Examen Algebră liniară, Seria 11
17.01.2022

La rezolvarea problemelor veţi folosi următorii parametri:
M = numărul de litere din primul vostru nume de familie
N = numărul de litere din primul vostru prenume.
De exemplu, pentru *Ionescu Ana-Maria* avem $M = 7$ şi $N = 3$.

Scrieţi pe prima pagină cu rezolvări valorile parametrilor voştri:
$M = \dots,\quad N = \dots.$
Timp pentru rezolvarea problemelor şi încărcarea soluţiilor în MoodleUB: 3 ore

(1) (2,25 pct.) În $\mathbb{R}^4$ considerăm vectorii

$$u_1 = \begin{pmatrix} 1 \\ M+2 \\ 2N+1 \\ N \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ N-1 \\ M+2 \\ N-1 \end{pmatrix}, u_3 = \begin{pmatrix} 7 \\ M+2N \\ 2N+2M+5 \\ 3N-2 \end{pmatrix}, u_4 = \begin{pmatrix} -1 \\ 2M-N+3 \\ 4N-M \\ N+1 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 4 \\ 3 \\ N+1 \end{pmatrix}$$

şi subspaţiul vectorial $V = \langle u_1, u_2, u_3, u_4 \rangle$.
(a) Aflaţi $\dim V$ şi o bază $\mathcal{B}$ pentru V.
(b) Completaţi $\mathcal{B}$ la o bază pentru $\mathbb{R}^4$.
(c) Aflaţi $pr_V(w)$ proiecţia ortogonală a lui w pe V şi $||pr_V(w)||$.
(d) Este w in V? Argumentaţi.

(2) (1,5 pct.) Pentru orice $n \in \mathbb{N}, n \geq 2$ notăm

$$A_n = \begin{pmatrix} 2 & N & 0 & \dots & \dots & 0 \\ N & 2 & N & 0 & \dots & 0 \\ N & N & 2 & N & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ N & N & \dots & N & 2 & N \\ N & N & \dots & N & N & 2 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}) \text{ şi } d_n = \det(A_n).$$

(a) Calculaţi d_2, d_3, d_4.
(b) Stabiliţi dacă matricea A_3 este inversabilă, iar în caz afirmativ găsiţi-i inversa.
(c) Arătaţi că are loc relaţia $d_n = 2d_{n-1} + N^2(N-2)\cdot d_{n-3}$, pentru orice $n \geq 5$.

(3) (2,75 pct.) Fie aplicaţia liniară $T : \mathbb{R}^3 \to \mathbb{R}^3$ dată de

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x-2y-4z \\ y \\ 12x-4y-7z \end{pmatrix}, \text{ pentru orice } x, y, z \in \mathbb{R}.$$

(a) Determinaţi subspaţiul $\operatorname{Ker} T \cap \operatorname{Im} T$.

Subiectele continuă pe pagina următoare...

(b) Verificaţi dacă $w = \begin{pmatrix} 3 \\ M-1 \\ N+1 \end{pmatrix}$ este vector propriu pentru T.

(c) Aflaţi valorile proprii ale lui T.

(d) Găsiţi o bază $\mathcal{B}$ in $\mathbb{R}^3$ astfel încât matricea lui T în baza $\mathcal{B}$ să fie diagonală. Argumentaţi dacă poate fi $\mathcal{B}$ aleasă să fie ortonormată.

(4) (1,5 pct.) Fie v_1, v_2, v_3 vectori din $\mathbb{R}$-spaţiul vectorial V. Arătaţi că dacă v_1, v_2, v_3 sunt liniar independenţi, atunci şi $Mv_1 + v_2$, $Mv_2 + v_3$, $Mv_3 + v_1$ sunt liniar independenţi.

Este reciproca adevărată? Argumentaţi.

(5) (1 pct.) Fie $A \in \mathcal{M}_{m,n}(\mathbb{R})$ o matrice de rang r. Arătaţi că există două matrici $B \in \mathcal{M}_{m,r}(\mathbb{R})$, $C \in \mathcal{M}_{r,n}(\mathbb{R})$ ambele de rang r astfel încât $A = BC$.

Se acordă 1 punct din oficiu.
Justificaţi toate răspunsurile date, arătând calculele efectuate.

