SEMINAR VIII
ANALIZĂ

Deriv., dif. funcțiilor compuse

$u, v : E \subseteq \mathbb{R} \to \mathbb{R}$, $u(x), v(x) \in F$; $f : F \to \mathbb{R}$ $\Rightarrow$ $f(x) = f(u(x))$

$$f'(x) = \frac{\partial f}{\partial u}\cdot\frac{du}{dx} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dx}$$

$$df(x) = f'(x)\,dx$$

$u, v : E \subseteq \mathbb{R} \to \mathbb{R}$, $u(x,y), v(x,y) \in F$; $f : F \to \mathbb{R}$ $\Rightarrow$ $f(x,y) = f(u(x,y), v(x,y))$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

Ex. 1 Calc. deriv. și dif. fct. $f(x)$:

$f(x) = f(\underbrace{1+x^2}_{u(x)}, \underbrace{\sin x}_{v(x)})$, $x \in \mathbb{R}$ unde $f(u,v) = 2u\cdot v$

<u>Sol 1</u>: $f(x) = 2(1+x^2)\cdot\sin x$ $(2uv)$

$f'(x) = 4x\sin x + 2(1+x^2)\cos x$, $x \in \mathbb{R}$

$df(x) = (4x\sin x + 2(1+x^2)\cos x)\,dx$

<u>Sol 2</u>: (formula) $f'(x) = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} =$

$= 2v\cdot 2x + 2\cdot u\cos x = 2\sin x\, 2x + 2(1+x^2)\cos x =$

$= 4x\sin x + 2(1+x^2)\cos x.$

Ex2. Deriv. parțiale + dif. funcției

$f(x,y) = f(\underbrace{x^2+y^2}_{u(x,y)}, \underbrace{x-y}_{v(x,y)})$, $f : \mathbb{R}^2 \to \mathbb{R}$, $f \in C^1(\mathbb{R}^2)$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}(x^2+y^2, x-y)\cdot 2x + \frac{\partial f}{\partial v}(x^2+y^2, x-y)\cdot 1$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(x^2+y^2, x-y)\cdot 2y + \frac{\partial f}{\partial v}(x^2+y^2, x-y)\cdot(-1).$$

(Ex. 3) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy \cdot \varphi(x^2-y^2)$ verifică:

$$xy^2 \cdot \frac{\partial f}{\partial x} + x^2y \frac{\partial f}{\partial y} = (x^2+y^2) f(x,y)$$

$$\frac{\partial f}{\partial x}(x,y) = y \cdot \varphi(x^2-y^2) + xy \cdot \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x} = y \cdot \varphi(x^2-y^2) + xy \frac{\partial \varphi}{\partial u}(x^2-y^2) \cdot 2x =$$

$$= y\varphi(x^2-y^2) + 2x^2y \frac{\partial \varphi}{\partial u}(x^2-y^2)$$

$$\frac{\partial f}{\partial y}(x,y) = x \cdot \varphi(x^2-y^2) + xy \cdot \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial y} = x\varphi(x^2-y^2) - 2y^2x \frac{\partial \varphi}{\partial u} \cdot (x^2-y^2)$$

$$\Rightarrow xy^2 \boxed{\frac{\partial f}{\partial x}} + x^2y \boxed{\frac{\partial f}{\partial y}} = \left[xy^3 \varphi(x^2-y^2) + 2x^3y^3 \frac{\partial \varphi}{\partial u}(x^2-y^2)\right] +$$

$$+\left[x^3y\, \varphi(x^2-y^2) - 2x^3y^3 \frac{\partial \varphi}{\partial u}(x^2-y^2)\right] =$$

$$= \underbrace{xy\varphi(x^2-y^2)}(y^2+x^2) = f(x,y) \cdot (x^2+y^2).$$

(Ex. 4) Calculați derivatele parțiale ale funcției $f(x,y) = \ln(u^2+v)$, unde $\begin{cases} u(x,y) = e^{x+y^2} \\ v(x,y) = x^2+y \end{cases}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{1}{u^2+v} \cdot 2u \cdot e^{x+y^2} + \frac{1}{u^2+v} \cdot 2x =$$

