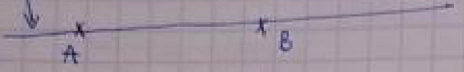
Geometrie - Curs 2

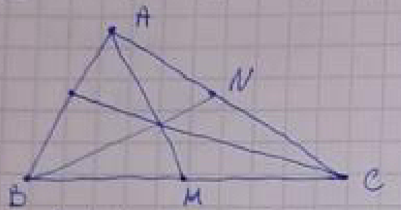
2: Submulțime ...

Seminar 3

Af({A, B})



$Af(\{A,B\}) = \{\alpha A + \beta B \mid \alpha + \beta = 1,\ \alpha, \beta \in K\}$



$G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$

2: Să demonstrăm că medianele AM, BN, CP sunt concurente în G.

$M = \frac{1}{2}B + \frac{1}{2}C$

$P = \frac{1}{2}A + \frac{1}{2}B$

$N = \frac{1}{2}A + \frac{1}{2}C.$

$\cancel{x/5 - 1 = y}$

$y/5 - 1 =$

$\cancel{\frac{\left(\frac{x}{5} - 1\right)}{5} - 1}$

$\left(\left(\left(\frac{x}{5} - 1\right) : 5 - 1\right) : 5 - 1\right)$

Tripleți pitagoreice:

(3, 4, 5), (5, 12, 13)

$\begin{cases} m = a^2 - b^2 \\ n = 2ab \\ p = a^2 + b^2 \end{cases} \Rightarrow (a^2 - b^2, 2ab, a^2 + b^2)$

$ax + by = c$

$a = \frac{m}{n}$

$b = \frac{p}{n}$

$c = \frac{q}{n}$, $m, n, q \in \mathbb{Z}$

$mx + py = q$

$x^2 + y^2 = 1,\ x, y \in \mathbb{Q}$

$x = \frac{m}{p}$

$y = \frac{n}{p}$

$\Rightarrow m^2 + n^2 = p^2$

Demonstr. : Să demonstr. că $G \in AM$.

$\longrightarrow$ $G \in AM \iff \exists \alpha, \beta \in \mathbb{R}$, $\alpha+\beta=1$. a.î. $G=\alpha A+\beta M$.

$$G=\alpha A+\beta\left(\frac{1}{2}B+\frac{1}{2}C\right).$$

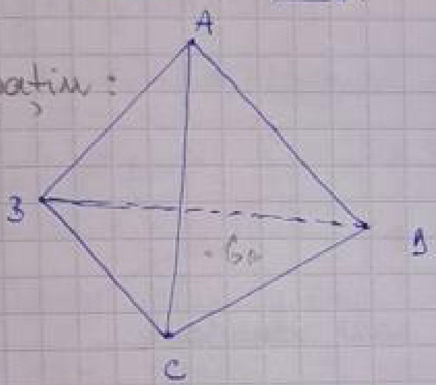
Iau $\alpha=\frac{1}{3}$

$\beta=\frac{2}{3}$ $\Rightarrow$ $G=\frac{1}{3}A+\frac{2}{3}\left(\frac{1}{2}B+\frac{1}{2}C\right)$

$$G=\frac{1}{3}A+\frac{1}{3}B+\frac{1}{3}C.$$

Analog $G \in BN$.
$G \in CP$. $\square$.

În spațiu:



$$G=\frac{1}{4}A+\frac{1}{4}B+\frac{1}{4}C+\frac{1}{4}D.$$

$$G_A=\frac{1}{3}B+\frac{1}{3}C+\frac{1}{3}D$$

$$G_B=\frac{1}{3}A+\frac{1}{3}C+\frac{1}{3}D$$

$$G_C=\frac{1}{3}A+\frac{1}{3}B+\frac{1}{3}D.$$

$$G_D=\frac{1}{3}A+\frac{1}{3}B+\frac{1}{3}C$$

Thm: AG_A, BG_B, CG_C, DG_D sunt concurente.

Dem: Arăt că $G \in AG_A$

$$G=\frac{1}{4}A+\frac{3}{4}G_A.$$

Analog pt $\begin{cases} G \in BG_B \\ G \in CG_C \\ G \in DG_D. \end{cases}$

2) $A_1A_2 \ldots A_6$ hexagon convex arbitrar.

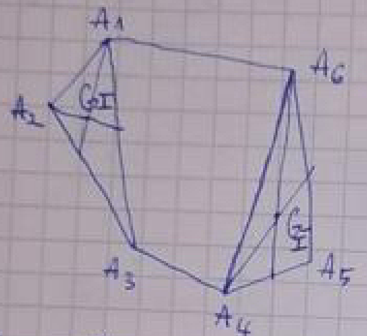
$I \subset \{1, \ldots, 6\}$ card $I = 3$

$\overline{I} = \{1, \ldots, 6\} \setminus I$

Fie G_I centru greut al Δ cu vf. în I.

$G_{\overline{I}}$ centru greut al Δ cu vf. în $\overline{I}$.

Sol:



$$G_I = \frac{1}{3}A_1 + \frac{1}{3}A_2 + \frac{1}{3}A_3$$

$$G_{\overline{I}} = \frac{1}{3}A_4 + \frac{1}{3}A_5 + \frac{1}{3}A_6$$

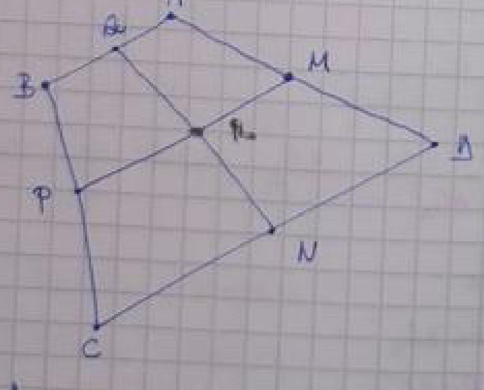
$$G = \frac{1}{2}G_I + \frac{1}{2}G_{\overline{I}}$$ ✓

Thm: Toate dreptele $G_I G_{\overline{I}}$ sunt concurente.

Fie $G = \frac{1}{6}A_1 + \frac{1}{6}A_2 + \cdots + \frac{1}{6}A_6$.

Arat că $G \in G_I G_{\overline{I}}$.

3) Fie ABCD patrulater. M, N, P, Q mijloacele lat.



Dem că QN și PM se înjumătățesc.

$\{R\} = QN \cap PM$ este mijlocul lui QN și PM.

$AB = \{\alpha A + \beta B \mid \alpha + \beta = 1,\ \alpha, \beta \in \mathbb{R}\}$

Segmentul $[AB] = \{\alpha A + \beta B \mid \alpha, \beta \in [0,1]\}$

$[0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$.

$(\mathbb{C}, \leq)$: $\begin{cases} x \leq y \iff x+z \leq y+z, \forall z \in \mathbb{C} \\ x \leq y \iff x \cdot z \leq y \cdot z, \forall z \in \mathbb{C} \end{cases}$

$\Downarrow$

$\boxed{t^2 \geq 0, \forall t}$.

$R = \frac{1}{2}A + \frac{1}{2}B$

$N = \frac{1}{2}C + \frac{1}{2}D$

Fie $I = \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D$. $\left(I = \frac{1}{2}R + \frac{1}{2}N\right)$

5) In $A = \mathbb{R}^3$ (cu structura afină canonică) considerăm

$A = (1,0,1)$ $C = (0,1,3)$.

$B = (2,0,0)$

- Sunt afin indep.?
- Sunt sistem de generatori pt A?

Sol: $\boxed{\{P_0, \dots, P_n\} \text{ afin indep.} \iff \{\overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_n}\} \text{ lin indep.}}$

$\{\overrightarrow{BA}, \overrightarrow{BC}\}$ lin independ.?

$\overrightarrow{BA} = (1,0,1) - (2,0,0) = (-1,0,1)$

$\boxed{\begin{array}{l} V = \mathcal{A};\ \varphi : V \times V \to V \\ \quad \varphi(u,v) = v - u \end{array}}$

$\overrightarrow{BC} = (-2,1,3)$

$\{(-1,0,1), (-2,1,3)\}$ lin indep?

$\iff \operatorname{rg}\begin{bmatrix} -1 & -2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} = 2$.

$\begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = -1 \neq 0 \implies$ sunt lin ind.!

Altfel, se observă că $\nexists\ t$ a.î. $(-1,0,1) = t \cdot (-2,1,3)$.

~~Sau:~~

$\{A,B,C\}$ sistem afin de generat $\iff \{\overrightarrow{BA}, \overrightarrow{BC}\}$ sist. generat.

Aici avem 3 puncte $\implies$ nu sunt sist. de generat.

$\mathbb{R}^3$ (generează maxim un plan).

6) Pb.:

Fie $A=\mathbb{R}^2$ cu structura canon afină.

Fie $R_0 = \{O, \underbrace{\{e_1,e_2\}}_{\text{baza canon.}}\}$

$O=(0,0)$

$e_1=(1,0)$

$e_2=(0,1)$

Fie $R_2 = (O_1, \{f_1,f_2\})$ $O_1=(1,-1)$

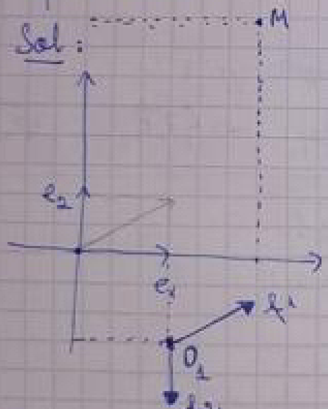
$f_1=e_1+e_2$

$f_2=-e_2$

a) Este R_2 reper cartezian? DA (are un punct și o bază).

b) Fie $M(2,3)$. Determ. coord. carteziene ale lui M în raport cu R_0 și în raport cu R_2.

Sol:



• în raport cu R_0:

$\vec{OM} = (2,3) = 2e_1+3e_2$

$\Rightarrow$ coordon $(2,3)$.

• în raport cu R_2:

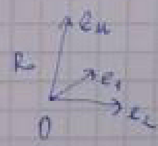
$\vec{O_1M} = (2,3)-(1,-1) = (1,4)$

coord lui M în raport cu R_2 (y_1,y_2).

$\vec{O_1M} = (1,4) = y_1 f_1 + y_2 f_2 \Rightarrow$

$\Rightarrow (1,4) = y_1(1,1) + y_2(0,-1)$

$\Rightarrow \begin{cases} y_1 = 1 \\ y_1 - y_2 = 4 \end{cases} \Rightarrow y_2=-3 \Rightarrow M$ are coordon. $(1,-3)$.



