1 Ecuații afine scalare

b) $tx' - x = t^2 e^t$

$t \neq 0$ pp $t > 0$ $\overset{:t}{\Rightarrow}$ $x' = te^t + \frac{x}{t}$ $\Leftrightarrow$ $x' = \underbrace{\frac{1}{t}}_{a(\cdot)} x + \underbrace{te^t}_{b(\cdot)}$

Ec. lin. asoc. $\bar{x}' = \frac{1}{t}\bar{x}$ $\Rightarrow$ $\bar{x}(t) = c\, e^{\int \frac{1}{t} dt} = c\, e^{\ln t} = ct$ $c \in \mathbb{R}$

Caut sol. de forma $x(t) = c(t) \cdot t$ *

$x(\cdot)$ sol $\Rightarrow$ $(c(t) \cdot t)' = x'(t)$

$c'(t) \cdot t + \cancel{c(t) \cdot 1} = \cancel{\frac{1}{t}} \cancel{c(t)} \cdot \cancel{t} + te^t$

$c'(t) t = te^t$ $\quad | : t \neq 0$

$c'(t) = e^t$ $\Rightarrow c(t) = e^t + k$ $\quad k \in \mathbb{R}$

$\Rightarrow x_k(t) = (e^t + k) \cdot t$ ~înlocuim în x *

Verificare: $tx' = t((e^t + k) \cdot t)' =$

$= t(e^t \cdot t + (e^t + k) \cdot 1) = e^t t^2 + e^t t + kt$

$x + t^2 e^t = (e^t + k) t + t^2 e^t = e^t \cdot t + kt + t^2 e^t$ $\Big\}$ ✓

c) $x' + \frac{2}{t} x = t^3$ $\Rightarrow x' = -\frac{2}{t} x + t^3$

Ec. lin asoc $\bar{x}' = -\frac{2}{t}\bar{x}$ $\rightarrow \bar{x}(t) = c \cdot e^{-\int \frac{2}{t} dt} = c\, e^{-2\ln t} = ct^{-2}$ $c \in \mathbb{R}$

Caut sol de forma $x(t) = c(t) \cdot t^{-2}$

$x(\cdot)$ sol $\Rightarrow (c(t) \cdot t^{-2})' = x'(t)$

$c'(t) \cdot t^{-2} \cancel{- 2c(t) t^{-3}} = \cancel{-\frac{2}{t} \cdot c(t) \cdot t^{-2}} + t^3$

$c'(t) \cdot t^{-2} = t^3$ $\quad | \cdot t^2$

$c'(t) = t^5$ $\Rightarrow c(t) = \frac{1}{6} t^6 + k$ $\quad k \in \mathbb{R}$

$\Rightarrow x_k(t) = \left(\frac{1}{6} t^6 + k\right) \cdot t^{-2} = \frac{1}{6} t^4 + kt^{-2}$

Verificare

$$\begin{cases} x' + \frac{2}{t}x = \left(\frac{1}{6}t^4 + kt^{-2}\right)' + \frac{2}{t}\cdot\left(\frac{1}{6}t^4 + kt^{-2}\right) = \frac{2}{3}t^3 - \cancel{2kt^{-1}} + \frac{1}{3}t^3 + \cancel{2kt^{-3}} = t^3 \\ t^3 \end{cases}$$

✓

Ecuații de tip Bernoulli

2. b) $x' = -\frac{x}{t} + \frac{1}{t^2x^2} \iff x' = \underbrace{-\frac{1}{t}}_{a(t)}x + \underbrace{\frac{1}{t^2}}_{b(t)}\cdot x^{-2}$

Ec. lin asoc. $\bar{x}' = a(t)\bar{x} = -\frac{1}{t}\bar{x} \Rightarrow \bar{x}(t) = c\cdot e^{-\int\frac{1}{t}dt} = ct^{-1}$ $c\in\mathbb{R}$

Caut sol de forma $x(t) = c(t)\cdot t^{-1}$

$x(\cdot)$ sol $\Rightarrow (c(t)\cdot t^{-1})' = x'(t)$

$$c'(t) = b(t)e^{(\alpha-1)A(t)}\cdot c^\alpha(t)$$

$$c'(t) = \frac{1}{t^2}\cdot e^{-3\ln t}\cdot c^{-2}(t)$$

$$c'(t) = t^{-5}\cdot c^{-2}(t)$$

$$\frac{dc}{dt} = t^{-5}\cdot c^{-2}$$

$$dc\cdot c^2 = \frac{dt}{t^5} \to \int c^2 dc = \int t^{-5}dt \iff \frac{c^3(t)}{3} = -\frac{1}{4t^4} + k \Rightarrow$$

$$\Rightarrow c^3 = -\frac{3}{4t^4} \Rightarrow c_k(t) = -\frac{3}{4t^4} + k$$

$$x(t) = c(t)\cdot t^{-1} = \left(-\frac{3}{4t^4} + k\right)\cdot t^{-1} = -\frac{3}{4}\cdot t^{-5} + k\cdot t^{-1}$$

