Facultatea de Matematica si Informatica 19 Ianuarie 2021
Universitatea din Bucuresti

Examen final

Disciplina: Ecuatii cu derivate partiale
Tipul examinarii: Examen
Nume student: ______________________________
Seria 31: Grupele 311, 312 ______________________________
Timp de lucru : 3 ore si 15 min (incluzand atasarea rezolvarilor pe Moodle)

Acest examen contine 5 probleme (toate obligatorii).

Examenul este individual. Nu uitati sa va salvati foile cu rezolvarile subiectelor intr-un singur fisier de tip PDF in timp util astfel incat sa va incadrati in cele 3 ore si 15 minute pentru incarcarea fisierului pe platforma Moodle.

Salvati fisierul PDF creat cu numele vostru (Nume_Prenume_Grupa.pdf).

Pentru elaborarea lucrarii scrise puteti folosi orice materiale ajutatoare.

Pentru redactare tineti cont de urmatoarele sugestii:

- Daca folositi o teorema fundamentala, rezultat cunoscut, etc **indicati** acest lucru si explicati cum se poate aplica rezultatul respectiv.
- **Organizati-va munca** intr-un mod coerent pentru a avea toti de castigat ! Incercati ca la crearea fisierului PDF fiecare problema sa fie redactata in ordinea aparitiei pe foaia cu subiecte. Ideal ar fi ca si subpunctele sa fie redactate in ordine. Daca nu stiti a rezolva un anumit subpunct scrieti numarul subpunctului si lasati liber.
- Raspunsurile corecte dar argumentate incomplet (din punct de vedere al calculelor/explicatiilor) vor primi punctaj partial.

Barem: P1 (2p) + P2 (1.5p)+ P3 (2p) +P4 (1.5p)+P5 (2p) + 1p oficiu= **10p** (Plus eventual BONUS acolo unde este cazul in functie de activitatea/temele din timpul semestrului).

Pentru orice nelamuriri scrieti-mi la adresa cristian.cazacu@fmi.unibuc.ro, sau lasati un mesaj pe chat-ul grupei creat pe Microsoft Teams.

Rezultatele finale vor fi postate pe Moodle si Microsoft Teams in cel mai scurt timp posibil, dar dupa proba orala.

Problema 1. *(2p).*

1). Calculati $\operatorname{div}(|x|^2 \cdot \nabla v(x))$*, unde* $v : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}$, $v(x) := |x|^{-\frac{5}{3}}$.

2). Sa se determine pentru ce valori $p \geq 1$ *are loc* $|v|^p \in L^1(B_1(0))$*, unde* $B_1(0)$ *este bila unitate din* $\mathbb{R}^4$.

3). Sa se determine pentru ce valori $p \geq 1$ *are loc* $\frac{|v(x)|^p}{|x|^2+1} \in L^1(\mathbb{R}^4 \setminus \overline{B_1(0)})$.

4). Dati exemplu de o functie strict subarmonica ($-\Delta u < 0$) *pe* $\mathbb{R}^2$ *care sa se anuleze pe dreapta* $x + 3y = 0$.

5). Consideram functia $u : B_1(0) \setminus \{0\} \to \mathbb{R}$ *data de*

$$u(x) = \left(\ln \frac{2}{|x|}\right)^{\frac{1}{2}}, \quad x = (x_1, x_2),$$

unde $B_1(0)$ *este bila unitate din* $\mathbb{R}^2$ *centrata in origine. Aratati ca*

$$-\Delta u(x) = \frac{u(x)}{4|x|^2 \ln^2(\frac{2}{|x|})}, \quad \forall x \in B_1(0) \setminus \{0\}.$$

Problema 2. *(1.5p). Se considera problema la limita*

$$(1) \qquad \begin{cases} u_{xx}(x,y) + 2u_{yy}(x,y) = 0, & (x,y) \in (0,1) \times (0,1) \\ u(x,0) = u(x,1) = 0, & x \in (0,1),\ y \in (0,1) \\ u(0,y) = \sin(2\pi y),\ u(1,y) = e^{-2\sqrt{2}\pi} \sin(2\pi y), & y \in (0,1). \end{cases}$$

1). Determinati solutia problemei (1) *cautand-o in variabile separate sub forma* $u(x,y) = A(x)B(y)$.

*2). * Aratati (folosind eventual metoda energetica) ca* (1) *are cel mult o solutie de clasa* C^2.

Problema 3. *(2p). Consideram urmatoarea problema de tip "unde"*

$$(2) \qquad \begin{cases} u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

unde $f, g \in C^2(\mathbb{R})$ *sunt functii date.*

1). Aratati ca daca $u = u(x,t)$ *este o functie de clasa* C^2 *atunci* u *verifica*

$$(\partial_t + 2\partial_x)(u_t(x,t) - 3u_x(x,t)) = u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t),$$

pe domeniul sau de definitie.

2). Rezolvati problema cu valori initiale satisfacuta de u *in* (2) *(scrieti forma generala a lui* u*) reducand-o la rezolvarea a doua ecuatii de transport (una omogena si alta neomogena).*

3). Folosind conditiile la $t = 0$ *deduceti solutia* u *a problemei* (2) *in cazul particular* $f(x) = \sin x$ *si* $g(x) = e^{-x}$.

