Proiect

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Exercitiul 1

Generati 100.000 de valori dintr-o variabila aleatoare folosind metoda transformarii inverse pentru repartitiile de mai jos:

a)

Repartitia logistica are densitatea de probabilitate $f(x) = \frac{1}{\beta} \cdot \frac{e^{-(x-\mu)/\beta}}{(1+e^{-(x-\mu)/\beta})^2}$ si functia de repartitie $F(x) = \frac{1}{1+e^{-(x-\mu)/\beta}}$

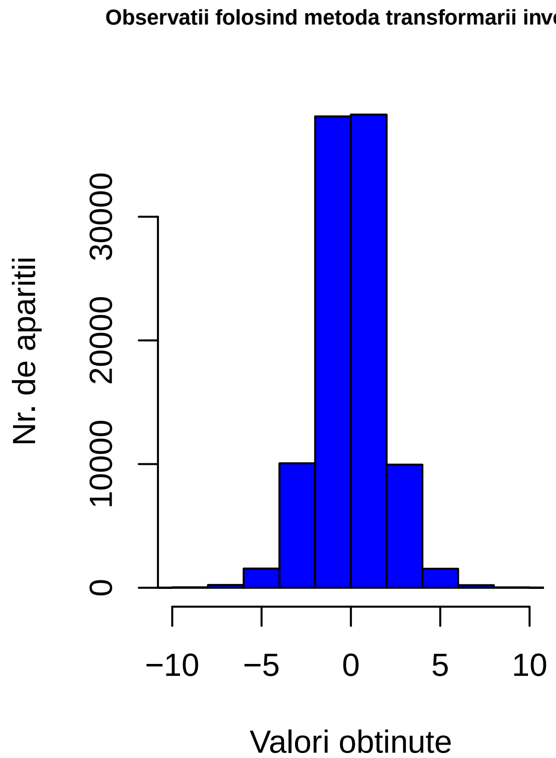
Solutie:

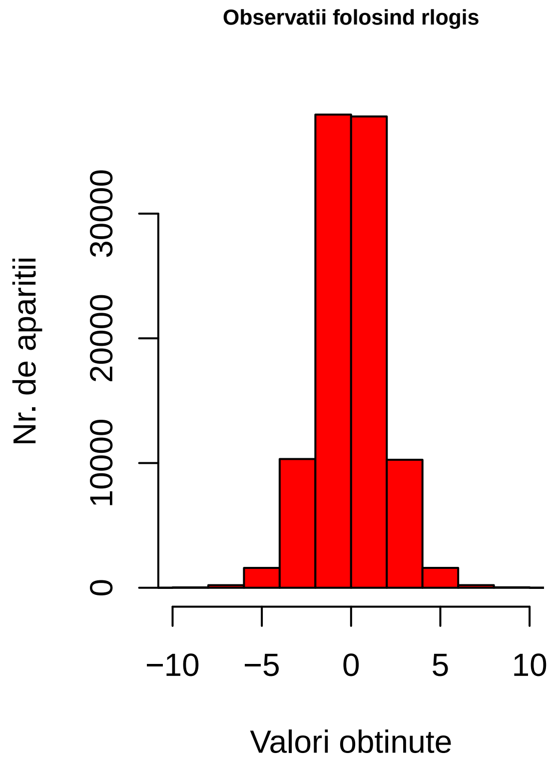
F(x)(functia de repartitie) este continua, deci calculam inversa ei:

$$y = F^{-1}(x) \implies F(y) = x \implies x = \frac{1}{1+e^{-(x-\mu)/\beta}} \implies$$
$$\implies x + xe^{-(y-\mu)/\beta} = 1 \implies 1 - x = xe^{-(y-\mu)/\beta} \implies$$
$$\implies \beta \cdot ln(x) - \beta ln(1-x) + \mu = y \implies y = \mu + \beta \cdot ln\left(\frac{x}{1-x}\right) \implies$$
$$\implies F^{-1}(x) = \mu + \beta ln\left(\frac{x}{1-x}\right) \implies F^{-1} = \mu - \beta \cdot ln(1-x) + \beta \ln(x)$$

```r
n <- 100000

valRepLogistica <- function(nr, miu, beta) {

  U <- runif(nr)
  return (miu-beta*log(1-U)+beta*log(U))
}
test1 <- valRepLogistica(n, 0, 1)
test2 <- rlogis(n)

par(mfrow=c(1,2)) #afisam 2 grafice pe o linie

hist(test1,
     main="Observatii folosind metoda transformarii inverse",
     xlab="Valori obtinute",
     ylab="Nr. de aparitii",
     xlim=c(-10, 10),
     cex.main=0.7,
```

```r
     col="blue")
hist(test2,
     main="Observatii folosind rlogis",
     xlab="Valori obtinute",
     ylab="Nr. de aparitii",
     xlim=c(-10, 10),
     cex.main=0.7,
     col="red")
```





b)