Verificare: $x' = \left(-\frac{3}{4}\cdot t^{-5} + kt^{-1}\right)' = -\frac{3}{4}\cdot(-5)\cdot t^{-6} - k\cdot t^{-2} =$

$$= 5\cdot\frac{3}{4}t^{-6} - k\cdot t^{-2}$$

$$= \frac{-\frac{3}{4}\cdot t^{-5} + k\cdot t^{-1}}{t} + \cancel{6}^{3}\cdot\frac{3}{\cancel{4}_2}t^{-6} - 2kt^{-2}$$

$$= \frac{x}{t} + t^{-2}\left(\frac{9}{2}t^{-4} - 2k\right)$$

$$\frac{1}{t^2x^2} = \frac{1}{t^2\left(\frac{9}{4}t^{-10} - \frac{3}{2}t^{-6} + k^2 \cdot t^{-2}\right)}$$

$$= \frac{9}{4} \cdot t^{-8} - \frac{3}{2}t^{-4} + k^2$$

c) $x' = x\cos t + x^2\cos t$

Ec. lin asoc. $\bar{x}' = \cos t \cdot \bar{x} \Rightarrow \bar{x}(t) = c \cdot e^{\int \cos t \, dt} = ce^{\sin t}$

Caut sol. de forma $x(t) = c(t)e^{\sin t}$

$x(\cdot)$ - sol $\Rightarrow c'(t) = b(t) \cdot e^{(\alpha-1)\sin t} \cdot c^{\alpha}(t)$

$$c'(t) = \cos t \cdot e^{\sin t} \cdot c^2(t)$$

$$\frac{dc}{dt} = \cos t \, e^{\sin t} \cdot c^2$$

$dc \cdot c^{-2} = \cos t \, e^{\sin t} \, dt \Rightarrow \int c^{-2} \, dc = \int \cos t \, e^{\sin t} \, dt$

$\Rightarrow -c^{-1} = e^{\sin t} + k \Rightarrow c_k(t) = \left(-e^{\sin t} - k\right)^{-1}$

$x(t) = c(t) \cdot e^{\sin t} \Rightarrow x_k(t) = \left(-e^{\sin t} - k\right)^{-1}.$

$$\begin{cases} x_k'(t) = \dfrac{e^{\sin t}\cos t}{(e^{\sin t}+k)^2} \\ x\cos t + x^2\cos t = \dfrac{\cos t}{-e^{\sin t} - k} + \dfrac{\cos t}{(e^{\sin t}+k)} = \dfrac{e^{\sin t}\cos t + (k+1)\cos t}{(e^{\sin t}+k)^2} \end{cases}$$

Problemă Cauchy:

$x'' = x' \ln x' \qquad x(0) = 0 \qquad x'(0) = 1$

S.V. $y = x' \Rightarrow y' = y \ln y$

$\frac{dy}{dt} = y \ln y \Rightarrow \frac{1}{y \ln y} dy = dt \Rightarrow \int \frac{1}{y \ln y} dy = \int dt \Rightarrow$

$\Rightarrow \ln(\ln y) = t + c \Rightarrow \ln y = e^{t+c} \Rightarrow y = e^{e^{t+c}} = e^{e^t \cdot e^c} = e^{k \cdot e^t}$

Revenim la S.V: $e^{ke^t} = x'$

Știm că $\begin{cases} x(0) = 0 \\ x'(0) = 1 \end{cases}$ $\Bigg\}$ $\Rightarrow e^{k \cdot e^0} = 1 \Rightarrow e^k = 1 \Rightarrow k = 0$

$\Rightarrow x' = 1 \Rightarrow x$ de forma $t + c$

$\left.\begin{matrix} x = t + c \\ x(0) = 0 \end{matrix}\right\} \Rightarrow 0 + c = 0 \Rightarrow c = 0 \Rightarrow x(t) = t$

Local Lipschitz în rap. cu a doua variabilă

$x' = 2e^{-t} + \ln(1 + x^2)$

$f(t,x) - f(t,y) = \cancel{2e^{-t}} + \ln(1 + x^2) - \cancel{2e^{-t}} - \ln(1 + y^2) = \ln \frac{1+x^2}{1+y^2}$