Problema 4. *(1.5p). Consideram problema Cauchy*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \frac{e^t}{e^{2t}+1} u(x,t) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R}. \end{cases} \tag{3}$$

1). Gasiti o functie $\phi : \mathbb{R} \to \mathbb{R}$ *astfel incat functia* $v(x,t) := u(x,t)\phi(t)$ *sa verifice ecuatia caldurii*

$$v_t(x,t) - v_{xx}(x,t) = 0, \quad \forall x \in \mathbb{R},\ \forall t > 0. \tag{4}$$

2). Scrieti problema Cauchy verificata de v *si determinati explicit solutia problemei* (3).

Problema 5. *(2p). Fie functia* $f : [-1,1] \to \mathbb{R}$, $f(x) = |x - \frac{1}{2}|$.

1). Explicitati functia f *si faceti graficul functiei* f.

2). Sa se determine punctele de derivabilitate ale lui f *pe intervalul (-1,1).*

3). Argumentati ca $f \in H^1(-1,1)$ *si calculati norma lui* f *in* $H^1(-1,1)$ *(precizati inainte norma cu care lucrati).*

4). Determinati $\alpha \in \mathbb{R}$ *astfel incat functia* $z : (0,1) \to \mathbb{R}$, $z(x) = x^\alpha$ *sa apartina lui* $H^1(0,1)$.

*5). * Determinati* $\alpha \in \mathbb{R}$ *astfel incat functia* $z : (1,\infty) \to \mathbb{R}$, $z(x) = \frac{x^\alpha}{1+x^3}$ *sa apartina lui* $W^{1,3}(1,\infty)$.

Velea Ana-Maria
grupa 312

PROBLEMA 1

1) $\text{div}(\underbrace{|x|^2}_{f} \cdot \underbrace{\nabla v(x)}_{F})$

$v: \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}, \quad v(x) = |x|^{-\frac{5}{3}}$

$n = 4$

$$\boxed{\text{div}(fF) = \nabla f \cdot F + f \cdot \text{div}\, F}$$

$$\text{div}(|x|^2 \cdot \nabla v(x)) = \nabla(|x|^2) \cdot \nabla v(x) + |x|^2 \cdot \text{div}(\nabla v(x))$$

$$= \boxed{\nabla(|x|^\lambda) = \lambda x |x|^{\lambda-2}}$$

$$\nabla |x|^2 = 2 \cdot x \cdot |x|^0 = 2x$$

$$\nabla v(x) = \nabla(|x|^{-\frac{5}{3}}) = -\frac{5}{3} \cdot x \cdot |x|^{-\frac{5}{3} - \overset{3}{2}} = -\frac{5}{3} \cdot x \cdot |x|^{-\frac{11}{3}}$$

$$\boxed{\text{div}(\nabla f) = \Delta f}$$

$$\text{div}(\nabla v(x)) = \Delta v(x) = \Delta(|x|^{-\frac{5}{3}})$$

$$\boxed{\Delta(|x|^\lambda) = \lambda(\lambda + n - 2) \cdot |x|^{\lambda - 2}}$$

$$\Delta(|x|^{-\frac{5}{3}}) = -\frac{5}{3} \cdot \left(-\frac{5}{3} + 4 - 2\right) \cdot |x|^{-\frac{5}{3} - \overset{3}{2}}$$

$$= -\frac{5}{3} \cdot \left(\overset{3}{2} - \frac{5}{3}\right) \cdot |x|^{-\frac{11}{2}} = -\frac{5}{3} \cdot \frac{1}{3} \cdot |x|^{-\frac{11}{3}}$$

$$= -\frac{5}{9} \cdot |x|^{-\frac{11}{3}}$$

Deci,

$$\operatorname{div}(|x|^2\cdot\nabla(v(x))) = 2x\cdot\frac{-5}{3}\cdot x\cdot|x|^{-\frac{11}{3}} + |x|^2\cdot\frac{-5}{9}\cdot|x|^{-\frac{11}{3}}$$

$$= -\frac{10}{3}|x|^2\cdot|x|^{-\frac{11}{3}} + |x|^2\cdot\frac{-5}{9}\cdot|x|^{-\frac{11}{3}}$$

$$= |x|^{-\frac{5}{3}}\cdot\left(-\frac{10}{3} - \frac{5}{9}\right) = \frac{-35}{9}\cdot|x|^{-\frac{5}{3}}$$

2) $p \geq 1$ $\quad |v|^p \in L^1(B_1(0))$

$B_1(0)$ – bila unitate din $\mathbb{R}^4$

$$|v|^p \in L^1(B_1(0)) \iff \int_{B_1 0} |v|^p < \infty$$

$$\int_{B_1(0)} |v|^p = \int_{B_1(0)} |x|^{-\frac{5}{3}}dx \overset{\text{formula}}{\underset{\text{co-arie}}{=}} \int_0^1\left(\int_{\partial B_R(0)} |x|^{-\frac{5p}{3}}d\sigma\right)dR$$

$$= \int_0^1\left(\int_{\partial B_r(0)} r^{-\frac{5p}{3}}d\sigma\right)dr = \int_0^1 r^{-\frac{5p}{3}}\left(\int_{\partial B_r(0)} d\sigma\right)dr$$

$$= \int_0^1 r^{-\frac{5p}{3}}|\partial B_r(0)|dr = \int_0^1 r^{-\frac{5p}{3}}\cdot \omega_4\cdot r^3 dr$$

$$= \omega_4 \int_0^1 r^{\frac{-5p+9}{3}} \, dr$$

$$= \omega_4 \cdot \left. \frac{r^{\frac{-5p+9}{3}+1}}{\frac{-5p+9}{3}+1} \right|_0^1$$

