TUTORIAT 2

1. Să se rezolve sistemul de ecuații liniare folosind MEGPPS și metoda substituției descendente:

$$\begin{cases} x_1 - x_2 + x_3 = 5 \\ x_1 + 5x_2 - x_3 = 8 \\ 2x_1 + x_2 + x_3 = 7 \end{cases} \Rightarrow A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 5 & -1 \\ 2 & 1 & 1 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$$

$$n = 3 \Rightarrow k = \overline{1,2}$$

Pentru $k=1$: $\bar{A}^{(1)} = [A^{(1)} \ \underline{b}^{(1)}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & 5 & -1 & 8 \\ 2 & 1 & 1 & 7 \end{array}\right]$

Căutăm maximul pe fiecare linie: $s_i = \max\limits_{i,j=\overline{1,3}} |a_{ij}^{(1)}|$

$i=\overline{1,3}$: $s_1 = \max\limits_{j=\overline{1,3}} |a_{1j}^{(1)}| = \max\{|1|, |-1|, |1|\} = 1$

$$s_2 = \max_{j=\overline{1,3}} |a_{2j}^{(1)}| = \max\{|1|, |5|, |-1|\} = 5$$

$$s_3 = \max_{j=\overline{1,3}} |a_{3j}^{(1)}| = \max\{|2|, |1|, |1|\} = 2$$

Împărțim fiecare element de pe coloana 1 la s_i, $i=\overline{1,3}$

$$\tilde{a}_{i1}^{(1)} = a_{i1}^{(1)} / s_i$$

$$\tilde{a}_{11}^{(1)} = a_{11}^{(1)} / s_1 = \frac{1}{1} = \underline{1}$$

$$\tilde{a}_{21}^{(1)} = a_{21}^{(1)} / s_2 = \underline{\frac{1}{5}}$$

$$\tilde{a}_{31}^{(1)} = a_{31}^{(1)} / s_3 = \frac{2}{2} = \underline{1}$$

Căutăm maximul pe coloana 1 : $\max\limits_{i=\overline{1,3}} |\tilde{a}_{i1}^{(1)}| = \max\{\underline{|1|}, |1/5|, \underline{|1|}\} =$

$= 1 = |\tilde{a}_{11}^{(1)}| = |\tilde{a}_{31}^{(1)}| \Rightarrow l \in [1, 3]$

Cum $1 \in [1, 3]$ $\Rightarrow$ nu interschimbăm linii în matricea $\bar{A}^{(1)}$

Matricea de permutare simplă : $P^{(1)} = I_3$ (1a)

$$P^{(1)} \cdot \bar{A}^{(1)} = P^{(1)} \cdot [A^{(1)} \; \underline{b}^{(1)}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & 5 & -1 & 8 \\ 2 & 1 & 1 & 7 \end{array}\right] = [\tilde{A}^{(1)} | \tilde{\underline{b}}^{(1)}] = \tilde{\bar{A}}^{(1)}$$

$\tilde{a}_{11}^{(1)} = 1 \neq 0 \Rightarrow$ putem aplica MEGPP

$i = \overline{2,3}$: $m_i^{(1)} = \tilde{a}_{i1}^{(1)} / \tilde{a}_{11}^{(1)}$

$m_2^{(1)} = \tilde{a}_{21}^{(1)} / \tilde{a}_{11}^{(1)} = \frac{1}{1} = 1$

$E_2 \leftarrow E_2 - m_2^{(1)} E_1$

$j = \overline{3,3}$: $a_{2j}^{(2)} = \tilde{a}_{2j}^{(1)} - m_2^{(1)} \tilde{a}_{1j}^{(1)}$

$a_{22}^{(2)} = \tilde{a}_{22}^{(1)} - m_2^{(1)} \tilde{a}_{12}^{(1)} = 5 - 1 \cdot (-1) = 5 + 1 = 6$

$a_{23}^{(2)} = \tilde{a}_{23}^{(1)} - m_2^{(1)} \tilde{a}_{13}^{(1)} = -1 - 1 \cdot 1 = -1 - 1 = -2$

