ECUAȚII DIFERENȚIALE
SEMINAR 1

5.10.2022.

SEMINAR - 1 pct

① • Să se determine soluția generală.

a) $tx' - x = t^2 e^t$

! b) $tx' = 2x + t^3 \cos t$

c) $x' + x \operatorname{tg} t = \frac{1}{\cos t}$

d) $x = t(x' - t\cos t)$

Ec. afine scalare: $x' = a(t)x + b(t)$ $\quad a(\cdot), b(\cdot) : J \subseteq \mathbb{R} \to \mathbb{R}$ cont.

Alg. (met. variației const.)

1. consideră ec. liniară asociată: $\bar{x}' = a(t)\bar{x}$
scrie sol. gen: $\bar{x}(t) = c \cdot e^{A(t)}$ $\quad (A'() = a())$

2 „variația" const.
caută sol de forma $x(t) = c(t) \cdot e^{A(t)}$
$x(\cdot)$ - sol a ec $\Rightarrow c'(t) = e^{-A(t)} b(t) \Rightarrow$

$$c(t) = \int e^{-A(t)} \cdot b(t).$$

REZOLVARE:

a) $tx' - x = t^2 e^t \Rightarrow x' = te^t + \frac{x}{t}$, $t > 0$.

$\bar{x}' = \frac{\bar{x}}{t} \Rightarrow \bar{x}(t) = c \cdot e^{\int \frac{1}{t} dt} = c \cdot e^{\ln t} = c \cdot t$, $c \in \mathbb{R}$

caut sol de forma: $x(t) = c(t) \cdot \underset{\shortparallel}{e^{A(t)}} = c(t) \cdot t$

$x()$ - sol $\Rightarrow (c(t) \cdot t)' = t \cdot e^t + \frac{c(t) \cdot \cancel{t}}{\cancel{t}}$.

$$(c(t) \cdot t)' = te^t + c(t)$$

$$c'(t) \cdot t + \cancel{c(t)} = te^t + \cancel{c(t)}$$

$$c'(t) = e^t \Rightarrow$$

$c(t) = \int e^t dt = e^t + b$, $b \in \mathbb{R}$

$\Rightarrow x(t) = (e^t + b) \cdot t$, $b \in \mathbb{R}$.

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a) $x' + \frac{x}{t} = \frac{1}{t^2 x^2}$

b) $x' = x\cos t + x^2 \cos t$

c) $x' - \frac{4}{t} x = t\sqrt{x}$

d) $x' = x^2 e^t - 2x$.

Ec. de tip Bernoulli: $x' = a(t)x + b(t)x^\alpha$, $\alpha \in \mathbb{R} \setminus \{0,1\}$

Algoritm

① considerăm ec. liniară asociată $\bar{x}' = a(t)\bar{x}$

scrie sol. gen: $\bar{x}(t) = c \cdot e^{A(t)}$

② "variația" const

caută sol de forma $\bar{x}(t) = c(t) \cdot e^{A(t)}$

$x(\cdot)$ sol ec $\Rightarrow$ $c'(t) = e^{(\alpha-1)A(t)} \cdot b(t) \cdot c^\alpha(t)$ – ec. cu var sep.

$\xrightarrow{\text{Alg} (*)} c(t) = \dots \rightarrow x(t) = \dots$

RESOLVARE:

d) $x' = x^2 e^t - 2x$

$\bar{x}' = -2\bar{x}$

$\bar{x}(t) = c \cdot e^{-2t}$, $c \in \mathbb{R}$

căutăm sol de forma: $\bar{x}(t) = c(t) \cdot e^{-2t}$

sol. ec. $\Rightarrow (c(t) \cdot e^{-2t})' = (c(t) \cdot e^{-2t})^2 \cdot e^t - 2 \cdot (c(t) \cdot e^{-2t})$

$c'(t) \cdot e^{-2t} + c(t) \cdot (-2e^{-2t}) = (c^2(t) \cdot e^{-4t}) e^t - 2c(t) \cdot e^{-2t}$

$c'(t) \cdot e^{-2t} \cancel{-2c(t) \cdot e^{-2t}} = c^2(t) \cdot e^{-3t} \cancel{- 2c(t) \cdot e^{-2t}}$

$c'(t) = \frac{c^2(t) \cdot e^{-3t}}{e^{-2t}} = c^2(t) e^{-t}$

$\Rightarrow \frac{dc}{dt} = c^2 \cdot e^{-t}$.

Alg (*) pt ec. cu var. sep.: $\frac{dx}{dt} = a(t) \cdot b(x)$

1. rez. ec. $b(x) = 0 \Rightarrow x_1, \dots, x_m$ rădăcini

scrie sol. staționare $\varphi_1(t) = x_1, \dots, \varphi_m(t) = x_m, \dots$

2. pe J_0 "separă" variabilele: $\frac{dx}{b(x)} = a(t) \cdot dt$

$J_0 = \{x \in J \mid b(x) \neq 0\}$

- integrează: $\int \frac{dx}{b(x)} = \int a(t)\,dt$

$$\underbrace{\int \frac{dx}{b(x)}}_{B(x)} = \underbrace{\int a(t)\,dt}_{A(t) + c,\ c \in \mathbb{R}}$$

$\hookrightarrow$ sol. gen. sub formă imp.

- inversează, dacă-i posibil: $x = \varphi(t, c)$

$\hookrightarrow$ sol. gen sub formă explic.

CONTINUARE REZOLVARE

$c^2 = 0 \to c = 0. \to c_0(t) = 0.$

$\frac{dc}{c^2} = dt \cdot e^{-t} \Rightarrow \int \frac{dc}{c^2} = \int e^{-t}\,dt \Rightarrow$

$\Rightarrow -c^{-1} = -e^{-t} + k,\ k \in \mathbb{R}.$

$\Rightarrow c(t) = \frac{1}{+e^{-t}+k},\ k \in \mathbb{R}$

$x_0(t) = 0.$

$x_k(t) - \frac{e^{-2t}}{e^{-t}+k},\ k \in \mathbb{R}.$

III

a) $x' = x^2 - tx - t, \quad \varphi_0(t) = t+1$ sol.

b) $tx' - (2t+1)x + x^2t^2 = 0, \quad \varphi_0(t) = t$ sol

c) $x' = -\frac{x^2}{3} - \frac{2}{3t^2}, \quad \varphi_0(t) - \frac{1}{t}$ sol.

Ecuații Riccati: $x' = a(t)x^2 + b(t)x + c(t)$

$\exists\ \varphi_0(\cdot)$ sol.

Algoritm: s.v. $y = x - \varphi_0(t)$ [$\forall\ x(\cdot)$ sol a ec. s.v. def. o nouă fct $y(\cdot)$ după reg: $y(t) = x(t) - \varphi_0(t)$]

$\Leftrightarrow x(t) = y(t) + \varphi_0(t)$

$x(\,)$ - sol. $\to$ ec. în y de tip Bernoulli $\to$ Alg Bernoulli

$\to y(t) = \ldots$

$x(t) = y(t) + \varphi_0(t)$

REZOLVARE:

a) $x' = x^2 - tx - t$ $\qquad \varphi_0(t) = t + 1$ soluție

S.V.: $y = x - \varphi_0(t) = x - t - 1 \Rightarrow y(t) = x(t) - t - 1 \Leftrightarrow$

$$x(t) = y(t) + t + 1.$$

$$(y(t)+t+1)' = (y(t)+t+1)^2 - t(y(t)+t+1) - t.$$

$$y'(t) + \cancel{1} = y^2(t) + \cancel{t^2} + \cancel{1} + 2y(t)\cdot t + \cancel{2t} + 2y(t) - ty(t) - \cancel{t^2} - \cancel{t} - \cancel{t}.$$

$$y'(t) = y^2(t) + ty(t) + 2y(t)$$

$$y' = y^2 + (t+2)y$$

$$\bar{y}' = (t+2)\bar{y}$$

$$\bar{y}(t) = c\cdot e^{\frac{t^2}{2}+2t}, \quad c \in \mathbb{R}.$$

căutăm sol de forma $y(t) = c(t)\cdot e^{\frac{t^2}{2}+2t}$

$$\left(c(t)\cdot e^{\frac{t^2}{2}+2t}\right)' = \left(c(t)\cdot e^{\frac{t^2}{2}+2t}\right)^2 + (t+2)\left(c(t)\cdot e^{\frac{t^2}{2}+2t}\right)$$

$$c'(t)\cdot e^{\frac{t^2}{2}+2t} + \cancel{c(t)\cdot e^{\frac{t^2}{2}+2t}\cdot(t+2)} = c^2(t)\cdot e^{t^2+4t} + \cancel{c(t)\cdot e^{\frac{t^2}{2}+2t}(t+2)}$$

$$c'(t) = c^2(t)\cdot e^{\frac{t^2}{2}+2t}.$$

$$\frac{dc}{dt} = c^2\cdot e^{\frac{t^2}{2}+2t}.$$

$c^2 = 0 \Rightarrow c = 0 \Rightarrow c_0(t) = 0.$ - sol. staționară

$$\frac{dc}{c^2} = e^{\frac{t^2}{2}+2t}\cdot dt$$

$$\int \frac{dc}{c^2} = \int e^{\frac{t^2}{2}+2t}\, dt.$$

$$-\frac{1}{c} = \int_0^t e^{\frac{s^2}{2}+2s}\, ds + k, \; k \in \mathbb{R} \Rightarrow c_k(t) = \frac{-1}{\int_0^t e^{\frac{s^2}{2}+2s}\, ds + k}, \; k \in \mathbb{R}$$

$$y_0(t) = 0.$$

$$y_k(t) = \frac{-e^{\frac{t^2}{2}+2t}}{\int_0^t e^{\frac{s^2}{2}+2s}\, ds + k}, \quad k \in \mathbb{R}$$

$x_0(t) = y_0(t) + \varphi_0(t) = 0 + t + 1 = t+1$

$$x_k(t) = y_k(t) + \varphi_0(t) = \frac{-e^{\frac{t^2}{2}+2t}}{\int_0^t e^{\frac{s^2}{2}+2s}\,ds + k} + t + 1, \quad k \in \mathbb{R}.$$

IV

a) $x' = \dfrac{x + \sqrt{tx}}{t}$

b) $2t^2 x' = t^2 + x^2$

c) $t x' - x - t e^{\frac{x}{t}}$

Ecuații omogene : $\dfrac{dx}{dt} = f\left(\dfrac{x}{t}\right)$ $\quad f(\cdot): D \subseteq \mathbb{R} \to \mathbb{R}$ cont.

Algoritm : S.V $\quad y = \dfrac{x}{t}$ [$\forall x(\cdot)$ sol s.v. def. fct $y(\cdot)$ după regula $y(t) = \dfrac{x(t)}{t}$]

$x(\cdot)$ sol $\longrightarrow \dfrac{dy}{dt} = \dfrac{f(y)-y}{t}$ ec. cu var. separabile $\to$

Algoritm ec. cu var. sep $\longrightarrow y(t) = \dots$

$\Rightarrow x(t) = t \cdot y(t)$.

REZOLVARE:

b) $2t^2 x' = t^2 + x^2$.

$$x' = \frac{t^2 + x^2}{2t^2} = \frac{x^2}{2t^2} + \frac{1}{2} = \frac{1}{2}\left(\frac{x}{t}\right)^2 + \frac{1}{2}.$$

S.V. $\quad y = \dfrac{x}{t} \Rightarrow y(t) = \dfrac{x(t)}{t} \Rightarrow x(t) = t \cdot y(t)$, $\quad t \neq 0$, $t > 0$!

$$(t \cdot y(t))' = \frac{1}{2}\left(\frac{\cancel{t} \cdot y(t)}{\cancel{t}}\right)^2 + \frac{1}{2}.$$

$$y(t) + t y'(t) = \frac{1}{2} y^2(t) + \frac{1}{2} \Rightarrow$$

$$\Rightarrow y'(t) = \frac{\frac{1}{2} y^2(t) - y(t) + \frac{1}{2}}{t} \Rightarrow$$

$$\Rightarrow y' = \frac{\frac{1}{2} y^2 - y + \frac{1}{2}}{t} \text{ - ec. cu var. separabile.}$$

$$\Rightarrow y' = \frac{(y-1)^2}{2t}.$$

$$\frac{dy}{dt} = \frac{(y-1)^2}{2t}$$

$(y-1)^2 = 0 \Rightarrow y = 1 \Rightarrow y_0(t) = 1$ - sol. staționară

$$\frac{dy}{(y-1)^2} = \frac{dt}{2t} \quad \Big| \int()$$

$$\int \frac{dy}{(y-1)^2} = \int \frac{dt}{2t}$$

$$\Downarrow$$

$$-\frac{1}{y-1} = \frac{1}{2}\ln t + k, \ k \in \mathbb{R}.$$

$$\Rightarrow y_k(t) = 1 - \frac{1}{\frac{1}{2}\ln t + k}, \ k \in \mathbb{R}.$$

$$x_0(t) = y_0(t) \cdot t = t$$

$$x_k(t) = y_k(t) \cdot t = t - \frac{t}{\frac{1}{2}\ln t + k}, \ k \in \mathbb{R}.$$

2.

- Fie $f(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t, x)$

$\varphi_1(\cdot) : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ sol.

$\varphi_2(\cdot) : (b,c) \subseteq \mathbb{R} \to \mathbb{R}$ sol. $\quad \lim\limits_{t \nearrow b} \varphi_1(t) = \lim\limits_{t \searrow b} \varphi_2(t) = x_0.$

Fie $\varphi(\cdot) : (a,c) \to \mathbb{R}$, $\varphi(t) = \begin{cases} \varphi_1(t), & t \in (a,b) \\ x_0, & t = b \\ \varphi_2(t), & t \in (b,c) \end{cases}$

Atunci $\varphi(\cdot)$ este soluție.

$\varphi(\cdot)\big|_{(a,b)} = \varphi_1(\cdot)$ soluție OK.

$\varphi(\cdot)\big|_{(b,c)} = \varphi_2(\cdot)$ soluție OK.

$$\underline{\varphi_s'(b)} = \lim_{\substack{t \to b \\ t < b}} \frac{\varphi(t) - \varphi(b)}{t - b} = \lim_{\substack{t \to b \\ t < b}} \frac{\varphi_1(t) - x_0}{t - b} \overset{L'H}{=}$$

$$= \lim_{\substack{t \to b \\ t < b}} \varphi_1'(t) \underset{\varphi_1 \text{ sol}}{=} \lim_{\substack{t \to b \\ t < b}} f(t, \varphi_1(t)) = f(b, x_0) =$$

$$= \underline{f(b, \varphi(b))}$$

analog $\varphi'_d(b) = \dots = f(b, \varphi(b))$

$\Rightarrow \exists\ \varphi'(b) = f(b, \varphi(b))$ ok.

12.10.2022.

ECUAȚII DIFERENȚIALE
SEMINAR 2

II Ecuații de ordin superior care admit reducerea ordinului

Ec. dif. de ordin m: $F(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^{m+1} \to \mathbb{R}$ $F(t,x,x',\dots,x^{(m)})=0$

Soluție: $\varphi(\cdot): J \subseteq \mathbb{R} \to \mathbb{R}$ m ori deriv $F(t,\varphi(t),\varphi'(t),\dots,\varphi^{(m)}(t))=0$

① $F(t, x^{(k)}, x^{(k+1)}, \dots, x^{(m)}) = 0$, $k \geq 1$

s.v: $y = x^{(k)}$ $\Rightarrow F(t,y,y',\dots,y^{(m-k)})=0$.

② $F\left(t, \frac{x'}{x}, \frac{x''}{x}, \dots, \frac{x^{(m)}}{x}\right)=0$.

s.v: $y=\frac{x'}{x}$ $\Rightarrow G(t,y,y',\dots,y^{(m-1)})=0$.

