Facultatea de Matematica si Informatica
Universitatea din Bucuresti

19 Ianuarie 2021

Examen final

Disciplina: Ecuatii cu derivate partiale
Tipul examinarii: Examen
Nume student: ____
Seria 31: Grupele 311, 312 _______________________________
Timp de lucru : 3 ore si 15 min (incluzand atasarea rezolvarilor pe Moodle)

Acest examen contine 5 probleme (toate obligatorii).

Examenul este individual. Nu uitati sa va salvati foile cu rezolvarile subiectelor intr-un singur fisier de tip PDF in timp util astfel incat sa va incadrati in cele 3 ore si 15 minute pentru incarcarea fisierului pe platforma Moodle.

Salvati fisierul PDF creat cu numele vostru (Nume_Prenume_Grupa.pdf).

Pentru elaborarea lucrarii scrise puteti folosi orice materiale ajutatoare.

Pentru redactare tineti cont de urmatoarele sugestii:

- Daca folositi o teorema fundamentala, rezultat cunoscut, etc **indicati** acest lucru si explicati cum se poate aplica rezultatul respectiv.
- **Organizati-va munca** intr-un mod coerent pentru a avea toti de castigat ! Incercati ca la crearea fisierului PDF fiecare problema sa fie redactata in ordinea aparitiei pe foaia cu subiecte. Ideal ar fi ca si subpunctele sa fie redactate in ordine. Daca nu stiti a rezolva un anumit subpunct scrieti numarul subpunctului si lasati liber.
- Raspunsurile corecte dar argumentate incomplet (din punct de vedere al calculelor/explicatiilor) vor primi punctaj partial.

Barem: P1 (2p) + P2 (1.5p)+ P3 (2p) +P4 (1.5p)+P5 (2p) + 1p oficiu= **10p** (Plus eventual BONUS acolo unde este cazul in functie de activitatea/temele din timpul semestrului).

Pentru orice nelamuriri scrieti-mi la adresa cristian.cazacu@fmi.unibuc.ro, sau lasati un mesaj pe chat-ul grupei creat pe Microsoft Teams.

Rezultatele finale vor fi postate pe Moodle si Microsoft Teams in cel mai scurt timp posibil, dar dupa proba orala.

Problema 1. *(2p).*

1). Calculati $\mathrm{div}(|x|^2 \cdot \nabla v(x))$, *unde* $v : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}$, $v(x) := |x|^{-\frac{5}{3}}$.

2). Sa se determine pentru ce valori $p \geq 1$ *are loc* $|v|^p \in L^1(B_1(0))$, *unde* $B_1(0)$ *este bila unitate din* $\mathbb{R}^4$.

3). Sa se determine pentru ce valori $p \geq 1$ *are loc* $\frac{|v(x)|^p}{|x|^2+1} \in L^1(\mathbb{R}^4 \setminus \overline{B_1(0)})$.

4). Dati exemplu de o functie strict subarmonica $(-\Delta u < 0)$ *pe* $\mathbb{R}^2$ *care sa se anuleze pe dreapta* $x + 3y = 0$.

5). Consideram functia $u : B_1(0) \setminus \{0\} \to \mathbb{R}$ *data de*

$$u(x) = \left(\ln \frac{2}{|x|}\right)^{\frac{1}{2}}, \quad x = (x_1, x_2),$$

unde $B_1(0)$ *este bila unitate din* $\mathbb{R}^2$ *centrata in origine. Aratati ca*

$$-\Delta u(x) = \frac{u(x)}{4|x|^2 \ln^2(\frac{2}{|x|})}, \quad \forall x \in B_1(0) \setminus \{0\}.$$

Problema 2. *(1.5p). Se considera problema la limita*

$$(1) \qquad \begin{cases} u_{xx}(x,y) + 2u_{yy}(x,y) = 0, & (x,y) \in (0,1) \times (0,1) \\ u(x,0) = u(x,1) = 0, & x \in (0,1),\ y \in (0,1) \\ u(0,y) = \sin(2\pi y),\ u(1,y) = e^{-2\sqrt{2}\pi} \sin(2\pi y), & y \in (0,1). \end{cases}$$

1). Determinati solutia problemei (1) *cautand-o in variabile separate sub forma* $u(x,y) = A(x)B(y)$.

*2). * Aratati (folosind eventual metoda energetica) ca* (1) *are cel mult o solutie de clasa* C^2.

