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Examen_Analiza...

15-06-2020 NUME ....GRUPA.........

EXAMEN LA ANALIZA MATEMATICA II

I. Sa se determine punctele de extrem local ale functiei $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x, y, z) = z^{12} - 12yz + 6y^2 + x^2 - 2x$$

si sa se precizeze natura lor.

II. Fie functia $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{x^2 y}{\sqrt{x^4 + y^{10}}} & \text{daca } (x,y) \neq (0,0) \\ 0 & \text{daca } (x,y) = (0,0) \end{cases}$$

Sa se calculeze derivatele partiale de ordinul intai ale functiei f si sa se studieze diferentiabilitatea functiei f.

III. Calculati integrala

$$\iint_D (y - 2xz)\,dx\,dy$$

unde D este multimea marginita de laturile triunghiului ABC cu $A(1,2)$, $B(3,-1)$ si $C(5,3)$.

IV. Calculati integrala

$$\iiint_V (z+2)\,dx\,dy\,dz$$

unde V este multimea marginita de planele $z = 1$, $z = 3$ si paraboloidul

$$\frac{x^2}{25} + \frac{y^2}{9} = z.$$

V. Studiati integrabilitatea Riemann a functiei $f : [0,4] \times [0,3] \times [0,2] \to \mathbb{R}$,

$$f(x,y,z) = \begin{cases} 2 & \text{daca } x = 1,\ y \in [0,3] \setminus \mathbb{Q} \\ 5 & \text{daca } x = 3,\ z \in [0,2] \cap \mathbb{Q} \\ 2x - y^2 & \text{altfel} \end{cases}$$

si in cazul in care este integrabila calculati

$$\iiint_V f(x,y,z)\,dx\,dy\,dz, \quad V = [0,4] \times [0,3] \times [0,2].$$

Nota. Timpul de lucru este de 2 ore. La subiectele **III** si **IV** nu trebuie sa justificati ca multimea pe care trebuie calculata integrala este masurabila Jordan si ca functiile sunt integrabile Riemann.

Fiecare subiect se noteaza cu note de la 1 la 10. Nota obtinuta la aceasta lucrare este media aritmetica a celor 5 note.

Rezolvarile trebuie scanate si trimise impreuna cu lista de subiecte sub forma unui **singur** fisier pdf.

Examen analiză matematică II

I $f: \mathbb{R}^3 \to \mathbb{R}$

$$f(x,y,z) = z^{12} - 12yz + 6y^2 + x^2 - 2x$$

Soluție

$\mathbb{R}^3$ este o mulțime deschisă.

f este o funcție de clasă C^2.

Căutăm punctele critice ale funcției f.

$$\frac{\partial f}{\partial x}(x,y,z) = 2x - 2 = 0 \Rightarrow 2x = 2 \Rightarrow x = 1$$

$$\frac{\partial f}{\partial z}(x,y,z) = 12z^{11} - 12y = 0 \Rightarrow 12(z^{11} - y) = 0 \Rightarrow z^{11} = y.$$

$$\frac{\partial f}{\partial y}(x,y,z) = -12z + 12y = 0 \Rightarrow z = y.$$

$z^{11} = y$ și $z = y \Rightarrow z^{11} = z \Rightarrow z = 0. \Rightarrow y = 0. \Rightarrow$

$\exists$ un punct critic al lui f $(1,0,0)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2. \qquad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = +12 \qquad \frac{\partial^2 f}{\partial x \partial z} = 0.$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = 12 \cdot 11 \cdot z^{10} \qquad \frac{\partial^2 f}{\partial y \partial z} = -12.$$

~~$H_f(1,0,0) = \left(\frac{\partial^2 f}{\partial x^2}(1,0,0) \; \frac{\partial^2 f}{\partial x \partial y}(1,0,0) \right.$~~

$$H_f(1,0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(1,0,0) & \frac{\partial^2 f}{\partial x \partial y}(1,0,0) & \frac{\partial^2 f}{\partial x \partial z}(1,0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(1,0,0) & \frac{\partial^2 f}{\partial y^2}(1,0,0) & \frac{\partial^2 f}{\partial y \partial z}(1,0,0) \\ \frac{\partial^2 f}{\partial z \partial x}(1,0,0) & \frac{\partial^2 f}{\partial z \partial y}(1,0,0) & \frac{\partial^2 f}{\partial z^2}(1,0,0) \end{pmatrix} \Rightarrow$$

$$H_f(1,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & -12 \\ 0 & -12 & 12 \cdot 11 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & -12 \\ 0 & -12 & 0 \end{pmatrix}$$

Observăm că $\Delta_1 = 2 > 0$.

