Data: 21 iunie 2023
Timp de lucru: 2h 30m
Punctaj total: 90p + 10p oficiu **Nume:** ______________________________

Examen **Analiză complexă**

Subiecte:

1. (a) (5 p) Scrieti seria Taylor în 0 pentru funcţia $f(z) = z^2 \cos z + \sin z$.
   (b) (5 p) Determinaţi dacă funcţia $f(x + iy) = 2x^2 - 4y^2 + 4ixy$ este olomorfă pe $\mathbb{C}$.
   (c) (5 p) Daţi exemplu de funcţie olomorfă cu pol de ordin 3 în punctul $z_0 = 1$, pentru care $\operatorname{res}(f, 1) = 2$.
   (d) (5 p) Daţi exemplu de doua funcţii olomorfe $f, g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ cu pol în 0, cu $\operatorname{res}(f, 0) = \operatorname{res}(g, 0) = 0$, astfel încât $\operatorname{res}(fg, 0) = 1$.
   (e) (5 p) Calculaţi
   $$\int_{|z-1|=1} (\overline{z} - 1) dz.$$

2. (a) (10 p) Calculaţi numărul soluţiilor ecuaţiei $z^5 + iz^3 - 4z + i = 0$ în $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.
   (b) (10 p) Demonstraţi că $\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2m+1} dz = 2\pi i\, \mathrm{C}_{2m+1}^m$, pentru orice $m \in \mathbb{Z}$, $m \geq 1$.

3. (a) (25 p) Calculaţi $\displaystyle\int_0^\infty \frac{1}{x^4 + x^2 + 1} dx$.

4. (10 p) Reprezentati grafic domeniul
   $$\Omega = \{z = x + iy \mid -1 < x - y < 1\}$$
   şi determinaţi o aplicaţie biolomorfă între Ω şi discul unitate.

5. (10 p) Considerăm polinomul $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, unde $n \in \mathbb{N}$, $a_k \in \mathbb{C}$ pentru orice $0 \leq k \leq n - 1$. Considerăm funcţia $Q(z) = z^n P(\frac{1}{z})$. Demonstraţi că:
   1. $\max\limits_{|z|=1} |Q(z)| = \max\limits_{|z|=1} |P(z)|$.
   2. $\max\limits_{|z|=1} |P(z)| \geq 1$.

1) ① $f(z) = z^2 \cos z + \sin(z)$

2 Scrie dezv. Taylor pt fiecare f in 0.

$z^2 = z^2$

$\cos z = \sum_{0}^{+\infty} \frac{(-1)^n \cdot z^{2n}}{(2n)!}$

$\Rightarrow z^2 \cos z = \sum_{0}^{+\infty} \frac{(-1)^n z^{2n+2}}{(2n)!}$

$\sin z = \sum_{0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

$f(z) = \sum_{0}^{+\infty} \overline{(-1)^n \frac{z^{2n+1}}{(2n+1)!} + \frac{(-1)^n z^{2n+2}}{2n!}}$ " + "

② f olom dacă $\frac{\partial}{\partial \bar{z}} f(z) = 0$

$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$

$\Rightarrow 2 \cdot \frac{\partial}{\partial \bar{z}} f(z) = \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)(2x^2 - 4y^2 + 4ixy) =$

$= (4x - 0 + 4iy) - 0 - \frac{1}{i} \cdot (-8y) - \frac{1}{i} \cdot 4ix) =$

$= 4x + 4iy + \frac{1}{i} 8y - 4x = 4iy - 8iy = -4iy \neq 0$

$\Rightarrow$ f nu e olom.

③ · Pol ord 3 in $1 \Rightarrow (z-1)^{-3}$

$res(f,1) = 2 \Rightarrow$ coef. lui $(z-1)^{-1}$ din Seria Laurent este 2

$\Rightarrow 2(z-1)^{-1}$

$$f(z) = (z-1)^{-3} + 2(z-1)^{-1}$$

④ $f(z) = \frac{1}{z^2} + \frac{1}{2}z = g(z)$ · cu pol in 0 și $\underbrace{res(f,0)}_{res(g,0)=0}$

$$f \cdot g = \left(\frac{1}{z^2} + \frac{1}{2}z\right)^2 = \frac{1}{z^4} + 2 \cdot \frac{1}{2}z \cdot \frac{1}{z^2} + \frac{1}{4}z^2 =$$

$$= \frac{1}{z^4} + \underbrace{\frac{1}{z} \cdot 1}_{res(f \cdot g, 0) = 1} + \frac{1}{4}z^2$$