Problema 3. *(2p). Consideram urmatoarea problema de tip "unde"*

$$(2) \qquad \begin{cases} u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

unde $f, g \in C^2(\mathbb{R})$ *sunt functii date.*

1). Aratati ca daca $u = u(x,t)$ *este o functie de clasa* C^2 *atunci* u *verifica*

$$(\partial_t + 2\partial_x)(u_t(x,t) - 3u_x(x,t)) = u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t),$$

pe domeniul sau de definitie.

2). Rezolvati problema cu valori initiale satisfacuta de u *in* (2) *(scrieti forma generala a lui* u*) reducand-o la rezolvarea a doua ecuatii de transport (una omogena si alta neomogena).*

3). Folosind conditiile la $t = 0$ *deduceti solutia* u *a problemei* (2) *in cazul particular* $f(x) = \sin x$ *si* $g(x) = e^{-x}$.

Problema 4. *(1.5p). Consideram problema Cauchy*

$$(3) \qquad \begin{cases} u_t(x,t) - u_{xx}(x,t) + \frac{e^t}{e^{2t}+1} u(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R}. \end{cases}$$

1). Gasiti o functie $\phi : \mathbb{R} \to \mathbb{R}$ *astfel incat functia* $v(x,t) := u(x,t)\phi(t)$ *sa verifice ecuatia caldurii*

$$(4) \qquad v_t(x,t) - v_{xx}(x,t) = 0, \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

2). Scrieti problema Cauchy verificata de v *si determinati explicit solutia problemei* (3).

Problema 5. *(2p). Fie functia* $f : [-1,1] \to \mathbb{R}$, $f(x) = |x - \frac{1}{2}|$.

1). Explicitati functia f *si faceti graficul functiei* f.

2). Sa se determine punctele de derivabilitate ale lui f *pe intervalul (-1,1).*

3). Argumentati ca $f \in H^1(-1,1)$ *si calculati norma lui* f *in* $H^1(-1,1)$ *(precizati inainte norma cu care lucrati).*

4). Determinati $\alpha \in \mathbb{R}$ *astfel incat functia* $z : (0,1) \to \mathbb{R}$, $z(x) = x^\alpha$ *sa apartina lui* $H^1(0,1)$.

*5). * Determinati* $\alpha \in \mathbb{R}$ *astfel incat functia* $z : (1,\infty) \to \mathbb{R}$, $z(x) = \frac{x^\alpha}{1+x^3}$ *sa apartina lui* $W^{1,3}(1,\infty)$.

BĂZDRĂGAN MARIA
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EXAMEN FINAL

PROBLEMA 1

1) $\operatorname{div}(|x|^2 \cdot \nabla v(x))$, $\quad v: \mathbb{R}^4 - \{0\} \to \mathbb{R}$,
$v(x) := |x|^{-5/3}$

$$\nabla v(x) = \nabla\left(|x|^{-\frac{5}{3}}\right)$$
$$= -\frac{5}{3} x |x|^{-\frac{5}{3}-2}$$
$$= -\frac{5}{3} x |x|^{-\frac{11}{3}}$$

$$\operatorname{div}(|x|^2 \cdot \nabla v(x)) = \operatorname{div}\left(|x|^2 \cdot \left(-\frac{5}{3}\right) x \cdot |x|^{-\frac{11}{3}}\right)$$
$$= \operatorname{div}\left(-\frac{5}{3} x |x|^{-\frac{11}{3}+2}\right) =$$
$$= \operatorname{div}\left(-\frac{5}{3} x |x|^{-\frac{5}{3}}\right)$$
$$= -\frac{5}{3} \operatorname{div}\left(x |x|^{-\frac{5}{3}}\right) =$$
$$= -\frac{5}{3}\left(4 - \frac{5}{3}\right)|x|^{-\frac{5}{3}}$$
$$= -\frac{5}{3} \cdot \frac{7}{3} |x|^{-\frac{5}{3}} \qquad = -\frac{35}{9} |x|^{-\frac{5}{3}}$$

2) $p = ?$, $p \geq 1$ a.î. $|v|^p \in L^1(B_1(0))$ - bila unitate din $\underline{\mathbb{R}^4}$

i.e. $\displaystyle\int_{B_1(0)} |v|^p \, dx < \infty$

$$\int_{B_1(0)} |v|^p \, dx = \int_{B_1(0)} |x|^{-\frac{5p}{3}} \, dx \;\underset{=}{\underline{\text{coarii}}}\; \int_0^1 \left(\int_{\partial B_s(0)} |x|^{-\frac{5p}{3}} \, d\sigma(t) \right) ds =$$
$$= \int_0^1 s^{-\frac{5p}{3}} \underbrace{\left(\int_{\partial B_s(0)} d\sigma(t) \right)}_{= |\partial B_s(0)|} ds \quad = *$$

$$|\partial B_s(0)| = \omega_n \cdot R^{n-1} = \omega_4 \cdot s^3 = \frac{2\pi^{\frac{4}{2}}}{\Gamma(\frac{4}{2})} s^3 = \frac{2\pi^2}{(2-1)!} s^3 = 2\pi^2 s^3$$

$$* = \int_0^1 s^{-\frac{5p}{3}} \cdot 2\pi^2 s^3 \, ds = 2\pi^2 \int_0^1 s^{\frac{9-5p}{3}} \, ds$$

