Seminarul 1 - 22.02.2023

Examen EDP 1 - 2023
seria 32

Problema 1

1\. $\nabla f = ?$ $\quad f(x,y,z) = (\cos(xy))^{xz}, \; (x,y,z) \in \mathbb{R}^3$

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ $\quad [f = g(x)^{h(x)}] = e^{\ln g(x)^{h(x)}} = e^{h(x)\ln g(x)}$

$\frac{\partial f}{\partial x} = ?$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{(xz)\ln\cos(xy)}\right) = e^{xz\ln(\cos(xy))}\cdot\left(z\ln(\cos(xy)) + \right.$$

$$\left. + xz\cdot\frac{1}{\cos(xy)}\cdot(-\sin(xy)\cdot y)\right) =$$

$$= e^{xz\ln(\cos(xy))}\cdot\left(z\ln(\cos(xy)) - xzy\cdot\frac{1}{\cos(xy)}\cdot\sin(xy)\right)$$

$$\frac{\partial f}{\partial y} = -x^2 z\cdot\cos(xy)^{xz-1}\cdot\sin(xy)$$

$$\frac{\partial f}{\partial z} = (\cos(xy))^{xz}\cdot\ln(\cos(xy))\cdot x$$

2\. $\operatorname{div}(x|x|^7), \; x \in \mathbb{R}^3 - \{0\}$ $\qquad (fg)' = f'g + fg'$

$$\operatorname{div}(x|x|^7) = \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(x_j|x|^7\right) = \qquad \operatorname{div}(Fg) = \underbrace{F'}_{\operatorname{div} F}g + F\underbrace{g'}_{\nabla g}$$

$$= \sum_{j=1}^{3}\left(1\cdot|x|^7 + x_j\cdot\underbrace{\frac{\partial}{\partial x_j}|x|^7}_{7\cdot|x|^6\cdot\underbrace{\frac{\partial}{\partial x_j}(|x|)}_{\frac{x_j}{|x|}}}\right) =$$

ne imaginam norma ca $\sqrt{x_1^2+\ldots+x_j^2+\ldots} = \frac{2x_j}{2\sqrt{\ldots}} = \frac{x_j}{|x|}$

$$\cancel{= 3\cdot|x|^7}$$

$$= 3\cdot|x|^7 + \sum_{j=1}^{3} 7x_j^2\cdot|x|^5 = 3|x|^7 + 7|x|^5\underbrace{\sum_{j=1}^{3}x_j^2}_{|x|^2} =$$

$$= 3|x|^7 + 7|x|^7 = 10|x|^7$$

3. $\Delta(x_5|x|^{-5}) = 0$, $(\forall) x \in \mathbb{R}^5 - \{0\}$

Var 1 (cea mai neplăcută) $\rightarrow$ cu def.

$$\Delta = \sum_{j=1}^{5} \frac{\partial^2}{\partial x_j^2}(x_5|x|^{-5}) = \dots$$ discuție: pt. $\begin{cases} j \neq 5, & x_5 \text{ iese în față} \\ j = 5, & \text{sumă de deriv.} \end{cases}$

Var. 2

$$\Delta(fg) = \Delta f\, g + f \Delta g + 2 \nabla f \nabla g$$

$f = x_5$, $g = |x|^{-5}$

$$\Delta(x_5|x|^{-5}) = \underbrace{\Delta(x_5)}_{=0}|x|^{-5} + x_5 \cdot \overbrace{\Delta(|x|^{-5})}^{=10|x|^{-7}} + 2\underbrace{\nabla(x_5)}_{\vec{e_5}} \cdot \underbrace{\nabla(|x|^{-5})}_{-5x|x|^{-7}} =$$

$$\Delta(x_5) \rightarrow \frac{\partial^2}{\partial x_1^2}(x_5) + \dots + \frac{\partial^2}{\partial x_5^2}(x_5)$$

funcție radială $f(x) = g(|x|)$, $g(r) = r^{-5}$

$$\Delta f(x) = g''(r) + \frac{n-1}{r} g'(r), \quad n = 5$$

$$= \left[(-5)\cdot(-6)\cdot r^{-7} + \frac{4}{r}\cdot(-5)\cdot r^{-6}\right]_{r=|x|}$$

