Problema 1. *(2p).*

1). Gasiti $\lambda \in \mathbb{R}$ *astfel incat* $\operatorname{div}(|x|^3 \cdot \nabla v(x)) = \lambda v(x)|x|$*, unde* $v : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}$, $v(x) := |x|^{-\frac{5}{4}}$.

2). Sa se determine pentru ce valori $p \geq 1$ *are loc* $|v|^p \in L^1(B_1(0))$*, unde* $B_1(0)$ *este bila unitate din* $\mathbb{R}^4$.

3). * *Sa se determine pentru ce valori* $p \geq 1$ *are loc* $(|x|^2 + e^{-|x|})|v|^p \in L^1(\mathbb{R}^4 \setminus \overline{B_1(0)})$.

4). Dati exemplu de o functie strict superarmonica ($-\Delta u > 0$*) pe* $\mathbb{R}^2$ *care sa se anuleze pe dreapta* $x + 3y - 1 = 0$.

5). Consideram functia $u : \mathbb{R}^2 \to \mathbb{R}$ *data de*

$$u(x) = \ln^2(1 + |x|^2), \quad x = (x_1, x_2).$$

Calculati $\Delta u(1, 1)$.

Problema 2. *(1.5p). Fie* $\Omega := \{(x, y) \in \mathbb{R}^2; \quad x^2 + y^2 < 9\}$ *si* $\partial\Omega$ *frontiera lui* Ω*. Fie problema*

$$(1) \qquad \begin{cases} -\Delta u(x, y) = \frac{3}{1+y^2}, & (x, y) \in \Omega \\ u(x, y) = 0, & (x, y) \in \partial\Omega \end{cases}$$

1). Aratati ca problema (1) *are cel mult o solutie* $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

2). Calculalti $\nabla u(0, 0)$.

3). Gasiti constanta C *astfel incat functia* $v(x, y) = C(x^2 + y^2)$ *sa verifice* $-\Delta v = 3$ *in* Ω.

4). Folosind eventual principiul de maxim pentru functii sub/super armonice sa se arate ca solutia problemei (1) *verifica*

$$0 < u(x, y) \leq \frac{27}{4}, \quad \forall (x, y) \in \Omega.$$

Problema 3. *(2p). Consideram urmatoarea problema de tip "unde"*

$$(2) \qquad \begin{cases} 2u_{tt}(x, t) + 5u_{tx}(x, t) - 3u_{xx}(x, t) = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}, \\ u_t(x, 0) = g(x), & x \in \mathbb{R}, \end{cases}$$

unde $f, g \in C^2(\mathbb{R})$ *sunt functii date.*

1). Aratati ca daca $u = u(x, t)$ *este o functie de clasa* C^2 *atunci* u *verifica*

$$(2\partial_t - \partial_x)(u_t(x, t) + 3u_x(x, t)) = 2u_{tt}(x, t) + 5u_{tx}(x, t) - 3u_{xx}(x, t),$$

pe domeniul sau de definitie.

2). Rezolvati problema cu valori initiale satisfacuta de u *in* (2) *(scrieti forma generala a lui* u*) reducand-o la rezolvarea a doua ecuatii de transport (una omogena si alta neomogena).*

3). Folosind conditiile la $t = 0$ *deduceti solutia* u *a problemei* (2) *in cazul particular* $f(x) = e^{-x}$ *si* $g(x) = \cos(2x)$.

Problema 4. *(1.5p) Consideram problema Cauchy*

$$(3)\qquad \begin{cases} u_t(x,t) - u_{xx}(x,t) + \frac{t^2}{t^2+2}u(x,t) = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = e^{-2x^2}, & x \in \mathbb{R}. \end{cases}$$

1). Gasiti o functie $\phi : \mathbb{R} \to \mathbb{R}$ astfel incat functia $v(x,t) := u(x,t)\phi(t)$ sa verifice ecuatia caldurii

$$(4)\qquad v_t(x,t) - v_{xx}(x,t) = 0, \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

2). Scrieti problema Cauchy verificata de v determinati explicit solutia problemei (3).

