

# Baze geometrice - 22 noiembrie

**Obs**  $f \in \text{End}(V)$  este diagonalizabil

**Obs**  $w \perp v_k, \forall k=1, n-1$  i.e.  $\langle w, v_k \rangle = 0, \forall k$

$k=1, n-1$

**Obs** Referul  $\{v_1, \dots, v_{n-1}, w\}$  este reper pozitiv orientat

**Obs** Determinant formal. Fie  $R = \{e_1, \dots, e_n\}$

reper ortonormat

$$w = \begin{pmatrix} v_1^1 & v_1^2 & \dots & v_1^{n-1} \\ \vdots & \vdots & & \vdots \\ v_{n-1}^1 & v_{n-1}^2 & \dots & v_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_{n-1} \end{pmatrix} = \begin{pmatrix} v_1^1 & v_1^2 & \dots & v_1^{n-1} \\ \vdots & \vdots & & \vdots \\ v_{n-1}^1 & v_{n-1}^2 & \dots & v_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

caz particular  $(\mathbb{R}^3, g_0)$  sp. vectorial euclidian, cu str. canonic.

$S = \{x, y, z\} \subset \mathbb{R}^3$  ;  $z = x \times y$  produs vectorial

1)  $S$  este SLD, at  $z = 0 \in \mathbb{R}^3$

2)  $S$  este SLI, at

$$\|z\|^2 = \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix}$$

$$\langle z, x \rangle = 0, \langle z, y \rangle = 0$$

3)  $\{x, y, z\}$  reper pozitiv orientat

**Def**  $R = \{e_1, e_2, e_3\}$  reper canonic (ortonormat)

$$\text{Obs } z = x \times y = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix}$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$x_1 y_2 - x_2 y_1$$

**Def** (produsul mixt)

$$z \wedge x \wedge y = \langle z, x \times y \rangle = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} =$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x \wedge y \wedge z$$

**Prop** 1)  $x \times y = -y \times x$

2)  $(x \times y) \times z = \langle x, z \rangle y - \langle y, z \rangle x$

3) Id. Jacobi:  $(x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$

•  $f$

**Obs**  $g: V \times V \rightarrow \mathbb{K}$  forma biliniară

•  $g_1: V \rightarrow V^*$

$w \rightarrow g(\cdot, w)$

$g_1(w): V \rightarrow \mathbb{K}, g_1(w)(u) = g(u, w)$

•  $g_2: V \rightarrow V^*$

$u \rightarrow g(u, \cdot)$

$g_2(u): V \rightarrow \mathbb{K}, g_2(u)(w) = g(u, w) \forall u, w \in V$

**Spatii vectoriale euclidiene**

$(V, +, \cdot)$  sp. vect. real,  $g: V \times V \rightarrow \mathbb{R}$  forma biliniară  $\rightarrow$  simetrică  
 $\rightarrow$  pozitiv definită

•  $g$  s.m. produs scalar

•  $(V, g) = (E, g) = (E, \langle \cdot, \cdot \rangle) = (E, (\cdot, \cdot))$  spatiu vectorial euclidian real

$E \subset (\mathbb{R}^n, g_0)$   $g_0: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, g_0(x, y) = x_1 y_1 + \dots + x_n y_n$  produs scalar canonic,  $\forall x, y \in \mathbb{R}^n$

•  $R = \{e_1, \dots, e_n\}$  reper în  $V$

a) ortonormal  $\Leftrightarrow \langle e_i, e_j \rangle = 0, \forall i \neq j$

b) ortonormat  $\Leftrightarrow \langle e_i, e_j \rangle = \delta_{ij}$

•  $R = \{e_1, \dots, e_n\} \xrightarrow{A} R' = \{e'_1, \dots, e'_n\}$  repere ortonormate

$$e'_i = \sum_{j=1}^n a_{ij} e_j, \quad i=1, \dots, n$$

$$A \in O(n) \Leftrightarrow A \cdot A^T = I_n$$

$S \subset R, R'$  sunt la fel orientate:  $A \in SO(n)$

**Produs vectorial**

$(E, \langle \cdot, \cdot \rangle)$  sp. vectorial euclidian real,  $\dim E = n$

$S = \{v_1, \dots, v_{n-1}\}$  sistem de vectori din  $E$ .

Definim vectorul  $w = v_1 \times \dots \times v_{n-1}$  numit produs vectorial mixt

1) Dacă  $S$  este SLD, at,  $w = 0$

2) Dacă  $S$  este SLI, atunci

$$\|w\|^2 = \begin{vmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \dots & \langle v_1, v_{n-1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle v_{n-1}, v_1 \rangle & \langle v_{n-1}, v_2 \rangle & \dots & \langle v_{n-1}, v_{n-1} \rangle \end{vmatrix}$$