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 12 \end{vmatrix} = 2 \cdot 12 - 0 = 24 > 0.$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 12 & -12 \\ 0 & -12 & 0 \end{vmatrix} = 2 \cdot \begin{vmatrix} 12 & -12 \\ -12 & 0 \end{vmatrix} = 2 \cdot (12 \cdot 0 - 12 \cdot 12) < 0$$

Deci $(1,0,0)$ nu este punct de extrem.

II $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{\sqrt{x^4+y^{10}}}, & \text{dacă } (x,y) \neq (0,0) \\ 0, & \text{dacă } (x,y) = (0,0) \end{cases}$$

Pentru $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy\sqrt{x^4+y^{10}} - \frac{1}{2\sqrt{x^4+y^{10}}} \cdot 4x^3 \cdot x^2 y}{x^4+y^{10}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^2 \cdot \sqrt{x^4+y^{10}} - \frac{1}{2\sqrt{x^4+y^{10}}} \cdot 10y^9 \cdot x^2 y}{x^4+y^{10}}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x-0} = \lim_{x \to 0} \frac{0-0}{x} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y-0} = \lim_{y \to 0} \frac{0-0}{y} = 0.$$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2xy\sqrt{x^4+y^{10}} - \frac{4x^5 y}{2\sqrt{x^4+y^{10}}}}{x^4+y^{10}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^2\sqrt{x^4+y^{10}} - \frac{10y^{10} x^2}{2\sqrt{x^4+y^{10}}}}{x^4+y^{10}}, & (x,y) \neq (0,0) \\ x^4+y^{10}, & (x,y) = (0,0) \end{cases}$$

Pe $\underbrace{\mathbb{R}^2 \setminus \{(0,0)\}}_{\text{deschis}}$ funcția f este ~~integrabilă~~ diferențiabilă deoarece pe $\mathbb{R}^2 \setminus \{(0,0)\}$ există derivate parțiale și sunt continue.

Dacă f ar fi diferențiabilă în punctul (0,0) atunci $\exists\, df_{(0)}: \mathbb{R}^2 \to \mathbb{R}$.

$$df(0,0)(u,v) = \left(\frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(0,0) \right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$= 0.$

Ar trebui ca $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - f(0,0) - df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} = 0.$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^2y}{\sqrt{x^4+y^{10}}} - 0 - 0}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^4+y^{10}} \cdot \sqrt{x^2+y^2}}$$

Fie $(x_n; y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ $\forall n \in \mathbb{N}$

Avem $\lim\limits_{n\to+\infty} (x_n, y_n) = (0,0)$

$$\lim_{n\to+\infty} \frac{x_n^2 y_n}{\sqrt{x_n^4+y_n^{10}}\sqrt{x_n^2+y_n^2}} = \lim_{n\to+\infty} \frac{\frac{1}{n^3}}{\sqrt{\frac{1}{n^4}+\frac{1}{n^{10}}}\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} =$$