$$= 30 r^{-7} + (-20) r^{-7} = 10 r^{-7}$$

sau $\Delta(|x|^{\lambda}) = \lambda(\lambda + n - 2)|x|^{\lambda - 2}$

$$= -5(-5+5-2)\cdot|x|^{-7} = 10|x|^{-7}$$

$$\nabla(|x|^{\lambda}) = \lambda|x|^{\lambda-2}\cdot x$$

$$\nabla(|x|^{-5}) = (-5)\cdot|x|^{-7}\cdot x = -5x|x|^{-7}$$

$$= 10x_5|x|^{-7} \cancel{+ \dots} + 2\cdot\vec{e_5}\cdot(-5x|x|^{-7}) =$$

$$= 10x_5|x|^{-7} - 10x_5|x|^{-7} = 0$$

4. funcția metodă $f: \mathbb{R}^4 \rightarrow \mathbb{R}$, $f(\lambda x) = \lambda^3 f(x)$, $(\forall) x \in \mathbb{R}^4$, $\lambda > 0$

$$x \cdot \nabla f(x) = 3f(x), \ (\forall) x \in \mathbb{R}^4$$ (relația lui Euler pt. fct. cu omogenitate 3)

$$f(x) = \begin{cases} x_1^3 \\ x_1^2 x_2 + x_2 x_4^2 \\ |x|^3 \end{cases}$$

1-d $f(x) \sim x^3$

$$x \cdot f'(x) = 3x^2 \cdot x = 3x^3 = 3f$$

$$f(\lambda x) = \lambda^3 f(x)$$

$$\frac{d}{d\lambda}\left[f(\lambda x)\right] = \frac{d}{d\lambda}\left(\lambda^3 \overbrace{f(x)}^{\text{const.}}\right) = 3\lambda^2 f(x)$$

$$\|$$

$$\frac{d}{d\lambda}\left[f(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4)\right] =$$

$$= \frac{\partial f}{\partial x_1}(\lambda x)\cdot \underbrace{x_1}_{\text{derivare în raport cu } \lambda} + \frac{\partial f}{\partial x_2}(\lambda x)\cdot x_2 + \dots = \nabla f(\lambda x)\cdot x$$

Deci $\nabla f(\lambda x)\cdot x = 3\lambda^2 f(x)$, $(\forall)\lambda, x$

Iau $\lambda = 1$ (alegere)

$\Rightarrow \nabla f(x)\cdot x = 3 f(x)$

Problema 2

$\Omega = (-1,1)\times(-1,1) \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u(x,y) = \dfrac{|x|}{1+x^2}, & (x,y)\in\Omega \\ u(x,y) = 0, & (x,y)\in\partial\Omega \end{cases}$$

1) $c = ?$ a.î. $v(x,y) = c(x^2+y^2)$ să verifice $-\Delta v = \frac{1}{2}$ în Ω

2) $0 \le u(x,y) \le \frac{1}{4}$ în Ω

1) $\Delta v = 4c = -\frac{1}{2} \Rightarrow c = -\frac{1}{8} \Rightarrow v(x,y) = -\frac{1}{8}(x^2+y^2)$

2) $-\Delta u \ge 0$ în Ω, deci u - superarmonică $\sim$ își atinge minimul pe $\partial\Omega$

Din ~~PMS~~ PSM $\Rightarrow \min_{\overline{\Omega}} u = \min_{\partial\Omega} u = 0 \Rightarrow u \ge 0$ în $\overline{\Omega}$

Pp. prin absurd că $(\exists)(x_0,y_0)\in\Omega$ a.î. $u(x_0,y_0) = 0$. (negare)

(x_0,y_0) punct de minim interior pt. u

~~Din~~ PTM $u \equiv ct. \Rightarrow \Delta u = 0$ ↯

$u > 0$ în Ω

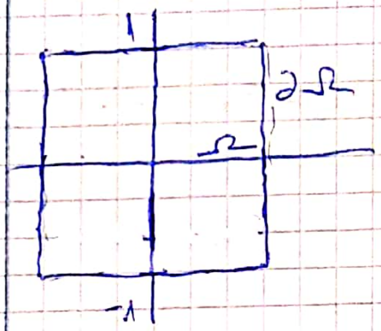
Fie $U := u - v \Rightarrow -\Delta U = -\Delta u + \Delta v =$

$$= \underbrace{\overbrace{\frac{|x|}{1+x^2}}^{\le \frac{1}{2}} - \frac{1}{2}}_{\le 0} \text{ în } \Omega$$

Deci U subarmonică

$$\Rightarrow \max_{\overline{\Omega}} U = \max_{\partial\Omega} U = \max_{\partial\Omega} \{0 - v\} = \max_{(x,y)\in\partial\Omega} \frac{1}{8}(x^2+y^2) = \frac{1}{8}(1^2+1^2) = \frac{1}{4}$$

$\hookrightarrow u$ pe $\partial\Omega$ este 0 (condiție)



Deci $\max_{\overline{\Omega}} U = \frac{1}{4} \Rightarrow U \leq \frac{1}{4}$ în $\overline{\Omega}$