$$= \frac{2e^{2(x+y^2)}}{e^{2(x+y^2)} + x^2+y} + \frac{2x}{e^{2(x+y^2)} + x^2+y}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{2u}{u^2+v} \cdot 2y \cdot e^{x+y^2} + \frac{1}{u^2+v} \cdot 1 = \frac{4ye^{2(x+y^2)}+1}{e^{2(x+y^2)}+x^2+y}$$

5) Dem. că $f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = \varphi(\underbrace{xy}_{u(x,y,z)}, \underbrace{x^2+y^2-z^2}_{v(x,y,z)})$, $\varphi \in C^1(\mathbb{R}^2)$

este o soluție a ec:

$$xz\boxed{\frac{\partial f}{\partial x}} - yz\boxed{\frac{\partial f}{\partial y}} + (x^2-y^2)\boxed{\frac{\partial f}{\partial z}} = 0.$$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial \varphi}{\partial u}(xy, x^2+y^2-z^2)\,y + \frac{\partial \varphi}{\partial v}(xy, x^2+y^2-z^2)\,2x$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial \varphi}{\partial u}(xy, x^2+y^2-z^2)\,x + \frac{\partial \varphi}{\partial v}(xy, x^2+y^2-z^2)\,2y$$

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial z} = \frac{\partial \varphi}{\partial u}(xy, x^2+y^2-z^2)\,0 + \frac{\partial \varphi}{\partial v}(xy, x^2+y^2-z^2)(-2z)$$

$$\Rightarrow xz\boxed{\frac{\partial f}{\partial x}} - yz\boxed{\frac{\partial f}{\partial y}} + (x^2-y^2)\boxed{\frac{\partial f}{\partial z}} = 0. \Rightarrow$$

$$\Rightarrow \cancel{xyz\frac{\partial \varphi}{\partial u}(x,y,z)} + 2x^2z\frac{\partial \varphi}{\partial v}(xy, x^2+y^2-z^2) - \cancel{xyz\frac{\partial \varphi}{\partial u}(\dots)} - 2y^2z\frac{\partial \varphi}{\partial v}(\dots)$$

$$-2z(x^2-y^2)\frac{\partial \varphi}{\partial v}(\dots) =$$

$$= (2x^2z - 2y^2z - 2z(x^2-y^2))\frac{\partial \varphi}{\partial u}(\dots) =$$

$$= \underbrace{(\cancel{2x^2z} - \cancel{2y^2z} - \cancel{2x^2z} + \cancel{2y^2z})}_{0}\frac{\partial \varphi}{\partial u}(\dots) = 0\cdot\frac{\partial \varphi}{\partial u}(\dots) = 0.\ (*)$$

$\boxed{\text{Ex.6}}$ $a \in \mathbb{R}$, $g, h \in C^2(\mathbb{R})$

Dem. că $f(x,y) = g(\underbrace{x-ay}_{u(x,y)}) + h(\underbrace{x+ay}_{v(x,y)})$ verifică ec:

$$\frac{\partial^2 f}{\partial y^2} - a^2\frac{\partial^2 f}{\partial x^2} = 0.$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial x} = \overbrace{\frac{\partial g}{\partial u}(x-ay)}^{g'(x-ay)}\cdot 1 + \overbrace{\frac{\partial h}{\partial v}(x+ay)}^{h'(x+ay)}\cdot 1$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial y} = \underbrace{\frac{\partial g}{\partial u}(x-ay)}_{g'(x-ay)}(-a) + \underbrace{\frac{\partial h}{\partial v}(x+ay)}_{h'(x+ay)}\cdot a$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = g''(x-ay)\cdot 1 + h''(x+ay)\cdot 1$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = g''\underbrace{(x-ay)\cdot a^2 + h''(x+ay)\, a^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial y^2}} - a^2\cdot\frac{\partial^2 f}{\partial x^2} = g''(x-ay)a^2 + h''(x+ay)a^2 - a^2\left(g''(x-ay) + h''(x+ay)\right) = 0.$$

POLINOM TAYLOR

(Cazul multidimensional)

1. Scrieți polinomul Taylor (gr. 2) pt. funcțiile:

$f:\mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 - 2y^2 + 3xy + \cancel{2y}$, și pct $(1,2)$

Ce des?