- $\frac{9-5p}{3} = -1 \iff 9-5p = -3 \iff p = \frac{12}{5}$ — NU

  $\int_0^1 \frac{1}{s} \, ds = \ln s \Big|_0^1$ ← $\ln(0) \to \infty$

- $\frac{9-5p}{3} > -1 \iff 9-5p > -3 \iff p < \frac{12}{5}$ — DA

  $$\left.\begin{array}{l} \int_0^1 s^{\frac{9-5p}{3}} \, ds = \frac{s^{\frac{12-5p}{3}}}{\frac{12-5p}{3}} \Big|_0^1 \\ \left.\begin{array}{l} p \geq 1 \\ p < \frac{12}{5} \end{array}\right\} \Rightarrow \frac{12-5p}{3} > 0 \end{array}\right\} \Rightarrow \int_0^1 s^{\frac{9-5p}{3}} \, ds < \infty$$

- $\frac{9-5p}{3} < -1 \iff p > \frac{12}{5}$ — NU

  $$\left.\begin{array}{l} \int_0^1 s^{\frac{9-5p}{3}} \, ds = \frac{s^{\frac{12-5p}{3}}}{\frac{12-5p}{3}} \\ \left.\begin{array}{l} p \geq 1 \\ p > \frac{12}{5} \end{array}\right\} \Rightarrow \frac{12-5p}{3} < 0 \end{array}\right\} \Rightarrow \int_0^1 s^{\frac{9-5p}{3}} \, ds \to \infty$$

$$\left.\begin{array}{l} \Rightarrow p \in \cancel{(\frac{12}{5}; +\infty)}\ (-\infty; \frac{12}{5}) \\ p \geq 1 \end{array}\right\} \Rightarrow p \in [1; \frac{12}{5})$$

3) $p = ?$, $p \geq 1$ a.î. $\frac{|v(x)|^p}{|x^2|+1} \in L^1(\mathbb{R}^4 - \overline{B_1(0)})$

i.e. $\int_{\mathbb{R}^4 - B_1(0)} \frac{|v(x)|^p}{|x^2|+1} \, dx < \infty \iff \int_{\mathbb{R}^4 - B_1(0)} \frac{|x|^{-\frac{5p}{3}}}{|x^2|+1} \, dx < \infty$

$|x|^2 \leq |x|^2+1 < 2|x|^2, \ \forall |x| > 1 \Rightarrow \int_{\mathbb{R}^4 - B_1(0)} \frac{|x|^{-\frac{5}{3}p}}{|x^2|+1} \, dx \sim \int_{\mathbb{R}^4 - B_1(0)} \frac{|x|^{-\frac{5}{3}p}}{|x|^2}$

$$\int_{\mathbb{R}^4 - B_1(0)} |x|^{\frac{-5p-6}{3}} dx \overset{\text{coarea}}{=} \int_1^{\infty} \left(\int_{\partial B_s(0)} s^{\frac{-5p-6}{3}} d\sigma(t) \right) ds = \int_1^{\infty} \left(s^{\frac{-5p-6}{3}} \left(\int_{\partial B_s(0)} d\sigma(t) \right) \right) ds =$$

$$= 2\pi^2 \int_1^{\infty} s^{\frac{-5p-6+9}{3}} ds = 2\pi \int_1^{\infty} s^{\frac{3-5p}{3}} ds$$

- $\frac{3-5p}{3} = -1 \Leftrightarrow 3-5p = -3 \Leftrightarrow p = \frac{6}{5}$ - NU

$$\int_1^{\infty} s \, ds \to \infty$$

- $\frac{3-5p}{3} > -1 \Leftrightarrow 3-5p > -3 \Leftrightarrow p < \frac{6}{5}$ - NU

$$\int_1^{\infty} s^{\frac{3-5p}{3}} ds = \frac{s^{poz}}{poz} \to \infty$$

- $\frac{3-5p}{3} < -1 \Leftrightarrow p > \frac{6}{5}$ - DA

$$\int_1^{\infty} s^{\frac{3-5p}{3}} ds = \frac{s^{neg}}{neg} = \frac{1}{s^{poz.}} < \infty$$

$$\left.\begin{matrix} \Rightarrow p > \frac{6}{5} \\ p \geq 1 \end{matrix}\right\} \Rightarrow p \in \left(\frac{6}{5}; +\infty \right)$$