$\text{OBS } \|x\|^2 = \|x\| \cdot \|y\| \cdot \cos \varphi = \frac{| \langle x, y \rangle |}{\|y\|} \Rightarrow \|x\| \cdot \|y\| \cdot \cos \varphi = \frac{| \langle x, y \rangle |}{\|y\|}$   
 $= \|x\|^2 \cdot \|y\|^2 \cdot \sin^2 \varphi$   
 $\|x \times y\| = \|x\| \cdot \|y\| \cdot \sin \varphi$   
 A paralelogram  
 aplicatii  
 1)  $A_{A_1 A_2 A_3} = \frac{1}{2} \| \overrightarrow{A_1 A_2} \times \overrightarrow{A_1 A_3} \| = \frac{1}{2} \left\| \begin{pmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{pmatrix} \right\|$   
 $= \frac{1}{2} \sqrt{ \begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_3-x_1 & z_3-z_1 \end{vmatrix}^2 + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}^2 }$   
 2)  $\{u, v, w\}$  sistem linear independent (LI)  $\Rightarrow \{u, v, w\}$  reprezintă în  $\mathbb{R}^3$   
 $\hat{e} = \{u, v, u \times v\}$  reprezintă pozitiv orientat  
 $R, R'$  sunt la fel orientate  $u \times v$   
 $\hat{V}_{\text{paralelipiped}} = \|u \times v\| \cdot \|w\| \cos \alpha = \langle w, u \times v \rangle = w_1 u_2 v_3 - w_2 u_1 v_3 + w_3 u_1 v_2 - w_4 u_2 v_3 + w_5 u_1 v_4 - w_6 u_2 v_4 + w_7 u_3 v_4 - w_8 u_1 v_5 + w_9 u_2 v_5 - w_{10} u_3 v_5 + w_{11} u_4 v_5 - w_{12} u_1 v_6 + w_{13} u_2 v_6 - w_{14} u_3 v_6 + w_{15} u_4 v_6 - w_{16} u_1 v_7 + w_{17} u_2 v_7 - w_{18} u_3 v_7 + w_{19} u_4 v_7 - w_{20} u_1 v_8 + w_{21} u_2 v_8 - w_{22} u_3 v_8 + w_{23} u_4 v_8 - w_{24} u_1 v_9 + w_{25} u_2 v_9 - w_{26} u_3 v_9 + w_{27} u_4 v_9 - w_{28} u_1 v_{10} + w_{29} u_2 v_{10} - w_{30} u_3 v_{10} + w_{31} u_4 v_{10} - w_{32} u_1 v_{11} + w_{33} u_2 v_{11} - w_{34} u_3 v_{11} + w_{35} u_4 v_{11} - w_{36} u_1 v_{12} + w_{37} u_2 v_{12} - w_{38} u_3 v_{12} + w_{39} u_4 v_{12} - w_{40} u_1 v_{13} + w_{41} u_2 v_{13} - w_{42} u_3 v_{13} + w_{43} u_4 v_{13} - w_{44} u_1 v_{14} + w_{45} u_2 v_{14} - w_{46} u_3 v_{14} + w_{47} u_4 v_{14} - w_{48} u_1 v_{15} + w_{49} u_2 v_{15} - w_{50} u_3 v_{15} + w_{51} u_4 v_{15} - w_{52} u_1 v_{16} + w_{53} u_2 v_{16} - w_{54} u_3 v_{16} + w_{55} u_4 v_{16} - w_{56} u_1 v_{17} + w_{57} u_2 v_{17} - w_{58} u_3 v_{17} + w_{59} u_4 v_{17} - w_{60} u_1 v_{18} + w_{61} u_2 v_{18} - w_{62} u_3 v_{18} + w_{63} u_4 v_{18} - w_{64} u_1 v_{19} + w_{65} u_2 v_{19} - w_{66} u_3 v_{19} + w_{67} u_4 v_{19} - w_{68} u_1 v_{20} + w_{69} u_2 v_{20} - w_{70} u_3 v_{20} + w_{71} u_4 v_{20} - w_{72} u_1 v_{21} + w_{73} u_2 v_{21} - w_{74} u_3 v_{21} + w_{75} u_4 v_{21} - w_{76} u_1 v_{22} + w_{77} u_2 v_{22} - w_{78} u_3 v_{22} + w_{79} u_4 v_{22} - w_{80} u_1 v_{23} + w_{81} u_2 v_{23} - w_{82} u_3 v_{23} + w_{83} u_4 v_{23} - w_{84} u_1 v_{24} + w_{85} u_2 v_{24} - w_{86} u_3 v_{24} + w_{87} u_4 v_{24} - w_{88} u_1 v_{25} + w_{89} u_2 v_{25} - w_{90} u_3 v_{25} + w_{91} u_4 v_{25} - w_{92} u_1 v_{26} + w_{93} u_2 v_{26} - w_{94} u_3 v_{26} + w_{95} u_4 v_{26} - w_{96} u_1 v_{27} + w_{97} u_2 v_{27} - w_{98} u_3 v_{27} + w_{99} u_4 v_{27} - w_{100} u_1 v_{28} + w_{101} u_2 v_{28} - w_{102} u_3 v_{28} + w_{103} u_4 v_{28} - w_{104} u_1 