Repartitia Cauchy are densitatea de probabilitate $f(x) = \frac{1}{\pi\sigma} \cdot \frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}$ si functia de repartitie $F(x) = \frac{1}{2} + \frac{1}{\pi} \cdot arctan\left(\frac{x-\mu}{\sigma}\right)$

Solutie:

F(x) (functia de repartitie) este continua, deci calculam inversa ei:

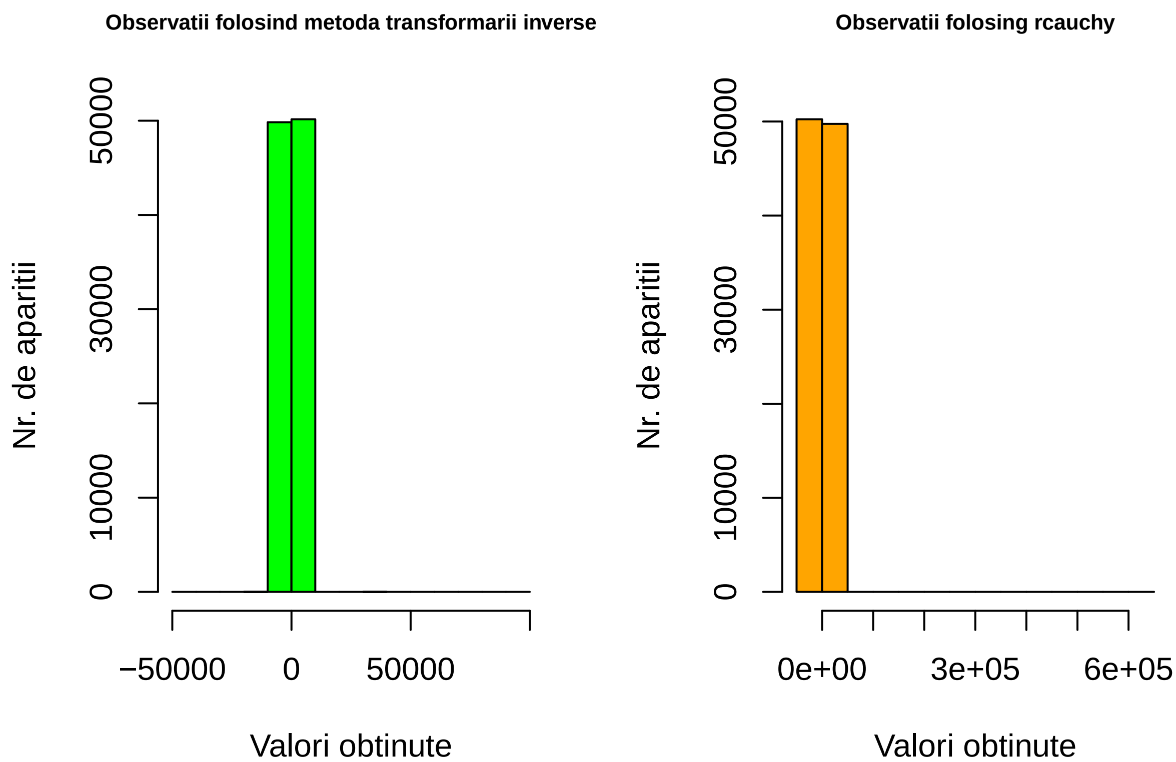
$$y = F^{-1}(x) \implies F(y) = x \implies x = \frac{1}{2} + \frac{1}{\pi} \cdot arctan\left(\frac{y-\mu}{\sigma}\right) \implies$$

$$\implies x - \frac{1}{2} = \frac{1}{\pi}arctan\left(\frac{y-\mu}{\sigma}\right) \implies \pi\left(x - \frac{1}{2}\right) = arctan(\frac{y-\mu}{\sigma}) \implies$$

$$\implies tan\left(\pi\left(x-\frac{1}{2}\right)\right) = \frac{y-\mu}{\sigma} \implies \sigma tan\left(\pi\left(x-\frac{1}{2}\right)\right) = y - \mu \implies y = \mu + tan\left(\pi\left(x-\frac{1}{2}\right)\right) \implies$$

$$F^{-1}(x) = \mu + tan\left(\pi\left(x - \frac{1}{2}\right)\right)$$

```r
n <- 100000

valorRepCauchy <- function(nr, miu, sigma){

  U <- runif(nr) #10000 de observatii dintr-o uniforma
  return(miu+sigma*tan(pi*(U-1/2))) #inversa functiei date
}

test1 <- valorRepCauchy(n,0,1) #observatii metoda inversei
test2 <- rcauchy(n,0,1) #observatii metoda rcauchy

par(mfrow=c(1,2)) #afisam 2 grafice pe o linie

hist(test1,
     main="Observatii folosind metoda transformarii inverse",
     xlab="Valori obtinute",
     ylab="Nr. de aparitii",
     cex.main=0.7,
     col="green")

hist(test2,
     main="Observatii folosing rcauchy",
     xlab="Valori obtinute",
     ylab="Nr. de aparitii",
     cex.main=0.7,
     col="orange")
```