Problema 5. *(2p). Fie functia $f : [-1,1] \to \mathbb{R}$, $f(x) = 1 - |x - \frac{1}{2}|$.*

1). Explicitati functia f si faceti graficul functiei f.

2). Sa se determine punctele de derivabilitate ale lui f pe intervalul $(-1,1)$.

3). Argumentati ca $f \in H^1(-1,1)$ si calculati norma lui f in $H^1(-1,1)$ (precizati inainte norma cu care lucrati). Este f in $H_0^1(-1,1)$?

4). * Determinati $\alpha \in \mathbb{R}$ astfel incat functia $z : (0,1) \to \mathbb{R}$, $z(x) = (1-x)^\alpha$ sa apartina lui $W^{1,4}(0,1)$.

*5). * Determinati $\alpha \in \mathbb{R}$ astfel incat functia $z : (1,\infty) \to \mathbb{R}$, $z(x) = \frac{x^\alpha}{1+x^2}$ sa apartina lui $H^1(1,\infty)$.*

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Examen EDP
Seria 32

22.01.2021

Pb.1

1) $v(x) = |x|^{-\frac{5}{4}}$

$$\nabla v(x) = \nabla\left(|x|^{-\frac{5}{4}}\right) = -\frac{5}{4} \cdot x \cdot |x|^{-\frac{5}{4}-2} =$$
$$= -\frac{5}{4} x |x|^{-\frac{13}{4}}$$

$$\operatorname{div}(|x|^3 \cdot \nabla v(x)) = \operatorname{div}\left(-\frac{5}{4} \cdot x \cdot |x|^{-\frac{1}{4}}\right) =$$
$$= -\frac{5}{4} \operatorname{div}\left(x|x|^{-\frac{1}{4}}\right) =$$
$$= -\frac{5}{4}\left(4 + \left(-\frac{1}{4}\right)\right)|x|^{-\frac{1}{4}} =$$
$$= -\frac{5}{4} \cdot \frac{15}{4} |x|^{-\frac{1}{4}} = \frac{-75}{16} |x|^{-\frac{5}{4}} \cdot |x| \Rightarrow$$
$$\Rightarrow \lambda = -\frac{75}{16}$$

2) $p \geq 1$

$$|v|^p \in L_1(B_1(0))$$
$$\Updownarrow$$
$$\int_{B_1(0)} |v|^p \, dx < \infty \Leftrightarrow \int_{B_1(0)} |x|^{-\frac{5p}{4}} \, dx < \infty$$

Aplicam formula Co-Ariei $\Rightarrow$

$$\int_{B_1(0)} |x|^{-\frac{5p}{4}} \, dx = \int_0^1 \left(\int_{\partial B_1(0)} |x|^{-\frac{5p}{4}} \, d\sigma \right) dr =$$

$$= \int_0^1 r^{-5p/4} \Big(\underbrace{\int_{\partial B_1(0)} d\sigma}_{= \omega_4 \cdot r^3} \Big) dr =$$

$$= \omega_4 \cdot \int_0^1 r^{\frac{-5p}{4}} \cdot r^3 \, dr =$$

$$= \omega_4 \int_0^1 r^{\frac{12-5p}{4}} dr =$$

$$= \omega_4 \cdot \begin{cases} \dfrac{r^{\frac{12-5p}{4}+1}}{\frac{12-5p}{4}+1} \Big|_0^1, & \dfrac{12-5p}{4} \neq -1 \\ \ln r \Big|_0^1, & \dfrac{12-5p}{4} = -1 \end{cases}$$

