Facultatea de Matematica si Informatica
Universitatea din Bucuresti

30 Ianuarie 2023

Examen EDP I

Disciplina: Ecuatii cu derivate partiale
Tipul examinarii: Examen (scris)
Nume student: ______________________________
Seria 31
Timp de lucru : 3 ore

Acest examen contine 4 probleme (toate obligatorii).

Examenul este individual. Nu uitati sa va scrieti numele si prenumele in casuta de mai sus. La final veti aduce atat rezolvarile subiectelor cat si foile de concurs si le vom capsa.
Pentru elaborarea lucrarii scrise puteti folosi ca materiale ajutatoare o singura foaie fata-verso, format A4, scrisa cu pix/stilou si personalizata (nu se accepta copie xerox).

Pentru redactare tineti cont de urmatoarele sugestii:

- Daca folositi o teorema fundamentala, rezultat cunoscut, etc **indicati** acest lucru si explicati cum se poate aplica rezultatul respectiv.
- **Organizati-va munca** intr-un mod coerent pentru a avea toti de castigat ! La returnarea rezolvarilor va rog ca fiecare problema sa fie redactata in ordinea aparitiei pe foaia cu subiecte. Ideal ar fi ca si subpunctele sa fie redactate in ordine. Daca nu stiti a rezolva un anumit subpunct scrieti numarul subpunctului si lasati liber.
- Raspunsurile corecte dar argumentate incomplet (din punct de vedere al calculelor/explicatiilor) vor primi punctaj partial.

Barem: P1 (2.5p) + P2 (2p)+ P3 (2.5p) +P4 (2p) + 1p oficiu= **10p** (Plus eventual BONUS acolo unde este cazul in functie de activitatea din timpul semestrului).

Pentru orice nelamuriri scrieti-mi la adresa cristian.cazacu@fmi.unibuc.ro.

Rezultatele finale vor fi anuntate pe email sefilor de grupe in cel mai scurt timp posibil.

ATENTIE! Copiatul sau greseli fundamentale de scoala de generala/liceu, cum ar fi explicitarea modulului, scoaterea de sub integrala a unei functii neconstante inafara integralei, derivarea/integrarea defectuoasa a functiilor de tip exponentiala, sin, cos, etc..sunt motive automate de picare a examenului independent de punctajul cumulat.

Problema 1. *(2.5p).*

1). Calculati gradientul functiei $f(x,y,z) = (xz)^{\sin(yz)}$ *pe domeniul maxim de definitie pentru* $(x,y,z) \in \mathbb{R}^3$.

2). Calculati $\mathrm{div}((x_1, x_2, 2x_3)|x|^5)$, *unde* $x \in \mathbb{R}^3 \setminus \{0\}$.

3). Aratati ca $\Delta(x_4|x|^{-4}) = 0$, $\forall x \in \mathbb{R}^4 \setminus \{0\}$.

4). Aratati ca daca o functie neteda $f : \mathbb{R}^4 \to \mathbb{R}$ *verifica* $f(\lambda x) = \lambda^4 f(x)$ *pentru orice* $x \in \mathbb{R}^4$ *si orice* $\lambda > 0$, *atunci*

$$x \cdot \nabla f(x) = 4f(x), \quad \forall x \in \mathbb{R}^4.$$

Problema 2. *(2p). Fie* $\Omega := (-1,1) \times (0,1) \subset \mathbb{R}^2$ *si notam cu* $\partial\Omega$ *frontiera lui* Ω. *Consideram problema*

$$(1) \qquad \begin{cases} -\Delta u(x,y) = \frac{|\cos x|}{4+(\cos x)^2}, & (x,y) \in \Omega \\ u(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}$$

1). Gasiti constanta C *astfel incat functia* $v(x,y) = C(x^2+y^2)$ *sa verifice* $-\Delta v = \frac{1}{4}$ *in* Ω.