v_{29} + w_{105} u_2 v_{29} - w_{106} u_3 v_{29} + w_{107} u_4 v_{29} - w_{108} u_1 v_{30} + w_{109} u_2 v_{30} - w_{110} u_3 v_{30} + w_{111} u_4 v_{30} - w_{112} u_1 v_{31} + w_{113} u_2 v_{31} - w_{114} u_3 v_{31} + w_{115} u_4 v_{31} - w_{116} u_1 v_{32} + w_{117} u_2 v_{32} - w_{118} u_3 v_{32} + w_{119} u_4 v_{32} - w_{120} u_1 v_{33} + w_{121} u_2 v_{33} - w_{122} u_3 v_{33} + w_{123} u_4 v_{33} - w_{124} u_1 v_{34} + w_{125} u_2 v_{34} - w_{126} u_3 v_{34} + w_{127} u_4 v_{34} - w_{128} u_1 v_{35} + w_{129} u_2 v_{35} - w_{130} u_3 v_{35} + w_{131} u_4 v_{35} - w_{132} u_1 v_{36} + w_{133} u_2 v_{36} - w_{134} u_3 v_{36} + w_{135} u_4 v_{36} - w_{136} u_1 v_{37} + w_{137} u_2 v_{37} - w_{138} u_3 v_{37} + w_{139} u_4 v_{37} - w_{140} u_1 v_{38} + w_{141} u_2 v_{38} - w_{142} u_3 v_{38} + w_{143} u_4 v_{38} - w_{144} u_1 v_{39} + w_{145} u_2 v_{39} - w_{146} u_3 v_{39} + w_{147} u_4 v_{39} - w_{148} u_1 v_{40} + w_{149} u_2 v_{40} - w_{150} u_3 v_{40} + w_{151} u_4 v_{40} - w_{152} u_1 v_{41} + w_{153} u_2 v_{41} - w_{154} u_3 v_{41} + w_{155} u_4 v_{41} - w_{156} u_1 v_{42} + w_{157} u_2 v_{42} - w_{158} u_3 v_{42} + w_{159} u_4 v_{42} - w_{160} u_1 v_{43} + w_{161} u_2 v_{43} - w_{162} u_3 v_{43} + w_{163} u_4 v_{43} - w_{164} u_1 v_{44} + w_{165} u_2 v_{44} - w_{166} u_3 v_{44} + w_{167} u_4 v_{44} - w_{168} u_1 v_{45} + w_{169} u_2 v_{45} - w_{170} u_3 v_{45} + w_{171} u_4 v_{45} - w_{172} u_1 v_{46} + w_{173} u_2 v_{46} - w_{174} u_3 v_{46} + w_{175} u_4 v_{46} - w_{176} u_1 v_{47} + w_{177} u_2 v_{47} - w_{178} u_3 v_{47} + w_{179} u_4 v_{47} - w_{180} u_1 v_{48} + w_{181} u_2 v_{48} - w_{182} u_3 v_{48} + w_{183} u_4 v_{48} - w_{184} u_1 v_{49} + w_{185} u_2 v_{49} - w_{186} u_3 v_{49} + w_{187} u_4 v_{49} - w_{188} u_1 v_{50} + w_{189} u_2 v_{50} - w_{190} u_3 v_{50} + w_{191} u_4 v_{50} - w_{192} u_1 v_{51} + w_{193} u_2 v_{51} - w_{194} u_3 v_{51} + w_{195} u_4 v_{51} - w_{196} u_1 v_{52} + w_{197} u_2 v_{52} - w_{198} u_3 v_{52} + w_{199} u_4 v_{52} - w_{200} u_1 v_{53} + w_{201} u_2 v_{53} - w_{202} u_3 v_{53} + w_{203} u_4 v_{53} - w_{204} u_1 v_{54} + w_{205} u_2 v_{54} - w_{206} u_3 v_{54} + w_{207} u_4 v_{54} - w_{208} u_1 v_{55} + w_{209} u_2 v_{55} - w_{210} u_3 v_{55} + w_{211} u_4 v_{55} - w_{212} u_1 v_{56} + w_{213} u_2 v_{56} - w_{214} u_3 v_{56} + w_{215} u_4 v_{56} - w_{216} u_1 v_{57} + w_{217} u_2 v_{57} - w_{218} u_3 v_{57} + w_{219} u_4 v_{57} - w_{220} u_1 v_{58} + w_{221} u_2 v_{58} - w_{222} u_3 v_{58} + w_{223} u_4 v_{58} - w_{224} u_1 v_{59} + w_{225} u_2 v_{59} - w_{226} u_3 v_{59} + w_{227} u_4 v_{59} - w_{228} u_1 v_{60} + w_{229} u_2 v_{60} - w_{230} u_3 v_{60} + w_{231} u_4 v_{60} - w_{232} u_1 v_{61} + w_{233} u_2 v_{61} - w_{234} u_3 v_{61} + w_{235} u_4 v_{61} - w_{236} u_1 v_{62} + w_{237} u_2 v_{62} - w_{238} u_3 v_{62} + w_{239} u_4 v_{62} - w_{240} u_1 v_{63} + w_{241} u_2 v_{63} - w_{242} u_3 v_{63} + w_{243} u_4 v_{63} - w_{244} u_1 v_{64} + w_{245} u_2 v_{64} - w_{246} u_3 v_{64} + w_{247} u_4 v_{64} - w_{248} u_1 