4) $x + 3y = 0$

$u : \mathbb{R}^2 \to \mathbb{R}$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

→ caut $u(x,y) = (x+3y) \cdot$ ceva → se anulează pe dr $x+3y=0$

→ fie ceva $= c$ - ct.

$$\left.\begin{matrix} \frac{\partial u}{\partial x} = c & \frac{\partial^2 u}{\partial x^2} = 0 \\ \frac{\partial u}{\partial y} = 3c, & \frac{\partial^2 u}{\partial y^2} = 0 \end{matrix}\right\} \Rightarrow -\Delta u \not< 0 \leftarrow \text{nu merge cu ceva} = ct$$

→ fie ceva $= c(x-3y) \Rightarrow u(x,y) = (x^2 - 9y^2) \cdot c$

$$\left.\begin{matrix} \frac{\partial u}{\partial x} = 2cx & \frac{\partial^2 u}{\partial x^2} = 2c \\ \frac{\partial u}{\partial y} = -18cy & \frac{\partial^2 u}{\partial y^2} = -18c \end{matrix}\right\} \Rightarrow \left.\begin{matrix} -\Delta u = 16c \\ \text{aleg } c = -1 \end{matrix}\right\} \Rightarrow -\Delta u < 0$$

$$\boxed{\Rightarrow u(x,y) = 9y^2 - x^2}$$

5) $u: B_1(0)-\{0\} \to \mathbb{R}$

$$u(x) = \left(\ln \frac{2}{|x|}\right)^{\frac{1}{2}}, \quad x = (x_1, x_2)$$

$$-\Delta u(x) = \frac{u(x)}{4|x|^2 \ln^2\left(\frac{2}{|x|}\right)}, \quad \forall x \in B_1(0)-\{0\}$$

• obs că $u(x)$ = fc. radială și atunci $u(x) = g(|x|)$,

$$g(t) = \left(\ln \frac{2}{t}\right)^{\frac{1}{2}}$$

$\frac{2}{t} \geq 1 \iff t \leq 2$, $t > 0$ $\} \Rightarrow t \in [0,2]$ $\{|x|<1$ ✓

• $\Delta u(x) = g''(|x|) + \frac{n-1}{|x|} \cdot g'(|x|), \quad \forall x \in B_1(0)-\{0\}$, $n=2$

$$g'(t) = \frac{1}{2}\left(\ln \frac{2}{t}\right)^{-\frac{1}{2}} \cdot \frac{t}{2} \cdot 2 \cdot (-1) \cdot t^{-2} =$$

$$= -\frac{t^{-1}}{2} \ln\left(\frac{2}{t}\right)^{-\frac{1}{2}}$$

$$g''(t) = -\frac{1}{2} \cdot (-1) \cdot t^{-2} \ln\left(\frac{2}{t}\right)^{-\frac{1}{2}} + \left(-\frac{t^{-1}}{2}\right)\left(-\frac{1}{2}\right) \ln\left(\frac{2}{t}\right)^{-\frac{3}{2}} \frac{t}{2} \cdot 2 \cdot (-1) \cdot t^{-2} =$$

$$= \frac{1}{2} t^{-2} \ln\left(\frac{2}{t}\right)^{-\frac{1}{2}} + \frac{1}{2} \cdot \left(-\frac{t^{-2}}{2}\right) \cdot \ln\left(\frac{2}{t}\right)^{-\frac{3}{2}}$$

$$g''(t) + \cancel{g'(t)} \frac{g'(t)}{t} =$$

$$= \frac{1}{2} t^{-2} \ln\left(\frac{2}{t}\right)^{-\frac{1}{2}} \left(1 - \frac{1}{2} \ln\left(\frac{2}{t}\right)^{-1}\right) - \frac{1}{2} t^{-2} \ln\left(\frac{2}{t}\right)^{-\frac{1}{2}} =$$

$$= \frac{1}{2} t^{-2} \ln\left(\frac{2}{t}\right)^{-\frac{1}{2}} \cdot \left(-\frac{1}{2}\right) \ln\left(\frac{2}{t}\right)^{-1} =$$

$$= -\frac{1}{4} t^{-2} \ln\left(\frac{2}{t}\right)^{-\frac{3}{2}} = \frac{-1}{4t^2 \ln\left(\frac{2}{t}\right)\sqrt{\ln\left(\frac{2}{t}\right)}} = \frac{-\left(\ln\left(\frac{2}{t}\right)\right)^{\frac{1}{2}}}{4t^2 \ln^2\left(\frac{2}{t}\right)} =$$

$$= \frac{-u(x)}{4|x|^2 \ln^2\left(\frac{2}{|x|}\right)} = \Delta u(x)$$

$$\Rightarrow -\Delta u(x) = \frac{u(x)}{4|x|^2 \ln^2\left(\frac{2}{|x|}\right)}$$