③ Ec. autonome $F(x,x',x'',\dots,x^{(m)})=0$.

s.v: $x'=y(x)$ $\Rightarrow G(x,y,y',\dots,y^{(m-1)})=0$

④ Ec. de tip Euler

$F(t^0 x, tx', t^2x'', \dots, t^m x^{(m)})=0$.

s.v. $|t|=e^s \longrightarrow G(y,y',\dots,y^{(m)})=0$

Ecuații liniare de ordinul al doilea cu coef. constanți.

$$x''+ax'+bx=0 \quad a,b\in\mathbb{R}$$

Ec. caracteristică: $\lambda^2+a\lambda+b=0 \; < \begin{matrix}\lambda_1\\ \lambda_2\end{matrix}$

- dacă $\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1\neq\lambda_2 \Rightarrow$ sol. gen: $x(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$, $c_1,c_2\in\mathbb{R}$
- dacă $\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1=\lambda_2=\lambda \Rightarrow$ sol gen: $x(t)=c_1e^{\lambda t}+c_2 t\cdot e^{\lambda t}$, $c_1,c_2\in\mathbb{R}$
- dacă $\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$, $\beta>0$ $\Rightarrow$ sol. gen: $x(t)=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t$

Să se determine sol. generală:

I
a) $x^{IV} + x''' = 0$

b) $x'' = 1 - \frac{x'}{t}$ $\quad y = x' \quad y' = 1 - \frac{y}{t}$ ec. afină scalară

c) $2tx'x'' = (x')^2 - 1 \Rightarrow y = x' \Rightarrow y' = \frac{y^2-1}{2ty}$ ec. cu var. separabile (mai simplă)

$y' = \frac{1}{2t} y - \frac{1}{2t} \cdot \frac{1}{y}$ ec. Bernoulli

a) $x^{IV} + x''' = 0$

$y = x''' \Rightarrow y' + y = 0 \Rightarrow y' = -y \Rightarrow y(t) = c \cdot e^{-t}, \ c \in \mathbb{R}$

$x''' = y = c e^{-t} \Rightarrow x'' = -c e^{-t} + c_1, \ c_1 \in \mathbb{R}$

$\Rightarrow x' = c \cdot e^{-t} + c_1 t + c_2, \ c_2 \in \mathbb{R}$

$\Rightarrow x = -c e^{-t} + c_1 \frac{t^2}{2} + c_2 t + c_3, \ c_3 \in \mathbb{R}$

II
a) $t^2 x x'' - (x - tx')^2 = 0$

b) $txx'' + t(x')^2 - xx' = 0$.

a) $t^2 x \cdot x'' - (x - tx')^2 = 0 \ | : x^2 \quad \Leftrightarrow t^2 x \cdot x'' - x^2 + 2xtx' - t^2 x'^2 = 0$.

observație: $x(t) = 0$ - sol

$$\frac{t^2 \cdot x''}{x} - 1 + 2 \cdot \frac{tx'}{x} - t^2 \left(\frac{x'}{x}\right)^2 = 0.$$

S.V. $y = \frac{x'}{x} \Rightarrow y(t) = \frac{x'(t)}{x(t)}$

$\Downarrow$

$x'(t) = y(t) \cdot x(t) \quad | ()'$

$x''(t) = y'(t) \cdot x(t) + y(t) \cdot x'(t) \quad | : x(t)$

$$\frac{x''(t)}{x(t)} = y'(t) + y^2(t).$$

$$\frac{x''}{x} = y^2 + y'$$

$\Rightarrow t^2 \cdot (y' + y^2) - 1 + 2ty - t^2 y^2 = 0$

$t^2 y' + \cancel{t^2 y^2} + 2ty - \cancel{t^2 y^2} - 1 = 0 \Rightarrow y' = \frac{1 - 2ty}{t^2}, \ t \neq 0$

Presupunem că $t > 0$.

$\Downarrow$

$y' = \frac{1}{t^2} - \frac{2y}{t}$ - ec. afină scalară

$$\bar{y}' = -\frac{2}{t}\bar{y}$$

$$\Rightarrow \bar{y}(t) = c \cdot e^{\int -\frac{2}{t} dt} = c \cdot e^{-2\ln t} = c \cdot t^{-2}$$

$$y(t) = \frac{c(t)}{t^2}$$

$$\left(\frac{c(t)}{t^2}\right)' = \frac{1}{t^2} - \frac{2 \cdot c(t)}{t^3}$$

$$\frac{c'(t) \cdot t^2 - \cancel{c(t) \cdot 2t}}{t^4} = \frac{t^3 - \cancel{2t c(t)}}{t^4}$$

$$c'(t) = 1 \to c(t) = t + k, \ k \in \mathbb{R}$$

$$\Rightarrow y(t) = \frac{1}{t} + \frac{k}{t^2}, \ k \in \mathbb{R}$$

$$y = \frac{x'}{x} \Rightarrow \frac{x'}{x} = \frac{1}{t} + \frac{k}{t^2}, \ k \in \mathbb{R}$$

$$\Rightarrow x' = \left(\frac{1}{t} + \frac{k}{t^2}\right) \cdot x$$

$$x(t) = c \cdot e^{\int \frac{1}{t} + \frac{k}{t^2} dt} = c \cdot e^{\ln t - k \cdot \frac{1}{t}} = c \cdot t \cdot e^{-\frac{k}{t}}, \ c, k \in \mathbb{R}.$$

III $\begin{cases} a) \ x x'' + 1 = (x')^2 \\ b) \ x x'' + (x')^2 = 0. \end{cases}$

b) $x x'' + (x')^2 = 0.$

Metoda 1: $(: x^2 \to ②)$

Metoda 2: ec. autonomă

$x' = y(x)$

se caută o funcție $y(\cdot)$ a.î. $x'(t) = y(x(t)), \ \forall t$

$$x'(t) = y(x(t))$$

$$x''(t) = y'(x(t)) \cdot x'(t) = y'(x(t)) \cdot y(x(t))$$

$$\Rightarrow x'' = y'(x) \cdot y(x)$$

$$\Rightarrow x y'(x) \cdot y(x) + y^2(x) = 0.$$

$$y(x)\left(x y'(x) + y(x)\right) = 0 \Rightarrow \begin{cases} y(x) = 0. \\ x y'(x) + y(x) = 0 \end{cases} \Rightarrow y'(x) = -\frac{y(x)}{x}$$

$y(x) = c \cdot e^{\int -\frac{1}{x} dx} = c \cdot e^{-\ln x} = \frac{c}{x}, \quad c \in \mathbb{R}$

① $\Rightarrow x' = 0 \Rightarrow x(t) = k, \; k \in \mathbb{R}$

② $\Rightarrow x' = y(x) = \frac{c}{x} \Rightarrow \frac{dx}{dt} = \frac{c}{x} \Rightarrow x\,dx = c \cdot dt \; | \int$

$\frac{x^2}{2} = ct + k_1, \; k_1 \in \mathbb{R}$

$\Rightarrow x(t) = \pm \sqrt{2ct + 2k_1}, \quad c, k_1 \in \mathbb{R}$

metoda 1:

$x x'' + (x')^2 = 0 \quad | : x^2$

$\frac{x''}{x} + \left(\frac{x'}{x}\right)^2 = 0$

$y = \frac{x'}{x} \Rightarrow y(t) = \frac{x'(t)}{x(t)}$

$x'(t) = y(t) \cdot x(t) \; | \; ()'$

x''

a) $t^2x'' - tx' - 3x = 0$

b) $t^2x'' + tx' + x = 0$

c) $t^2x'' - 2tx' + 2x = 0$

Ec. Euler:

$$\text{S.V: } |t| = e^s \begin{cases} t = e^s, & \text{dacă } t > 0 \\ t = -e^s, & \text{dacă } t < 0 \end{cases}$$

Fie $t > 0 \to$ S.V. $t = e^s$ $\forall x(\cdot)$ sol. a ec. s.v. def funcția $y(\cdot)$ după regula $y(s) = x(e^s)$

$x(t) \cancel{\Rightarrow} x(e^s)$ – NU!!

$$x(e^s) = y(s) \iff x(t) = y(\ln t)$$

a) $e^{2s}x''(e^s) - e^s x'(e^s) - 3x(e^s) = 0.$

$$x(e^s) = y(s) \mid ()'$$

$$x'(e^s) \cdot e^s = y'(s) \Rightarrow \underline{x'(e^s)} = \frac{y'(s)}{e^s} = \underline{y'(s) \cdot e^{-s}} \mid ()'$$

$$x''(e^s) \cdot e^s = y''(s) \cdot e^{-s} + y'(s) \cdot (-e^{-s}) \Rightarrow$$

$$x''(e^s) = y''(s) \cdot e^{-2s} - y'(s) \cdot e^{-2s}$$

$$\Rightarrow e^{2s}\left(y''(s)e^{-2s} - y'(s) \cdot e^{-2s}\right) - e^s\left(y'(s) \cdot e^{-s}\right) - 3y(s) = 0$$

$$y''(s) - y'(s) - y'(s) - 3y(s) = 0.$$

$$y''(s) - 2y'(s) - 3y(s) = 0.$$

$y'' - 2y' - 3y = 0$. ec. liniară de ord. al II-lea cu coef. const.

$$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \begin{cases} -1 \\ 3. \end{cases}$$

$$y(s) = c_1 e^{-s} + c_2 e^{3s}, \ c_1, c_2 \in \mathbb{R}.$$

$$\underline{\underline{x(t)}} = y(\ln t) = c_1 \cdot e^{-\ln t} + c_2 e^{\ln 3} = \underline{\underline{\frac{c_1}{t} + c_2 \cdot t^3}}, \ c_1, c_2 \in \mathbb{R}.$$

ECUAȚII DIFERENȚIALE
SEMINAR 3

19.10.2022

• $\frac{dx}{dt} = f(t, x)$ $\qquad$ $f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

T. Peano (E.L)

★ $D = \mathring{D}$, $f(\cdot,\cdot)$ cont. $\Rightarrow$ E.L. pe D $(\forall (t_0, x_0) \in D, \exists J_0 \in \mathcal{V}(t_0), \exists \varphi(\cdot): J_0 \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0)$

T. Cauchy - Lipschitz (E.U.L)

★ $D = \mathring{D}$, $f(\cdot,\cdot)$-cont, local Lipschitz (II) $\Rightarrow$ E.U.L. pe D $(\forall (t_0, x_0) \in D$, $\exists J_0 \in \mathcal{V}(t_0)$, $\exists!$ $\varphi(\cdot): J_0 \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0)$

Să se studieze E.U.L a sol:

1) $x' = \alpha(t)$ $\qquad$ $\alpha(t) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$

$f(t, x) = \alpha(t)$ $\qquad$ $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

arătăm că ec. nu are prop. E.L. în niciun punct

Pp. prin R.A. că $\exists (t_0, x_0) \in \mathbb{R} \times \mathbb{R}$, $\varphi: J_0 \to \mathbb{R}$ sol. ai $\varphi(t_0) = x_0$.

$\left.\begin{matrix} \varphi\text{- deriv. și } \varphi'(t) = \alpha(t) \to \text{are prop. lui Darboux} \\ \varphi(t_0) = x_0 \to \text{nu are prop lui Darboux} \end{matrix}\right\} \Rightarrow$ ∡e.

2) $x' = \operatorname{sgn} x$ $\qquad$ $\operatorname{sgn} x = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

$f(t, x) = \operatorname{sign}(x)$ $\qquad$ $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

• Fie $D = \mathbb{R} \times \mathbb{R}^*$ - mulț deschisă, $f|_D$ - fct continuă

fct $f|_D$ este de clasă $C^1(\text{II}) \Rightarrow f|_D$ - local Lipchitz (II)

$\xRightarrow{\text{Th. Cauchy-Lipschitz}}$ E.U.L. pe D.

? E.U.L în pct. de forma $(t_0, 0)$, $t_0 \in \mathbb{R}$

?. $\varphi : J_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol. $\varphi(t_0) = 0 \Rightarrow \varphi'(t) = \operatorname{sign}(\varphi(t)), \forall t \in \mathbb{R}$

$\varphi(t_0) = 0$

Obs: $\varphi_0(t) = 0$ verif. cele 2 cond.

$\Rightarrow$ E.L.

Pp. $\exists\ \varphi_1 : J_1 \to \mathbb{R}$ sol. $\varphi_1(t_0) = 0$.

$\varphi_1 \neq \varphi_0 \Rightarrow \exists\ t_1$ aî $\varphi_1(t_1) \neq 0$.

$\varphi_1(t_1) > 0$

φ_1 - sol. $\Rightarrow \varphi_1'(t) = \operatorname{sign}(\varphi_1(t)), \forall t \in \mathbb{R} \Rightarrow$ (*)

$\varphi_1(t_0) = 0$.

(*) $\Rightarrow t = t_0 \Rightarrow \varphi_1'(t_0) = \operatorname{sgn} \varphi_1(t_0) = 0$.

$t = t_1 \Rightarrow \varphi_1'(t_1) = \operatorname{sgn} \underbrace{\varphi_1(t_1)}_{>0} = 1$

φ_1' are prop. lui Darboux $\Rightarrow$ pt $\frac{1}{2022} \in (0,1) \Rightarrow \exists\ t_2 \in (t_0, t_1)$ aî

$\varphi_1'(t_2) = \frac{1}{2022} \notin \{-1, 0, 1\}$ Fals.

$\Rightarrow$ sol. unică

3) $x' = 3x^{\frac{2}{3}} + a,\ a \in \mathbb{R}$.

- $f(t,x) = 3x^{\frac{2}{3}} + a$, $f : \underbrace{\mathbb{R} \times \mathbb{R}}_{\text{mulț. deschisă}} \to \mathbb{R}$, f - fct continuă

$\xRightarrow{\text{T. Peano}}$ EL pe $\mathbb{R} \times \mathbb{R}$

- fie $g(x) = x^{\frac{2}{3}}$

$g'(x) = \frac{2}{3} \cdot x^{-\frac{1}{3}} = \frac{2}{3} \cdot \frac{1}{\sqrt[3]{x}}$

g de clasă C^1 pe $\mathbb{R}^* \Rightarrow f$ de clasă $C^1(\mathrm{II})$ pe $\mathbb{R} \times \mathbb{R}^* \Rightarrow f$ local Lipschitz (II) pe $\mathbb{R} \times \mathbb{R}^*$ $\xRightarrow{\text{T. Cauchy-Lipschitz}}$ E.U.L pe $\mathbb{R} \times \mathbb{R}^*$

- Arătăm că g nu este local Lipschitz în $x = 0$.

$g : G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ s.m. local Lipschitz în $x_0 \in G$ dacă $\exists\ r > 0, L > 0$ aî $\|g(x) - g(y)\| \leq L\|x, y\|$, $\forall\ x, y \in B_r(x_0) \cap G$

BLON - nu e fct. Lipschitz

g nu e local Lipschitz în x_0, dacă $\forall r>0, \forall L>0, \exists x_{r,L}, y_{r,L} \in B_r(x_0) \cap G$ a.î. $\|g(x_{r,L}) - g(y_{r,L})\| > L\|x_{r,L} - y_{r,L}\|$.

$\forall m \in \mathbb{N}^*, \exists x_m, y_m$ din $B_{\frac{1}{m}}(x_0) \cap G$ a.î. $\|g(x_m) - g(y_m)\| > m\|x_m - y_m\|$

$\begin{pmatrix} r \to \frac{1}{m} \\ L \to m \end{pmatrix}$

$(y_m = x_0, \forall m)$

$\forall m \in \mathbb{N}^*, \exists (x_m) \subset B_{\frac{1}{m}}(x_0) \cap G$ a.î. $\|g(x_m) - g(x_0)\| > m\|x_m - x_0\|$

Revenind, $g(x) = x^{\frac{2}{3}}$; $G = \mathbb{R}$; $x_0 = 0$.

? $x_m \subset B_{\frac{1}{m}}(0)$ a.î. $|x_m^{\frac{2}{3}} - 0| > m|x_m - 0|$

$\Downarrow$ $|x_m| < \frac{1}{m}$

$\Updownarrow$ $|x_m^{\frac{2}{3}}| > m \cdot |x_m| \quad | \, ()^3$

$\Rightarrow x_m^{\cancel{2}} > m^3 |x_m|^{\cancel{3}}$

$\Rightarrow |x_m| < \frac{1}{m^3}$

Fie $x_m = \frac{1}{m^4} \Rightarrow$ Ok.

$x' = 3x^{\frac{2}{3}} + a, \; a \in \mathbb{R}$

? U.L. în $(t_0, 0)$, $t_0 \in \mathbb{R}$

ec. cu var. sep. $\frac{dx}{dt} = a(t) \cdot b(x)$

$a(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}$
$b(\cdot): J \subseteq \mathbb{R} \to \mathbb{R}$ | cont.
$J_0 = \{x \in J \mid b(x) \neq 0\}$

• Prop (Exist. și unicitatea locală a sol.)

1. $\forall (t_0, x_0) \in I \times J, \exists I_0 \in \mathcal{V}(t_0) \; \exists \varphi(\cdot): I_0 \to J$ sol. cu $\varphi(t_0) = x_0$.

$\boxed{2.}$ $\forall (t_0, x_0) \in I \times J_0, \exists I_0 \in \mathcal{V}(t_0) \; \exists! \varphi(\cdot): I_0 \to J$ sol. cu $\varphi(t_0) = x_0$

$a(t) \equiv 1$

$b(x) = 3x^{\frac{2}{3}} + a$

$I = \mathbb{R}$

$J = \mathbb{R}$

$J_0 = \{x \in \mathbb{R} \mid 3x^{\frac{2}{3}} + a \neq 0\}$

$3x^{\frac{2}{3}} = -a \Rightarrow x^{\frac{2}{3}} = \frac{-a}{3} \;|\; ()^3 \Rightarrow x^2 = \frac{-a^3}{27} \Rightarrow x = \pm\sqrt{\frac{-a^3}{27}}$.