Ca integrala de mai sus să fie finită, nu vrem ca r să coboare la numitor și să dăm peste ~~e~~ $\frac{1}{0} = \infty$

$\Rightarrow$ trebuie ca $\frac{-5p+9}{3}+1 > 0 \Rightarrow -5p+12 > 0 \Rightarrow$

$-5p > -12 \Rightarrow p < \frac{12}{5}$

$\Rightarrow p \in [1, \frac{12}{5})$

3). $p \geq 1$

$$\frac{|v(x)|^p}{|x|^2+1} \in L^1(\mathbb{R}^4 \setminus \overline{B_1(0)}) \iff$$

$$\int_{\mathbb{R}^4 \setminus \overline{B_1(0)}} \frac{|v(x)|^p}{|x|^2+1} \, dx < \infty$$

$$I = \int_{\mathbb{R}^4 \setminus B_1(0)} \frac{|v(x)|^p}{|x|^2+1}\,dx = \int_{\mathbb{R}^4 \setminus B_1(0)} \frac{|x|^{-\frac{5p}{3}}}{|x|^2+1}\,dx$$

$$\underset{\text{co-arii}}{\overset{\text{formula}}{=}} \int_1^\infty \left(\int_{\partial B_R(0)} \frac{|x|^{-\frac{5p}{3}}}{|x|^2+1}\,d\sigma \right) dr =$$

$$\int_1^\infty \left(\int_{\partial B_R(0)} \frac{r^{-\frac{5p}{3}}}{r^2+1}\,d\sigma \right) dr =$$

$$\int_1^\infty \left(\frac{r^{-\frac{5p}{3}}}{r^2+1} \left(\int d\sigma \right) \right) dr = \int_1^\infty \frac{r^{-\frac{5p}{3}}}{r^2+1} \cdot \underbrace{|\partial B_r(0)|}_{\omega_4 \cdot r^3}\,dr$$

$$= \int_1^\infty \frac{r^{-\frac{5p}{3}}}{r^2+1} \cdot \omega_4 \cdot r^3\,dr = \omega_4 \cdot \int_1^\infty \frac{r^{-\frac{5p}{3}}}{r^2+1}\, r^3\,dr$$

Arătăm că $r^2+1 \sim r^2$

$$\left.\begin{array}{l} \lim\limits_{r\to 1} \dfrac{r^2+1}{r^2} = 2 > 0 \\ \lim\limits_{r\to\infty} \dfrac{r^2+1}{r^2} = 1 > 0 \end{array}\right\} \Rightarrow r^2+1 \sim r^2$$

Deci, $\omega_4 \int_1^\infty \frac{r^{-\frac{5p}{3}}}{r^2+1} \cdot r^3\,dr \sim \omega_4 \cdot \underbrace{\int_1^\infty \frac{r^{-\frac{5p}{3}}}{r^2}\, r^3\,dr}_{J}$

$$I = \omega_4 \cdot \int_1^{\infty} \frac{r^{-\frac{5p}{3}}}{r^2} \cdot r^3 dr =$$

$$\omega_4 \int_1^{\infty} r^{-\frac{5p}{3}} \cdot r \, dr = \omega_4 \cdot \int_1^{\infty} r^{\frac{-5p+3}{3}} dr$$

$$= \omega_4 \cdot \frac{r^{\frac{-5p+3}{3}+1}}{\frac{-5p+3}{3}+1} \Bigg|_{r=1}^{r=\infty}$$

$$= \omega_4 \left(\cancel{3} \frac{r^{\frac{-5p+6}{3}}}{\frac{-5p+6}{3}} \Bigg|_{r=1}^{r=\infty} \right)$$

$$= \omega_4 \cdot \frac{3 \cdot r^{\frac{-5p+6}{3}}}{-5p+6} \Bigg|_{r=1}^{r=\infty})$$

Ca rez. de mai sus să fie finit trebuie ca

$$\frac{-5p+6}{3} < 0 \iff -5p+6 < 0 \iff 5p > 6$$

$$\iff \boxed{p > \frac{6}{5}}$$

Deci, pentru ca I finită, trebuie ca $p > \frac{6}{5}$

4) $x+3y=0 \Rightarrow x=-3y$

~~Vrem~~ funcție strict subarmonică $(-\Delta U<0)$ pe $\mathbb{R}^2$

care să se anuleze pe dreapta $x+3y=0$

Vrem o funcție cu $-\Delta U<0$ și $x=-3y$

$$x=-3y \Rightarrow x^2=9y^2 \Leftrightarrow 9y^2-x^2=0$$

luăm $u:\mathbb{R}^2 \to \mathbb{R}$ a.î. $u(x,y)=9y^2-x^2$

$u_x=-2x$

$u_{xx}=-2$

$u_y=18y$

$u_{yy}=18$

$\Delta U=u_{xx}+u_{yy}=-2+18=16$

$-\Delta U=-16<0 \Rightarrow u$ e subarmonică

$\left.\begin{array}{l} \text{și } u=(3y-x)(3y+x) \\ d: x+3y=0 \end{array}\right\} \Rightarrow u$ se anulează pe d

