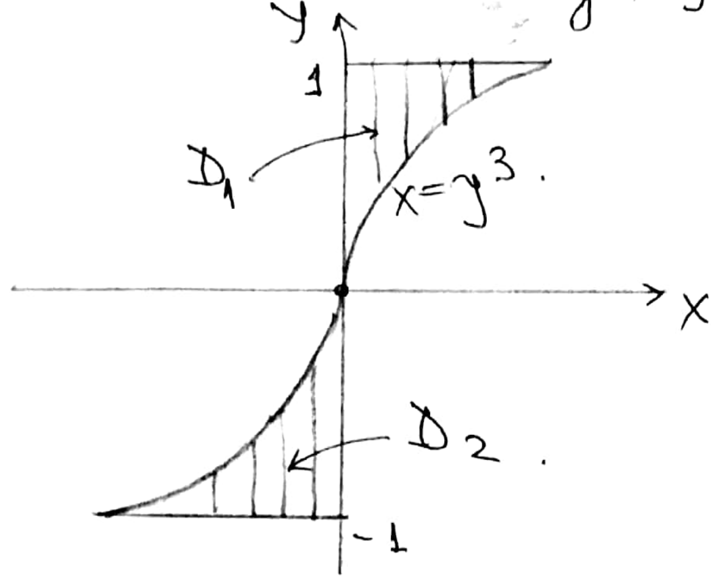
1) $\iint\limits_D e^{y^4}\,dx\,dy$, D este mulțimea mărginită de curbele $x=y^3$, $y=1$, $y=-1$, $x=0$.



$D=D_1\cup D_2$.

$D_1=\{(x,y)\in\mathbb{R}^2 \mid 0\leq y\leq 1;\ 0\leq x\leq y^3\}\in J(\mathbb{R}^2)$

continue.

$D_2=\{(x,y)\in\mathbb{R}^2 \mid -1\leq y<0;\ y^3\leq x\leq 0\}\in J(\mathbb{R}^2)$

$D=D_1\cup D_2\in J(\mathbb{R}^2)$; $f:D\to\mathbb{R}$, $f(x,y)=e^{y^4}$ - integrabilă

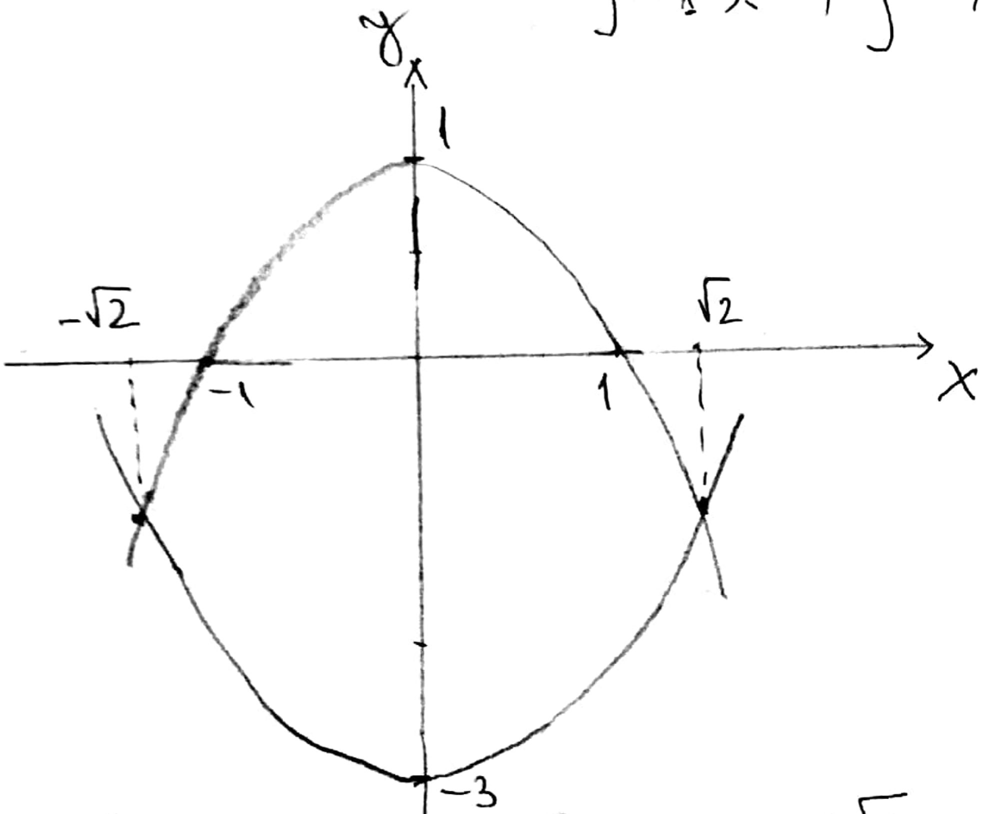
$\lambda(D_1\cap D_2)=0$.

