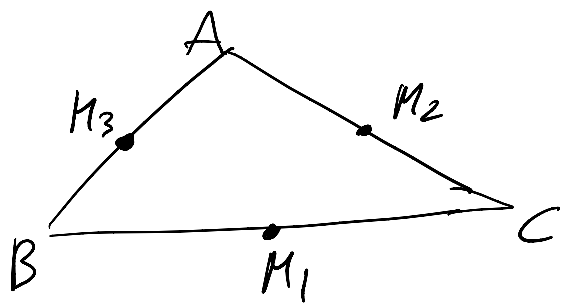
Seminarul 4 111

Tuesday, October 26, 2021 1:44 PM

1. $M_1=(-2,1)$, $M_2=(2,3)$ și $M_3=(-4,-1)$ mijloacele laturilor unui triunghi ΔABC

$\rightsquigarrow$ A, B, C?



$$\underbrace{x_{M_1}=\frac{x_B+x_C}{2},\quad y_{M_1}=\frac{y_B+y_C}{2}}$$

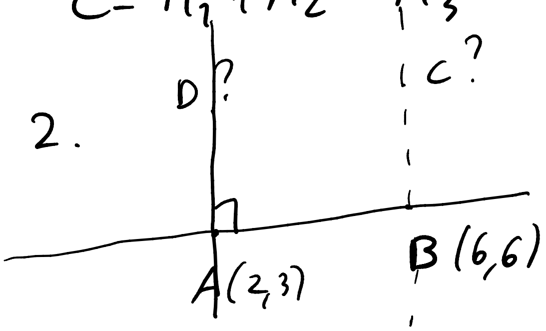
$$\begin{cases} M_1=\frac{1}{2}B+\frac{1}{2}C \\ M_2=\frac{1}{2}A+\frac{1}{2}C \\ M_3=\frac{1}{2}A+\frac{1}{2}B \end{cases}$$

$$-M_1+M_2+M_3=A \Rightarrow A=-(-2,1)+(2,3)+(-4,-1)$$
$$=(0,2)$$

$$B=M_1-M_2+M_3=(-2,1)-(2,3)+(-4,-1)=(-8,-3).$$

$$C=M_1+M_2-M_3=(-2,1)+(2,3)-(-4,-1)=(4,5).$$

2.



Caut C, D a.î. $ABCD$ pătrat.

punctul $A \in AB$

$$AB: \frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A} \iff AB: \frac{x-2}{4}=\frac{y-3}{3}$$

$(4,3)$ un vector director

Aleg $d \perp AB$, $d \ni A \Rightarrow d$ are $(3,-4)$ ca vector director

$$d: \frac{x-2}{3}=\frac{y-3}{-4} \iff 4x+3y-17=0$$

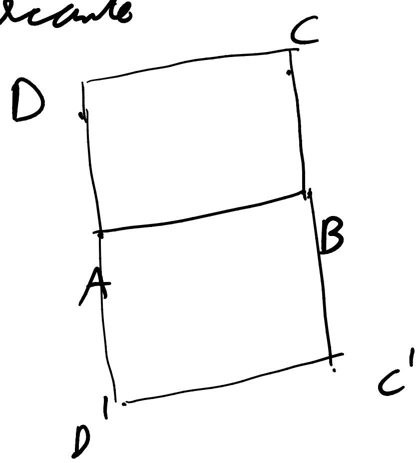
$$d: \frac{x-2}{3} = \frac{y-3}{-4} \iff 4x+3y-17=0$$

Caut $D \in d$ a.î. $\|AD\| = \|AB\| = \sqrt{(x_B-x_A)^2+(y_B-y_A)^2} =$

$$= \sqrt{4^2+3^2} = 5.$$

$\downarrow$

2 variante



Var 1 $\begin{cases} \|AD\| = 5 \\ D \in d \end{cases} \iff \begin{cases} (x_D-2)^2+(y_D-3)^2 = 25 \\ 4x_D+3y_D-17=0 \end{cases} \Rightarrow y_D = \frac{17-4x_D}{3}$