• $M \rightarrow \vec{OM} = x_1e_1+\dots+x_ne_n$

$\Rightarrow (x_1,\dots,x_n)$ coordon.

7) Care

a) $x_1 =$

b) $x_2 =$

c) $x_3 =$

d) $x_4 =$

e) parabolă $x_5 =$

a) $x=$

d) A

7) Care submulțimi sunt subspații afine? (în $\mathbb{R}^2$).

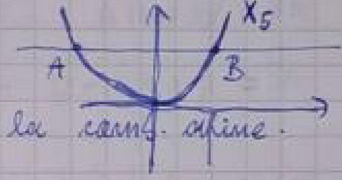
a) $X_1 = \{(0,0)\}$ ✓

b) $X_2 = \{(1,2)\}$ ✓

c) $X_3 = [AB]$, $A=(1,1)$ $B=(2,2)$ X nu e închis la combin. afine.

cerc d) $X_4 = \{(x,y) \mid x^2+y^2=1\}$ X nu e închis la combin. afine.

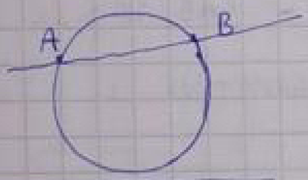
e) parabola $X_5 = \{(x, x^2) \mid x \in \mathbb{R}\}$ X nu e închisă la comb. afine.



a) $X = \{P\}$ e subsp. afin ✓

$\operatorname{dir}(X) = \{\vec{PP}\} = \{0\}$

d)



$U, V \subset W$

$U+V = \{u+v \mid u \in U, v \in V\}$ ✓ la subspații vectoriale.

$A', A'' \subset A$.

$A' + A'' = \{\alpha P + \beta Q \mid \alpha+\beta=1;\ P \in A',\ Q \in A''\}$.

$$\boxed{A''' = \bigcup_{\substack{P \in A' \\ Q \in A''}} PQ}$$

$\vec{OM} = x_1 e_1 + \dots + x_n e_n$

$\Rightarrow (x_1, \dots, x_n)$ cartez.

Seminar 4

1) Fie $A = R^3$ cu str. afină canonică. Care sunt subsp. afine?

a) $A' = \{(x,y,z) \mid x+y+z = 0\}$ ✓ plan

b) $A' = \{(x,y,z) \mid x^2+y^2-z^2 = \sqrt{2}z\}$ $\not\equiv$ X

c) $A' = \{(x,y,z) \mid x+y-z \geq 0\}$. X "semispațiu"

d) $A' = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$. $\not\equiv$ X sferă

$P = (0,0,-\sqrt{2}) \in A'$

$Q = (0,0,0) \in A'$

$\frac{1}{2}P + \frac{1}{2}Q = \left(0,0,-\frac{\sqrt{2}}{2}\right) \in A'$?

Verificam în ecuație $\Rightarrow \notin A'$.

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2) Care sunt plane? $(A = R^3)$

a) $A' = \{(t, 2t, \frac{s}{t^2+1}) \mid t,s \in R\}$ $\not\equiv$ ✓

b) $A' = \{(t^2, 2t^2, s) \mid t,s \in R\}$ X

c) $A' = \{(t+1, 1, 2) \mid t \in R\}$ $\not\equiv$ X dreaptă

~~d) A'~~

$(x_0, y_0, z_0) \in A'$

$x_0 \geq 0$, $\forall$ punct.

$\forall P, Q \in A'$

$R = \alpha P + \beta Q$. (negativ x-ul)

$P = (1,2,0)$ $\Rightarrow$ ~~$2t+(-2)=$~~

$Q = (2,4,0)$ $\quad 3P - 2Q = (-1, ,) \notin A'$.

$A' = \{(x,y,z) \mid y-2x = 0\}$

$\subset$

$\supset$ $P \in A' \Rightarrow P(x, 2x, z)$.

Caut t, s a.î. $P = \left(t, 2t, \frac{s}{t^2+1}\right)$

$t = x$
$2t = x$ $\Rightarrow t = x$

$\frac{s}{t^2+1} = z$ $\Rightarrow s = (x^2+1)z$

4) Care sunt drepte?

a) $A' = \{(t, 2t, a-1) \mid a, t \in \mathbb{R}\}$ ✗ plan

b) $A' = \{(t^3, 2t^3, 0) \mid t \in \mathbb{R}\}$ ~~plan~~ ✓ dreaptă

c) $A' = \{(t+1, 1, 2) \mid t \in \mathbb{R}\}$ ✓

d) $A' = \{(x, y, z) \mid x^2 = 0\}$ ✗ plan

$\to A' = \{(x, 2x, 0) \mid x \in \mathbb{R}\}$

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a) $A' = \{(s^4, t, s+t) \mid t, s \in \mathbb{R}\}$ ✗ ~~plan~~

b) $A' = \{(x, y, z) \mid x^3 = y^3,\ x = 2z+1\}$ ✓

c) $A' = \{(t+1, 1, 2-t) \mid t \in \mathbb{R}\}$ ✓ ~~plan~~ dreaptă

d) $A' = \{(x, y, z) \mid (x-y+z)^3 = -1\}$ ✗ plan

$\to x^3 = y^3 \Rightarrow x = y \Rightarrow A' = \{(x, x, z) \mid x = 2z+1\}$

$\to x - y + z = -1$

5) $A = \mathbb{C}^3$ cu struct. afină canonică.

Fie punctele $A = (1, 0, 2)$ $B = (1, 1, 1)$ $C = (-2, 2, 3)$

$D = (1, 1, 1)$ $E = (1, 0, -1)$ $F = (1, 2, 3)$

a) Să se determine int. dintre DE și (ABC).

b) Ec. planului $\ni F$ și care e paralel cu (ABC).

a) $\overrightarrow{DE} = (0, -1, -2)$

$$DE: \begin{cases} x = 1 \\ y = -t + 1 \\ z = -2t + 1 \end{cases}, t \in \mathbb{R}$$

$\overrightarrow{AP} \in \operatorname{dir}(ABC) = \langle \overrightarrow{AB}, \overrightarrow{AC} \rangle$

$\Leftrightarrow \overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ depend.

$\overrightarrow{AP} = (x-1, y, z-2)$

$\overrightarrow{AB} = (0, 1, -1)$

$\overrightarrow{AC} = (-3, +2, 1)$

$$\begin{vmatrix} x-1 & y & z-2 \\ 0 & 1 & -1 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \boxed{x-1} & 0 & -3 \\ \boxed{y} & 1 & 2 \\ \boxed{z-2} & -1 & 1 \end{vmatrix} = 0$$

$(x-1) \cdot 3 + y(-3) + (z-2)3 = 0 \quad |:3$

$x - 1 + y + z - 2 = 0$

$(ABC): x + y + z = 3$

$1 + 0 + 2 = 3$ ✓

$(DE) \cap (ABC)$ ~~$x+y+z=3$~~ $\Leftrightarrow \exists t \in \mathbb{R}$ a.î. $1 + (-t+1) + (-2t+1) = 3$

$\Rightarrow t = 0$

$\Rightarrow (1, 1, 1) = B$.

b) $\pi: x + y + z = 6$

A', A'' de ac. dimensiuni

$A' \parallel A'' \Leftrightarrow \operatorname{dir} A' = \operatorname{dir} A''$

$A': Ax + B = 0$

$\operatorname{dir}(A'): Ax = 0$

$\Rightarrow \operatorname{dir}(ABC): x + y + z = 0$

π ~~[illegible]~~ $: x + y + z + b = 0$

$F \in \pi \Rightarrow \pi:$ ~~$x+y+z$~~ $1 + 2 + 3 + b = 0$, $b = -6$

c) Ec. ec. dreptei prin A și e paralelă cu dreapta DE.

$$(DE): \begin{cases} x = 1 \\ y = -t + 1 \\ z = -2t + 1 \end{cases}$$

$$(DE): \begin{cases} x = 0 \\ y = -t \\ z = -2t \end{cases}$$

$$d: \begin{cases} x = 1 \\ y = -t \\ z = -2t + 2 \end{cases}$$

Ex: Fie A sp. afin / R.

$A_1, A_2 \subseteq A$ subsp.

Demonstr. sau dați un contraexemplu:

$$A_1 \vee A_2 = \underbrace{\{ R = aP + bQ \mid P \in A_1, Q \in A_2, a+b=1, a,b \in \mathbb{R} \}}_{X}$$ ~~sp. afin~~

$$\underbrace{A_1 \vee A_2}_{Y} = Af(A_1 \cup A_2)$$

$$= \{ P = \alpha_1 P_1 + \dots + \alpha_n P_n + \beta_1 Q_1 + \dots + \beta_n Q_n, \}$$

$P_1, \dots, P_n \in A_1$, $Q_1, \dots, Q_n \in A_2$

$\alpha_1 + \dots + \alpha_n + \beta_1 + \dots + \beta_n = 1$

Dem: $X \supset Y$

Fie $P \in Y \Rightarrow P = \underbrace{\alpha_1 P_1 + \dots + \alpha_n P_n}_{R} + \underbrace{\beta_1 Q_1 + \dots + \beta_n Q_n}$; $\alpha_1 + \dots + \alpha_n + \beta_1 + \dots + \beta_n = 1$.

$s_1 = \alpha_1 + \dots + \alpha_n$

$s_2 = \beta_1 + \dots + \beta_n$

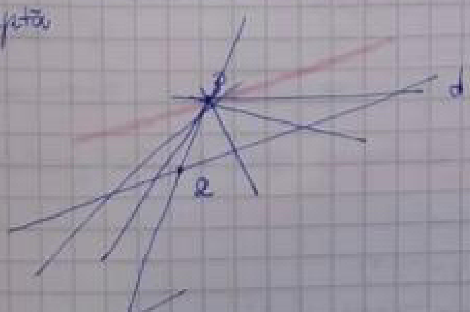
$$= s_1 \left(\underbrace{\frac{\alpha_1}{s_1} P_1 + \dots + \frac{\alpha_n}{s_1} P_n} \right) + s_2 \left(\frac{\beta_1}{s_2} Q_1 + \dots + \frac{\beta_n}{s_2} Q_n \right) =$$

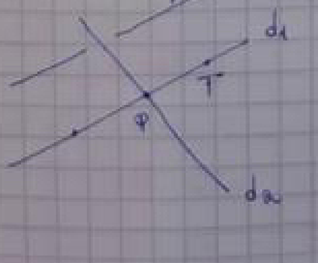
$$= \frac{\alpha_1 + \dots + \alpha_n}{s_1} = \frac{s_1}{s_1} = 1$$

$A_1 = \{P\}$ Contraexemplu:

$A_2 = d$ dreaptă

$P \notin d$





Ex: Fie $\mathbb{R}^4$ sp. afin cu str. afină canonică.