[redacted]

Examen algebră liniară 113

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$M=5$

$N=4$

(1) $u_1 = \begin{pmatrix} 1 \\ 7 \\ 9 \\ 4 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ 3 \\ 7 \\ 3 \end{pmatrix}, u_3 = \begin{pmatrix} 7 \\ 13 \\ 23 \\ 10 \end{pmatrix}, u_4 = \begin{pmatrix} -1 \\ 9 \\ 11 \\ 5 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 5 \end{pmatrix}$

$V = \langle u_1, u_2, u_3, u_4 \rangle$

(a) Verific liniar independența lui u_1, u_2, u_3, u_4

$$\begin{pmatrix} 1 & 3 & 7 & -1 \\ 7 & 3 & 13 & 9 \\ 9 & 7 & 23 & 11 \\ 4 & 3 & 10 & 5 \end{pmatrix} \xrightarrow[\substack{L_3' = L_3 - 9L_1 \\ L_4' = L_4 - 4L_1}]{L_2' = L_2 - 7L_1} \begin{pmatrix} 1 & 3 & 7 & -1 \\ 0 & -18 & -36 & 16 \\ 0 & -20 & -40 & 20 \\ 0 & -9 & -18 & 9 \end{pmatrix} \xrightarrow[L_2 \leftrightarrow L_3]{L_3' = -\frac{1}{20}L_3}$$

$$\begin{pmatrix} 1 & 3 & 7 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & -18 & -36 & 16 \\ 0 & -9 & -18 & 9 \end{pmatrix} \xrightarrow[\substack{L_3' = L_3 + 18L_2 \\ L_4' = L_4 + 9L_2}]{L_1' = L_1 - 3L_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_3' = -\frac{1}{2}L_3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[L_2' = L_2 + L_3]{L_1' = L_1 - 2L_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{am 3 pivoți} \Rightarrow$$

$\Rightarrow u_1, u_2, u_4$ sunt liniar independenți $\Rightarrow u_3$ e o combinație liniară de $u_1, u_2, u_4 \Leftrightarrow u_3 \in \langle u_1, u_2, u_4 \rangle$

$\Rightarrow B = \langle u_1, u_2, u_4 \rangle$ bază pt. $V \Rightarrow \dim B = 3$

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(b) Verific dacă B poate fi completată cu e_1:

$$\begin{pmatrix} 1 & 3 & -1 & 1 \\ 7 & 3 & 9 & 0 \\ 9 & 7 & 11 & 0 \\ 4 & 3 & 5 & 0 \end{pmatrix} \xrightarrow[\substack{L_3' = L_3 - 9L_1 \\ L_4' = L_4 - 4L_1}]{L_2' = L_2 - 7L_1} \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -18 & 16 & -7 \\ 0 & -20 & 20 & -9 \\ 0 & -9 & 9 & -4 \end{pmatrix} \xrightarrow{L_2' = -\frac{1}{18}L_2}$$

$$\begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -8/9 & 7/18 \\ 0 & -20 & 20 & -9 \\ 0 & -9 & 9 & -4 \end{pmatrix} \xrightarrow[\substack{L_3' = L_3 + 20L_2 \\ L_4' = L_4 + 9L_2}]{L_1' = L_1 - 3L_2} \begin{pmatrix} 1 & 0 & 5/3 & -1/6 \\ 0 & 1 & -8/9 & 7/18 \\ 0 & 0 & 20/9 & -11/9 \\ 0 & 0 & 1 & -1/2 \end{pmatrix} \xrightarrow{L_4' = L_4 - \frac{9}{20}L_3}$$

$$\begin{pmatrix} 1 & 0 & 5/3 & -1/6 \\ 0 & 1 & -8/9 & 7/18 \\ 0 & 0 & 20/9 & -11/9 \\ 0 & 0 & 0 & 1/20 \end{pmatrix}$$