⑤ $\{|z-1| = 1\} \Rightarrow \gamma : [0; 2\pi] \to \mathbb{C}$,

$\gamma(\theta) = 1 + e^{i\theta}$

$$\int_\gamma (\bar{z} - 1)\, dz = \int_0^{2\pi} \left(\overline{(1+e^{i\theta})} - 1\right) i e^{i\theta}\, d\theta =$$

$$= \int_0^{2\pi} i\, e^{-i\theta} \cdot e^{i\theta}\, d\theta = 2\pi i$$

$$= \int_{|z|=1} C_{2m+1}^{0} \frac{1}{z^{2m+1}} + C_{2m+1}^{1} \frac{1}{z^{2m-1}} + \dots + C_{2m+1}^{m-1} \frac{1}{z^{3}} +$$

$$+ C_{2m+1}^{m} \frac{1}{z} \, dz = I = \int_{|z|=1} f(z)\, dz$$

Din Th. reziduurilor $\Rightarrow I = 2\pi i \cdot \operatorname{res}(f, 0) =$

$$= 2\pi i \cdot C_{2m+1}^{m}$$

2) ① $\{z \,/\, 1 < |z| < 2\} \Rightarrow$

$f(z) = z^5$, $P(z) = z^5 + iz^3 - 4z + i$

Notez $Z_A(P)$ nr. de zerouri din mulț. A.

$\Rightarrow Z_{\{z/1\le|z|<2\}}^{(P(z))} = Z_{\{z/|z|<2\}}^{(P(z))} - Z_{\{z/|z|=1\}}^{(P(z))}$

Observ că $|z^5| = 2^5$ în $\{|z|<2\}$

$|iz^3 - 4z + i| \le |z^3| + 4|z| + 1 = 8+8+1=17 < 2^5$

Th. Rouché $\Longrightarrow$ $Z_{\{|z|<2\}}^{P(z)} = 5$ (z^5 are 5 răd. în disc.)

fct. olom.

Discul e simplu conex.

Obs. că $|-4z| = 4$ în $\{|z| \le 1\}$

$|z^5 + iz^3 + i| \le |z^5| + |iz^3| + |i| = 3 < 4$

Analog, tot din Rouché, $Z_{\{|z|<1\}}^{P(z)} = 1$

$\Rightarrow Z_{\{\dots\}}(P(z)) = 5-1=4$

②. $\int_{|z|=1} \left(z+\frac{1}{z}\right)^{2m+1} dz = ?$

$$P(z) = \left(z+\frac{1}{z}\right)^{2m+1} = \sum_{k=0}^{2m+1} C_{2m+1}^{k} z^{k} \cdot \left(\frac{1}{z}\right)^{2m+1-k} =$$

$$= \sum_{k=0}^{2m+1} C_{2m+1}^{k} z^{2k-(2m+1)}$$

Dar observ că $\underbrace{\sum_{k=m+1}^{2m+1} C_{2m+1}^{k} z^{2k-(2m+1)}}$ $= C_{\cdot}^{\cdot} z + C_{\cdot}^{\cdot} z^{3} \dots$ olomorfă,

deci integrala pe $|z|=1$ din este 0 pt că [illegible] $\{z \,/\, |z| \leq 1\}$ simplu conex și Th. Cauchy.

Deci $\int_{|z|=1} P(z)\,dz = \int_{|z|=1} \sum_{k=0}^{m} C_{2m+1}^{k} z^{2k-(2m+1)} dz =$

