I. (a) Arătați că funcția $\phi:\mathbb{R}\to\mathbb{R}$, $\phi(x)=\frac{1}{2}\left(x^2+\frac{1}{2}\right)$ are două puncte fixe, x^* și x^{**}, a.î. $0<x^*<1<x^{**}$. Scrieți metoda iterativă indusă de funcția de punct fix ϕ pentru șirul $\{x_n\}_{n\geq 0}$.

(b) Arătați că termenii șirului de la punctul (a) satisfac relația $x_{n+1}-x^*=\frac{1}{2}(x_n+x^*)\cdot(x_n-x^*)$, $n\geq 0$

(c) Arătați că dacă $0\leq x_0<x^*$, atunci $\lim\limits_{n\to\infty} x_n=x^*$.

(d) Determinați $\lim\limits_{n\to\infty} x_n$ pentru $x_0\in\mathbb{R}\setminus[0,x^{**}]$.

II. Fie $f:[-1,1]\to\mathbb{R}$, $f(x)=\frac{2}{1+x^2}$

(a) Determinați polinomul de interpolare Hermite $H_3(x)$, $x\in[-1,1]$, asociat fct. f și nodurilor de interpolare $x_0=-1$, $x_1=1$.

(b) Calculați $I(f)=\int_{-1}^{1} f(x)\,dx$

(c) Calculați $\int_{-1}^{1} H_3(x)\,dx$.

(d) Calculați cuadratura Simpson asociată fct. f și nodurilor $y_0=-1$, $y_0=0$ și $y_2=1$, i.e. $I_2(f)$.

III. Fie fct. pondere $w:(0,\infty)\to\mathbb{R}$, $w(x)=e^{-x}$.

(a) Determinați sistemul de polinoame ortogonale în raport cu produsul scalar din $L^2_w(0,\infty)$, $\{\varphi_0,\varphi_1,\varphi_2\}\subset\mathbb{P}_2$.

(b) Determinați cea mai bună aproximare polinomială, $p_2\in\mathbb{P}_2$, în norma $\|\cdot\|_{L^2_w(0,\infty)}$ a fct. $f:[0,\infty)\to\mathbb{R}$, $f(x)=e^{-x}$.

25.1.2023.

EXAMEN - 25.01.2023

I. (a) $\phi: \mathbb{R} \to \mathbb{R}$, $\phi(x) = \frac{1}{2}\left(x^2 + \frac{1}{2}\right)$

Caut puncte fixe: $\phi(x) = x$

$\frac{1}{2}\left(x^2 + \frac{1}{2}\right) = x$

$x^2 + \frac{1}{2} = 2x$

$x^2 - 2x + \frac{1}{2} = 0$

$\Delta = 4 - 2 = 2$

$x_{1,2} = \frac{2 \pm \sqrt{2}}{2} \Rightarrow x^* = \frac{2-\sqrt{2}}{2}$; $x^{**} = \frac{2+\sqrt{2}}{2}$

$0 < x^* < 1 < x^{**} < 2$

Metoda iterativă indusă de ϕ:

$\begin{cases} \cancel{x_0 \in [0,1]}\ x_0 \in \mathbb{R} \\ x_n = \phi(x_{n-1}) \end{cases}$, pt. $\cancel{x \in [0,1]}$ $x \in \mathbb{R}$ $\quad$ $\cancel{\begin{cases} x_0 \in [1,2] \\ x_n = \phi(x_{n-1}) \end{cases}, \text{pt. } x \in [1,2]}$

(b) $x_{n+1} - x^* = \frac{1}{2}(x_n + x^*)(x_n - x^*)$

$x_{n+1} = \phi(x_n) \Rightarrow \phi(x_n) - x^* = \frac{1}{2}\left(x_n^2 - (x^*)^2\right)$

$\frac{1}{2}\left(x_n^2 + \frac{1}{2}\right) - x^* = \frac{1}{2} \cdot x_n^2 - \frac{1}{2}(x^*)^2$

$\frac{1}{2}x_n^2 + \frac{1}{4} - x^* = \frac{1}{2} \cdot x_n^2 - \frac{1}{2}(x^*)^2$

$\frac{1}{4} - \frac{2-\sqrt{2}}{2} = -\frac{1}{2} \cdot \frac{4 - 4\sqrt{2} + 2}{4}$

