Tema 1

ONUȚU RADU

Grupa 312

1. X - v. a. continuă, densitatea de probabilitate:

$$f(x) = \begin{cases} A \cdot \sin x & , x \in [0, \pi] \\ 0 & , \text{în rest} \end{cases}$$

a) $A = ?$

f - densitatea de probabilitate $\Rightarrow f(x) \geq 0, \forall x \in \mathbb{R}$

$A \cdot \sin x \geq 0, \quad \forall x \in [0, \pi]$

$(\sin x \geq 0, \forall x \in [0, \pi]) \Rightarrow A \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx + \int_{\pi}^{\infty} f(x)\,dx =$$

$$= 0 + \int_{0}^{\pi} A \cdot \sin x\,dx + 0$$

$$= A \cdot (-\cos x)\Big|_0^{\pi}$$

$$= A \cdot (-\cos \pi + \cos 0)$$

$$= 2A$$

$$2A = 1 \Rightarrow A = \frac{1}{2} \geq 0$$

b) $\mathbb{P}(X < \frac{\pi}{3}) = \int_{-\infty}^{\frac{\pi}{3}} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\frac{\pi}{3}} f(x)dx =$

$= 0 + \int_{0}^{\frac{\pi}{3}} \frac{1}{2} \cdot \sin x \, dx = -\frac{1}{2} (\cos x)\Big|_0^{\frac{\pi}{3}}$

$= -\frac{1}{2}\left(\frac{1}{2} - 1\right)$

$= \frac{1}{4}$

$\mathbb{P}(X < \frac{\pi}{4} \mid X > \frac{\pi}{6}) = \cancel{\int_{-\infty}^{\frac{\pi}{4}} f(x)dx + \int_{\frac{\pi}{6}}^{\infty}}$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f(x)dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x \, dx = -\frac{1}{2}(\cos x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} =$

$= -\frac{1}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3} - \sqrt{2}}{4}$

c) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{0} x \cdot f(x)dx + \int_{0}^{\pi} x \cdot f(x)dx +$

$+ \int_{\pi}^{\infty} x \cdot f(x)dx =$

$= 0 + \int_0^{\pi} x \cdot f(x)\,dx + 0 = \int_0^{\pi} x \cdot \sin x \, dx =$

$= x \cdot (-\cos x)\Big|_0^{\pi} - \int_0^{\pi} -\cos x \, dx = \pi + \sin x \Big|_0^{\pi}$

$= \pi$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_{-\infty}^{0} x^2 \cdot f(x)\,dx + \int_{0}^{\pi} x^2 \cdot f(x)\,dx +$$

$$+ \int_{\pi}^{\infty} x^2 \cdot f(x)\,dx =$$

$$= \int_0^{\pi} x^2 \cdot \sin x\,dx = x^2 (-\cos x)\Big|_0^{\pi} - \int_0^{\pi} 2x \cdot (-\cos x)\,dx =$$

$$= \pi^2 + 2\int_0^{\pi} x \cos x\,dx = \pi^2 + 2\left(x \sin x\Big|_0^{\pi} - \int_0^{\pi} \sin x\,dx\right) =$$

$$= \pi^2 + 2\left(-(-\cos x)\Big|_0^{\pi}\right) = \pi^2 + 2\cos\pi - 2\cos 0$$

$$= \pi^2 - 4$$

$$\operatorname{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \pi^2 - 4 - \pi^2$$

$$= \pi^2 - \pi^2 - 4$$

$$= -4$$

d) $$f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}\sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

$$f(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}\sin t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$$

$$x \in [0, \pi] \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^{x} \frac{1}{2}\sin t\,dt =$$

$$= -\frac{1}{2}\cos t\Big|_0^{x} = -\frac{1}{2}(\cos x - 1) = \frac{1-\cos x}{2}$$

$x > \pi \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{\pi} \frac{1}{2}\sin t\,dt + \int_{\pi}^{x} 0\,dt$