2). Folosind eventual "Principiile de Maxim" studiate (comparati eventual u cu v, etc.) si deduceti ca

$$0 < u(x,y) \leq \frac{1}{8}, \quad \forall (x,y) \in \Omega.$$

Problema 3. *(2.5p). Consideram urmatoarea problema de tip "unde"*

$$(2) \qquad \begin{cases} 3u_{tt}(x,t) + 8u_{tx}(x,t) - 3u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

unde $f, g \in C^2(\mathbb{R})$ *sunt functii date.*

1). Aratati ca daca $v = v(x,t)$ *este o functie de clasa* C^2 *atunci*

$$(3) \qquad (3\partial_t - \partial_x)(v_t(x,t) + 3v_x(x,t)) = 3v_{tt}(x,t) + 8v_{tx}(x,t) - 3v_{xx}(x,t), \quad \forall x, \forall t.$$

2). Rezolvati problema cu valori initiale (2) *satisfacuta de u (scrieti forma generala a lui u) reducand-o eventual la rezolvarea a doua ecuatii de transport (una omogena si alta neomogena).*

Problema 4. *(2p). Consideram problema Cauchy*

$$(4) \qquad \begin{cases} u_t(x,t) - 3u_x(x,t) - u_{xx}(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

unde $u_0 : \mathbb{R} \to \mathbb{R}$ *este o functie continua si marginita.*

1). Fie functia $v : \mathbb{R} \to \mathbb{R}$ *astfel incat functia* $v(x,t) := u(x-3t, t)$. *Aratati ca v verifice ecuatia*

$$(5) \qquad v_t(x,t) - v_{xx}(x,t) = 0, \quad \forall x \in \mathbb{R},\ \forall t > 0.$$

2). Determinati u in problema (4) *pentru* $u_0(x) = \cos^2(2x)$.

Examen EDP 1 - 31-2023

Problema 1

1) $f(x,y,z) = (xz)^{\sin(yz)}$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\frac{\partial f}{\partial x} = \sin(yz)\cdot(xz)^{\sin(yz)-1}\cdot z$

$\frac{\partial f}{\partial y} = (xz)^{\sin(yz)}\cdot \ln(xz)\cdot \cos(yz)\cdot z$

$\frac{\partial f}{\partial z} = \ln(xz)\cdot(xz)^{\sin(yz)}\cdot y\cdot\cos(yz) + \sin(yz)\cdot(xz)^{\sin(yz)-1}\cdot x =$

$= (xz)^{\sin(yz)}\left(y\ln(xz)\cdot\cos(yz) + x\sin(yz)\cdot\frac{1}{xz}\right) =$

$= (xz)^{\sin(yz)}\left(y\ln(xz)\cdot\cos(yz) + \frac{\sin(yz)}{z}\right)$

2) $\operatorname{div}\left((x_1,x_2,2x_3)|x|^5\right)$ Știm că $|x| = \left(\sum_{j=1}^{3} x_j^2\right)^{1/2} \Rightarrow |x|^5 = \left(\sum_{j=1}^{3} x_j^2\right)^{\frac{5}{2}} \overset{not.}{=} a$

$\Rightarrow (x_1,x_2,2x_3)|x|^5 = (x_1 a, x_2 a, 2x_3 a)$

$\operatorname{div}\left((x_1,x_2,2x_3)|x|^5\right) = \operatorname{div}(x_1\cdot a, x_2\cdot a, x_3\cdot a) = \frac{\partial}{\partial x_1}(x_1\cdot a) + \frac{\partial}{\partial x_2}(x_2\cdot a) + \frac{\partial}{\partial x_3}(x_3\cdot 2\cdot a)$

$\bullet\ \frac{\partial}{\partial x_1}(x_1\cdot a) = a + x_1\cdot\frac{\partial}{\partial x_1}(a) = \left(\sum_{j=1}^{3} x_j^2\right)^{5/2} + x_1\cdot\frac{5}{2}\cdot\left(\sum_{j=1}^{3} x_j^2\right)^{3/2}\cdot 2x_1 =$

