Seminar EDP - Ecuații cu derivate parțiale

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October 21, 2020

Cuprins

1 Seminar 1

Exercițiul 1. Se dă ecuația afină scalară (omogenă) : $y'(x) + y(x)cosx = cosx$.

Soluție. Pasul 1 : $\overline{y}'(x) + \overline{y}(x)cosx = 0 \Longleftrightarrow \overline{y}'(x) = -\overline{y}(x)cosx$.

$$\overline{y}(x) = 0 \text{ este o suluție staționară.}$$

$$\overline{y}(x) \neq 0 \Rightarrow \frac{\overline{y}'(x)}{\overline{y}(x)} = -cosx \xrightarrow{\int} \int \frac{\overline{y}'(x)}{\overline{y}(x)} dx = \int -cosx dx$$

$$ln|\overline{y}(x)| = -sinx + C_1 \Big|^{exp} \Rightarrow e^{ln|\overline{y}(x)|} = e^{-sinx+C_1} \Rightarrow \overline{y}(x) = \pm e^{-sinx+C_1}$$

$$\overline{y}(x) = C_2 e^{-sinx}, C_2 = \pm e^{C_1}$$

Pasul 2 : Se aplică metoda variației constantelor (MVC). Caut $y(x)$ de forma $y(x) = C(x)e^{-sinx}$.

$$C'(x)e^{-sinx} - cosxC(x)e^{-sinx} + cosxC(x)e^{-sinx} = cosx \Rightarrow$$

$$C'(x) = cosxe^{sinx} \Rightarrow C(x) = \int cosxe^{sinx} dx = e^{sinx} + \overline{C} \Rightarrow$$

$$y(x) = \left(e^{sinx} + \overline{C}\right)e^{-sinx} = \underline{1 + \overline{C}e^{-sinx}}$$

Exercițiul 2. Se dă următoarea ecuație : $y'' - 5y' + 6y = \begin{cases} e^x & (1) \\ e^{2x} & (2) \end{cases}$.

Soluție. Alegem ecuația (1) : $y'' - 5y' + 6y = e^x$.

Pasul 1 : Considerăm ecuația : $\overline{y}'' - 5\overline{y}' + 6\overline{y} = 0$. Ecuația caracteristică asociată este : $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 2; \lambda_2 = 3 \Rightarrow \overline{y}(x) = C_1e^{2x} + C_2e^{3x}, y = \overline{y} + y_{part,neomo}; y_{part,neomo}?$

Pasul 2 : MVC : $y(x) = C_1(x)e^{2x} + C_2(x)e^{3x}$. Pentru determinarea $C_1(x)$ și $C_2(x)$ avem sistemul următor :

$$\begin{pmatrix} e^{2x} & e^{3x} \\ (e^{2x})' & (e^{3x})' \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^x \end{pmatrix}$$

$$\begin{cases} e^{2x}C_1'(x) + e^{3x}C_2'(x) = 0 & (3) \\ 2e^{2x}C_1'(x) + 3e^{3x}C_2'(x) = e^x & (4) \end{cases} \xrightarrow{(4)-2(3)} e^{3x}C_2'(x) = e^x \Rightarrow$$

$$C_2'(x) = e^{-2x} \xrightarrow{\int} C_2(x) = \int e^{-2x}dx = -\frac{1}{2}e^{-2x} + \widetilde{C_1}$$

$$e^{2x}C_1'(x) + e^{3x}e^{-2x} = 0 \Rightarrow C_1'(x) = -e^{-x} \xrightarrow{\int} C_1(x) = \int -e^{-x}dx = e^{-x} + \widetilde{C_2}$$

$$y(x) = (e^{-x} + \widetilde{C_2})e^{2x} + \big(-\frac{1}{2}e^{-2x} + \widetilde{C_1}\big)e^{3x} \Rightarrow$$

$$y(x) = \frac{e^x}{2} + \widetilde{C_2}e^{2x} + \widetilde{C_1}e^{3x}; \widetilde{C_1}, \widetilde{C_2} \in \mathbb{R}$$

În colcluzie $\frac{e^x}{2}$ este o soluție particulară a ecuației neomogene, iar $\widetilde{C_2}e^{2x} + \widetilde{C_1}e^{3x}$ este soluția generală a ecuației omogene.