$= \frac{1}{2}(-\cos t)\Big|_0^{\pi} = \frac{1}{2}(1+1) = 1$

$$\Rightarrow F(x) = \begin{cases} 0, & x < 0 \\ \frac{1-\cos x}{2}, & 0 \le x \le \pi \\ 1, & x > \pi \end{cases}$$

e) $F(m_e) = \frac{1}{2}$

$\int_0^{m_e} \frac{1}{2}\sin t\,dt = \frac{1}{2}$

$\frac{1}{2}(-\cos t)\Big|_0^{m_e} = \frac{1}{2}$

$-\cos m_e + 1 = 1$

$-\cos m_e = 0$

$m_e = \frac{\pi}{2}$

$$f(x) = \begin{cases} \frac{1}{2}\sin x, & x \in [0, \pi] \\ 0, & \hat{\imath}n\ rest \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{2}\cos x, & x \in [0, \pi] \\ 0, & \hat{\imath}n\ rest \end{cases}$$

x	$-\infty$	0	$\frac{\pi}{2}$	π	∞
$f'(x)$		$+ + + + 0 - - -$			
$f(x)$	$0 \longrightarrow$	$0 \nearrow$	$1 \searrow$	$0 \longrightarrow$	0

$\Rightarrow m_0 = \frac{\pi}{2}$

2. X - v.a., densitatea de probabilitate:

$$f(x) = \begin{cases} k \cdot (e^{-x} + e^{x}) & , x \in [0, 1] \\ 0 & , \text{în rest} \end{cases}$$

a) $k = ?$

f - densitatea de probabilitate $\Rightarrow \begin{cases} f(x) \geq 0 \ , \ \forall x \in \mathbb{R} \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases}$

$k \cdot (e^{-x} + e^{x}) \geq 0 \Rightarrow k \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx =$$

$$= \int_{0}^{1} k \cdot (e^{-x} + e^{x})\,dx = -k \cdot e^{-x} + k \cdot e^{x} \Big|_0^1$$

$$= -\frac{k}{e} + k \cdot e + k - k = -\frac{k}{e} + k \cdot e = \frac{-k + k \cdot e^2}{e}$$

$$\frac{-k + k \cdot e^2}{e} = 1 \Leftrightarrow -k + k \cdot e^2 = e \Leftrightarrow k(e^2 - 1) = e$$

$$k = \frac{e}{e^2 - 1} > 0$$

b) $$f(t) = \begin{cases} 0 & , t < 0 \\ k(e^{-t} + e^{t}) & , 0 \le t \le 1 \\ 0 & , t > 1 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$$

$$x \in [0,1] \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} k\cdot(e^{-t} + e^{t})\,dt$$

$$= k(-e^{-t} + e^{t})\Big|_{0}^{x} = k(-e^{-x} + e^{x} + 1 - 1)$$

$$= k(e^{x} - e^{-x})$$

$$x > 1 \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} k(e^{-t} + e^{t})\,dt + \int_{1}^{x} 0\,dt =$$

$$= k(-e^{-t} + e^{t})\Big|_{0}^{1} = k(\cancel{e} - \frac{1}{e} + e + 1 - 1)$$

$$= \frac{e}{e^2 - 1} \cdot \left(\frac{e^2 - 1}{e}\right)$$

$$= 1$$

$$\Rightarrow F(x) = \begin{cases} 0 & , x < 0 \\ k(e^{x} - e^{-x}) & , x \in [0,1] \\ 1 & , x > 1 \end{cases}$$

$\mathbb{P}(X < \frac{1}{2} |$ …

$= k \cdot \int_{\frac{1}{4}}^{\frac{1}{2}} e^{-x}$ …

$= k \cdot (-\frac{1}{\sqrt{e}}$ …

$= \frac{e}{e^2 - 1}($ …

$= \frac{e}{(e-1)(e}$ …

c) $\mathbb{E}(X) =$ …

$= k(-x \cdot$ …

$= k[-\frac{1}{e}$ …

$= k(-$ …

$= k(-$ …

$= \frac{e}{e^2-}$ …

$$\mathbb{P}\left(X<\frac{1}{2} \mid X>\frac{1}{4}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx =$$

$$= k\cdot\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-x}+e^{x}\,dx = k\cdot\left(-e^{-x}+e^{x}\right)\Big|_{\frac{1}{4}}^{\frac{1}{2}} =$$

$$= k\cdot\left(-\frac{1}{\sqrt{e}}+\sqrt{e}+\frac{1}{e^{\frac{1}{4}}}-e^{\frac{1}{4}}\right)$$

$$= \frac{e}{e^2-1}\left(\frac{-1+e}{\sqrt{e}}+\frac{1-e}{\sqrt[4]{e}}\right)$$

$$= \frac{e}{(e-1)(e+1)}\cdot\left(\frac{e-1}{\sqrt{e}}+\frac{-e+1}{\sqrt[4]{e}}\right) = \frac{e}{\sqrt{e}\,(e+1)}-\frac{e}{\sqrt[4]{e}\,(e+1)}$$