$= \left(\sum_{j=1}^{3} x_j^2\right)^{5/2} + 5x_1^2\left(\sum_{j=1}^{3} x_j^2\right)^{3/2} = |x|^5 + 5x_1^2|x|^3$

$\bullet\ \frac{\partial}{\partial x_2}(x_2\cdot a) = |x|^5 + 5x_2^2|x|^3$

$\bullet\ \frac{\partial}{\partial x_3}(2x_3\cdot a) = 2a + 2x_3\cdot\frac{\partial}{\partial x_3}(a) = 2\left(\sum_{j=1}^{3} x_j^2\right)^{5/2} + 2x_3\cdot\frac{5}{2}\left(\sum_{j=1}^{3} x_j^2\right)^{3/2}\cdot 2x_3 =$

$= 2|x|^5 + 10x_3^2|x|^3$

Deci $\operatorname{div}\left((x_1, x_2, x_3)|x|^5\right) = |x|^5 + 5x_1^2|x|^3 + |x|^5 + 5x_2^2|x|^3 + 2|x|^5 +$

$+ 10x_3^2|x|^3 = 4|x|^5 + 5|x|^3(x_1^2 + x_2^2 + 2x_3^2) = 4|x|^5 + 5|x|^3(|x|^2 + x_3^2) =$

$= \underbrace{4|x|^5 + 5|x|^5} + x_3^2 \cdot 5 \cdot |x|^3 = 9|x|^5 + 5x_3^2|x|^3.$

3) $\Delta(x_4 |x|^{-4}) = 0$ # $\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \nabla g.$

$$\Delta(x_4 |x|^{-4}) = x_4 \Delta(|x|^{-4}) + \underbrace{|x|^{-4}\Delta x_4}_{=0} + 2\nabla x_4 \nabla(|x|^{-4}).$$

$$\Delta(|x|^{-4}) = -4(-4-2+4)|x|^{-6} = 8|x|^{-6}$$

x_4

$$\nabla x_4 = \vec{e_4} \qquad \nabla(|x|^{-4}) = -4|x|^{-6}\cdot x$$

Donc $\Delta(x_4|x|^{-4}) = 8x_4|x|^{-6} + 2\,\vec{e_4}\cdot(-4)\cdot|x|^{-6}\cdot x =$

$$= 8x_4|x|^{-6} - 8x_4|x|^{-6} = 0.$$

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4) $f:\mathbb{R}^4\to\mathbb{R},\ f(\lambda x)=\lambda^4 f(x) \Rightarrow x\cdot\nabla f(x)=4f(x)$

$$\frac{d}{d\lambda}[f(\lambda x)]=\frac{d}{d\lambda}[\lambda^4 f(x)]$$

$$\frac{d}{d\lambda}[f(\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4)]=4\lambda^3 f(x)$$

$$\frac{\partial f}{\partial x_1}(\lambda x)\cdot x_1+\dots+\frac{\partial f}{\partial x_4}(\lambda x)\cdot x_4=4\lambda^3 f(x)$$

$$\nabla f(\lambda x)\cdot x=4\lambda^3 f(x)$$

Aleg $\lambda=1 \Rightarrow x\cdot\nabla f(x)=4f(x)$.

Problema 2 $\Omega=(-1,1)\times(0,1)\subset\mathbb{R}^2$, $\begin{cases}-\Delta u(x,y)=\dfrac{|\cos x|}{4+(\cos x)^2}, & (x,y)\in\Omega\\ u(x,y)=0, & (x,y)\in\partial\Omega\end{cases}$

1) $v(x,y)=C(x^2+y^2)$, $-\Delta v=\frac{1}{4}$ în Ω

$\Delta v=4C$, $\Delta v=-\frac{1}{4} \Rightarrow 4C=-\frac{1}{4}\Rightarrow \boxed{C=-\frac{1}{16}}$

2) $0<u(x,y)\le\frac{1}{8}$, $(\forall)\ (x,y)\in\Omega$.