Fie $A_1 = \{(x, y, z) \mid x+y=1, \ x-y=0\} \longrightarrow x=\frac{1}{2}, \ y=\frac{1}{2}$

$A_2 = \{(x, y, z) \mid y+z=-1, \ y-z=0\} \longrightarrow y=-\frac{1}{2}, \ z=-\frac{1}{2}$

Se cere ec. lui $A_1 \vee A_2$.

$\dim A_1 = \dim A_2 = 2.$

$\dim(A_1 \vee A_2) > 2$

$\operatorname{dir}(A_1) = \begin{cases} x+y=0 \\ x-y=0 \end{cases} \Rightarrow x=y=0$

$\left.\begin{array}{l} \operatorname{dir}(A_1) = \{(0,0,z,t) \mid t, z \in \mathbb{R}\} \\ \operatorname{dir}(A_2) = \{(x,0,0,t) \mid x, t \in \mathbb{R}\} \end{array}\right\}$ intersect. după $(0,0,0,t)$

$\dim(\operatorname{dir}(A_1) + \operatorname{dir}(A_2)) = \underbrace{2+2-1}_{\text{(Grassmann)}} = 3$

$$\boxed{\operatorname{dir}(A_1 \vee A_2) = \operatorname{dir}(A_1) + \operatorname{dir}(A_2) + \langle \overrightarrow{O_1O_2} \rangle, \quad O_1 \in A_1, \ O_2 \in A_2.}$$

$\langle \overrightarrow{O_1O_2} \rangle \neq 0 \Rightarrow \dim(A_1 \vee A_2) = 4$

$\Rightarrow A_1 \vee A_2 = \mathbb{R}^4$

$\Rightarrow$ <u>ec: $0=0$.</u>

7) Fie $A = \mathbb{R}^2$ cu str. afină canon.

$f: A \to A$

$f(x_1, x_2) = (2x_1 + x_2 - 1, \ x_1 + 2x_2 - 3)$ e <u>transf. afină</u>.

<u>Sol</u>: $P, Q \in A$.

$P(x_1, y_1) \quad Q(x_2, y_2)$

Fie $\alpha, \beta \in \mathbb{R}$ cu $\alpha + \beta = 1$.

Demonstr. că $f(\alpha P + \beta Q) = \alpha f(P) + \beta(f(Q))$

$f(\alpha x_1 + \beta x_2, \ \alpha y_1 + \beta y_2) = (2\alpha x_1 + 2\beta x_2 + \alpha y_1 + \beta y_2 - 1, \ \alpha x_1 + \beta x_2 + 2\alpha y_1 + 2\beta y_2 - 3)$

$\alpha f(x_1, y_1) + \beta f(x_2, y_2) = \ldots$

$\rightarrow$ sunt egale.

2) Fie $A = \mathbb{R}^3$ și $f: A \to A$ (cu redactare)

$f(x_1, x_2, x_3) = (x_1 + x_3, x_1 + x_2, x_2 - x_3 - 1)$

a) Arătați că f e transf. afină și decideți dacă e bijectivă.

b) Fie punctele $P(2, -1, 1)$, $Q(1, 0, 2)$.

Determ. imag. prin f a dreptei PQ ; $f(PQ) = ?$.

c) Există drepte $d \subset A$ a.î. $f(d) \parallel d$? Justificați.

cu $f(d)$ dreaptă.

Sol: a) Def $\to$ transf. afină: $f(x) = Ax + B$.

$\det(A) \neq 0 \Rightarrow$ f bijectivă.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Avem $A \cdot X + B = f(x)$ ✓ $\Rightarrow$ f e afină

$$\det(A) = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + 1 \cdot (-1)^4 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 + 1 = 0 \Rightarrow f \text{ nu e biject}$$

Def b) Dacă $f(P) = f(Q) = T$, Atunci Im $f(PQ) = \{T\}$

Dem: Fie $R \in PQ$; $R = \alpha P + (1-\alpha) Q$

$\Rightarrow f(R) = \alpha f(P) + (1-\alpha) f(Q) = \alpha T + (1-\alpha) T = T$.

Sol: $f(Q) = f(1, 0, 2) = (3, 1, -3)$

$f(P) = f(2, -1, 1) = (3, 1, -3)$

$f(P) = f(Q) \Rightarrow f(PQ) = T$; $T(3, 1, -3)$.

Def $\to$ c) Dacă d există, urma liniară T_f va păstra direcția.

$T_f(\operatorname{dir}(d)) = \operatorname{dir}(d)$.

$\operatorname{dir}(d) = \langle v \rangle$

$\Rightarrow T(u) = \lambda \cdot u$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$

$\Rightarrow$ Există astfel de drepte $\Leftrightarrow$ Tf are măcar un vector propriu asociat unei valori proprii nenule.

Ne uităm la polinomul caracteristic.

e) Determ. imaginea lui f, $Im(f)$.

f) Fie punctele $A=(1,0,1)$ $B=(1,1,1)$, $C=(2,1,1)$.

Determ. $f((ABC))$

Sol: e)

Def → [În op. afine nu avem nucleu ($Ker(f)$).

Im f nu e punct (nu e constanta)

Im f nu e plan (f nu e bijectivă)

Rangul matricei e dimens. imaginii.

Def → [$f(x)=AX+B$, atunci $\dim(Im(f)) = rg(A)$

Dem: $\dim(Im(f)) = \dim(Im(T_f)) = rg(A)$.

$$A=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$\det = 0 \Rightarrow f$ nu e biject.

rang $A = 2 \Rightarrow \dim(Im(f)) = 2 \Rightarrow Im(f) = $ plan

Def → [Dacă $T: V \to W$ apl. lin și $\{e_1, ..., e_n\}$ sist. generat pt V

$\Rightarrow \{T(e_1), ..., T(e_n)\}$ sist. gener. pt. imaginea $T(V)$.

Def → [Dacă $\mathcal{E}: \mathcal{A}_1 \to \mathcal{A}_2$ apl. afină și $\{P_1, ..., P_n\}$ sist. afin de gener.

$\Rightarrow \{\mathcal{E}(P_1), ..., \mathcal{E}(P_n)\}$ sist. afin de gener. pt $\mathcal{E}(\mathcal{A}_1)$.

$Im(f)$ e un plan, generat de pct $\underbrace{f(0,0,0)}_{A}$, $\underbrace{f(1,0,0)}_{B}$,

$\underbrace{f(0,1,0)}_{C}$, $\underbrace{f(0,0,1)}_{D}$.

(ABC):

$A: f(0,0,0) = (0,0,-1)$

$B = f(1,0,0) = (1,1,-1)$

$C: f(0,1,0) = (0,1,0)$

$D = f(0,0,1) = (1,0,-2)$

(ABC): Sunt A, B, C, D afin ind?

$\Leftrightarrow \vec{xy}$ și $\vec{xz}$ indep.; $\vec{xy} = (1,1,0)$, $\vec{xz} = (0,1,1)$ sunt indep.

$$\begin{vmatrix} y_1 & 1 & 0 \\ y_2 & 1 & 1 \\ y_3+1 & 0 & 1 \end{vmatrix} = 0$$

$\Leftrightarrow y_1 - y_2 + y_3 = 0$

$\Rightarrow \operatorname{Im}(f) = y_1 - y_2 + y_3 = 0$

la matrice $l_3 - l_2 + l_1 = 0$

$y_3 - y_2 + y_1 + 1 = 0$ ✓

<u>Sau</u> : (metoda algoritmică)

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_3 = y_1 \\ x_1 + x_2 = y_2 \\ x_2 - x_3 - 1 = y_3 + 1 \end{cases}$$

$\Rightarrow$ pt. (y_1, y_2, y_3) e imagine $\Leftrightarrow$ sist e compatibil.

$\Leftrightarrow rg(A) = rg(A \text{ extins})$

$$A \text{ extinsă} = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}} & \begin{matrix} 1 \\ 0 \end{matrix} & \begin{matrix} y_1 \\ y_2 \end{matrix} \\ \begin{matrix} 0 & 1 \end{matrix} & -1 & y_3+1 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & y_1 \\ 1 & 1 & y_2 \\ 0 & 1 & y_3+1 \end{vmatrix} = 0$$

f) $\vec{MN} = (0, 1, 0)$

$\vec{NP} = (1, 1, 0)$

$\Rightarrow M, N, P$ determină un plan.

$(MNP): x_3 = 1$

utilizând descrierea parametrică:

$(MNP) = \{ (t, s, 1) \mid t, s \in \mathbb{R} \}$

$f(t, s, 1) = (t+1, t+s, s-2)$ $t \in \mathbb{R}$, $s \in \mathbb{R}$

~~$f(t, s, 1) =$~~

$f(MNP) = \{ (t+1, t+s, s-2) \mid t, s \in \mathbb{R} \}$.

Descrierea parametrică a planului.

$$f(MNP): \begin{cases} y_1 = t+1 \\ y_2 = t+s \\ y_3 = s-2 \end{cases} \Rightarrow t = y_1 - 1$$

$$s = y_3 + 2$$

$$y_2 = t + s = \ldots$$

$$f(MNP) = y_1 - y_2 + y_3 - 1 = 0.$$

g) Există plane π a.î. $f(\pi)$ să fie plan și $f(\pi) \parallel \pi$?

<u>Sol</u>: π = orice plan $\parallel \operatorname{Im}(f)$

lucrăm pt $f(x_1, x_2, x_3) = (x_1 + x_3,\ x_1 + x_2,\ x_2 + x_3 - 1)$

Aici, determinantul $\det(A) \neq 0$.

