$\boxed{\text{EDP - 2019}}$

$\boxed{\text{Pb 2}}$ $\Omega := \{(x,y) \in \mathbb{R}^2 ;\ x^2+y^2 < 4\}$

$$\begin{cases} -\Delta u(x,y) = 3\cos y & (x,y) \in \Omega \\ u(x,y) = 0 & (x,y) \in \partial\Omega \end{cases}$$

① Fie u_1, u_2 2 soluții și consider $U = u_1 - u_2$

$$\begin{cases} -\Delta U = -\Delta u_1 + \Delta u_2 = 3\cos y - 3\cos y = 0 & \text{în } \Omega \\ U|_{\partial\Omega} = u_1|_{\partial\Omega} - u_2|_{\partial\Omega} = 0 - 0 = 0 & \text{pe } \Omega \end{cases} \xRightarrow{PM}$$

$$\Rightarrow \underbrace{\min_{\partial\Omega} U}_{0} \le U(z) \le \underbrace{\max_{\partial\Omega} U}_{0} \Rightarrow U \equiv 0 \text{ în } \overline{\Omega}$$

$\Rightarrow u_1 = u_2$

② $v(x,y) = C(x^2+y^2)$

$-\Delta v = 3$

$$\left.\begin{aligned} v_x(x,y) &= 2Cx \\ v_{xx}(x,y) &= 2C \\ v_y(x,y) &= 2Cy \\ v_{yy}(x,y) &= 2C \end{aligned}\right\} \Rightarrow \Delta v = v_{xx} + v_{yy} = 4C$$

$-\Delta v = 3 \Leftrightarrow -4C = 3$

$\Leftrightarrow C = -\frac{3}{4}$

③ Fie $u = w - v$

$u(x,y) = w(x,y) - v(x,y)$

$$\begin{cases} -\Delta u = -\Delta w + \Delta v = 3 - 3 = 0 \\ u|_{\partial\Omega} = w|_{\partial\Omega} - v|_{\partial\Omega} = 0 - v|_{\partial\Omega} = \\ \quad = +\frac{3}{4}(x^2+y^2)\Big|_{x^2+y^2=4} = 3 \end{cases} \xRightarrow{PM}$$

$$\Rightarrow \underbrace{\min_{\partial\Omega} u}_{3} \le u(x,y) \le \underbrace{\max_{\partial\Omega} u}_{3} \Rightarrow u(x,y) = 3 \text{ pe } \overline{\Omega}$$

$\Rightarrow w = u + v = 3 - \frac{3}{4}(x^2+y^2)$

④ Pp că $\exists x_0 \in \Omega$ a.î. $u(x_0) > 3$

Fie $\Omega_1 := \{x \in \Omega \mid u(x) > 3\}$ → nevidă din pp

$= u^{-1}((3,\infty))$ - deschisă

$\partial\Omega_1 = \{x \in \Omega \mid u(x) = 3\}$

Fie $\boxed{w_1 = +u + v}$, unde $v = +\frac{3}{4}(x^2+y^2)$

$$\begin{cases} -\Delta w_1 = -\Delta u - \Delta v = +3\cos y - 3 \le 0 \\ w_1|_{\partial\Omega} = u|_{\partial\Omega} + v|_{\partial\Omega} = 0 + \frac{3}{4}(x^2+y^2)\Big|_{x^2+y^2=4} = +3 \end{cases} \xRightarrow{PM}$$

$\Rightarrow w(z) \le \underbrace{\max_{\partial\Omega} w}_{3}$, $\forall z \in \partial\Omega \Rightarrow u + v \le 3 \Rightarrow$

$\Rightarrow u(x,y) \le 3 - v(x,y) \le 3$ (1)

$-5 \le x$

$\boxed{w_2 = u - v}$, $v = \frac{+3}{4}(x^2+y^2)$

$$\begin{cases} -\Delta w_1 = -\Delta u + \Delta v = 3\cos y + 3 \ge 0 \\ w_1\big|_{\partial\Omega} = -3 \end{cases} \xrightarrow{PM}$$

$\Rightarrow \underbrace{\min_{\partial\Omega} w}_{= -3} \le w(x,y) \Rightarrow -3 \le u - v \Rightarrow$

$\Rightarrow \underbrace{-3 + v}_{\ge -3} \le u$ (2)

$\Rightarrow -3 \le u(z) \le 3$

Pb 1 $u(x) = |x|^{-\frac{3}{7}}$

$\Delta u(x) = g''(|x|) + \frac{4-1}{|x|} \cdot g'(|x|)$

$g(|x|) = u(x) = |x|^{-\frac{3}{7}}$

$g'(|x|) = -\frac{3}{7}|x|^{-\frac{3}{7}-1} = -\frac{3}{7}|x|^{-\frac{10}{7}}$

$g''(|x|) = \frac{30}{49}|x|^{-\frac{10}{7}-1} = \frac{30}{49}|x|^{-\frac{17}{7}}$