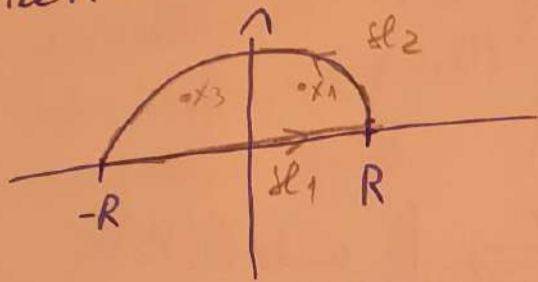
$$\cancel{= \sum_{k=0}^{m} C_{2m+1}^{k} \int z^{2k-(2m+1)}} dz = \cancel{\sum_{k=0}^{m-1} C_{2m+1}^{k} \int z^{2k-(2m+1)}} dz +$$

$$\cancel{+ C_{2m+1}^{k}}$$

3). $i = \int_0^{+\infty} \frac{1}{x^4+x^2+1}\,dx$, x^4+x^2+1 fct pară $\Rightarrow i = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{x^4+x^2+1}\,dx$

$f(x) = \frac{1}{x^4+x^2+1}$

Fie $R>0$. Consider conturul:



$\gamma_1 : [-R, R] \to \mathbb{C}$

$\gamma_1(t) = t$

$\gamma_2 : [0, \pi] \to \mathbb{C}$

$\gamma_2(\theta) = Re^{i\theta}$,

Γ = tot conturul

$x^4+x^2+1 = (x^2+x+1)(x^{\cancel{4}\,2}-x+1)$

$\Delta_1 = 1-4 = -3 \Rightarrow x_{1,2} = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\pi/3}$

$x_{3,4} = \frac{-1 \pm i\sqrt{3}}{2} = e^{\pm i 2\pi/3}$

$\Rightarrow$ Pt R suf. de mare, x_1, x_3 se află în int conturului.

Din Th. Rez. $\Rightarrow \int_\Gamma f(z)\,dz = 2\pi i\left(\text{res}(f, x_1) + \text{res}(f, x_3)\right)$

$f(z) = \frac{1}{(z-x_1)(z-x_2)(z-x_3)(z-x_4)}$

$\text{res}(f, x_1) = \lim_{z \to x_1} (z-x_1) f(z) = \frac{1}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$

$\text{res}(f, x_3) = \lim_{z \to x_3} (z-x_3) f(z) = \frac{1}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}$

Dar $\int_\Gamma f(z)\,dz = \underbrace{\int_{\gamma_1} f(z)\,dz}_{i_1} + \underbrace{\int_{\gamma_2} f(z)\,dz}_{i_2}$

$i_1 = \int_{-R}^{R} \frac{1}{t^4+t^2+1}\,dt = 2\int_0^R \frac{1}{t^4+t^2+1}$

$\lim_{R\to+\infty} i_1 = 2\,i$

$|i_2| \le l(\gamma_2)\cdot \max_{\gamma_2} \left|\frac{1}{z^4+z^2+1}\right|$ unde $l(\gamma_2)$ = = lung. curbei = πR

$\Rightarrow |i_2| \le \pi R \cdot \max_{\gamma_2} \left|\frac{1}{z^4+z^2+1}\right|$

Dar $|z^4+z^2+1| \ge |z^4| - |z^2+1| \ge |z^4| - |z^2| - 1$

$\Rightarrow \frac{1}{|z^4+z^2+1|} \le \frac{1}{|z^4|-|z^2|-|1|} \Rightarrow \max_{\gamma_2} \left|\frac{1}{z^4+z^2+1}\right| \le \frac{1}{R^4-R^2-1}$

$\Rightarrow |i_2| \le \frac{\pi R}{R^4-R^2-1} \xrightarrow{R\to+\infty} 0$

$\Rightarrow \lim_{R\to+\infty} \int_\Gamma f(z)\,dz = \underbrace{\lim_{R\to+\infty} i_1}_{2i} + 0$

Rămâne să calculez $\int_\Gamma f(z)$ cu Th. Rez.

$$\left.\begin{aligned}2x_1 &= 1+i\sqrt{3}\\ 2x_2 &= 1-i\sqrt{3}\\ 2x_3 &= -1+i\sqrt{3}\\ 2x_4 &= -1-i\sqrt{3}\end{aligned}\right\} \Rightarrow \left.\begin{aligned}2(x_1-x_2) &= 2i\sqrt{3}\\ 2(x_1-x_3) &= 2\\ 2(x_1-x_4) &= 2+2i\sqrt{3}\end{aligned}\right\} \text{pt } res(f, x_1)$$