Avem $-\Delta u\ge 0$ în Ω, deci u este supraarmonică $\xRightarrow{\text{PSM}}$

$\Rightarrow \min\limits_{\overline{\Omega}} u=\min\limits_{\partial\Omega} u \Rightarrow u\ge 0$ în $\overline{\Omega}$

Pp. că $(\exists)\ (x_0,y_0)\in\Omega$ a.î. $u(x_0,y_0)=0 \Rightarrow (x_0,y_0)$ este punct ~~de~~ minim de interior pt. u

$\xRightarrow{\text{PTM}}$ ~~u este supr~~ $u\equiv ct. \Rightarrow \Delta u=c$ $\cancel{\text{,}}$, deci $\boxed{u>0 \text{ în } \Omega}$ (1)

Iau $v(x,y)=-\frac{1}{16}(x^2+y^2)$, deci $\Delta v=-\frac{1}{4}$ în Ω

$$v/_{\partial v}=-\frac{1}{16}(x^2+y^2)$$

Fie $U=u-v \Rightarrow -\Delta U=-\Delta u+\Delta v=\underbrace{\frac{|\cos x|}{4+(\cos x)^2}}_{\le\frac{1}{4}}-\frac{1}{4}\le 0$ în Ω

Avem $-\Delta U\le 0$ în Ω, deci U este subarmonică $\Rightarrow$

$\Rightarrow \max\limits_{\overline{\Omega}} U=\max\limits_{\partial\Omega} U=\max\limits_{\partial\Omega}\left\{0+\frac{1}{16}(x^2+y^2)\right\}=\max\limits_{\partial\Omega}\left\{\frac{1}{16}(x^2+y^2)\right\}=$

$=\frac{1}{16}\cdot 2=\frac{1}{8}$

$\max U=\frac{1}{8}\Rightarrow U\le\frac{1}{8}\Rightarrow u\le\frac{1}{8}+\overset{\le 0}{v}$ în $\Omega \Rightarrow \boxed{u\le\frac{1}{8}}$ (2) $\quad$ (1) + (2) ✓

Problema 3 $\begin{cases} 3u_{tt}(x,t) + 8u_{tx}(x,t) - 3u_{xx}(x,t) = 0, & x\in\mathbb{R},\ t>0 \\ u(x,0) = f(x), & x\in\mathbb{R} \\ u_t(x,0) = g(x), & x\in\mathbb{R} \end{cases}$

1) $v = v(x,t) \in C^2$

$(3\partial_t - \partial_x)(v_t(x,t) + 3v_x(x,t)) = 3v_{tt}(x,t) + \overbrace{9v_{xt}(x,t) - v_{tx}(x,t)}^{\text{T. Schwartz}} - 3v_{xx}(x,t) = 3v_{tt}(x,t) + 8v_{tx}(x,t) - 3v_{xx}(x,t),\ (\forall)\ x,t$

2) Notez $v(x,t) = u_t(x,t) + 3u_x(x,t)$

Iau ecuația de transport omogenă: $\begin{cases} 3v_t(x,t) - 1v_x(x,t) = 0 \\ v(x,0) = u_t(x,0) + 3u_x(x,0) = g(x) + 3f'(x) \end{cases}$

Avem $\underbrace{(v_x, v_t)}_{\nabla v} \cdot \underbrace{(-1,3)}_{\bar a} = 0$, deci $\frac{\partial v}{\partial \bar a} = 0 \Rightarrow v$ este const. pe direcția lui $\bar a$