$\Delta u(x) = \frac{30}{49}|x|^{-17/7} + \frac{3}{|x|} \cdot \frac{-3}{7}|x|^{-\frac{10}{7}}$

$= \frac{30}{49}|x|^{-17/7} - \frac{9}{7}|x|^{-\frac{17}{7}} = |x|^{-17/7} \cdot \frac{30-63}{49}$

$= \frac{-33}{49}|x|^{-17/7}$

Ex3

$$\begin{cases} u_{tt} - 3u_{tx} - 4u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

1) să se verifice

$$(\partial_t + \partial_x)(u_t(x,t) - 4u_x(x,t)) = \\ = u_{tt} - 3u_{tx} - 4u_{xx}$$

2) Să se rez pb. „tt”, cu 2 ec. de transport (omog și neomog)

3) $t=0$ deduceți sol u a pb în cazul $f(x) = \sin x$

?? $g(x) = e^{-x}$ sau $2e^{-x^2}$

Nu mai știu exact.

Ex4

$$?\begin{cases} u_t(x,t) - u_{xx}(x,t) + u_t(x,t) = 0. \\ u(x,0) = e^{-x^2} \end{cases}$$

Era și ceva cu $\frac{e^t}{e^t+1}$ Nu mai știu unde

1) $\phi = ?$ ec. căldurii Pb Cauchy verificată v, $v(0,\frac{1}{2}) = ?$

2)

3) Sol explicită a pb.

Fie $w(s) = u(x-4s,\ t+s)$

$$w'(s) = u_t(x-4s,\ t+s) + u_x(x-4s,\ t+s)\cdot(-4)$$
$$= u_t(x-4s,\ t+s) - 4u_x(x-4s,\ t+s) =$$
$$= g(x-4s+t+s) - 4f'(x-4s+t+s)$$
$$= g(x+t-3s) - 4f'(x+t-3s)$$

$$\underbrace{\int_0^{3s} w'(\tau)\,d\tau}_{M_s} = \int_0^{3s} g(x+t-\tau)\,d\tau - 4\int_0^{3s} f'(x+t-\tau)\,d\tau \quad (*)$$

$$M_s = w(3s) - w(0) = u(x-12s,\ \underbrace{t+3s}_{\overset{\text{trebuie}}{=\,0}}) - u(x,t)$$

Iau $s = \frac{-t}{3}$

$$M_s = u(x+4t,\ 0) - u(x,t) = f(x+4t) - u(x,t)$$

$(*)$ devine pt $s = \frac{-t}{3}$

$$f(x+4t) - u(x,t) = \underbrace{\int_0^{-t} g(x+t-\tau)\,d\tau}_{J_1} - 4\underbrace{\int_0^{-t} f'(x+t-\tau)\,d\tau}_{J_2}$$

$$u(x,t) = f(x+4t) - J_1 + 4J_2$$

$$J_2 = \int_0^{-t} \left(-f(x+t-\tau)\right)'\,d\tau = -f(x+2t) + f(x+t)$$

$$J_1 = \int_0^{-t} g(x+t-\tau)\,d\tau \xlongequal[\substack{-d\tau = dz \\ \tau = 0 \Rightarrow z = x+t \\ \tau = -t \Rightarrow z = x+2t}]{x+t-\tau = z} \int_{x+t}^{x+2t} -g(z)\,dz$$

Deci, $u(x,t) = f(x+4t) - 4f(x+2t) + 4f(x+t) + \int_{x+t}^{x+2t} g(z)\,dz$

Ex1 $v : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ $v(x) = |x|^{-\frac{4}{3}}$

1) $\mathrm{div}\,(|x| \cdot \nabla u(x)) = ?$

2) Să se găsească $p \geq 1$, $v(x) \in L_1(B_1(0))$

3) Să se găsească $p \geq 1$, $v(x) \in L_1(\mathbb{R}^5 \setminus \overline{B_1(0)})$

4) Să se găsească un exemplu de fc. $(-\Delta u < 0)$ pe $\mathbb{R}^2$ ca dreapta $x + 3y$ să fie 0.

5) $u(x) = \left(\ln \frac{2}{|x|}\right)^{\frac{1}{2}}$

$$-\Delta u(x) = \frac{u(x)}{4|x|^2 \ln^2\left(\frac{2}{|x|}\right)}$$

Ex2
$$\begin{cases} u_{xx}(x,y) + 2u_{yy}(x,y) = 0 \\ u(x,0) = u(x,1) = 0 \\ u(0,y) = \sin(2\pi y) \\ u(1,y) = e^{-2\sqrt{2}\pi} \sin(2\pi y) \end{cases}$$

1) $u(x,y) = A(x) \cdot B(y)$

2) Unicitatea folosind metoda energetică