$$\Rightarrow \iint\limits_D e^{y^4}dx\,dy=\iint\limits_{D_1} e^{y^4}dx\,dy+\iint\limits_{D_2} e^{y^4}dx\,dy$$

$$\iint\limits_{D_1} e^{y^4}dx\,dy=\int_0^1\left(\int_0^{y^3}e^{y^4}dx\right)dy=\int_0^1 y^3e^{y^4}dy=\frac{1}{4}e^{y^4}\Big|_0^1=\frac{1}{4}(e-1)$$

$$\iint\limits_{D_2} e^{y^4}dx\,dy=\frac{1}{4}(e-1)\Rightarrow \iint\limits_D e^{y^4}dy=\frac{1}{2}(e-1)$$

2) $\iint\limits_D x(y-1)\,dx\,dy$; D este mărginită de curbele $y=1-x^2$, $y=x^2-3$.



$1-x^2=x^2-3 \Rightarrow 2x^2=4 \Rightarrow x^2=2$, $x=\pm\sqrt{2}$.

$D=\{(x,y)\in\mathbb{R}^2 \mid -\sqrt{2}\le x\le\sqrt{2},\ x^2-3\le y\le 1-x^2\}\in J(\mathbb{R}^2)$

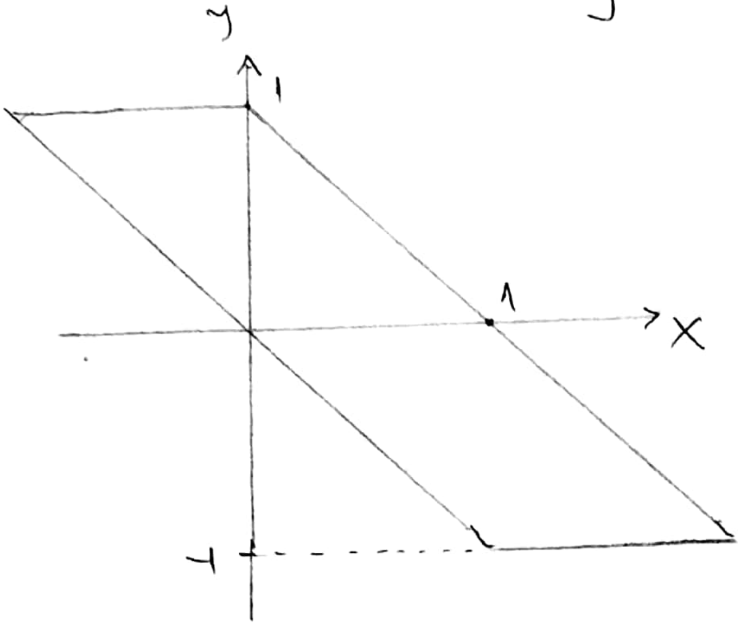
$$\iint\limits_D x(y-1)\,dx\,dy=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\int_{x^2-3}^{1-x^2}x(y-1)\,dy\right)dx=$$

$$=\int_{-\sqrt{2}}^{\sqrt{2}} x\left(\frac{y^2}{2}-y\right)\Bigg|_{y=x^2-3}^{y=1-x^2}dx=\int_{-\sqrt{2}}^{\sqrt{2}}\left[\frac{1}{2}x(1-x^2)^2-\frac{1}{2}x(x^2-3)^2\right]dx$$

$$-\int_{-\sqrt{2}}^{\sqrt{2}}\left[x(1-x^2)-x(x^2-3)\right]dx=\int_{-\sqrt{2}}^{\sqrt{2}}(\ldots)\,dx=\ldots$$

3) $\iint\limits_D \arcsin\sqrt{x+y}\,dx\,dy$, D este mărginit de dreptele

$x+y=0$, $x+y=1$, $y=1$ și $y=-1$.