Avem nevoie ca această cantitate să fie finită,

deci $\frac{12-5p}{4} + 1 > 0 \iff 16 - 5p > 0 \iff$

$5p < 16 \iff p < \frac{16}{5} \Rightarrow$

$\Rightarrow p \in [1, \frac{16}{5})$

4) $-\Delta u > 0$, să se anuleze pe $x + 3y - 1 = 0$,

i.e. $x = 3y + 1 \iff x^2 = 9y^2 + 6y + 1$

Fie $\underline{u(x,y) = x^2 - 9y^2 + 6y + 1}$

$u_x(x,y) = 2x$, $u_{xx}(x,y) = 2$

$u_y(x,y) = -18y + 6$, $u_{yy}(x,y) = -18$

$\Rightarrow \Delta u(x,y) = u_{xx} + u_{yy} = -16 \iff$

$-\Delta u(x,y) = 16 > 0$,

deci funcția luată corespunde cerinței.

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5) $u: \mathbb{R}^2 \to \mathbb{R}$ $\quad u(x) = \ln^2(1+|x|^2) \Rightarrow$

putem observa că $u(x) = g(|x|)$, i.e. u este o funcție radială și definim $g: (0,\infty) \to \mathbb{R}$, $g(r) = \ln^2(1+r^2) \Rightarrow$

formula laplacianului pt. fct. cu simetrie radială

$$\Rightarrow \Delta u(x) = g''(x) + \frac{n-1}{|x|} g'(x)$$

$$g'(x) = 2 \cdot \ln(1+x^2) \cdot \frac{1}{1+x^2} \cdot 2x = \frac{4x \cdot \ln(1+x^2)}{1+x^2}$$

$$g''(x) = \left(\frac{4x}{1+x^2} \cdot \ln(1+x^2)\right)'$$

$$\left(\frac{4x}{1+x^2}\right)' = \frac{4(1+x^2) - 4x \cdot 2x}{(1+x^2)^2} = \frac{4(1-x^2)}{(1+x^2)^2}$$

$$(\ln(1+x^2))' = \frac{1}{1+x^2} \cdot 2x = \frac{2x}{1+x^2}$$

$$g''(x) = \frac{4(1-x^2)\ln(1+x^2)}{(1+x^2)^2} + \frac{4x \cdot 2x}{(1+x^2)^2} =$$

$$= \frac{4(1-x^2)\ln(1+x^2) \cdot 8x^2}{(1+x^2)^2}$$

$$\Delta u(x) = \frac{4(1-|x|^2)\ln(1+|x|^2) \cdot 8|x|^2}{(1+|x|^2)^2} + \frac{1}{|x|} \cdot \frac{4|x| \ln(1+|x|^2)}{(1+|x|^2)}$$

$$\Delta u(x) = \frac{4(1-|x|^2)\ln(1+|x|^2) \cdot 8|x|^2}{(1+|x|^2)^2} + \frac{4\ln(1+|x|^2)}{1+|x|^2}$$

$\Delta u(1,1) = \ldots$

$$\Delta u(1,1) = \frac{-4\ln(3)\cdot 16}{9} + \frac{{}^{3)}4\ln 3}{3} = -\frac{16\cdot 4\ln 3 - 3\cdot 4\ln 3}{3} = -\frac{13}{3}\cdot 4\cdot \ln 3$$

$$|(1,1)| = \sqrt{1^2+1^2} = \sqrt{2} \Rightarrow |(1,1)|^2 = 2$$

Pb. 2

$$(1)\begin{cases} -\Delta u(x,y) = \frac{3}{1+y^2}, & (x,y)\in\Omega \\ u(x,y) = 0, & (x,y)\in\partial\Omega \end{cases}$$

unde $\Omega = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<9\}$ și $\partial\Omega$ fr.