Dc. $a > 0 \Rightarrow$ ec. nu are răd.

Dc. $a < 0 \Rightarrow$ ec. are 2 răd: $\pm\sqrt{\frac{-a^3}{27}}$.

Dc. $a = 0 \Rightarrow$ o. răd. $= 0$.

$$J_0 = \begin{cases} \mathbb{R}, & a > 0 \\ \mathbb{R} \setminus \left\{\pm\sqrt{\frac{-a^3}{27}}\right\}, & a < 0 \\ \mathbb{R}^*, & a = 0. \end{cases}$$

$(t_0, 0) \in \mathbb{R} \times J_0$?

1. dc. $a > 0$, $(t_0, 0) \overset{?}{\in} \mathbb{R} \times \mathbb{R}$ DA $\overset{PROP}{\Longrightarrow}$ E.U.L în $(t_0, 0)$
2. dc. $a < 0$, $(t_0, 0) \overset{?}{\in} \mathbb{R} \times \left(\mathbb{R} \setminus \left\{\pm\sqrt{\frac{-a^3}{27}}\right\}\right)$ DA $\overset{PROP}{\Longrightarrow}$ E.U.L în $(t_0, 0)$
3. dc. $a = 0$, $(t_0, 0) \overset{?}{\in} \mathbb{R} \times \mathbb{R}^*$ NU $\Longrightarrow$ nu putem folosi prop. anterioară.

pt. $a = 0 \Rightarrow x' = 3x^{\frac{2}{3}}$ – ec. cu var. separabile

? U.L. în $(t_0, 0)$, $t_0 \in \mathbb{R}$.

$\frac{dx}{dt} = 3x^{\frac{2}{3}}$ $\qquad 3x^{\frac{2}{3}} = 0 \Rightarrow x = 0 \Rightarrow x_1(t) \equiv 0$ sol. staționară

$\frac{dx}{3x^{\frac{2}{3}}} = dt \Rightarrow \int \frac{1}{3x^{\frac{2}{3}}} \, dx = \int dt$

$$\frac{1}{3} \cdot \frac{x^{\frac{1}{3}}}{\frac{1}{3}} = t + k, \ k \in \mathbb{R}$$

$$x^{\frac{1}{3}} = t + k, \ k \in \mathbb{R}$$

$\Rightarrow x(t) = (t + k)^3, \ k \in \mathbb{R}$

$x(t_0) = 0$.

$\Rightarrow (t_0 + k)^3 = 0$

$\Downarrow$

$t_0 + k = 0 \Rightarrow$

$k = -t_0$

$\Rightarrow x_2(t) = (t - t_0)^3$

$\Rightarrow x_1(t) \neq x_2(t) \Rightarrow$ nu avem unicitate locală în $(t_0, 0)$

4) $x' = 4 + \sqrt[3]{x-4t}$

$f(t,x) = 4 + \sqrt[3]{x-4t}$ $\quad f(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, cont $\overset{\text{Peano}}{\Longrightarrow}$ E.L. pe $\mathbb{R}\times\mathbb{R}$

$\frac{\partial f}{\partial x}(t,x) = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(x-4t)^2}}$

mulț. închisă

$D_1 = \{(t,x) \in \mathbb{R}\times\mathbb{R},\ x-4t=0\} = \{(t,4t) \mid t\in\mathbb{R}\}$

$D = \mathbb{R}\times\mathbb{R} \setminus D_1$ deschisă pt că D_1 - închisă

$f|_D$ $C^1(\text{II}) \Rightarrow$ local Lipschitz (II) $\overset{\text{T.C-L}}{\Longrightarrow}$ E.U.L pe D!

f nu este local Lipschitz (II) în $(t_0, 4t_0)$

? U.L în $(t_0, 4t_0)$

$x' = 4 + \sqrt[3]{x-4t}$

$x' - 4 = \sqrt[3]{x-4t}$

S.V. $y = x - 4t$ $(y(t) = x(t) - 4t) \Rightarrow y' = \sqrt[3]{y}$ - ec. cu var. sep.

$\Rightarrow$ neunicitate.

ECUAȚII DIFERENȚIALE

SEMINAR 4

26.10.2022.

$\star$ $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$ $\quad f(\cdot,\cdot): D \subset \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$

T. Peano (pt. ec. de ordin superior)

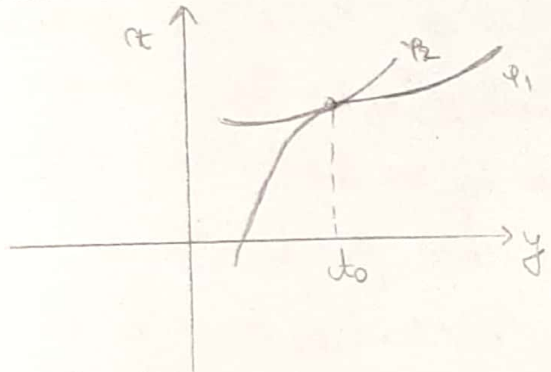
$D = \mathring{D}$, $f(\cdot,\cdot)$ cont $\Rightarrow$ E.L pe D ($\forall (t_0, (x_0, x_0', \dots x_0^{n-1})) \in D$, $\exists J_0 \in \mathcal{V}(t_0)$, $\exists \varphi(\cdot): J_0 \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0'$, $\dots$, $\varphi^{(n-1)}(t_0) = x_0^{n-1}$)

T. Cauchy-Lipschitz (pt ec de ord sup)

$D = \mathring{D}$, $f(\cdot,\cdot)$ cont, local Lipschitz (II) $\Rightarrow$ E.U.L pe D ($\forall (t_0, (x_0, x_0', \dots x_0^{n-1}))$, $\exists J_0 \in \mathcal{V}(t_0)$, $\exists! \varphi(\cdot): J_0 \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0'$, $\dots$, $\varphi^{(n-1)}(t_0) = x_0^{n-1}$).

1. Să se studieze posibilitatea ca graficele a 2 sol. diferite să fie tangente pt ec:
a) $x' = t^2 + x^4$
b) $x'' = t^2 + x^4$
c) $x''' = t^2 + x^4$.

SOLUȚIE:



$\varphi_1: J_1 \to \mathbb{R}$
$\varphi_2: J_2 \to \mathbb{R}$ $\Big|$ $\exists t_0 \in J_1 \cap J_2$ a.î.
$\varphi_1(t_0) = \varphi_2(t_0)$
$\varphi_1'(t_0) = \varphi_2'(t_0)$

a) $x' = t^2 + x^4$

$f(t,x) = t^2 + x^4$, $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, cont., local Lipschitz (II) $\Longrightarrow$ E.U.L pe $\mathbb{R}\times\mathbb{R}$
$\Rightarrow \forall (t_0, x_0) \in \mathbb{R}\times\mathbb{R}$, $\exists! \varphi(\cdot): J_0 \in \mathcal{V}(t_0) \to \mathbb{R}$, sol cu $\varphi(t_0) = x_0$ $\Rightarrow$ NU (Cauchy-Lipschitz)

b) $x'' = t^2 + x^4$

$f(t, (x_1, x_2)) = t^2 + x_1^4$, $f: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ cont, local Lipschitz (II) $\Rightarrow$ E.U.L pe $\mathbb{R}\times\mathbb{R}^2$ $\Rightarrow \forall (t_0, (x_0, x_0')) \in \mathbb{R}\times\mathbb{R}^2$, $\exists! \varphi(\cdot): J_0 \in \mathcal{V}(t_0) \to \mathbb{R}^2$, sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0'$ $\Rightarrow$ NU

c) $x''' = t^2 + x^4$

$f(t, (x_1, x_2, x_3)) = t^2 + x_1^4$, $f: \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ cont, local Lipschitz (II) $\Rightarrow$ E.U.L. pe $\mathbb{R}\times\mathbb{R}^3$ $\Rightarrow \forall (t_0, (x_0, x_0', x_0^2)) \in \mathbb{R}\times\mathbb{R}^3$, $\exists! \varphi(\cdot): J_0 \in \mathcal{V}(t_0) \to \mathbb{R}^3$, sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0'$, $\varphi''(t_0) = x_0''$ $\Rightarrow$

dacă $\varphi_1''(t_0) = a$, $\varphi_2''(t_0) = b$, $a \neq b \Rightarrow$ DA

2. $\forall m \in \mathbb{N}$, să se det. α_m, α_m = nr. sol. posibile ale urm. pb:

$$x^{(m)} = t + x^3, \quad x(0) = 1, \quad x'(0) = 0.$$

SOLUȚIE:

* $m=0 \Rightarrow x = t + x^3 \qquad x(0) = 1 \qquad x'(0) = 0.$

$x(t) = t + x^3(t), \forall t$

$t=0 \Rightarrow x(0) = 0 + x^3(t)$

$1 = 0 + 1^3$ (A)

$x'(t) = 1 + 3x^2(t) \cdot x'(t)$

$t=0 \Rightarrow 0 = 1 + 3 \cdot 1 \cdot 0$

$0 = 1$ $\cancel{\text{Æ}}$ $\Rightarrow \alpha_0 = 0$

* $m=1 \Rightarrow x' = t + x^3 \qquad x(0) = 1 \qquad x'(0) = 0$

$x'(t) = t + x^3(t), \forall t$

$t=0 \Rightarrow 0 = 0 + 1^3$ Æ. $\Rightarrow \alpha_1 = 0$

* $m=2 \Rightarrow x'' = t + x^3 \quad x(0) = 1, \quad x'(0) = 0$

$f(t, (x_1, x_2)) = t + x_1^3$, $f: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, cont, local Lips (ii) $\to$ EU.L pe $\mathbb{R} \times \mathbb{R}^2$,

$\exists!$ $\varphi(\cdot): J_0 \in \mathcal{V}(0) \to \mathbb{R}$ sol. pt care $\varphi(0) = 1$, $\varphi'(0) = 0 \Rightarrow \alpha_2 = 1$

* $m=3 \Rightarrow x''' = t + x^3, \quad x(0) = 1, \quad x'(0) = 0$

$f(t, (x_1, x_2, x_3)) = t + x_1^3$, $f: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, cont, local Lips (ii) $\Rightarrow$

$\forall a \in \mathbb{R}$, $\exists$ $\varphi_a: J_A \in \mathcal{V}(0) \to \mathbb{R}$ sol. care satisf. $\varphi_a(0) = 1$, $\varphi_a'(0) = 0$, $\varphi_a''(0) = a$

dar $a \in \mathbb{R} \Rightarrow \alpha_3 = \infty$

* pt $m \geq 3 \Rightarrow \alpha_m = \infty$.

3. Fie $f_1(\cdot,\cdot): \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$, $f_2(\cdot): \mathbb{R}^k \to \mathbb{R}^k$ def.

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} = f_2(x_2) \end{cases}$$

a) Să se arate că dacă $f_1(\cdot,\cdot)$ cont, local Lipschitz (I), dacă $f_2(\cdot)$ local Lipschitz at. ea admite EUL.

b) Să se arate că $\exists$ $f_1^0(\cdot,\cdot)$ cont, local Lipschitz (I) și $\exists$ $f_2^0(\cdot)$ local Lipschitz a.î. $f^0(\cdot,\cdot) - (f_1^0(\cdot,\cdot), f_2^0(\cdot))$ nu este local Lipschitz.

...utie:

a) • $f(t,(x_1,x_2)) = (f_1(x_1,x_2), f_2(x_2))$, $f(\cdot,\cdot): \mathbb{R}\times\mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$

• E.U.L $\Rightarrow \forall (t_0,x_0)\in \mathbb{R}\times\mathbb{R}^{m+k}$, $\exists J_0 \in \mathcal{V}(t_0)$, $\exists! \varphi: J_0 \to \mathbb{R}^{m+k}$ sol cu $\varphi(t_0) = x_0$

$x_0 = (x_0^1, x_0^2)$

$\varphi = (\varphi_1, \varphi_2)$ ai $\varphi_1(t_0) = x_0^1$

$\varphi_2(t_0) = x_0^2$

* fie Pb. Cauchy (2) $\frac{dx_2}{dt} = f_2(x_2)$, $x_2(t_0) = x_0^2$

• fie $g(t,x_2) = f_2(x_2)$, $g: \mathbb{R}\times\mathbb{R}^k \to \mathbb{R}^k$, $\mathbb{R}\times\mathbb{R}^k$ - mulț deschisă, fct g - continuă (fiindcă f_2-continuă (local Lipschitz)), local Lipschitz (II)

$\xRightarrow[\text{Lipschitz}]{\text{T.Cauchy}}$ $\exists J_0 \in \mathcal{V}(t_0)$, $\exists! \varphi_2: J_0 \to \mathbb{R}^k$ sol a pb Cauchy (2)

$(\varphi_2(t_0) = x_0^2)$ (*)

* fie Pb. Cauchy (1): $\frac{dx_1}{dt} = f_1(x_1, \underset{x_2}{\underset{\parallel}{\varphi_2(t)}})$; $x_1(t_0) = x_0^1$

• fie $h(t,x_1) = f_1(x_1, \varphi_2(t))$,

$h: \underset{\text{ca să fie mulț desch}}{\overset{\circ}{J_0}}\times\mathbb{R}^m \to \mathbb{R}^m$ - deschisă, h - continuă (comp. de fct continue), local Lipschitz (II)

$\xRightarrow[\text{Lipschitz}]{\text{T.Cauchy}}$ $\exists \mathcal{J}_0 \in \mathcal{V}(t_0) \subset \overset{\circ}{J_0}$, $\exists! \varphi_1: \mathcal{J}_0 \to \mathbb{R}^m$ sol a pb. Cauchy (1)

$(\varphi_1(t_0) = x_0^1)$ (**)

din (*) și (**) $\Rightarrow$ $\forall (t_0, \overbrace{(x_0^1, x_0^2)}^{x_0}) \in \mathbb{R}\times\mathbb{R}^{m+k}$, $\exists \mathcal{J}_0 \in \mathcal{V}(t_0)$, $\exists! \varphi(\varphi_1,\varphi_2)$: $\mathcal{J}_0 \to \mathbb{R}$ sol a sist ai $\varphi(t_0) = (\varphi_1(t_0), \varphi_2(t_0)) = (x_0^1, x_0^2) = x_0$ (adică E.U.L)

b) $m = k = 1$

$f_2(x_2) = x_2$

$f_1(x_1, x_2) = x_1^2 + g(x_2)$

$g(x_2) = \sqrt[3]{x_2}$; g - nu e local Lipschitz în $x_2 = 0$ (unde se anulează radical)

$f(\cdot,\cdot) = (f_1(\cdot), f_2(\cdot))$ nu este local Lipschitz

ECUAȚII DIFERENȚIALE
SEMINAR 5

2.11.2022

$f(\cdot,\cdot): J\times\mathbb{R}^n \to \mathbb{R}^n$

a) Disipativitate (D) $\exists\, r>0,\ \exists\, a(\cdot): J\to\mathbb{R}_+$ cont

$$|\langle x, f(t,x)\rangle| \le a(t)\,\|x\|^2$$

b) Creștere liniară (C.L) $\exists\, r>0,\ \exists\, a(\cdot): J\to\mathbb{R}_+$ cont

$$\|f(t,x)\| \le a(t)\,\|x\| \quad \forall\, t\in J,\ x\in\mathbb{R}^n,\ \|x\|>r$$

c) Creștere afină (C.A) $\exists\, r>0,\ \exists\, a(\cdot), b(\cdot): J\to\mathbb{R}_+$ cont

$$\|f(t,x)\| \le a(t)\|x\| + b(t) \quad \forall\, t\in J,\ x\in\mathbb{R}^n,\ \|x\|>r$$

T. (E.G):

Fie $f(\cdot,\cdot): J\times\mathbb{R}^n\to\mathbb{R}^n$ cont. cu (D) $\frac{dx}{dt} = f(t,x)$

Atunci $f(\cdot,\cdot)$ admite E.G. pe $J\times\mathbb{R}^n$ ($\forall\,(t_0,x_0)\in J\times\mathbb{R}^n\ \exists\, \varphi(\cdot): J\to\mathbb{R}^n$ sol $\varphi(t_0)=x_0$)

Ex. 1.