$D=\{(x,y) \mid -1\le y\le 1;\ -y\le x\le -y+1\}\in J(\mathbb{R}^2)$

$f: D\to\mathbb{R}$, $f(x,y)=\arcsin\sqrt{x+y}$; continuă

f integrabilă pe D (continuă și mărginită)

$$\iint\limits_D \arcsin\sqrt{x+y}\,dx\,dy=\int_{-1}^{1}\left(\int_{-y}^{-y+1}\arcsin\sqrt{x+y}\,dx\right)dy$$

$$\int_{-y}^{-y+1}\arcsin\sqrt{x+y}\,dx=\int_{0}^{1}\arcsin\sqrt{u}\,du.$$

$u=x+y$ $\qquad x=-y \qquad u=0$

$du=dx.$ $\qquad x=-y+1 \qquad u=1$

$$\int_0^1 \arcsin\sqrt{u}\, du.$$

$\arcsin\sqrt{u} = t \qquad t \in [0, \frac{\pi}{2}].$

$\sqrt{u} = \sin t \Rightarrow u = \sin^2 t. \qquad t = 0 \leftrightarrow u = 0$

$du = 2\sin t \cos t\, dt. \qquad t = \frac{\pi}{2} \leftrightarrow u = 1.$

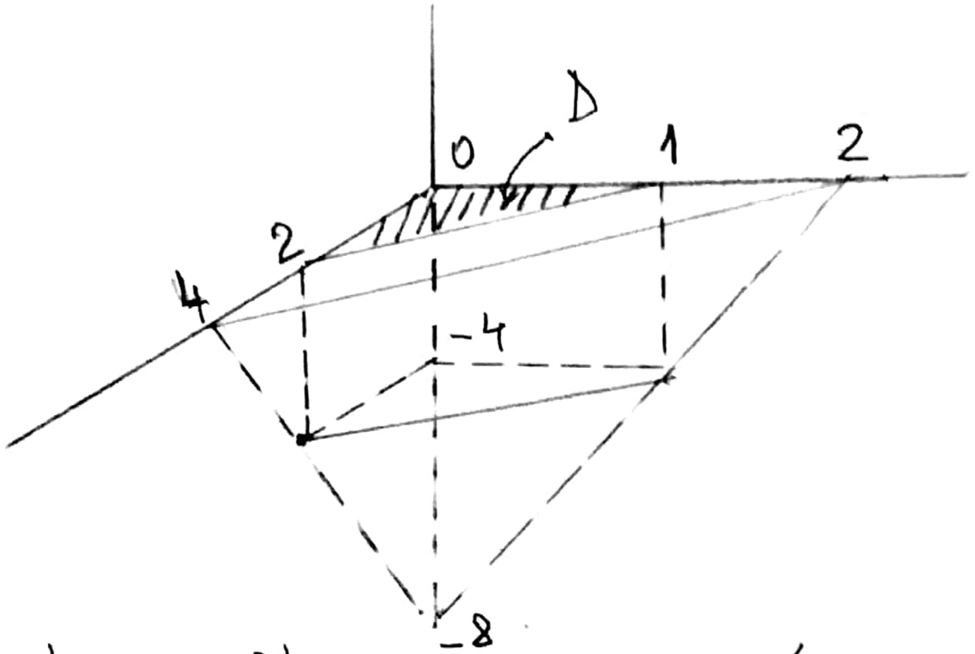
$$\int_0^1 \arcsin\sqrt{u}\, du = \int_0^{\frac{\pi}{2}} t \cdot 2\sin t \cos t\, dt = \int_0^{\frac{\pi}{2}} t \sin 2t\, dt$$

$$= -\frac{t\cos 2t}{2}\Big|_0^{\frac{\pi}{2}} + \underbrace{\int_0^{\frac{\pi}{2}} \frac{\cos 2t}{2}\, dt}_{0} = \frac{\pi}{4}.$$

$$\int_{-y}^{-y+1} \arcsin\sqrt{x+y}\, dx = \frac{\pi}{4}.$$

$$\iint_D \arcsin\sqrt{x+y}\, dx\, dy = \int_{-1}^{1} \frac{\pi}{4}\, dy = \frac{\pi}{2}.$$