Observatii folosind metoda transformarii inverse
Nr. de aparitii
0
10000
30000
50000
−50000
0
50000
Valori obtinute
Observatii folosing rcauchy
Nr. de aparitii
0
10000
30000
50000
0e+00
3e+05
6e+05
Valori obtinute

Exercitiul 2

a)

Alegem functia $h(x) = e^{\frac{-(x-2)^2}{8}}(\sin^2 2x - cos^2x \cdot sin^2 3x + 5)$

```
h <- function(x){
  exp(-(x-2)^2/8) * (sin(2*x)^2 - 2*cos(x)^2 * sin(3*x)^2 + 5)
}
```

$$N(2,4) = \mu\sigma^2$$

$$Densitatea : \frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(\frac{x-2}{2})^2}$$

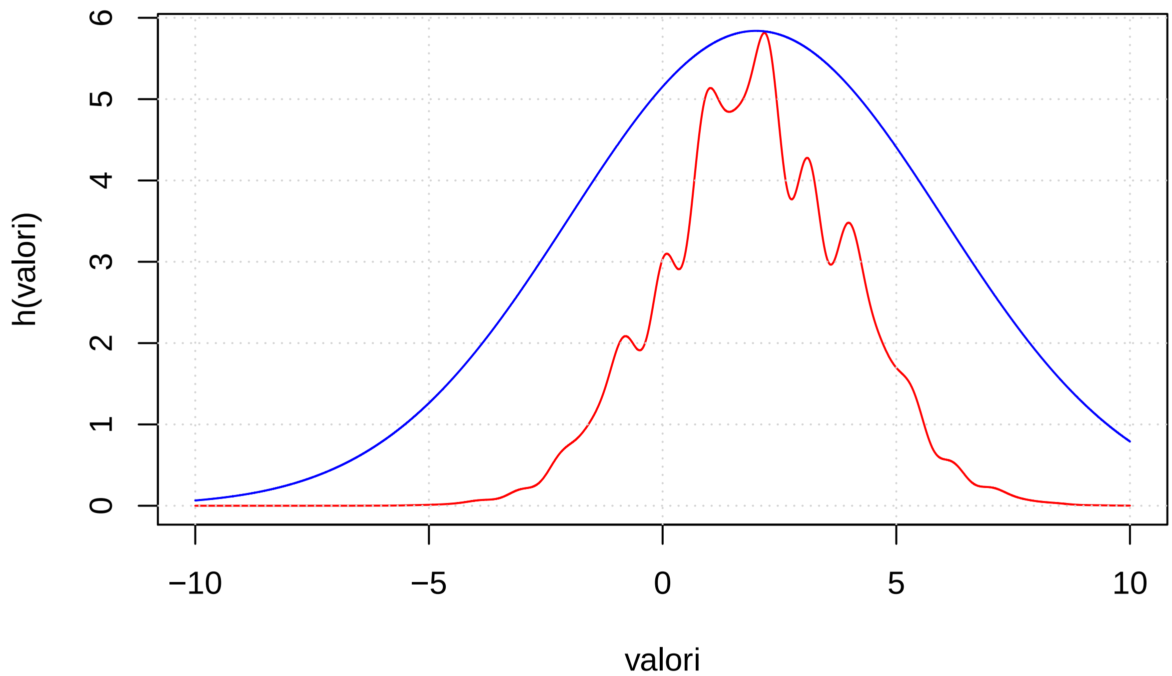
$$\frac{f}{g} = 2\sqrt{2\pi}\lambda$$

$$unde\lambda = (\sin^2 2x - cos^2x \cdot sin^2 3x + 5)$$

```
raport <- function(x) {
  2*sqrt(2*pi) * (sin(2*x)^2 - 2*cos(x)^2 * sin(3*x)^2 + 5)
}
```

Cautam M cu optimise() si inmultim g() cu M pentru a margini pe f

```
valori <- seq(-10,10,0.0005)
plot(valori, h(valori), type="l", col="red")
grid(nx=NULL, col="lightgray", lty="dotted",lwd=par("lwd"),equilogs=TRUE)

M <- optimise(raport, c(-10,10), maximum = TRUE)
lines(valori, dnorm(valori,2,4)*M[[2]]*2, col = "blue")
```



b)

Folosind metoda respingerii vom genera 100000 observatii:

```
valoriRetinute <- c()
n <- 100000
contor <- 0
i <- 1
while( i <= n ) {
  u <- runif(1,0,1)
  x <- rnorm(1,2,4)
  if( u <= h(x)/(M[[2]]*dnorm(x,2,4))) {
    valoriRetinute[contor] <- x
    contor <- contor + 1
  }
  i <- i + 1
}
```

c)

Rata de acceptare se calculeaza astfel:

```r
p <- contor/n
print("Rata de acceptare:")
```

```
## [1] "Rata de acceptare:"
```

```r
print(p)
```

```
## [1] 0.65019
```

Reprezentam histograma:

```r
hist(valoriRetinute, breaks = 100, freq = FALSE, col = "turquoise",
     xlab = "Valori acceptate",
     ylab = "Densitate",
     main = "Histrograma valorilor acceptate",
     cex.main = 0.7)
```