v_{65} + w_{249} u_2 v_{65} - w_{250} u_3 v_{65} + w_{251} u_4 v_{65} - w_{252} u_1 v_{66} + w_{253} u_2 v_{66} - w_{254} u_3 v_{66} + w_{255} u_4 v_{66} - w_{256} u_1 v_{67} + w_{257} u_2 v_{67} - w_{258} u_3 v_{67} + w_{259} u_4 v_{67} - w_{260} u_1 v_{68} + w_{261} u_2 v_{68} - w_{262} u_3 v_{68} + w_{263} u_4 v_{68} - w_{264} u_1 v_{69} + w_{265} u_2 v_{69} - w_{266} u_3 v_{69} + w_{267} u_4 v_{69} - w_{268} u_1 v_{70} + w_{269} u_2 v_{70} - w_{270} u_3 v_{70} + w_{271} u_4 v_{70} - w_{272} u_1 v_{71} + w_{273} u_2 v_{71} - w_{274} u_3 v_{71} + w_{275} u_4 v_{71} - w_{276} u_1 v_{72} + w_{277} u_2 v_{72} - w_{278} u_3 v_{72} + w_{279} u_4 v_{72} - w_{280} u_1 v_{73} + w_{281} u_2 v_{73} - w_{282} u_3 v_{73} + w_{283} u_4 v_{73} - w_{284} u_1 v_{74} + w_{285} u_2 v_{74} - w_{286} u_3 v_{74} + w_{287} u_4 v_{74} - w_{288} u_1 v_{75} + w_{289} u_2 v_{75} - w_{290} u_3 v_{75} + w_{291} u_4 v_{75} - w_{292} u_1 v_{76} + w_{293} u_2 v_{76} - w_{294} u_3 v_{76} + w_{295} u_4 v_{76} - w_{296} u_1 v_{77} + w_{297} u_2 v_{77} - w_{298} u_3 v_{77} + w_{299} u_4 v_{77} - w_{300} u_1 v_{78} + w_{301} u_2 v_{78} - w_{302} u_3 v_{78} + w_{303} u_4 v_{78} - w_{304} u_1 v_{79} + w_{305} u_2 v_{79} - w_{306} u_3 v_{79} + w_{307} u_4 v_{79} - w_{308} u_1 v_{80} + w_{309} u_2 v_{80} - w_{310} u_3 v_{80} + w_{311} u_4 v_{80} - w_{312} u_1 v_{81} + w_{313} u_2 v_{81} - w_{314} u_3 v_{81} + w_{315} u_4 v_{81} - w_{316} u_1 v_{82} + w_{317} u_2 v_{82} - w_{318} u_3 v_{82} + w_{319} u_4 v_{82} - w_{320} u_1 v_{83} + w_{321} u_2 v_{83} - w_{322} u_3 v_{83} + w_{323} u_4 v_{83} - w_{324} u_1 v_{84} + w_{325} u_2 v_{84} - w_{326} u_3 v_{84} + w_{327} u_4 v_{84} - w_{328} u_1 v_{85} + w_{329} u_2 v_{85} - w_{330} u_3 v_{85} + w_{331} u_4 v_{85} - w_{332} u_1 v_{86} + w_{333} u_2 v_{86} - w_{334} u_3 v_{86} + w_{335} u_4 v_{86} - w_{336} u_1 v_{87} + w_{337} u_2 v_{87} - w_{338} u_3 v_{87} + w_{339} u_4 v_{87} - w_{340} u_1 v_{88} + w_{341} u_2 v_{88} - w_{342} u_3 v_{88} + w_{343} u_4 v_{88} - w_{344} u_1 v_{89} + w_{345} u_2 v_{89} - w_{346} u_3 v_{89} + w_{347} u_4 v_{89} - w_{348} u_1 v_{90} + w_{349} u_2 v_{90} - w_{350} u_3 v_{90} + w_{351} u_4 v_{90} - w_{352} u_1 v_{91} + w_{353} u_2 v_{91} - w_{354} u_3 v_{91} + w_{355} u_4 v_{91} - w_{356} u_1 v_{92} + w_{357} u_2 v_{92} - w_{358} u_3 v_{92} + w_{359} u_4 v_{92} - w_{360} u_1 v_{93} + w_{361} u_2 v_{93} - w_{362} u_3 v_{93} + w_{363} u_4 v_{93} - w_{364} u_1 v_{94} + w_{365} u_2 v_{94} - w_{366} u_3 v_{94} + w_{367} u_4 v_{94} - w_{368} u_1 v_{95} + w_{369} u_2 v_{95} - w_{370} u_3 v_{95} + w_{371} u_4 v_{95} - w_{372} u_1 v_{96} + w_{373} u_2 v_{96} - w_{374} u_3 v_{96} + w_{375} u_4 v_{96} - w_{376} u_1 v_{97} + w_{377} u_2 v_{97} - w_{378} u_3 v_{97} + w_{379} u_4 v_{97} - w_{380} u_1 v_{98} + w_{381} u_2 v_{98} - w_{382} u_3 v_{98} + w_{383} u_4 v_{98} - w_{384} u_1 v_{99} + w_{385} u_2 v_{99} - w_{386} u_3 v_{99} + w_{387} u_4 v_{99} - w_{388} u_1 v_{100} + w_{389} u_2 v_{100} - w_{390} u_3 v_{100} + w_{391} u_4 v_{100}$