5/ $u(x) = \cancel{\text{the}} \left(\ln \frac{2}{|x|}\right)^{\frac{1}{2}}$, $x = (x_1, x_2)$

$B_1(0)$ este bila unitate din $\mathbb{R}^2$ centrată în origine

Atunci:

$$-\Delta u(x) = \frac{u(x)}{4|x|^2 \ln^2\left(\frac{2}{|x|}\right)}, \quad \forall x \in B_1(0) \setminus \{0\}$$

Sol u e funcție radială $\Rightarrow \exists g$ a.î.

$$u(x) = g(|x|) = g(\sqrt{x^2+y^2}) = g(r)$$

facem $|x| \overset{not}{=} r$

$$g(|x|) = \ln\left(\frac{2}{|x|}\right)^{\frac{1}{2}} = \ln\left(\frac{2}{r}\right)^{\frac{1}{2}}$$

$$g'(r) = \frac{1}{2}\left(\ln\frac{2}{r}\right)^{-\frac{1}{2}} \cdot \left(\ln\left(\frac{2}{r}\right)\right)' = \frac{1}{2}\left(\ln\frac{2}{r}\right)^{-\frac{1}{2}} \cdot \frac{r}{2} \cdot \left(\frac{2}{r}\right)' = \frac{1}{2}\left(\ln\frac{2}{r}\right)^{-\frac{1}{2}} \cdot \frac{r}{2} \cdot \frac{(-2)}{r^2} = \frac{-1}{2r} \ln\left(\frac{2}{r}\right)^{-\frac{1}{2}}$$

$$g''(r) = \frac{-1}{2} \cdot \frac{-1}{r^2}\left(\ln\frac{2}{r}\right)^{-\frac{1}{2}} + \frac{-1}{2r} \cdot \left(-\frac{1}{2}\right) \left(\ln\frac{2}{r}\right)^{-\frac{3}{2}} \cdot \ln\left(\frac{2}{r}\right)' = \frac{1}{2r^2} \cdot \left(\ln\frac{2}{r}\right)^{-\frac{1}{2}} + \frac{1}{4r}\left(\ln\frac{2}{r}\right)^{-\frac{3}{2}} \cdot \frac{r}{2} \cdot \frac{-2}{r^2}$$

7

$$= \frac{1}{2r^2}\left(\ln\frac{2}{r}\right)^{-\frac{1}{2}} - \frac{1}{4r^2}\left(\ln\frac{2}{r}\right)^{-\frac{3}{2}}$$

$$g''(r) + \frac{2-1}{r}g'(r) = \cancel{\frac{1}{2r^2}\left(\ln\frac{2}{r}\right)^{-\frac{1}{2}}} - \frac{1}{4r^2}\left(\ln\frac{2}{r}\right)^{-\frac{3}{2}}$$

$$+ \cancel{\frac{1}{r}\cdot\frac{-1}{2r}\left(\ln\frac{2}{r}\right)^{-\frac{1}{2}}}$$

$$= -\frac{1}{4r^2}\left(\ln\frac{2}{r}\right)^{-\frac{3}{2}}$$

Deci Laplacianul este

$$\Delta u(x) = g''(|x|) + \frac{1}{|x|}\cdot g'(|x|)$$

$$= -\frac{1}{4|x|^2}\left(\ln\frac{2}{|x|}\right)^{-\frac{3}{2}} = \frac{-1}{4|x|^2}\cdot\left(\ln\frac{2}{|x|}\right)^{-\frac{1}{2}}\cdot$$

$$\left(\ln\frac{2}{|x|}\right)^{-2} = \frac{-1}{4|x|^2}\cdot u(x)\cdot\ln\left(\frac{2}{|x|}\right)^{-2} =$$

$$-\frac{1}{4|x|^2}\cdot\frac{u(x)}{\left(\ln\frac{2}{|x|}\right)^2}$$

$$\Rightarrow -\Delta u(x) = \frac{u(x)}{4|x|^2\cdot\left(\ln\frac{2}{|x|}\right)^2}, \forall x\in B_1(0)\setminus\{0\}$$

PROBLEMA 3

problema de tip undă

$$\begin{cases} u_{tt}(x,t) - u_{tx}(x,t) - 6u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$f, g \in C^2(\mathbb{R})$ sunt funcții date

1) dacă $u = u(x,t)$ este o funcție de clasă C^2, atunci u verifică

$(\partial_t + 2\partial_x)(u_t(x,t) - 3u_x(x,t)) = u_{tt}(x,t) - u_{tx}(x,t)$

$-6u_{xx}(x,t)$, pe domeniul său de definiție

$(\partial_t + 2\partial_x)(u_t(x,t) - 3u_x(x,t)) =$

$\partial_t u_t(x,t) - 3\partial_t u_x(x,t) + 2\partial_x u_t(x,t) -$

$6\partial_x u_x(x,t) \underset{\substack{u \\ \text{de clasă } C_2}}{=} u_{tt}(x,t) - 3u_{xt}(x,t) + 2u_{tx}(x,t)$

derivatele parțiale comută -

$(u_{xt} = u_{tx})$

$-6u_{xx}(x,t) = u_{tt}(x,t) - u_{xt}(x,t) - 6u_{xx}(x,t)$

a) $(\partial_t + 2\partial_x)(\partial_t - 3\partial_x)u$

→ ~~a~~ rezolvarea a două ecuații de transport:

$$(1)\begin{cases} v_t + 2v_x = 0 \\ v(x,0) = u_t(x,0) - 3u_x(x,0) = g(x) - 3f'(x) \end{cases}$$