$v(x,t) = v\left(\frac{t}{3}(-1,3) + \left(x + \frac{t}{3}, 0\right)\right) = v\left(x + \frac{t}{3}, 0\right) = g\left(x+\frac{t}{3}\right) + 3f'\left(x+\frac{t}{3}\right)$

Iau ecuația de transport neomogenă:

$\begin{cases} 1\cdot u_t(x,t) + 3\cdot u_x(x,t) = g\left(x+\frac{t}{3}\right) + 3f'\left(x+\frac{t}{3}\right) \\ u(x,0) = f(x) \end{cases}$

Fixăm x și t. Fie $\boxed{w(s) = u(x+3s, t+s)}$, $s\in\mathbb{R}$

$w'(s) = 3u_x(\underbrace{x+3s}_{x'}, \underbrace{t+s}_{t'}) + u_t(x+3s, t+s)$

$w'(s) = 3f'\left(x+3s+\frac{t+s}{3}\right) + g\left(x+3s+\frac{t+s}{3}\right) = 3f'\left(x+\frac{t}{3}+\frac{10s}{3}\right) + g\left(x+\frac{t}{3}+\frac{10s}{3}\right)$

Deci $\begin{cases} w'(s) = g\left(x+\frac{t}{3}+\frac{10}{3}s\right) + 3f'\left(x+\frac{t}{3}+\frac{10}{3}s\right) \quad * \\ w(0) = u(x,t) \\ w(-t) = \cancel{u(x,0) = f(x)}\ u(x-3t, 0) = f(x-3t) \end{cases}$

Integrăm $(*) \Rightarrow w(0) - w(-t) = \int_{-t}^{0}\left[g\left(x+\frac{t}{3}+\frac{10}{3}s\right) + 3f'\left(x+\frac{t}{3}+\frac{10}{3}s\right)\right]ds =$

$\frac{x+\frac{t}{3}+\frac{10}{3}s = \tau}{d\tau = ds\cdot\frac{10}{3},\ ds = \frac{3}{10}d\tau}$ $\int_{x-3t}^{x+\frac{t}{3}}\left[g(\tau) + 3f'(\tau)\right]\cdot\frac{3}{10}\,d\tau = \frac{3}{10}\int_{x-3t}^{x+\frac{t}{3}} g(\tau)\,d\tau +$

$+ \frac{9}{10}\left(\cancel{f(x-3t)}\ \left(f\left(x+\frac{t}{3}\right) - f(x-3t)\right)\right)$

$\underset{=w(0)}{u(x,t)} = \frac{3}{10}\int_{x-3t}^{x+\frac{t}{3}} g(\tau)\,d\tau + \underbrace{f(x-3t) - \frac{9}{10}f(x-3t)}_{\frac{1}{10}f(x-3t)} + \frac{9}{10}f\left(x+\frac{3}{t}\right)

Problema 4 $\begin{cases} u_t(x,t) - 3u_x(x,t) - u_{xx}(x,t) = 0, & x\in\mathbb{R}, t>0 \\ u(x,0) = u_0(x), & x\in\mathbb{R} \end{cases}$

1) $v:\mathbb{R}\to\mathbb{R}$ a.î. $v(x,t) := u(x-3t,t) \Rightarrow v_t(x,t) - v_{xx}(x,t) = 0$

~~Calculez $v_t(x,t)$ și $v_{xx}(x,t)$.~~

$v_t(x,t) = u_x(x-3t,t)\cdot(-3) + u_t(x-3t,t)\cdot 1 =$

$= (-3u_x + u_t)(x-3t,t) \overset{\text{din ip.}}{=} u_{xx}(x-3t,t) = v_{xx}(x,t)$

Deci $v_t(x,t) - v_{xx}(x,t) = 0$.