**Teorema (Cauchy-Bunyakowski-Schwarz)**

$(E, \langle \cdot, \cdot \rangle)$  spațiu euclidian real,  $x, y \in E$   
 $\Rightarrow |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$   
 $\Leftrightarrow \{x, y\}$  sunt s.l.  
 sau  
 1. dacă  $x=0$  sau  $y=0$   
 $|\langle x, y \rangle| = 0$   
 $\|x\| \cdot \|y\| = 0 \Rightarrow |\langle x, y \rangle| = \|x\| \cdot \|y\|$   
 2. dacă  $x \neq 0$  și  $y \neq 0$   
 - fix  $\lambda \in \mathbb{R}$   
 $\langle x + \lambda y, x + \lambda y \rangle \geq 0, \forall \lambda \in \mathbb{R}$   
 $\lambda^2 \|y\|^2 + 2\lambda \langle x, y \rangle + \|x\|^2 \geq 0, \forall \lambda \in \mathbb{R}$   
 $\Delta \leq 0 \Rightarrow 4\langle x, y \rangle^2 - 4\|x\|^2 \|y\|^2 \leq 0 \Rightarrow$   
 $\Rightarrow |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$   
 $\Leftrightarrow |\langle x, y \rangle| = \|x\| \cdot \|y\|$ . Cum  $\{x, y\}$  este s.l.  
 $\exists \lambda \in \mathbb{R}$  aș.  $\langle x + \lambda y, x + \lambda y \rangle = 0$  def  
 $x + \lambda_0 y = 0 \Rightarrow \{x, y\}$  s.l.  
 $\Leftrightarrow \{x, y\}$  este s.l. sau  $|\langle x, y \rangle| = \|x\| \cdot \|y\|$   
 $\exists a \in \mathbb{R}^n$  aș.  $y = ax$   $|\langle x, y \rangle| = |\langle x, ax \rangle| = |a| \|x\|^2$   
 $\|ax\| = \sqrt{\langle ax, ax \rangle} = \sqrt{a^2 \langle x, x \rangle} = |a| \|x\|$

**OBS** Dacă un produs scalar  $\langle \cdot, \cdot \rangle$  se declare un spațiu ortogonal

OBS  $(V, \langle \cdot, \cdot \rangle_v) \rightarrow (V^*, \langle \cdot, \cdot \rangle_{v^*})$   
 $R = \{e_1, \dots, e_n\}$  reprezintă ortogonal în  $V$   
 $R^* = \{e_1^*, \dots, e_n^*\}$  reprezintă ortogonal în  $V^*$   
 $f_i^*: V \rightarrow \mathbb{R}$  apl. liniară  
 $f_i^*(e_j) = \delta_{ij}$   
 $\langle f, g \rangle_{v^*} = \langle \sum_{i=1}^n f_i e_i^*, \sum_{j=1}^n g_j e_j^* \rangle = \sum_{i,j=1}^n f_i g_j \langle e_i^*, e_j^* \rangle = \sum_{i,j=1}^n f_i g_j \delta_{ij} = \sum_{i=1}^n f_i g_i$   
 $\langle \cdot, \cdot \rangle_{v^*}$  forme biliniară sim. + poz. def  
 a)  $\langle f, g \rangle_{v^*} = \langle g, f \rangle_{v^*}$   
 b)  $\langle f, f \rangle_{v^*} = \sum_{i=1}^n f_i^2$   
 $\langle f, f \rangle_{v^*} > 0, \forall f \neq 0, \langle f, f \rangle_{v^*} = 0 \Leftrightarrow f_i = 0 \Leftrightarrow f = 0$   
 $\|f\|_{v^*} = \sqrt{\langle f, f \rangle_{v^*}}$

**Example**  
 $(\mathbb{R}^3, g_0)$  sp. v. e., str. canonic  
 $u = (1, 2, -1)$   
 $v = (0, 1, 2)$   
 $w = (1, 1, 2)$

a)  $u \times v$   
 b)  $w \wedge u \wedge v$   
 c)  $\frac{4}{3} = ?$  aș.  $\{u, v, z\}$  reprezintă ortogonal, pozitiv orientat  
 80  
 a)  $\det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix} = 2 \Rightarrow \{u, v\}$  sunt l.i.  
 $u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = (5, -2, 1)$   
 c)  $w \wedge u \wedge v = \langle w, u \times v \rangle = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 5 - 2 + 1 = 4$   
 $\langle (1, 1, 2), (5, -2, 1) \rangle = 5 - 2 + 1 = 4 = \sqrt{\text{paralelip. } u, v, w}$



**Teoremă** (procedeu de diagonalizare Gram-Schmidt)

$\det A^{-1} = 1 \Rightarrow \det A = \frac{1}{\det A^{-1}} = 1$

$(E, \langle \cdot, \cdot \rangle)$  sp euclidian real

$B = \begin{pmatrix} \frac{1}{\|e_1\|} & 0 & 0 \\ 0 & \frac{1}{\|e_2\|} & \dots \\ 0 & \dots & \frac{1}{\|e_n\|} \end{pmatrix} \det B = \frac{1}{\|e_1\| \cdot \dots \cdot \|e_n\|} > 0$