a) $CL \Leftrightarrow CA$

b) $C.L \Rightarrow D$

c) $n=1 \quad D \Rightarrow C.L.$

d) $n>1 \quad D \not\Rightarrow C.L.$

a) „$\Rightarrow$" $b(t)=0$ – Evident

„$\Leftarrow$" $\exists\, r>0,\ \exists\, a(\cdot), b(\cdot): J\to\mathbb{R}_+$ cont. cu $\|f(t,x)\| \le a(t)\cdot\|x\| + b(t)$,

$$\forall\, t\in J,\ x\in\mathbb{R}^n,\ \|x\|>r$$

$$\|f(t,x)\| \le \|x\|\left(a(t) + \frac{b(t)}{\|x\|}\right) \le \|x\|\underbrace{\left(a(t) + \frac{b(t)}{r}\right)}_{c(t)} \le \|x\|\cdot c(t)$$

b) $\exists\, r>0,\ \exists\, a(\cdot): J\to\mathbb{R}_+$ cont cu $\|f(t,x)\| \le a(t)\cdot\|x\|,\ \forall\, t\in J,\ x\in\mathbb{R}^n,$
$\|x\|>r$

$$|\langle f(t,x), x\rangle| \le \|f(t,x)\|\cdot\|x\| \le \|a(t)\|x\|\|\cdot\|x\| = a(t)\cdot\|x\|^2 \Rightarrow D.$$

C.S: $|\langle x,y\rangle| \le \|x\|\cdot\|y\|,\ x,y\in\mathbb{R}^n$

c) $n=1$: $\exists\, r>0,\ \exists\, a(\cdot): J\to\mathbb{R}_+$ cont. cu $|x\cdot f(t,x)| \le a(t)\cdot|x|^2,\ \forall\, t\in J,\ x\in\mathbb{R}$
$|x|>r$

$$|x|\cdot|f(t,x)| \le a(t)\cdot|x|^2 \Rightarrow |f(t,x)| \le a(t)\cdot|x|,\ \forall\, t\in J\ \forall\, x\in\mathbb{R}$$

$\Rightarrow$ C.L. ($n=1$).

d) $n=2$ $\quad f=?$ a.î. $\langle x, f(t,x)\rangle = 0$.

$$f(t,(x_1,x_2)) = (f_1(t,(x_1,x_2)), f_2(t,(x_1,x_2)))$$

$$\langle x, f(t,x)\rangle = \langle (x_1,x_2), (f_1(t,(x_1,x_2)), f_2(t,(x_1,x_2)))\rangle =$$

$$= x_1 f_1(t,(x_1,x_2)) + x_2 f_2(t,(x_1,x_2))$$

NU luăm $f(t,(x_1,x_2)) = (-x_2, x_1)$

E OK $\|f(t,x)\| = \|(-x_2+x_1)\| = \sqrt{x_1^2+x_2^2} = \|x\|$

luăm $f(t,(x_1,x_2)) = (-x_2, x_1)\cdot x_1$

$\|f(t,x)\| = \sqrt{x_1^2 x_2^2 + x_1^4} = |x_1|\cdot\|x\|$

Sp. că f are C.L $\Rightarrow \exists r>0, \exists a(\cdot): \mathbb{R}\to\mathbb{R}_+$ cont. a.î. $\|f(t,x)\| \le a(t)\cdot\|x\|$, $\forall t\in J, x\in\mathbb{R}^2 = (x_1,x_2)$ a.î. $\|x\|>r$.

$|x_1|\cdot\cancel{\|x\|} \le a(t)\cdot\cancel{\|x\|}$, $\forall t\in\mathbb{R}, \forall x\in\mathbb{R}^2$, $\|x\|>r$

$|x_1| \le a(t)$, $\forall t\in\mathbb{R}$, $\forall x\in\mathbb{R}^2 \overset{\shortparallel}{} (x_1,x_2)$

Fie $t=25 \Rightarrow |x_1|\le a(25)$, $\forall x_1\in\mathbb{R}$ contr. $\mathcal{L}e$.

$\boxed{\text{Ex. 2.}}$ Fie ec. $\begin{cases} x_1' = -x_2^3 \\ x_2' = x_2^2 x_1 \end{cases}$

a) Să se arate că $\forall\, \varphi(\cdot)$ sol $\exists\, c\in\mathbb{R}$ a.î. $\|\varphi(t)\| \equiv c$

b) Să se arate că ec. admite E.G. sol.

$\boxed{\text{Ex. 3}}$ Fie ec. $\begin{cases} x_1' = x_1 x_2 \\ x_2' = -2x_1^4 \end{cases}$

a) Să se arate că $\forall\ \varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol, $\exists\ c \in \mathbb{R}$ a.î. $\varphi_1^4(t) + \varphi_2^2(t) \equiv c$.

b) Să se arate că ec. admite E.G. sol.

a) Fie $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol $\Leftrightarrow$ $\varphi_1'(t) = \varphi_1(t)\varphi_2(t)$

$\varphi_2'(t) = -2\varphi_1^4(t)$

Fie $g(t) = \varphi_1^4(t) + \varphi_2^2(t)$

$g'(t) = 4\varphi_1^3(t)\varphi_1'(t) + 2\varphi_2(t)\varphi_2'(t)$

$= 4\varphi_1^3(t)\cdot\varphi_1(t)\varphi_2(t) + 2\varphi_2(t)\cdot(-2)\varphi_1^4(t)$

$= 4\varphi_1^4(t)\varphi_2(t) - 4\varphi_1^4(t)\varphi_2(t) = 0 \Rightarrow \exists\ k \in \mathbb{R}$ a.î. $g(t) \equiv k$;

b) $f(t, (x_1, x_2)) = (x_1 x_2, -2x_1^4)$

$f(\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^2$ cont.

$|\langle x, f(t,x)\rangle| = |\langle (x_1,x_2), (x_1 x_2, -2x_1^4)\rangle| = |x_1^2 x_2 - 2x_2 x_1^4|$?

f are (D) $\Rightarrow \exists\ r > 0$, $\exists\ a(\cdot) : \mathbb{R} \to \mathbb{R}_+$ a.î. $|x_1^2 x_2 - 2x_1^4 x_2| \le a(t)(x_1^2 + x_2^2)$

$\forall\ t \in \mathbb{R}, x \in \mathbb{R}^2,$

$\|x\| > r$.

Fie $t = 0$, $x_1 = x_2 = m \in \mathbb{N}$ avem $|m^3 - 2m^5| \le 2a(0)\, m^2$, $\forall\ m \in \mathbb{R}$, $m\sqrt{2} > r$

$\Rightarrow \underbrace{|m - 2m^3|}_{\to\ \infty} \le \underbrace{2a(0)}_{\text{ct}}$ Fals.

b) E.G: $\forall (t_0, x_0) \in \mathbb{R}\times\mathbb{R}^2$, $\exists\ \varphi(\cdot): \mathbb{R} \to \mathbb{R}^2$ sol $\varphi(t_0) = x_0$.

(ca la curs)

fie $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^2$ – mulț deschisă, f. cont $\xRightarrow{\text{T. Peano}}$ $\exists\ J_0 \in \mathcal{V}(t_0)$ și

$\exists\ \varphi_0(\cdot): J_0 \to \mathbb{R}^2$, $\varphi_0(t_0) = x_0$.

T. exist. sol. max $\Rightarrow$ $\exists\ \underbrace{\varphi: J \to \mathbb{R}^2}_{\text{interval}}$, sol. max. $\varphi \succ \varphi_0 \Rightarrow \varphi(t_0) = \varphi_0(t_0) = x_0$

Prop (int. de def. al sol max) $\Longrightarrow$ J – interval deschis $= (a, b)$

Arătăm că $a = -\infty$ și $b = +\infty$.

- $b = +\infty$

Pp. că $b < +\infty$; $t_0 \in (a,b)$? $\underset{\text{compactă}}{D_0} \subset \mathbb{R}\times\mathbb{R}^2$ a.î. $(t, \varphi(t)) \in D_0$, $\forall t \in [t_0, b)$

T. asupra prelungirii soluțiilor

Fie $f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t,x)$;

fie $\varphi(\cdot): (a,b) \to \mathbb{R}^n$ soluție.

Atunci: 1) $\varphi(\cdot)$ admite o prelungire strictă la dreapta $\Leftrightarrow b < +\infty$, $\exists\ t_0 \in (a,b)$ $\exists\ D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0, \forall t \in (t_0, b)$.

2) analog, stânga

admițând că am găsit D_0 $\xRightarrow[\text{sol.}]{\text{T. as. pre.}}$ φ admite o prelungire strictă la dreapta

Ⅎ (maximalitatea lui φ).

$\Rightarrow b = +\infty$

(„despărțire" de curs)

conform pct. a) ptr. $\varphi(\cdot)$ $\exists\ c \in \mathbb{R}$ a.î. $\varphi_1^4(t) + \varphi_2^2(t) \equiv c$

$\Rightarrow \varphi_1^4(t) \le c, \forall t \Rightarrow \varphi_1^2(t) \le \sqrt{c}, \forall t$.

$\varphi_2^2(t) \le c, \forall t$

$\Rightarrow \varphi_1^2(t) + \varphi_2^2(t) \le c + \sqrt{c}, \forall t$

$\|\varphi(t)\|^2 \le c + \sqrt{c}, \forall t$

$\Rightarrow \|\varphi(t)\| \le \underbrace{\sqrt{c+\sqrt{c}}}_{k}, \forall t \Rightarrow \|\varphi(t)\| \le k, \forall t$

$\Rightarrow \varphi(t) \in \overline{B}_k(0)$

fie $D_0 = [t_0, b] \times \overline{B}_k(0)$ $\Rightarrow$ $(t, \varphi(t)) \in D_0, \ \forall t \in [t_0, b)$ ■

$\boxed{\text{Ex 4.}}$ Fie ec. $\frac{dx_i}{dt} = \sum_{j,k=1}^{n} a_{ij} x_j x_k \quad i=\overline{1,n}$

TEMĂ $a_{ijk} = -a_{kji}, \ \forall i,j,k = \overline{1,n}.$

a) Să se arate că $\forall \varphi(\cdot)$ sol. $\exists c \in \mathbb{R}$ $\|\varphi(t)\| \equiv c$.

b) Să se arate că admite E.G. a sol.

ECUAȚII DIFERENȚIALE

SEMINAR 6

9.11.2022

$\boxed{1}$ $A(\cdot): J \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ caut $\frac{dx}{dt} = A(t)x$

$m < n$ $\quad \varphi_1(\cdot), \dots, \varphi_m(\cdot)$ sol.

$$\det\left((\varphi_{ij}(t))_{i,j=\overline{1,m}} \right) \neq 0, \forall t \in J$$

$$X(t) := col(\varphi_1(t), \dots, \varphi_m(t), e_{m+1}, \dots, e_n) \qquad e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$$

Să se arate că:

a) $X(t)$ este inversabila $\forall t$

b) s.v. $x = X(t)y \longrightarrow \frac{dy}{dt} = B(t)y$

c) $B(t) = col(0, \dots, 0, b_{m+1}(t), \dots, b_n(t))$

a) $$X(t) = \left(\begin{array}{ccc|ccc} \varphi_{11}(t) & \dots & \varphi_{m1}(t) & 0 & \dots & 0 \\ \varphi_{12}(t) & & \varphi_{m2}(t) & 0 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_{1m}(t) & & \varphi_{mm}(t) & 1 & & \vdots \\ \hline \varphi_{1m+1}(t) & & \varphi_{m\,m+1}(t) & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_{1n}(t) & & \varphi_{mn}(t) & 0 & & 1 \end{array} \right)$$

sau - regula lui Laplace
- dezvoltând după ult. coloană $\Rightarrow$

$$\det(X(t)) = \det\left((\varphi_{ij}(t))_{i,j=1,m} \right) \neq 0, \forall t$$

b) $x = X(t)y$, $\forall x(\cdot)$ sol. a ec. s.a. def.

$y(\cdot)$ după regula $x(t) = X(t)y(t) \Leftrightarrow y(t) = X(t)^{-1}x(t)$

$$x(\cdot) \text{ sol} \Rightarrow (X(t)y(t))' = A(t)\cdot x(t)y(t)$$

$$X'(t)y(t) + X(t)y'(t) = A(t)x(t)y(t)$$

$$X(t)y'(t) = (A(t)x(t) - X'(t))y(t)$$

$$y'(t) = \underbrace{X^{-1}(t)\left[-X'(t) + A(t)x(t)\right]}_{B(t)} y(t)$$

c) $B(t) = X^{-1}(t)\left[-X'(t) + A(t)X(t)\right]$

$= X^{-1}(t)\left[-col\left(\varphi_1'(t),\dots,\varphi_m'(t),0\dots,0\right) + A(t)\cdot col\left(\varphi_1(t),\dots,\varphi_m(t),e_{m+1},\dots,e_n\right)\right]$

$= X^{-1}(t)\left[col\left(\varphi_1'(t),\dots,\varphi_m'(t),0,\dots,0\right) + col\left(A(t)\varphi_1(t),\dots,A(t)e_n\right)\right]$

$= X^{-1}(t)\left[col\left(A(t)\varphi_1(t)-\varphi_1'(t),\dots,A(t)\varphi_m(t)-\varphi_m'(t),\dots,A(t)e_n\right)\right]$

$= X^{-1}(t)\left[col\left(0,0,\dots,0,A(t)e_{m+1},\dots,A(t)e_n\right)\right]$

$= col\left(0,0,\dots,0,X^{-1}(t)A(t)e_{m+1},\dots,X^{-1}(t)A(t)e_n\right)$

$\boxed{2.1}$ Fie ec. $\begin{cases} x' = y - tx \\ y' = (1-t^2)x + ty \end{cases}$ cu sol. $\varphi_1(t) = \begin{pmatrix} t \\ t^2+1 \end{pmatrix}$

$\varphi_2(\cdot)$ ai $\{\varphi_1(\cdot), \varphi_2(\cdot)\}$ sistem fundamental de soluții.

<u>Metoda I</u>

$A(t) = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}$

$A(\cdot): J \subseteq \mathbb{R} \to L(\mathbb{R}^m, \mathbb{R}^m)$ cont $\frac{dx}{dt} = A(t)x$

$S_{A(\cdot)} = \{\varphi(\cdot): J \to \mathbb{R}^m;\ \varphi(\cdot) \text{ sol a ec.}\}$

$\varphi_1(\cdot), \dots, \varphi_m(\cdot) \in S_{A(\cdot)}$

* **WRONSKIANUL** as. sol. $W_{\varphi_1, \dots, \varphi_m}(\cdot): J \to \mathbb{R}$

$$W_{\varphi_1, \dots, \varphi_m}(t) = \det[\,\mathrm{col}(\varphi_1(t), \dots, \varphi_m(t))]$$

* **T. Liouville**: $W_{\varphi_1, \dots \varphi_m}(t) = W_{\varphi_1, \dots, \varphi_m}(t_0) \cdot e^{\int_{t_0}^{t} Tr(A(s))ds}, \forall t, t_0 \in J$

Metoda a II-a (folosind T. Liouville)

Fie $\varphi_2(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$

$\varphi_2(\cdot)$ sol $\Rightarrow \begin{cases} a'(t) = b(t) - t \cdot a(t) & (1) \\ b'(t) = (1-t^2)a(t) + t \cdot b(t) \end{cases}$

Fie $W(t) = \det(\,\mathrm{col}(\varphi_1(t), \varphi_2(t))) = \det \begin{pmatrix} t & a(t) \\ t^2+1 & b(t) \end{pmatrix} = tb(t) - (t^2+1)a(t)$

$W(t) = W(t_0) \cdot e^{\int_{t_0}^{t} Tr(A(s))ds} = W(t_0) \cdot e^{\int_{t_0}^{t} 0 ds} = W(t_0), \forall t, t_0 \in J$

W const $\Rightarrow \exists\, c \in \mathbb{R}$ a.î. $W(t) = c \Leftrightarrow tb(t) - (t^2+1)a(t) = c$

$$b(t) = \frac{c + (t^2+1)a(t)}{t}$$

în (1) $\Rightarrow a'(t) = \frac{c + (t^2+1)a(t)}{t} - ta(t) \Rightarrow a'(t) = \frac{c + a(t)}{t} \Rightarrow$

$$a' = \frac{a+c}{t}, t > 0;\quad a' = \frac{a}{t} + \frac{c}{t} \text{ ec. afină}$$

considerăm ec. lin. asoc. $\bar{a}' = \frac{\bar{a}}{t} \Rightarrow \bar{a}(t) = k \cdot e^{\int \frac{1}{t} dt} = k \cdot e^{\ln t} = k \cdot t$

$a(t) = k(t) \cdot t$

$$(k(t)\cdot t)' = \frac{k(t)\cdot t}{t} + \frac{c}{t} \Rightarrow k'(t)\cdot t + \cancel{k(t)} = \cancel{k(t)} + \frac{c}{t}$$

$$k'(t) = \frac{c}{t^2} \Rightarrow k(t) = -\frac{c}{t} + d$$

$$\Rightarrow a(t) = -c + dt \Rightarrow b(t) = \frac{c + (t^2+1)(-c+dt)}{t} = -ct + d(t^2+1)$$

$$\Rightarrow \varphi_2(t) = \begin{pmatrix} -c + dt \\ -ct + d(t^2+1) \end{pmatrix} = -c\begin{pmatrix} 1 \\ t \end{pmatrix} + d\begin{pmatrix} t \\ t^2+1 \end{pmatrix}$$

Metoda a III-a

$$\begin{cases} x' = y - tx \\ y' = (1-t^2)x + ty \end{cases} \Rightarrow y(t) = x'(t) + tx(t)$$

$$\Downarrow$$

$$(x'(t) + tx(t))' = (1-t^2)x(t) + t\cdot(x'(t) + tx(t))$$

$$x''(t) + \cancel{x(t)} + \cancel{tx'(t)} = \cancel{x(t)} - \cancel{t^2x(t)} + \cancel{tx'(t)} + \cancel{t^2x(t)}$$

$$x''(t) = 0 \Rightarrow x'(t) = c,\ c \in \mathbb{R}$$

$$x(t) = ct + k,\ c, k \in \mathbb{R}$$

$$\Rightarrow y(t) = c + ct^2 + tk$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ct + k \\ c + ct^2 + tk \end{pmatrix} = c\begin{pmatrix} 1 \\ t \end{pmatrix} + k\begin{pmatrix} t \\ t^2+1 \end{pmatrix},\ c, k \in \mathbb{R}$$

- Dacă $Tr(A) \neq 0 \Rightarrow$ met II, III au mers.

$$\begin{cases} x' = y + tx \\ y' = (1-t^2)x + ty \end{cases}$$

$$y(t) = x'(t) - tx(t)$$

$$(x'(t) - tx(t))' = (1-t^2)x(t) + t(x'(t) - tx(t))$$

$$x''(t) - x(t) - tx'(t) = (1-t^2)x(t) + tx'(t) - t^2x(t)$$

$$x''(t) - 2tx'(t) + 2(t^2-1)x(t) = 0$$

$$x'' - 2tx' + 2(t^2-1)x = 0.$$

ECUAȚII DIFERENȚIALE

SEMINAR 7

16.11.2022.