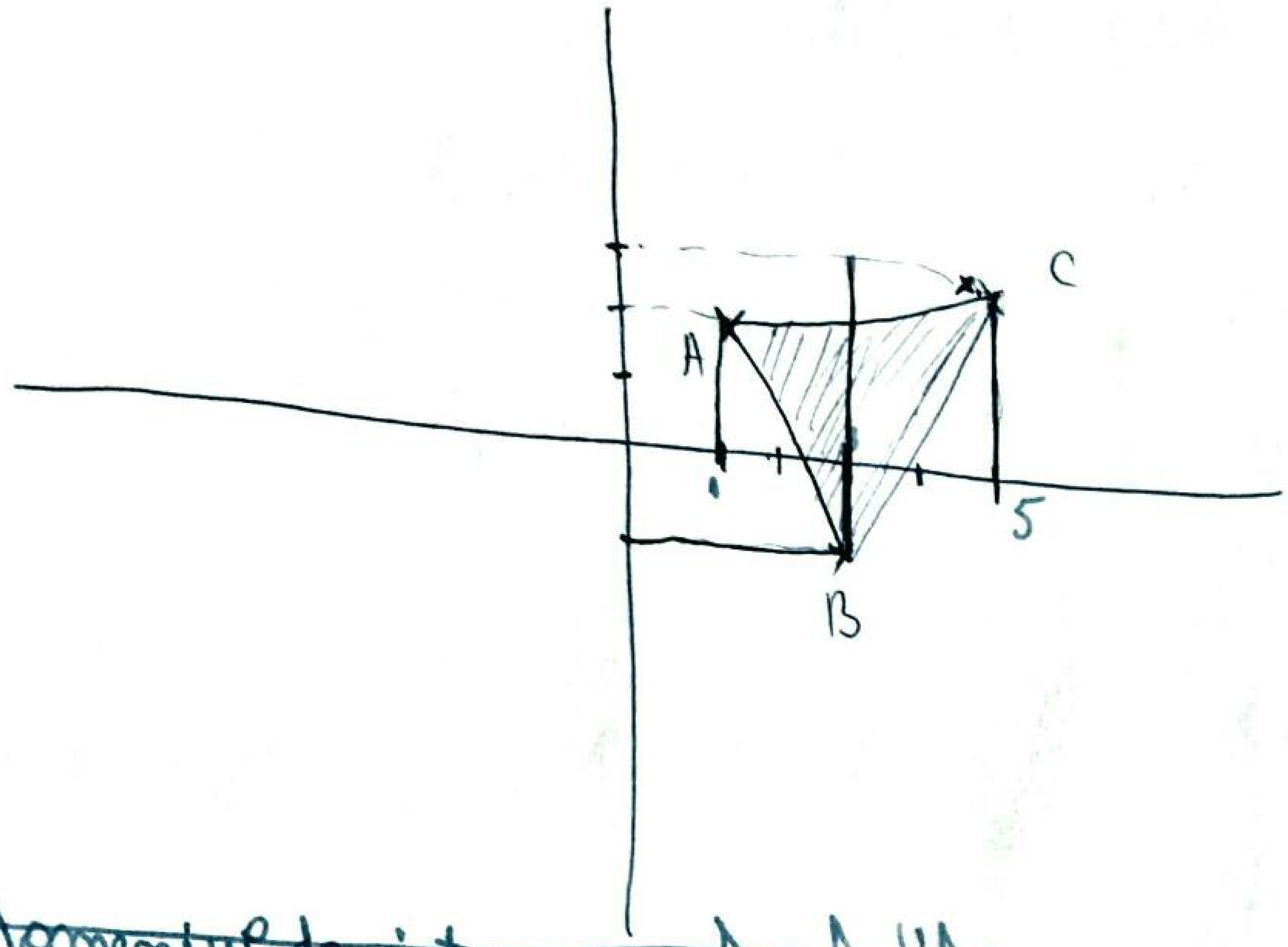
$$= \lim_{n\to+\infty} \frac{\frac{1}{n^3}}{\sqrt{\frac{1}{n^4}\left(1+\frac{1}{n^6}\right)} \cdot \sqrt{\frac{2}{n^2}}} = \lim_{n\to+\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^2} \cdot \sqrt{1+\frac{1}{n^6}} \cdot \frac{\sqrt{2}}{n}} = \lim_{n\to+\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^3} \cdot \sqrt{1+\frac{1}{n^6}} \cdot \sqrt{2}}$$

$= \lim\limits_{n\to+\infty} \dfrac{1}{\cancel{n^3}} \cdot \dfrac{\cancel{n^3}}{\sqrt{1+\frac{1}{n^6}} \cdot \sqrt{2}} = \dfrac{1}{\sqrt{2} \cdot \sqrt{1}} = \dfrac{1}{\sqrt{2}} \neq 0.$, deci f nu e diferențiabilă în (0,0).

III

$$\iint_D (y - 2xz)\,dx\,dy$$

$A(1,2)$ $B(3,-1)$ $C(5,3)$



~~Domeniul de integrare $D = D_1 \cup D_2$.~~

~~$D_1 = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 3,\ \le y$~~

AB: $\frac{x-1}{3-1} = \frac{y-2}{-1-2} \Leftrightarrow \frac{x-1}{2} = \frac{y-2}{-3} \Rightarrow -3x+3 = 2y-2 \Rightarrow$

$-3x - 2y = -2 - 3 \Rightarrow -3x - 2y = -5.$

AC: $\frac{x-1}{5-1} = \frac{y-2}{3-2} \Rightarrow \frac{x-1}{4} = \frac{y-2}{1} \Rightarrow x-1 = 4y-8 \Rightarrow$

$x - 1 - 4y = -8 \Rightarrow x - 4y = -8 + 1 \Rightarrow x - 4y = -7.$

BC: $\frac{x-3}{5-3} = \frac{y+1}{3+1} \Rightarrow \frac{x-3}{2} = \frac{y+1}{4} \Rightarrow 4x - 12 = 2y + 2 \Rightarrow$

$4x - 2y = 14.$

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Domeniul de integrare este $D = D_1 \cup D_2$

$$D_1 = \{ (x,y) \in \mathbb{R}^2 \mid 1 \le x \le 3 ,\ \frac{5}{2} - \frac{3}{2}x \le y \le \frac{7}{4} + \frac{x}{4} \}.$$

$$D_2 = \{ (x,y) \in \mathbb{R}^2 \mid 3 \le x \le 5 ,\ -7+2x \le y \le \frac{7}{4} + \frac{x}{4}$$

$$\lambda(D_1 \cup D_2) = \lambda(D_1) + \lambda(D_2) - \lambda(D_1 \cap D_2)$$

$D_1 \cap D_2 = \{3\} \times [-1, 3] \Rightarrow \lambda(D_1 \cap D_2) = 0$, deci