Calculam constanta (aproximata) pentru normalizarea functiei:

```r
normF <- 1/(M[[2]]*p)
print("normF")
```

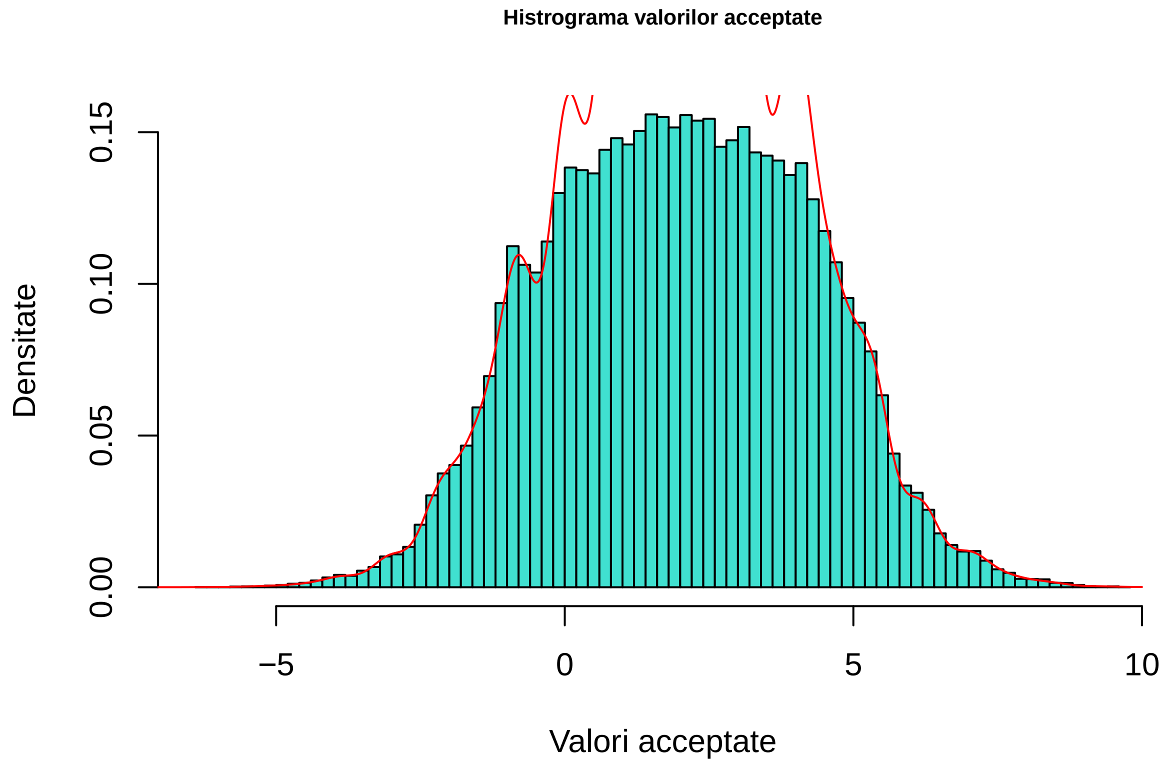
```
## [1] "normF"
```

```r
print(normF)
```

```
## [1] 0.05253583
```

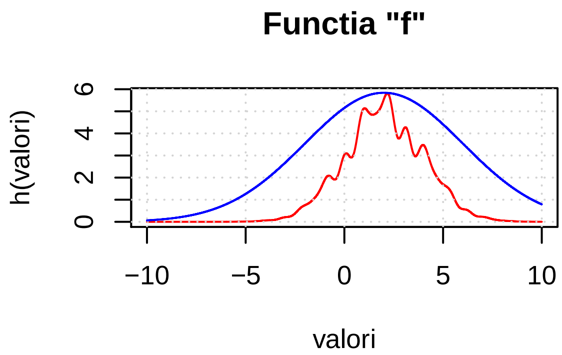
Trasam graficul normalizat peste histograma

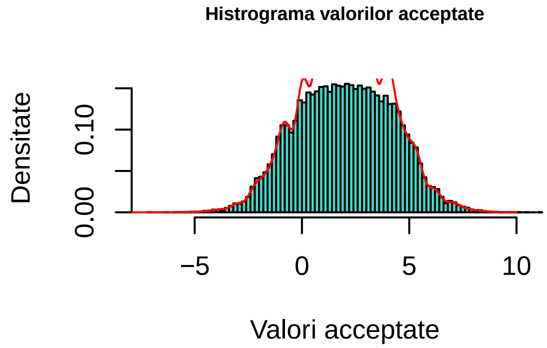
```r
lines(valori, normF*h(valori), col = "red", type="l")
```