4) $\iiint\limits_V (x+y)\,dx\,dy\,dz$, $V=\left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, \begin{array}{l}2x+4y-z\leq 8,\ x,y\geq 0,\\ z\leq -4\end{array}\right\}$



$$pr_{xOy} V = \{(x,y)\in\mathbb{R}^2 \mid 2x+4y\leq 4,\ x,y\geq 0\}$$

$$= \{(x,y)\in\mathbb{R}^2 \mid 0\leq y\leq 1,\ 0\leq x\leq 2-2y\} \in J(\mathbb{R}^2)$$

$$V = \{(x,y,z)\in\mathbb{R}^3 \mid 2x+4y-8\leq z\leq -4,\ (x,y)\in D\}$$

$$\iiint\limits_V (x+y)\,dx\,dy\,dz = \iint\limits_D \left(\int_{2x+4y-8}^{-4} (x+y)\,dz\right)dx\,dy$$

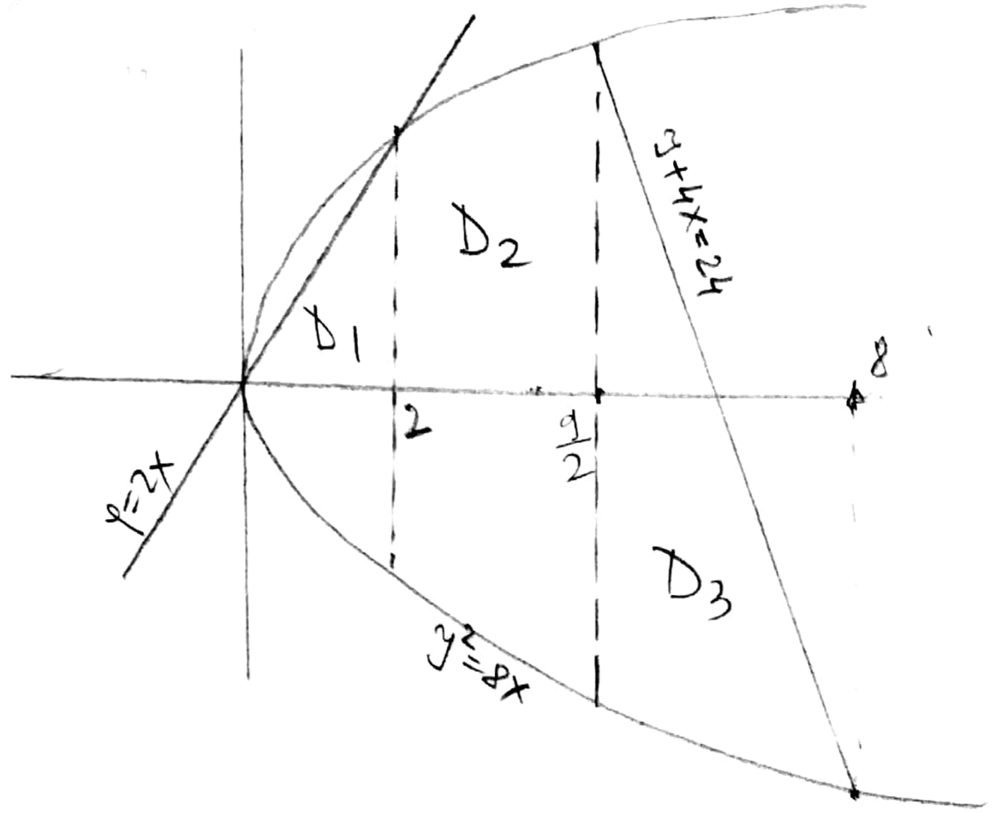
$$= \iint\limits_D (x+y)(-4-2x-4y+8)\,dx\,dy$$

$$= \iint\limits_D (x+y)(4-2x-4y)\,dx\,dy$$

$$= \int_0^1 \left(\int_0^{2-2y} (4x - 2x^2 - 4xy + 4y - 2xy - 4y^2)\, dx \right) dy$$

$$= \int_0^1 \left(\int_0^{2-2y} (-2x^2 - 4y^2 - 6xy + 4x + 4y)\, dx \right) dy$$

$$= \int_0^1 \left[-\frac{2x^3}{3} \Big|_{x=0}^{x=2-2y} - 4y^2 x \Big|_{x=0}^{x=2-2y} - 3x^2 y \Big|_{x=0}^{x=2-2y} + 2x^2 \Big|_{x=0}^{x=2-2y} + 4yx \Big|_{x=0}^{x=2-2y} \right] dy$$



$$\iint_D \frac{1}{\sqrt{x}}\,dx\,dy\ ;\quad D=\{(x,y)\in\mathbb{R}^2 \mid y^2\le 8x,\ y\le 2x,\ y+4x\le 24\}$$