$\Rightarrow$ am pivot pe fiecare coloană

$\Rightarrow u_1, u_2, u_4, e_1$ sunt liniar independenți

$\Rightarrow B = \{u_1, u_2, u_4, e_1\}$ bază pt. $\mathbb{R}^4$

(d) $w \in V \Leftrightarrow \exists\, a,b,c,d \in \mathbb{R}$ a.î. $w = au_1 + bu_2 + cu_3 + du_4$

$$\left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 7 & 3 & 13 & 9 & 4 \\ 9 & 7 & 23 & 11 & 3 \\ 4 & 3 & 10 & 5 & 5 \end{array}\right) \xrightarrow[\substack{L_3' = L_3 - 9L_1 \\ L_4' = L_4 - 4L_1}]{L_2' = L_2 - 7L_1} \left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 0 & -18 & -36 & 16 & -31 \\ 0 & -20 & -40 & 20 & -42 \\ 0 & -9 & -18 & 9 & -15 \end{array}\right) \xrightarrow[L_3 \leftrightarrow L_2]{L_3' = -\frac{1}{20}L_3}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 0 & 1 & 2 & -1 & 21/10 \\ 0 & -18 & -36 & 16 & -31 \\ 0 & -9 & -18 & 9 & -15 \end{array}\right) \xrightarrow[\substack{L_3' = L_3 + 18L_2 \\ L_4' = L_4 + 9L_2}]{L_1' = L_1 - 3L_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -6/5 \\ 0 & 1 & 2 & -1 & 21/10 \\ 0 & 0 & 0 & -2 & 34/5 \\ 0 & 0 & 0 & 0 & -15 + \frac{9 \cdot 21}{10} \end{array}\right)$$

ultima ecuație devine $0 \cdot a + 0 \cdot b + 0 \cdot c + 0 \cdot d = -15 + \frac{9 \cdot 21}{10}$ Fals

$\Rightarrow \nexists\, a,b,c,d \in \mathbb{R}$ a.î. $w = au_1 + bu_2 + cu_3 + du_4 \Leftrightarrow$

$\Leftrightarrow w \notin V$

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(2) $A_n = \begin{pmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & & & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{pmatrix}$

$\det(A_n) = d_n$

(a) $d_2 = \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = -12$

$d_3 = \begin{vmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{vmatrix} = 8 + 64 - 32 - 32 = 8$

$d_4 = \begin{vmatrix} 2 & 4 & 0 & 0 \\ 4 & 2 & 4 & 0 \\ 4 & 4 & 2 & 4 \\ 4 & 4 & 4 & 2 \end{vmatrix} = 2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{vmatrix} + 4 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{vmatrix}$

$= 2 \cdot d_3 - 4 \cdot (16 + 64 - 64 - 32)$

$= 16 + 4 \cdot 16$

$= 5 \cdot 16 = 80$

(b) $\left(\begin{array}{ccccc|ccccc} 2 & 4 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 4 & 4 & 4 & \dots & 2 & 0 & 0 & 0 & \dots & 1 \end{array}\right) \xrightarrow{L'_1 = \frac{1}{2} L_1} \left(\begin{array}{ccccc|ccccc} 1 & 2 & 0 & \dots & 0 & 1/2 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & & & & & & \\ 4 & 4 & 4 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{array}\right)$

$\cancel{L'_2 = L_2 - 4L_1}$

$\cancel{L'_n = L_n - 4L_1}$

$\xrightarrow{\substack{L'_2 = L_2 - 4L_1 \\ \vdots \\ L'_n = L_n - 4L_1}}$

$\left(\begin{array}{ccccc|ccccc} 1 & 2 & 0 & \dots & 0 & 1/2 & 0 & 0 & \dots & 0 \\ 0 & -6 & 4 & \dots & 0 & -2 & 1 & 0 & \dots & 0 \\ 0 & -4 & -14 & \dots & 0 & -2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & & & & & & \\ 0 & -4 & 4 & \dots & 4 & -2 & 0 & 0 & \dots & 2 \end{array}\right)$

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(c) $d_n = \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{\substack{\cancel{d_n = 0} \\ n \text{ coloane}}} = 2 \cdot (-1)^{1+1} \cdot \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} +$

$+ 4 \cdot (-1)^{1+2} \underbrace{\begin{vmatrix} 4 & 4 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & 0 & \dots & 0 \\ 4 & 4 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} \underset{C_1' = C_1 - C_2}{=}$

$= 2 \cdot d_{n-1} \cancel{+} - 4 \cdot \underbrace{\begin{vmatrix} 0 & 4 & 0 & \dots & 0 \\ 2 & 2 & 4 & \dots & 0 \\ 0 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} = 2d_{n-1} - 4 \cdot 4 \cdot (-1)^{1+2} \cdot$