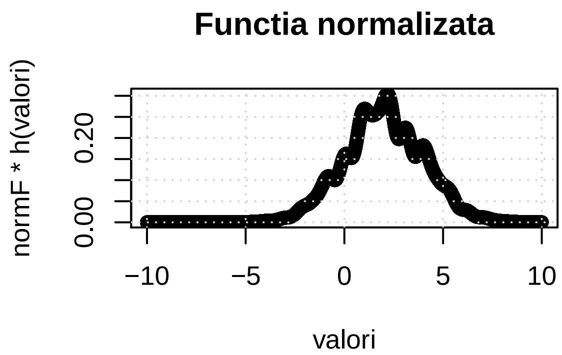



```r
source("Ex2.R")
```

```
## [1] "Rata de acceptare:"
## [1] 0.65183
## [1] "normF"
## [1] 0.05240365
```


Functia "f"
h(valori)
valori


Histrograma valorilor acceptate
Densitate
Valori acceptate


Functia normalizata
normF * h(valori)
valori

Exercitul 3

Se scrie minunata functia $h(x) = (1 - x^2)^{\frac{3}{2}}$ in R Vom calcula valoarea integralei prin pe intervalul [0,1] prin $\frac{1}{n}\sum_{i=1}^{n} h(x_i)$ unde x_i este repartizata uniform pe [0,1].

Vom lua un numar de observatii si le vom retine pentru a le pune in grafic.

```
valInt = c()
valInt[0] = 0
for(i in 1:N) {
  Xn = runif(1,0,1)
  Sn = Sn + funMonteCarlo(Xn) # fac suma, adunand h(Xn), unde h este funtia de mai devreme
  valInt[i] = Sn/i           # pastram valorile pentru a desena graficul
}
```

Astfel ajungem la o aproximare a valorii, o putem compara cu cea optinuta de algoritmul din R:

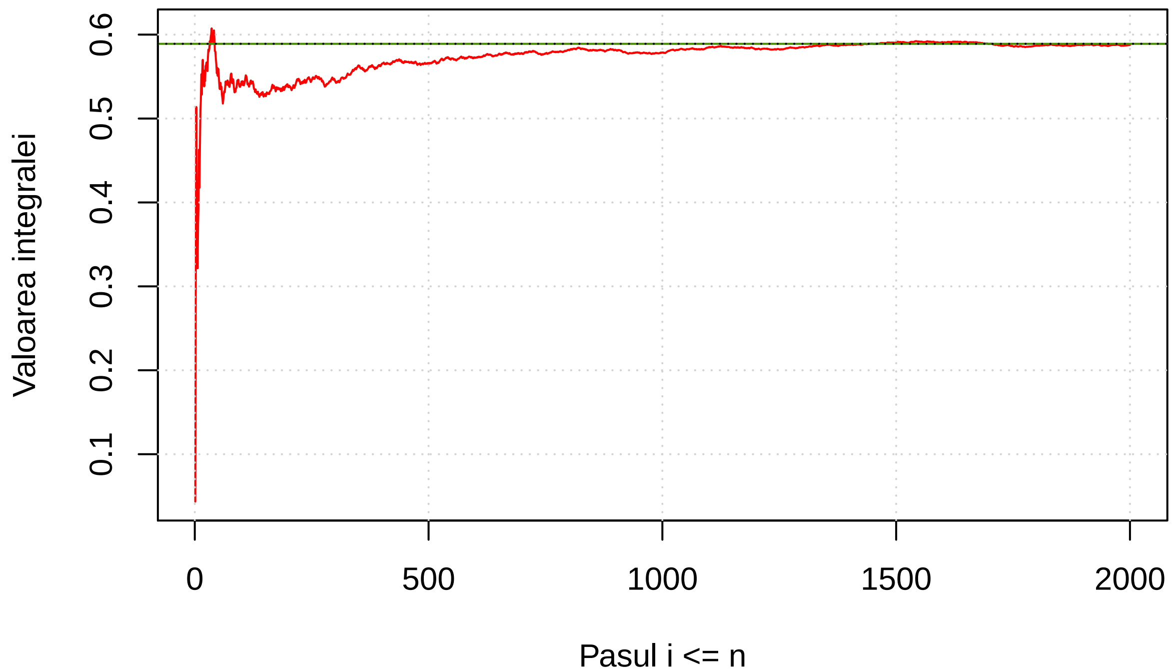
```
valIntMC = Sn/N          #valoarea integralei prin metoda numerica
valR = integrate(funMonteCarlo, 0, 1) # practic si asta tot prin metoda numerica e
```

Valoarea analitica

$$\int (1 - x^2)^{\frac{3}{2}} = \frac{1}{8}\left(x\sqrt{1 - x^2} + 3\sin^{-1} x\right)$$

Iar $\int_0^1 (1 - x^2)^{\frac{3}{2}} = 0.58905$

```
## [1] "Valoare aproximata: 0.587641"
## [1] "Valoare data de functia integrate: 0.589049"
```



Valoarea integralei
0.1
0.2
0.3
0.4
0.5
0.6
0
500
1000
1500
2000
Pasul i <= n

Exercitiul 4

Pentru repartitia Poisson

Se logaritmeaza functia de repartitie $\frac{\lambda e^{-\lambda}}{x!}$ si de ia a 2-a derivata

```
lnf <- expression(log(lambda^x*exp(-lambda)/factorial(x)))
deriv1 <- D(lnf,'lambda')
deriv2 <- D(deriv1,'lambda
```