$\frac{1 - 4 + 2\sqrt{2}}{4} = \frac{-6 + 4\sqrt{2}}{8} \Leftrightarrow \frac{-3 + 2\sqrt{2}}{4} = \frac{-3 + 2\sqrt{2}}{4}$ Adev. $\forall n \geq 0$

(c) $0 \le x_0 < x^*$

Fie $\phi : [0, x^*) \to \mathbb{R}$

$\phi'(x) = x$

x	0			x^*
$\phi'(x)$	$+$	$+$	$+$	$+$
$\phi(x)$	$\frac{1}{4}$	$\nearrow$		$\frac{2-\sqrt{2}}{2}$

Pornesc de la: $0 \le x_0 < x^*$ | Aplic ϕ

$$\phi(0) \le x_1 < x^* \quad | \text{Aplic } \phi$$

$$\phi(\phi(0)) \le x_2 < x^*$$

$$\vdots$$

$$\phi(\phi(\dots(0))) \le x_n < x^*$$

ϕ - strict crescătoare pe $[0, x^*)$

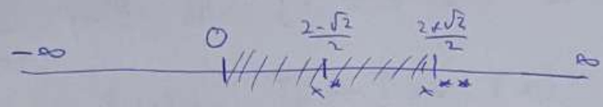
$\phi([0, x^*)) \subset [0, x^*) \iff [\frac{1}{4}, x^*) \subset [0, x^*)$ $\Big| \Rightarrow$

$\Rightarrow$ dacă $n \to \infty$ și ϕ s. $\nearrow$ $\Big| \Rightarrow \phi(\underbrace{\phi(\dots(0))}_{\substack{\text{de } n \text{ ori} \\ n \to \infty}}) = \phi(x^*) = x^*$

Din Th. Cleștelui cu

$x^* \le x_n < x^*$ $\Longrightarrow \lim\limits_{n \to \infty} x_n = x^*$

(d) $x_0 \in \mathbb{R} \setminus [0, x^{**})$



Fie $\phi : (-\infty, \infty) \setminus [0, x^{**})$

x	$-\infty$	0	$\frac{1}{4}$	$\frac{2-\sqrt{2}}{2}$		$\frac{2+\sqrt{2}}{2}$	∞
$\phi'(x)$	$- - - - - -$	0	$+ + + + + + + +$		$+$	$+ + + +$	
$\phi(x)$	$\infty \searrow$	$\frac{1}{4}$	$\nearrow$			$\frac{2+\sqrt{2}}{2} \nearrow$	∞

~~Pt. $x \in (-\infty, 0)$: ϕ - descrescătoare~~

Pt. $x \in (-\infty, 0)$ și $\phi(x) \in \left(\frac{1}{4}, \frac{2+\sqrt{2}}{2}\right)$

$\Rightarrow \lim\limits_{n \to \infty} x_n = x^*$ sau x^{**}

Pt. $x \in (-\infty, 0) \cup (x^{**}, \infty)$ și $\phi(x) \notin \left(\frac{1}{4}, \frac{2+\sqrt{2}}{2}\right)$

$\Rightarrow \lim\limits_{n \to \infty} x_n = \infty$

II. $f:[-1,1]\to\mathbb{R}$, $f(x)=\frac{2}{1+x^2}$; ~~$x_0=-1$, $x_1=1$~~

(a) $x\in[-1,1]$; nodurile de interpolare: $x_0=-1$, $x_1=1$

$$f'(x)=\frac{-2\cdot 2x}{(1+x^2)^2}=\frac{-4x}{(1+x^2)^2}$$

$$f(-1)=1\ ;\ f'(-1)=1$$

$$f(1)=1\ ;\ f'(1)=-1$$

$$\underset{2m+1}{H_3}(x)=\sum_{k=0}^{1}\left[H_{1,k}(x)\cdot f(x_k)+K_{1,k}\cdot f'(x_k)\right]$$

$$=H_{1,0}(x)\cdot f(x_0)+K_{1,0}\cdot f'(x_0)+H_{1,1}\cdot f(x_1)+K_{1,1}\cdot f'(x_1)$$

