$

$$\tilde{a}_{21}^{(1)} = 0/10 = 0$$

$$\tilde{a}_{31}^{(1)} = 10/10 = 1$$

$$\max_{i=\overline{1,3}} |\tilde{a}_{i1}^{(1)}| = \max\{|1|, |0|, |1|\} = 1 \Rightarrow \tilde{a}_{e1}^{(1)} \in \left\{\begin{array}{l} \tilde{a}_{11}^{(1)} \\ \tilde{a}_{31}^{(1)} \end{array}\right. \Rightarrow e \in \{1, 3\}$$

$1 \in \{1,3\} \Rightarrow$ nu e nevoie să interschimbăm linii.

$\tilde{a}_{11}^{(1)} = 34 \neq 0 \Rightarrow$ putem aplica MEG fără pivotare.

$i=\overline{2,3}$, $m_i^{(1)} = \tilde{a}_{i1}^{(1)} / \tilde{a}_{11}$

- $m_2^{(1)} = 0/34 = 0$

  $(E_2) \to (E_2)$

- $m_3^{(1)} = 10/34 = \frac{5}{17}$

  $(E_3 - m_3^{(1)} E_1) \to (E_3)$

  $$\tilde{a}_{32}^{(2)} = \tilde{a}_{32}^{(1)} - m_3^{(1)} \tilde{a}_{12}^{(1)} = 0$$

  $$\tilde{a}_{33}^{(2)} = \tilde{a}_{33}^{(1)} - m_3^{(1)} \tilde{a}_{13}^{(1)} = 4 - \frac{5}{17} \cdot 10 = \frac{68-50}{17} = \frac{18}{17}$$

$\tilde{a}_{31}^{(2)} = 0$

$\tilde{b}_3^{(2)} = \tilde{b}_3^{(1)} - m_3^{(1)} b_1^{(1)} = 2 - \frac{5}{17} \cdot 14 = \frac{34 - 70}{17} = -\frac{36}{17}$

Am obținut:

PASUL $k=2$

$$\bar{A}^{(2)} = \left[\begin{array}{ccc|c} 34 & 0 & 10 & 14 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & \frac{18}{17} & -\frac{36}{17} \end{array}\right]$$

$i = \overline{2,3}$, $s_i = \max\limits_{j=\overline{2,3}} |a_{ij}^{(1)}|$

$$s_2 = \max_{j=\overline{2,3}} |a_{2j}^{(1)}| = 10$$

$$s_3 = \max_{j=\overline{2,3}} |a_{3j}^{(1)}| = \frac{18}{17}$$

$i = \overline{2,3}$, $\tilde{a}_{i2}^{(2)} = a_{i2}^{(2)} / s_i$

$$\tilde{a}_{22}^{(2)} = 10 / 10 = 1$$

$$\tilde{a}_{32}^{(2)} = 0 \cdot \frac{17}{18} = 0$$

$\max\limits_{i=\overline{2,3}} |\tilde{a}_{i2}^{(2)}| = \max\{|1|, |0|\} = 1 \Rightarrow \ell = 1 \Rightarrow$

$\Rightarrow$ nu e nevoie să interschimbăm liniile.

$$\tilde{\bar{A}}^{(2)} = \left[\begin{array}{ccc|c} 34 & 0 & 10 & 14 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & \frac{18}{17} & -\frac{36}{17} \end{array}\right]$$

$\tilde{a}_{22}^{(2)} = 10 \neq 0 \Rightarrow$ putem aplica MEGPP. $\Rightarrow$

$$\Rightarrow \bar{A}^{(3)} = \left[\begin{array}{ccc|c} 34 & 0 & 10 & 14 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & \frac{18}{17} & -\frac{36}{17} \end{array}\right]$$

Rezolvăm sist. obținut prin met. subs. descendente:

$$\begin{cases} 34x_1 + 10x_3 = 14 \\ 10x_2 = 0 \\ \frac{18}{17}x_3 = -\frac{36}{17} \end{cases} \Rightarrow \boxed{x_3 = -2}$$