$b_2^{(2)} = \tilde{b}_2^{(1)} - m_2^{(1)} \tilde{b}_1^{(1)} = 8 - 1 \cdot 5 = 8 - 5 = 3$

$\{ a_{21}^{(2)} = 0 \}$

$m_3^{(1)} = \tilde{a}_{31}^{(1)} / \tilde{a}_{11}^{(1)} = \frac{2}{1} = 2$

$E_3 \leftarrow E_3 - m_3^{(1)} E_1$

$j = \overline{2,3}$: $a_{3j}^{(2)} = \tilde{a}_{3j}^{(1)} - m_3^{(1)} \tilde{a}_{1j}^{(1)}$

$a_{32}^{(2)} = \tilde{a}_{32}^{(1)} - m_3^{(1)} \tilde{a}_{12}^{(1)} = 1 - 2 \cdot (-1) = 1 + 2 = 3$

$a_{33}^{(2)} = \tilde{a}_{33}^{(1)} - m_3^{(1)} \tilde{a}_{13}^{(1)} = 1 - 2 \cdot 1 = 1 - 2 = -1$

$$b_3^{(2)} = \tilde{b}_3^{(1)} - m_3^{(1)} \tilde{b}_1^{(1)} = 7 - 2 \cdot 5 = 7 - 10 = -3$$

$$\{ a_{31}^{(2)} = 0 \}$$

Pentru $k=2$, am obținut: $\overline{A}^{(2)} = [A^{(2)} \mid b^{(2)}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 6 & -2 & 3 \\ 0 & 3 & -1 & -3 \end{array}\right]$

Matricea care transformă $\tilde{\overline{A}}^{(1)}$ în $\overline{A}^{(2)}$ este: $M^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (1b)

Are loc relația $M^{(1)} \cdot P^{(1)} \cdot \overline{A}^{(1)} = \overline{A}^{(2)}$ (1)

cu $M^{(1)}$ și $P^{(1)}$ date de (1a) și (1b)

$$\overline{A}^{(2)} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 6 & -2 & 3 \\ 0 & 3 & -1 & -3 \end{array}\right]$$

Căutăm maximul pe fiecare linie: $\Delta_i = \max_{j=\overline{2,3}} |a_{ij}^{(2)}|$

$i=\overline{2,3}$: $\Delta_2 = \max_{j=\overline{2,3}} |a_{2j}^{(2)}| = \max\{|6|, |-2|\} = 6$

$\Delta_3 = \max_{j=\overline{2,3}} |a_{3j}^{(2)}| = \max\{|3|, |-1|\} = 3$

Împărțim elementele de pe coloana 2 la Δ_i, $i=\overline{2,3}$

$$\tilde{a}_{i2}^{(2)} = a_{i2}^{(2)} / \Delta_i$$

$$\tilde{a}_{22}^{(2)} = a_{22}^{(2)} / \Delta_2 = 6/6 = 1$$

$$\tilde{a}_{32}^{(2)} = a_{32}^{(2)} / \Delta_3 = 3/3 = 1$$

Căutăm maximul pe coloana 2: $\max_{i=\overline{2,3}} |\tilde{a}_{i2}^{(2)}| = \max\{|1|, |1|\} = 1 =$

$= |\tilde{a}_{22}^{(2)}| = |\tilde{a}_{32}^{(2)}| \Rightarrow l \in \{2,3\}$

cum $2 \in \{3,3\}$ $\Rightarrow$ nu interschimbăm linii în matricea $\overline{A^{(2)}}$

Matricea de permutare simplă: $P^{(2)} = I_3$ (2a)

$$P^{(2)} \overline{A^{(2)}} = \overline{\tilde{A}^{(2)}} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 6 & -2 & 3 \\ 0 & 3 & -1 & -3 \end{array}\right]$$

$\tilde{a}_{22}^{(2)} = 6 \neq 0 \rightarrow$ putem aplica MEGPP

$i = \overline{3,3}$: $m_i^{(2)} = \tilde{a}_{i2}^{(2)} / \tilde{a}_{22}^{(2)} \Rightarrow m_3^{(2)} = \tilde{a}_{32}^{(2)} / \tilde{a}_{22}^{(2)} = \frac{3}{6} = \frac{1}{2}$