$D_1 \cup D_2$ este măsurabilă Jordan și este o mulțime compactă.

f continuă pe $D_1 \cup D_2 \Rightarrow f$ este mărginită, prin urmare f este integrabilă Riemann.

$$\iint_D (y - 2xz)\,dx\,dy = \iint_{D_1} (y - 2xz)\,dx\,dy + \iint_{D_2} (y - 2xz)\,dx\,dy$$

$$\cancel{\iint_{D_1}} = \int_1^3 \left(\int_{\frac{5}{2} - \frac{3}{2}x}^{\frac{7}{4} + \frac{x}{4}} (y - 2xz)\,dy \right) dx + \int_3^5 \left(\int_{-7+2x}^{\frac{7}{4} + \frac{x}{4}} (y - 2xz)\,dy \right) dx$$

$$\cancel{\int_1^3 \int_{\frac{5}{2} - \frac{3}{2}x}^{\frac{7}{4} + \frac{x}{4}} \frac{y^2}{2} - 2xzy}$$

$$\iint_{D_1} f(x,y) = \int_3^1 \frac{y^2}{2} - 2xzy \Big|_{\frac{5}{2} - \frac{3}{2}x}^{\frac{7}{4} + \frac{x}{4}} = \int^1 \left[\frac{\left(\frac{7}{4} + \frac{x}{4}\right)^2}{2} - 2xz\left(\frac{7}{4} + \frac{x}{4}\right) - \right.$$

$$\left. - \frac{\left(\frac{5}{2} - \frac{3}{2}x\right)^2}{2} + 2xz \cdot \left(\frac{5}{2} - \frac{3}{2}x\right) \right]^3 dx = \int_1^3 \left(\frac{\frac{49}{16} + \frac{x^2}{16} + \frac{14x}{16}}{2} - 2\frac{7xz}{4} - \frac{2x^2z}{4} \right.$$

$$\left. - \frac{25}{4} - \frac{30x}{4} + \frac{9}{4}x^2 + 5xz - 3x^2z \right) dx$$

$$\int_3^5 \left(\int_{-7+2x}^{\frac{7}{4}+\frac{x}{4}} (y-2xz)\,dy \right) dx = \int_3^5 \frac{y^2}{2} - 2xzy \Big|_{-7+2x}^{\frac{7}{4}+\frac{x}{4}} \, dy$$

$$\cancel{= \int_3^5 \frac{y^2}{2}} = \frac{1}{2}\int_3^5 y^2 - 4xzy \Big|_{-7+2x}^{\frac{7}{4}+\frac{x}{4}} = \frac{1}{2}\int_3^5 \left(\frac{7}{4}+\frac{x}{4}\right)^2 - 4xz\cdot\left(\frac{7}{4}+\frac{x}{4}\right) -$$

$$-(-7+2x)^2 + 4\cdot xz\cdot(-7+2x)\,dx = \frac{1}{2}\int_3^5 \left(\frac{7}{4}+\frac{x}{4}\right)^2 - \cancel{4z}\,\cancel{7z}\,7zx - x^2z -$$

$$-(-7+2x)^2 - 28xz + 8x^2z \, dx$$

$$\int_3^1 \left(\int_{\frac{5}{2}-\frac{3}{2}x}^{\frac{7}{4}+\frac{x}{4}} (y-2xz)\,dy \right) dx = \int_3^1 \frac{y^2}{2} - 2xzy \Big|_{\frac{5}{2}-\frac{3}{2}x}^{\frac{7}{4}+\frac{x}{4}} =$$