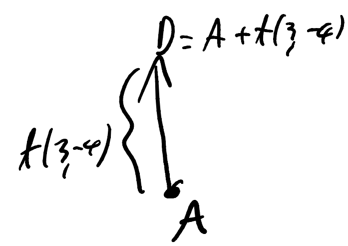
$$(x_D-2)^2 + \left(\frac{17-4x_D}{3} - 3\right)^2 = 25$$

ec de grad 2 $\to x_D \to y_D \to C$

$\to x_D' \to y_D' \to C'$

Var 2 $D \in d \Rightarrow D = A + t_D(3,-4)$ pt un $t_D \in \mathbb{R}$

și $\|\vec{AD}\| = 5$



Dar $\vec{AD} = t_D(3,-4)$!

(în coordonate $\vec{AD} = (x_D - x_A, y_D - y_A) = (x_A + 3t_D - x_A, y_A - 4t_D - y_A)$

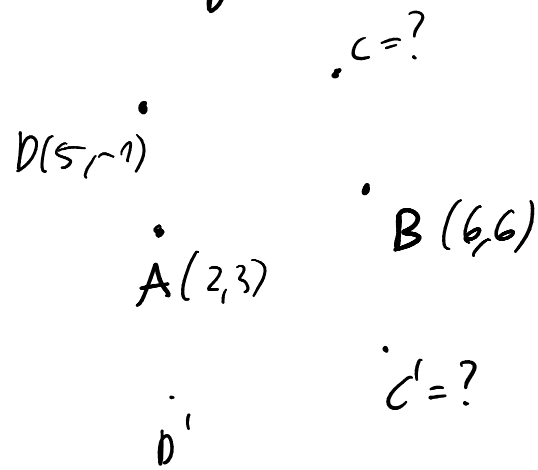
$= t_D(3,-4)$

$$\|\vec{AD}\| = \|t_D(3,-4)\| = |t_D|\,\|(3,-4)\| = 5|t_D| = 5$$

$$\|\vec{AD}\| = \|t_D(3,-4)\| = |t_D|\,\|(3,-4)\| = 5|t_D| = 5$$

$$\Rightarrow t_D = 1 \Rightarrow D = A + 1\cdot(3,-4) = (2,3)+1(3,-4) = (5,-1)$$

$$t_{D'} = -1 \Rightarrow D' = A + (-1)(3,-4) = (-1,7).$$



Var 1 $\frac{1}{2}A + \frac{1}{2}C = \frac{1}{2}D + \frac{1}{2}B \Rightarrow C$

Var 2 $\vec{AC} = \vec{AB} + \vec{AD}$

$$\Rightarrow \vec{AC} = (4,3) + (3,-4) = (7,-1)$$

$$\Rightarrow C = A + \vec{AC} = (2,3)+(7,-1) = (9,2).$$

La fel pt C'...

Exc 3 $\mathcal{F} = \{ d_m : \underbrace{(m^2+2m+4)}_{\neq 0}x - \underbrace{(2m^2+3m+5)}y - (m+3) = 0 \mid m \in \mathbb{R} \}$

mulțime de drepte.

Este $\mathcal{F}$ un fascicul de drepte?

def $\Rightarrow$ mulțimea tuturor dreptelor care trec printr-un punct comun.

Q1 Trec toate d_m prin același punct? Dacă da, care?

Var 1 Aleg arbitrar două drepte:

$$d_0:\ 4x-5y-3=0 \quad \Rightarrow d_0\cap d_1=\{(2,1)\}$$
$$d_1:\ 7x-10y-4=0$$

Verific: $(2,1)\in d_m,\ \forall m$?

$$(m^2+2m+4)\cdot 2-(2m^2+3m+5)\cdot 1-(m+3)=0,\quad \forall m\in\mathbb{R}$$

Var 2 d_m trec prin același pct (x_0,y_0) $\Leftrightarrow$

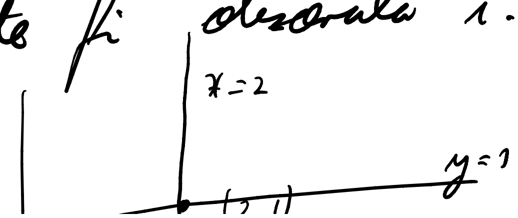
$$(m^2+2m+4)x_0-(2m^2+3m+5)y_0-(m+3)=0,\quad \forall m\in\mathbb{R}$$