PROBLEMA 2

$$\begin{cases} u_{xx}(x,y)+2u_{yy}(x,y)=0 & (x,y)\in(0,1)\times(0,1) \\ u(x,0)=u(x,1)=0 & x\in(0,1),\ y\in(0,1) \\ u(0,y)=\sin(2\pi y) & \\ u(1,y)=e^{-2\sqrt{2}\pi}\sin(2\pi y) & y\in(0,1) \end{cases}$$

1) $u(x,y)$ a.î. $u(x,y)=A(x)B(y)$

$$u_{xx}(x,y)=A''(x)B(y)$$
$$u_{yy}(x,y)=A(x)B''(y)$$

$$\Rightarrow u_{xx}+u_{yy}=A''(x)B(y)+A(x)B''(y)=0 \ \Big|\cdot\frac{1}{A(x)B(y)}$$

$$\Rightarrow \frac{A''(x)}{A(x)}+2\frac{B''(y)}{B(y)}=0$$

$$\Rightarrow \frac{A''(x)}{A(x)}=-2\frac{B''(y)}{B(y)}=ct=c$$

$$\begin{cases} A''(x)=A(x)\cdot c \\ B''(y)=-\frac{c}{2}B(y) \end{cases} \Leftrightarrow \begin{cases} A''(x)-cA(x)=0 \\ B''(y)+\frac{c}{2}B(y)=0 \end{cases}$$

$$\left.\begin{array}{l} A(x)B(0)=A(x)B(1)=0 \\ A(x)\neq 0 \end{array}\right\}\Rightarrow B(0)=B(1)=0$$

$$\left.\begin{array}{l} A(0)B(y)=\sin(2\pi y) \\ A(1)B(y)=e^{-2\sqrt{2}\pi}\sin(2\pi y) \end{array}\right\}\Rightarrow \left.\begin{array}{l} B(y)=\frac{\sin 2\pi y}{A(0)} \\ A(0)\neq 0 \end{array}\right\}\Rightarrow B'(y)=\frac{2\pi\cos(2\pi y)}{A(0)}$$

$$\Rightarrow B''(y)=-\frac{4\pi^2\sin(2\pi y)}{A(0)}$$

$$\Rightarrow \frac{B''(y)}{B(y)}=-4\pi^2$$

$$\frac{A''(x)}{A(x)}+2\frac{B''(y)}{B(y)}=0$$

$$\Rightarrow \frac{A''(x)}{A(x)}=8\pi^2 \Rightarrow A''(x)=8\pi^2A(x) \qquad (\lambda^2-(2\sqrt{2}\pi)^2=0)$$

$$\Rightarrow A(x)=c_1e^{2\sqrt{2}\pi x}+c_2e^{-2\sqrt{2}\pi x}$$

$$A(0)=c_1+c_2$$
$$B(y)=\frac{\sin(2\pi y)}{c_1+c_2}$$

$A(1)\cdot B(y) = e^{-2\sqrt{2}\pi} \sin(2\pi y)$

~~A(x)~~ aleg $c_1 = 0$, $c_2 = 1$ $\Big\} \Rightarrow A(x) = e^{-2\sqrt{2}\pi x}$

$B(x) = \sin(2\pi y)$

$\Rightarrow u(x) = e^{-2\sqrt{2}\pi x} \cdot \sin(2\pi y)$

2) Fie u_1, u_2 - soluții și $U = u_1 - u_2$

arătam că $U \equiv 0$

$$\begin{cases} U_{xx} + 2U_{yy} = 0 \\ U(x,0) = U(x,1) = 0 \\ U(0,y) = u_1(0,y) - u_2(0,y) = 0 \\ U(1,y) = u_1(1,y) - u_2(1,y) = 0 \end{cases} \Big\}^* \quad U/_{\partial\Omega} = 0$$

$$0 = \int_\Omega U \Delta U \, dx\, dy = \underbrace{\int_{\partial\Omega} U \frac{\partial U}{\partial \nu} \, d\sigma}_{\overset{*}{=} 0} - \int_\Omega \nabla U \nabla u$$