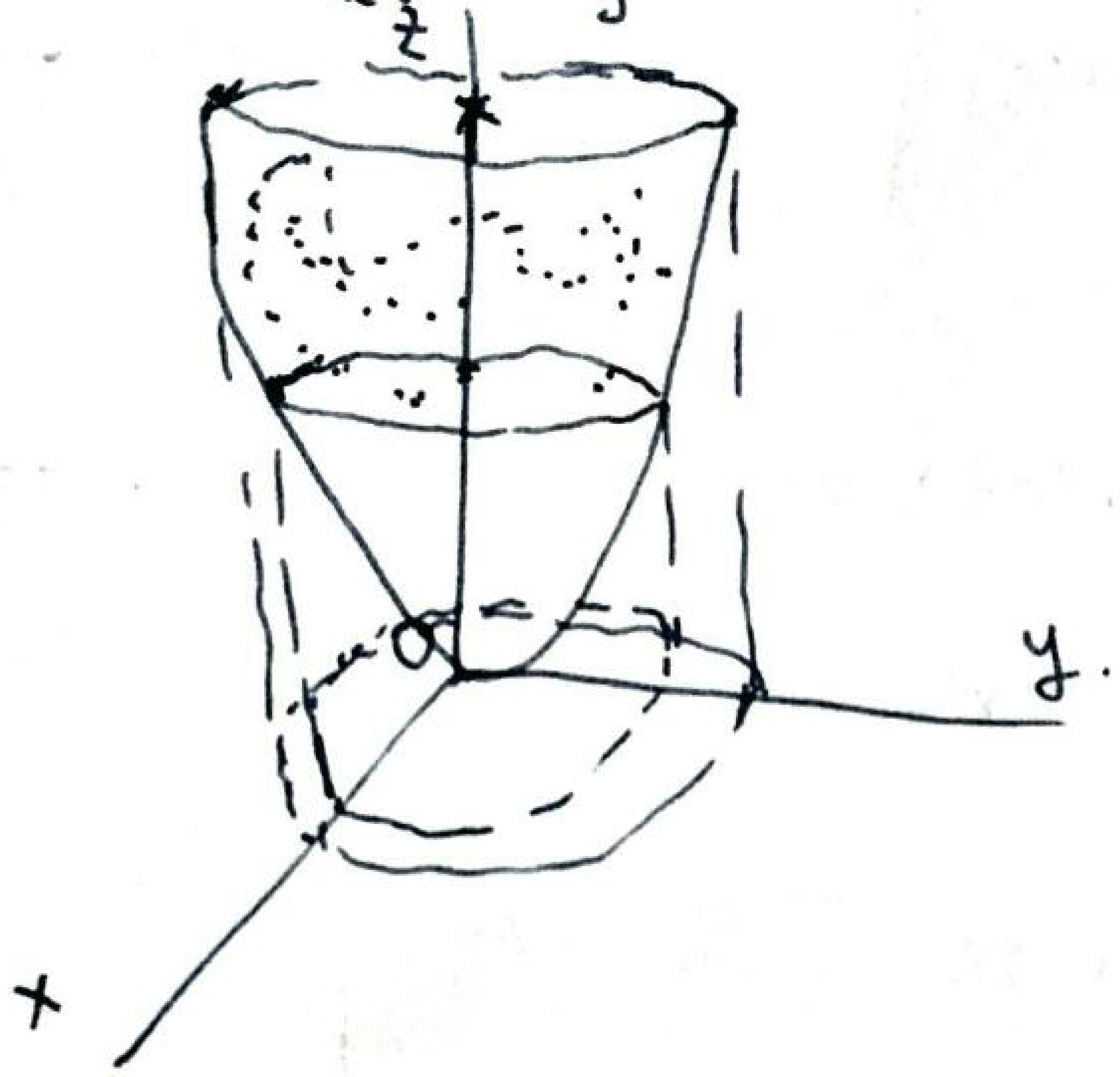
$$= \frac{1}{2}\int_3^1 y^2 - 4xzy \Big|_{\frac{5}{2}-\frac{3}{2}x}^{\frac{7}{4}+\frac{x}{4}} = \frac{1}{2}\int_3^1 \left(\frac{7}{4}+\frac{x}{4}\right)^2 - 4xz\left(\frac{7}{4}+\frac{x}{4}\right) -$$

$$-\left(\frac{5}{2}-\frac{3}{2}x\right)^2 + 4xz\left(\frac{5}{2}-\frac{3}{2}x\right)$$