c) $$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = k\cdot\int_{0}^{1} x\cdot\left(e^{-x}+e^{x}\right)dx =$$

$$= k\left(x\cdot\left(-e^{-x}+e^{x}\right)\Big|_0^1 - \int_0^1 -e^{-x}+e^{x}\,dx\right) =$$

$$= k\left[-\frac{1}{e}+e-\left(e^{-x}+e^{x}\right)\Big|_0^1\right] =$$

$$= k\left(-\frac{1}{e}+e-\left(\frac{1}{e}+e-1-1\right)\right)$$

$$= k\left(-\frac{2}{e}+2\right)$$

$$= \frac{e}{e^2-1}\left(\frac{-2+2e}{e}\right) = \frac{-2(1-e)}{(e-1)(e+1)} = \frac{2}{e+1}$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx =$$

$$= k\int_0^1 x^2(e^{-x}+e^{x})\,dx = k\left(x^2\cdot(-e^{-x}+e^{x})\Big|_0^1 - \int_0^1 2x(-e^{-x}+e^{x})\,dx\right.$$

$$= k\left(-\frac{1}{e}+e-2\left[x(e^{-x}+e^{x})\Big|_0^1 - \int_0^1 e^{-x}+e^{x}\,dx\right]\right) =$$

$$= k\left(-\frac{1}{e}+e-2\left[\frac{1}{e}+e-\left(-e^{-x}+e^{x}\right)\Big|_0^1\right]\right) =$$

$$= k\left(-\frac{1}{e}+e-2\left[\frac{1}{e}+e-\left(-\frac{1}{e}+e+1-1\right)\right]\right) =$$

$$= k\left(-\frac{1}{e}+e-2\left(\frac{2}{e}\right)\right)$$

$$= k\left(-\frac{1}{e}+e-\frac{4}{e}\right) = k\left(-\frac{5}{e}+e\right) = \frac{e}{e^2-1}\cdot\frac{e^2-5}{e}$$

$$= \frac{e^2-5}{e^2-1}$$

$$\Rightarrow Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{e^2-5}{e-1} - \frac{4}{(e+1)^2} =$$

$$= \frac{(e^2-5)(e^2+2e+1)-4e+4}{(e-1)(e+1)^2} = \frac{e^4+2e^3+e^2-5e^2-10e-5 \quad {}^{+4e+4}}{(e^2-1)(e+1)}$$

$$= \frac{e^4+2e^3-4e^2-10e-5}{(e^2-1)(e+1)}$$

3. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} k \cdot x^{a-1} \cdot (1-x)^{b-1} & , x \in [0,1) \\ 0 & , \text{în rest} \end{cases}$

$a, b > 0$

a) $k = ?$

f - densitatea de probabilitate $\Rightarrow \begin{cases} f(x) \geq 0, & \forall x \in \mathbb{R} \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases}$

$k \cdot \cancel{x}^{a-1} \cdot (1-x)^{b-1} \geq 0 \Rightarrow k \geq 0$

$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 k \cdot x^{a-1} \cdot (1-x)^{b-1}\,dx = k \cdot \beta(a, b)$

$k \cdot \beta(a+b) = 1 \Rightarrow k = \frac{1}{\beta(a, b)}$

b) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = k \int_0^1 x^a \cdot (1-x)^{b-1}\,dx$

$= k \cdot \beta(a+1, b)$

$= \frac{1}{\beta(a,b)} \cdot \frac{\Gamma(a+1) \cdot \Gamma(b)}{\Gamma(a+b+1)} = \frac{\Gamma(a+b)}{\Gamma(a) \cdot \Gamma(b)} \cdot \frac{a \cdot \Gamma(a) \cdot \Gamma(b)}{(a+b)\,\Gamma(a+b)}$

$= \frac{a}{a+b}$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = k \int_0^1 \cancel{x}^{a+1} \cdot (1-x)^{b-1}\,dx$