$\underbrace{\begin{vmatrix} 2 & 4 & 0 & 0 & \dots & 0 \\ 0 & 2 & 4 & 0 & \dots & 0 \\ 0 & 4 & 2 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-2 \text{ coloane}} = 2d_{n-1} + 16 \cdot 2 \cdot (-1)^{1+1} \cdot \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-3 \text{ coloane}} =$

$= 2d_{n-1} + 32 d_{n-3} \iff d_n = 2d_{n-1} + 4^2(4-2) d_{n-3}, \quad \forall n \geq 5$

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(2)(b) $A_3 = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{pmatrix}$

$\det A_3 = d_3 = 8 \neq 0 \Rightarrow A_3$ - inversabilă

$^tA_3 = \begin{pmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \Rightarrow A_3^* = \begin{pmatrix} -12 & -8 & 16 \\ 8 & 4 & -8 \\ 8 & 8 & -12 \end{pmatrix}$

$A_3^{-1} = \frac{1}{\det A_3} \cdot A_3^* = \begin{pmatrix} -3/2 & -1 & 2 \\ 1 & 1/2 & -1 \\ 1 & 1 & -3/2 \end{pmatrix}$

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(3) $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x - 2y - 4z \\ y \\ 12x - 4y - 7z \end{pmatrix} = \overset{A}{\overset{\|}{\begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(a) $\operatorname{Ker} T = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0_{\mathbb{R}^3} \right\}$

$$\Leftrightarrow \begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\cancel{7 \; -2 \; -4} \quad \cancel{12}$$

$$\begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix} \xrightarrow{L_1' = \frac{1}{7} L_1} \begin{pmatrix} 1 & -2/7 & -4/7 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix}$$

$$\xrightarrow{L_3' = L_3 - 12 L_1} \begin{pmatrix} 1 & -2/7 & -4/7 \\ 0 & 1 & 0 \\ 0 & -4/7 & -1/7 \end{pmatrix} \xrightarrow[L_3' = L_3 + \frac{4}{7} L_2]{L_1' = L_1 + \frac{2}{7} L_2}$$

$$\begin{pmatrix} 1 & 0 & -4/7 \\ 0 & 1 & 0 \\ 0 & 0 & -1/7 \end{pmatrix} \xrightarrow{L_3' = -7 L_3} \begin{pmatrix} 1 & 0 & -4/7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{L_1' = L_1 + \frac{4}{7} L_3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow a = b = c = 0 \Rightarrow \operatorname{Ker} T = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \dim(\operatorname{Ker}(T)) = 0$$

$\Rightarrow T$-injectivă

din Th. rang-defect $\Rightarrow \dim \mathbb{R}^3 = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T))$

$\Rightarrow \dim(\operatorname{Im}(T)) = 3 \Rightarrow T$ surjectivă $\Rightarrow \operatorname{Im} T = \mathbb{R}^3$

$\operatorname{Im} T = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \exists \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ a.î. } T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow x \cdot C_1(A) + y \cdot C_2(A) + z \cdot C_3(A) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \langle C_1(A), C_2(A), C_3(A) \rangle$$

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dim Ker T am arătat că $C_1(A), C_2(A), C_3(A)$ sunt liniar independenți $\Rightarrow < C_1(A), C_2(A), C_3(A) >$ bază pt. Im T

$\Rightarrow \dim(\operatorname{Im} T) = 3$; $\operatorname{Ker} T = \{0_{\mathbb{R}^3}\}$; $\operatorname{Im} T = \mathbb{R}^3$

$\operatorname{Ker} T \cap \operatorname{Im} T = \{x \in \operatorname{Ker} T$ și $x \in \operatorname{Im} T\}$

$$\left.\begin{aligned} x &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ x &= a \cdot C_1(A) + b \cdot C_2(A) + c \cdot C_3(A) \end{aligned}\right| \Rightarrow a = b = c = 0$$

$\Rightarrow \operatorname{Ker} T \cap \operatorname{Im} T = \operatorname{Ker} T = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