IV

$$\iiint_V (z+2)\,dx\,dy\,dz$$

$z=1$, $z=3$ $\quad \dfrac{x^2}{25}+\dfrac{y^2}{9}=z$

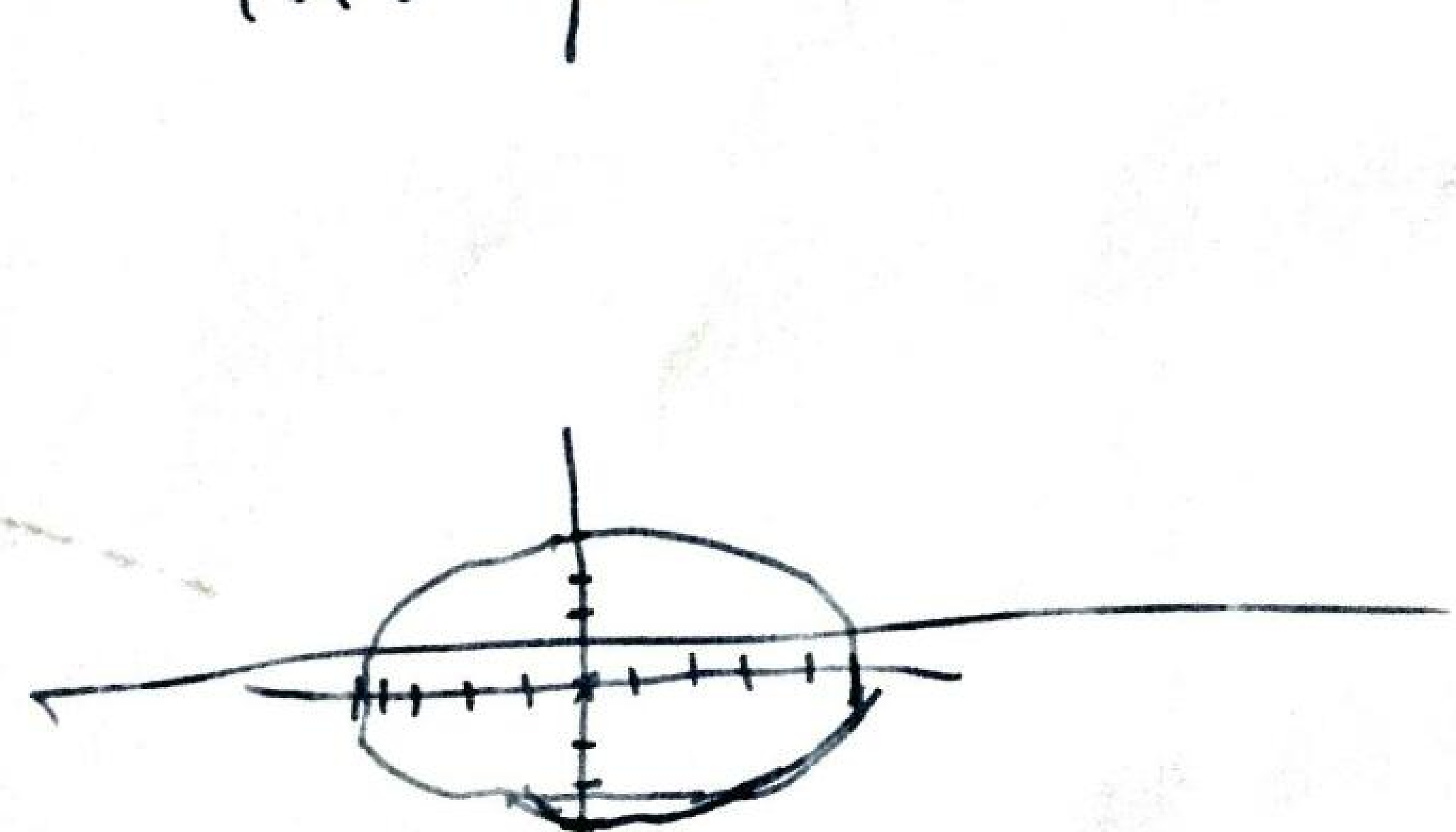


Determinăm intersecția dintre paraboloid și planele $z=1$ și $z=3$.

$$\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{9}=z \\ z=1 \end{cases} \Rightarrow \begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{9}=1 \\ z=1 \end{cases}$$ (este o elipsă)