Se retine derivata ca functie pentru a o folosi in formula pentru MIRC:

```
d2 <- parse(text = deparse(deriv2))
derivata2 <- function(lambda, x){ eval(d2[1]) }
```

Functia frcpois:

```
frcpois <- function(n,lambda,esantion) {
  lnf1 <- expression(log(lambda^x*exp(-lambda)/factorial(x)))
  deriv1 <- D(lnf1,'lambda')
  deriv2 <- D(deriv1,'lambda')
  # din anumite motive trebuie trecuta in text, apoi reparsata in R
  d2 <- parse(text = deparse(deriv2))
  derivata2 <- function(lambda, x){ eval(d2[1]) }

  #rezultatul derivatei a doua

  MIRC <- 1/(-n*mean(eval(derivata2(lambda,esantion))))
  return(MIRC)
}
```

Generam un esantion cu functia rpois() pentru cuntia frcexp()

```
X <- rpois(n,1)
MIRC <- frcpois(n,1,X)
```

Pentru repartitia Exponentiala

Analog:

Se logaritmeaza functia de repartitie $\lambda e^{-\lambda x}$ si de ia a 2-a derivata

```
lnf <- expression(log(lambda*exp(-lambda*x)))
deriv1 <- D(lnf,'lambda')
deriv2 <- D(deriv1,'lambda
```

Functia frcexp:

```r
frcexp <- function(n,lambda, esantion) {
  lnf <- expression(log(lambda*exp(-lambda*x)))         #logaritmam functia de repartitie Exponentiala
  deriv1 <- D(lnf,'lambda')                            #derivata 1 a functiei de mai sus
  deriv2 <- D(deriv1,'lambda')                          #derivata a 2-a a functiei de mai sus

  # ceva mai simplu
  d2 <- parse(text = deparse(deriv2))
  derivata2 <- function(lambda, x){ eval(d2[1]) }

  #rezultatul derivatei a doua
  MIRC <- 1/(-n*mean(eval(derivata2(lambda,esantion))))
  return(MIRC)
}
```

Generam un esantion cu functia rexp() pentru cuntia frcexp()

```r
X <- rexp(n)
MIRC <- frcexp(n,1, X)
```

```r
source("Ex4.R")
```

```
## [1] "Poisson"
## [1] 0.001013171
## [1] "Exponentiala"
## [1] 0.001
```

Exercitiul 6

a)

Pentru repartitiile logistica si respectiv Cauchy (vezi problema 1) construiti functia de verosimilitate pentru parametrul μ considerand ca parametrii β si respectiv σ sunt cunoscuti (alegeti valori potrivite pentru acestia).

Solutie:

- Fie y o observatie repartizata uniform si functia de **repartitie logistica**:

$$F(x) = \frac{1}{1 + e^{-(x-\mu)/\beta}}$$

F (functia de repartitie) este continua, deci calculam inversa acesteia:

$$y = F^{-1}(x) \implies F(y) = x \implies x = \frac{1}{1 + e^{-(x-\mu)/\beta}} \implies$$
$$\implies x + xe^{-(y-\mu)/\beta} = 1 \implies 1 - x = xe^{-(y-\mu)/\beta} \implies$$
$$\implies \beta \cdot ln(x) - \beta ln(1-x) + \mu = y \implies y = \mu + \beta \cdot ln\left(\frac{x}{1-x}\right) \implies$$
$$\implies F^{-1}(x) = \mu + \beta ln\left(\frac{x}{1-x}\right) \implies F^{-1} = \mu - \beta \cdot ln(1-x) + \beta \ln(x)$$

Calculam functia de verosimilitate:

$$L(x_1, x_2, ..., x_n; \theta) = \prod_{i=1}^{n} \frac{e^{-(x-\mu)/\beta}}{\beta(1 + e^{-(x-\mu)/\beta})^2}$$

```r
n <- 100  # pentru esantion

# repartitia logistica
rlogistic <- function(n,miu,beta){
  U <- runif(n) # vector cu n elemente
  X <- miu+beta*log(U)-beta*log(1-U)           #inversa functiei de repartitie
  return(X)
}

g <- rlogistic(n,0,1)      # n repartitii
L <- 1;
beta <- 10;

# functie de verosimilitate pt rep logistica
ver1 <- function(miu)
{
  for(i in 1:n)
  {
    L <- L*exp(-(g[i]-miu)/beta)/beta*(1+exp(-(g[i]-miu)/beta))^2;
  }
  return(L)
}
```

- Fie y o observatie repartizata uniform si functia de repartitie Cauchy:

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \cdot arctan\left(\frac{x-\mu}{\sigma}\right)$$