$T(\operatorname{dir}(\pi)) = \operatorname{dir}(\pi)$

$\operatorname{dir}(\pi)$ = 2 dimensională

Dacă am avea 2 vectori proprii $T(e_1) = \lambda_1 e_1$, atunci orice $T(e_2) = \lambda_2 e_2$

plan de direcție generată de ei ar satisface relația.

$$\operatorname{dir}(\pi) = \langle e_1, e_2 \rangle \Rightarrow \operatorname{dir}(f(\pi)) = \langle f(e_1), f(e_2) \rangle = \langle \lambda_1 e_1, \lambda_2 e_2 \rangle = \langle e_1, e_2 \rangle.$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

P_A are gradul 3 $\Rightarrow$ are o rădăcină reală

$\Rightarrow \exists \lambda (\neq 0)$ $P_A(\lambda) = 0$

$\Rightarrow$ A are un vector propriu

Metodă ec. grad 3:

$$\boxed{x^3 - 6x + 1 = 0}$$

$x = m + n$

$x^3 = m^3 + n^3 + 3m^2 n + 3mn^2$

$$x^3 = \underbrace{m^3 + n^3}_{q} + \underbrace{3mn}_{p}\underbrace{(m+n)}_{x}$$

Rezolvăm (algebric - liniar): $x^3 = px + q$

$x^3 = 6x - 1$

$$\Rightarrow \begin{cases} p = 6 \\ q = -1 \end{cases} \Rightarrow \begin{cases} m^3 + n^3 = -1 \\ 3mn = 6 \\ mn = 2 \end{cases}$$

$$A \sim \begin{pmatrix} e_1 & 0 & 0 \\ 0 & \boxed{\text{Bloc}} & \\ 0 & \boxed{\text{Jordan}} & \end{pmatrix}$$

e_1, e_2, e_3

$f(e_2)$ $f(e_3)$

Dacă are 2 răd. complexe $\Rightarrow$ A diagon. m.$\mathbb{C}$ $\overset{3}{\Rightarrow}$ $m^3 + \frac{8}{m^3} = -1$

Ar exista $w \in \mathbb{C}^3$ vect. propriu pt A

$$\boxed{t^2 + 8 + t = 0}$$

$Aw = \lambda w$

$= \{R_\alpha \mid \alpha \in [0, 2\pi)\} \cup \{S_\alpha \mid \alpha \in [0, 2\pi)\}$

$\uparrow$

simetria față de dreapta d a.î.
$\angle(Ox, d) = \frac{\alpha}{2}$.

• $n = 3$. și mai departe ?

a)

Seminar 5 — Vivi — Victor Vulețescu

1) Care dintre fct. următoarele sunt tr. afine bijective:

a) $f(x, y) = (x+7, y-7)$ ✓ e afină ; $\det(A) = 1$ e biject.

b) $f(x,y) = (x^3+7, 2y)$ X nu e afină

c) $f(x,y) = (7, x+y)$ X e afină ; nu e biject

d) $f(x,y) = (7x+2, 2x+7)$ X e afină ; nu e biject

Def $\rightarrow$ [$f: K^n \to K^n$ trf. afină $\Leftrightarrow$
f e de forma
$f(x) = A \cdot x + B$; A, B matrice
$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

a) $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ translație

$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 7 \\ -7 \end{pmatrix}$

c) $f(x) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

d) $f(x) = \begin{pmatrix} 7 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

Def $\rightarrow$ [f biject $\Leftrightarrow$ A inversabilă
$\Leftrightarrow \det(A) \neq 0$.

25.III.2022

Seminar 6

Algoritmul G-S

Orientare

Fixăm V = sp. vectorial peste $\mathbb{R}$, $\dim_{\mathbb{R}}(V) = n$.

Două baze B_1, B_2 ale lui V sunt la fel orientate dacă $\det A > 0$ (unde A = matricea de trecere de la B_1 la B_2)

Obs: Fie $\mathcal{B} = \{ B' \in V \mid B' = \text{bază a lui } V \}$

Definim relația „$\sim$" pe $\mathcal{B}$ prin $B_1 \sim B_2$ dacă sunt la fel orientate

Prop: • „$\sim$" este o relație de echivalență

• $\mathcal{B}/_{\sim}$ are exact două elemente.

Dem: • Reflexiv.: $B \sim B$: matricea de trecere I_n, $\det > 0$.

• Sim : $B_1 \sim B_2$; arăt că $B_2 \sim B_1$

$\Downarrow$

$\det(A_{B_1 B_2}) > 0$; dar $A_{B_2 B_1} = A^{-1}_{B_1 B_2}$

$$\det(A_{B_2 B_1}) = \frac{1}{\det(A_{B_1 B_2})} > 0$$

• Tranzitiv. : $B_1 \overset{A}{\sim} B_2$

$B_2 \overset{B}{\sim} B_3$

$\overline{\quad B_1 \underset{BA}{\sim} B_3 \quad}$

$\det(BA) = \det B \cdot \det A > 0$.

• $\text{Card}(\mathcal{B}/_{\sim}) = 2$

↳ Există o bază B a lui $V \Rightarrow [B_0] \in \mathcal{B}/_{\sim}$

deci $|\mathcal{B}/_{\sim}| \geq 1$.

Dacă $B_0 = (e_1, \dots, e_n)$; $B_1 = (-e_1, e_2, \dots, e_n)$

$$\Rightarrow A = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & I_{n-1} \end{array}\right)$$

$\det A = -1 < 0$

$B_1 \not\sim B_0 \Rightarrow \mathrm{Card}\,(\mathcal{B}/_\sim) = 2$

$\qquad [B_0], [B_1]$

$\qquad$ clase de echiv.

Fie $[B] \in \mathcal{B}/_\sim$

- $B \sim B_0 \Rightarrow [B] = [B_0]$
- $B \not\sim B_0 \Rightarrow \det(A_{B_0 B}) < 0$

Însă $B_0 \not\sim B_1$, $\det(A_{B_0 B_1}) < 0$

Dar atunci $A_{B B_1} = \underset{<0}{A_{B_0 B_1}} \cdot \underset{<0}{A_{B B_0}}$

$$\det(A_{B B_1}) > 0 \Rightarrow [B] = [B_1]$$

Def: Un element din $\mathcal{B}/_\sim$ s.n. orientare a lui V.

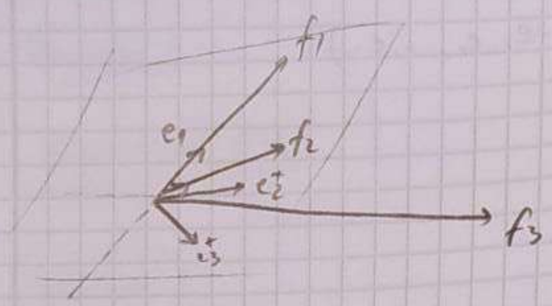
Pe $\mathbb{R}^n$ definim orientarea pozitivă ca fiind clasa de echivalență $[B]$, unde B = baza canonică.

$\boxed{\text{Algoritmul G-S}}$

Fie $(V, \langle , \rangle)$ sp. vectorial euclidian, $\mathcal{B}_0 = \{f_1, \dots, f_n\}$ bază arbitrară a lui V.

Atunci $\exists!$ o bază ORTONORMATĂ $\{e_1, \dots, e_n\}$ a lui V caracterizată de:

- $\forall i = \overline{1, n}$, subspațiile vectoriale $\langle f_1, \dots, f_i \rangle$ și $\langle e_1, \dots, e_i \rangle$ coincid
- și, în plus, bazele $(f_1, \dots, f_i)$ și $(e_1, \dots, e_i)$ sunt la fel orientate.



$e_1 = \frac{f_1}{\|f_1\|}$

Ip că am construit $e_1, \dots, e_{i-1}$

Construim pe e_i.

Caut un vector $g_i \perp$ pe toți $e_1, \dots, e_{i-1}$

$$g_i \in \langle f_1, \dots f_{i-1}, f_i \rangle = \langle e_1, \dots, e_{i-1}, f_i \rangle$$

$$g_i = f_i + \sum_{j=1}^{i-1} \alpha_j e_j$$

$$g_i \perp e_k, \quad k = \overline{1, i-1}$$

$$\langle g_i, e_k \rangle = \langle f_i, e_k \rangle + \sum_{j=1}^{i-1} \alpha_i \langle e_j, e_k \rangle = \langle f_i, e_k \rangle + \alpha_k \underbrace{\langle e_k, e_k \rangle}_{\|e_k\|^2 = 1}$$

$$\boxed{\alpha_k = -\langle f_i, e_k \rangle}$$

$$\boxed{\text{Deci, } g_i = f_i - \sum_{j=1}^{i-1} \langle f_i, e_j \rangle e_j}$$

$$e_i = \frac{1}{\|g_i\|} g_i$$

Q1: $\|g_i\| \neq 0$? Dacă $\|g_i\| = 0 \Rightarrow g_i = 0 \Rightarrow f_i \in \langle e_1, \dots, e_{i-1} \rangle = \langle f_1, \dots f_{i-1} \rangle$
$\Rightarrow \{f_1, \dots, f_{i-1}, f_i\}$ liniar dependent ↯

Q2: $\langle e_1, \dots, e_i \rangle$ la fel orientată cu $\langle e_1, \dots, e_{i-1}, g_i \rangle$.

$$A = \begin{pmatrix} I_n & \\ & \frac{1}{\|g_i\|} \end{pmatrix}$$

$$g_i = f_i - \sum \langle \dots \rangle e_i$$

① Să se ortonormeze cu G-S baza:

$f_1 = (1,1,1)$, $f_2 = (0,-1,1)$, $f_3 = (0,0,-1)$ $\in \mathbb{R}^3$

$(\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$

$\| \underbrace{(x_1, x_2, x_3)}_{v} \| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Sol: $e_1 = \frac{f_1}{\|f_1\|} = \frac{1}{\sqrt3}(1,1,1) = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$

$\|f_1\| = \|(1,1,1)\| = \sqrt{1^2+1^2+1^2} = \sqrt3$

$g_k = f_k - \sum_{j=1}^{k-1} \langle f_k, e_j \rangle e_j$

$g_2 = f_2 - \langle f_2, e_1 \rangle e_1$

$g_2 = (0,-1,1) - \frac{1}{\sqrt3} \langle (0,-1,1), \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) \rangle (1,1,1)$

$= (0,-1,1) - \frac{1}{3}(0 \cancel{-1} + \cancel{1})(1,1,1)$

$= (0,-1,1)$

$e_2 = \frac{g_2}{\|g_2\|}$

$e_2 = \frac{1}{\sqrt2}(0,-1,1)$

$g_3 = f_3 - (\langle f_3, e_1 \rangle e_1 + \langle f_3, e_2 \rangle e_2)$

$g_3 = (0,0,-1) - \left(\frac{1}{3} \langle (0,0,-1), (1,1,1) \rangle (1,1,1) + \frac{1}{2} \langle (0,0,-1), (0,-1,1) \rangle \cdot \right.$

$\left. \cdot (0,-1,1) \right)$

$g_3 = (0,0,-1) - \left(\frac{1}{3} \cdot (-1)(1,1,1) + \frac{1}{2}(-1)(0,-1,1) \right) \ldots$

$\ldots$

$g_3 = \left(\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}\right) = \frac{1}{6}(2,-1,-1)$

$e_3 = \frac{g_3}{\|g_3\|} = \frac{1}{\sqrt6}(2,-1,-1)$

$\|g_3\| = \frac{1}{6}\|(2,-1,-1)\| = \frac{1}{\sqrt6}$

Verificare: $\begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \quad A^T A = I_3$$