Vreau $\ln \frac{1+x^2}{1+y^2} \leq L(x-y)$

$$\lim_{y \to x} \frac{\ln(1+x^2) - \ln(1+y^2)}{x-y} \leq L$$

$$\frac{2x}{1+x^2} \leq L$$

$\Rightarrow$ Th. suprem $\Rightarrow \leq 1$

Pb. sol. maximală Cauchy ~ easy version

$f(\cdot, \tau) : I(\tau) \subset \mathbb{R} \to \mathbb{R} \quad \tau \in \mathbb{R}$

$x' = x^2 - x\cos t + \sin t + 1 \qquad x(\tau) = 0$

$D_2 \varphi(t, 0) = ?$

Fie $\alpha_f(\cdot,\cdot,\cdot) : D_f \to \mathbb{R}$ curentul maximal C^1

$\varphi(t, \tau) = \alpha_f(t, \tau, 0) \quad \Big| \frac{\partial}{\partial \tau}$

$x'' = 2x - \cos t$

$\tau = 0 \longrightarrow D_2 \varphi(t, 0) = D_2 \alpha_f(t, 0, 0)$ am luat $\tau = 0$ mai sus

$t \to D_2 \alpha_f(t, 0, 0)$ sol: $\underbrace{y' = D_2 f(t, \alpha_f(t,0,0))\, y \qquad y(0) = -f(0,0)}_{\text{din teorie}}$

$$\left.\begin{aligned} &D_2 f(t, x) = 2x - \cos t \\ &\alpha_f(t, 0, 0) = \varphi(t, 0) = \operatorname{tg} t \end{aligned}\right\} \Rightarrow$$

$\Rightarrow y' = (2\operatorname{tg} t - \cos t)\, y \qquad y(0) = -f(0,0) = -(0 - 0\cdot\cos 0 + \sin 0 + 1) = -1$

$y(t) = c\, e^{\int (2\operatorname{tg} t - \cos t)\, dt} = c\, e^{-2\ln|\cos t| - \sin t}$

$$\left.\begin{aligned} &\Rightarrow y(t) = c \cdot \frac{1}{\cos^2 t}\, e^{-\sin t} \\ &y(0) = -1 \end{aligned}\right\} \Rightarrow c \cdot \frac{1}{\cos^2 0}\, e^{-\sin 0} = -1$$

$c \cdot \frac{1}{1} \cdot e^0 = -1 \Rightarrow c = -1$

$\Rightarrow y(t) = -\frac{e^{-\sin t}}{\cos^2 t} = -\frac{1}{\cos^2 t\, e^{\sin t}}$

Pb. sol maximală Cauchy ~ exam model

$f(\cdot,\lambda): I(\lambda) \subseteq \mathbb{R} \to \mathbb{R}, \lambda \in \mathbb{R}$

$x' = x^2 + \frac{x}{t} + \lambda(x^2+1) \qquad x(\lambda+1) = \lambda - 1$.

$D_2 \varphi(t,0) = ?$

$f(t,x,\lambda) = x^2 + \frac{x}{t} + \lambda(x^2+1) \qquad f(\cdot,\cdot,\cdot): \underbrace{\mathbb{R}^* \times \mathbb{R} \times \mathbb{R}}_{\text{deschisă}} \to \mathbb{R} \quad f \text{ cont } C^1$

Fie $\alpha_f(\cdot,\cdot,\cdot,\cdot): D_f \to \mathbb{R}$ curentul maximal parametrizat C^1

$\varphi(t,\lambda) = \alpha_f(t, \lambda+1, \lambda-1, \lambda) \quad \Big| \frac{\partial}{\partial\lambda} \Rightarrow$

$\begin{cases} x' = f(t,x,\lambda) \\ x(\underbrace{\lambda+1}_{\tau}) = \underbrace{\lambda-1}_{\xi} \end{cases}$ ~ din ip .