Alegem ecuația (2) : $y'' - 5y' + 6y = e^2x$. Pasul 1 este identic cu cel de la (1).

$$\begin{pmatrix} e^{2x} & e^{3x} \\ (e^{2x})' & (e^{3x})' \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{2x} \end{pmatrix}$$

$$\begin{cases} e^{2x}C_1'(x) + e^{3x}C_2'(x) = 0 & (3) \\ 2e^{2x}C_1'(x) + 3e^{3x}C_2'(x) = e^{2x} & (4) \end{cases} \xrightarrow{(4)-2(3)} e^{3x}C_2'(x) = e^{2x} \Rightarrow$$

$$C_2'(x) = e^{-x} \xrightarrow{\int} C_2(x) = \int e^{-x}dx = -e^{-x} + \widetilde{C_1}$$

$$e^{2x}C_1'(x) + e^{3x}e^{-x} = 0 \Rightarrow C_1'(x) = -1 \xrightarrow{\int} C_1(x) = \int -1dx = -x + \widetilde{C_2}$$

$$y(x) = (-x + \widetilde{C_2})e^{2x} + (-e^{-x} + \widetilde{C_1})e^{3x} \Rightarrow$$
$$y(x) = -e^{2x}(x+1) + \widetilde{C_2}e^{2x} + \widetilde{C_1}e^{3x}; \widetilde{C_1}, \widetilde{C_2} \in \mathbb{R}$$

În colcluzie $-e^{2x}(x+1)$ este o soluție particulară a ecuației neomogene, iar $\widetilde{C_2}e^{2x} + \widetilde{C_1}e^{3x}$ este soluția generală a ecuației omogene.

Exercițiul 3. Se dă următoarea ecuație : $y' + \frac{2}{x}y = x^3$.

Soluție. <u>Pasul 1</u> : $\overline{y}' + \frac{2}{x}\overline{y} = 0 \Rightarrow \overline{y}' = -\frac{2}{x}\overline{y} \Rightarrow \overline{y} = 0$ este soluție staționară.

$$\overline{y} \neq 0 \Rightarrow \frac{\overline{y}'}{\overline{y}} = -\frac{2}{x} \xrightarrow{\int} \int \frac{\overline{y}'(x)}{\overline{y}(x)}dx = \int -\frac{2}{x}dx \Rightarrow ln|\overline{y}| = -2ln|x| + C$$

$$\xrightarrow{exp} e^{-2ln|x|+C} = e^{ln|\overline{y}|} \Rightarrow \overline{y(x)} = \frac{\widetilde{C}}{x^2}; \widetilde{C} \in \mathbb{R}$$

<u>Pasul 2</u> : MVC : Căutăm $y(x) = \frac{C(x)}{x^2}$. $\left(\frac{C(x)}{x^2}\right)' + \frac{2}{x}\left(\frac{C(x)}{x^2}\right) = x^3 \Rightarrow$

$$\frac{C'(x)}{x^2} - \frac{2C(x)}{x^3} + \frac{2C(x)}{x^3} = x^3 \Rightarrow C'(x) = x^5 \xrightarrow{\int} C(x) = \int x^5dx = \frac{x^6}{6} + c$$

$$\Rightarrow y(x) = \left(\frac{x^6}{6} + c\right)\frac{1}{x^2} = \frac{x^4}{6} + \frac{c}{x^2}; c \in \mathbb{R}$$

Ecuații de tip Euler :

$$\sum_{i=1}^{n} a_i x^i y^{(i)} = f(x); a_i = ct.$$

Exercițiul 4. Se dă următoarea ecuație : $x^2y'' + xy' - y = x^2, x > 0$.