$f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\quad (a,b)\in A \quad f \in C^3(A)$ $\qquad a \qquad b \qquad (a+b)^2 \quad (a+b)^3$

$$T_3((x,y),(a,b)) = f(a,b) + \frac{1}{1!}\left[\boxed{\frac{\partial f}{\partial x}(a,b)(x-a)} + \boxed{\frac{\partial f}{\partial y}(a,b)(y-b)}\right] +$$

$$+\frac{1}{2!}\left[\boxed{\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2} + \boxed{2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b)} + \boxed{\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2}\right] +$$

$a^2 \qquad 2ab \qquad b^2$

$$+\frac{1}{3!}\left[\boxed{\frac{\partial^3 f}{\partial x^3}(a,b)(x-a)^3} + \boxed{3\cdot\frac{\partial^3 f}{\partial x^2 \partial y}(a,b)(x-a)^2(y-b)} + \boxed{3\cdot\frac{\partial^3 f}{\partial x \partial y^2}(a,b)(x-a)(y-b)^2} +\right.$$

$a^3 \qquad 3a^2b \qquad 3ab^2$

$$\left.+\boxed{\frac{\partial^3 f}{\partial y^3}(a,b)(y-b)^3}\right]$$

b^3.

$\frac{\partial f}{\partial x}(x,y) = 2x + \boxed{3y}$ $\qquad \frac{\partial f}{\partial y}(x,y) = -4y + \boxed{3x}$ $\qquad \frac{\partial^2 f}{\partial x^2}(x,y) = 2$ $\qquad \frac{\partial^2 f}{\partial y^2}(x,y) = -4.$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = \boxed{3}.$

$$T_2((x,y),(a,b)) = f(1,2) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(1,2)(x-1)\right) + \frac{\partial f}{\partial y}((1,2)(y-2)) +$$

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(1,2)(x-1)^2 + \frac{\partial^2 f}{\partial y^2}(1,2)(y-2)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,2)(x-1)(y-2)\right) =$$

$$= -1 + 8x - 8 - 8y + 10 + (x-1)^2 - 2(y-2)^2 + 3(x-1)(y-2) =$$

$$= x^2 - 2y^2 \ldots$$

Extremele funcțiilor de mai multe variabile

$\mathbb{R}^2$.

Fie $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \ldots$

1. Rez. sistemul: $\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0. \end{cases}$ (puncte critice/staționare)

$\left.\begin{array}{l} \Delta_1 = - \\ \Delta_2 = + \end{array}\right| (x_i, y_i)$ punct max

$\Delta_2 = - \Rightarrow (x_i, y_i)$ nu e pct de extrem

2. $H(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}$

3. $H(x_i, y_i) = \begin{array}{c} \Delta_1 \quad \Delta_2 \\ \begin{pmatrix} \boxed{\cdot} & \cdot \\ \cdot & \cdot \end{pmatrix} \end{array}$

4. $\left.\begin{array}{l} \Delta_1 + \\ \Delta_2 + \end{array}\right| \Rightarrow (x_i, y_i)$ puncte min

$\left.\begin{array}{l} \Delta_1 - \\ \Delta_2 + \end{array}\right| \Rightarrow (x_i, y_i)$ puncte max

$\left.\begin{array}{l} \Delta_2 - \\ \Delta_2 = 0 ? \end{array}\right| \Rightarrow (x_i, y_i)$ nu e punct de extrem