$E_3 \leftarrow E_3 - m_3^{(2)} E_2$

$j = \overline{3,3}$: $a_{3j}^{(3)} = \tilde{a}_{3j}^{(2)} - m_3^{(2)} \tilde{a}_{2j}^{(2)}$

$a_{33}^{(3)} = \tilde{a}_{33}^{(2)} - m_3^{(2)} \tilde{a}_{23}^{(2)} = -1 - \frac{1}{2} \cdot (-2) = -1 + 1 = 0$

$b_3^{(3)} = \tilde{b}_3^{(2)} - m_3^{(2)} \tilde{b}_2^{(2)} = -3 - \frac{1}{2} \cdot 3 = -3 - \frac{3}{2} = \frac{-6-3}{2} = \frac{-9}{2}$

$\{ a_{32}^{(3)} = 0 \}$

Am obținut: $\overline{A}^{(3)} = [A^{(3)} \mid \underline{b}^{(3)}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 6 & -2 & 3 \\ \underline{0} & \underline{0} & \underline{0} & \boxed{-9/2} \end{array}\right] = [U \ \underline{\tilde{b}}]$

sistem incompatibil

Matricea care transformă $\overline{\tilde{A}^{(2)}}$ în $\overline{A^{(3)}}$ este $M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$ (2b)

Are loc relația $M^{(2)} \cdot P^{(2)} \cdot \overline{A^{(2)}} = [U \ \underline{\tilde{b}}]$ (2)

cu $M^{(2)}$ și $P^{(2)}$ date de (2a) și (2b)

Din relațiile (1) și (2) obținem $M^{(2)}P^{(2)}M^{(1)}P^{(1)}[A^{(1)}|\underline{b}^{(1)}]=[U\ \underline{\tilde{b}}]$

Sistemul $A\cdot\underline{x}=\underline{b}$ a devenit de forma $U\cdot\underline{x}=\underline{\tilde{b}}$:

$$\begin{cases} x_1-x_2+x_3=5 \\ 6x_2-2x_3=3 \\ 0=-\frac{9}{2} \Rightarrow \text{sistem incompatibil} \end{cases}$$

Deci nu există soluții.

2. Să se rezolve sistemul de ecuații liniare folosind MEGPT și metoda substituției descendente:

$$\begin{cases} x_1-x_2+x_3=5 \\ 7x_1+5x_2-x_3=8 \\ 2x_1+x_2+x_3=7 \end{cases} \Rightarrow A=\begin{bmatrix} 1 & -1 & 1 \\ 7 & 5 & -1 \\ 2 & 1 & 1 \end{bmatrix} \quad \underline{b}=\begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$$

$\Rightarrow n=3,\ k=\overline{1,2}$

Pentru $k=1$: $\overline{A^{(1)}}=[A^{(1)}|\underline{b}^{(1)}]=\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 7 & 5 & -1 & 8 \\ 2 & 1 & 1 & 7 \end{array}\right]$

Căutăm maximul din A: $7=|a_{21}^{(1)}|=|a_{\ell m}^{(1)}| \Rightarrow \begin{cases} \ell=2\neq 1=k \\ m=1=k \end{cases}$

$E_1\longleftrightarrow E_2$: Matricea de permutare simplă $P^{(1)}=P_{12}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (1a)

$C_1\longleftrightarrow C_1$: Matricea de permutare simplă $Q^{(1)}=I_3$ (1b)

$$P^{(1)}[A^{(1)}|\underline{b}^{(1)}]=\left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

$$[P^{(1)}A^{(1)}Q^{(1)}|P^{(1)}\underline{b}^{(1)}]=\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & 1 & 7 \end{array}\right] \end{array}=[\tilde{A}^{(1)}|\underline{\tilde{b}}^{(1)}]=\overline{\tilde{A}^{(1)}}$$