$= \beta(a+2, b)$

$$\mathbb{E}(X^2) = \frac{1}{\beta(a,b)} \cdot \beta(a+2,b)$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot \frac{\Gamma(a+2)\cdot\Gamma(b)}{\Gamma(a+b+2)}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot \frac{a\cdot(a+1)\cdot\Gamma(a)\cdot\Gamma(b)}{(a+b+1)\cdot(a+b)\cdot\Gamma(a+b)}$$

$$= \frac{a(a+1)}{(a+b+1)(a+b)}$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \frac{a^2+a}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2}$$

$$= \frac{(a^2+a)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)}$$

$$= \frac{a^3+a^2b+a^2+ab-a^3-a^2b-a^2}{(a+b)^2\cdot(a+b+1)}$$

$$= \frac{ab}{(a+b)^2\cdot(a+b+1)}$$

$$f(x) = \begin{cases} k\cdot x^{a-1}\cdot(1-x)^{b-1}, & x\in[0,1) \\ 0, & \text{în rest} \end{cases}$$

$$f'(x) = \begin{cases} k\cdot(a-1)\cdot x^{a-2}\cdot(b-1)\cdot(1-x)^{b-2}, & x\in[0,1) \\ 0, & \text{în rest} \end{cases}$$

+ (pt. $a,b \geq 1$ sau $a,b < 1$)

− (pt. $a > 1$ și $b < 0$ sau invers)

x	$-\infty$	0	$\frac{1}{2}$	1	∞
$f'(x)$		0	$+ + + +$ $- - - - -$	0	
$f(x)$		0	$\nearrow$ $\searrow$	0	

$\cancel{k\cdot(a-1)\cdot(b-1)}$

$k\cdot\left(\frac{1}{2}\right)^{a+b-2}$

$\Rightarrow m_0 = \frac{1}{2}$

$$m_\lambda = \int_{-\infty}^{\infty} x^\lambda \cdot f(x)\,dx = k\int_0^1 x^{a+\lambda-1}\cdot(1-x)^{b-1}\,dx$$

$$= \frac{1}{\beta(a,b)}\cdot\beta(a+\lambda,b) = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}\cdot\frac{\Gamma(a+\lambda)\cdot\Gamma(b)}{\Gamma(a+b+\lambda)} =$$

$$= \frac{(a+\lambda-1)\cdot(a+\lambda-2)\ldots(a+\lambda-\lambda)\cdot\Gamma(a)\cdot\Gamma(b)\cdot\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)\cdot(a+b+\lambda-1)\cdot(a+b+\lambda-2)\ldots(a+b+\lambda-\lambda)\cdot\Gamma(a+b)} =$$

$$= \frac{(a+\lambda-1)(a+\lambda-2)\ldots(a+\lambda-\lambda)}{(a+b+\lambda-1)(a+b+\lambda-2)\ldots(a+b+\lambda-2)}$$

c) ~~$f(t) =$~~ ~~$t<0$~~

$a = 2;\ b = 3 \Rightarrow k = \frac{1}{\beta(2,3)} = \frac{\Gamma(2+3)}{\Gamma(2)\cdot\Gamma(3)} = \frac{4!}{1!\cdot 2!} = 3\cdot 4$

$= 12$

$$\Rightarrow f(t) = \begin{cases} 0, & t<0 \\ 12\cdot t\cdot(1-t)^2, & t\in[0,1) \\ 0, & t\geq 1 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$x<0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$$

$$x\in[0,1) \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} 12\,t\cdot(1-t)^2\,dt =$$

$$= 12\int_0^x t - 2t^2 + t^3\,dt = 12\left(\frac{t^2}{2} - \frac{2}{3}t^3 + \frac{t^4}{4}\right)\Big|_0^x =$$

$$= 12\left(\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right) = 6x^2 - 8x^3 + 3x^4$$

$$x>1 \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^1 12\,t\cdot(1-t)^2\,dt + \int_1^x 0\,dt =$$

$$= 12\,\beta(2,3) = 12\cdot\frac{\Gamma(2)\cdot\Gamma(3)}{\Gamma(5)} = 12\cdot\frac{1!\cdot 2!}{4!} = 1$$

$$\Rightarrow F(x) = \begin{cases} 0 & , \ x < 0 \\ 6x^2 - 8x^3 + 3x^4 & , \ x \in [0,1) \\ 1 & , \ x \geq 1 \end{cases}$$

$$\mathbb{P}\left(X < \frac{1}{2}\right) = \int_{-\infty}^{0} 0 \, dx + \int_{0}^{\frac{1}{2}} 12x \cdot (1-x)^2 \, dx =$$

$$= \int_{0}^{\frac{1}{2}} 12 \cdot (x - 2x^2 + x^3) \, dx$$

$$= 12\left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right)\Big|_0^{\frac{1}{2}} = 6x^2 - 8x^3 + 3x^4 \Big|_0^{\frac{1}{2}}$$