1) <u>Cel mult o soluție</u>, $u \in C^2(\Omega)\cap C(\overline{\Omega})$

Pp. prin absurd că u_1, u_2 soluții diferite ale problemei (1)

Fie $U = u_1 - u_2 \Rightarrow$

$$\Delta U = \Delta u_1 - \Delta u_2 = \frac{3}{1+y^2} - \frac{3}{1+y^2} = 0$$

$$U\big|_{\partial\Omega} = u_1\big|_{\partial\Omega} - u_2\big|_{\partial\Omega} = 0 - 0 = 0$$

$\Delta U = 0 \Rightarrow U$ armonică $\xRightarrow[\text{de maxim}]{\text{Principiul}}$

$$\left.\begin{cases} \max\limits_{\overline{\Omega}} U = \max\limits_{\partial\Omega} U \\ \min\limits_{\overline{\Omega}} U = \min\limits_{\partial\Omega} U \end{cases} \quad \text{dar } U\big|_{\partial\Omega} = 0 \right\} \Rightarrow$$

$$\Rightarrow U \equiv 0 \Rightarrow u_1 = u_2$$

CONTRADICȚIE

2) $\nabla u(0,0)$

3) $v(x,y) = C(x^2+y^2)$

$-\Delta v = 3$ în Ω

$v_{xx}(x,y) = 2C$ $\quad$ $(v_x(x,y) = 2xC,$

$v_{yy}(x,y) = 2C$ $\quad$ $v_y(x,y) = 2yC)$

$\Delta v(x,y) = v_{xx}(x,y) + v_{yy}(x,y) = 4C$

$\Rightarrow -\Delta v(x,y) = -4C$ $\Bigg| \Rightarrow$

$\Rightarrow \boxed{C = -\frac{3}{4}}$

4) $0 < u(x,y) \leq \frac{27}{4}$ $\forall (x,y) \in \Omega$

$-\Delta u(x,y) = \frac{3}{1+y^2} > 0$ $\Rightarrow$

$(y^2 \geq 0 \Rightarrow 1+y^2 > 0)$

$\Rightarrow$ u este funcție superarmonică $\Rightarrow$

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$\Rightarrow$ (Principiul de maxim) $\min\limits_{\overline{\Omega}} u = \min\limits_{\partial\Omega} u$

dar $u\big|_{\partial\Omega} = 0 \Rightarrow u(x,y) \geq 0$

~~Putem scrie~~

Am demonstrat la seminar că

$$|u(x,y)| \leq \frac{R^2}{4} \cdot \max_{(x,y)\in B_R(0)} |f(x,y)|,$$

În cazul nostru $R=3$, $f(x,y) = \frac{3}{1+y^2}$,

~~net~~ putând ignora modulul deoarece $u \geq 0$

Așadar, $|u(x,y)| \leq \frac{9}{4} \cdot \max\limits_{(x,y)\in B_3(0)} \left(\frac{3}{1+y^2}\right)$

$\max\limits_{(x,y)\in B_3(0)} \left(\frac{3}{1+y^2}\right) = 3$ (când $y=0$)

deci $0 \leq u(x,y) \leq \frac{9}{4} \cdot 3 = \frac{27}{4}$

$\forall (x,y) \in \Omega$

Problema 3 (2)
$$\begin{cases} 2u_{tt}(x,t) + 5u_{tx}(x,t) - 3u_{xx}(x,t) = 0 & x\in\mathbb{R}, t>0 \\ u(x,0) = f(x), & x\in\mathbb{R} \\ u_t(x,0) = g(x), & x\in\mathbb{R} \end{cases}$$

1) $u \in C^2$

$(2\partial_t - \partial_x)(u_t(x,t) + 3u_x(x,t)) =$

$= 2\cdot u_{tt}(x,t) + 2\cdot 3\cdot u_{xt}(x,t) - u_{tx}(x,t) - 3u_{xx}(x,t) =$

$= 2\cdot u_{tt}(x,t) + 6u_{xt}(x,t) - u_{tx}(x,t) - 3u_{xx}(x,t)$

$u \in C^2 \Rightarrow$ (Schwarz) $u_{xt} = u_{tx}$, deci

$(2\partial_t - \partial_x)(u_t(x,t) + 3u_x(x,t)) =$

$= 2u_{tt}(x,t) + (6-1)u_{tx}(x,t) - 3u_{xx}(x,t) =$

$= 2u_{tt}(x,t) + 5u_{tx}(x,t) - 3u_{xx}(x,t)$, q.e.d.