$\Rightarrow D_2\varphi(t,\lambda) = D_2\alpha_f(t,\lambda+1,\lambda-1,\lambda) + D_3\alpha_f(t,\lambda+1,\lambda-1,\lambda) + D_4\alpha_f(t,\lambda+1,\lambda-1,\lambda)$

Noi vrem $\varphi(t,0)$ deci luăm $\lambda = 0$

$D_2\varphi(t,0) = D_2\alpha_f(t,1,-1,0) + D_3\alpha_f(t,1,-1,0) + D_4\alpha_f(t,1,-1,0)$

- $t \mapsto D_2\alpha_f(t,-1,1,0)$ sol $\begin{cases} y' = D_2 f(t, \alpha_f(t,1,-1,0), 0)y \\ y(1) = -f(1,-1,0) = -(1-1+0) = 0 \end{cases}$
- $t \mapsto D_3\alpha_f(t,-1,1,0)$ sol matriceală $\begin{cases} y' = D_2 f(t, \alpha_f(t,1,-1,0), 0) \\ y(1) = I_4 \end{cases}$
- $t \to D_4\alpha_f(t,-1,1,0)$ sol: $\begin{cases} z' = D_2 f(t,\alpha_f(t,1,-1,0),0)z + D_3 f(t,\alpha_f(t,1,-1,0),0) \\ z(1) = 0 \end{cases}$

$D_2 f(t,x,\lambda) = 2x + \frac{1}{t} + 2x\lambda = 2x(\lambda+1) + \frac{1}{t} \qquad \frac{\partial f}{\partial x}$

$D_3 f(t,x,\lambda) = x^2+1 \qquad \frac{\partial f}{\partial \lambda}$

$\alpha_f(t,1,-1,0) = ?$ înlocuim în ec inițială cu $\lambda = 0$

$x' = x^2 + \frac{x}{t} + 0(x^2+1) = x^2 + \frac{x}{t}$ ec. Bernoulli

$$\left.\begin{aligned} &x(\lambda+1)=\lambda-1\\ &\lambda=0\end{aligned}\right\} \Rightarrow x(1)=-1$$

Băgăm la calculator $x(t)=-\dfrac{2t}{t^2+1}=\alpha_f(t,1,-1,0)$

- $$\left.\begin{aligned} &y'=D_2f(t,\alpha_f(t,1,-1,0),0)y=D_2f\left(t,-\frac{2t}{t^2+1},0\right)y\\ &D_2f(t,x,\lambda)=2x(\lambda+1)+\frac{1}{t}\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow y'=\left(\frac{-4t}{t^2+1}(0+1)+\frac{1}{t}\right)\cdot y=\frac{t^2-4t^2+1}{t(t^2+1)}\cdot y=\frac{1-3t^2}{t(t^2+1)}\cdot y$$ ec. lin. scalară

$$y(t)=c\cdot e^{\int\frac{1-3t^2}{t^3+t}dt}=c\cdot e^{\ln t-2\ln(t^2+1)}=c\cdot\frac{e^{\ln t}}{e^{2\ln(t^2+1)}}=c\cdot\frac{t}{(t^2+1)^2}\quad c\in\mathbb{R}$$

$$y(1)=0\Rightarrow c\cdot\frac{1}{4}=0\Rightarrow c=0\Rightarrow y(t)\equiv 0$$

- $$\left.\begin{aligned} &y'(t)=c\cdot\frac{t}{(t^2+1)^2}\\ &y(1)=1\end{aligned}\right\}\Rightarrow\frac{c}{4}=1\Rightarrow c=4\Rightarrow y(t)=\frac{4t}{(t^2+1)^2}$$

- $$\left.\begin{aligned} &D_3f(t,\alpha_f(t,1,-1,0),0)=D_3f\left(t,\frac{-2t}{t^2+1},0\right)\\ &D_3f(t,x,\lambda)=x^2+1\end{aligned}\right\}=\frac{4t^2}{(t^2+1)^2}+1$$

$$\left.\begin{aligned} &z'=\frac{1-3t^2}{t(t^2+1)}\cdot z+\frac{4t^2}{(t^2+1)^2}+1\\ &z(1)=0\end{aligned}\right\}\overset{\text{Wolfram}}{\Rightarrow} z(t)=\frac{t(t^4+12t^2+4\ln t-13)}{4(t^2+1)^2}$$

$$D_2\varphi(t,0)=D_2\alpha_f(t,1,-1,0)+D_3\alpha_f(t,1,-1,0)+D_4\alpha_f(t,1,-1,0)$$

$$=0+\frac{4t}{(t^2+1)^2}+\frac{t(t^4+12t^2+4\ln t-13)}{4(t^2+1)^2}=$$

$$=\frac{t(16+t^4+12t^2+4\ln t-13)}{4(t^2+1)^2}$$

$$=\frac{t(t^4+12t^2+4\ln t+3)}{4(t^2+1)^2}$$

Pb sol. maximală Cauchy

$x' = x^3 + tx \qquad x(0) = \xi$

$D_2 \varphi(t, 0) = ?$

$\varphi(\cdot, \xi) : I(\xi) \subset \mathbb{R} \to \mathbb{R}$ sol max a pb

$f(t, x) = x^3 + tx \qquad f(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