F (functia de repartitie) este continua, deci calculam inversa acesteia:

$$y = F^{-1}(x) \implies F(y) = x \implies x = \frac{1}{2} + \frac{1}{\pi} \cdot arctan\left(\frac{y-\mu}{\sigma}\right) \implies$$
$$\implies x - \frac{1}{2} = \frac{1}{\pi} arctan\left(\frac{y-\mu}{\sigma}\right) \implies \pi\left(x - \frac{1}{2}\right) = arctan(\frac{y-\mu}{\sigma}) \implies$$
$$\implies tan\left(\pi\left(x-\frac{1}{2}\right)\right) = \frac{y-\mu}{\sigma} \implies \sigma tan\left(\pi\left(x-\frac{1}{2}\right)\right) = y - \mu \implies y = \mu + tan\left(\pi\left(x-\frac{1}{2}\right)\right) \implies$$
$$F^{-1}(x) = \mu + tan\left(\pi\left(x-\frac{1}{2}\right)\right)$$

Calculam functia de verosimilitate:

$$L(x_1, x_2, ..., x_n; \theta) = \prod_{i=1}^{n} \frac{1}{\sigma n} \cdot \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}$$

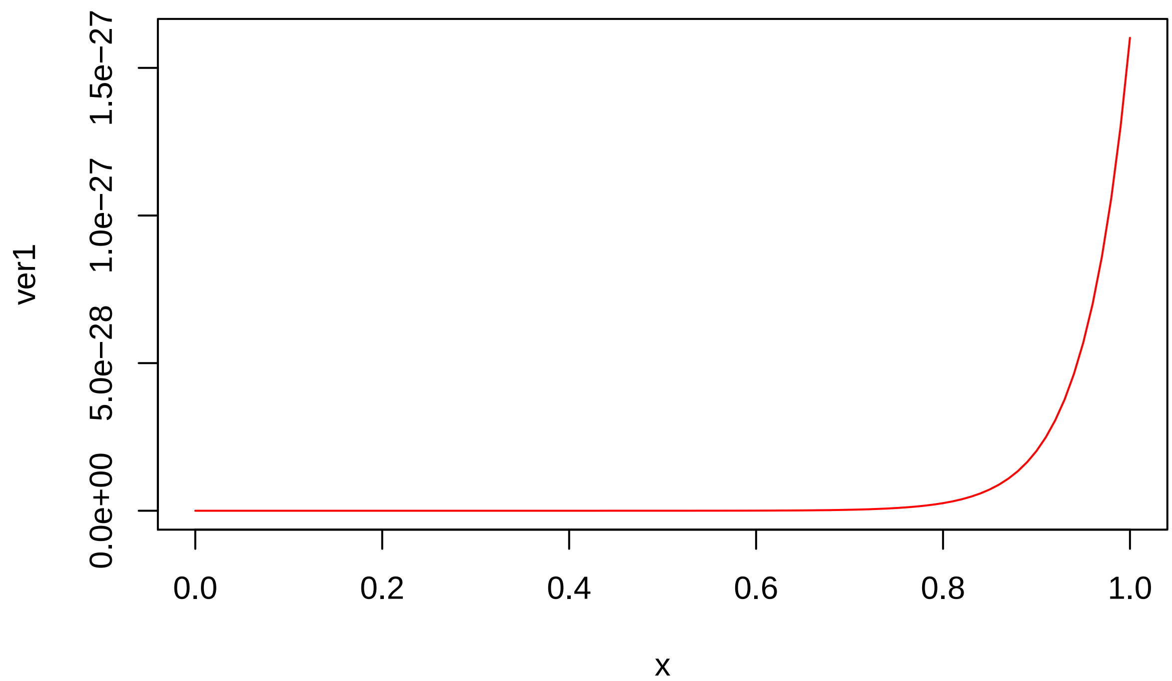
```
rCauchy1 <- function(n,miu,sigma){
  U <- runif(n)
  X <- miu+sigma*tan(pi*(2*U-1/2))
  return(X)
}


f <- rCauchy1(n,0,1)
sigma <- 1;

# functia de verosimilitate pentru rep Cauchy
ver2 <- function(miu){
  for(i in 1:n){
    L <- L*(1/(pi*sigma))*1/(1+((f[i]-miu)/sigma)^2)
  }
  return(L)
}
```

b)

Reprezentati grafic functiile de verosimilitate pentru cele doua cazuri si folosind functia optimise determinati o estimare pentru μ in baza unui esantion de dimensiune 1000 pe care l-ati construit prealabil. Explicati modul in care ati generat valorile din esantion. Comentati si interpretati rezultatele.