② Fie $\langle (x_1, x_2), (y_1, y_2) \rangle = \alpha x_1 y_1 + \beta x_1 y_2 + \gamma x_2 y_1 + \delta x_2 y_2$
cu $\beta = \gamma$

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

$\langle , \rangle$ produs scalar $\Leftrightarrow \begin{cases} \alpha > 0 \\ \alpha\delta - \beta^2 > 0 \end{cases}$

Dem: „$\Rightarrow$" $\underbrace{\langle (1,0), (1,0) \rangle}_{\alpha > 0} > 0$

$$\langle (x_1, x_2), (x_1, x_2) \rangle = \alpha x_1^2 + 2\beta x_1 x_2 + \delta x_2^2 =$$
$$= \alpha \left(x_1^2 + 2\frac{\beta}{\alpha} x_1 x_2 + \left(\frac{\beta}{\alpha}\right)^2 x_2^2\right) - \frac{\beta^2}{\alpha} x_2^2 + \delta x_2^2$$
$$= \underbrace{\alpha}_{>0} \left(x_1 + \frac{\beta}{\alpha} x_2\right)^2 + \left(\frac{\delta\alpha - \beta^2}{\alpha}\right) x_2^2 \geq 0, \quad \forall x_1, x_2$$
$$\Updownarrow$$
$$\delta\alpha - \beta^2 > 0$$

01.IV.2022

Seminar 7

① $\mathbb{R}^3$ cu str. afină canonică: $A(1,0,1)$, $B(2,1,1)$, $C(0,2,3)$, $M(1,1,1)$.
Atunci AM este: a) bisectoarea lui $\widehat{BAC}$

~~b)~~ mediana din A în $\triangle ABC$

c) înălțimea din A în $\triangle ABC$

Nu, pt că $M \neq \frac{1}{2}B + \frac{1}{2}C$

Dacă $M = (1, \frac{3}{2}, 2) \Rightarrow$ AM mediană în $\triangle ABC$

AM bisectoare $\overset{?}{\Leftrightarrow}$ $\triangle ABC$ isoscel în A

$\|\vec{AB}\| = \sqrt{1+1+0} = \sqrt{2}$

$\|\vec{AC}\| = \sqrt{1+4+...} > \sqrt{2}$ $\Big\} \Rightarrow$ Nu e isoscel.

28.03.2022

Aplicații afine
Clase speciale de aplicații afine

Def: Fie $(A_i, V_i/k, \varphi_i)$, $i=\overline{1,2}$ – 2 spații afine.
O apl $\tau: A_1 \to A_2$ s.n. apl. afină (transf. afină) dacă:

$\exists\ O \in A$, a.î. $T_\tau: V_1 \to V_2$ apl. liniară

$T_\tau(\overrightarrow{OP}) = \overrightarrow{\tau(O)\tau(P)}$, $\forall P \in A$

$T_\tau \to$ urma apl. liniare

- $\tau: \underset{R_1}{A_1} \to \underset{R_2}{A_2}$ apl. afină

$\to Y = AX + B$ (ec. liniare).

- $\tau: \underset{R}{A} \to \underset{R}{A}$ apl. afină bijectivă

$\to Y = AX + B$ (ec. liniare)

$\underline{\det A \neq 0}$!

Aplicație: (seminar 6).

$\tau: \mathbb{R}^3 \to \mathbb{R}^3$, $\tau(O) = (1,0,0)$ $\qquad \tau(A_2) = (1,2,0)$

$\tau(A_1) = (3,0,0)$ $\qquad \tau(A_3) = (1,-1,1)$

a) ec. apl. τ
b) pct. fixe

Soluție: a) $Y = AX + B$.

$P(x_1, x_2, x_3)$
$\tau(P)(y_1, y_2, y_3)$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + B$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (A) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + B \Rightarrow B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix}3\\0\\0\end{pmatrix} = \left(\underset{\substack{a_{11}\\a_{21}\\a_{31}}}{\underline{A}}\right)\begin{pmatrix}1\\0\\0\end{pmatrix} + B \Rightarrow \begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\0\end{pmatrix} = \left(\underset{\substack{a_{12}\\a_{22}\\a_{32}}}{\underline{A}}\right)\begin{pmatrix}0\\1\\0\end{pmatrix} + B \Rightarrow \begin{pmatrix}a_{12}\\a_{22}\\a_{32}\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\-1\\1\end{pmatrix} = \left(\underset{\substack{a_{13}\\a_{23}\\a_{33}}}{\underline{A}}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} + B \Rightarrow \begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix}2 & 0 & 0\\0 & 2 & -1\\0 & 0 & 1\end{pmatrix}, \quad B = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad Y = AX + B.$$

$\det(A) = 4 \neq 0 \Rightarrow \tau$ - transf. afină bijectivă

b) $\tau(P) = P \Leftrightarrow X = AX + B \Leftrightarrow y_i = x_i, \ \forall i = \overline{1,3}$.

$$Y = AX + B \Rightarrow \begin{cases} y_1 = 2x_1 + 1 \\ y_2 = 2x_2 - x_3 \\ y_3 = x_3 \end{cases} \rightarrow \begin{cases} x_1 = 2x_1 + 1 \Rightarrow -x_1 = 1 \Rightarrow x_1 = -1 \\ x_2 = 2x_2 - x_3 \Rightarrow x_2 = x_3 \ (=t) \\ x_3 = x_3 \end{cases}$$

$\mathcal{P} = \{P \in \mathbb{R}^3 \mid \tau(P) = P\} = \{(-1, t, t) \mid t \in \mathbb{R}\}$

mulț. pct. fixe este dr. $= \{(-1,0,0) + t\underbrace{(0,1,1)}_{\text{vector director}} \mid t \in \mathbb{R}\}$

de ec $\longrightarrow \dfrac{x_1+1}{0} = \dfrac{x_2}{1} = \dfrac{x_3}{1} = t$

$\operatorname{dir} \mathcal{P} = \langle (0,1,1) \rangle$

Clase speciale

- Fie $(A, V/_{K}, \varphi)$ spațiu afin.

  $\tau : A \to A$ s.n. <u>translație</u> dacă:

  $$(\exists) O \in A, (\forall) P \in A \text{ ai } \underbrace{\overrightarrow{\tau(O)\tau(P)}}_{T_\tau(\overrightarrow{OP})} = \overrightarrow{OP}$$

  $T_\tau = \mathbb{1}_V$ - apl. identică

  $\downarrow$

  urma.

- • Fie $O \in A$, $k \in K^*$.

  $h_O^k : A \to A$, $h_O^k(P) = P'$

  $\overrightarrow{OP'} = k\overrightarrow{OP}$, $\forall P \in A$

  s.n. <u>omotetie</u> de centru O și putere k.

$O h_O^k(P) = k\overrightarrow{OP} + (1-k)\overrightarrow{OO}$, $\forall P \in A$

$$\boxed{h_O^k(P) = kP + (1-k)O, \ \forall P \in A}$$

Considerăm: $\mathcal{H}_O$ - mulț. omotetiilor de centru O și diverse puteri $k \in K^*$.

Fie $h_O^{k_1}, h_O^{k_2} \to$ 2 omotetii de același centru și puteri k_1, k_2

$$(h_O^{k_2} \circ h_O^{k_1})(P) = h_O^{k_2}(h_O^{k_1}(P)) = h_O^{k_2}(k_1P + (1-k_1)O) =$$
$$= k_2k_1P + k_2(1-k_1)O + (1-k_2)O$$
$$= k_2k_1P + (1-k_1k_2)O = h_O^{k_1k_2}(P).$$

$h_O^1 \to$ elem. neutru

$(h_O^k)^{-1} = h_O^{k^{-1}} = h_O^{\frac{1}{k}}$

$(\mathcal{H}_O, \circ)$ - grup.

$f : \mathcal{H}_O \to K^*$, $f(h_O^k) = k$, $\forall h_O^k \in \mathcal{H}_O$.

$\hookrightarrow$ izom. de grupuri; $(\mathcal{H}_O, \circ) \simeq (K^*, \cdot)$

Mult. omotetiilor de centre și puteri diferite.

Fie $h_{O_1}^{k_1}, h_{O_2}^{k_2}$ - două omotetii

$$\left(h_{O_2}^{k_2} \circ h_{O_1}^{k_1}\right)(P) = h_{O_2}^{k_2}\left(h_{O_1}^{k_1}(P)\right) = h_{O_2}^{k_2}\left(k_1 P + (1-k_1) O_1\right)$$

$$= k_2 k_1 P + k_2(1-k_1) O_1 + (1-k_2) O_2$$

caz 1. dc. $k_1 k_2 \neq 1$ $\Rightarrow k_2 k_1 P + \underbrace{\left(\frac{k_2(1-k_1)}{1-k_1k_2} O_1 + \frac{1-k_2}{1-k_1k_2} O_2\right)}_{\text{not. } O} \cdot (1-k_1k_2)$

$$O \overset{\text{not}}{=} \frac{k_2(1-k_1)}{1-k_1k_2} O_1 + \frac{1-k_2}{1-k_1k_2} O_2.$$

$$\left(h_{O_2}^{k_2} \circ h_{O_1}^{k_1}\right)(P) = k_2 k_1 P + (1-k_1k_2) O = h_O^{k_1k_2}(P)$$

caz 2. dc. $k_1 k_2 = 1$

$$\left(h_{O_2}^{k_2} \circ h_{O_1}^{k_1}\right)(P) = P + (k_2-1) O_1 + (1-k_2) O_2 \overset{\text{not}}{=} P'$$

$$\overrightarrow{PP'} = (k_2-1)\underbrace{\overrightarrow{PO_1}}_{= -\overrightarrow{O_1P}} + (1-k_2)\overrightarrow{PO_2} = (1-k_2)\overrightarrow{O_1O_2} \rightarrow \text{vector constant}$$

$\Downarrow$

$$\Rightarrow \left(h_{O_2}^{k_2} \circ h_{O_1}^{k_1}\right) = t_{(1-k_2)\overrightarrow{O_1O_2}}$$

$\hookrightarrow$ translație.

În concluzie, mult. omotetiilor de diverse centre și puteri nu mai formează o structură de grup.

$\boxed{\text{Apl}}$ Fie $A_1A_2A_3A_4$ - tetraedru oarecare;

$G_i, i=\overline{1,4}$ - centrul de greutate al feței tetraedrului care se opune vârfului $A_i, i=\overline{1,4}$.

Arătați că tetraedrul $G_1G_2G_3G_4$ este imaginea tetraedrului $A_1A_2A_3A_4$ printr-o omotetie.