Fie $\alpha_f(\cdot,\cdot,\cdot) : D_f \to \mathbb{R}$ curentul maximal parametrizat

$\varphi(t, \xi) = \alpha_f(t, 0, \xi) \qquad \Big| \frac{\partial}{\partial \xi}$

$D_2 \varphi(t, \xi) = D_3 \alpha_f(t, 0, \xi)$

$\xi = 0 \Rightarrow D_2 \varphi(t, 0) = D_3 \alpha_f(t, 0, 0)$

$D_3 \alpha_f(t, 0, 0)$ sol a ec $\Rightarrow \begin{cases} y' = D_2 f(t, \alpha_f(t, 0, 0)) \cdot y \\ y(0) = 1 \end{cases}$

$D_2 f(t, x) = 3x^2 + t$

$\alpha_f(t, 0, 0) = ?$

$x' = x^3 + tx \qquad x(0) = 0$ ec. Bernoulli

Ec. lin asoc. $\quad \bar{x}' = t\bar{x} \Rightarrow \bar{x}(t) = c e^{\frac{t^2}{2}}$

$\left.\begin{array}{l} x(t) = c(t) e^{\frac{t^2}{2}} \\ x(0) = 0 \end{array}\right\} \Rightarrow c(0) = 0 \Rightarrow$

$x(\cdot)$ sol $\Rightarrow \left(c(t) \cdot e^{\frac{t^2}{2}}\right)' = x'(t) \iff$

$\iff c'(t) \cdot e^{\frac{t^2}{2}} + \cancel{c(t) e^{\frac{t^2}{2}}} = c(t)^3 e^{\frac{3t^2}{2}} + \cancel{t c(t) e^{\frac{t^2}{2}}}$

$\Rightarrow c'(t) = c^3(t) e^{t^2}$

$\frac{dc}{dt} = c^3 e^{t^2}$

I. $c^3 \equiv 0 \Rightarrow c(t) \equiv 0 \Rightarrow x(t) \equiv 0$ Ok

II. $\frac{dc}{c^3} = dt\, e^{t^2} \Rightarrow \int \frac{1}{c^3} dc = \int e^{t^2} dt \Rightarrow -\frac{1}{c^2} = \int e^{t^2} dt + k$ ~~abs~~

$\Rightarrow \alpha_f(t, 0, 0) \equiv 0 \Rightarrow \begin{cases} y' = ty \\ y(0) = 1 \end{cases} \Rightarrow \frac{dy}{dt} = ty$

I $y \equiv 0$ ~~sol~~

II $\frac{dy}{y} = t\,dt \Rightarrow \ln y = \frac{t^2}{2} + k \Rightarrow y = e^{\frac{t^2}{2} + k}$

$y(0) = 1 \Rightarrow e^k = 1 \Rightarrow k = 0 \Rightarrow y(t) = e^{\frac{t^2}{2}}$

$\Rightarrow D_2 \varphi(t,0) = e^{\frac{t^2}{2}}$

Pb sol. maximală Cauchy

$x' = x^2 + \lambda t x^3 - x \qquad x(0) = \lambda$

$D_2 \varphi(t,0) = ?$

$f(t,x,\lambda) = x^2 + \lambda t x^3 - x$

$\alpha_f(\cdot,\cdot,\cdot)$ curent maximal param

$\varphi(t,\lambda) = \alpha_f(t,0,(\lambda,\lambda))$

$\lambda = 0 \Rightarrow D_2 \varphi(t,0) = D_3 f(t,0,(0,0)) + D_4 \alpha_f(t,0,(0,0))$

$$\begin{cases} y' = D_2 f(t, \alpha_f(t,0,(0,0)),0)\, y \\ y(0) = -f(0,0,0) = 0 \end{cases}$$

$D_2 f(t,x,\lambda) = 2x + 3\lambda x^2 - 1$

$D_3 f(t,x,\lambda) = t x^3$

$\alpha_f(t,0,(0,0)) = ? \qquad x' = x^2 - x \quad x(0) = 1$

$\bar{x}' = -\bar{x}$, sol de forma $x(t) = c(t) \cdot e^{\int -1 dt} = c(t) e^{-t}$

$x(t) = c(t) e^{-t} \qquad |(\,)'$

$c'(t) e^{-t} \cancel{- c(t) e^{-t}} = c^2(t) e^{-2t} \cancel{- c(t) e^{-t}}$

$\Rightarrow c'(t) = c^2(t) e^{-t}$

$\frac{dc}{dt} = c^2 e^{-t} \Leftrightarrow \int c^{-2} dc = \int e^{-t} dt \Leftrightarrow -\frac{1}{c} = -e^{-t} + k$