$$L_{1,0}(x)=\frac{x-x_1}{x_0-x_1}=\frac{x-1}{-1-1}=\frac{1-x}{2}\ ;\ L'_{1,0}(x)=-\frac{1}{2}$$

$$L_{1,1}(x)=\frac{x-x_0}{x_1-x_0}=\frac{x+1}{2}\ ;\ L'_{1,1}(x)=\frac{1}{2}$$

$$H_{1,0}(x)=\left[L_{1,0}(x)\right]^2\cdot\left[1-2L'_{1,0}(x_0)(x-x_0)\right]$$

$$=\frac{1-2x+x^2}{4}\cdot\left(1-2\cdot\left(-\frac{1}{2}\right)\cdot(x+1)\right)$$

$$=\frac{x^2-2x+1}{4}\cdot(x+2)=\frac{(x-1)^2\cdot(x+2)}{4}$$

$$H_{1,1}(x)=\left[L_{1,1}(x)\right]^2\cdot\left[1-2L'_{1,1}(x_1)\cdot(x-x_1)\right]$$

$$=\frac{(x+1)^2}{4}\cdot(1-x+1)=\frac{(x+1)^2\cdot(2-x)}{4}.$$

$$K_{1,0}(x) = [L_{1,0}(x)]^2 \cdot (x - x_0)$$
$$= \frac{(x-1)^2}{4} \cdot (x+1)$$

$$K_{1,1}(x) = [L_{1,1}(x)]^2 \cdot (x - x_1)$$
$$= \frac{(x+1)^2}{4} \cdot (x-1)$$

$$\Rightarrow H_3(x) = \frac{(x-1)^2 \cdot (x+2)}{4} \cdot 1 + \frac{(x-1)^2 \cdot (x+1)}{4} \cdot 1 +$$
$$+ \frac{(x+1)^2 \cdot (2-x)}{4} \cdot 1 + \frac{(x+1)^2 \cdot (x-1)}{4} \cdot (-1)$$

$$\cancel{H_3(x) = (x^2 - 2x + 1)(x+2) + (x - 2x + 1}$$

$$H_3(x) = \frac{(x^2 - 2x + 1)(\overbrace{x + 2 + x + 1}^{2x+3}) + (x^2 + 2x + 1)(\overbrace{2 - x + 1 - x}^{-2x+3})}{4}$$

$$H_3(x) = \frac{2x^3 - 4x^2 + 2x + 3x^2 - 6x + 3 + -2x^3 - 4x^2 - 2x + 3x^2 + 6x + 3}{4}$$

$$H_3(x) = \frac{-2x^2 + 6}{4} = \frac{3 - x^2}{2}$$

(b) $I(f) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{2}{1+x^2}\,dx = 2 \cdot \operatorname{arctg} x \Big|_{-1}^{1}$
$$= 2 \cdot (2 \operatorname{arctg} 1)$$
$$= \pi$$

$$\cancel{c)\ \int_{-1}^{1} H_3(x)\,dx = \int_{-1}^{1} \frac{3-x^2}{2}\,dx = \frac{3}{2}x - \frac{x^3}{6}\Big|_{-1}^{1} = \frac{3}{2} - \frac{1}{6} - \left(-\frac{3}{2} + \frac{1}{6}\right)}$$

~~pară => 0~~

$$\cancel{= \frac{6}{2} - \frac{2}{6}}$$

(c) $\displaystyle\int_{-1}^{1} H_3(x)\,dx = \int_{-1}^{1} \frac{3-x^2}{2}\,dx = \frac{3}{2}x - \frac{x^3}{6}\Big|_{-1}^{1} = \frac{3}{2} - \frac{1}{6} + \frac{3}{2} - \frac{1}{6}$

$$= \frac{6}{2} - \frac{2}{6} = \frac{16}{6} = \frac{8}{3}$$

(d) $\overset{a}{y_0 = -1} \quad \overset{\frac{a+b}{2}}{y_1 = 0} \quad \overset{b}{y_2 = 1}$

$$I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$= \frac{2}{6}\left(f(-1) + 4f(0) + f(1) \right)$$