$\Rightarrow u - v \leq \frac{1}{4} \Rightarrow u \leq \frac{1}{4} + v$ în $\overline{\Omega}$

dar $v \leq 0 \Rightarrow u \leq \frac{1}{4}$ în $\overline{\Omega}$

Problema 3 problemă de tip „unde"

$$(1)\begin{cases} 2u_{tt}(x,t) + 3u_{tx}(x,t) - 2u_{xx}(x,t) = 0, & x\in\mathbb{R}, t>0 \\ u(x,0) = f(x), & x\in\mathbb{R} \\ u_t(x,0) = g(x), & x\in\mathbb{R} \end{cases}, \text{ unde } f,g\in C^2(\mathbb{R})$$

1) $v = v(x,t)$ funcție de clasă C^2

$$(2\partial_t - \partial_x)(v_t(x,t) + 2v_x(x,t)) = 2v_{tt}(x,t) + 3v_{tx}(x,t) - 2v_{xx}(x,t)$$

$(\forall) x, (\forall) t$

2) integrați pb. (1)

1) $(2\partial_t - \partial_x)(\overbrace{v_t(x,t) + 2v_x(x,t)}) = 2v_{tt}(x,t) + \cancel{2}\, v_{xt}(x,t) - v_{tx}(x,t) - 2v_{xx}(x,t) \overset{T.\ Schwartz}{=\!=} 2v_{tt}(x,t) + 3v_{tx}(x,t) - 2v_{xx}(x,t)$

2) Notez $v(x,t) := u_t(x,t) + 2u_x(x,t)$

$$\overset{1)}{\Rightarrow}\begin{cases} 2v_t(x,t) - v_x(x,t) = 0, \ x\in\mathbb{R}, t>0 \\ v(x,0) = u_t(x,0) + 2u_x(x,0) = g(x) + 2f'(x) \end{cases} \rightarrow \text{ecuație de transport omogenă}$$

$$(v_x, v_t)\cdot(-1,2) = 0$$

$$\underbrace{\quad}_{\nabla v} \cdot \underbrace{\quad}_{\overline{a}} = 0$$

$\frac{\partial v}{\partial \overline{a}} = 0 \Rightarrow v$ const. pe direcția $\overline{a}$

$$v(x,t) = v\left(\frac{t}{2}\underbrace{(-1,2)}_{\overline{a}} + \left(x+\frac{t}{2}, 0\right)\right) = v\left(x+\frac{t}{2}, 0\right) = g\left(x+\frac{t}{2}\right) + 2f'\left(x+\frac{t}{2}\right)$$

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$$\begin{cases} u_t(x,t) + 2u_x(x,t) = g\left(x+\frac{t}{2}\right) + 2f'\left(x+\frac{t}{2}\right) \\ u(x,0) = f(x) \end{cases}$$

$\rightarrow$ nu mai adăugăm $u_t(x,0) = g(x)$ căci e infiltrată în prima

ec. de transport neomogenă

Fie $w(s) := u(x+2s, t+s)$
$s \in \mathbb{R}$

$$(*)\ \left[w'(s) := u_x(\overbrace{x+2s}^{x'}, \overbrace{t+s}^{t'}) \cdot 2 + u_t(x+2s, t+s) \cdot 1 \right.$$

$$= g\left(x+2s+\frac{t+s}{2}\right) + 2f'\left(x+2s+\frac{t+s}{2}\right)$$

$$= \boxed{g\left(x+\frac{t}{2}+\frac{5s}{2}\right) + 2f'\left(x+\frac{t}{2}+\frac{5s}{2}\right)}$$

$w(0) = u(x,t)$

$w(-t) = u(x-2t, 0) = f(x-2t)$ $\rightarrow$ $u(x,t) - f(x-2t)$

$$\int_{-t}^{0} (*) \Rightarrow \boxed{w(0) - w(-t)} = \int_{-t}^{0} \left[g\left(x+\frac{t}{2}+\frac{5s}{2}\right) + 2f'\left(x+\frac{t}{2}+\frac{5s}{2}\right)\right] ds \ \underset{d\tau = \frac{5}{2} ds}{\overset{x+\frac{t}{2}+\frac{5s}{2} = \tau}{=}} \int_{x-2t}^{x+\frac{t}{2}} \left[g(\tau) + 2f'(\tau)\right] \cdot \frac{2}{5}\, d\tau =$$

$$= \frac{2}{5}\int_{x-2t}^{x+\frac{t}{2}} g(\tau)\, d\tau + \frac{4}{5}\left(f\left(x+\frac{t}{2}\right) - f(x-2t)\right)$$

$$u(x,t) = \frac{2}{5}\int_{x-2t}^{x+\frac{t}{2}} g(\tau)\, d\tau + \frac{4}{5} f\left(x+\frac{t}{2}\right) + \frac{1}{5} f(x-2t)$$