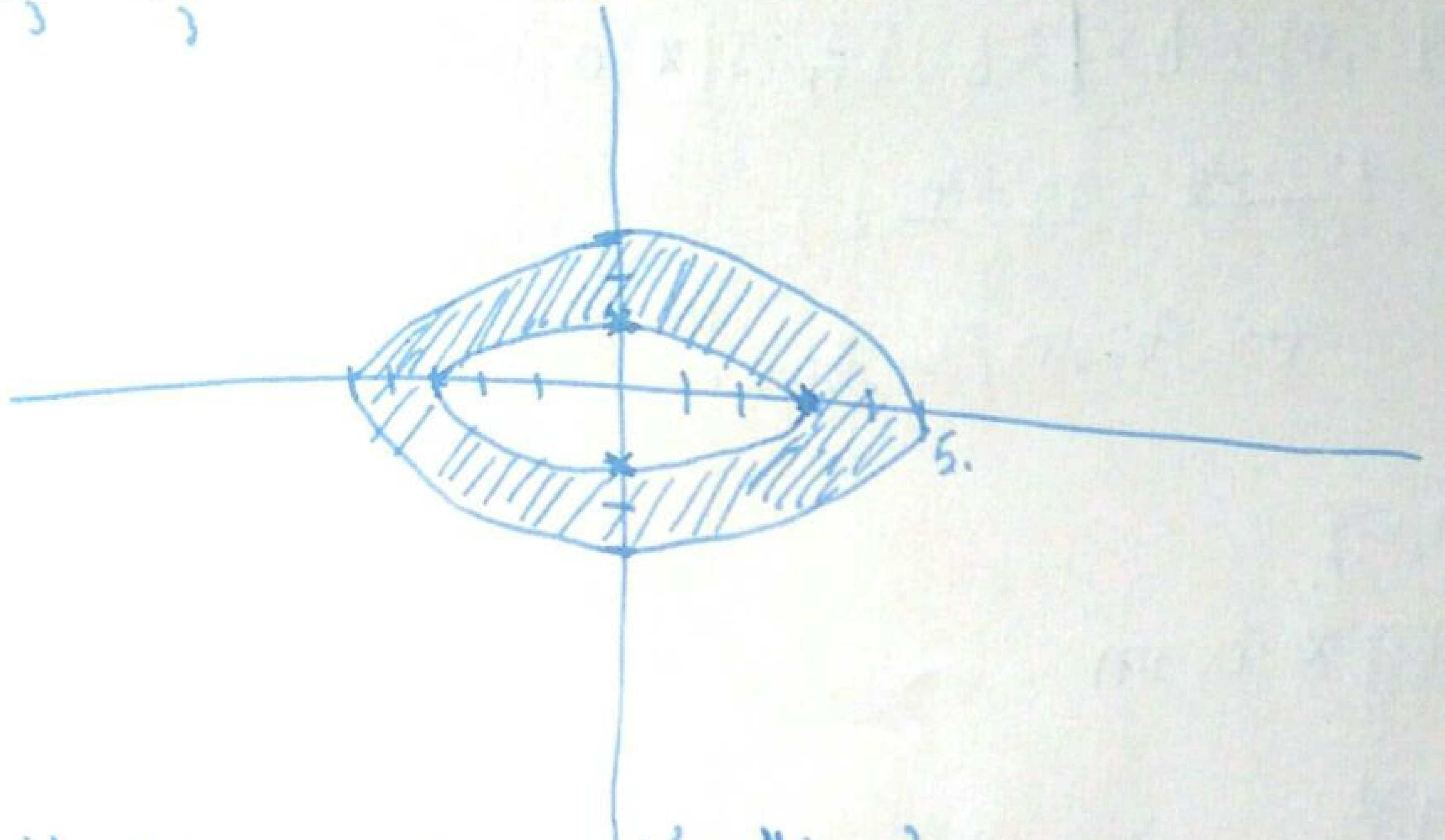
$$\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{9}=3 \\ z=3. \end{cases}$$ (este o elipsă)

$\cancel{Pr_{xOy} V =}$



$$\frac{x^2}{25}+\frac{y^2}{9}=1 \iff \frac{x^2}{5^2}+\frac{y^2}{3^2}=1.$$

$$\frac{x^2}{\frac{25}{3}}+\frac{y^2}{\frac{9}{3}}=1 \Rightarrow \frac{x^2}{\left(\frac{5}{\sqrt{3}}\right)^2}+\frac{y^2}{(\sqrt{3})^2}=1$$



$$pr_{xOy} V = \left\{ (x,y)\in\mathbb{R}^2 \,\middle|\, 1 \le \frac{x^2}{25}+\frac{y^2}{9} \le 3 \right\}.$$

$$\cancel{pr_{xOy} V = \left\{ (x,y)\in\mathbb{R}^2 \,\middle|\, 3 \le \frac{x^2}{25}+\frac{y^2}{9} \le 1 \right.}$$

$$\cancel{pr_{xOy} V = \left\{ (x,y)\in\mathbb{R}^2 \,\middle|\, \frac{x^2}{25}+\frac{y^2}{9} \le 1 \text{ şi } \frac{x^2}{25}+\frac{y^2}{9} \ge 3 \right\}}.$$

$$\cancel{pr_{(xOy)} V = \left\{ (x,y)\in\mathbb{R}^2 \,\middle|\, \frac{x^2}{25}+\frac{y^2}{9} \le 1 \right\}}.$$

$$V = \left\{ (x,y,z)\in\mathbb{R}^3 \,\middle|\, \frac{x^2}{25}+\frac{y^2}{9} \le z \le 3 \right\}.$$

$$\iiint_V (z+2)\,dx\,dy\,dz = \iint_D \left(\int_{\frac{x^2}{25}+\frac{z^2}{9}}^{3} (z+2)\,dz \right) dx\,dy =$$

$$= \iint_D \left.\frac{z^2}{2}+2\cdot z\right|_{\frac{x^2}{25}+\frac{z^2}{9}}^{3} = \iint_D \left(\frac{\left(\frac{x^2}{25}+\frac{z^2}{9}\right)^2}{2} + 2\cdot\left(\frac{x^2}{25}+\frac{y^2}{9}\right)^{2} - \frac{9}{2} - 6 \right) dx\,dy$$