$$= \frac{1}{3}(1 + 8 + 1) = \frac{10}{3}$$

III. $w : (0,\infty) \to \mathbb{R},\ w(x) = e^{-x}$

(a) $\varphi_0(x) = 1$

$$\varphi_1(x) = (x - a_1)\cdot\varphi_0(x) = x - a_1$$

$$a_1 = \frac{\langle x\cdot\varphi_0, \varphi_0\rangle_w}{\langle \varphi_0, \varphi_0\rangle_w} = \frac{\langle x, 1\rangle_w}{\langle 1, 1\rangle_w} = \frac{\int_0^\infty e^{-x}\cdot x\cdot 1\,dx}{\int_0^\infty e^{-x}\,dx} =$$

$$a_1 = \frac{-e^{-x}\cdot x\Big|_0^\infty - \int_0^\infty -e^{-x}\,dx}{-e^{-x}\Big|_0^\infty} = \frac{\int_0^\infty e^{-x}\,dx}{1} = 1$$

$$\Rightarrow \varphi_1(x) = x - 1$$

$$\varphi_2(x) = (x - a_2)\cancel{x}\cdot\varphi_1(x) - b_2\cdot\varphi_0(x)$$

$$\varphi_2(x) = (x - a_2)\cdot(x-1) - b_2\cdot$$

$$a_2 = \frac{\langle x\cdot\varphi_1 , \varphi_1\rangle_w}{\langle \varphi_1, \varphi_1\rangle_w} = \frac{\langle x^2 - x , x-1\rangle_w}{\langle x-1, x-1\rangle_w} =$$

$$= \frac{\int_0^\infty e^{-x}\cdot(x^2-x)\cdot(x-1)\,dx}{\int_0^\infty e^{-x}\cdot(x-1)^2\,dx} = \frac{3}{1} = 3$$

$$b_2 = \frac{\langle x\cdot\varphi_1 , \varphi_0\rangle_w}{\langle \varphi_0, \varphi_0\rangle_w} = \frac{\langle x^2 - x , 1\rangle_w}{\langle 1, 1\rangle_w} =$$

$$= \frac{\int_0^\infty e^{-x}\cdot(x^2-x)\,dx}{\int_0^\infty e^{-x}\,dx} = \frac{1}{1} = 1$$

$$\Rightarrow \varphi_2(x) = (x-3)(x-1) - 1 = x^2 - 3x - x + 3 - 1$$
$$= x^2 - 4x + 2$$

$$\Rightarrow \{\varphi_0, \varphi_1, \varphi_2\} \rightarrow \{1, x-1, x^2-4x+2\}$$

(b) $p_2 \in \mathbb{P}_2$ in $\|\cdot\|_{L^2_w(0,\infty)}$ $\qquad p_2(x) = a_2 x^2 + a_1 x + a_0$

$f : [0,\infty) \to \mathbb{R};\ f(x) = e^{-x}$

Luăm $q \in \{1,\ x-1,\ x^2-4x+2\}$

- $\langle f(x) - p_2(x), 1 \rangle_w = 0$

$$\int_0^\infty f(x) - p_2(x)\,dx = 0$$

$$\int_0^\infty f(x)\,dx - \int_0^\infty p_2(x)\,dx = 0$$

$$\int_0^\infty p_2(x)\,dx = 1$$

- $\langle f(x) - p_2(x),\ x-1 \rangle_w = 0$

$$\int_0^\infty \left(e^{-x} - p_2(x)\right)(x-1)\,dx = 0$$

$$\int_0^\infty e^{-x}\cdot(x-1)\,dx - \int_0^\infty p_2(x)\cdot(x-1)\,dx = 0$$

$$\int_0^\infty p_2(x)\cdot(x-1)\,dx = 0$$

- $\langle f(x) - p_2(x),\ x^2-4x+2 \rangle_w = 0$

$$\int_0^\infty e^{-x}\cdot(x^2-4x+2)\,dx - \int_0^\infty p_2(x)\cdot(x^2-4x+2)\,dx = 0$$

$$\int_0^\infty p_2(x)\cdot(x^2-4x+2)\,dx = 0$$

$$\int_0^{\infty} p_2(x)\,dx = 1$$

$$\int_0^{\infty} a_2x^2 + a_1x + a_0\,dx = 1$$

$$\frac{a_2}{3}\cdot x^3 + \frac{a_1}{2}\cdot x^2 + a_0 \cdot x \Big|_0^{\infty} = 1$$