Algoritm (cazul val. proprii simple) $\frac{dx}{dt} = Ax$, $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

1. Rezolvă ec. caracteristică $\det(A - \lambda I_n) = 0$
$\to \sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ distincte
2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ a.î. $(A - \lambda I_n) u_\lambda = 0$.
Scrie sol. $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$
3. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$ caută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ a.î. $(A - \lambda I_n) u_\lambda = 0$
Scrie sol. $\varphi_\lambda(t) = \mathrm{Re}(e^{\lambda t} \cdot u_\lambda)$
$\varphi_{\bar{\lambda}}(t) = \mathrm{Im}(e^{\lambda t} \cdot u_\lambda)$
4. Renumerotează $\{\varphi_\lambda(\cdot)\}_{\lambda \in \sigma(A)} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sist. fund. de sol.
scrie sol. generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ $c_i \in \mathbb{R}$, $i = \overline{1,n}$

- Să se det. sol. generală

1) $\begin{cases} x' = x + z - y \\ y' = x + y - z \\ z' = 2x - y. \end{cases}$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I_n) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = (1-\lambda)^2(-\lambda) + 2 - 1 - 2(1-\lambda) - 1 + \cancel{\lambda} - \cancel{\lambda}$$
$$= (1-\lambda)^2 \cdot (-\lambda) + \cancel{2} - 1 - \cancel{2} + 2\lambda - 1$$
$$= (1-\lambda)^2(-\lambda) - 2(1-\lambda)$$
$$= (1-\lambda)[(1-\lambda)(-\lambda) - 2]$$
$$= (1-\lambda)(-\lambda + \lambda^2 - 2)$$
$$\Rightarrow \det(A - \lambda I_n) = 0$$

I $\lambda^2 - \lambda - 2 = 0$

$\Delta = b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot (-2) = 1 + 8 = 9$

$\lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{1 \pm 3}{2} \Rightarrow$ $\lambda_1 = 2$, $\lambda_2 = -1$

II $1 - \lambda = 0 \Rightarrow \lambda_3 = 1$

$\boxed{\lambda_1 = 1}$

Caut $u \in \mathbb{R}^3 \setminus \{0\}$ ai $(A - I_3)u = 0$; fie $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -b + c = 0 \Rightarrow b = c \\ a - c = 0 \Rightarrow a = c \\ 2a - b - c = 0 \end{cases} \Rightarrow u = \begin{pmatrix} a \\ a \\ a \end{pmatrix} \underset{\text{pt } a=1}{=} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\varphi_1(t) = e^{\lambda t} \cdot u = e^{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\boxed{\lambda_2 = -1}$

Caut $u \in \mathbb{R}^3 \setminus \{0\}$ ai $(A + I_3)u = 0$; fie $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2a - b + c = 0 \\ a + 2b - c = 0 \\ 2a - b + c = 0 \end{cases} \Rightarrow 3a + b = 0 \Rightarrow b = -3a$$

$$\Rightarrow c = -5a$$

$$\Rightarrow u = \begin{pmatrix} a \\ -3a \\ -5a \end{pmatrix} \underset{\text{pt } a=-1}{=} \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$$

$$\varphi_2(t) = e^{-t} \cdot \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$$

$\boxed{\lambda_3 = 2}$

Caut $u \in \mathbb{R}^3 \setminus \{0\}$ ai $(A - 2I_3)u = 0$; fie $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -a-b+c=0 \\ a-b-c=0 \\ 2a-b-2c=0 \end{cases} \Rightarrow b=0 \Rightarrow c=a \qquad \Rightarrow u=\begin{pmatrix} a \\ 0 \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\varphi_3(t)=e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\varphi(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+c_3\varphi_3(t)=c_1e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+c_2e^{-t}\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}+c_3e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$=\begin{pmatrix} c_1e^{t}-c_2e^{-t}+c_3e^{2t} \\ c_1e^{t}+3c_2e^{-t} \\ c_1e^{t}+5c_2e^{-t}+c_3e^{2t} \end{pmatrix} \Rightarrow \begin{cases} x(t)=c_1e^{t}-c_2e^{-t}+c_3e^{2t} \\ y(t)=c_1e^{t}+3c_2e^{-t} \\ z(t)=c_1e^{t}+5c_2e^{-t}+c_3e^{2t} \end{cases}$$

sol. generală cu $c_1, c_2, c_3 \in \mathbb{R}$

2) $$\begin{cases} x'=2x+y \\ y'=x+3y-z \\ z'=2y+3z-x \end{cases}$$

$$A=\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\det(A-\lambda I_n)=\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix}=(3-\lambda)^2(2-\lambda)+1+2(2-\lambda)-(3-\lambda)$$

$$=(3-\lambda)^2(2-\lambda)+1+4-2\lambda-3+3\lambda$$

$$=(3-\lambda)^2(2-\lambda)+(2+\lambda)$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)^2+1\right]=0 \Rightarrow \begin{cases} \lambda_1=2 \\ \lambda_2=3+i \\ \lambda_3=3-i \end{cases}$$

$\boxed{\lambda=2}$

caut $u\in\mathbb{R}^3\setminus\{0\}$ a.i. $(A-2I_3)u=0$; $u=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} b=0 \\ a+b-c=0 \Rightarrow a-c=0 \Rightarrow a=c \\ -a+2b+c=0 \end{cases}$$

$$\Rightarrow u=\begin{pmatrix} a \\ 0 \\ a \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \varphi_1(t) = e^{2t}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$\boxed{\lambda = 3+i}$

Caut $u \in \mathbb{C}^3\setminus\{0\}$ ai $(A-(3+i)I_3)u = 0$; $u = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\begin{pmatrix}-1-i & 1 & 0\\ 1 & -i & -1\\ -1 & 2 & -i\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{cases}-(1+i)a + b = 0 \Rightarrow b = a(1+i)\\ a - ib - c = 0 \Rightarrow c = a - ib \Rightarrow\\ -a + 2b - ic = 0\end{cases}$$

$$\Rightarrow c = a - i(a+ai) = a - ai + a = 2a - ai$$

$$\Rightarrow u = \begin{pmatrix}a\\a(1+i)\\a(2-i)\end{pmatrix}$$

$$\tilde{\varphi}(t) = e^{(3+i)t}\cdot\begin{pmatrix}1\\1+i\\2-i\end{pmatrix} = e^{3t}\cdot e^{it}\begin{pmatrix}1\\1+i\\2-i\end{pmatrix} = e^{3t}\cdot(\cos t + i\sin t)\begin{pmatrix}1\\1+i\\2-i\end{pmatrix} =$$

$$= e^{3t}\begin{pmatrix}\cos t + i\sin t\\ \cos t - \sin t + i(\cos t + \sin t)\\ 2\cos t + \sin t + i(2\sin t - \cos t)\end{pmatrix} = \underbrace{e^{3t}\begin{pmatrix}\cos t\\ \cos t - \sin t\\ 2\cos t + \sin t\end{pmatrix}}_{\varphi_2(t)} + i\cdot\underbrace{e^{3t}\begin{pmatrix}\sin t\\ \cos t + \sin t\\ 2\sin t - \cos t\end{pmatrix}}_{\varphi_3(t)}$$

3) $\begin{cases}x' = x - 2y\\ y' = 3y + x.\end{cases}$

Metoda I

$$A = \begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}$$

$$\det(A - \lambda I_2) = 0 \Rightarrow \begin{vmatrix}1-\lambda & -2\\ 1 & 3-\lambda\end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) + 2 = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$\Delta = b^2 - 4ac = 16 - 4 \cdot 1 \cdot 5 = -4$

$$\Rightarrow \lambda_{1;2} = \frac{-b \pm i\sqrt{|\Delta|}}{2a} = \frac{4 \pm 2i}{2} \Rightarrow \begin{array}{l} \lambda_1 = 2+i \\ \lambda_2 = 2-i \end{array}$$

$\boxed{\lambda = 2+i}$

Caut $u \in \mathbb{C}^2 \setminus \{0\}$ a.î. $(A - (2+i)I_2)u = 0$

$$\begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -a - ai - 2b = 0 \\ a + b - bi = 0 \end{cases} \Rightarrow a = (i-1)b \Rightarrow u = \begin{pmatrix} (i-1)b \\ b \end{pmatrix} = \begin{pmatrix} i-1 \\ 1 \end{pmatrix}$$

$$\tilde{\varphi}(t) = e^{(2+i)t} \cdot \begin{pmatrix} i-1 \\ 1 \end{pmatrix} = e^{2t} \cdot e^{it}\begin{pmatrix} i-1 \\ 1 \end{pmatrix} = e^{2t}(\cos t + i \sin t)\begin{pmatrix} i-1 \\ 1 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} -\cos t - \sin t + i(\cos t - \sin t) \\ \cos t + i \sin t \end{pmatrix}$$

$$= \underbrace{e^{2t}\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix}}_{} + i \cdot \underbrace{e^{2t}\begin{pmatrix} -\sin t + \cos t \\ \sin t \end{pmatrix}}_{}$$

Metoda a II-a.

$$\begin{cases} x' = x - 2y \\ y' = 3y + x \end{cases} \Rightarrow x = y' - 3y$$

$$(y' - 3y)' = y' - 3y - 2y$$

$$y'' - 3y' = y' - 5y \Rightarrow y'' - 4y' + 5y = 0$$

(ec. liniară de gr. al II-lea cu coef. constanți)

$\lambda^2-4\lambda+5=0 \Rightarrow \lambda_1 = 2+i$
$\lambda_2 = 2-i$

Sol. generală: $\begin{cases} y(t)= c_1\cdot e^{2t}\cos(1\cdot t)+c_2e^{2t}\sin(1\cdot t), \quad c_1,c_2\in\mathbb{R} \\ x(t) = 2c_1e^{2t}\cos(t)-c_1e^{2t}\sin t+2c_2e^{2t}\sin(t)+c_2e^{2t}\cos(t) \end{cases}$

4) Să se găsească sol. generală a sistemului

$$\begin{cases} x' = \frac{x}{t}-\frac{y}{t} \\ y' = \frac{y}{t}-\frac{4x}{t}, \quad t>0. \end{cases}$$

efectuând s.v. $t=e^s$.

$t=e^s$, $\forall x(\cdot), y(\cdot)$ sol. s.v. def. $u(\cdot), v(\cdot)$ după regula:

$$\begin{cases} u(s)=x(e^s) \\ v(s)=y(e^s) \end{cases} \Leftrightarrow \begin{cases} x(t)=u(\ln(t)) \\ y(t)=v(\ln(t)) \end{cases}$$

$$\Rightarrow \begin{cases} x'(e^s) = \frac{x(e^s)}{e^s}-\frac{y(e^s)}{e^s} \\ y'(e^s) = \frac{y(e^s)}{e^s}-\frac{4x(e^s)}{e^s} \end{cases} \Bigg| \; (*)$$

$x(e^s)=u(s) \;|\,()' \Rightarrow x'(e^s)\cdot e^s = u'(s) \Rightarrow x'(e^s)=\frac{u'(s)}{e^s}$

$y(e^s)=v(s) \;|\,()' \Rightarrow y'(e^s)=\frac{v'(s)}{e^s}$

$$(*) \begin{cases} \frac{u'(s)}{e^s} = \frac{u(s)}{e^s}-\frac{v(s)}{e^s} \\ \frac{v'(s)}{e^s} = \frac{v(s)}{e^s}-\frac{4u(s)}{e^s} \end{cases} \Leftrightarrow \begin{cases} u'(s)=u(s)-v(s) \\ v'(s)=v(s)-4u(s) \end{cases} (=)$$

$$\Leftrightarrow \begin{cases} u'=u-v \\ v'=v-4u \end{cases} \rightsquigarrow \begin{cases} \text{Algorit} \\ \text{sau} \\ \text{ec. lin de gr. II} \end{cases}$$

$v=u-u'$

$u-u''=u-u'-4u \Rightarrow u''-2u'-3u=0$

$\lambda^2-2\lambda-3=0 \Rightarrow \Delta=b^2-4ac=4-4\cdot1\cdot(-3)=16$

$\lambda_{1,2}=\frac{+2\pm\sqrt{16}}{2} \Rightarrow \lambda_1=3$
$\lambda_2=-1$

Sol. generală : $\begin{cases} u(s) = c_1 e^{3s} + c_2 \cdot e^{-s} \\ v(s) = c_1 \cdot e^{3s} + c_2 \cdot e^{-s} - 3c_1 e^{3s} + c_2 e^{-s} = -2c_1 e^{3s} + 2c_2 e^{s} \end{cases}$ $\Rightarrow$, $c_1, c_2 \in \mathbb{R}$

$x(t) = u(\ln(t)) = c_1 \cdot e^{3\ln t} + c_2 \cdot e^{-\ln t} = c_1 t^3 + c_2 t^{-1}$

$y(t) = v(\ln(t)) = -2c_1 \cdot e^{3\ln t} + 2c_2 \cdot e^{\ln t} = -2c_1 t^3 + 2c_2 t^{-1}$, $c_1, c_2 \in \mathbb{R}$

TEMĂ

5. $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\frac{dx}{dt} = \frac{1}{t} Ax$, $t \neq 0$.

Să se arate că s.v. $|t| = e^s$ conduce la ec. $\frac{dy}{ds} = Ay$.

23.11.2022.

ECUAȚII DIFERENȚIALE
SEMINAR 8

Algoritm

$\frac{dx}{dt} = Ax \qquad A \in L(\mathbb{R}^n, \mathbb{R}^n)$

1. Rez. ec. caracteristică : $\det(A - \lambda I_n) = 0 \longrightarrow \sigma(A): (\lambda, m_\lambda)$
2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = 1$
   Caută $u \in \mathbb{R}^n \setminus \{0\}$ $(A - \lambda I_n) u = 0$.
   Scrie sol. $\varphi_\lambda(t) = e^{\lambda t} \cdot u$
3. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = m > 1$
   Caută $\{P_0^{\lambda 1}, \ldots, P_0^{\lambda m}\} \subset \ker(A - \lambda I_n)^m$ lin. indep (în $\mathbb{R}^n$)
   Scrie $P_j^{\lambda l} = \frac{1}{j!}(A - \lambda I_n)^j P_0^{\lambda l}$ $\quad j = \overline{1, m-1}$, $\quad l = \overline{1, m}$
   Scrie sol. $\varphi_{\lambda l}(t) = e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda l} \cdot t^j$, $\quad l = \overline{1, m}$
4. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = 1$
   Caută $u \in \mathbb{C}^n \setminus \{0\}$ a.î. $(A - \lambda I_n) u = 0$
   Scrie sol. $\varphi_\lambda(t) = \mathrm{Re}(e^{\lambda t} \cdot u)$
   $\varphi_{\bar\lambda}(t) = \mathrm{Im}(e^{\lambda t} \cdot u)$
5. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = m > 1$
   Caută $\{P_0^{\lambda 1}, \ldots, P_0^{\lambda m}\} \subset \ker(A - \lambda I_n)^m$ lin. indep. (în $\mathbb{C}^n$)
   Scrie $P_j^{\lambda l} = \frac{1}{j!}(A - \lambda I_n)^j P_0^{\lambda l}$
   Scrie sol. $\varphi_{\lambda l}(t) = \mathrm{Re}\left(e^{\lambda t} \sum_{j=0}^{m-1} P_j^{\lambda l} t^j\right)$ $\quad l = \overline{1, m}$
   $\varphi_{\bar\lambda l}(t) = \mathrm{Im}\left(e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda l} \cdot t^j\right)$
6. Renumerotează $\{\varphi_{\lambda l}(\cdot)\}_{\substack{\lambda \in \sigma(A) \\ l = \overline{1,m}}} = \{\varphi_1(\cdot), \ldots, \varphi_n(\cdot)\} \subset S_A$
   sist. fund. de sol.