Soluție. Schimbare de var : $x = e^s, s \in \mathbb{R}; v(s) \overset{def}{=} y(e^s) = y(x) = v(lnx)$.

$$y'(x) = \frac{1}{x}v'(lnx) \Rightarrow xy' = v'(s)$$

$$y''(x) = \left(\frac{1}{x}v'(lnx)\right)' = \frac{-1}{x^2}v'(lnx) + \frac{1}{x^2}v''(lnx) \Rightarrow x^2y'' = v''(lnx) - v'(lnx) = v''(s) - v'(s)$$

$$v''(s) - v'(s) + v'(s) - v(s) = e^{2s} \Rightarrow v''(s) - v(s) = e^{2s} \xrightarrow{ec.car.asc.}$$

$$\overline{v}''(s) - \overline{v}'(s) = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1 \Rightarrow \overline{v}(s) = C_1e^{\lambda_1 s} + C_2e^{\lambda_2 s} = C_1e^s + C_2e^{-s}$$

$$\begin{pmatrix} e^s & e^{-s} \\ e^s & -e^{-s} \end{pmatrix} \begin{pmatrix} C_1'(s) \\ C_2'(s) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{2s} \end{pmatrix}$$

$$\begin{cases} e^sC_1'(s) + e^{-s}C_2'(s) = 0 & (5) \\ e^sC_1'(s) - e^{-s}C_2'(s) = e^{2s} & (6) \end{cases} \xrightarrow{(5)+(6)} 2e^sC_1'(s) = e^{2s} \xrightarrow{\int} C_1'(s) = \int \frac{e^s}{2}ds = \frac{e^s}{2} + \widetilde{C_1}$$

$$C_2'(s) = \frac{-e^sC_1'(s)}{e^{-s}} = \frac{-e^{2s}e^2}{2} = -\frac{e^{3s}}{2} \xrightarrow{\int} C_2(s) = \int -\frac{e^{3s}}{2}ds = -\frac{e^{3s}}{6} + \widetilde{C_2}$$

$$v(s) = \left(\frac{e^s}{2} + \widetilde{C_1}\right)e^s + \left(-\frac{e^{3s}}{6} + \widetilde{C_2}\right)e^{-s} \xrightarrow{x=e^s} y(x) = \frac{x^2}{3} + x\widetilde{C_1} + \frac{1}{x}\widetilde{C_2}$$

Exercițiul 5. Calculați derivatele parțiale de ordin I pe domeniu maxim de definiție pentru :

(a). $f(x,y) = ln\left(cos\frac{y}{x}\right)$

(b). $f(x,y) = (x+y^2)^{\sqrt{\frac{x}{y}}}$

(c). $f(x,y) = tg(xarcsiny)$

Soluție. (b). Se rescrie funcția astfel : $f(x,y) = (x+y^2)^{\sqrt{\frac{x}{y}}} = e^{ln(x+y^2)^{\sqrt{\frac{x}{y}}}} = e^{\sqrt{\frac{x}{y}}ln(x+y^2)}$.
Cunoaștem că : $\left(e^{f(x,y)g(x,y)}\right)' = \left(f(x,y)g(x,y)\right)'e^{f(x,y)g(x,y)}$, unde $f(x,y) = \sqrt{\frac{x}{y}}, g(x,y) = ln(x+y^2)$. Astfel rezultă că $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{\sqrt{\frac{x}{y}}ln(x+y^2)}\right) = \left(e^{\sqrt{\frac{x}{y}}ln(x+y^2)}\right)\frac{\partial}{\partial x}\left(\sqrt{\frac{x}{y}}ln(x+y^2)\right) =$

$$(x+y^2)^{\sqrt{\frac{x}{y}}}\left(\frac{\sqrt{\frac{x}{y}}}{x+y^2} - \frac{ln(x+y^2)}{2y\sqrt{\frac{x}{y}}}\right)$$

Iar pentru $\dfrac{\partial f}{\partial y}$ avem : $\dfrac{\partial f}{\partial y} = \left(e^{\sqrt{\frac{x}{y}}ln(x+y^2)}\right)\dfrac{\partial}{\partial y}\left(\sqrt{\frac{x}{y}}ln(x+y^2)\right) =$

$$(x+y^2)^{\sqrt{\frac{x}{y}}}\left(\frac{2y\sqrt{\frac{x}{y}}}{x+y^2} - \frac{3\sqrt{\frac{x}{y}}ln(x+y^2)}{2y}\right)$$

(c). $\dfrac{\partial f}{\partial x} = \dfrac{arcsiny}{cos^2(xarcsiny)}; \dfrac{\partial f}{\partial y} = \dfrac{\frac{\partial}{\partial y}xarcsiny}{cos^2(xarcsiny)} = \dfrac{x}{\sqrt{1-y^2}}\dfrac{1}{cos^2(xarcsiny)} = \dfrac{x}{cos^2(xarcsiny)\sqrt{1-y^2}}$

2 Seminar 2

Operatori diferențiali :

Definiție 2.1. Fie $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ se numește câmp scalar. Fie $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ se numește câmp vectorial , $F = (F_1, \dots, F_n), F_i = F_i(x_1, \dots, x_n)$.