↗ ec. de transport omogenă

$$(2)\begin{cases} u_t - 3u_x = g(x-2t) - 3f'(x-2t) \\ u(x,0) = f(x) \end{cases}$$

↗ ec. de transport neomogenă

pentru prima ecuație:

$v_t + 2v_x = (2,1)\cdot(v_x, v_t) \Rightarrow v$ e constantă

pe direcția $(2,1)$

$v(x,t) = v(t(2,1) + (x-2t, 0)) = v(x-2t, 0)$

$= g(x-2t) - 3f'(x-2t)$

pentru a doua ecuatie folosim metoda curbelor caracteristice:

fie

$$\Gamma : \begin{cases} x = s \\ y = 0 \\ u(x,y) = f(s) \end{cases}$$

Caut curbele caracteristice $(X(\tau,s), Y(\tau,s), Z(\tau,s))$ care taie Γ în punctul de coordonată curbilinie s

$$\begin{cases} X_\tau(\tau,s) = -3 \\ Y_\tau(\tau,s) = 1 \\ Z_\tau(\tau,s) = g(X(\tau,s) - 2y(\tau,s)) - 3f'(X(\tau,s) \\ -2Y(\tau,s)) \\ X(0,s) = s \\ Y(0,s) = 0 \\ Z(0,s) = f(s) \end{cases}$$

condițiile inițiale

$X_\tau(\tau,s) = -3 \Rightarrow$ integrăm $\Rightarrow$

$X(\tau,s) = -3\tau + C(s)$

Dar $X(0,s) = s$

$\Rightarrow \boxed{C(s) = s}$ $\Rightarrow X(\tau,s) = -3\tau + s$

$Y_\tau(\tau,s) = 1$

$\Rightarrow Y(\tau,s) = \tau + \cancel{}\, \tilde{C}(s)$

Dar $Y(0,s) = 0$

$\Rightarrow Y(\tau,s) = \cancel{}\ 0 + \tilde{C}(s)$

$\Rightarrow \boxed{\tilde{C}(s) = 0}$

$Y(\tau,s) = \tau$

$$Z_{\tau'}(\tau',s) = g(-3\tau'+s, -2\tau') - 3f'(-3\tau'+s-2\tau')$$

$$= g(-5\tau'+s) - 3f'(-5\tau'+s)$$

integrăm de la 0 la τ $d\tau'$

$$\underbrace{\int_0^\tau Z_{\tau'}(\tau',s)\,d\tau'}_{\substack{\shortparallel \\ Z(\tau,s) - \underbrace{Z(0,s)}_{=f(s)}}} = \int_0^\tau g(-5\tau'+s)\,d\tau' + \int_0^\tau f'(-5\tau+s)$$

$d\tau'$

FACEM S.V. $-5\tau'+s = p$ $\quad dp = -5\,d\tau'$

$\tau' \rightsquigarrow 0 \Rightarrow p \to s$

$\tau' \rightsquigarrow \tau \Rightarrow p \to -5\tau+s$

OBȚINEM

$$z(\tau, s) - f(s) = -\frac{1}{5}\int_{s}^{-5\tau+s} g(p)\,dp + -\frac{1}{5}\int_{s}^{} f'(p)\,dp$$

$$z(\tau, s) = -\frac{1}{5}\left(f(-5\tau+s) - f(s)\right) - \frac{1}{5}\int_{s}^{-5\tau+s} g(p)\,dp + f(s)$$

$$z(\tau, s) = \frac{6f(s) - f(-5\tau+s)}{5} - \frac{1}{5}\int_{s}^{-5\tau+s} g(p)\,dp$$

$\tau \to t$

$s = x+5t$

$$u(x,t) = \frac{6f(x+5t) - f(x)}{5} - \frac{1}{5}\int_{x+5t}^{x} g(p)\,dp$$

3) $t=0$

$f(x) = \sin x$

$g(x) = e^{-x}$

$$u(x,t) = \frac{6\sin(x+5t) - \sin x}{5} - \frac{1}{5}\int_{x+5t}^{x} e^{-p}\,dp$$

$$= \frac{6\sin(x+5t) - \sin x}{5} - \frac{1}{5}\left(-e^{-p}\Big|_{x+5t}^{x}\right)$$

$$u(x,0) = \frac{6\sin x - \sin x}{5} + \frac{1}{5}\left(e^{-x} - e^{-x}\right)$$

$= \sin x \Rightarrow$ este în regulă

PROBLEMA 4

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \dfrac{e^t}{e^{2t}+1}\, u(x,t) = 0, x \in \mathbb{R}, t > 0 \\ u(x,0) = e^{-x^2}, \quad x \in \mathbb{R} \end{cases}$$

1) $\Phi : \mathbb{R} \to \mathbb{R}$

$v(x,t) = u(x,t) \cdot \Phi(t)$

care să verifice $v_t(x,t) - v_{xx}(x,t) = 0, \forall x \in \mathbb{R}, \forall t > 0$