$R = \{f_1, \dots, f_m\}$  reper în  $E \Rightarrow \exists R' = \{e_1, \dots, e_n\}$  reper ortogonal în  $E$  at  $\exists \{f_1, \dots, f_m\} = \{p_1 e_1, \dots, p_m e_m\}, \forall i = \overline{1, m}$

$R, R', R''$  sunt repere la fel orientate

Solu aplicăm o mărime inductivă

Def  $(E, \langle \cdot, \cdot \rangle)$  sp vectorial euclidian real.

- a)  $x \in E$   
 $x^\perp = \{y \in E / \langle x, y \rangle = 0\}$
- b)  $U \subseteq E$  subsp  $u$   
 $U^\perp = \{x \in E / \langle x, y \rangle = 0, \forall y \in U\}$

Prop a)  $x \in E \Rightarrow x^\perp \subseteq E$  subspațiu vectorial  
b)  $U \subseteq E$  subsp rect  $\Rightarrow U^\perp \subseteq E$  subspațiu vectorial

demonstratie

- a) Fie  $u, v \in x^\perp \Rightarrow \alpha u + \beta v \in x^\perp$   
 $\langle \alpha u + \beta v, x \rangle = \alpha \langle u, x \rangle + \beta \langle v, x \rangle = 0 + 0 = 0$
- b) Fie  $u, v \in U^\perp \Rightarrow \alpha u + \beta v \in U^\perp$   
 $x \in U$  arbitrar  
 $\langle \alpha u + \beta v, x \rangle = \alpha \langle u, x \rangle + \beta \langle v, x \rangle = 0 + 0 = 0$

Prop  
 $U, W \subseteq E$  subsp rect  
 $U \subseteq W \Rightarrow W^\perp \subseteq U^\perp$

Solu  
fie  $x \in W^\perp \Rightarrow \langle x, y \rangle = 0, \forall y \in W$   
 $U \subseteq W \Rightarrow \langle x, z \rangle = 0, \forall z \in U \Rightarrow x \in U^\perp \Rightarrow W^\perp \subseteq U^\perp$

**Teoremă**  $(E, \langle \cdot, \cdot \rangle)$  sp rect euclidian real

$U \subseteq E$  subsp rect  
 $\Rightarrow E = U \oplus U^\perp$   
 $(U^\perp = \text{complementul ortogonal al lui } U)$

Solu  
fie  $x \in U \cap U^\perp \Rightarrow x \in U$   
 $x \in U^\perp \Rightarrow \langle x, x \rangle = 0 \xRightarrow{f.o.z} x = 0$

$U, U^\perp \subseteq E$  subsp rect  $\Rightarrow U \oplus U^\perp \subseteq E$  subsp rect  
Sau că  $E \subseteq U \oplus U^\perp$

fie  $R = \{e_1, \dots, e_k\}$  reper ortogonal în  $U$

fie  $v \in E$   
 $v' = v - \sum_{i=1}^k \langle v, e_i \rangle e_i$   
sau că  $v' \in U^\perp$

$\langle v', e_i \rangle = \langle v, e_i \rangle - \sum_{j=1}^k \langle v, e_j \rangle \langle e_j, e_i \rangle$   
 $\frac{d}{d_1} \langle e_i, e_i \rangle = \langle v, e_i \rangle - \langle v, e_i \rangle = 0$

$R' \xrightarrow{A} R$   
 $A^{-1} = \begin{pmatrix} \frac{\langle f_1, e_1 \rangle}{\langle e_1, e_1 \rangle} & \dots & \frac{\langle f_1, e_n \rangle}{\langle e_n, e_n \rangle} \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$

$f_1 = e_1$   
 $e_2 = f_2 + \alpha e_1$   
 $\langle e_1, e_2 \rangle = 0 \Rightarrow \langle f_2 + \alpha e_1, e_1 \rangle = 0$   
 $\langle f_2, e_1 \rangle + \alpha \langle e_1, e_1 \rangle = 0$   
 $\alpha = - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle}$   
 $e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1$

$f_1 = e_1$   
 $f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2$   
 $\exists \{f_1, f_2\} = \{p_1 e_1, e_2\}$   
Prindem  $P_k : \{e_1, \dots, e_k\}$  sistem ortogonal at  
 $\exists \{f_1, \dots, f_k\} = \{p_1 e_1, \dots, p_k e_k\}, \forall i = \overline{1, k}$

Construim  $e_{k+1} = f_{k+1} + \sum_{i=1}^k \alpha_{k+1,i} e_i$   
 $\langle e_{k+1}, e_j \rangle = 0, \forall j = \overline{1, k}$   
 $\langle f_{k+1}, e_j \rangle + \sum_{i=1}^k \alpha_{k+1,i} \langle e_i, e_j \rangle = 0$   
 $\alpha_{k+1,j} = - \frac{\langle f_{k+1}, e_j \rangle}{\langle e_j, e_j \rangle}, \forall j = \overline{1, k}$   
 $e_{k+1} = f_{k+1} - \sum_{j=1}^k \frac{\langle f_{k+1}, e_j \rangle}{\langle e_j, e_j \rangle} e_j$