~~(b) $w = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$~~

(c) $P_A(x) = \det(x \cdot I_3 - A)$

$$\det\left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix}\right) = \det\begin{pmatrix} x-7 & 2 & 4 \\ 0 & x-1 & 0 \\ -12 & 4 & x+7 \end{pmatrix} =$$

$$= \begin{vmatrix} x-7 & 2 & 4 \\ 0 & x-1 & 0 \\ -12 & 4 & x+7 \end{vmatrix} = (x-1)(x^2-49) + 48(x-1) =$$

$$= (x-1)(x^2-49+48)$$

$$= (x-1)(x^2-1)$$

$$= (x-1)^2(x+1)$$

$\Rightarrow$ valorile proprii sunt $\lambda_1 = 1$, $\lambda_2 = -1$

(b) $\lambda_1 = 1 \Rightarrow (A - 1 \cdot I_3) v_1 = 0_{\mathbb{R}^3}$

$$\begin{pmatrix} 6 & -2 & -4 \\ 0 & 0 & 0 \\ 12 & -4 & -8 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & -4 \\ 12 & -4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_1' = \frac{1}{6} L_1} \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} \\ 12 & -4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_2' = L_2 - 12 L_1}$$

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$$\begin{pmatrix} 1 & -1/3 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 - \frac{1}{3}y_1 - \frac{2}{3}z_1 = 0$$

x_1 – nec. principală

y_1, z_1 – nec. secundare; $y_1 = s$, $z_1 = t$, $s, t \in \mathbb{R}$

$$\Rightarrow x_1 = \frac{1}{3}y_1 + \frac{2}{3}z_1$$

$$x_1 = \frac{1}{3}s + \frac{2}{3}t$$

$$V_1 = \left\{ s\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

$$\lambda_2 = -1 \Rightarrow (A + I_3) v_2 = 0_{\mathbb{R}^3}; \ v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -2 & -4 \\ 0 & 2 & 0 \\ 12 & -4 & -6 \end{pmatrix} \xrightarrow{L_1' = \frac{1}{8}L_1} \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 2 & 0 \\ 12 & -4 & -6 \end{pmatrix} \xrightarrow{L_3' = L_3 - 12L_1}$$

$$\begin{pmatrix} 1 & -1/4 & -1/2 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{L_2' = \frac{1}{2}L_2} \begin{pmatrix} 1 & -1/4 & -1/2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow[L_3' = L_3 + L_2]{L_1' = L_1 + \frac{1}{4}L_2}$$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow y_2 = 0 \Rightarrow x_2 - \frac{1}{2}z_2 = 0$$

$z_2 = \alpha, \alpha \in \mathbb{R} \Rightarrow x_2 = \frac{1}{2}\alpha$

$$V_2 = \left\{ \alpha\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

$$w = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \notin V_2$$

pt. $s = 4$ și $t = 5 \Rightarrow v_1 = \begin{pmatrix} 14/3 \\ 4 \\ 5 \end{pmatrix} \Rightarrow w \notin V_1 \Rightarrow$

w nu este vector propriu pt. T

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(5) v_1, v_2, v_3 - liniar independenți <=>
<=> ~~a~~ $\exists a,b,c \in \mathbb{R}$ a.î. $av_1+bv_2+cv_3=0_V$

$5v_1+v_2, 5v_2+v_3, 5v_3+v_1$ - lin. indep. <=>
<=> $\exists x,y,z \in \mathbb{R}$ a.î.

$5xv_1+xv_2+5y\,v_2+y\,v_3+5zv_3+zv_1=0_V$

$v_1(5x+z)+v_2(x+5y)+v_3(y+5z)=0_V$

$$\begin{cases} 5x+0+z=0 \\ x+5y+0=0 \\ 0+y+5z=0 \end{cases}$$

$$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{L_1 <-> L_2} \begin{pmatrix} 1 & 5 & 0 \\ 5 & 0 & 1 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{L_2'=L_2-5L_1} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -25 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\xrightarrow{L_2 <-> L_3} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & -25 & 1 \end{pmatrix} \xrightarrow[L_3'=L_3+25L_2]{L_1'=L_1-5L_2} \begin{pmatrix} 1 & 0 & -25 \\ 0 & 1 & 5 \\ 0 & 0 & 126 \end{pmatrix} \xrightarrow{L_3'=\frac{1}{126}L_3}$$

$$\begin{pmatrix} 1 & 0 & -25 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[L_2'=L_2-5L_3]{L_1'=L_1+25L_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} => x=y=z=0$$

=> $\exists x,y,z \in \mathbb{R}$ a.î. $v_1(5x+z)+v_2(x+5y)+v_3(y+5z)=0_V$