Sol

$$v_x(x,t) = u_x(x,t) \cdot \Phi(t)$$

$$v_{xx}(x,t) = u_{xx}(x,t) \cdot \Phi(t)$$

$$v_t(x,t) = u_t(x,t) \cdot \Phi(t) + u(x,t) \cdot \Phi'(t)$$

$$0 = v_t(x,t) - v_{xx}(x,t) = u_t(x,t) \cdot \Phi(t) + u(x,t) \cdot \Phi'(t) - u_{xx}(x,t) \cdot \Phi(t)$$

$$= \Phi(t) \cdot \left(u_t(x,t) - u_{xx}(x,t) \right) + u(x,t) \cdot \Phi'(t)$$

$$= \Phi(t) \cdot \frac{-e^t}{e^{2t}+1}\, u(x,t) + u(x,t) \cdot \Phi'(t)$$

$$= u(x,t) \left[\Phi(t) \cdot \frac{-e^t}{e^{2t}+1} + \Phi'(t) \right]$$

Cum $u(x,t) \neq 0$

$$\Rightarrow \Phi'(t) = \Phi(t) \cdot \frac{e^t}{e^{2t}+1} \Rightarrow \frac{d\Phi}{\Phi} = \frac{e^t}{e^{2t}+1}$$

$$\Rightarrow \ln \Phi = \int \frac{e^{t'}}{e^{2t'}+1} dt \overset{s.v.\ e^{t'}=y}{\underset{e^{t'}dt=dy}{=}}$$

$$= \int \frac{1}{y^2+1} dy = \operatorname{arctg} y + K = \operatorname{arctg} e^t + K$$

$$\Phi(t) = e^{\operatorname{arctg} e^t + K}$$

luăm $\underline{\Phi(t) = e^{\operatorname{arctg} e^t}}$

2.

Problema Cauchy

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = 0 \\ v(x,0) = u(x,0) \cdot \Phi(0) = e^{-x^2} \cdot e^{\operatorname{arctg} 1} \\ \qquad = e^{-x^2} \cdot e^{\frac{\pi}{4}} = e^{-x^2+\frac{\pi}{4}} \end{cases}$$

Din formula ecuației căldurii

$$v(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{(x-y)^2}{4t}} \cdot e^{-y^2+\frac{\pi}{4}} dy$$

$$= \frac{e^{\frac{\pi}{4}}}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t} - y^2} \, dy$$

Facem S.V. $\frac{-x+y}{2\sqrt{t}} \to z \Rightarrow y = 2\sqrt{t}\, z + x$

$\cancel{dy = \frac{-1}{2\sqrt{t}} dz}$ $\quad dy = 2\sqrt{t}\, dz$

integrala devine

$$\frac{e^{\frac{\pi}{4}}}{2\sqrt{\pi t}} \cdot \int_{\mathbb{R}} e^{-z^2} \cdot e^{-(2\sqrt{t} z + x)^2} \cdot 2\sqrt{t}\, dz$$

$$= \frac{e^{\frac{\pi}{4}}}{\cancel{2}\sqrt{\pi \cancel{t}}} \cdot \int_{\mathbb{R}} e^{-z^2} \cdot e^{-x^2 + 4\sqrt{t} z + 4tz^2} \, dz$$

$$= \frac{e^{\frac{\pi}{4}}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2} \cdot e^{-x^2} \cdot e^{4tz^2 + 4\sqrt{t} z}$$

$$= \frac{e^{\frac{\pi}{4}} \cdot e^{-x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2(1-4t) + 4\sqrt{t} z} \, dz$$

$$= \frac{e^{-x^2}\cdot e^{\frac{\pi}{4}}}{\sqrt{\pi}} \int_R e^{-z^2(1-4t)+4\sqrt{t}z-\frac{4t}{1-4t}+\frac{4t}{1-4t}}\, dz$$

$$= \frac{e^{-x^2}\cdot e^{\frac{\pi}{4}}}{\sqrt{\pi}} \int_R e^{-\left(z\sqrt{1-4t}-\frac{2\sqrt{t}}{\sqrt{1-4t}}\right)^2+\frac{4t}{1-4t}}\, \cancel{dt}$$

facem s.v. $z\sqrt{1-4t}-\dfrac{2\sqrt{t}}{\sqrt{1-4t}}=w$

$$dz = \frac{1}{\sqrt{1-4t}}\, dw$$

$$\frac{e^{-x^2}\cdot e^{\frac{\pi}{4}}\cdot e^{\frac{4t}{1-4t}}}{\sqrt{\pi}} \int_R e^{-w^2} dw\cdot \frac{1}{\sqrt{1-4t}}$$

$$= \frac{e^{-x^2}\cdot e^{\frac{\pi}{4}}\cdot e^{\frac{4t}{1-4t}}}{\sqrt{\pi}\cdot\sqrt{1-4t}}\cdot \underbrace{\int_R e^{-w^2} dw}_{\sqrt{\pi}}$$

$$= \frac{e^{\left(-x^2+\frac{\pi}{4}\right)}\cdot e^{\frac{4t}{1-4t}}}{\sqrt{1-4t}} \Rightarrow v(x,y) = \frac{e^{-x^2}\cdot e^{\frac{\pi}{4}}\cdot e^{\frac{4t}{1-4t}}}{\sqrt{1-4t}}$$

$$u(x,y) = \frac{v(x,y)}{\Phi(t)} = \frac{e^{-x^2} \cdot e^{\frac{\pi}{4}} \cdot e^{\frac{4t}{1-4t}}}{\sqrt{1-4t} \cdot e^{\operatorname{arctg} e^t}}$$