$\Rightarrow c(t) = \frac{1}{e^{-t} + k} \Rightarrow x(t) = \frac{e^{-t}}{e^{-t} - k}$

$x(0) = 1 \Rightarrow k = 0 \qquad x(t) = \alpha_f(t, 0, (0,0)) = 1$

$$\begin{cases} y' = D_2 f(t,1,0) = (2\cdot 1 + 3\cdot 0\cdot t\cdot 1 - 1)y = y \Rightarrow y(t) = c_1 e^t \\ y(0) = 0 \quad \Rightarrow y(t) = 0 \end{cases}$$

$$\begin{cases} z' = D_2 f(t,1,0)\cdot z + D_3 f(t,1,0) = z + t \Rightarrow z(t) = c_1 e^t - t - 1 \\ z(0) = 0 \quad \Rightarrow c_1 = 1 \Rightarrow z(t) = e^t - t - 1 \end{cases}$$

$\Rightarrow D_2 \varphi(t, 0) = y(t) + z(t) = 0 + e^t - t - 1 = e^t - t - 1$

Pb. sol. maximală Cauchy

$x' = x^2 + \lambda t x - \lambda \quad x(0) = 0$

$D_2 \varphi(t, 0) = ?$

$f(t, x, \lambda) = x^2 + \lambda t x - \lambda$

$\varphi(t, \lambda) = \alpha_f(t, 0, \lambda, \lambda)$ ~ curentul maximal

$\lambda = 0 \Rightarrow \varphi(t, 0) = \alpha_f(t, 0, 0, 0)$

$$\begin{cases} y' = D_2 f(t, \alpha_f(t, 0, 0, 0), 0) \cdot y \\ y(0) = -f(0, 0, 0) = 0 \end{cases}$$

$$\begin{cases} z' = D_2 f(t, \alpha_f(t, 0, 0, 0), 0) \cdot z + D_3 f(t, \alpha_f(t, 0, 0, 0), 0) \\ z(0) = 0 \end{cases}$$

$D_2 f(t, x, \lambda) = 2x + \lambda t$ ~ deriv f după x

$D_3 f(t, x, \lambda) = tx - 1$ ~ deriv f după λ

$\alpha_f(t, 0, 0, 0) = ?$

înlocuim $\lambda = 0$ în ec: $x' = x^2 + 0tx + 0 \Rightarrow x' = x^2$

Wolfram $\Rightarrow x(t) = \dfrac{1}{c - t}$

$x(0) = 0 \Rightarrow \dfrac{1}{c} = 0 \quad \cancel{\alpha} \quad \Rightarrow x(t) \equiv 0 \Rightarrow \alpha_f(t, 0, 0, 0) = 0$ (*)

$$\begin{cases} y' = D_2 f(t, \overset{(*)}{\underset{x}{0}}, \underset{\lambda}{0}) \cdot y = 2 \cdot \underset{x}{0} + \underset{\lambda}{0} \cdot t = 0 \\ y(0) = 0 \end{cases} \Rightarrow y(t) = 0$$

$$\begin{cases} z' = D_2 f(t, 0, 0) \cdot z + D_3 f(t, 0, 0) = (2 \cdot 0 + 0 \cdot t) z + t \cdot 0 - 1 = -1 \\ z(0) = 0 \end{cases}$$

$\Rightarrow z(t) = -t$

$D_2 \varphi(t, 0) = y(t) + z(t) = 0 - t = -t$

Pb. parametrizare

$$(p-q)(x-y)+\frac{p^2}{2}-z=0, \quad y=1, z=\frac{(x-1)^2}{2}$$

este:

$$x(t,\sigma)=1+(\sigma-1)e^{2t},\ y(t,\sigma)=1+(\sigma-1)e^{t}-(\sigma-1)e^{2t},\ z(t,\sigma)=\frac{(\sigma-1)^2e^{2t}}{2}$$

$p=\frac{\partial z}{\partial x}(x,y)$

$q=\frac{\partial z}{\partial y}(x,y)$

$F((x,y),z,(p,q))=(p-q)(x-y)+\frac{p^2}{2}-z=0$

$\begin{cases} x=\sigma \\ y=1 \\ z=\frac{(\sigma-1)^2}{2} \end{cases}$ $\quad \alpha(\sigma)=\begin{pmatrix}\sigma\\1\end{pmatrix}$ $\quad \beta(\sigma)=\frac{(\sigma-1)^2}{2}$