Rezolvare: fie G_4 - centrul de greutate al $\Delta A_1A_2A_3$.

$$G_4 = \frac{1}{3}A_1 + \frac{1}{3}A_2 + \frac{1}{3}A_3.$$

G - centrul de greutate al tetraedrului $A_1A_2A_3A_4$

$$G = \frac{1}{4}A_1 + \frac{1}{4}A_2 + \frac{1}{4}A_3 + \frac{1}{4}A_4$$

$\Rightarrow G = \frac{3}{4} G_4 + \frac{1}{4} A_4$

Analog, rezultă că $G = \frac{3}{4} G_i + \frac{1}{4} A_i$, $i = \overline{1,4}$.

$$\overrightarrow{GG} = \frac{3}{4}\overrightarrow{GG_i} + \frac{1}{4}\overrightarrow{GA_i}$$

$$\overrightarrow{GG} = \vec{0}$$

$$\Downarrow$$

$$\overrightarrow{GG_i} = -\frac{1}{3}\overrightarrow{GA_i}, \ \forall i = \overline{1,4}.$$

$$h_O^k(P) = P' \Leftrightarrow \overrightarrow{OP'} = k\overrightarrow{OP}$$

$$h_G^{-1/3}(A_i) = G_i, \forall i = \overline{1,4}$$

Tetr. $G_1G_2G_3G_4$ este imaginea lui $A_1A_2A_3A_4$ printr-o omotetie de centru G și raport $k = -1/3$.

Apl: Ec: a) unei translații.

b) unei omotetii

c) simetria față de un punct

Rezolvare: $\tau: \mathcal{A} \to \mathcal{A}$ transf. afină, $\mathcal{R}_c = (O, \{e_1, \dots, e_n\})$ → reper cartezian

$P \in \mathcal{A}$ $(x_1, x_2, \dots, x_m)_{\mathcal{R}_c}$

$\tau(P)$ $(y_1, y_2, \dots, y_m)$

a) τ translație de vector $v = (v_1, \dots, v_m)$

$\overrightarrow{P\tau(P)} = \vec{v} \Rightarrow y_i - x_i = v_i, \forall i = \overline{1,m}$.

$\Rightarrow y_i = x_i + v_i, \forall i = \overline{1,m}$.

$$Y = AX + B; \quad A = I_m; \quad B = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

b) Fie $O \in \mathcal{A}$, $k \in K^*$.

$O = (a_1, \dots, a_n)$

$h_O^k(P) = P'$ a.î. $\boxed{\overrightarrow{OP'} = k\overrightarrow{OP}}$ (*).

$P(x_1, \dots, x_m)$

$h_O^k(P) = P'(y_1, \dots, y_m)$ $\mathcal{R}_c$

din $\circledast \Rightarrow$ $y_i - a_i = k(x_i - a_i), \forall i = \overline{1,n}$.

$y_i = k x_i + (1-k) a_i, \forall i = \overline{1,n}$

$Y = AX + B, \quad A = k I_n, \quad B = \begin{pmatrix} \vdots \\ a_i(1-k) \\ \vdots \end{pmatrix}$

c) fie $O \in A$, $S_O(P) = P' \Rightarrow O = \frac{1}{2}P + \frac{1}{2}P'$.

R_c $O(a_1, \dots, a_n)$
$P(x_1, \dots, x_n)$
$P'(y_1, \dots, y_n)$

$a_i = \frac{1}{2}x_i + \frac{1}{2}y_i, \forall i = \overline{1,n}$

$y_1 = 2a_i - x_i, \forall i = \overline{1,n}$

$Y = AX + B, \quad A = -I_n, \quad B = 2\begin{pmatrix} \vdots \\ a_i \\ \vdots \end{pmatrix}$

Apl. În $\mathbb{R}^3$ cu str. af. canonică se consideră pct:

$$\begin{cases} A_0(1,-1,2) \\ A_1(1,0,1) \\ A_2(1,0,-1) \end{cases}$$

Să se scrie ec:

a) $h^3_{A_0}$

b) S_{A_0}

c) $S_{A_0} \circ h^3_{A_0}(e)$, $e = \frac{2}{3}A_1 + \frac{1}{3}A_2$

Rezolvare: a) fie $P(x_1, x_2, x_3)$ R_c

$P' = h^3_{A_0}(P)(y_1, y_2, y_3)$ R_c

$\overrightarrow{A_0P'} = 3\overrightarrow{A_0P}$

$(y_1 - 1, y_2 + 1, y_3 - 2) = 3(x_1 - 1, x_2 + 1, x_3 - 2)$

$$\begin{cases} y_1 - 1 = 3x_1 - 3 \\ y_2 + 1 = 3x_2 + 3 \\ y_3 - 2 = 3x_3 - 6 \end{cases} \Rightarrow \begin{cases} y_1 = 3x_1 - 2 \\ y_2 = 3x_2 + 2 \\ y_3 = 3x_3 - 4 \end{cases} \Rightarrow \begin{array}{l} Y = AX + B. \\ A = 3I_3 \\ B = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} \end{array}$$

b) $P(x_1, x_2, x_3)$ $\mathcal{R}_c$

$P' = S_{A_0}(P)\ (y_1, y_2, y_3)$

$A_0 = \frac{1}{2}P + \frac{1}{2}P' \Rightarrow P' = 2A_0 - P = 2(1,-1,2) - (x_1, x_2, x_3) = (2-x_1, -2-x_2, 4-x_3)$

$$\Rightarrow \begin{cases} y_1 = 2 - x_1 \\ y_2 = -2 - x_2 \\ y_3 = 4 - x_3 \end{cases} \qquad Y = AX + B$$

$A = -I_3$

$B = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

c) $C = \frac{2}{3}A_1 + \frac{1}{3}A_2 = \frac{2}{3}(0,1,0) + \frac{1}{3}(1,0,-1) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

$S_{A_0} \circ h^3_{A_0}(C) = S_{A_0}(h^3_{A_0}(C)) = S_{A_0}\left(h^3_{A_0}\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)\right) =$

$= S_{A_0}(-1, 4, -5) = (3, -6, 9).$

... Proiecții

Apl. Fie $\mathbb{R}^3$ cu str. afină canonică..

a) Det. proiecția p_1 pe planul $\pi: x_1 - 2x_2 + 2x_3 - 1 = 0$, paralelă cu direcția $V_1 = \langle(1,1,1)\rangle$.

b) Det. $\approx$ p_2 pe dreapta $d: \frac{x_1 - 1}{2} = \frac{x_2 + 1}{-1} = \frac{x_3 - 1}{0}$, $\parallel$ cu direcția $V_2 = \langle(1,0,0), (1,2,1)\rangle$.

Rez: a) Fie $P(a,b,c)$.

$d: \frac{x_1 - a}{1} = \frac{x_2 - b}{1} = \frac{x_3 - c}{1} \overset{not}{=} t$

$$d \cap \pi. \begin{cases} x_1 - 2x_2 + 2x_3 - 1 = 0 \\ x_1 = a + t \\ x_2 = b + t \\ x_3 = c + t, \ t \in \mathbb{R} \end{cases} \rightarrow a + t - 2b - \cancel{2t} + 2c + \cancel{2t} - 1 = 0.$$

$\boxed{t = 1 - a + 2b - 2c}$

$$\begin{cases} x_1 = t + a = 2b - 2c + 1 \\ x_2 = t + b = -a + 3b - 2c + 1 \\ x_3 = t + c = -a + 2b - c + 1. \end{cases}$$

$p_1(a,b,c) = (2b - 2c + 1, -a + 3b - 2c + 1, -a + 2b - c + 1)$

$Y = AX + B$, $A = \begin{pmatrix} 0 & 2 & -2 \\ -1 & 3 & -2 \\ -1 & 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

! verificare: $A^2 = A$ (pt. o proiecție)

↳ A mat. idempotentă $\Leftrightarrow$ p_1 - proiecție.

b) fie $P(a,b,c)$

$$\pi: \begin{vmatrix} x_1 - a & x_2 - b & x_3 - c \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0. \Leftrightarrow$$

$\Leftrightarrow \pi: 2x_3 - 2c - x_2 + b = 0. \ | \cdot (-1). \Leftrightarrow$

$\Leftrightarrow \pi: x_2 - 2x_3 - b + 2c = 0.$

$$\pi \cap d: \begin{cases} x_2 - 2x_3 - b + 2c = 0. \\ x_1 = 2t + 1 \\ x_2 = -t - 1 \\ x_3 = 1., \ t \in \mathbb{R}. \end{cases}$$

$\Rightarrow -t - 1 - 2 - b + 2c = 0$

$\Rightarrow t = -b + 2c - 3.$

$x_1 = -2b + 4c - 6 + 1 = -2b + 4c - 5$

$x_2 = +b - 2c + 3 - 1 = +b - 2c + 2$

$x_3 = 1.$

$\Rightarrow p_2(a,b,c) = (-2b + 4c - 5, b - 2c + 2, 1)$

$Y = AX + B$, $A = \begin{pmatrix} 0 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$

verificare: $\underline{A^2 = A}$ $\Leftrightarrow$ p_2 proiecție

⑧

Scanned with CamScanner

• Det. simetriile aferente proiecțiilor aut.

$s = 2p - id.$

Ex $\longrightarrow s_2^{(a,b,c)} = 2p_2 - id = 2(__, __, __) - (a,b,c) = $ calcule

$s_1(a,b,c) = 2p_1(a,b,c) - \underset{\substack{\parallel \\ (a,b,c)}}{id} =$

$\boxed{\text{Ap}}$: Fie $\mathbb{R}^3$ cu str. afină canonică

$\tau: \mathbb{R}^3 \to \mathbb{R}^3, \tau(x_1,x_2,x_3) = (x_1+2x_2-2,\ x_1+x_2-x_3+1,\ 2x_1+3x_2-x_3+2)$

Considerăm $d: \frac{x_1-1}{1} = \frac{x_2-1}{2} = \frac{x_3-1}{0} \overset{not}{=} t$

$\pi: x_1 + x_2 - 2x_3 - 1 = 0.$

Det. a) $\tau(d)$
b) $\tau(\pi)$

Rez: a) $d: \begin{cases} x_1 = t+1 \\ x_2 = 2t+1 \\ x_3 = 1 \end{cases}, t \in \mathbb{R}.$

$\tau(d) = \{(t+1+4t+2-1,\ t+1+2t+1-\cancel{1}+\cancel{1},\ 2t+2+6t+3-1+2) \mid t \in \mathbb{R}\}$

$\tau(d) = \{(5t+2,\ 3t+2,\ 8t+6) \mid t \in \mathbb{R}\}$

$= \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \exists\, t \in \mathbb{R}$ a.î. $\begin{cases} x_1 = 5t+2 \\ x_2 = 3t+2 \\ x_3 = 8t+6. \end{cases}\}$