2.1 Operatori diferențiali pentru câmpuri scalare

2.1.1 Operatorul de derivare parțială de ordin superior D^α

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$$

Exemplu : $f(x_1, x_2, x_3) = cos(x_1^2x_2) + x_3^2$

$D^{(0,2,1)}f = \dfrac{\partial^3 f}{\partial x_2^2\partial x_3} = \dfrac{\partial^2}{\partial x_2^2}\left(\dfrac{\partial f}{\partial x_3}\right) = \dfrac{\partial^2}{\partial x_2^2}(2x_3) = 0$

$D^{(2,1,0)}f = \dfrac{\partial^3 f}{\partial x_1^2\partial x_2} = \dfrac{\partial}{\partial x_1^2}\left(\dfrac{\partial f}{\partial x_x}\right) = \dfrac{\partial}{\partial x_1^2}(x_1^2sin(x_1^2x_2)) = \dfrac{\partial}{\partial x_1}(2x_1sin(x_1^2x_2)+2x_1^3x_2cos(x_1^2x_2)) =$ $2sin(x_1^2x_2)+4x_1^2x_2cos(x_1^2x_2)+6x_1^2x_2cos(x_1^2x_2)-4x_1^4x_2^2sin(x_1^2x_2) = 10x_1^2x_2cos(x_1^2x_2)+sin(x_1^2x_2)(2-4x_1^4x_2^2)$

2.1.2 Operatorul gradient ∇

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

Exemplu : $f(x_1, x_2, x_3) = x_3sin\dfrac{x_1}{x_2}; \nabla f = \left(\dfrac{x_3}{x_2}cos\dfrac{x_1}{x_2}, -\dfrac{x_1x_3}{x_2^2}cos\dfrac{x_1}{x_2}, sin\dfrac{x_1}{x_2}\right)$

Demonstrăm că $\nabla(|x|^\lambda) = \lambda x|x|^{\lambda-2}, x \neq 0, \lambda \in \mathbb{R}$.

$\dfrac{\partial}{\partial x_1}(|x|) = \dfrac{2x_1}{2|x|} = \dfrac{x_1}{|x|} \Rightarrow \dfrac{\partial}{\partial x_i}(|x|) = \dfrac{x_i}{|x|} \Rightarrow \dfrac{\partial}{\partial x_i}(|x|^\lambda) = \dfrac{x_i}{|x|}\lambda|x|^{\lambda-1} = \lambda x_i|x|^{\lambda-2} \Rightarrow$

$$\nabla(|x|^\lambda) = (\lambda x_1|x|^{\lambda-2}, \dots, \lambda x_n|x|^{\lambda-2}) = \lambda x|x|^{\lambda-2}$$

2.1.3 Operatorul Laplacian Δ

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = Tr(H_f)$$

, unde $H_f = \left[\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=\overline{1,n}}$ matricea Hesiană a funcției f, iar $Tr(H_f)$ urma matricei Hesiene.

Exemplu : Demonstrăm că $\Delta(|x|^\lambda) = \lambda(\lambda + n - 2)|x|^{\lambda-2}, x \neq 0, \lambda \in \mathbb{R}$.

$$\frac{\partial^2}{\partial x_i^2}(|x|^\lambda) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}(|x|^\lambda)\right) = \frac{\partial}{\partial x_i}\left(\lambda x_i |x|^{\lambda-2}\right) = \lambda\left((\lambda-2)x_i^2|x|^{\lambda-4} + |x|^{\lambda-2}\right) \Rightarrow$$

$$\Delta(|x|^\lambda) = \sum_{i=1}^{n} \lambda\left((\lambda-2)x_i^2|x|^{\lambda-4} + |x|^{\lambda-2}\right) = \lambda(\lambda-2)|x|^{\lambda-4}\sum_{i=1}^{n} x_i^2 + n|x|^{\lambda-2} =$$

$$\lambda(\lambda-2)|x|^{\lambda-4}|x|^2 + n|x|^{\lambda-2} = \lambda(\lambda-2)|x|^{\lambda-2} + n|x|^{\lambda-2} = \lambda(\lambda+n-2)|x|^{\lambda-2}$$

$$\Delta(|x|^\lambda) = 0 \Leftrightarrow \begin{cases} \lambda = 0 \Rightarrow & |x|^\lambda = |x|^0 = 1 \\ \lambda = 0 \Rightarrow & |x|^\lambda = |x|^{2-n} \end{cases}$$