$$\begin{cases} y^2=8x \\ y=2x \end{cases} \Rightarrow \begin{cases} 4x^2=8x \\ y=2x \end{cases} \Rightarrow \quad \begin{array}{l} x_1=0\ ,\ y_1=0 \\ x_2=2\ ,\ y_2=4. \end{array}$$

$$\begin{cases} y^2=8x \\ y+4x=24 \end{cases} \Rightarrow \begin{cases} x=\dfrac{y^2}{8} \\ y+\dfrac{y^2}{2}-24=0 \end{cases} \Leftrightarrow \begin{cases} x=\dfrac{y^2}{8} \\ y^2+2y-48=0 \end{cases} \Rightarrow \begin{cases} y_1=-8,\ x_1=8 \\ y_2=6,\ x_2=\dfrac{9}{2} \end{cases}$$

Fie :

$D_1=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 2,\ -\sqrt{8x}\le y\le 2x\}$ $\quad\lambda(D_1\cap D_2)=0$

$D_2=\{(x,y)\in\mathbb{R}^2 \mid 2\le x\le \frac{9}{2},\ -\sqrt{8x}\le y\le \sqrt{8x}\}$ $\quad\lambda(D_2\cap D_3)=0$

$D_3=\{(x,y)\in\mathbb{R}^2 \mid \frac{9}{2}\le x\le 8,\ -\sqrt{8x}\le y\le 24-4x\}$

$$\iint_D \frac{1}{\sqrt{x}}\,dx\,dy = \iint_{D_1} \frac{1}{\sqrt{x}}\,dx\,dy + \iint_{D_2} \frac{1}{\sqrt{x}}\,dx\,dy + \iint_{D_3} \frac{1}{\sqrt{x}}\,dx\,dy$$

$$\iint_{D_1} \frac{1}{\sqrt{x}}\,dx\,dy = \int_0^2 \left(\int_{-\sqrt{8x}}^{2x} \frac{1}{\sqrt{x}}\,dy \right) dx = \int_0^2 \frac{1}{\sqrt{x}} \left(2x + \sqrt{8x}\right) dx$$

$$= \int_0^2 \left(2\sqrt{x} + 2\sqrt{2}\right) dx = \frac{4}{3} x\sqrt{x} \Big|_0^2 + 2\sqrt{2}\cdot 2 = \frac{20\sqrt{2}}{3}$$

$$\iint_{D_2} \frac{1}{\sqrt{x}}\,dx\,dy = \int_2^{\frac{9}{2}} \left(\int_{-\sqrt{8x}}^{\sqrt{8x}} \frac{1}{\sqrt{x}}\,dy \right) dx = \int_2^{\frac{9}{2}} 4\sqrt{2x} \cdot \frac{1}{\sqrt{x}}\,dx$$

$$= \int_2^{\frac{9}{2}} 4\sqrt{2}\,dx = 10\sqrt{2}$$

$$\iint_{D_3} \frac{1}{\sqrt{x}}\,dx\,dy = \int_{\frac{9}{2}}^{8} \left(\int_{-\sqrt{8x}}^{24-4x} \frac{1}{\sqrt{x}}\,dy \right) dx = \int_{\frac{9}{2}}^{8} \frac{24 - 4x + \sqrt{8x}}{\sqrt{x}}\,dx$$

$$= \int_{\frac{9}{2}}^{8} \left(\frac{24}{\sqrt{x}} - 4\sqrt{x} + \sqrt{8} \right) dx = 48\sqrt{x} \Big|_{\frac{9}{2}}^{8} - \frac{8}{3} x\sqrt{x} \Big|_{\frac{9}{2}}^{8} + \sqrt{8}x \Big|_{\frac{9}{2}}^{8}$$