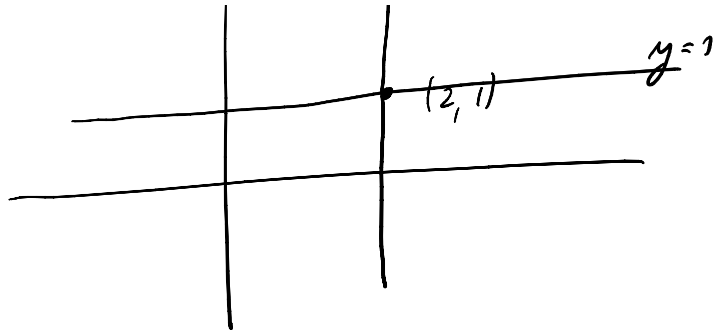
$$\Leftrightarrow (x_0-2y_0)m^2+(2x_0-3y_0-1)m+(4x_0-5y_0-3)=0,\quad \forall m\in\mathbb{R}$$

$$\Leftrightarrow \begin{cases} x_0-2y_0=0\\ 2x_0-3y_0-1=0\\ 4x_0-5y_0-3=0\end{cases} \Longleftrightarrow (x_0,y_0)=(2,1).$$

Dar, $d_m:\ (\underbrace{m^2+2m+4}_{>0,\ \forall m\in\mathbb{R}\ (\Delta<0)})x-(2m^2+3m+5)y-(m+3)=0$ nu are vector normal $(0,1)$! $\Leftrightarrow$ nu are vector director $(1,0)$.

$\Rightarrow d_m$ nu poate fi orizontală i.e. d_m nu poate fi $y=1$!





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4. $M(3,3)$ $\quad d_1: x+2y-4=0$

$d_2: 3x+y-2=0$ $\qquad d_2 \perp d_3$

$d_3: x-3y-4=0$

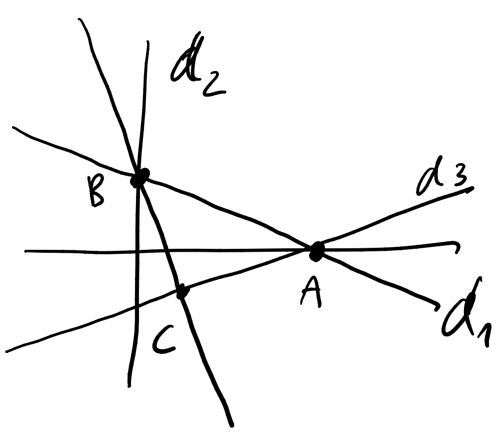
$\{A\}=d_1\cap d_3,\quad \{B\}=d_1\cap d_2,\quad \{C\}=d_2\cap d_3$

0) $A, B, C = ?$

$A: \begin{cases} x+2y-4=0 \\ x-3y-4=0 \end{cases} \Rightarrow A=(4,0)$

$B: \begin{cases} x+2y-4=0 \\ 3x+y-2=0 \end{cases} \Rightarrow B=(0,2)$

$C: \begin{cases} x-3y-4=0 \\ 3x+y-2=0 \end{cases} \Rightarrow C=(1,-1)$



$d_1 = AB,\ d_2 = BC,\ d_3 = AC$

a) $\mathcal{A}_{\Delta ABC}$ = cu det (vezi curs joi)

$A(4,0), B(0,2), C(1,-1)$

$\|AB\| = 2\sqrt{5}$

$\|AC\| = \sqrt{10}$

T. Pitagora

ΔABC dreptunghic în C!

$\|AB\| = 2\sqrt{5}$

$\|AC\| = \sqrt{10}$ $\Longrightarrow$ (Reciproca Pitagora) ΔABC dreptunghic în C!