$\Downarrow$

$\tau(d): \underbrace{\frac{x_1-2}{5} = \frac{x_2-2}{3} = \frac{x_3-6}{8}}_{\text{dr. afină.}}$

temă m_2: cu 2 pcte dif.

b) $\pi = \{(\alpha, \beta, \frac{1}{2}(\alpha+\beta-1) \mid \alpha, \beta \in \mathbb{R}\}$

04.04.2022

Seminar 8

1. Apl. afine
2. Sp. afine euclidiene

1 $\boxed{Apl.}$ Fie $\pi_1 \parallel \pi_2$

$\mathcal{C}_1(O_1, R_1) \subset \pi_1$

$\mathcal{C}_2(O_2, R_2) \subset \pi_2$ a.î. $O_1O_2 \perp \pi_1, \pi_2$

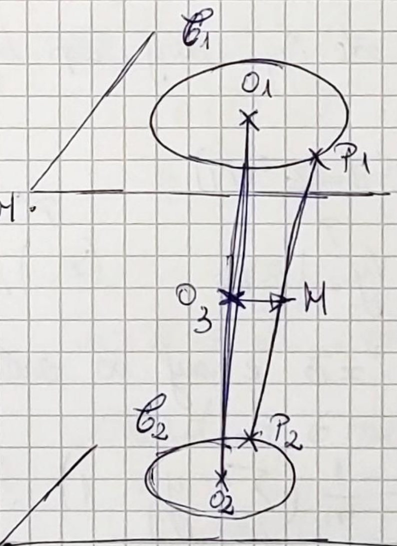
$P_1 \in C_1,\ P_2 \in C_2 \longrightarrow$ 2 pct variabile

$M = \frac{1}{2}P_1 + \frac{1}{2}P_2$

$L_g\ M = ?$

$$O_3 \nearrow\ 0 = \frac{1}{2}\overrightarrow{O_3O_1} + \frac{1}{2}\overrightarrow{O_3O_2}$$

Rez:



$\langle u, v \rangle = \|u\| \cdot \|v\| \cdot \cos(\widehat{u, v})$

Fie $O_3 = \frac{1}{2}O_1 + \frac{1}{2}O_2$

$$\overrightarrow{O_3M} = \overrightarrow{O_3O_2} + \overrightarrow{O_2P_2} + \overrightarrow{P_2M}$$

$$\overrightarrow{O_3M} = \overrightarrow{O_3O_1} + \overrightarrow{O_1P_1} + \overrightarrow{P_1M}$$

$$2\overrightarrow{O_3M} = \overrightarrow{O_1P_1} + \overrightarrow{O_2P_2}$$

$$M = \frac{1}{2}P_1 + \frac{1}{2}P_2 \Longrightarrow 0 = \frac{1}{2}\overbrace{\overrightarrow{MP_1}}^{-\overrightarrow{P_1M}} + \frac{1}{2}\overbrace{\overrightarrow{MP_2}}^{-\overrightarrow{P_2M}} \Rightarrow \overrightarrow{P_1M} + \overrightarrow{P_2M} = 0$$

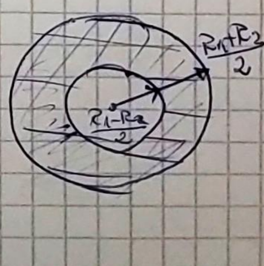
$$4\|\overrightarrow{O_3M}\|^2 = \langle 2\overrightarrow{O_3M}, 2\overrightarrow{O_3M} \rangle = \langle \overrightarrow{O_1P_1} + \overrightarrow{O_2P_2}, \overrightarrow{O_1P_1} + \overrightarrow{O_2P_2} \rangle =$$

$$= \underbrace{\langle \overrightarrow{O_1P_1}, \overrightarrow{O_1P_1} \rangle}_{R_1^2} + \underbrace{\langle \overrightarrow{O_2P_2}, \overrightarrow{O_2P_2} \rangle}_{} + 2\langle \underbrace{\overrightarrow{O_1P_1}}_{R_1}, \underbrace{\overrightarrow{O_2P_2}}_{R_2} \rangle =$$

$$= R_1^2 + R_2^2 + 2R_1R_2\underbrace{\cos\alpha}_{\in[-1,1]} \Longrightarrow$$

$$\Rightarrow \frac{(R_1-R_2)^2}{4} \le \|\overrightarrow{O_3M}\|^2 \le \frac{(R_1+R_2)^2}{4}$$

$$\left(\frac{R_1-R_2}{2}\right)^2 \le \|\overrightarrow{O_3M}\|^2 \le \left(\frac{R_1+R_2}{2}\right)^2$$



(2) Fie $(A, V_{/\mathbb{K}}, \varphi)$ sp afin (dim $A = n$)

$R_{af} = \{A_0, \dots, A_n\}$

$\tau : A \to A$ endomorfism afin

$\tau(P) = P'$

$y_i = x_i - \frac{1}{n}\left(\sum_{j=1}^{n} x_j - 1\right), (\forall)\ i = \overline{1,n}$

$(x_i)_{i=\overline{1,n}}$, $(y_i)_{i=\overline{1,n}}$ → coord P, resp. P' în R_{af}

---

a) τ idempotentă

b) Det. m pct. ale lui τ

c) Stab. dc. τ este inj., surj., resp. bij.

Rez: $\tau(P) = P'$

$\tau(P') = P'' \ (= \tau^2(P))$

$\overset{P}{(x_i)_{i=\overline{1,n}}}$ $\overset{P'}{(y_i)_{i=\overline{1,n}}}$ $\overset{P''}{(z_i)_{i=\overline{1,n}}}$ → coord.

a)

Pt. a dem. că $\tau^2 = \tau$ e suf. să arătăm că $z_i = y_i, \forall\ i = \overline{1,n}$

$$z_i = y_i - \frac{1}{n}\left(\sum_{j=1}^{n} y_j - 1\right) (\forall)\ i = \overline{1,n} =$$

$$= y_i - \frac{1}{n}\left(\sum\left[x_j - \frac{1}{n}\left(\sum_{k=1}^{n} x_k - 1\right)\right] - 1\right)$$

$$= y_i - \frac{1}{n}\left[\sum_{j=1}^{n} x_j - \frac{1}{n} \cdot n\left(\sum_{k=1}^{n} x_k - 1\right) - 1\right] =$$

$$= y_i - \frac{1}{n}\left[\cancel{\sum_{j=1}^{n} x_j} - \left\{\cancel{\sum_{k=1}^{n} x_k} + \cancel{1}\right\} - \cancel{1}\right] = y_i, \forall\ i = \overline{1,n} \Rightarrow$$

$\Rightarrow \tau^2(P) = \tau(P)\ \forall P \Rightarrow \tau$ - idempotentă

b) P - pct. fix pt. $\tau \Leftrightarrow \tau(P) = P$

$\Rightarrow y_i = x_i \ \forall i=\overline{1,n}$

$$y_i = x_i - \frac{1}{n}\left(\sum_{j=1}^{n} x_j - 1\right) \quad \forall i=\overline{1,n}$$

$$\cancel{x_i} = \cancel{x_i} - \frac{1}{n}\left(\sum_{j=1}^{n} x_j - 1\right) \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{n} x_j - 1 = 0 \longrightarrow x_1 + x_2 + \dots + x_n - 1 = 0$$

n-dim

$\longrightarrow \underbrace{\mathcal{H} = \{P / \tau(P) = P\}}_{\text{hiperplan afin}}$ $(\dim \mathcal{H} = n-1)$

c) Inj.:

Fie $P \in \mathcal{A} \setminus \mathcal{H}$

$\tau(P) = P'$ $(P \neq P')$

$\underbrace{\tau^2(P)}_{\overset{\shortparallel}{\tau(P)}} = \tau(P') \longrightarrow$ nu este injectivă

Surj.: $\operatorname{Im} \tau = \mathcal{H}$ (mult. pctelor fixe) $\longrightarrow$ nu este surjectivă

Evident că τ nu este nici bijectivă.

3) $\{A_0, A_1, \dots, A_n\}$ reper afin al unui sp. afin n-dim $(\mathcal{A})$

Det. ec. morfismului afin care duce A_0 în A_1, A_1 în A_2, …, A_n în A_0.

Rez: $\tau : \mathcal{A} \to \mathcal{A}$ endomorfism afin

Urma endomorfismului afin
$$\begin{cases} \tau(A_0) = A_1 \\ \tau(A_1) = A_2 \\ \vdots \\ \tau(A_n) = A_0 \end{cases}$$

$$\underset{(x_i)_{i=\overline{1,n}}}{P} \longrightarrow \underset{(y_i)_{i=\overline{1,n}}}{\tau(P)}$$

$\sum_{i=1}^{n} x_i \overrightarrow{A_0A_i}$ $\overrightarrow{A_0\tau(P)} = \overrightarrow{\tau(A_n)\tau(P)} = T(\overrightarrow{A_nP}) = T(\overrightarrow{A_nA_0} + \overrightarrow{A_0P}) = T(\overrightarrow{A_0P}) - T(\overrightarrow{A_0A_n}) = \sum_{i=1}^{n} x_i T(\overrightarrow{A_0A_i}) - T(\overrightarrow{A_0A_n}) =$

$$= \sum_{i=1}^{n} x_i \overrightarrow{A_1 A_{i+1}} - \overrightarrow{A_1 A_0} = \sum_{i=1}^{n-1} x_i \overrightarrow{A_1 A_{i+1}} + (x_n - 1)\overrightarrow{A_1 A_0} =$$

$$= \sum_{i=1}^{n-1} x_i (\overrightarrow{A_1 A_0} + \overrightarrow{A_0 A_{i+1}}) + (x_n - 1)\overrightarrow{A_1 A_0}$$

$$= -\sum_{i=1}^{n-1} x_i \overrightarrow{A_0 A_1} + \sum_{i=1}^{n-1} x_i \overrightarrow{A_0 A_{i+1}} + (1 - x_n)\overrightarrow{A_0 A_1} =$$

$$= \left(1 - \sum_{i=1}^{n} x_i\right) \overrightarrow{A_0 A_1} + \sum_{i=2}^{n} x_{i-1} \overrightarrow{A_0 A_i}$$

Ec. G
$$\begin{cases} y_1 = 1 - \sum_{i=1}^{n} x_i \\ y_2 = x_1 \\ y_3 = x_2 \\ \vdots \\ y_n = x_{n-1} \end{cases}$$

$$Y = AX + B$$

$$A = \begin{pmatrix} -1 & -1 & \dots & -1 \\ 1 & 1 & & 0 \\ & & 1 & 1 & \\ & & & & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -1 & -1 & \dots & -1 \\ 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & \ddots \end{bmatrix}$$