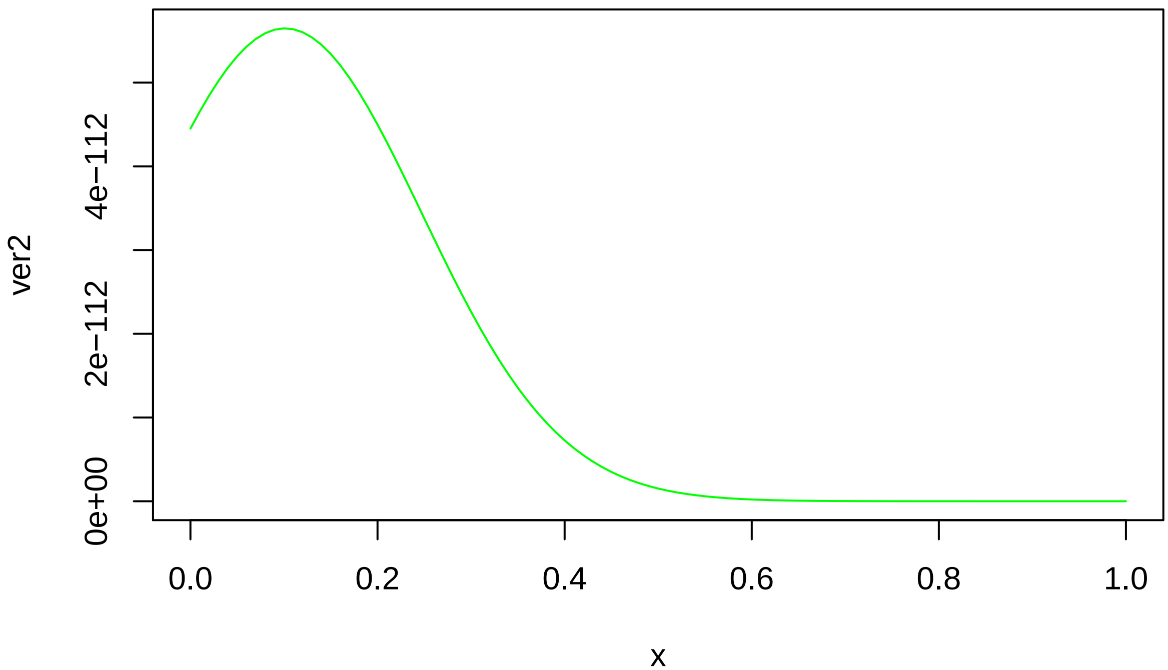
Graficul pentru functia de verosimilitate a repartitiei logistice:

```
plot(ver1, col="red")
```



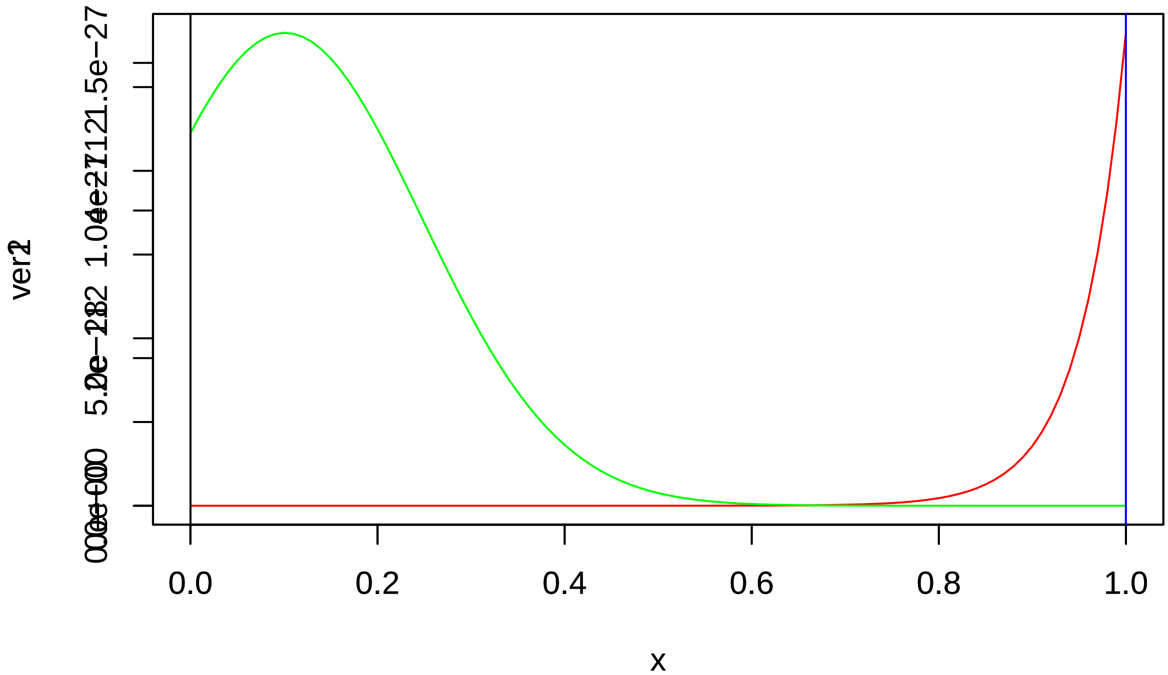
Graficul pentru functia de verosimilitate a repartitiei Cauchy:

```
plot(ver2, col="green")
```



Estimare pentru µ (folosind functia optimise):

```
miu_optim1 <- optimise(ver1,lower=0,upper=1)
miu_optim2 <- optimise(ver2,lower=0,upper=1)
```


0.0
0.2
0.4
0.6
0.8
1.0
x
ver1
ver2