$f_1 = e_1$   
 $f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2$   
 $\vdots$   
 $f_i = \frac{\langle f_i, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle f_i, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 + \dots + \frac{\langle f_i, e_{i-1} \rangle}{\langle e_{i-1}, e_{i-1} \rangle} e_{i-1} + e_i$   
 $\vdots$   
 $f_{k+1} = \frac{\langle f_{k+1}, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots + \frac{\langle f_{k+1}, e_k \rangle}{\langle e_k, e_k \rangle} e_k + e_{k+1}$

$\exists \{f_1, \dots, f_k\} = \{p_1 e_1, \dots, p_k e_k\}, \forall i = \overline{1, k+1}$

Construim recursiv  
 $f_n = \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots + \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1} + e_n$  și arădem

$\exists \{e_1, \dots, e_n\} = \{p_1 f_1, \dots, p_n f_n\}, i = \overline{1, n}$   
 $R = \{f_1, \dots, f_n\} \xrightarrow{A} R' = \{e_1, \dots, e_n\} \xrightarrow{B} R'' = \left\{ \frac{e_1}{\|e_1\|}, \dots, \frac{e_n}{\|e_n\|} \right\}$   
reper  $B$  reper ortogonal reper ortonormat

$f_1 = e_1$   
 $f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + f_2 e_2$   
 $\vdots$   
 $f_n = \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots + \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1} + e_n$

$$\langle v, e_k \rangle = \langle v, e_k \rangle - \sum_{i=1}^k \langle v, e_i \rangle \langle e_i, e_k \rangle = 0$$

$$\underbrace{\langle v, e_k \rangle}_{\langle v, e_k \rangle} - \sum_{i=1}^k \underbrace{\langle v, e_i \rangle \langle e_i, e_k \rangle}_{d_{ik}} = 0$$

$$\Rightarrow \langle v, e_j \rangle = 0, \forall j = \overline{1, k} \Rightarrow \langle v, x \rangle = 0, \forall x \in U^\perp$$

$$v = \sum_{i=1}^k \langle v, e_i \rangle e_i + \underbrace{v'}_{v' \in U^\perp} \in U \oplus U^\perp$$

$$E = U \oplus U^\perp$$

Example

1)  $(\mathbb{R}^3, g_0)$  sp. vect. e. real cu str. canonică,  
 $x = (1, 2, -1)$

a)  $x^\perp$

b) Rep. ort. normat în  $x^\perp$

Sol.

$$x^\perp = \{y \in \mathbb{R}^3 / \langle x, y \rangle = 0\} = \{y \in \mathbb{R}^3 / y_1 + 2y_2 - y_3 = 0\}$$

$$y = (y_1, y_2, y_3) \quad y_3 = y_1 + 2y_2$$

$$(y_1, y_2, y_3) = (y_1, y_2, y_1 + 2y_2) = y_1 \underbrace{(1, 0, 1)}_{f_1} + y_2 \underbrace{(0, 1, 2)}_{f_2}$$

deci  $x^\perp =$

$$R = \{f_1, f_2\} \text{ rep. în } x^\perp$$

Aplicăm proced. de ortogonalizare Gram-Schmidt!

$$e_1 = f_1 = (1, 0, 1)$$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (0, 1, 2) - \frac{2}{2} \cdot (1, 0, 1) = (-1, 1, 1)$$

$$\langle f_2, e_1 \rangle = \langle f_2, f_1 \rangle = \langle (0, 1, 2), (1, 0, 1) \rangle = 2$$

$$\langle e_1, e_1 \rangle = \langle (1, 0, 1), (1, 0, 1) \rangle = 2$$

$$\langle e_1, e_1 \rangle$$

$$e_2 = (0, 1, 2) - (1, 0, 1) = (-1, 1, 1)$$

$$R' = \{e_1 = (1, 0, 1), e_2 = (-1, 1, 1)\} \text{ rep. ortogonal}$$

$$R'' = \{e_1' = \frac{1}{\sqrt{2}}(1, 0, 1), e_2' = \frac{1}{\sqrt{3}}(-1, 1, 1)\} \text{ rep. ort. normat în } x^\perp$$

$$\mathbb{R}^3 = \langle x \rangle \oplus x^\perp$$

$$\tilde{R} = \left\{ \underbrace{\frac{1}{\sqrt{6}}(1, 2, -1)}_{R_1}, \underbrace{\frac{1}{\sqrt{2}}(1, 0, 1)}_{R_2}, \underbrace{\frac{1}{\sqrt{3}}(-1, 1, 1)}_{R_3} \right\}$$

$R_1$   
 $\downarrow$   
 $\langle x \rangle$

$R_2$  rep. ort. normat  
 $\downarrow$   
 $x^\perp$

2)  $(\mathbb{R}^3, g_0)$

$$U = \{x \in \mathbb{R}^3 / x_1 + x_2 + x_3 = 0\}$$

a)  $U^\perp$

b) Rep.  $R = R_1, R_2$  ort. normat în  $\mathbb{R}^3$  ar.  $R_1$  rep. ort. în  $U$   
 $R_2 \perp U^\perp$