$$= \frac{3}{2} - 1 + \frac{3}{16} = \frac{24 - 16 + 3}{16} = \frac{11}{16}$$

$$\mathbb{P}\left(X > \frac{1}{3}\right) = \int_{\frac{1}{3}}^{1} 12x \cdot (1-x)^2 \, dx = 6x^2 - 8x^3 + 3x^4 \Big|_{\frac{1}{3}}^{1} =$$

$$= 6 - 8 + 3 - \frac{6}{9} + \frac{8}{27} - \frac{3}{81}$$

$$= 1 - \frac{2}{3} + \frac{8}{27} - \frac{1}{27} = 1 - \frac{2}{3} + \frac{7}{27} = \frac{27 - 18 + 7}{27}$$

$$= \frac{16}{27}$$

$$\mathbb{P}\left(X \leq \frac{1}{2} \,\middle|\, X > \frac{1}{4}\right) = \int_{\frac{1}{2}}^{\frac{1}{4}} 12x \cdot (1-x)^2 \, dx = 6x^2 - 8x^3 + 3x^4 \Big|_{\frac{1}{2}}^{\frac{1}{4}} =$$

$$= \frac{6}{16} - \frac{8}{64} + \frac{3}{256} - \frac{6}{4} - \frac{8}{8} + \frac{3}{16} =$$

$= \frac{144 - 32 + 3 - 384 - 256}{256}$

$= \frac{112 + 3 - 640}{256} = \frac{-525}{256}$

24. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} k \cdot x \cdot e^{\frac{-x^2}{2a^2}} & , x \geq 0, k \in \mathbb{R}, a > 0 \\ 0 & , x < 0 \end{cases}$

a) $k = ?$

f - densitatea de probabilitate $\Rightarrow \begin{cases} f(x) \geq 0 & , \forall x \in \mathbb{R} \\ \int_{-\infty}^{\infty} f(x)dx = 1 \end{cases}$

$k \cdot x \cdot e^{\frac{-x^2}{2a^2}} \geq 0 \Rightarrow k \geq 0$

$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} k \cdot x \cdot e^{\frac{-x^2}{2a^2}} dx = \cancel{k \cdot e^{\frac{1}{2a}} \int_0^{\infty} x \cdot e^{-x^2} dx}$

$t = \frac{-x^2}{2a^2}$

$\cancel{= k \int_0^{\infty}}$

$dt = \frac{-2x}{2a^2} dx = \frac{-x}{a^2} dx \Rightarrow dx = \frac{a^2}{-x} dt$

$= k \int_0^{\infty} e^t \cdot x \cdot -(a^2) dt = -ka^2 e^t \Big|_0^{\infty} = -ka^2 e^{\frac{-x^2}{2a^2}} \Big|_0^{\infty}$

$= +ka^2$

$k \cdot a^2 = 1 \Rightarrow k = \frac{1}{a^2}$

b) $f(t) = \begin{cases} 0 & , t < 0 \\ \frac{1}{a^2} \cdot t \cdot e^{\frac{-t^2}{2a^2}} & , t \geq 0 \end{cases}$

$F(x) = \int_{-\infty}^{x} f(t)\,dt$

$x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$

$x \geq 0 \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} \frac{1}{a^2} \cdot t \cdot e^{\frac{-t^2}{2a^2}}\,dt$

$= \frac{1}{a^2} \cdot \left(-a^2 \cdot e^{\frac{-x^2}{2a^2}} + a^2\right) = -e^{\frac{-x^2}{2a^2}} + 1$

$\Rightarrow F(x) = \begin{cases} 0 & , x < 0 \\ -e^{\frac{-x^2}{2a^2}} + 1 & , x \geq 0 \end{cases}$

c) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \frac{1}{a^2} \cdot \int_{0}^{\infty} x^2 \cdot e^{\frac{-x^2}{2a^2}}\,dx = \frac{1}{a^2} \cdot \frac{\sqrt{\pi} \cdot a^3}{\sqrt{2}}$

$= \frac{\sqrt{\pi} \cdot a}{\sqrt{2}}$

$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \frac{1}{a^2} \cdot \int_{0}^{\infty} x^3 \cdot e^{\frac{-x^2}{2a^2}}\,dx = \frac{1}{a^2} \cdot 2a^4$

$= 2a^2$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 2a^2 - \frac{\pi \cdot a^2}{2}$