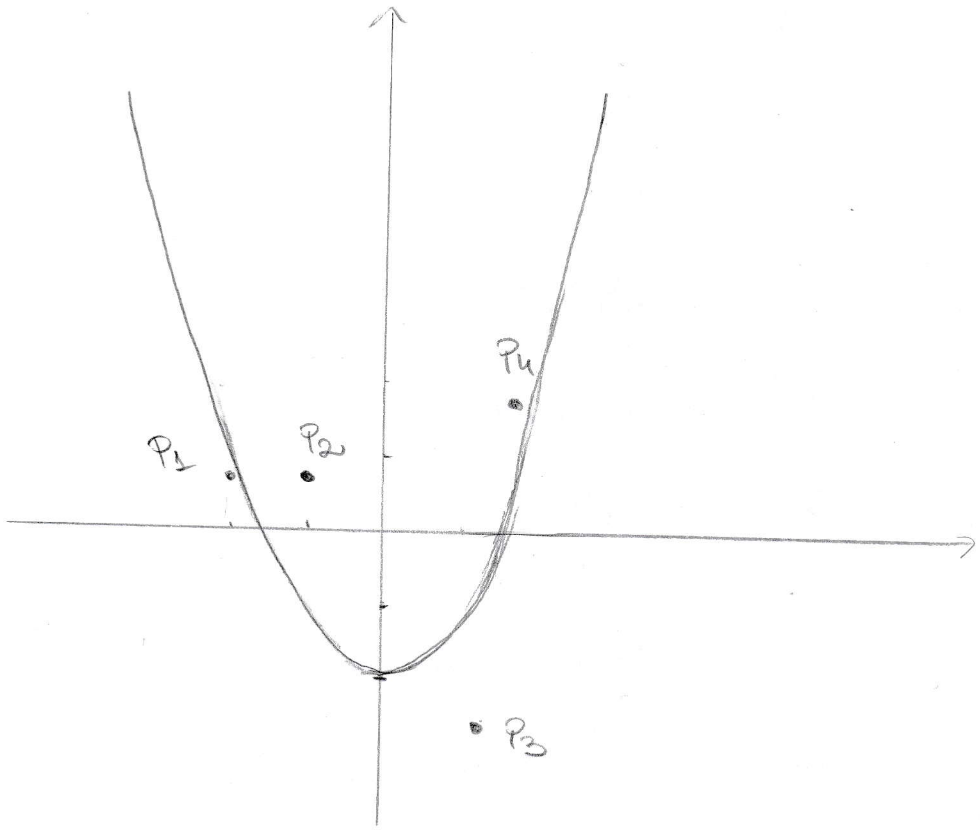
$$\boxed{x_2 = 0}$$

$$x_1 = \frac{14 - 10x_3}{34} = \frac{14+20}{34} = 1 \Rightarrow$$

$$\Rightarrow \boxed{x_1 = 1}$$

$\Rightarrow \underline{x} = (1 \;\; 0 \;\; -2)$

Deci am obținut parabola de ecuație: $x^2 - 2 = 0$



Subiectul 3

Prin metoda Gram-Schmidt clasică avem că:

$$\begin{cases} r_{jk} = q_j^T a_k, \quad j=\overline{1,k-1}, \quad k=\overline{1,n} \\ r_{kk} = \| a_k - \sum_{j=1}^{k-1} q_j r_{jk} \|_2 \\ q_k = \frac{1}{r_{kk}} \left(a_k - \sum_{j=1}^{k-1} q_j \cdot r_{jk} \right) \end{cases} \Rightarrow$$

$$\Rightarrow q_k r_{kk} = a_k - \sum_{j=1}^{k-1} q_j r_{jk}$$

$$\left. \begin{array}{l} r_{jk} = q_j^T a_k \Rightarrow \cancel{q_k r_{jk} = q_k q_j^T a_k = 0} \\ q_j r_{jk} = q_j q_j^T a_k = Q_j a_k \end{array} \right\} \Rightarrow$$

$$\Rightarrow q_k r_{kk} = a_k - \sum_{j=1}^{k-1} Q_j a_k = \left[I_n - \sum_{j=1}^{k-1} Q_j \right] a_k.$$