PROBLEMA 5

$f : [-1,1] \to \mathbb{R}, \quad f(x) = \left|x - \frac{1}{2}\right|$

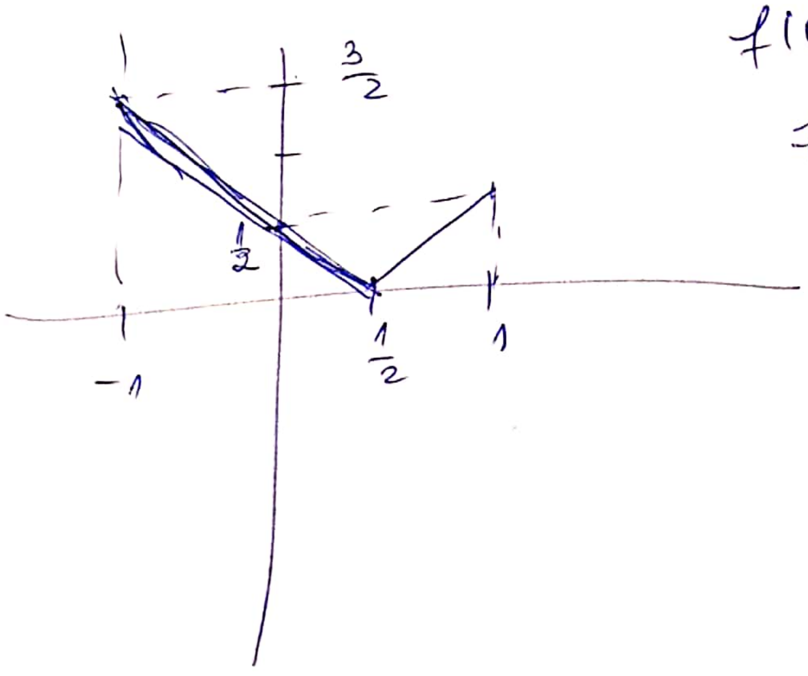
1) $f(x) = \begin{cases} x - \frac{1}{2}, & \text{dacă } x - \frac{1}{2} \geq 0 \\ \frac{1}{2} - x, & \text{dacă } \frac{1}{2} - x < 0 \end{cases} =$

$\begin{cases} x - \frac{1}{2}, & x \geq \frac{1}{2} \\ \frac{1}{2} - x, & x < \frac{1}{2} \end{cases}$

$f(-1) = \frac{3}{2}$

$f(0) = \frac{1}{2}$

$f(1) = \frac{1}{2}$



2/ f e continuă pe $[-1, \frac{1}{2})$

$$\left.\begin{array}{l} f \text{ cont pe } [\frac{1}{2}, 1) \\ f(\frac{1}{2}) = \lim\limits_{\substack{x \to \frac{1}{2} \\ x < \frac{1}{2}}} f(x) = \lim\limits_{\substack{x \to \frac{1}{2} \\ x > \frac{1}{2}}} f(x) = 0 \end{array}\right\} \Rightarrow f \text{ e cont pe } [-1, 1]$$

punctul problemă pt derivare este $x_0 = \frac{1}{2}$

$$l_s = \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{x \to \frac{1}{2} \\ x < \frac{1}{2}}} \frac{\frac{1}{2} - x - 0}{x - \frac{1}{2}} = -1$$

$$l_d = \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{x \to \frac{1}{2} \\ x > \frac{1}{2}}} \frac{x - \frac{1}{2} - 0}{x - \frac{1}{2}} = 1$$

$l_s \neq l_d \Rightarrow f$ nu e deriv. în $\frac{1}{2} \Rightarrow f$ e derivabilă pe $(-1, 1) \setminus \{\frac{1}{2}\}$

3/ $f \in H^1(-1, 1)$

$$H^1(-1,1) = \{ u \in L^2_{(-1,1)} \mid \exists\, u' \in L^2(-1,1) \}$$

$$= \{ u \in L^2(-1,1) \mid \exists\, g \in L^2(-1,1) \text{ a.î.}$$

$$\int_{-1}^{1} u f' dx = - \int_{-1}^{1} g f dx \;,\; \forall f \in C_c^\infty(-1,1) \}$$

$$\int_{-1}^{1} |x - \tfrac{1}{2}| \cdot f' dx = \underbrace{\int_{-1}^{\frac{1}{2}} (\tfrac{1}{2} - x) \cdot f' dx}_{I_1} + \int_{\frac{1}{2}}^{1} (x - \tfrac{1}{2}) f' dx$$

$$I_1 = \int_{-1}^{1} (\tfrac{1}{2} - x) f' dx = (\tfrac{1}{2} - x) \cdot f(x) \Big|_{-1}^{\frac{1}{2}}$$

$$- \int_{-1}^{\frac{1}{2}} - f dx$$

PROBLEMA 2

$$\begin{cases} u_{xx}(x,y) + 2u_{yy}(x,y) = 0 & , (x,y) \in (0,1)\times(0,1) \\ u(x,0) = u(x,1) = 0 & x \in (0,1), y \in (0,1) \\ u(0,y) = \sin(2\pi y) & \\ u(1,y) = e^{-2\sqrt{2}\pi} \sin(2\pi y) & \end{cases} \Big\} y \in (0,1)$$

a) Soluția problemei în variabile separate sub forma

$u(x,y) = A(x)\cdot B(y)$

~~$u(x,0) = 0$~~

$u(x,0) = 0 \Rightarrow A(x)\cdot B(0) = 0$, $\forall x \in (0,1)$

Arătăm că $A(x) \not\equiv 0$ (← nu-mi convine, deoarece $u(x,y) = A(x)\cdot B(y)$ și $u(x,y) \not\equiv 0$!)