$= \frac{a^2(4-\pi)}{2}$

$$m_\lambda = \int_{-\infty}^{\infty} x^\lambda \cdot f(x)\,dx = \frac{1}{a^2}\int_0^{\infty} x^{\lambda+1} \cdot e^{\frac{-x^2}{2a^2}}\,dx =$$

d) $$\mathbb{P}(X<2a) = \frac{1}{a^2}\int_0^{2a} x \cdot e^{-\frac{x^2}{2a^2}}\,dx = \frac{1}{a^2}\cdot\left(a^2 - e^{-2}\cdot a^2\right)$$

$$= 1 - \frac{1}{e^2}$$

$$\mathbb{P}(X>a) = \frac{1}{a^2}\int_a^{\infty} x \cdot e^{\frac{-x^2}{2a^2}}\,dx = \frac{1}{a^2}\cdot\frac{a^2}{\sqrt{e}} = \frac{1}{\sqrt{e}}$$

$$\mathbb{P}(X \leq 4a \mid X>2a) = \frac{1}{a^2}\int_{2a}^{4a} x \cdot e^{\frac{-x^2}{2a^2}}\,dx = \frac{1}{a^2}\left(e^{-2}\cdot a^2 - e^{-8}\cdot a^2\right)$$

$$= \frac{1}{e^2} - \frac{1}{e^8}$$

e) $$\mu_3 = \int_{-\infty}^{\infty}\left(x - \frac{\sqrt{\pi}\cdot a}{\sqrt{2}}\right)^3 \cdot f(x)\,dx = \frac{1}{a^2}\int_0^{\infty}\left(x - \frac{\sqrt{\pi}\cdot a}{\sqrt{2}}\right)^3 \cdot x \cdot e^{\frac{-x^2}{2a^2}}\,dx$$

$$= \frac{\sqrt{\pi}\,(\pi-3)\,a^5}{\sqrt{2}} \cdot \frac{1}{a^2}$$

$$= \frac{\sqrt{\pi}\,(\pi-3)\,a^3}{\sqrt{2}}$$

$$F(m_e) = \frac{1}{2}$$

$$\frac{1}{a^2}\int_0^{m_e} t \cdot e^{\frac{-t^2}{2a^2}}\,dt = \frac{1}{2}$$

$$\frac{1}{a^2}\left(a^2 - a^2 e^{-\frac{m_e^2}{2a^2}}\right) = \frac{1}{2}$$

$$1 - e^{\frac{-m_e^2}{2a^2}} = \frac{1}{2}$$

$$e^{\frac{-m_e^2}{2a^2}} = \frac{1}{2}$$

$$\ln \frac{-m_e^2}{2a^2} = \ln \frac{1}{2}$$

$$-m_e^2 = -\ln 2 \cdot 2a^2$$

$$m_e^2 = a^2 \cdot \ln 4$$

$$m_e = a\sqrt{\ln 4}$$

$$f(x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{a^2} \cdot x \cdot e^{\frac{-x^2}{2a^2}} & , x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} 0 & , x < 0 \\ \dfrac{a^2 - x^2}{a^4 e^{\frac{x^2}{2a^2}}} & , x \geq 0 \end{cases}$$

x	$-\infty$		0		a		∞
$f'(x)$			$\frac{1}{a^2}$	$+ + +$	0	$- - - -$	
$f(x)$	0	$\longrightarrow$	0	$\nearrow$	$\frac{1}{a} \cdot \frac{1}{\sqrt{e}}$	$\searrow$	

$\Rightarrow m_0 = a$

f) $\mathbb{P}(0 < X < a\sqrt{2\ln 4}) = \int_0^{a\sqrt{2\ln 4}} \frac{1}{a^2} \cdot x \cdot e^{\frac{-x^2}{2a^2}} dx =$

$= \frac{1}{a^2} \cdot (a^2 - e^{-\ln 4} \cdot a^2) = 1 - \frac{1}{e^{\ln 4}}$

5. X_n - v.a., densitatea de probabilitate:

$$f_n(x) = \begin{cases} p \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}} & , x \geq 0 \quad , p \in \mathbb{R}, n \in \mathbb{N}^* \\ 0 & , x < 0 \end{cases}$$

a) $p = ?$

f_n - densitatea de probabilitate $\Rightarrow \begin{cases} f_n(x) \geq 0, \forall x \in \mathbb{R} \\ \int_{-\infty}^{\infty} f_n(x)\,dx = 1 \end{cases}$

$p \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}} \geq 0 \Rightarrow p \geq 0$

$\int_{-\infty}^{\infty} f_n(x)\,dx = \int_0^{\infty} p \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}} dx =$