$\Delta U = U_{xx} + U_{yy} = -U_{yy}$

$\Rightarrow U \Delta U = -U \cdot U_{yy} = 0 \longrightarrow$ nu știu cum

$\Rightarrow |\nabla U|^2 = 0 \Rightarrow \nabla U = 0$

$\Rightarrow U = ct$, $U(x,0) = 0$ $\Big\} \Rightarrow U \equiv 0$

PROBLEMA 3

$$\begin{cases} u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$f, g \in C^2(\mathbb{R})$ fc. date

1) $u = u(x,t)$ de clasă C^2, atunci

$$(\partial_t + 2\partial_x)(u_t(x,t) - 3u_x(x,t)) = u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t)$$

$$\partial_t(u_t - 3u_x) + 2\partial_x(u_t - 3u_x) =$$

$$= u_{tt} - 3u_{xt} + 2u_{tx} - 6u_{xx}$$

$$\left.\begin{matrix} \\ u \in C^2 \end{matrix}\right\} \xrightarrow{\text{Lema lui Sch}} -3u_{xt} + 2u_{tx} =$$

$$= -3u_{tx} + 2u_{tx} = -u_{tx}$$

$$= u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t)$$

2) $v = u_t - 3u_x$

$v(x,0) = u_t(x,0) - 3u_x(x,0) =$

$= g(x) - 3f'(x)$

$$\begin{cases} v_t(x,t) + 2v_x(x,t) = 0 \\ v(x,0) = g(x) - 3f'(x) \end{cases}$$

$$1 \cdot v_t + 2 \cdot v_x = 0$$

$$\Rightarrow \nabla v \cdot (2,1) = 0$$

$$\Rightarrow v(\alpha(2,1) + \beta(a,b)) = v(\beta(a,b))$$

$v(x,0) = g(x) - 4f'(x)$

$(x,t) = (x,0) + (0,t) = (x,0) + (t+t, t) + (-t-t, 0) =$

$= (x,0) + t(2,1) + (-2t, 0) = (x-2t, 0) + t(2,1)$

$$v(x,t) = v((x-2t, 0) + t(2,1))$$

$$= v(x-2t, 0) =$$

$$= g(x-2t) - 3f'(x-2t)$$

$$\begin{cases} u_t(x,t) - 3u_x(x,t) = g(x-2t) - 3f'(x-2t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

$$w(\tau) = u(x-3\tau, t+\tau)$$

$$w'(\tau) = u_x(x-3\tau, t+\tau)\cdot(-3) + u_t(x-3\tau, t+\tau)\cdot(1)$$

$$= g(x-2t-4\tau) - 3f'(x-t-4\tau) \quad \Big| \int_0^s$$

$$w(s) - w(0) = \int_0^s g(x-t-4\tau)\,d\tau - 3\int_0^s f'(x-t-4\tau)\,d\tau$$

$$w(s) = u(x-3s, t+s)$$

$$w(0) = u(x,t)$$

aleg $s=-t$ a.i. $w(-t) = u(x+3t, 0) = f(x+3t)$

$$\Rightarrow u(x,t) = f(x+3t) - \int_0^{-t} g(x-t-4\tau)\,d\tau + 3\int_0^{-t} f'(x-t-4\tau)\,d\tau$$

3) $t=0$

$f(x) = \sin x$

$g(x) = e^{-x}$

$$\cancel{u(x,t) = \sin(x) - \int_0^{-t} e^{t-x-4\tau}\,d\tau + 3\int_0^{s} \cos(x-4\tau)\,d\tau}$$

$$\cancel{= \sin(x) + 3\sin(x-4\tau)\cdot\left(-\frac{1}{4}\right)\Big|_0^s}$$

$$= \cancel{\sin(x) +}$$

$$u(x,t) = \sin(x+3t) - \int_0^0 + 3\int_0^0$$

$$= \sin(x+3t)$$

$$= \sin x$$

PROBLEMA 4

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \dfrac{e^t}{e^{2t}+1} u(x,t) = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = e^{-x^2} & x \in \mathbb{R} \end{cases}$$

1) $\phi : \mathbb{R} \to \mathbb{R}$ a.î. $v(x,t) := u(x,t)\phi(t)$ verifică $v_t(x,t) - v_{xx}(x,t) = 0, \forall x \in \mathbb{R}, t \geq 0$