2) ne folosim de 1) și prima ec. din (2) devine

$(2\partial_t - \partial_x)(u_t + 3u_x) = 0$

$(2\partial_t - \partial_x)\underbrace{(\partial_t + 3\partial_x)u}_{\text{not } v} = 0$

Fie $v = (\partial_t + 3\partial_x)u = u_t(x,t) + 3u_x(x,t)$

prima ec. de transp. $\begin{cases} 2v_t - v_x = 0 \\ v(x,0) = u_t(x,0) + 3u_x(x,0) = g(x) + 3f'(x). \end{cases}$

a doua $\begin{cases} u_t(x,t) + 3u_x(x,t) = g(x) + 3f'(x) \\ u(x,0) = f(x) \end{cases}$

$2v_t - v_x = 0 \Rightarrow (v_x, v_t)\begin{pmatrix}-1\\2\end{pmatrix} = 0 \Rightarrow$

$\Rightarrow (\nabla v)\cdot a = 0 \Rightarrow \dfrac{\partial v}{\partial a} = 0, \quad a = (-1,2)$

Deci v este constantă pe direcția $(-1,2)$:

$v(x,t) = v(t(-1,2) + (x+t, -t)) =$

$= v(x+t, -t) =$

$= v\left(\frac{t}{2}(-1,2) + \left(x+\frac{t}{2}, 0\right)\right) =$

$= v\left(x+\frac{t}{2}, 0\right) =$

$= g\left(x+\frac{t}{2}\right) + 3f'\left(x+\frac{t}{2}\right)$

Fixăm pe x și t și definim

$w(s) = v\left(x+\frac{s}{2}, t+s\right)$

$w'(s) = v_x\left(x+\frac{s}{2}, t+s\right)\cdot \dfrac{\partial\left(x+\frac{s}{2}\right)}{\partial s} +$

$+ v_t\left(x+\frac{s}{2}, t+s\right)\cdot \dfrac{\partial(t+s)}{\partial s} =$

$= \frac{1}{2} v_x\left(x+\frac{s}{2}, t+s\right) + v_t\left(x+\frac{s}{2}, t+s\right) =$

$\left(\cancel{= \frac{1}{2} g\left(x+\frac{s}{2}\right) + f'\left(x+\frac{s}{2}\right)}\right)$

$\left(\cancel{\int_0^{-t} w'(s)\,ds = \frac{1}{2}\int_0^{-t} g\left(x+\frac{s}{2}\right)}\right)$

$= \frac{1}{2} g\left(x+\frac{s}{2}+\frac{t+s}{2}\right) + f'\left(x+\frac{s}{2}+\frac{t+s}{2}\right) =$

$= \frac{1}{2} g\left(x+s+\frac{t}{2}\right) + f'\left(x+s+\frac{t}{2}\right)$

Integrăm de la 0 la $-t$ după s

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$$\Rightarrow \int_0^{-t} w'(s)\,ds = \int_0^{-t}\left(\frac{1}{2}g\left(x+s+\frac{t}{2}\right) + f'\left(x+s+\frac{t}{2}\right)\right)ds \Rightarrow$$

$\Rightarrow$ S.V.

$x+s+\frac{t}{2} = z$

$ds = dz$

$s=0 \Rightarrow z = x+\frac{t}{2}$

$s=-t \Rightarrow z = x-\frac{t}{2}$

$$w(-t) - w(0) = \frac{1}{2}\int_{x+\frac{t}{2}}^{x-\frac{t}{2}} g(z)\,dz + f(z)\Big|_{x+\frac{t}{2}}^{x-\frac{t}{2}} \Rightarrow$$

$$\Rightarrow v\left(x-\frac{t}{2}, 0\right) - v(x,t) = \frac{1}{2}\int_{x+\frac{t}{2}}^{x-\frac{t}{2}} g(z)\,dz + f(z)\Big|_{x+\frac{t}{2}}^{x-\frac{t}{2}}$$