$\Rightarrow$ interschimbăm coloane

$\tilde{a}_{11}^{(1)}=7\neq 0 \Rightarrow$ putem aplica MEGPP

$i=\overline{2,3}$: $m_i^{(1)} = \tilde{a}_{i1}^{(1)} / \tilde{a}_{11}^{(1)}$

$m_2^{(1)} = \tilde{a}_{21}^{(1)} / \tilde{a}_{11}^{(1)} = \frac{1}{7}$

$E_2 \leftarrow E_2 - m_2^{(1)} E_1$

$j=\overline{2,3}$: $a_{2j}^{(2)} = \tilde{a}_{2j}^{(1)} - m_2^{(1)} \tilde{a}_{1j}^{(1)}$

$$a_{22}^{(2)} = \tilde{a}_{22}^{(1)} - m_2^{(1)} \tilde{a}_{12}^{(1)} = -1 - \frac{1}{7}\cdot 5 = -1 - \frac{5}{7} = \frac{-7-5}{7} = -\frac{12}{7}$$

$$a_{23}^{(2)} = \tilde{a}_{23}^{(1)} - m_2^{(1)} \tilde{a}_{13}^{(1)} = 1 - \frac{1}{7}\cdot(-1) = 1 + \frac{1}{7} = \frac{7+1}{7} = \frac{8}{7}$$

$$b_2^{(2)} = \tilde{b}_2^{(1)} - m_2^{(1)} \tilde{b}_1^{(1)} = 5 - \frac{1}{7}\cdot 8 = 5 - \frac{8}{7} = \frac{35-8}{7} = \frac{27}{7}$$

$\{ a_{21}^{(2)} = 0 \}$

$m_3^{(1)} = \tilde{a}_{31}^{(1)} / \tilde{a}_{11}^{(1)} = \frac{2}{7}$

$E_3 \leftarrow E_3 - m_3^{(1)} E_1$

$j=\overline{2,3}$: $a_{3j}^{(2)} = \tilde{a}_{3j}^{(1)} - m_3^{(1)} \tilde{a}_{1j}^{(1)}$

$$a_{32}^{(2)} = \tilde{a}_{32}^{(1)} - m_3^{(1)} \tilde{a}_{12}^{(1)} = 1 - \frac{2}{7}\cdot 5 = 1 - \frac{10}{7} = \frac{7-10}{7} = -\frac{3}{7}$$

$$a_{33}^{(2)} = \tilde{a}_{33}^{(1)} - m_3^{(1)} \cdot \tilde{a}_{13}^{(1)} = 1 - \frac{2}{7}\cdot(-1) = 1 + \frac{2}{7} = \frac{7+2}{7} = \frac{9}{7}$$

$$b_3^{(2)} = \tilde{b}_3^{(1)} - m_3^{(1)} \cdot \tilde{b}_1^{(1)} = 7 - \frac{2}{7}\cdot 8 = 7 - \frac{16}{7} = \frac{49-16}{7} = \frac{33}{7}$$

Pentru K=2, am obținut: $\tilde{A}^{(2)} = \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 0 & -12/7 & 8/7 & 27/7 \\ 0 & -3/7 & 9/7 & 33/7 \end{array}\right]$

Matricea care transformă $\tilde{A}^{(1)}$ în matricea $\tilde{A}^{(2)}$ este $M^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1/7 & 1 & 0 \\ -2/7 & 0 & 1 \end{bmatrix}$

(vc)

Are loc relația $M^{(1)} P^{(1)} \left[A^{(1)}\ Q^{(1)}\ \underline{b}^{(1)} \right] = \left[A^{(2)} \mid \underline{b}^{(2)} \right]$ (1)

cu $P^{(1)}$, $Q^{(1)}$, $M^{(1)}$ luate de (1a), (1b), (1c)

$$\overline{A^{(2)}} = \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 0 & -12/7 & 8/7 & 27/7 \\ 0 & -3/7 & 9/7 & 33/7 \end{array}\right]$$

Căutăm maximul: $\max\limits_{i,j=\overline{2,3}} |a_{ij}^{(2)}| = 12/7 = |a_{22}^{(2)}| = |a_{\ell m}^{(2)}| \rightarrow \begin{cases} \ell = 2 = k \\ m = 2 = k \end{cases}$