Scrie sol. gen: $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ $\quad c_i \in \mathbb{R}$, $i = \overline{1, n}$

Să se determine soluția generală:

$$1)\begin{cases} x' = x - y + z \\ y' = x + y - z \\ z' = 2z - y \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I_n) = 0 \iff \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \iff$$

$$(1-\lambda)^2(2-\lambda) - 1 - (1-\lambda) + 2 - \lambda = 0$$

$$(1-\lambda)^2(2-\lambda) \cancel{-1} \cancel{-1} + \cancel{\lambda} + \cancel{2} \cancel{-\lambda} = 0 \iff (1-\lambda)^2(2-\lambda) = 0$$

$$\Rightarrow 2-\lambda = 0 \Rightarrow \lambda_1 = 2,\ m_{\lambda_1} = 1$$

$$\Rightarrow (1-\lambda)^2 = 0 \Rightarrow 1-\lambda = 0 \Rightarrow \lambda_2 = 1,\ m_{\lambda_2} = 2.$$

- Pt. $\lambda_1 = 2$ caut $u \in \mathbb{R}^3 \setminus \{0\}$ a.î. $(A - 2I_n)u = 0$.

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a - b + c = 0 \\ a - b - c = 0 \\ -b = 0 \Rightarrow b = 0 \end{cases} \Rightarrow \begin{cases} -a + c = 0 \\ a - c = 0 \end{cases} \Rightarrow a = c$$

$$\Rightarrow u = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} \underset{\text{pt } a=1}{\Longrightarrow} u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\varphi_1(t) = p_1(t) = e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$$

- Pt. $\lambda_2 = 1$, $m_{\lambda_2} = 2$

$P_0 = ?$ a.î. $(A - I_3)^2 P_0 = 0$; $P_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -a-b+2c=0 \Rightarrow c = \frac{a+b}{2}$$

$$\Rightarrow P_0 = \begin{pmatrix} a \\ b \\ \frac{a+b}{2} \end{pmatrix}$$

- pt $a=2$, $b=0$ $\Rightarrow P_{0_1} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$
- pt $a=0$, $b=2$ $\Rightarrow P_{02} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

} lin. indep.

$$P_j = \frac{1}{j!}(A - I_3)^j P_0 \qquad j = \overline{1, m-1}, \ m = 2$$

$$P_1^1 = (A - I_3) \cdot P_{0_1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_1^2 = (A - I_3) \cdot P_{02} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\varphi_2(t) = e^t (P_0^1 + t P_1^1) = e^t \begin{pmatrix} 2+t \\ t \\ 1+t \end{pmatrix}$$

$$\varphi_3(t) = e^t (P_0^2 + t P_1^2) = e^t \begin{pmatrix} -t \\ 2-t \\ -t+1 \end{pmatrix}$$

2) $\begin{cases} x' = 4x - y \\ y' = 3x + y - z \\ z' = x + z \end{cases}$

$$A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det(A - \lambda I_3) = 0 \iff \begin{vmatrix} 4-\lambda & -1 & 0 \\ 3 & 1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\iff (4-\lambda)(1-\lambda)^2 + 1 + 3(1-\lambda) = 0$

$\iff (4-\lambda)(1-\lambda)^2 + (4-3\lambda) = 0$

$\iff (4-\lambda)(1-2\lambda+\lambda^2) + 4-3\lambda = 0$

$\iff 4 - 8\lambda + \underline{4\lambda^2} - \lambda + \underline{4\lambda^2} \, \underline{-\lambda^3} + 4 - 3\lambda = 0$

$\iff -\lambda^3 + 8\lambda^2 - 12\lambda + 8 = 0$

$\iff (\lambda - 2)^3 = 0 \Rightarrow \lambda = 2, \; m = 3.$

$P_0 = ?$ ai $(A - 2I_3)^3 P_0 = 0.$

$$\begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow (A - 2I_3)^3 P_0 = O_3 \quad , \forall P_0 \in \mathbb{R}$

$$\underbrace{P_0^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad P_0^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad P_0^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{lui indep.}}$$

$$P_j = \frac{1}{j!}(A - J_3)^j P_0 \quad , \; j = \overline{1,2}$$

$$\begin{cases} P_1^1 = (A - 2J_3)P_0^1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ P_1^2 = (A-2J_3)P_0^2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \\ P_1^3 = (A-2J_3)P_0^3 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \end{cases} \qquad \begin{cases} P_2^1 = (A-2J_3)^2 P_0^1 = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} \\ P_2^2 = (A-2J_3)^2 P_0^2 = \begin{pmatrix} -1/2 \\ -1 \\ -1/2 \end{pmatrix} \\ P_2^3 = (A-2J_3)^2 P_0^3 = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} \end{cases}$$

$$\begin{cases} \varphi_1(t) = e^{2t}(P_0^1 + tP_1^1 + t^2 P_2^1) = e^{2t}\begin{pmatrix} 1+2t+t^2/2 \\ 3t+t^2 \\ t+t^2/2 \end{pmatrix} \\ \varphi_2(t) = e^{2t}(P_0^2 + tP_1^2 + t^2P_2^2) = e^{2t}\begin{pmatrix} -t^2/2 - t \\ 1-t-t^2 \\ -t^2/2 \end{pmatrix} \\ \varphi_3(t) = e^{2t}(P_0^3 + tP_1^3 + t^2P_2^3) = e^{2t}\begin{pmatrix} t^2/2 \\ -t+t^2 \\ 1-t+\frac{t^2}{2} \end{pmatrix} \end{cases}$$

TEMĂ: 3) $\begin{cases} x_1' = x_4 \\ x_2' = -8x_3 - 16x_1 \\ x_3' = x_2 \\ x_4' = x_3 \end{cases}$

4) $\begin{cases} x' = 2y - 3x \\ y' = y - 2x \end{cases}$

MI

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$$

$$\det(A - \lambda J_2) = 0 \iff \begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (-3-\lambda)(1-\lambda) + 4 = 0$$

$$-3 + 3\lambda - \lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0$$

$$\Rightarrow \lambda + 1 = 0 \Rightarrow \lambda = -1, \; m_\lambda = 2$$

$P_0 = ?$ a.î. $(A + J_2)^2 P_0 = 0$

$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2 \Rightarrow (A+J_2)^2 P_0 = O_2, \; \forall P_0 \in \mathbb{R}$$

$P_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $P_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ - lin. indep.

$$P_j = \frac{1}{j!}(A + I_2)^j P_0 \quad , \; j = 1$$

$$P_1^1 = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$P_1^2 = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\varphi_1(t) = e^{-t}(P_0^1 + tP_1^1) = e^{-t}\begin{pmatrix} 1-2t \\ -2t \end{pmatrix}$$

$$\varphi_2(t) = e^{-t}(P_0^2 + t \cdot P_1^2) = e^{-t}\begin{pmatrix} 2t \\ 1+2t \end{pmatrix}$$

M II

b) $\begin{cases} x' = \frac{3}{t}x - \frac{y}{t} \\ y' = \frac{4}{t}x - \frac{y}{t}, \ t>0. \end{cases}$

Metoda I : s.v. $t=e^s$ (seminar 7)

Metoda II : reducere

$\Rightarrow y = -tx' + 3x$

$(-tx+3x)' = \frac{4}{t}x - \frac{3x-tx'}{t}$

$-tx''+3x' = \frac{4x}{t} + \frac{tx'-3x}{t}$

$-t^2x''(t) + 2tx'(t) = x(t) + tx'(t)$

$-t^2x''(t) + tx'(t) - x(t) = 0$

$t^2x'' - tx' + x = 0. \rightarrow$ ec. de tip Euler

s.v. $t = e^s$

$\forall x(\cdot)$ sol. s.v. def. $y(\cdot)$: $y(s) = x(e^s) \Leftrightarrow x(t) = y(\ln t)$

$\Downarrow$

GREȘIT! MOD

$e^{2s}x''(e^s) - e^s x'(e^s) + x(e^s) = 0$

$x(e^s) = y(s)$

$x'(e^s)\, e^s = y'(s) \rightarrow x'(e^s) = \frac{y'(s)}{e^s}$

$e^s \cdot x''(e^s) = \frac{y''(s)\cdot e^s - y'(s)\cdot e^s}{e^{2s}}$

$x''(e^s) = (y''(s)\cdot e^{-s} + y'(s) e^{-s})\cdot e^{-s}$

$e^{2s}\cdot (y''(s)\, e^{-s} + y'(s) e^{-s} + y'(s) e^{-s}) e^{-s} - e^s y'(s)\cdot e^{-s} + y(s) = 0.$

$y''(s) - 2y'(s) + y(s) = 0$

$\lambda^2 - 2\lambda + 1 = 0$, $\lambda = 1$, $m = 2$

$y(s) = c_1 e^s + c_2 s\cdot e^s$

$x(t) = y(\ln(t)) = c_1 e^{\ln t} + c_2 \ln t\, e^{\ln t}$

$= c_1 t + c_2 t \ln t$, $c_1, c_2 \in \mathbb{R}$

$y(t) = -tx'(t) + 3x(t) = -t(c_1 + c_2 \ln t + c_2) + 3\cdot(c_1 t + c_2 t \ln t)$

ECUAȚII DIFERENȚIALE

SEMINAR 9

7.12.2022.

Algoritm (Ec. afine pe $\mathbb{R}^n$) $x' = A(t)x + b(t)$ $A(\cdot): J \to L(\mathbb{R}^n, \mathbb{R}^n)$, $b(\cdot): J \to \mathbb{R}^n$ / cont

1. Consideră ec. lin. asociată:

$$\bar{x}' = A(t)\bar{x}$$

Determină $\{\bar{\varphi}_1(\cdot), \dots, \bar{\varphi}_n(\cdot)\}$ sist. fundamental de sol.

<u>Obs</u>: Dacă $A(t) \equiv A \in L(\mathbb{R}^n, \mathbb{R}^n) \to$ vezi Algoritm

Scrie sol. generală $\bar{x}(t) = \sum_{i=1}^{n} c_i \bar{\varphi}_i(t)$ $c_i \in \mathbb{R}$, $i = \overline{1,n}$

2. „Variația constantelor"

Caut sol de forma: $x(t) = \sum_{i=1}^{n} c_i(t) \bar{\varphi}_i(t)$

$x(\cdot)$ sol $\Rightarrow \sum_{i=1}^{n} c_i'(t) \bar{\varphi}_i(t) = b(t) \Rightarrow c_i'(t) = \dots$

$\Rightarrow c_i(t) = \dots$

$\Rightarrow x(t) = \dots$

Algoritm (ec. liniare de ordin superior cu coef. constanți)

$$x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)} \qquad a_j \in \mathbb{R}, \ j = \overline{1,n}$$

1. Rezolvă ec. caracteristică

$$\lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j} \to \sigma(a_1, \dots, a_n), \ (\lambda, m_\lambda)$$

2. Scrie sol $\varphi_\lambda^j(t) = \begin{cases} t^{j-1} e^{\lambda t} & \lambda \in \sigma(a_1, \dots, a_n) \subset \mathbb{R}, \ j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \cos \beta t & \lambda = \alpha + i\beta \in \sigma(a_1, \dots, a_n) \\ t^{j-1} e^{\alpha t} \sin \beta t & \beta > 0 \end{cases}$

3. Renumerotează $\{\varphi_\lambda^j(\cdot)\}_{\lambda \in \sigma(a_1, \dots, a_n)} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sist. fundam de sol.

Scrie sol. generală: $x(t) = \sum_{i=1}^{n} c_i \bar{\varphi}_i(t)$, $c_i \in \mathbb{R}, i = \overline{1,n}$

Algoritm (ec. afine de ordin superior)

$$x^{n} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t), \quad a_1(\cdot), \dots, a_n(\cdot), b(\cdot): J \to \mathbb{R} \text{ cont.}$$

1. Consideră ec. lin. asociată: $\bar{x}^{(n)} = \sum_{i=1}^{n} a_j(t) \bar{x}^{(n-j)}$

Det. $\{\bar{\varphi}_1(\cdot), \dots, \bar{\varphi}_n(\cdot)\}$ sist fundam. de sol.

<u>Obs</u>: Dacă sol. generală $\bar{x}(t) = \sum_{i=1}^{n} c_i \varphi_i(t) \quad c_i \in \mathbb{R}, i = \overline{1,n}$

2. „Variația constantelor"

Caută sol. de forma $x(t) = \sum_{i=1}^{n} c_i(t) \bar{\varphi}_i(t)$

Rezolvă sistemul algebric:

$$\begin{cases} \sum_{i=1}^{n} c_i'(t) \bar{\varphi}_i(t) = 0 \\ \sum_{i=1}^{n} c_i'(t) \bar{\varphi}_i'(t) = 0 \\ \vdots \\ \sum_{i=1}^{n} c_i'(t) \bar{\varphi}_i^{(n-2)}(t) = 0 \\ \sum_{i=1}^{n} c_i'(t) \cdot \bar{\varphi}_i^{(n-1)}(t) = b(t) \end{cases} \Rightarrow c_i'(t) = \dots, \forall i = \overline{1,n} \Rightarrow c_i(t) = \dots, \forall i = \overline{1,n} \Rightarrow x(t) = \dots$$

$\boxed{\text{Ex.}}$ Să se det. sol. generală:

① $\begin{cases} x' = 2x - 4y \\ y' = x - 3y + 3e^t \end{cases}$

$$\begin{cases} \bar{x}' = 2\bar{x} - 4\bar{y} \\ \bar{y}' = \bar{x} - 3\bar{y} \end{cases} \Rightarrow \bar{x} = \bar{y}' + 3\bar{y}$$

$$\Rightarrow (\bar{y}' + 3\bar{y})' = 2(\bar{y}' + 3\bar{y}) - 4\bar{y}$$

$$\bar{y}'' + 3\bar{y}' = 2\bar{y}' + 6\bar{y} - 4\bar{y}$$

$$\bar{y}'' + 3\bar{y}' = 2\bar{y} + 2\bar{y}$$

$$\bar{y}'' + 5\bar{y}' - 2\bar{y} = 0$$

$$\Rightarrow \lambda^2 + 5\lambda - 2 = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 1$$

$$\bar{y}(t) = c_1 \cdot e^{-2t} + c_2 \cdot e^t, \ c_1, c_2 \in \mathbb{R} \Rightarrow \bar{x}(t) = 3c_1 e^{-2t} + 3c_2 e^t + (-2c_1 e^{-2t} + c_2 e^t)$$

$$\bar{x}(t) = c_1 e^{-2t} + 4c_2 e^t$$

Căutăm sol. de forma $\begin{cases} x(t) = c_1(t) e^{-2t} + 4c_2(t) e^t \\ y(t) = c_1(t) e^{-2t} + c_2(t) e^t. \end{cases}$

$$\Rightarrow \cancel{c_1(t)(-2e^{-2t})} + c_1'(t) e^{-2t} + \cancel{4c_2(t) e^t} + 4c_2'(t) e^t = \cancel{2c_1(t) e^{-2t}} + \cancel{8c_2(t) e^t}$$
$$-4c_1(t) e^{-2t} - 4c_2(t) e^t$$

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$$\left[\; \cancel{c_1(t)(-2e^{-2t})} + c_1'(t)e^{-2t} + \cancel{c_2(t)e^t} + c_2'(t)e^t = \cancel{c_1(t)e^{-2t}} + \cancel{4c_2(t)e^t} - \cancel{3c_1(t)e^{-2t}} - \cancel{3c_2(t)e^t} + 3e^t \right.$$

$$\begin{cases} c_1'(t)e^{-2t} + 4c_2'(t)e^t = 0 \\ c_1'(t)e^{-2t} + c_2'(t)e^t = 3e^t \end{cases} \Rightarrow 3c_2'(t)e^t = -3e^t$$

$$\Rightarrow \begin{cases} c_2'(t) = -1 \Rightarrow c_2(t) = -t + k_2 \\ c_1'(t) = 4e^{3t} \Rightarrow c_1(t) = \frac{4}{3}e^{3t} + k_1, \; k_1, k_2 \in \mathbb{R} \end{cases}$$

$$\Rightarrow \begin{cases} x(t) = \frac{4}{3}e^t + k_1 e^{-2t} + 4(-t + k_2)e^t \\ y(t) = \frac{4}{3}e^t + k_1 e^{-2t} + (-t + k_2)e^t \end{cases}, \; k_1, k_2 \in \mathbb{R}$$