$$\Rightarrow v(x,t) = -\frac{1}{2}\int_{x+\frac{t}{2}}^{x-\frac{t}{2}} g(z)\,dz - f\left(x-\frac{t}{2}\right) + f\left(x+\frac{t}{2}\right) + g\left(x-\frac{t}{2}\right) + 3f'\left(x-\frac{t}{2}\right)$$

$$v(x,t) = \frac{1}{2}\int_{x-\frac{t}{2}}^{x+\frac{t}{2}} g(z)\,dz + f\left(x+\frac{t}{2}\right) + g\left(x-\frac{t}{2}\right) - f\left(x-\frac{t}{2}\right) + 3f'\left(x-\frac{t}{2}\right)$$

3) Dacă aș fi calculat corect la 2) credeți-mă că l-aș fi făcut și pe 3), dar de acum asta e ☺

Problema 4 / Pb Cauchy

$$(3)\begin{cases} u_t(x,t) - u_{xx}(x,t) + \frac{t^2}{t^2+2}u(x,t) = 0, & x\in\mathbb{R}, t>0 \\ u(x,0) = e^{-2x^2}, & x\in\mathbb{R} \end{cases}$$

1) $v(x,t) = u(x,t)\Phi(t)$, $\Phi:\mathbb{R}\to\mathbb{R}$

(4) $v_t(x,t) - v_{xx}(x,t) = 0.$

$$\left.\begin{array}{l} v_t(x,t) = u_t(x,t)\Phi(t) + u(x,t)\Phi'(t) \\ v_{xx}(x,t) = \Phi(t)\cdot u_{xx}(x,t) \end{array}\right\} \Rightarrow$$

$\Rightarrow (4) \Leftrightarrow u_t(x,t)\Phi(t) + u(x,t)\Phi'(t) - \Phi(t)u_{xx}(x,t) = 0$

$\Leftrightarrow \Phi(t)\underbrace{(u_t(x,t) - u_{xx}(x,t))}_{0-\frac{t^2}{t^2+2}u(x,t)} + u(x,t)\Phi'(t) = 0$

$\Phi(t)\left(-\frac{t^2}{t^2+2}\right)u(x,t) = -u(x,t)\Phi'(t)$

$\Phi(t)\frac{t^2}{t^2+2} = \Phi'(t)$

$\frac{\Phi'(t)}{\Phi(t)} = \frac{t^2}{t^2+2} \Leftrightarrow \int\frac{\Phi'(t)}{\Phi(t)}dt = \int\frac{t^2}{t^2+2}dt \Leftrightarrow$

$\Leftrightarrow \ln(\Phi(t)) + \mathscr{C} = \int 1\,dt - 2\cdot\int\frac{1}{t^2+2}dt \Leftrightarrow$

$\ln(\Phi(t)) + \mathscr{C}_1 = t - 2\cdot\int\frac{1}{2(\frac{t^2}{2}+1)}dt =$

$= t - \frac{2}{\sqrt{2}}\operatorname{arctg}\left(\frac{t}{\sqrt{2}}\right) + \mathscr{C}_2$

Luăm $\mathscr{C}_1 = \mathscr{C}_2 = 0$

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$\Rightarrow \ln(\Phi(t)) = t - \sqrt{2} \cdot \operatorname{arctg}\left(\frac{t}{\sqrt{2}}\right)$

$\Phi(t) = e^{t - \sqrt{2} \cdot \operatorname{arctg}\left(\frac{t}{\sqrt{2}}\right)}$

2) $\begin{cases} v_t(x,t) - v_{xx}(x,t) = 0 \\ v(x,0) = \underbrace{\Phi(0)}_{=1} \cdot u(x,0) = u(x,0) = e^{-2x^2} \end{cases}$