Fie $\varphi: (0;+\infty)\times(0,2\pi) \to \mathbb{R}^2 \setminus \{(x,0) \mid x\ge 0\}$, $\varphi(r,\theta) = (5r\cos\theta, 3r\sin\theta)$

φ difeomorfism, $J\varphi(r,\theta) = a\cdot b r = 15r$.

$x = 5r\cos\theta$

$y = 3r\sin\theta$

$(x,y) = \varphi(r,\theta) \in \Delta \setminus \{F \cap \Delta \cup [\frac{5}{\sqrt{3}}; 5] \times \{0\}\}$

$\Leftrightarrow \begin{cases} 1 \le \frac{25r^2\cos^2\theta}{25} + \frac{9r^2\sin^2\theta}{9} \le 3 \\ \theta \in (0;2\pi) \text{ (din figură)} \end{cases} \Leftrightarrow \begin{cases} 1 \le r^2 \le 3 \\ \theta \in (0,2\pi) \end{cases}$

$\Leftrightarrow r \in (1,\sqrt{3})$.

$A = (1,\sqrt{3}) \times (0,2\pi)$

$$\iint_\Delta \left(\frac{\left(\frac{x^2}{25}+\frac{y^2}{5}\right)^2}{2} + 2\left(\frac{x^2}{25}+\frac{y^2}{5}\right) - \frac{9}{2} - 6 \right) dx\,dy =$$

$$= \iint_{\varphi(A)} \frac{\left(\frac{x^2}{25}+\frac{y^2}{5}\right)^2}{2} + 2\left(\frac{x^2}{25}+\frac{y^2}{5}\right) - \frac{9-12}{2} \right) dx\,dy =$$

$$= \int_0^{2\pi}\left(\int_1^{\sqrt{3}} \left(\frac{r^2}{2} + 2r + \frac{3}{2}\right)15\,dr\right) d\theta = \cancel{\int_0^{2\pi}\left(\int_1^{\sqrt{3}}\right.}$$

$$= 15\int_0^{2\pi} \left(\frac{r^3}{6} + r^2 + \frac{3}{2}r\right)\Bigg|_1^{\sqrt{3}} d\theta = 15\int_0^{2\pi}\left(\frac{3\sqrt{3}}{6} + 3 + \frac{3}{2}\sqrt{3} - \frac{1}{6} - 1 - \frac{3}{2}\right)$$

$$= 15\int_0^{2\pi} \frac{12\sqrt{3}}{6} + 2 - \frac{1}{6} - \frac{3}{2}\, d\theta = 15\int_0^{2\pi} \frac{12\sqrt{3}}{6} + \frac{12-1-9}{6}\, d\theta$$

$$= \int_0^{2\pi} 2\sqrt{3} + \frac{11-9}{6}\, d\theta = \int_0^{2\pi} 15\left(2\sqrt{3} + \frac{1}{3}\right) d\theta = 15\left(2\sqrt{3} + \frac{1}{3}\right)\theta\Big|_0^{2\pi} = \cancel{15}\left(2\sqrt{3} + \frac{1}{3}\right)\Big|^{2\pi}$$

$= 15\cdot(2\sqrt{3} + \frac{1}{3})2\pi$

V $f:[0,4]\times[0,3]\times[0,2]\to\mathbb{R}$.

$$f(x,y,z)=\begin{cases} 2 & x=1,\ y\in[0,3]\setminus\mathbb{Q} \\ 5 & x=3,\ z\in[0,2]\cap\mathbb{Q} \\ 2x-y^2 & \text{altfel.} \end{cases}$$

Observăm că $[0,4]\times[0,3]\times[0,2]$ este o mulțime măsurabilă Jordan.

f este mărginită $-9\le f(x,y,z)\le 8$, $|f(x,y,z)|\le 8$.

Pentru primele 2 ramuri ale funcției, mulțimea punctelor de discontinuitate este $D_f\subset(\{1\}\times[0,3]\setminus\mathbb{Q}\times[0,2])\cup$
$\cup\{3\}\times[0,3]\times[0,2]\cap\mathbb{Q})$, deci D_f este inclusă într-o mulțime numărabilă.

Mulțimea punctelor de discontinuitate este neglijabilă Lebesgue.

Din criteriul lui Lebesgue $\Rightarrow$ f este integrabilă Riemann.