$\mathbb{R}^3$. Fie $f:\mathbb{R}^3 \to \mathbb{R}$

1. Rez. sist
$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 0 \end{cases}$$

2. $$H(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

$\left.\begin{matrix}\Delta_1 + \\ \Delta_2 + \\ \Delta_3 +\end{matrix}\right| \Rightarrow (x_i, y_i, z_i)$ pct. min

$\left.\begin{matrix}\Delta_1 - \\ \Delta_2 + \\ \Delta_3 -\end{matrix}\right| \Rightarrow (x_i, y_i, z_i)$ pct max

$f:\mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = 3x^2y + y^3 + 12x - 15y + 11$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 6xy - 12 = 0 \Rightarrow xy = 2 \Rightarrow x = 2/y \\ \frac{\partial f}{\partial y} = 0 \Rightarrow 3x^2 + 3y^2 - 15 = 0 \Rightarrow x^2 + y^2 = 5 \Rightarrow \frac{4}{y^2} + y^2 = 5 \Rightarrow t = y^2 \end{cases}$$

$4 + y^4 - 5y^2 = 0 \Rightarrow t^2 - 5t + 4 = 0$

$\Delta = 25 - 16 = 9 \quad t_1 = 4 \quad t_2 = 1$

$y^2 = 4$ | $y^2 = 1$

$y = \pm 2$ | $y = \pm 1$

$x = \pm 1$ | $y = \pm 2$

$(1,2)$; $(-1,-2)$; $(-2,1)$; $(2,1)$ - puncte critice

$$H(x,y) = \begin{pmatrix} 6y & 6x \\ 6x & 6y \end{pmatrix} \Rightarrow H(1,2) = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$$

$\Delta_1 = 12 > 0$ $\quad$ $(1,2)$ - pct de min local strict

$\Delta_2 > 0$ -

④. $f:\mathbb{R} \to \mathbb{R}$ $f(x,$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2(6- \\ \frac{\partial f}{\partial y} = 0 \Rightarrow 2yx^3(6-x- \end{cases}$$

$\cdot(-1,-2) = \begin{pmatrix} -12 \\ -6 \end{pmatrix}$

$(-1,-2) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix}$ $\Delta_1 < 0$, $\Delta_2 > 0$ | punct de max local

② . $f : \mathbb{R} \to \mathbb{R}$ $f(x,y) = x^3y^2(6-x-y)$, $x,y > 0$.

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2(6-x-y) + x^3y^2(-1) = 0 \Rightarrow x^2y^2(18-3x-3y-x) = 0 \\ \frac{\partial f}{\partial y} = 0 \Rightarrow 2yx^3(6-x-y) + x^3y^2(-1) = 0 \Rightarrow x^3y(12-2x-3y) = 0. \end{cases}$$

$$\Rightarrow \begin{cases} 18-4x-3y = 0 \\ 12-2x-3y = 0 \end{cases} \Rightarrow \begin{cases} 4x+3y = 18 \\ 2x+3y = 12 \end{cases} -$$

$2x = 6 \Rightarrow x = 3 \Rightarrow y = 2$. $(3,2)$ punct critic

$H(x,y) = \begin{pmatrix} 2xy^2(18-4x-3y) - 4x^2y^2 & 2x^2y(18-4x-3y) - 3x^2y^2 \\ 2x^2y(18-4x-3y) - 3x^2y^2 & 2xy^2(18-4x-3y) - 4x^2y^2 \end{pmatrix}$

Hesiana

$= \begin{pmatrix} 2xy^2(18-6x-3y) & x^2y(36-8x-9y) \\ x^2y(36-8x-9y) & 2xy^2(18-6x-3y) \end{pmatrix}$

$H(3,2) = \begin{pmatrix} -24\cdot 6 & -18\cdot 6 \\ -18\cdot 6 & -24\cdot 6 \end{pmatrix}$

$\Delta_1 < 0$, $\Delta_2 > 0$ $(3,2)$ max local.