② $\begin{cases} x' = 2x - y + e^t \\ y' = 3y - 2x + e^{-t} \end{cases}$ Temă

③ $x''' + 3x'' + 3x' + x = e^{-t}$

$$\bar{x}''' + 3\bar{x}'' + 3\bar{x}' + \bar{x} = 0.$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0.$$

$$(\lambda + 1)^3 = 0 \Rightarrow \lambda = -1, \; m_\lambda = 3.$$

$$\begin{cases} \varphi_1(t) = e^{-t} \\ \varphi_2(t) = t\,e^{-t} \\ \varphi_3(t) = t^2 \cdot e^{-t} \end{cases} \Rightarrow \bar{x}(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 \cdot e^{-t}, \; c_1, c_2, c_3 \in \mathbb{R}.$$

$$x(t) = c_1(t)e^{-t} + c_2(t)te^{-t} + c_3(t)\cdot t^2 \cdot e^{-t}$$

$$\begin{cases} c_1'(t)e^{-t} + c_2'(t)t\cdot e^{-t} + c_3'(t)\,t^2 \cdot e^{-t} = 0 \\ -c_1'(t)e^{-t} + c_2'(t)(e^{-t} - te^{-t}) + c_3'(t)\cdot(2te^{-t} - t^2e^{-t}) = 0 \\ c_1'(t)e^{-t} + c_2'(t)(-2e^{-t} + te^{-t}) + c_3'(t)(2e^{-t} - 4te^{-t} + t^2e^{-t}) = e^{-t} \end{cases}$$

$$\begin{cases} c_1'(t) + c_2'(t)t + c_3'(t)t^2 = 0 \\ -c_1'(t) + c_2'(t)(1-t) + c_3'(t)(2t - t^2) = 0 \\ c_1'(t) + c_2'(t)(-2+t) + c_3'(t)(2 - 4t + t^2) = 1 \end{cases} \Big\} \; (+)$$

$$\Rightarrow \begin{cases} c_2'(t) + c_3'(t)t^2 = 0 \\ -c_2'(t) + c_3'(t)(2 - 2t) = 1 \end{cases}$$

$\Rightarrow 2c_3'(t) = 1 \Rightarrow c_3'(t) = \frac{1}{2} \Rightarrow c_3(t) = \frac{1}{2}t + k_3$

$$\begin{cases} \Rightarrow c_2'(t) = -t \Rightarrow c_2(t) = -\frac{t^2}{2} + k_2 \\ \Rightarrow c_1'(t) = \frac{t^2}{2} \Rightarrow c_1(t) = \frac{t^3}{6} + k_1 \end{cases}, \quad k_1, k_2, k_3 \in \mathbb{R}$$

$\Rightarrow x(t) = e^{-t}\left(\frac{t^3}{6} + k_1\right) + t \cdot e^{-t}\left(\frac{t^2}{2} + k_2\right) + t^2 e^{-t}\left(\frac{t}{2} + k_3\right)$

④. $x^{IV} + x''' = \cos t$ Temă.

⑤. $t^2x'' - 3tx' + 5x = t^2$ ($t^2x'' - 3tx' + 5x = 0 \to$ ec. de tip Euler)

Considerăm ec. afină asociată: $t^2\bar{x}'' - 3t\bar{x}' + 5\bar{x} = 0$ - Euler

$|t| = e^s$; pp. $t > 0$. $\Rightarrow$ s.v. $t = e^s$.

$e^{2s}\bar{x}''(e^s) - 3e^s\bar{x}'(e^s) + 5\bar{x}(e^s) = 0$.

$\forall\, \bar{x}(\cdot)$ sol $\Rightarrow$ s.v. def. $y(\cdot)$ după regula: $y(s) = \bar{x}(e^s) \Leftrightarrow$
$\bar{x}(t) = y(\ln t)$

• $\bar{x}(e^s) = y(s)$ | derivăm

$\Rightarrow \bar{x}'(e^s) \cdot e^s = y'(s) \Rightarrow \bar{x}'(e^s) = \frac{y'(s)}{e^s} = y'(s) \cdot e^{-s}$

• $\bar{x}'(e^s) = y'(s) \cdot e^{-s}$ | derivăm

$\Rightarrow \bar{x}''(e^s) \cdot e^s = y''(s)e^{-s} - y'(s) \cdot e^{-s} \Rightarrow \bar{x}''(e^s) = -y'(s)e^{-2s} + y''(s) \cdot e^{-2s}$

$\Rightarrow e^{2s}\left(-y'(s)e^{-2s} + y''(s)e^{-2s}\right) - 3e^s\left(y'(s) \cdot e^{-s}\right) + 5y(s) = 0$.

$-y'(s) + y''(s) - 3y'(s) + 5y(s) = 0$.

$y''(s) - 4y'(s) + 5y(s) = 0$.

$\Rightarrow y'' - 4y' + 5y = 0$

ec. caracteristică: $\lambda^2 - 4\lambda + 5 = 0$. $\Delta = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 5 = 16 - 20 = -4$

$(\lambda - 2)^2 + 1 = 0 \Rightarrow (\lambda - 2)^2 = -1 \Rightarrow \lambda_1 = 2 + i$
$\lambda_2 = 2 - i$

$y(s) = c_1 \cdot e^{2s} \cdot \cos(1 \cdot s) + c_2 e^{2s} \sin(1 \cdot s)$

$\bar{x}(t) = y(\ln t) = c_1 e^{2\ln t} \cdot \cos(1 \cdot \ln t) + c_2 e^{2\ln t} \sin(1 \cdot \ln t)$

$\Rightarrow \bar{x}(t) = t^2 c_1 \cos(\ln t) + c_2 \cdot t^2 \sin(\ln t)$, $c_1, c_2 \in \mathbb{R}$.

Pas 2 - variația constantelor

căutăm sol. de forma: $x(t) = c_1(t)t^2\cos(\ln t) + c_2(t)t^2\sin(\ln t)$

$$\begin{cases} c_1'(t)t^2\cos(\ln t) + c_2'(t)t^2\sin(\ln t) = 0 \\ c_1'(t)(2t\cos(\ln t) - t\sin(\ln t)) + c_2'(t)[2t\sin(\ln t) + t\cos(\ln t)] = t^2 \end{cases}$$

$$\begin{cases} c_1'(t)\cos(\ln t) + c_2'(t)\sin(\ln t) = 0 \\ c_1'(t)(2\cos(\ln t) - \sin(\ln t)) + c_2'(t)(2\sin(\ln t) + \cos(\ln t)) = t \end{cases}$$

$$\begin{cases} c_1'(t)\cos(\ln t) + c_2'(t)\sin(\ln t) = 0 & \mid \cdot \sin \\ -c_1'(t)\sin(\ln t) + c_2'(t)\cos(\ln t) = t & \mid \cdot \cos \end{cases} \quad \begin{array}{|l} \cos \\ -\sin \end{array}$$

$\Rightarrow c_2'(t)\sin^2(\ln t) + c_2'(t)\cos^2(\ln t) = t\cos(\ln t)$

$\Rightarrow c_2'(t) = t\cos(\ln t)$

$\Rightarrow c_1'(t) = -t\sin(\ln(t))$

$\Rightarrow c_2(t) = \int_1^t s\cos(\ln s)\,ds + k_2$

$\Rightarrow c_1(t) = \int_1^t -s\sin(\ln s)\,ds + k_1$, k_1, k_2 - constante reale.

⑥ $t^2x'' - tx' + x = t$ Temă

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$x' = f(t,x)$ $\quad f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, cont, $C^1(\mathrm{II})$

$\alpha_f(\cdot,\cdot,\cdot) : D_f \to \mathbb{R}^n$ curentul maximal C^1

$\forall (\tau,\xi) \in D$, $\alpha_f(\cdot,\tau,\xi) : J(\tau,\xi) = (t^-(\tau,\xi), t^+(\tau,\xi)) \to \mathbb{R}^n$ sol. maximală unică a pb. $C(f,\tau,\xi)$

$\alpha_f(\cdot,\tau,\xi) = \varphi_{\tau,\xi}(\cdot)$ $\quad \begin{cases} D_1\alpha_f(t,\tau,\xi) = f(t, \alpha_f(t,\tau,\xi)) \\ \alpha(\tau,\tau,\xi) = \xi \end{cases}$

$t \mapsto D_2\alpha_f(t,\tau,\xi) \left(= \frac{\partial}{\partial \tau} \alpha_f(t,\tau,\xi)\right) \in L(\mathbb{R},\mathbb{R}^n) \simeq \mathbb{R}^n$ sol a „ec. în variații"

$y' = D_2 f(t, \alpha_f(t,\tau,\xi))y$ $\quad$ cu $y(\tau) = -f(\tau,\xi)$

$t \mapsto D_3\alpha_f(t,\tau,\xi) \left(= \frac{\partial}{\partial \xi} \alpha_f(t,\tau,\xi)\right) \in L(\mathbb{R}^n,\mathbb{R}^n) \simeq M_n(\mathbb{R})$ sol. matriceală

$y' = D_2 f(t, \alpha_f(t,\tau,\xi))y$ $\quad$ cu $y(\tau) = I_n$

$x' = f(t,x,\lambda)$ $\quad f(\cdot,\cdot,\cdot) : D = \mathring{D} \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont $C^1(\mathrm{II},\mathrm{III})$

$\alpha_f(\cdot,\cdot,\cdot,\cdot) : D_f \to \mathbb{R}^n$ curentul maximal parametrizat

$t \mapsto D_2\alpha_f(t,\tau,\xi,\lambda)$ sol $\quad y' = D_2 f(t, \alpha_f(t,\tau,\xi,\lambda),\lambda)y$

$y(\tau) = -f(\tau,\xi,\lambda)$

$t \mapsto D_3\alpha_f(t,\tau,\xi,\lambda)$ sol. matriceală $\quad y' = D_2 f(t, \alpha_f(t,\tau,\xi,\lambda),\lambda)y$

$y(\tau) = I_n$

$t \mapsto D_4\alpha_f(t,\tau,\xi,\lambda) \left(= \frac{\partial}{\partial \lambda} \alpha_f(t,\tau,\xi,\lambda)\right) \in L(\mathbb{R}^k,\mathbb{R}^k)$ sol. matriceală pt „ec. în variații pt. parametrii"

$z' = D_2 f(t, \alpha_f(t,\tau,\xi,\lambda),\lambda)z + D_3 f(t, \alpha_f(t,\tau,\xi,\lambda),\lambda)$ cu

$z(\tau) = 0$.

$\boxed{\text{Ex. 1.}}$ Fie $\varphi(\cdot,\xi) : J(\xi) \subseteq \mathbb{R} \to \mathbb{R}$, $\xi \in \mathbb{R}$, sol. maximală a pb.

$x' = x^2 - tx - t$, $\quad x(-1) = \xi$

a) Să se arate că $\varphi(t,0) = t+1$

b) Să se calculeze $D_2\varphi(t,0) = ?$

Rezolvare:

a) $\varphi(\cdot,0)$ sol. maximală a pb. $x' = x^2 - tx - t$, $x(-1)=0$.

Fie $\psi(t) = t+1$, $\psi(\cdot): \mathbb{R} \to \mathbb{R}$:

- $\psi'(t) = \psi^2(t) - t\psi(t) - t$
- $\psi(-1) = 0$.

Fie $f(t,x) = x^2 - tx - t$, $f(\cdot,\cdot): \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ cont, $C^1(\text{ii})$
$\downarrow$
(local Lipschitz(ii))

$\xrightarrow{T.C-L} \exists\, U.L \overset{T}{\iff} U.G \Rightarrow \varphi(\cdot,0) = \psi(\cdot)$ $\boxed{\text{OK}}$

b) Fie $\alpha_f(\cdot,\cdot,\cdot): D_f \to \mathbb{R}$ curentul maximal C^1

$$\varphi(t,\xi) = \alpha_f(t,-1,\xi) \quad \Big| \frac{\partial}{\partial \xi}$$

$$D_2\varphi(t,\xi) = D_3\alpha_f(t,-1,\xi)$$

$$\xi = 0 \Rightarrow D_2\varphi(t,0) = \underbrace{D_3\alpha_f(t,-1,0)}$$

$t \to D_3\alpha_f(t,-1,0)$ sol. a ec. var. $y' = D_2 f(t, \alpha_f(t,-1,0))\,y$
cu $y(\tau) = y(-1) = 1$

$$\left.\begin{aligned} &D_2 f(t,x) = 2x - t \\ &\alpha_f(t,-1,0) = \varphi(t,0) \overset{a)}{=} t+1. \end{aligned}\right| \Rightarrow$$

$$y' = (2(t+1) - t)\,y,\quad y(-1) = 1$$

$$y' = (t+2)\,y,\quad y(-1) = 1.$$

$$y(t) = c\cdot e^{\frac{t^2}{2} + 2t}$$

$$y(-1) = c \cdot e^{-\frac{3}{2}} = 1 \Rightarrow c = e^{+\frac{3}{2}}$$

$$D_2\varphi(t,0) = D_3\alpha_f(t,-1,0) = y(t) = e^{\frac{t^2}{2} + 2t + \frac{3}{2}}$$

$\boxed{\text{Ex. 2.}}$ Fie $\varphi(\cdot,\tau): J(\tau) \subseteq \mathbb{R} \to \mathbb{R}$, sol. maximală a pb.

$$x' = x^2 - x\cos t + \sin t + 1,\quad x(\tau) = 0$$

a) Să se arate că $\varphi(t,0) = \operatorname{tg} t$

b) Să se calculeze $D_2\varphi(t,0) = ?$

a) $\varphi(\cdot,0)$ sol. maximală a pb. $x' = x^2 - x\cos t + \sin t + 1$, $x(\tau) = 0$.
$x(0) = 0$.

Fie $\psi(t) = tg\, t$, $\psi(\cdot)\ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

- $\psi'(t) = \psi^2(t) - \psi(t)\cos t + \sin t + 1$

$$\frac{1}{\cos^2 t} = \frac{\sin^2 t}{\cos^2 t} - \cancel{\sin t} + \cancel{\sin t} + \overset{\cos^2 t}{1}$$

$$\frac{1}{\cos^2 t} = \frac{1}{\cos^2 t} \quad (A)$$

- $\psi(0) = 0$

Fie $f(t,x) = x^2 - x\cos t + \sin t + 1$, $f(\cdot,\cdot): \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ cont, $C^1(\mathrm{II})$

$\Downarrow$

(local Lipschitz(II))

$\xrightarrow{T.C-L} E\ (U.L) \overset{T}{\Longleftrightarrow} U.G \Rightarrow \varphi(\cdot,0) = \psi(\cdot)$ $\boxed{Ok.}$

b) Fie $\alpha f(\cdot,\cdot,\cdot)\ D_f \to \mathbb{R}$ curentul maximal C^1

$\varphi(t,\tau) = \alpha f(t,\tau,0) \ \Big|\ \frac{\partial}{\partial \tau}$

$D_2 \overset{\varphi}{f}(t,\tau) = D_2\alpha f(t,\tau,0)$

$\tau = 0 \Rightarrow D_2 f(t,0) = \underline{D_2 \alpha f(t,0,0)}$

$t \mapsto D_2\alpha f(t,0,0)$ sol. a unui pb.: $y' = D_2 f(t, \alpha f(t,0,0))y$

cu $y(\tau) = y(0) = -f(0,0)$

$D_2 f(t,x) = 2x - \cos t$

$\alpha f(t,0,0) = \varphi(t,0) \overset{a)}{=} tg\, t$ $\Big| \Rightarrow y' = (2tg\,t - \cos t)y$; $y(0) = -f(0,0) = -1$

$$y(t) = c \cdot e^{\int 2tg\,t - \cos t\, dt} = c\cdot e^{2\int tg\,t\,dt - \int \cos t\, dt}$$

$$= c \cdot e^{-2\ln|\cos t| - \sin t} = c \cdot \frac{1}{\cos^2 t} \cdot e^{-\sin t}$$

$y(0) = -1 \Rightarrow c = -1 \Rightarrow y(t) = -e^{-\sin t} \cdot \frac{1}{\cos^2 t}$.

$D_2 f(t,0) = y(t) = -e^{-\sin t} \cdot \frac{1}{\cos^2 t}$.