$\begin{cases} F(\alpha(\sigma),\beta(\sigma),(p,q))\equiv 0 \\ (p,q)\,\Delta\alpha(\sigma)\equiv\Delta\beta(\sigma) \end{cases}$

$\begin{cases} (p-q)(\sigma-1)+\frac{p^2}{2}-\frac{(\sigma-1)^2}{2}=0 \\ (p,q)\cdot\begin{pmatrix}1\\0\end{pmatrix}=\left(\frac{(\sigma-1)^2}{2}\right)'=\left(\frac{\sigma^2-2\sigma+1}{2}\right)'=\sigma-1 \Rightarrow p=\sigma-1 \end{cases}$

înlocuiesc $p=\sigma-1$ în ec 1:

$p(\sigma-1)-q(\sigma-1)+\cancel{\frac{(\sigma-1)^2}{2}}-\cancel{\frac{(\sigma-1)^2}{2}}=0$

$(\sigma-1)^2-q(\sigma-1)=0$

$q=\sigma-1$

$\gamma(\sigma)=\begin{pmatrix}\sigma-1\\\sigma-1\end{pmatrix}$

$$\begin{cases} x' = \dfrac{\partial F}{\partial p} = x - y + p \\ y' = \dfrac{\partial F}{\partial q} = y - x \\ z' = p \cdot x' + q \cdot y' = px - py + p^2 + qy - qx \\ \qquad = (p-q)x - (p-q)y + p^2 \\ \qquad = (p-q)(x-y) + p^2 \\ p' = -\dfrac{\partial F}{\partial x} - p\dfrac{\partial F}{\partial z} = q - p + p = q \\ q' = -\dfrac{\partial F}{\partial y} - q\dfrac{\partial F}{\partial z} = p - q + q = p \end{cases}$$

$x(0) = \sigma$

$y(0) = 1$

$\alpha(\sigma) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$

$z(0) = \dfrac{(\sigma-1)^2}{2} = \beta(\sigma)$

$\left.\begin{aligned} p(0) &= \sigma - 1 \\ q(0) &= \sigma - 1 \end{aligned}\right\} \delta(\sigma)$

$$\begin{cases} x' = x - y + p & x(0) = \sigma \\ y' = y - x & y(0) = 1 \\ z' = (p-q)(x-y) + p^2 & z(0) = \dfrac{(\sigma-1)^2}{2} \\ p' = q & p(0) = \sigma - 1 \\ q' = p & q(0) = \sigma - 1 \end{cases}$$

$\left.\begin{aligned} p' &= q \\ q' &= p \end{aligned}\right\} \Rightarrow q'' = q \Rightarrow q'' - q = 0 \Rightarrow q(t) = c_1 e^{t} + c_2 e^{-t}$

$q(0) = \sigma - 1 \Rightarrow c_1 e^0 + c_2 e^0 = \sigma - 1 \Rightarrow c_1 + c_2 = \sigma - 1$

$p(t) = q'(t) = c_1 e^{t} - c_2 e^{-t}$

$p(0) = \sigma - 1 \Rightarrow c_1 e^0 - c_2 e^0 = \sigma - 1 \Rightarrow c_1 - c_2 = \sigma - 1 \quad (+)$

$2c_1 = 2\sigma - 2 \Rightarrow c_1 = \sigma - 1 \Rightarrow c_2 = 0$

$$\Rightarrow \begin{cases} p(t) = (\sigma - 1)e^{t} \\ q(t) = (\sigma - 1)e^{t} \end{cases}$$

$$x' = x - y + (\sigma - 1)e^t$$
$$y' = y - x$$

$$x' + y' = (\sigma - 1)e^t \qquad (+) \qquad | \int$$

$$x + y = (\sigma - 1)e^t + k$$

$$x(t) + y(t) = (\sigma - 1)e^t + k$$

$$t = 0 \Rightarrow x(0) + y(0) = (\sigma - 1)e^0 + k$$

$$\sigma + 1 = \sigma - 1 + k \Rightarrow k = 2$$

$$x(t) = (\sigma - 1)e^t - y(t) + 2$$

$$\begin{aligned} x' &= x - y + (\sigma - 1)e^t \\ &= x - y + (\sigma - 1)e^t + 2 - 2 \\ &= x + x - 2 \\ &= 2x - 2 \end{aligned}$$