$$\left.\begin{aligned}2(x_3-x_1) &= -2\\ 2(x_3-x_2) &= -2+2i\sqrt{3}\\ 2(x_3-x_4) &= 2i\sqrt{3}\end{aligned}\right\} \text{pt } res(f, x_3)$$

$$\Rightarrow res(f, x_1) = \frac{8}{2i\sqrt{3}\cdot 2\cdot(2+2i\sqrt{3})} = \frac{1}{i\sqrt{3}(1+i\sqrt{3})} =$$

$$= \frac{1}{i\sqrt{3}-3} =$$

$$= \frac{3+i\sqrt{3}}{-3-9} = \frac{3+i\sqrt{3}}{-12}$$

$$res(f, x_3) = \frac{8}{(-2)(-2+2i\sqrt{3})(2i\sqrt{3})} = \frac{1}{(-1)(-1+i\sqrt{3})\cdot i\sqrt{3}} =$$

$$= \frac{1}{(1-i\sqrt{3})i\sqrt{3}} = \frac{1}{i\sqrt{3}+3} = \frac{i\sqrt{3}-3}{-3-9} = \frac{i\sqrt{3}-3}{-12}$$

$$2\pi i \sum res = \cancel{2i\sqrt{3}} \cancel{-\frac{3+i\sqrt{3}}{12} - \frac{i\sqrt{3}-3}{12} = -}$$

$$2\pi i\left(-\frac{3+i\sqrt{3}}{12} - \frac{i\sqrt{3}-3}{12}\right) = -2\pi i\left(\frac{3+i\sqrt{3}+i\sqrt{3}-3}{12}\right) =$$

$$= -2\pi i \cdot \frac{2i\sqrt{3}}{12} = 4\pi\frac{\sqrt{3}}{12} = \frac{\sqrt{3}}{3}\pi.$$

$$\Rightarrow \boxed{I = \frac{\sqrt{3}}{6}\pi}$$

5) $P(z)=z^n+\dots+a_1z+a_0$

$Q(z)=z^n\cdot P(\frac{1}{z})$

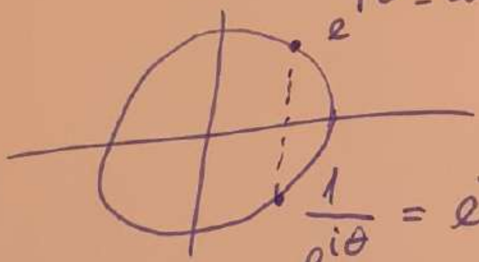
① $M=\max\limits_{|z|=1}|Q(z)|=\max\limits_{|z|=1}\underbrace{|z^n|}_{=1\text{ pe cerc.}}\cdot|P(\frac{1}{z})|=\max\limits_{|z|=1}|P(\frac{1}{z})|=$

Notez $g(z)=\frac{1}{z} \Rightarrow M=\max\limits_{|z|=1}|P(g(z))|$

$g:\{|z|=1\}\longrightarrow\{|z|=1\}$

$g(\{z/|z|=1\})=\{z/|z|=1\}$ pt. că g duce fiecare pct. al cercului unitate în sim. său față de Ox:

$e^{i\theta}=\cos\theta+i\sin\theta$



$\frac{1}{e^{i\theta}}=e^{-i\theta}=\cos(\theta)+i\sin(-\theta)=\cos(\theta)-i\sin(\theta)$

$\Rightarrow M=\max\limits_{w\in Im(g)}|P(w)|=\max\limits_{w\in\{|z|=1\}}|P(w)|$ Q.E.D.

② Din ①, $\max\limits_{|z|=1}|P(z)|=\max\limits_{|z|=1}|Q(z)|=M$

$Q(z)=z^n\left(\frac{1}{z^n}+a_{n-1}\frac{1}{z^{n-1}}\cdots\frac{a_1}{z}+a_0\right)=$

$=1+a_{n-1}z+\dots+a_0z^n.$

Observă că $Q(0)=1$ și că $0 \in \{z \mid |z| \le 1\}$

Din Princ. max. modulului $\Longrightarrow$ Q își atinge maximul pe frontieră, i.e.:

$$1 \le \max_{|z|\le 1} Q(z) = \max_{|z|=1} Q(z)$$

$$\max_{|z|} Q(z) \ge 1 \qquad \text{Q.E.D.}$$