$v_t(x,t) = u_t(x,t)\phi(t) + u(x,t) \cdot \phi'(t)$
$v_{xx}(x,t) = u_{xx}(x,t) \cdot \phi(t)$

$$u_t(x,t)\phi(t) + u(x,t)\cdot\phi'(t) - u_{xx}(x,t)\cdot\phi(t) = 0 \;|\; : \phi(t)$$
$$u_t(x,t) - u_{xx}(x,t) + \frac{\phi(t)}{\phi(t)} \cdot u(x,t) = 0$$

avem $\dfrac{\phi'(t)}{\phi(t)} = \dfrac{e^t}{e^{2t}+1} \;|\; \int$

$$\ln \phi(t) = \int \frac{e^t}{e^{2t}+1} dt = \operatorname{arctg}(e^t) + c$$

$\Rightarrow \phi(t) = c e^{\operatorname{arctg}(e^t)}$

$v(x,0) = u(x,0)\cdot\phi(0) \Rightarrow e^{-x^2} = e^{-x^2} \cdot c \cdot e^{\operatorname{arctg}(e^0 = 1)}$

$\Leftrightarrow 1 = c \cdot e^{\pi/4} \Rightarrow c = e^{-\pi/4}$

$\Rightarrow \boxed{\phi(t) = e^{-\pi/4} \cdot e^{\operatorname{arctg}(e^t)}}$

2) $v(x,0) = u(x,0)\cdot\phi(0)$
$= e^{-x^2} \cdot e^{\operatorname{arctg} 1 - \frac{\pi}{4}} = e^{-x^2} \cdot e^{\frac{\pi}{4} - \frac{\pi}{4}} = e^{x^2}$

pb. Cauchy: $\begin{cases} v_t(x,t) - v_{xx}(x,t) = 0 \\ v(x,0) = e^{x^2} \end{cases}$

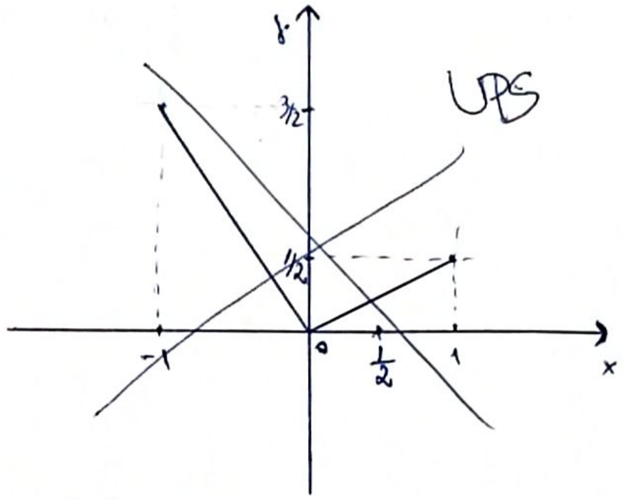
Nucleul căldurii: $K(x,t) = \dfrac{1}{\sqrt{4\pi t}} \cdot e^{\frac{-|x|^2}{4t}}, \; t > 0, x \in \mathbb{R}$

$v(x,t) = (K(\cdot,t) * u_0)(x) = \displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \cdot e \;\ldots$

PROBLEMA 5

$f : [-1,1] \to \mathbb{R}$

$f(x) = |x - \frac{1}{2}|$



1) $f(x) = \begin{cases} x - \frac{1}{2}, & x \geq \frac{1}{2} \\ \frac{1}{2} - x, & x < \frac{1}{2} \end{cases}$

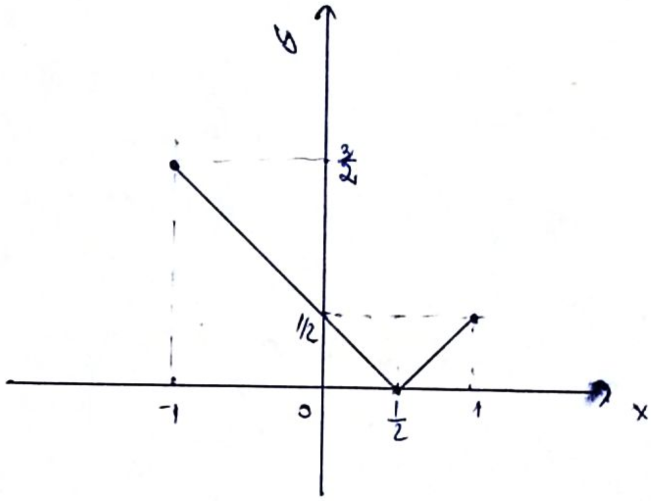
2) f derivabilă pe $(-1, \frac{1}{2}) \cup (\frac{1}{2}; 1)$ - evident, fc. elementară