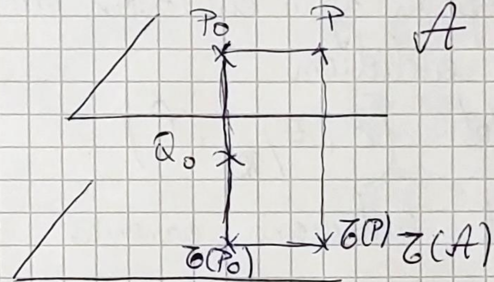
(4) Fie $(\mathcal{A}, V/K, \varphi)$ sp. afin, char $K \neq 2$

$\underset{\text{subsp. afin}}{\mathcal{A}' \subset \mathcal{A}}$ și $\tau : \mathcal{A} \to \mathcal{A}$ endomorfism

$T : V \to V$ urma endomorfismului τ

Atunci $\mathcal{A}'' = \{ Q = \frac{1}{2}P + \frac{1}{2}\tau(P) \mid P \in \mathcal{A}' \} \underset{\text{subsp. afin}}{\subset} \mathcal{A}$

Rez:



Fie $Q_0 \in \mathcal{A}''$, $Q_0 = \frac{1}{2}P_0 + \frac{1}{2}\tau(P_0)$, $P_0 \in \mathcal{A}'$

$$V'' = \{ \overrightarrow{Q_0Q} \mid Q \in \mathcal{A}'' \} \underset{\text{subsp. vect}}{\subset} V$$

$$\cancel{\overrightarrow{Q_0Q} = \frac{1}{2}\overrightarrow{P_0Q} + \frac{1}{2}\overrightarrow{\tau(P_0)Q}}$$

$$\overrightarrow{Q_0Q} = \frac{1}{2}\overrightarrow{Q_0P} + \frac{1}{2}\overrightarrow{Q_0\tau(P)} = \frac{1}{2}(\overrightarrow{Q_0P_0} + \overrightarrow{P_0P}) +$$

$$+ \frac{1}{2}(\overrightarrow{Q_0\tau(P_0)} + \overrightarrow{\tau(P_0)\tau(P)}) = \frac{1}{2}\underbrace{(\overrightarrow{Q_0P_0} + \overrightarrow{Q_0\tau(P_0)})}_{=0} + \frac{1}{2}(\overrightarrow{P_0P} + \overrightarrow{\tau(P_0)\tau(P)})$$

$$= \frac{1}{2}(\overrightarrow{P_0P} + T(\overrightarrow{P_0P}))$$

$$V'' = \{ \frac{1}{2}\underbrace{(\overrightarrow{P_0P} + T(\overrightarrow{P_0P}))}_{v+T(v)} \mid P \in \mathcal{A}' \} \underset{\text{subsp. vect.}}{\subset} V \Longrightarrow$$

$$\Longrightarrow \mathcal{A}'' \underset{\text{subsp. afin}}{\subset} \mathcal{A}$$

$\boxed{\text{Temă}}$: ABCD patrulater M

Fie M_1 simetricul (M) său în raport cu mijl. segm. [AB]

M_2 simetricul lui M_1 în rap. cu mijl. [BC]

M_3 simetricul lui M_2 în rap. cu mijl. [CD].

Arătați că M este simetricul pct. M_3 în rap. cu mijl.
[AD].

<u>Indicație</u> $\mathcal{R}_{af} = \{A, B, D\}$ $\longrightarrow A(0,0)$, $\longrightarrow B(1,0)$, $\longrightarrow D(0,1)$

$C(c_1, c_2)$

<u>Sp. afine euclidiene</u>

<u>Def</u> S.n. spațiu afin euclidian un sp. afin asociat unui sp. vect. euclidian

Not. $(\mathcal{E}, E_{/\mathbb{R}}, \varphi)$

Struct. euclidiană canonică

$(E_{/\mathbb{R}}, \underbrace{\langle , \rangle}_{\text{produs scalar canonic}})$ $\quad \langle , \rangle : E \times E \to \mathbb{R}$

formă biliniară simetrică pozitiv definită

$(\mathbb{R}^n, \mathbb{R}^n_{/\mathbb{R}}, \varphi)$ cu struct. afină canonică

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_{i=1}^{n} x_i \cdot y_i$$

(5) Fie A,B,C 3 pcte necoliniare din planul geom.

CU

$$\langle \vec{OA}, \vec{OB} \rangle = \frac{1}{2} (\|OA\|^2 + \|OB\|^2 - \|AB\|^2)$$

$$\|\vec{AB}\|^2 = \langle \vec{AB}, \vec{AB} \rangle = \langle \underbrace{\vec{AO}}_{-\vec{OA}} + \vec{OB}, \underbrace{\vec{AO}}_{-\vec{OA}} + \vec{OB} \rangle =$$

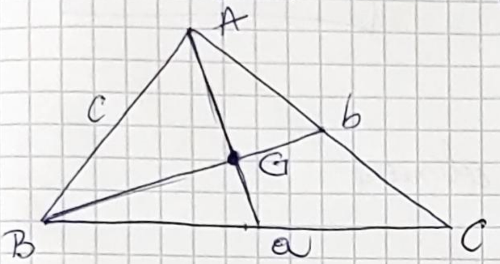
$$= \langle \vec{OB} - \vec{OA}, \vec{OB} - \vec{OA} \rangle = \langle \vec{OB}, \vec{OB} \rangle + \langle \vec{OA}, \vec{OA} \rangle - 2 \langle \vec{OA}, \vec{OB} \rangle$$

$$= \|\vec{OB}\|^2 + \|\vec{OA}\|^2 - 2 \langle \vec{OA}, \vec{OB} \rangle \Rightarrow$$

$$\langle \vec{OA}, \vec{OB} \rangle = \frac{1}{2} [\|\vec{OA}\|^2 + \|\vec{OB}\|^2 - \|AB\|^2]$$

⑥ G – c.g. al $\triangle ABC$

$$3\|\vec{OG}\|^2 = \|\vec{OA}\|^2 + \|OB\|^2 + \|OC\|^2 - \frac{1}{3}(a^2+b^2+c^2)$$



Rez:

G – c.g. al $\triangle ABC \Rightarrow$

$\Rightarrow G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$

$$3\vec{OG} = \vec{OA} + \vec{OB} + \vec{OC}$$

$$\underbrace{\langle 3\vec{OG}, 3\vec{OG}\rangle}_{9\|\vec{OG}\|^2} = \langle \vec{OA}+\vec{OB}+\vec{OC}, \vec{OA}+\vec{OB}+\vec{OC}\rangle =$$

$$= \langle \vec{OA}, \vec{OA}\rangle + \langle \vec{OB}, \vec{OB}\rangle + \langle \vec{OC}, \vec{OC}\rangle +$$

$$+ 2\left[\langle \vec{OA}, \vec{OB}\rangle + \langle \vec{OA}, \vec{OC}\rangle + \langle \vec{OB}, \vec{OC}\rangle\right] =$$

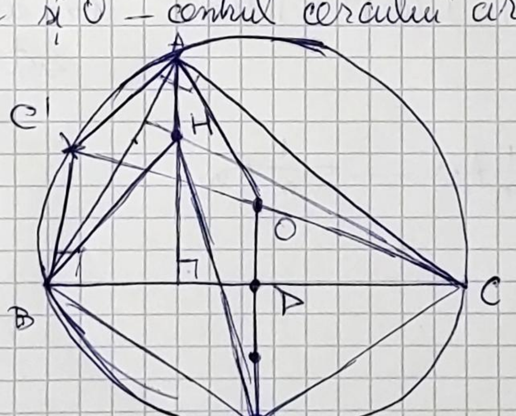
$$= \|\vec{OA}\|^2 + \|\vec{OB}\|^2 + \|\vec{OC}\|^2 + \left(\|\vec{OA}\|^2 + \|\vec{OB}\|^2 - \|AB\|^2 + \right.$$

$$\left. + \|\vec{OA}\|^2 + \|\vec{OC}\|^2 - \|\vec{AC}\|^2 + \|\vec{OB}\|^2 + \|\vec{OC}\|^2 - \|\vec{BC}\|^2\right) =$$

$$= 3\|\vec{OA}\|^2 + 3\|\vec{OB}\|^2 + 3\|\vec{OC}\|^2 - \left(\|AB\|^2 + \|\vec{AC}\|^2 + \|\vec{BC}\|^2\right)$$

$$3\|\vec{OG}\|^2 = \|\vec{OA}\|^2 + \|\vec{OB}\|^2 + \|\vec{OC}\| - \frac{1}{3}(a^2+b^2+c^2)$$

⑦ Arătați că $\boxed{\vec{OH} = \vec{OA} + \vec{OB} + \vec{OC}}$, unde H – ortocentrul $\triangle ABC$ și O – centrul cercului circ. $\triangle ABC$.



Rez:

$|BD| = |DC|$

O' – sim. pt. O în rap. cu dr. BC

$\vec{OB} + \vec{OC} = 2\vec{OD} = \vec{OO'}$

C' – sim. pt. C în rap. cu O

OD – linie mijlocie în $\triangle BCC' \longrightarrow OD = \frac{1}{2}BC'$

$HAC'B$ – paralelogram $\Rightarrow$ $\vec{OD} = \frac{1}{2}\vec{C'B} =$

$\Rightarrow BC' = HA$ $(\vec{C'B} = \vec{AH})$ $= \frac{1}{2}\vec{AH} \Rightarrow$

$\Rightarrow 2\overrightarrow{OD} = \overrightarrow{AH} \Rightarrow \overrightarrow{OO'} = \overrightarrow{AH} \Rightarrow AOO'H$ — paralelogram $\Rightarrow$

$\Rightarrow \overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OO'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \Rightarrow \boxed{\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}$

Dreapta lui Euler

$O, G, H \longrightarrow$ coliniare

O — centru c. circ; G — c.g.; H — ortocentru

G — c.g. al $\triangle ABC \Rightarrow G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C \Rightarrow$

$\Rightarrow 3\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$

$3\overrightarrow{OG} = \overrightarrow{OH} \Rightarrow$ O, G, H coliniare $\Rightarrow$ dr. lui Euler

Th. Steiner:

S.n. moment de inerție al unui sistem de puncte $S = \{A_1, \dots, A_n\}$ în rap. cu un punct M nr. real

$$I(M) = \sum_{i=1}^{n} \|\overrightarrow{MA_i}\|^2$$

<u>Th</u>: $I(M) = I(G) + n\|\overrightarrow{MG}\|^2$ G — echibaricentrul sist. S

$$G = \frac{1}{n}A_1 + \dots + \frac{1}{n}A_n$$