Rezultă că funcția $|x|^{2-n}$, care este și funcție armonică, este soluția fundamentală a Laplacianului pentru $n \geq 3$.

2.2 Operatori diferențiali pentru câmpuri vectoriale

2.2.1 Operatorul divergentă *div*

$$divF = \nabla \cdot F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = Tr(J_f)$$

, unde $J_f = \left[\dfrac{\partial F_i}{\partial x_j}\right]_{i,j=\overline{1,n}}$ matricea Jacobiană a funcției f, iar $Tr(J_f)$ urma matricei Jacobiene.

Exemplu :

$$div(x|x|^\lambda) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(x|x|^\lambda) = \sum_{i=1}^{n}\left(|x|^\lambda + \lambda x_i^2 |x|^{\lambda-2}\right) = n|x|^\lambda + \lambda|x|^{\lambda-2}\sum_{i=1}^{n} x_i^2 = n|x|^\lambda + \lambda|x|^\lambda$$

$$\Rightarrow div(x|x|^\lambda) = |x|^\lambda(n+\lambda)$$

$$div(x|x|^\lambda) = 0 \Leftrightarrow \lambda = -n$$

2.2.2 Operatorul rotațional *rot*

$$rotF = \begin{vmatrix} e_1 & e_2 & e_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right)e_1 + \left(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}\right)e_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)e_3$$

$$\frac{\partial F_i}{\partial x_j} \xleftarrow{not} F_{ij} \Rightarrow rotF = (F_{32} - F_{23}, F_{13} - F_{31}, F_{21} - F_{12})$$

Mai sus este definit operatorul rotațional în $\mathbb{R}^3$, dar se poate defini și în dimensiuni mai mari.

$$rotF = \nabla \times F = \sum_{i=1}^{n}\left(\frac{\partial F_{i+2}}{\partial x_{i+1}} - \frac{\partial F_{i+1}}{\partial x_{i+2}}\right)$$

2.3 Proprietăți : Legături între ∇, Δ, rot, div

(1) $\nabla(f_1, f_2) = f_1\nabla f_2 + f_2\nabla f_1$

(2) $\Delta f = div(\nabla f)$

(3) $\Delta(f_1, f_2) = f_1\Delta f_2 + f_2\Delta f_1 + 2\nabla f_1 f_2$

(4) $div(fF) = \nabla f \cdot F + f \cdot divF$

(5) $div(rotF) = 0; (n = 3)$

(6) $rot(\nabla f) = 0; (n = 3)$

(7) ∇, Δ, rot, div operatori liniari

Demonstrație. (1)

$$\nabla(f_1, f_2) = \left(\frac{\partial}{\partial x_1}(f_1, f_2), \ldots, \frac{\partial}{\partial x_n}(f_1, f_2)\right) =$$
$$\left(f_1\frac{\partial f_2}{\partial x_1} + f_2\frac{\partial f_1}{\partial x_1}, \ldots, f_1\frac{\partial f_2}{\partial x_n} + f_2\frac{\partial f_1}{\partial x_n}\right) =$$
$$\left(f_1\frac{\partial f_2}{\partial x_1}, \ldots, f_1\frac{\partial f_2}{\partial x_n}\right) + \left(f_2\frac{\partial f_1}{\partial x_1}, \ldots, f_2\frac{\partial f_1}{\partial x_n}\right) =$$
$$f_1\left(\frac{\partial f_2}{\partial x_1}, \ldots, \frac{\partial f_2}{\partial x_n}\right) + f_2\left(\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_n}\right) =$$
$$f_1\nabla f_2 + f_2\nabla f_1$$

(2)

$$div(\nabla f) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\nabla f_i = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = \Delta f$$