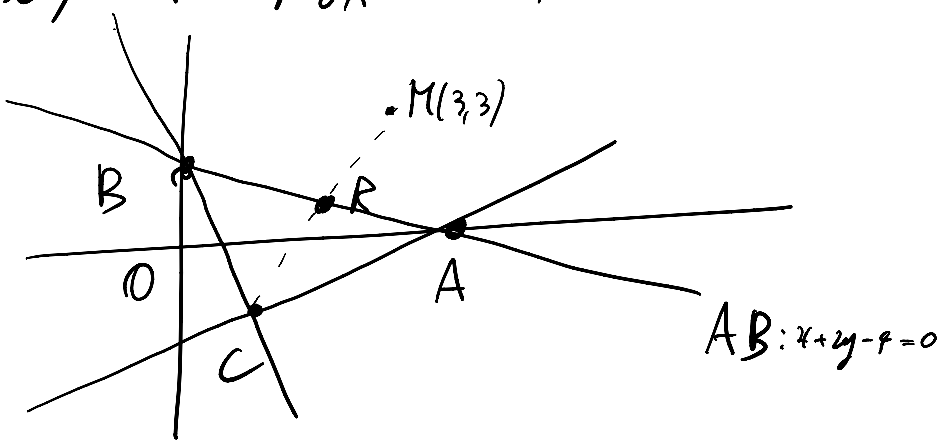
$\|BC\| = \sqrt{10}$

(de altfel, puteam observa că $d_2 \perp d_3$)

$$\Rightarrow \mathcal{A}_{\Delta ABC} = \frac{\|AC\| \cdot \|BC\|}{2} = \frac{10}{2} = 5.$$

Var 2 $\mathcal{A}_{\Delta ABC} = \dfrac{\|BC\| \cdot \operatorname{dist}(A, BC)}{2}$

b) $P = pr_{OA} M$, $Q = pr_{OB} M$, $R = pr_{AB} M$



$pr_{OA} M = (3,0)$

(OA e axa Ox)

$pr_{OB} M = (0,3)$

(OB e axa Oy)

$pr_{AB} M = ?$ $R = M + t_R(1,2) \in AB$

$\Rightarrow R(3 + t_R, 3 + 2t_R) \in AB \Leftrightarrow (3 + t_R) + 2(3 + 2t_R) - 4 = 0$

$\Leftrightarrow 5t_R + 5 = 0 \Rightarrow t_R = -1 \Rightarrow R = (3-1, 3-2) = (2,1).$

Dem că P, Q, R coliniare.
$(3,0), (0,3), (2,1)$

$PQ: \dfrac{x-3}{0-3} = \dfrac{y}{3-0} \Leftrightarrow 3x + 3y - 9 = 0 \Leftrightarrow x + y - 3 = 0$

Ușor $R(2,1) \in PQ$.

PQ: $\frac{x-3}{0-3} = \frac{y-0}{3-0} \iff$ ...

Clar $R(2,1) \in PQ$.

Altfel: $\vec{PQ} = (-3, 3)$

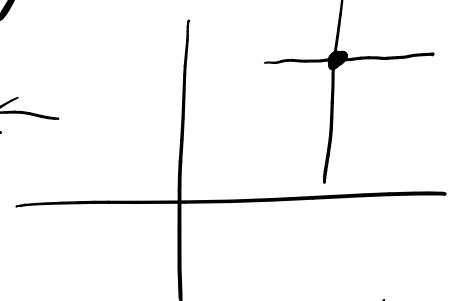
$\vec{PR} = (-1, 1) = \frac{1}{3}\vec{PQ}$

c) Ecuația fasciculului determinat de AB și PQ:

Var1 Curs: $\mathcal{F} = \{ \alpha(x+2y-4) + \beta(x+y-3) = 0 \mid \alpha, \beta \in \mathbb{R} \} \leftarrow$

Var2 $AB \cap PQ = \{R\} \Rightarrow \mathcal{F}$ - fasciculul dreptelor care trec prin R.