şi $f'(x) = \begin{cases} 1, & x \in (\frac{1}{2}, 1) \\ -1, & x \in (-1, \frac{1}{2}) \end{cases}$

cum $\lim\limits_{\substack{x \to 1/2 \\ x < 1/2}} f'(x) \neq \lim\limits_{\substack{x \to 1/2 \\ x > 1/2}} f'(x) \Rightarrow$ f nu e derivabilă în $x = \frac{1}{2}$

+ se observă din grafic (o inf. de tangente la $x = \frac{1}{2}$)

$\Rightarrow$ f deriv. pe $(-1; 1) - \{\frac{1}{2}\}$



3) • $f \in H^1(-1,1)$

$f \in L^2(-1,1)$ i.e. $\int_{-1}^{1} |f(x)|^2 dx < \infty \iff \int_{-1}^{1} |x - \frac{1}{2}|^2 dx < \infty$

f e discontinuă în $x = \{\frac{1}{2}\}$ - mulţime finită $\Big\} \Rightarrow f \in L^2(-1,1)$

• $g(x) = \begin{cases} 1, & x \in (\frac{1}{2}, 1] \\ -1, & x \in [-1, \frac{1}{2}) \\ 0, & x = \frac{1}{2} \end{cases}$

fie $\varphi \in C_c^\infty(-1,1) \Rightarrow \varphi(1) = \varphi(-1) = 0$

$\int_{-1}^{1} f(x)\varphi'(x)\,dx = \int_{-1}^{1/2} (\frac{1}{2} - x)\varphi'(x)\,dx + \int_{1/2}^{1} (x - \frac{1}{2})\varphi'(x)\,dx =$

$= (1/2 - x)\varphi(x)\Big|_{-1}^{1/2} - \int_{-1}^{1/2} (-1)\varphi(x)\,dx + (x - \frac{1}{2})\varphi(x)\Big|_{1/2}^{1} - \int_{1/2}^{1} \varphi'(x)\,dx =$

$= \int_{-1}^{1/2} \varphi(x)\,dx - \int_{1/2}^{1} \varphi(x) = \int_{-1}^{1} g(x) \cdot \varphi(x)\,dx \Rightarrow$ g e deriv. slabă a lui f

• $g \in L^2(0,1) \Leftrightarrow \exists \int_{-1}^{1} |g(x)|^2 dx < \infty$

$$\int_{-1}^{1} 1 \, dx = 2 < \infty \Rightarrow g \in L^2(0,1)$$

$\Rightarrow f \in H^1(-1,1)$

4) $\alpha \in \mathbb{R}$ a.î. $f : (0,1) \to \mathbb{R}, f(x) = x^\alpha \in H^1(0,1)$

$$\begin{cases} f(x) \in L^2(0,1) \\ \exists f'(x) \text{ deriv. slabă} \\ f'(x) \in L^2(0,1) \end{cases}$$

$$\int_0 |f(x)|^2 dx = \int_0 x^{2\alpha} dx$$

• $\alpha = -\frac{1}{2} \Rightarrow \int_0^1 x^{2\alpha} = \ln x \Big|_0^1 \to \infty$

• $\alpha \neq -\frac{1}{2} \Rightarrow \int_0^1 x^{2\alpha} = \frac{x^{2\alpha+1}}{2\alpha+1} \Big|_0^1$

$f'(x) = \alpha \cdot x^{\alpha-1} \in L^2(0,1) \Leftrightarrow \int_0^1 x^{2\alpha-2} dx < \infty$

• $\alpha = \frac{1}{2} \Rightarrow \int_0^1 x^{2\alpha-2} dx = \ln x \Big|_0^1 \to \infty$

• $\alpha \neq \frac{1}{2} \Rightarrow \int_0^1 x^{2\alpha-2} dx = \frac{x^{2\alpha-1}}{2\alpha-1} \Big|_0^1 \Rightarrow g \in L^2(0,1) \Leftrightarrow \alpha \neq \frac{1}{2}$

$f(x) \in H^1(0,1)$, $\forall \alpha \in \mathbb{R} - \{\pm \frac{1}{2}\}$

5) la fel, doar că $\int_1^\infty \left(\frac{x^\alpha}{1+x^3}\right)^2 dx < \infty$

$\int_1^\infty \frac{x^{2\alpha}}{(1+x^3)^2} dx \sim \int_1^\infty x^{2\alpha-6} dx$ nu mai am timp!

↳ crit. comp.

$\Leftrightarrow 6 - 2\alpha > 1 \Leftrightarrow 2\alpha < 5 \Rightarrow \alpha < \frac{5}{2}$

$f'(x) = \ldots$