Deci, $\boxed{B(0) = 0}$

$u(x,1) = A(x)\cdot B(1)$, $\forall x \in (0,1) \Rightarrow$ Analog, $\boxed{B(1) = 0}$

$u(0,y) = \sin 2\pi y \Rightarrow A(0)\cdot B(y) = \sin 2\pi y \Rightarrow \boxed{A(0) = \frac{\sin 2\pi y}{B(y)}}$

$u(1,y) = A(1)\cdot B(y) = e^{-2\sqrt{2}\pi} \sin(2\pi y)$, $\forall y \in (0,1)$

$\Rightarrow \boxed{A(1) = \frac{e^{-2\sqrt{2}\pi} \sin(2\pi y)}{B(y)}}$

Deci, $\boxed{A(1) = e^{-2\sqrt{2}\pi} \cdot A(0)}$ / $A(0) \neq 0$, deoarece $A(0)\cdot B(y) = \sin 2\pi y$, $\forall y$

$\cancel{\to B y}$ $\Rightarrow$ $\boxed{B(y) = \frac{\sin 2\pi y}{A(0)}}$

$\cancel{u(x,y)}$ $A(0)\cdot B(y) = \sin 2\pi y \Rightarrow$

$A(0)\cdot B'(y) = 2\pi \cdot \cos 2\pi y$

$\Rightarrow B'(y) = \frac{2\pi \cdot \cos 2\pi y}{A(0)}$

$\cancel{B''(y)} \Rightarrow B''(y) = \frac{2\pi}{A(0)} \cdot 2\pi \cdot (-\sin 2\pi y)$

$= \frac{-4\pi^2}{A(0)} \cdot \sin 2\pi y$

$u(x,y) = A(x)\cdot B(y)$

$u_x(x,y) = A_x(x)\cdot B(y)$

$u_{xx}(x,y) = A_{xx}(x)\cdot B(y) = A''(x)\cdot B(y)$

$u_{yy}(x,y) = A(x)\cdot B''(y)$

$u_{xx}(x,y) + 2u_{yy}(x,y) = 0 \Rightarrow$

$A''(x)\cdot B(y) + 2\cdot A(x)\cdot B''(y) = 0$

$B''(y) = \frac{-4\pi^2}{A(0)} \cdot \sin 2\pi y$

$B'(y) = \frac{2\pi \cdot \cos 2\pi y}{A(0)}$

Deci,

$$A''(x) \cdot \frac{\sin 2\pi y}{A(0)} + 2 \cdot A(x) \cdot \frac{-4\pi^2}{A(0)} \cdot \sin 2\pi y = 0 \Big/ \cdot A$$

Obs (! A(0) e diferit de 0) ~~deoarece~~

$$\Rightarrow A''(x) \cdot \sin 2\pi y \cancel{+2A(x)} - 8\pi^2 \cdot A(x) \cdot \sin 2\pi y = 0$$

$$\Rightarrow \left(A''(x) - 8\pi^2 \cdot A(x)\right) \sin 2\pi y = 0, \forall y$$

$$\Rightarrow A''(x) - 8\pi^2 \cdot A(x) = 0$$

Ec. caracteristică: $\lambda^2 - 8\pi^2 \cdot \cancel{1} = 0$

$$\lambda^2 = 8\pi^2$$

$$\underline{\lambda = \pm 2\sqrt{2}\pi}$$

$$\Rightarrow A(x) = \cancel{C_1 \cos A_1} \; C_1 \cdot e^{\lambda_1 x} + C_2 \cdot e^{-\lambda_2 x}$$

$$= C_1 \cdot e^{-2\sqrt{2}\pi x} + C_2 \cdot e^{2\sqrt{2}\pi x}$$

$$\Rightarrow A(0) = C_1 + C_2$$

și

$$A(1) = C_1 \cdot e^{-2\sqrt{2}\pi} + C_2 \cdot e^{2\sqrt{2}\pi}$$

Dar $A(1) = e^{-2\sqrt{2}\pi} \cdot A(0)$

$\Rightarrow$

$$C_1 \cdot e^{-2\sqrt{2\pi}} + C_2 \cdot e^{2\sqrt{2\pi}} = e^{-2\sqrt{2\pi}} \cdot (C_1 + C_2)$$

$$\Rightarrow \cancel{C_1 \cdot e^{-2\sqrt{2\pi}}} + C_2 \cdot e^{2\sqrt{2\pi}} = \cancel{e^{-2\sqrt{2\pi}} C_1} + e^{-2\sqrt{2\pi}} \cdot C_2$$

$$\Rightarrow C_2 \left(e^{2\sqrt{2\pi}} - e^{-2\sqrt{2\pi}}\right) = 0 \Rightarrow$$

$$\boxed{C_2 = 0}$$