Dar fasciculul dreptelor care trec prin $D(a,b)$ este



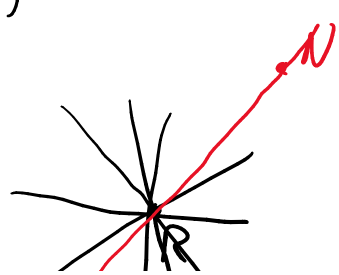
$\alpha(x+y-a-b) + \beta(x-y-a+b) \mid \alpha, \beta \in \mathbb{R} \leftarrow$

sau

$\alpha(\underset{\substack{\uparrow \\ \text{verticală}}}{x-a}) + \beta(\underset{\substack{\uparrow \\ \text{orizontală}}}{y-b}) \mid \alpha, \beta \in \mathbb{R}$

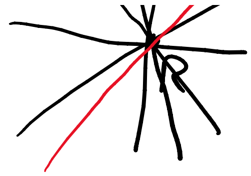
$\Rightarrow \mathcal{F} = \{ \alpha(x-2) + \beta(y-1) \mid \alpha, \beta \in \mathbb{R} \} \leftarrow$

d) Care este dreapta din $\mathcal{F}$ care trece prin $N(0,5)$?



Var1 RN: $\frac{x-2}{0-2} = \frac{y-1}{5-1}$

$0-2 \qquad 5-1$



$\iff 4(x-2) = -2(y-1)$

$4x+2y-10=0$

$2x+y-5=0$

Var 2 $\mathcal{F} = \{\alpha(x-2)+\beta(y-1)=0 \mid \alpha,\beta\in\mathbb{R}\}$

$N \in \nearrow \iff \alpha(0-2)+\beta(5-1)=0 \iff -2\alpha+4\beta=0$

$\iff \alpha=2\beta.$

$\Rightarrow d: \ 2\beta(x-2)+\cancel{\beta}(y-1)=0, \quad \beta\in\mathbb{R}$

$2(x-2)+y-1=0$

$2x+y-5=0.$

Ex 5. $d: 2x-5y-1=0$

$d_\alpha: \dfrac{x+1}{2} = \dfrac{y-2}{\alpha}, \quad \alpha\in\mathbb{R}$

a) Este $\{d_\alpha \mid \alpha\in\mathbb{R}\}$ un fascicul de drepte?

Obs $(-1,2)\in d_\alpha, \ \forall \alpha\in\mathbb{R}.$

d_α are vector director $(2,\alpha)$ ← neparalel cu $(0,1)$!

$\Rightarrow d_\alpha$ nu poate fi verticală i.e. $d_\alpha \neq x=-1.$

Dar dat (u,v), $u\neq 0 \Rightarrow \exists\, d_\alpha$ cu direcția (u,v):

$(u,v) = \dfrac{u}{2}\left(2, \dfrac{2v}{u}\right)$

$u\neq 0$

$u \neq 0$

b) Determinați $\alpha \in \mathbb{R}$, dacă există, a.î. $d_\alpha \parallel d$, $d_\alpha \perp d$.

Există

c)