Sol.

$$U = \{x \in \mathbb{R}^3 / \langle x, (1, 1, 1) \rangle = 0\} \text{ deci } U^\perp = \langle (1, 1, 1) \rangle$$

c) Rep. în  $U$

$$x_3 = -x_1 - x_2$$

$$(x_1, x_2, x_3) = (x_1, x_2, -x_1 - x_2) = x_1 \underbrace{(1, 0, -1)}_{f_1} + x_2 \underbrace{(0, 1, -1)}_{f_2}$$

$\{f_1, f_2\}$  rep. în  $U$

Aplicăm G-S

$$e_1 = f_1 = (1, 0, -1)$$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (0, 1, -1) - \frac{1}{2} (1, 0, -1) = (-\frac{1}{2}, 1, -\frac{1}{2}) = \frac{1}{2} (-1, 2, -1)$$

$\{e_1, e_2\}$  rep. ortogonal în  $U$

$\{\frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(-1, 2, -1)\}$  rep. ort. normat în  $U$

$$\text{OBS: } u = \alpha v, \alpha > 0, \frac{\|u\|}{\|v\|} = \frac{\alpha \|v\|}{\|v\|} = \alpha$$

$$\left\{ \frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(-1, 2, -1), \frac{1}{\sqrt{3}}(1, 1, 1) \right\} \text{ rep. în } \mathbb{R}^3$$

Temă

ex. 1.  $(\mathbb{R}^3, g)$  sp. vect.

$$g: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}, g = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 6 & 3 \\ 0 & 3 & 3 \end{pmatrix}$$

formă biliniară

matricea asociată lui  $g$  în raport cu rep. canonic

ad  $(\mathbb{R}^3, g)$  sp. vectorial euclidian

$$a) u = (1, 0, 1)$$

$$u^\perp = \{x \in \mathbb{R}^3 / g(x, u) = 0\} = ?$$

c) Rep. ort. normat în  $U^\perp$

ex. 2.  $Q: \mathbb{R}^3 \rightarrow \mathbb{R}, Q(x) = 2x_1x_2 - 8x_1x_3 + 8x_2x_3$

formă pătratică

a)  $Q = ?$  mat. asociată lui  $Q$  în raport cu rep. canonic

b)  $Q$  este definit formă polară  $g: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$  asociată lui  $Q$

i)  $\text{Ker } g = ?$

ii) este  $g$  nedegenerată?

c)  $Q$  reduce la formă canonică



**def**  $(E, \langle \cdot, \cdot \rangle)$  sp. v. real

$$f \in \text{End}(E)$$

$f$  m. ortogonală  $\Leftrightarrow \langle f(x), f(y) \rangle = \langle x, y \rangle, \forall x, y \in E$

**Obs.** Fie  $R = \{e_1, \dots, e_m\}$  reper. ortonormal în  $E$   
 $A =$  matricea asociată lui  $f$  în raport cu  $R$   
 $f(e_i) = \sum_{j=1}^m a_{ji} e_j$

$$\begin{aligned} \langle f(e_i), f(e_k) \rangle &= \left\langle \sum_{j=1}^m a_{ji} e_j, \sum_{k=1}^m a_{jk} e_k \right\rangle = \\ &= \sum_{j=1}^m \sum_{k=1}^m a_{ji} a_{jk} \underbrace{\langle e_j, e_k \rangle}_{\delta_{jk}} = \sum_{j=1}^m a_{ji} a_{jk} \end{aligned}$$

$$\begin{aligned} \langle f(e_i), f(e_k) \rangle &= \langle e_i, e_k \rangle = \delta_{ik} \\ \sum_{j=1}^m a_{ji} a_{jk} &= \delta_{ik}, \forall i, k = \overline{1, m} \end{aligned}$$

$A^T A = I_m \Rightarrow A \in O(m) \leftarrow$  este matrice ortogonală

**Obs.**  $f$  ortogonală  $\Leftrightarrow$  schimbare de reper. ortonormal  
 $f \in O(E) \Rightarrow A =$  matricea asoc. în rap. cu  $R = \{e_1, \dots, e_m\}$   
 $A \in O(m)$

$R = \{e_1, \dots, e_m\} \xrightarrow{A} R' = \{e'_1, \dots, e'_m\}$  reper. orton.

$$\begin{aligned} R &\xrightarrow{A} R' \\ \{e_i\}_{i=1, \dots, m} &\xrightarrow{A} \{e'_i\}_{i=1, \dots, m} \\ f(e_i) &= e'_i = \sum_{j=1}^m a_{ji} e_j \\ f &\in O(E) \end{aligned}$$

$$\begin{aligned} \langle f(e_i), f(e_k) \rangle &= \langle e'_i, e'_k \rangle = \delta_{ik} = \langle e_i, e_k \rangle \\ \Rightarrow \langle f(x), f(y) \rangle &= \langle x, y \rangle, \forall x, y \in E \end{aligned}$$