$\Rightarrow v(x,t) = (K(\cdot, t) * u_0)(x) =$

$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \cdot e^{-2x^2} \, dy =$

$= \frac{1}{\sqrt{4\pi t}} \cdot \int_{-\infty}^{\infty} e^{\frac{-(x^2 - 2xy + y^2)}{4t} + (-2x^2)} \, dy =$

$= \frac{1}{\sqrt{4\pi t}} \left(\int_{-\infty}^{0} e^{-\frac{(-x^2 + 2xy - y^2)}{4t} + \frac{-2x^2 \cdot 4t}{4t}} \, dy + \int_{0}^{\infty} e^{-\frac{(x^2 - 2xy + y^2) - 2xy \cdot 4t}{4t}} \, dy \right.$

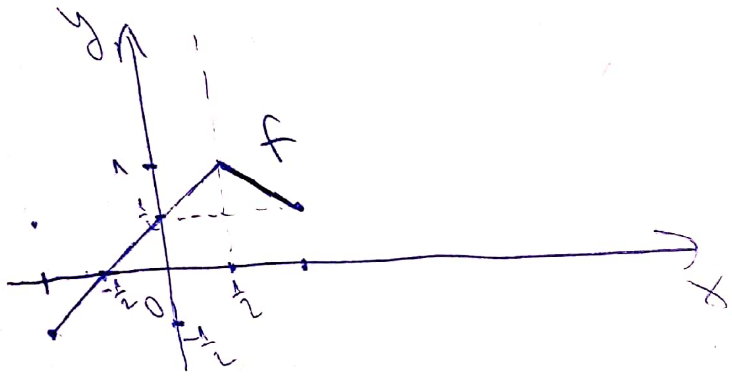
$= \ldots$

Problema 5) $f:[-1,1] \to \mathbb{R}$, $f(x) = 1 - |x - \frac{1}{2}|$

1) $|x-\frac{1}{2}| = \begin{cases} x-\frac{1}{2}, & x \geq \frac{1}{2} \\ \frac{1}{2}-x, & x < \frac{1}{2} \end{cases}$

$$f(x) = \begin{cases} 1 - x + \frac{1}{2}, & x \in [\frac{1}{2}, 1] \\ 1 - \frac{1}{2} + x, & x \in [-1, \frac{1}{2}) \end{cases}$$

$$f(x) = \begin{cases} \frac{3}{2} - x, & x \in [\frac{1}{2}, 1] \\ \frac{1}{2} + x, & x \in [-1, \frac{1}{2}) \end{cases}$$



2) f e continuă pe tot intervalul, singurul punct problematic fiind $-\frac{1}{2}$.

$f(-\frac{1}{2}) = 1$, $\lim\limits_{\substack{x \to \frac{1}{2} \\ x > \frac{1}{2}}} f(x) = \lim\limits_{\substack{x \to \frac{1}{2} \\ x < \frac{1}{2}}} f(x) = 1$ $\Rightarrow$

$(\frac{3}{2} - \frac{1}{2} = \frac{1}{2} + \frac{1}{2})$

f cont pe $[-1,1]$, iar pt derivare același punct este singurul problematic, dar acolo ~~nu~~ putem folosi de derivata slabă.

$(\underbrace{f'_s(-\frac{1}{2})}_{=1} \neq \underbrace{f'_d(-\frac{1}{2})}_{=-1}.)$

3) $f \in H^1(-1,1)$?

$H^1(-1,1) = W^{1,2}(-1,1) = \{ f \in L^2(0,2) \mid f'_w \in L^2(0,2) \}$

derivata slabă

Am arătat la 2) că f nu e derivabilă în $-\frac{1}{2}$, dar este derivabilă în sens slab cu derivata

$$f'_w(x) = \begin{cases} -1, & x \geq -\frac{1}{2} \\ 1, & x < \frac{1}{2} \end{cases}$$

Dem. că f'_w este derivată slabă:

Fie $\varphi \in C_c^\infty(-1,1) \Rightarrow \varphi(-1) = \varphi(1) = 0$ ($\varphi = 0$ pe fr.)