$\boxed{\text{Ex. 3.}}$ Fie $\varphi(\cdot,\lambda): J(\lambda) \subseteq \mathbb{R} \to \mathbb{R}$, $\lambda \in \mathbb{R}$, sol. maximală

$$x' = x^2 + \frac{x}{t} + \lambda(x^2+1), \quad x(\lambda+1) = \lambda - 1$$

$D_2\varphi(t,0) = ?$

Fie $f(t,x,\lambda) = x^2 + \frac{x}{t} + \lambda(x^2+1)$, $f(\cdot,\cdot,\cdot): \mathbb{R}^* \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ cont, $C^1(\text{II}, \text{III})$

Fie $\alpha f(\cdot,\cdot,\cdot,\cdot): D_f \to \mathbb{R}$ curent maximal parametrizat C^1

$$\varphi(t,\lambda) = \alpha f(t, \lambda+1, \lambda-1, \lambda) \quad \Big| \frac{\partial}{\partial\lambda}$$

$$D_2\varphi(t,\lambda) = D_2\alpha f(t,\lambda+1,\lambda-1,\lambda) + D_3\alpha f(t,\lambda+1,\lambda-1,\lambda) + D_4\alpha f(t,\lambda+1,\lambda+1,\lambda)$$

pt $\lambda = 0 \Rightarrow D_2\varphi(t,0) = \underbrace{D_2\alpha f(t,1,-1,0)} + \underbrace{D_3\alpha f(t,1,-1,0)} + \underbrace{D_4\alpha f(t,1,-1,0)}$

$t \to D_2\alpha f(t,1,-1,0)$ sol. a unui pb: $y' = D_2 f(t, \alpha f(t,1,-1,0), 0)y$

cu $y(1) = -f(1,-1,0) = 0$

$t \to D_3\alpha f(t,1,-1,0)$ sol a unui pb: $y' = D_2 f(t, \alpha f(t,1,-1,0), 0)y$

cu $y(1) = 1$

$t \to D_4\alpha f(t,1,-1,0)$ sol. a unui pb: $z' = D_2 f(t, \alpha f(t,1,-1,0),0)z + D_3 f(t, \alpha f(t,1,-1,0), 0)$

cu $z(1) = 0$.

$D_2 f(t,x,\lambda) = 2x + \frac{1}{t} + 2x\lambda$

$D_3 f(t,x,\lambda) = x^2 + 1$

$\alpha f(t,1,-1,0): \; x' = x^2 + \frac{x}{t} \qquad x(1) = -1$

$\bar{x}' = \frac{\bar{x}}{t} \Rightarrow \bar{x}(t) = c \cdot e^{\ln t} = c \cdot t$

$\Rightarrow x(t) = c(t) \cdot t \Rightarrow (c(t)\cdot t)' = c^2(t)\cdot t^2 + c(t)$

$$c'(t)\cdot t + \cancel{c(t)} = c^2(t)\cdot t^2 + \cancel{c(t)}$$

$$c'(t) = c^2(t)\cdot t$$

$\frac{dc}{dt} = t \cdot c^2$

- $c^2 = 0 \Rightarrow c = 0 \Rightarrow c(t) \equiv 0 \Rightarrow x(t) \equiv 0$ (NU pt. că $x(1) = -1$)
- $\frac{dc}{c^2} = t\,dt \Rightarrow -\frac{1}{c} = \frac{t^2}{2} + k,\ k \in \mathbb{R} \Rightarrow c(t) = \frac{-2}{t^2+k},\ k \in \mathbb{R}$

$x(t) = \frac{-2t}{t^2+k}$

$x(1) = -1 \Rightarrow \frac{-2}{1+k} = -1 \Rightarrow -1-k=-2$

$-k=-1 \Rightarrow \underline{k=1}.$

$\alpha_\varphi(t,1,-1,0) = \frac{-2t}{t^2+1}.$

TEMĂ:

① $\varphi(\cdot,\eta): J(\eta)\subseteq\mathbb{R}\to\mathbb{R}$, $\eta\in\mathbb{R}$ sol. maximală

$$x' = tx^2 - t^3 + 1 \quad , x(\eta)=0$$

a) Să se arate că $\varphi(t,0)\equiv t$

b) $D_2\varphi(t,0)=?$

② $\varphi(\cdot,\xi): J(\xi)\subseteq\mathbb{R}\to\mathbb{R}$, $\xi\in\mathbb{R}$ sol. maximală

$$x' = x + x^2 + tx^3, \quad x(0)=\xi$$

a) Să se arate că $\varphi(t,0)\equiv 0$

b) $D_2\varphi(t,0)=?$

③ $\varphi(\cdot,\lambda): J(\lambda)\subseteq\mathbb{R}\to\mathbb{R}$, $\lambda\in\mathbb{R}$ sol maximală

$$x' = \frac{x}{t} + 3tx^2 + \lambda, \quad x(1)=\lambda-1$$

$D_2\varphi(t,0)=?$

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Integrala primă

$\frac{dx}{dt} = f(t,x)$ $\quad f(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$\frac{dx_i}{dt} = f_i(t,x)$ $\quad f(\cdot,\cdot) = (f_1(\cdot,\cdot), \dots, f_n(\cdot,\cdot))$

$i = \overline{1,n}$

Def. $F(\cdot,\cdot) : D_0 \subset D \to \mathbb{R}$ s.n. integrală primă pt. ec. dacă: $\exists\, c_\varphi \in \mathbb{R}$ a.î. $F(t, \varphi(t)) \equiv c_\varphi$
$\forall \varphi(\cdot)$ sol., graph $\varphi(\cdot) \subset D_0$

Criteriul: $D = \mathring{D}$, f_i continuă, $D_0 = \mathring{D}_0$, $F(\cdot,\cdot)$ - dif.

$F(\cdot,\cdot)$ integrală primă $\Leftrightarrow \frac{\partial F}{\partial t}(t,x) + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t,x) \cdot f_i(t,x) = 0$, $\forall (t,x) \in D_0$

- $F_1(\cdot,\cdot), \dots, F_k(\cdot,\cdot) : D_0 \to \mathbb{R}$, $k < n$ dif, int. prime s.n. funcțional independente dacă

$$\text{rang}\left(\frac{\partial F_i}{\partial x_j}(t,x)\right)_{\substack{i=\overline{1,k} \\ j=\overline{1,k}}} = k \text{ (maxim)} \le n, \ \forall (t,x) \in D_0$$

- $F_1(\cdot,\cdot), \dots, F_n(\cdot,\cdot)$ - integrale prime funcțional indep.

$$\underbrace{F_i(t,x) = c_i}_{\text{sol. generală sub formă implicită}}, \quad i = \overline{1,n}, \quad c_i \in \mathbb{R}$$

Reducerea ordinului cu ajutorul int. prime

$F_1(\cdot,\cdot), \dots, F_k(\cdot,\cdot) : D_0 \to \mathbb{R}$, $k < n$, int. prime funcțional independente

Sp. $\det\left(\frac{\partial F_i}{\partial x_j}(t,x)\right)_{i,j=\overline{1,k}} \neq 0$, $\forall (t,x) \in D_0$.

(1) $\begin{cases} F_1(t, x_1, \dots, x_k, x_{k+1}, \dots, x_n) = c_1 \\ \vdots \\ F_k(t, x_1, \dots, x_k, x_{k+1}, \dots, x_n) = c_k \end{cases} \overset{T.F.I.}{\Longrightarrow} x_j = \varphi_j(t, x_{k+1}, \dots, x_n, c_1, \dots, c_k)$, $j = \overline{1,k}$

$$②\ \begin{cases} \dfrac{dx_{k+1}}{dt} = f_{k+1}(t, \varphi_1(t, x_{k+1}, \dots, x_n, c_1, \dots, c_k), \dots, \varphi_k(t, x_{k+1}, \dots, x_n, c_1, \dots, c_k), x_{k+1}, \dots, x_n) \\ \vdots \\ \dfrac{dx_n}{dt} = f_n(t, \varphi_1(t, x_{k+1}, x_n, c_1, \dots, c_k), \dots, \varphi_k(t, x_{k+1}, \dots x_n, c_1, \dots, c_k), x_{k+1}, \dots, x_n) \end{cases}$$

$\underline{n-1 \text{ ee.}}$

Ex. ① Fie ee. $\begin{cases} x' = \dfrac{x^2-2t}{y} & (=f_1(t,(x,y))) \\ y' = -x & (=f_2(t,(x,y))) \end{cases}$

a) Să se arate că $F_1(t,(x,y)) = t^2 + xy$ este integrală primă

b) Să se determine sol. generală.

c) $F_2(\cdot,\cdot) = ?$ a.î. $\{F_1(\cdot,\cdot), F_2(\cdot,\cdot)\}$ int. prime funcțional independente

$\boxed{\text{Rezolvare}}$

a) Aplicăm criteriul

$$\frac{\partial F_1}{\partial t}(t,(x,y)) + \frac{\partial F_1}{\partial x}(t,(x,y)) \cdot \frac{x^2-2t}{y} + \frac{\partial F_1}{\partial y}(t,(x,y)) \cdot (-x) =$$

$$= 2t + \cancel{y}\, \frac{x^2-2t}{\cancel{y}} + x \cdot (-x)$$

$$= 2t + \cancel{x^2} - 2t - \cancel{x^2}$$

$= 0. \Rightarrow F_1(t,(x,y))$ integrală primă

b) $F_1(t,(x,y)) = c \in \mathbb{R} \Rightarrow t^2 + xy = c. \Rightarrow x = \dfrac{c-t^2}{y}$

$$\Rightarrow y' = \frac{t^2-c}{y} \Rightarrow \frac{dy}{dt} = \frac{t^2-c}{y} \Rightarrow y\,dy = (t^2-c)\,dt$$

$$\frac{1}{2}y^2 = \frac{1}{3}t^3 - ct + k \Rightarrow y(t) = \pm\sqrt{2\left(\frac{1}{3}t^3 - ct + k\right)}$$

$c, k \in \mathbb{R}$

$$\Rightarrow x(t) = \frac{c-t^2}{\pm\sqrt{2\left(\frac{1}{3}t^3 - ct + k\right)}}$$

c)

$$k = \frac{1}{2}y^2_{(t)} - \frac{1}{3}t^3 + ct = \frac{1}{2}y^2(t) - \frac{1}{3}t^3 + (t^2 + x(t)y(t)) \cdot t$$

Fie $F_2(t,(x,y)) = \frac{1}{2}y^2 - \frac{1}{3}t^3 + (t^2+xy)\cdot t$ – integrală primă (din def.)

$$\begin{pmatrix} \frac{\partial F_1}{\partial x}(t,(x,y)) & \frac{\partial F_1}{\partial y}(t,(x,y)) \\ \frac{\partial F_2}{\partial x}(t,(x,y)) & \frac{\partial F_2}{\partial y}(t,(x,y)) \end{pmatrix} = \begin{pmatrix} y & x \\ yt & y+xt \end{pmatrix}$$

$$\det\begin{pmatrix} y & x \\ yt & y+xt \end{pmatrix} = y^2 + \cancel{yxt} - \cancel{yxt} = y^2 \neq 0, \ \forall (t,x,y) \in D$$

$$\begin{cases} x' = \frac{x^2-2t}{y} & | y \\ y' = -x & | x \end{cases} \ +$$

$$x'y + y'x = -2t$$

$$(xy)' + 2t = (xy+t^2)' = 0$$

$$\Rightarrow \exists\, c \in \mathbb{R} \text{ a.î. } x(t)y(t) + t^2 \equiv c$$

$$\Rightarrow F_1(t,(x,y)) = xy + t^2$$

Ex. ② Fie ec. $\begin{cases} x' = x^2 y \\ y' = \frac{y}{t} - xy^2 \end{cases}$

a) Să se arate că $F_1(t,(x,y)) = \frac{xy}{t}$ este integrală primă.

b) Să se det. sol. generală.

c) $F_2(\cdot,\cdot) = ?$ a.î. $\{F_1(\cdot,\cdot), F_2(\cdot,\cdot)\}$ sint prime funcț. indep.

$$x'y + xy' = \frac{xy}{t}$$

$$(xy)' = \frac{xy}{t}, \ t > 0.$$

s.v. $z = xy \Rightarrow z' = \frac{z}{t} \Rightarrow z(t) = c\cdot e^{\int \frac{1}{t} dt} = c e^{\ln t} = ct \quad c \in \mathbb{R}$

$$x(t)y(t) = c\cdot t \Rightarrow \frac{x(t)y(t)}{t} \equiv c \Rightarrow F_1 -$$

a) Aplicăm criteriul

$$\frac{\partial F_1}{\partial t}(t,(x,y)) + \frac{\partial F_1}{\partial x}(t,(x,y))\cdot f_1 + \frac{\partial F_1}{\partial y}(t,(x,y))\cdot f_2$$

$$= xy\cdot\left(-\frac{1}{t^2}\right) + \frac{y}{t}\cdot x^2 y + \frac{x}{t}\cdot\left(\frac{y}{t} - xy^2\right)$$

$$= \cancel{\frac{-xy}{t^2}} + \cancel{\frac{x^2y^2}{t}} + \cancel{\frac{xy}{t^2}} - \cancel{\frac{x^2y^2}{t}}$$

$= 0 \Rightarrow F_1$ integrală primă

b) $F_1(t,(x,y)) = c \in \mathbb{R}^2 \Rightarrow \frac{xy}{t} = c \Rightarrow xy = c\cdot t \Rightarrow y = \frac{c\cdot t}{x}$

$\Rightarrow x' = x^{\cancel{2}}\cdot\frac{c\cdot t}{\cancel{x}} = x\, c\cdot t \Rightarrow x' = x\cdot c\cdot t$ - ec. liniară scalară

$$\Rightarrow \begin{cases} x(t) = k\cdot e^{\int ct\,dt} = k\, e^{\frac{ct^2}{2}}, k\in\mathbb{R} \\ y(t) = \frac{c\cdot t}{k\cdot e^{\frac{ct^2}{2}}}, k\in\mathbb{R}, k\neq 0 \end{cases}$$

$\downarrow$

soluția generală.

c) $k = \frac{x(t)}{e^{\frac{ct^2}{2}}} = x(t)\cdot e^{\frac{-ct^2}{2}} = x(t)\cdot e^{-\frac{\frac{x(t)y(t)}{\cancel{t}}\cdot t^{\cancel{2}}}{2}}$

$= x(t)\cdot e^{-\frac{x(t)y(t)\cdot t}{2}}$

$F_2(t,(x,y)) = x\cdot e^{\frac{-xyt}{2}}$ - integrală primă

$$\begin{pmatrix} \frac{\partial F_1}{\partial x}(t,(x,y)) & \frac{\partial F_1}{\partial y}(t,(x,y)) \\ \frac{\partial F_2}{\partial x}(t,(x,y)) & \frac{\partial F_2}{\partial y}(t,(x,y)) \end{pmatrix} = \begin{pmatrix} \frac{y}{t} & \frac{x}{t} \\ e^{\frac{-xyt}{2}} - xe^{\frac{-xyt}{2}}\cdot\frac{1}{2}yt & -xe^{\frac{-xyt}{2}}\cdot\frac{xt}{2} \end{pmatrix}$$

$$\det = \frac{-x^2ye^{\frac{-xyt}{2}}}{2} - \frac{xe^{\frac{-xyt}{2}}}{t} + \frac{x^2ye^{\frac{-xyt}{2}}}{2} = -\frac{xe^{\frac{-xyt}{2}}}{t} \neq 0 \quad \forall (t,(x,y)) \in \Delta_0$$

unde $\Delta_0 = \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}$

$\Delta = \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}$

$\swarrow$

domeniu

Ex. 3 Fie ec. $\begin{cases} x' = \dfrac{t-x}{t+x+y} \\ y' = \dfrac{x-t}{t+x+y} \end{cases}$

b) Să se det. soluția generală.

a) Să se arate că $F_1(t,(x,y)) = x+y$ este integrală primă.

c) $F_2(\cdot,\cdot) = ?$ ai $\{F_1(\cdot,\cdot), F_2(\cdot,\cdot)\}$ int. prime funcțional indep.

de copiat

Temă: Fie ec. $\begin{cases} x' = 1 - \dfrac{1}{y} \\ y' = \dfrac{1}{x-t} \end{cases}$

a) Să se arate că $F_1(t,(x,y)) = (x-t)y$ este int. primă.

b) Să se det. sol. generală.

c) $F_2(\cdot,\cdot)$ ai $\{F_1(\cdot,\cdot), F_2(\cdot,\cdot)\}$ int prime funct. indep.

$x-1 = -\dfrac{1}{y}$

deriv. lui $x-t$

s.v. $z = x-t$

$$\begin{cases} z' = -\dfrac{1}{y} & \Big| \cdot y \\ y' = \dfrac{1}{z} & \Big| \cdot z \end{cases}$$

$z'y + y'z = 0$

$(yz)' = 0$

$\Rightarrow y(t)z(t) \equiv c$

$y(t)(x(t)-t) \equiv c$