$$= \frac{19\sqrt{2}}{3}$$

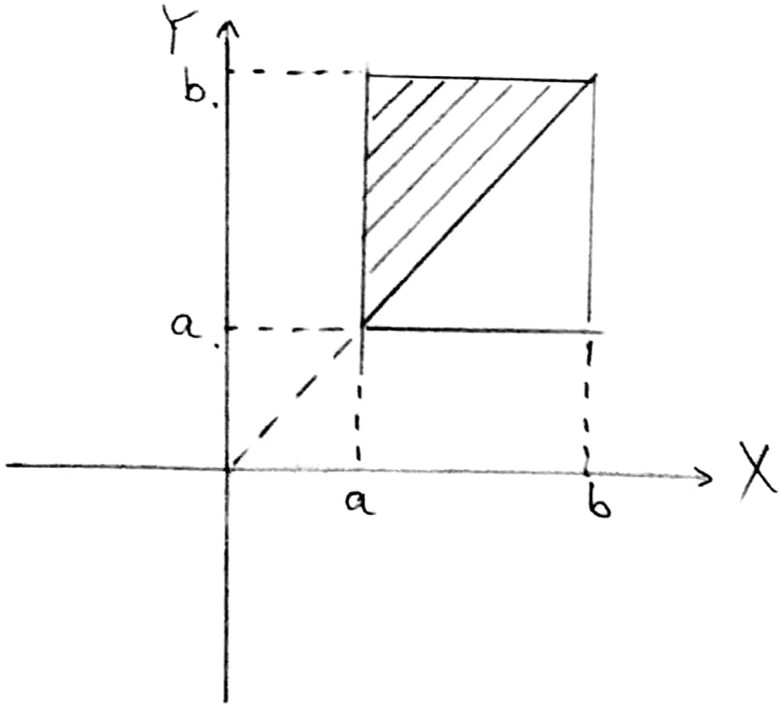
$$\iint_D \frac{1}{\sqrt{x}}\,dx\,dy = \frac{20\sqrt{2}}{3} + 10\sqrt{2} + \frac{19\sqrt{2}}{3} = 23\sqrt{2}.$$

Problemă. Fie $f : [a,b] \times [a,b] \to \mathbb{R}$ continuă. Arătați că

$$\int_a^b \left(\int_a^y f(x,y)\,dx \right) dy = \int_a^b \left(\int_x^b f(x,y)\,dy \right) dx$$

Soluție.

$$C = \{ (x,y) \in \mathbb{R}^2 \mid a \leq y \leq b,\ a \leq x \leq y \}$$
$$= \{ (x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ x \leq y \leq b \}$$



f continuă pe C

T. Fubini $\Rightarrow$ (de fapt consecință) $\int_a^b \left(\int_a^y f(x,y)\,dx \right) dy = \int_a^b \left(\int_x^b f(x,y)\,dy \right) dx$

$$\parallel$$

$$\iint_C f(x,y)\,dx\,dy$$

Problemă: Fie $J \subset \mathbb{R}^n$ un interval din $\mathbb{R}^n$, $f : J \longmapsto \mathbb{R}^+$ o funcție integrabilă a.î. $\int_J f(x)dx = 0$. Arătați că mulțimea $B = \{x \in J \mid f(x) > 0\}$ este neglijabilă Lebesgue.

Soluție. $\forall\, n \geq 1$, fie $B_n = \{x \in J \mid f(x) > \frac{1}{n}\}$

$B = \bigcup_{n=1}^{\infty} B_n$. Vom arăta că B_n este neglijabilă Lebesgue pentru orice $n \geq 1$.

Fixăm $n \in \mathbb{N}$, și fie $\varepsilon > 0$.

$\int_J f = 0 \Rightarrow \exists\, \mathcal{P} = \{J_1, J_2, \dots, J_p\}$ o descompunere a lui J

a.î. $$\frac{\varepsilon}{n} > S_{\mathcal{P}}(f) = \sum_{i=1}^{p} M_i \, vol(J_i)$$

unde $M_i = \sup\{f(x) \mid x \in J_i\}$

Fie $K = \{i \mid 1 \leq i \leq n,\ J_i \cap B_n \neq \emptyset\}$

Evident $B_n \subset \bigcup_{i \in K} J_i$

$$\frac{\varepsilon}{n} > S_{\mathcal{P}}(f) = \sum_{i=1}^{p} M_i \, vol(J_i) \geq \sum_{i \in K} M_i \, vol(J_i) >$$

$$> \frac{1}{n} \sum_{i \in K} vol(J_i)$$

$$\Rightarrow \left.\begin{array}{l} \sum_{i \in K} vol(J_i) < \varepsilon \\ B_n \subset \bigcup_{i \in K} J_i \end{array}\right\} \Rightarrow \lambda^*(B_n) < \varepsilon \text{ .. și } B_n \text{ este neglijabilă Lebesgue.}$$

Deoarece $B = \bigcup_{n \geq 1} B_n$ este o reuniune numărabilă de mulțimi neglijabile Lebesgue rezultă că B este neglijabilă Lebesgue. (Vezi Alte Exerciții - 1).