$$\int_{(-1,1)} f\varphi' dx = \int_{-1}^{1} f\varphi' dx = \int_{-1}^{\frac{1}{2}} f\varphi' dx + \int_{\frac{1}{2}}^{1} f\varphi' dx =$$

$$= f\left(\tfrac{1}{2}\right)\varphi\left(\tfrac{1}{2}\right) - \underbrace{f(-1)\varphi(-1)}_{0}$$

$$-\int_{-1}^{\frac{1}{2}} f'\varphi dx + f(1)\overset{0}{\varphi(1)} - f\left(\tfrac{1}{2}\right)\varphi\left(\tfrac{1}{2}\right) + \left(-\int_{\frac{1}{2}}^{+1} f'\varphi dx\right) =$$

$$= -\int_{-1}^{\frac{1}{2}} \underbrace{f'}_{1}\varphi dx + \int_{\frac{1}{2}}^{+1} \underbrace{f'}_{-1}\varphi dx = -\int_{-1}^{1} f'_w \varphi dx$$

unde $f'_w = \begin{cases} -1, & x \geq \frac{1}{2} \\ 1, & x < \frac{1}{2} \end{cases}$

$f'_w \in L^2(-1,1)$ (evident, $\int_{-1}^{1} |f'_w| dx = \int_{-1}^{1} 1 dx = 2 < \infty$

$f \in L^2(-1,1)$ deoarece

$$\int_{-1}^{1} |f(x)|^2 dx < \infty$$

$f(x)$ continuă, $(-1,1)$ mărginit

Așadar, $f \in H^1_0(-1,1)$

$$\|f\|_{H^1(-1,1)} = \|f\|_{L^2(-1,1)} + \|f'_w\|_{L^2(-1,1)} =$$

$$= \left(\int_{-1}^{1} |f(x)|^2 dx\right)^{\frac{1}{2}} + \underbrace{\left(\int_{-1}^{1} |f'_w(x)| dx\right)^{\frac{1}{2}}}_{\sqrt{2}} = \sqrt{2} + \frac{\sqrt{5}}{\sqrt{12}}$$

$$\int_{-1}^{1} |f(x)|^2 dx = \int_{-1}^{\frac{1}{2}} \left|\frac{1}{2}+x\right|^2 dx + \int_{\frac{1}{2}}^{1} \left|\frac{3}{2}-x\right|^2 dx =$$

$$= \int_{-1}^{-\frac{1}{2}} \left(-x-\frac{1}{2}\right)^2 dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}+x\right)^2 dx + \int_{\frac{1}{2}}^{1} \left(\frac{9}{4}-3x+x^2\right) dx =$$

$$= \int_{-1}^{-\frac{1}{2}} \left(x^2+x+\frac{1}{4}\right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x^2+x+\frac{1}{4}\right) dx + \frac{9}{4}\cdot\frac{1}{2} -$$

$$- \frac{3x^2}{2}\Big|_{\frac{1}{2}}^{1} + \frac{x^3}{3}\Big|_{\frac{1}{2}}^{1} =$$

$$= \int_{-1}^{\frac{1}{2}} \left(x^2+x+\frac{1}{4}\right) dx + \frac{9}{8} - \left(\frac{3}{2}-\frac{3}{8}\right) + \frac{1}{3} - \frac{1}{24} =$$

$$= \frac{x^3}{3}\Big|_{-1}^{\frac{1}{2}} + \frac{x^2}{2}\Big|_{-1}^{\frac{1}{2}} + \frac{1}{4}\cdot\frac{3}{2} + \cancel{\frac{9}{8}} - \cancel{\frac{9}{8}} + \frac{7}{24} =$$

$$= \frac{1}{24} + \frac{1}{3} + \frac{1}{8} - \frac{1}{2} + \frac{3}{8} + \frac{7}{24} =$$

$$= \frac{10}{24} = \frac{5}{12}$$