LU cu pivotare (PLU)

Să se determine factorizarea LU cu pivotare (PLU) a matricei

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

Matricea A nu admite factorizarea LU întrucât $a_{11} = 0$. De aceea, $(E_1) \leftrightarrow (E_2)$, i.e. înmulțim matricea A, la stânga, cu permutarea simplă $P^{(1)} = P_{12} := [\underline{e}^{(2)}\ \underline{e}^{(1)}\ \underline{e}^{(3)}\ \underline{e}^{(4)}]^T$:

$$P^{(1)}A = \left[\begin{array}{c|ccc} 1 & 1 & -1 & 2 \\ \hline 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{array}\right] = \begin{bmatrix} l_{11} & \underline{0} \\ \underline{L}_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & \underline{U}_{12}^T \\ \underline{0} & U_{22} \end{bmatrix}$$

- $l_{11} u_{11} = 1 \Rightarrow \boxed{l_{11} = 1}$ $\boxed{u_{11} = 1}$

- $l_{11} \underline{U}_{12} = [1 \ -1 \ 2] \Rightarrow \underline{U}_{12} = [1 \ -1 \ 2] \Rightarrow$

$$\boxed{u_{12} = 1 \quad u_{13} = -1 \quad u_{14} = 2}$$

- $\underline{L}_{21} u_{11} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \underline{L}_{21} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow$

$$\boxed{l_{21} = 0 \quad l_{31} = -1 \quad l_{41} = 1}$$

- $\underline{L}_{21} \underline{U}_{12} + L_{22} U_{22} = (P^{(1)} A)_{22}$

$L_{22} U_{22} = A_{22} - \underline{L}_{21} \underline{U}_{12} =: S_1$

$$= \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} [1 \ -1 \ 2]$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & -2 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

ie, pentru a putea continua factorizarea LU, trebuie ca liniile 1 și

3 ale complementului Schur să fie interschimbate. Prin urmare, trebuie ca în matricea $P^{(1)}A$ să interschimbăm $(E_2) \longleftrightarrow (E_4)$, ie înmulțire la stânga cu matricea permutare elementară

$$P^{(2)} = P_{24} = [\underline{e}^{(1)}\ \underline{e}^{(4)}\ \underline{e}^{(3)}\ \underline{e}^{(2)}]^T :$$

$$P^{(2)}P^{(1)}A = P^{(2)}\begin{bmatrix} l_{11} & \underline{0} \\ \underline{L}_{21} & L_{22} \end{bmatrix}\begin{bmatrix} u_{11} & \underline{U}_{12} \\ \underline{0} & U_{22} \end{bmatrix}$$

$$P^{(2)}\left[\begin{array}{c|ccc} 1 & 1 & -1 & 2 \\ \hline 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{array}\right] = P^{(2)}\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ -1 & & L_{22} & \\ 1 & & & \end{array}\right]$$

$$\left[\begin{array}{c|ccc} 1 & 1 & -1 & 2 \\ \hline 0 & & & \\ 0 & & U_{22} & \\ 0 & & & \end{array}\right] \Longrightarrow$$

$$P^{(2)}P^{(1)}A = \left[\begin{array}{c|ccc} 1 & 1 & -1 & 2 \\ \hline 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 1 & & & \\ -1 & & \underbrace{P^{(2)}L_{22}}_{=:\tilde{L}_{22}} & \\ 0 & & & \end{array}\right] \left[\begin{array}{c|ccc} 1 & 1 & -1 & 2 \\ \hline 0 & & & \\ 0 & & U_{22} & \\ 0 & & & \end{array}\right]$$

Am obtinut:

$$\left(P^{(2)}P^{(1)}A\right)_{22} = P^{(2)}\underline{L}_{21}\underline{U}_{12} + P^{(2)}L_{22}U_{22}$$

$$\Rightarrow \left(P^{(2)}L_{22}\right)U_{22} = \left(P^{(2)}P^{(1)}A\right)_{22} - \left(P^{(2)}\underline{L}_{21}\right)\underline{U}_{12}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} =: S_1$$

$$S_1 := \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 0 & 1 & 2 \\ 0 & -1 & 1 \end{array}\right] = \begin{bmatrix} l_{22} & \underline{0} \\ \underline{L}_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_{22} & \underline{U}_{23} \\ \underline{0} & U_{33} \end{bmatrix} \Rightarrow$$

$$\bullet\ l_{22} u_{22} = 1 \Rightarrow \boxed{l_{22} = 1 \quad \Big| \quad u_{22} = 1}$$

$$\bullet\ l_{22} \underline{U}_{23} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow \underline{U}_{23} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow$$

$$\boxed{u_{23} = 1 \quad \Big| \quad u_{24} = 0}$$

$$\bullet\ \underline{L}_{32} u_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{L}_{32} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\boxed{l_{32} = l_{42} = 0}$$

$$\bullet\ \underline{L}_{32} \underline{U}_{23} + L_{33} U_{33} = \left(P^{(2)} P^{(1)} A\right)_{33} \Rightarrow$$

$$L_{33} U_{33} = \left(P^{(2)} P^{(1)} A\right)_{33} - \underline{L}_{32} \underline{U}_{23} =$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} =: S_2$$

Am obținut

$$P^{(2)}P^{(1)}A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} =$$

$$= \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline -1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{L_{33}}} \\ 0 & 0 & & \end{array}\right]}_{=L} \underbrace{\left[\begin{array}{cc|cc} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{U_{33}}} \\ 0 & 0 & & \end{array}\right]}_{=U}$$

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$$S_2 := \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{33} & 0 \\ l_{43} & l_{44} \end{bmatrix}}_{L_{33}} \underbrace{\begin{bmatrix} u_{33} & u_{34} \\ 0 & u_{44} \end{bmatrix}}_{U_{33}} \Rightarrow$$

- $l_{33} u_{33} = 1 \Rightarrow \boxed{l_{33} = 1 \mid u_{33} = 1}$

- $l_{33} u_{34} = 2 \Rightarrow \boxed{u_{34} = 2}$
- $l_{43} u_{33} = -1 \Rightarrow \boxed{l_{43} = -1}$
- $l_{43} u_{34} + l_{44} u_{44} = 1 \Rightarrow$

$$l_{44} u_{44} = 1 - (-1)2 = 3 \Rightarrow \boxed{\begin{array}{c} l_{44} = 1 \\ \hline u_{44} = 3 \end{array}}$$

Am obtinut $PA = LU$, unde

$$P = P^{(2)} P^{(1)} = \left[\underline{e}^{(2)}\ \underline{e}^{(4)}\ \underline{e}^{(3)}\ \underline{e}^{(1)} \right]^T =$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix};$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$