2) $u_0(x) = \cos^2(2x)$

$u\mapsto v: \begin{cases} v_t(x,t) - v_{xx}(x,t) = 0 \\ v(x,0) = u(x,0) = u_0(x) = \cos^2(2x) \end{cases}$

$\boxed{ay^2 + by = a\left(y + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}}$

$v(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} \cdot \cos^2(2y)\, dy \underset{\frac{x-y}{2\sqrt{t}} = z \Rightarrow y = -2z\sqrt{t} + x}{\overset{S.V.}{=}}$

$= \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-z^2} \cdot \cos^2(\overbrace{2x - 4z\sqrt{t}}^{\text{„}x\text{"}}) \cdot (2\sqrt{t})\, dz =$

$= \frac{2\sqrt{t}}{\sqrt{4\pi}\cdot\sqrt{t}} \int_{\mathbb{R}} e^{-z^2} \cdot \frac{1 + \cos(4x - 8z\sqrt{t})}{2}\, dz =$

$= \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2}\, dz + \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} \cdot \cos(4x - 8z\sqrt{t})\, dz =$

$= \frac{1}{2\sqrt{\pi}} \cdot \sqrt{\pi} + \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} \cdot \cos(4x - 8z\sqrt{t})\, dz =$

$\boxed{\cos(4x - 8z\sqrt{t}) = \cos(4x)\cos(8z\sqrt{t}) + \sin(4x)\sin(8z\sqrt{t})}$

$= \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} \underbrace{e^{-z^2} \cdot \cos 4x \cdot \cos(8z\sqrt{t})}_{\text{funcție pară}}\, dz + \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} \cdot \sin(4x) \cdot \sin(8z\sqrt{t})\, dz$

$= \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \cdot \cos(4x) \int_{\mathbb{R}} \cos(8z\sqrt{t}) \cdot e^{-z^2}\, dz + \frac{1}{2\sqrt{\pi}} \cdot \sin(4x) \overbrace{\int_{\mathbb{R}} \underbrace{\sin(8z\sqrt{t})\, e^{-z^2}}_{\text{funcție impară}}\, dz}^{0}$

~~fie $f(z) = \cos(8z\sqrt{t}) \Rightarrow f(-z)$~~

$= \frac{1}{2} + \frac{2}{2\sqrt{\pi}} \cdot \cos(4x) \int_0^\infty \cos(\underbrace{8\sqrt{t}}_{8\sqrt{t} = a} z) \cdot e^{-z^2}\, dz = (*)$

fie $J(a) = \int_0^\infty e^{-z^2} \cdot \cos(az)\, dz$ – integrală parametrică în a.

$J'(a) = \int_0^\infty -z \cdot e^{-z^2} \cdot \sin(az)\, dz = \int_0^\infty \frac{1}{2}\left(e^{-z^2}\right)'_z \cdot \sin(az)\, dz =$

$$= \underbrace{\frac{1}{2} e^{-t^2} \sin(at) \Big|_0^\infty}_{=0} - \int_0^\infty \frac{1}{2} e^{-t^2} \cdot \cancel{a} \cos(at)\, dt = -\frac{a}{2} J(a)$$

$$\begin{cases} J'(a) = -\dfrac{a}{2} J(a) \\ J(0) = \dfrac{\sqrt{\pi}}{2} \end{cases}$$

~~ec. caracteristică~~

$$\cancel{\lambda + \frac{a}{2} = 0 \Rightarrow \lambda = -\frac{a}{2}}$$

$$\cancel{J(a) = c \cdot e^{-\frac{a}{2} \cdot t}}$$

$$J(a) = \frac{\sqrt{\pi}}{2} \cdot e^{-\frac{a^2}{4}}$$

$$\circledast = \frac{1}{2} + \frac{2}{2\sqrt{\pi}} \cdot \cos(4x) \cdot J(8\sqrt{t}) = \frac{1}{2} + \frac{2}{2\sqrt{\pi}} \cdot \cos(4x) \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-16t} \Rightarrow$$

$$\Rightarrow u(x,t) = \frac{1}{2} \left(1 + \cos(4x) \cdot e^{-16t}\right)$$