$$\left.\begin{aligned} &x' = 2x - 2 \Rightarrow x = ce^{2t} + 1 \\ &x(0) = \sigma \end{aligned}\right\} \Rightarrow c + 1 = \sigma \Rightarrow c = \sigma - 1$$

$$\Rightarrow x(t) = (\sigma - 1)e^{2t} + 1$$

$$\begin{aligned} y(t) &= (\sigma - 1)e^t + 2 - (\sigma - 1)e^{2t} - 1 \\ &= (\sigma - 1)e^t - (\sigma - 1)e^{2t} + 1 \end{aligned}$$

$$\begin{aligned} z' &= (p - q)(x - y) + p^2 \\ &= \underbrace{\left((\sigma - 1)e^t - (\sigma - 1)e^t\right)}_{0}\left((\sigma - 1)e^{2t} + 1 - (\sigma - 1)e^t + (\sigma - 1)e^{2t} - 1\right) + (\sigma - 1)^2 e^{2t} \\ &= (\sigma - 1)^2 e^{2t} \end{aligned}$$

Pb parametrizare

$$2xy - pq - z = 0, \quad x = 1, z = y$$

este:

$$x(t,\sigma) = e^{-t},\ y(t,\sigma) = \sigma e^{-t},\ z(t,\sigma) = \sigma e^{-2t}$$

$p = \frac{\partial z}{\partial x}(x,y)$

$q = \frac{\partial z}{\partial y}(x,y)$

$F((x,y), z, (p,q)) = 2xy - pq - z = 0$

$$\begin{cases} x = 1 \\ y = \sigma \\ z = y = \sigma \end{cases} \quad \alpha(\sigma) = \begin{pmatrix} 1 \\ \sigma \end{pmatrix}, \quad \beta(\sigma) = \sigma$$

$$\begin{cases} F(\alpha(\sigma), \beta(\sigma), (p,q)) = 0 \\ (p,q) \Delta \alpha(\sigma) = \Delta \beta(\sigma) \end{cases} \Rightarrow \begin{cases} 2 \cdot 1 \cdot \sigma - pq - \sigma = 0 \Leftrightarrow \sigma = pq \\ (p,q) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \Leftrightarrow q = 1 \end{cases}$$

înlocuim $q = 1$ în ec 1:

$\sigma = p$

$\Rightarrow \gamma(\sigma) = \begin{pmatrix} \sigma \\ 1 \end{pmatrix}$ ez. gg.

$$\begin{cases} x' = \frac{\partial F}{\partial p} = -q & x(0) = 1 \\ y' = \frac{\partial F}{\partial q} = -p & y(0) = \sigma \\ z' = p \cdot x' + q \cdot y' = -pq - qp = -2pq & z(0) = \sigma \\ p' = -\frac{\partial F}{\partial x} - p \frac{\partial F}{\partial z} = -2y + p = p - 2y & p(0) = \sigma \\ q' = -\frac{\partial F}{\partial y} - q \cdot \frac{\partial F}{\partial z} = -2x + q = q - 2x & q(0) = 1 \end{cases}$$

$x' = -q \quad | ( )' \Rightarrow \quad x'' = -q'$

$q' = q - 2x$

$\Rightarrow -x'' = -x' - 2x$

$x'' = x' + 2x$

$\Rightarrow x(t) = c_1 e^{-t} + c_2 e^{2t}$

$x(0) = 1 \Rightarrow c_1 + c_2 = 1$

$q = -x' = c_1 e^{-t} - 2c_2 e^{2t}$

$q(0) = 1 \Rightarrow c_1 - 2c_2 = 1$

$\overset{(-)\uparrow}{\Rightarrow} -3c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 1$

$\Rightarrow \begin{cases} x(t) = e^{-t} \\ q(t) = e^{-t} \end{cases}$

$y' = -p \quad \overset{( )'}{\Rightarrow} y'' = -p'$

$p' = p - 2y$

$\Rightarrow y'' = y' + 2y$

$y(t) = c_1 e^{-t} + c_2 e^{2t}$

$y(0) = \sigma \Rightarrow c_1 + c_2 = \sigma$

$p = -y' = c_1 e^{-t} - 2c_2 e^{2t}$

$p(0) = \sigma \Rightarrow c_1 - 2c_2 = \sigma$

$\overset{(-)\uparrow}{\Rightarrow} -3c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = \sigma$

$\begin{cases} y(t) = \sigma e^{-t} \\ p(t) = \sigma e^{-t} \end{cases}$

$z' = -2pq = -2 \cdot \sigma e^{-t} \cdot e^{-t} = -2\sigma e^{-2t} \quad | \int$

$z = \sigma e^{-2t}$