(3)

$$\Delta(f_1, f_2) = \sum_{i=1}^{n} \frac{\partial^2(f_1, f_2)}{\partial x_i^2} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial(f_1, f_2)}{\partial x_i}\right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_1\frac{\partial f_2}{\partial x_i} + f_2\frac{\partial f_1}{\partial x_i}\right) =$$
$$\sum_{i=1}^{n}\left(f_1\frac{\partial^2 f_2}{\partial x_i^2} + \frac{\partial f_2}{\partial x_i}\frac{\partial f_1}{\partial x_i} + f_2\frac{\partial^2 f_1}{\partial x_i^2} + \frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial x_i}\right) = f_1\sum_{i=1}^{n}\frac{\partial^2 f_2}{\partial x_i^2} + f_2\sum_{i=1}^{n}\frac{\partial^2 f_1}{\partial x_i^2} + 2\sum_{i=1}^{n}\frac{\partial f_1 \partial f_2}{\partial x_i^2} =$$
$$f_1\Delta f_2 + f_2\Delta f_1 + 2\nabla f_1 f_2$$

(4)

$$div(fF) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(f \cdot F) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}F + \frac{\partial F}{\partial x_i}f\right) = F\sum_{i=1}^{n}\frac{\partial f}{\partial x_i} + f\sum_{i=1}^{n}\frac{\partial F}{\partial x_i} = \nabla f \cdot F + f \cdot divF$$

(5)

$$div(rotF) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial F_{i+2}}{\partial x_{i+1}} - \frac{\partial F_{i+1}}{\partial x_{i+2}}\right) = \sum_{i=1}^{n}\left(\frac{\partial^2 F_{i+2}}{\partial x_i \partial x_{i+1}} - \frac{\partial^2 F_{i+1}}{\partial x_i \partial x_{i+2}}\right) =$$
$$\sum_{i=1}^{n}\frac{\partial^2 F_{i+2}}{\partial x_i \partial x_{i+1}} - \sum_{i=1}^{n}\frac{\partial^2 F_{i+1}}{\partial x_i \partial x_{i+2}} = \sum_{i=1}^{n}\frac{\partial^2 F_{i+2}}{\partial x_i \partial x_{i+1}} - \sum_{i=1}^{n}\frac{\partial^2 F_{i+2}}{\partial x_i \partial x_{i+1}} = 0$$

(6)

$$rot(\nabla f) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_{i+1}} (\nabla f_{i+2}) - \frac{\partial}{\partial x_{i+2}} (\nabla f_{i+1}) \right) =$$

$$\sum_{i=1}^{n} \left[\frac{\partial}{\partial x_{i+1}} \left(\frac{\partial f}{\partial x_{i+2}} \right) - \frac{\partial}{\partial x_{i+2}} \left(\frac{\partial f}{\partial x_{i+1}} \right) \right] = \sum_{i=1}^{n} \left(\frac{\partial^2 f}{\partial x_{i+1} \partial x_{i+2}} - \frac{\partial^2 f}{\partial x_{i+2} \partial x_{i+1}} \right) = 0$$

■

Pentru cazul când $n = 3$ avem :

$$rot(\nabla f) = \left(\frac{\partial \nabla f_3}{\partial x_2} - \frac{\partial \nabla f_2}{\partial x_3} \right) e_1 + \left(\frac{\partial \nabla f_3}{\partial x_1} - \frac{\partial \nabla f_1}{\partial x_3} \right) e_2 + \left(\frac{\partial \nabla f_2}{\partial x_1} - \frac{\partial \nabla f_1}{\partial x_2} \right) e_3 =$$

$$\left(\frac{\partial}{\partial x_2} \frac{\partial f}{\partial x_3} - \frac{\partial}{\partial x_3} \frac{\partial f}{\partial x_2} \right) e_1 + \left(\frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_3} - \frac{\partial}{\partial x_3} \frac{\partial f}{\partial x_1} \right) e_2 + \left(\frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_2} - \frac{\partial}{\partial x_2} \frac{\partial f}{\partial x_1} \right) e_3 =$$

$$\left(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_2} \right) e_1 + \left(\frac{\partial^2 f}{\partial x_1 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_1} \right) e_2 + \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1} \right) e_3 =$$

$$0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 = 0$$