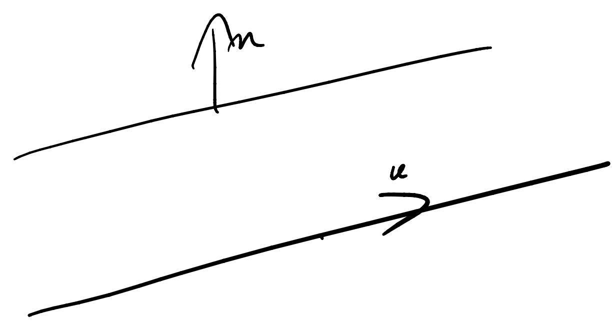
b) Caut α a.î. $d \parallel d_\alpha$

$d: 2x-5y-1=0$

$d_\alpha: \frac{x+1}{2} = \frac{y-2}{\alpha}$

normal la d — director pt d_α

$d \parallel d_\alpha \iff (2,-5)$ și $(2,\alpha)$ sunt ortogonali



$\iff \langle (2,-5),(2,\alpha) \rangle = 0$

$\| $

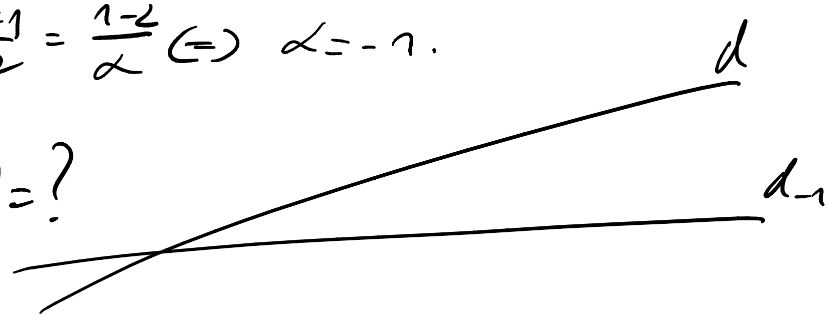
$4-5\alpha = 0 \iff \alpha = \frac{4}{5}$.

$d \perp d_\alpha \iff (2,-5)$ și $(2,\alpha)$ sunt proporționali $\iff \alpha = -5$.

d) Det $\alpha \in \mathbb{R}$ a.î. $(1,1) \in d_\alpha$. Calculați $\cos\sphericalangle(d, d_\alpha)$, α determinat înainte.

$(1,1) \in d_\alpha \iff \frac{1+1}{2} = \frac{1-2}{\alpha} \iff \alpha = -1$.

$\cos\sphericalangle(d, d_{-1}) = ?$



$\cos\sphericalangle(d, d_{-1}) = \cos(\sphericalangle$ dintre 2 vectori directori pt d și $d_{-1})$

$= \cos\sphericalangle((5,2),(2,-1)) = \frac{\langle (5,2),(2,-1) \rangle}{\|(5,2)\| \cdot \|(2,-1)\|} = \frac{8}{\sqrt{29}\cdot\sqrt{5}}$.

$$\cos \sphericalangle (v,w) = \frac{\langle v,w\rangle}{\|v\|\cdot\|w\|}$$

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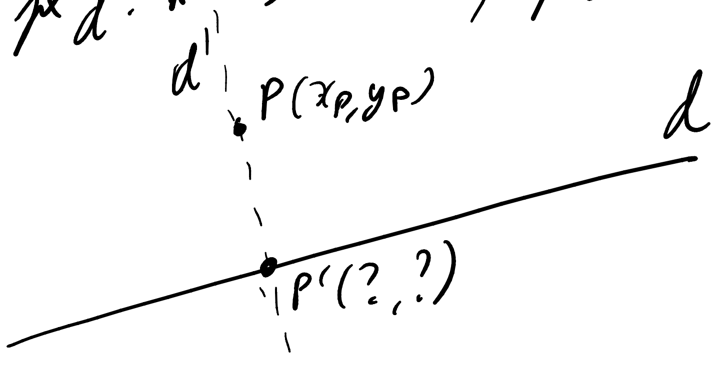
7. $d: x+2y+1=0$. Cartezian

$$pr_d:\mathbb{R}^2\to\mathbb{R}^2,\quad pr_d(x,y)=(?,?)$$



$$P' = P + t_0(1,2) = (x_p+t_0,\ y_p+2t_0)\in d$$

$$\Leftrightarrow 1(x_p+1t_0)+2(y_p+2t_0)+1=0 \Leftrightarrow t_0 = -\frac{x_p+2y_p+1}{5}$$

$$\Rightarrow P' = \left(x_p - \frac{x_p+2y_p+1}{5},\ y_p - 2\,\frac{x_p+2y_p+1}{5}\right)$$

$$= \left(\frac{4x_p-2y_p-1}{5},\ \frac{-2x_p+y_p-2}{5}\right)$$

$$\Rightarrow \boxed{pr_d(x,y) = \left(\frac{4x-2y-1}{5},\ \frac{-2x+y-2}{5}\right).}$$