$E_2 \leftrightarrow E_2$: Matricea de permutare simplă $P^{(2)} = I_3$ (2a)

$C_2 \leftrightarrow C_2$: Matricea de permutare simplă $Q^{(2)} = I_3$ (2b)

$$P^{(2)}[A^{(2)} | \underline{b}^{(2)}] = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 0 & -12/7 & 8/7 & 27/7 \\ 0 & -3/7 & 9/7 & 33/7 \end{array}\right] \end{array}$$

$$[P^{(2)} A^{(2)} Q^{(2)} \; P^{(2)} \underline{b}^{(2)}] = \downarrow = [\tilde{A}^{(2)} | \tilde{\underline{b}}^{(2)}] = \overline{\tilde{A}^{(2)}}$$

$\tilde{a}_{22}^{(2)} = -12/7 \neq 0 \rightarrow$ putem aplica MEGFP

$i = \overline{3,3}$: $m_i^{(2)} = \tilde{a}_{i2}^{(2)} / \tilde{a}_{22}^{(2)} \Rightarrow m_3^{(2)} = \tilde{a}_{32}^{(2)} / \tilde{a}_{22}^{(2)} = \dfrac{-\frac{3}{7}}{-\frac{12}{7}} = 1/4$

$E_3 \leftarrow E_3 - m_3^{(2)} E_2$

$j = \overline{3,3}$: $a_{3j}^{(3)} = \tilde{a}_{3j}^{(2)} - m_3^{(2)} \tilde{a}_{2j}^{(2)}$

$$a_{33}^{(3)} = \tilde{a}_{33}^{(2)} - m_3^{(2)} \tilde{a}_{23}^{(2)} = 1$$

$$b_3^{(3)} = \tilde{b}_3^{(2)} - m_3^{(2)} \tilde{b}_2^{(2)} = \frac{15}{4}$$

$\{ a_{32}^{(3)} = 0 \}$

Am obținut $\overline{A^{(3)}} = \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 0 & -12/7 & 8/7 & 27/7 \\ 0 & 0 & 1 & 15/4 \end{array}\right] = [U \; \tilde{\underline{b}}]$

Matricea care transformă $\overline{\tilde{A}^{(2)}}$ în $\overline{A^{(3)}}$ este $M^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{bmatrix}$ (2c)

Am loc relația $M^{(2)} P^{(2)} [A^{(2)} Q^{(2)} \underline{b}^{(2)}] = [A^{(3)} / \underline{b}^{(3)}] = \overline{A^{(3)}}$ (2)

cu $P^{(2)}$, $Q^{(2)}$, $M^{(2)}$ date de (2a), (2b), (2c)

Din relațiile (1) și (2) obținem: $M^{(2)} P^{(2)} M^{(1)} P^{(1)} [A \; Q^{(1)} Q^{(2)} \underline{b}] = [U \; \underline{\tilde{b}}]$

Sistemul $A \cdot \underline{x} = \underline{b}$ devine $U \cdot \underline{x} = \underline{\tilde{b}}$

$$\begin{cases} 7x_1 + 5x_2 - x_3 = 8 \\ -\frac{12}{7}x_2 + \frac{8}{7}x_3 = \frac{27}{7} \\ x_3 = \frac{15}{4} \end{cases}$$

$x_3 = \frac{15}{4}$

$-\frac{12}{7}x_2 = \frac{27}{7} - \frac{8}{7}x_3 = \frac{27}{7} - \frac{\cancel{8}^{2}}{7} \cdot \frac{15}{\cancel{4}_1} = \frac{27}{7} - \frac{30}{7} = -\frac{3}{7} \Rightarrow x_2 = \frac{1}{4}$

$7x_1 = 8 + x_3 - 5x_2 = \overset{4}{\cancel{8}} + \frac{15}{4} - \frac{5}{4} = \frac{42}{4} \Rightarrow x_1 = \frac{42}{7 \cdot 4} = \frac{3}{2}$

$(x_1, x_2, x_3) = (3/2, 1/4, 15/4)$