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Tutoriat 3 - Geometrie I

Ex 1. (Temă - tutoriat trecut) Fie $A(1,0)$, $B(3,1)$, $C(4,2)$. Să se afle izometria $g = S_A \circ S_B \circ S_C$.

SOL: $S_A : \begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow S_A : \begin{cases} x' = -x+2 \\ y' = -y \end{cases}$

$S_B : \begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \Leftrightarrow S_B : \begin{cases} x' = -x+6 \\ y' = -y+2 \end{cases}$

$S_C : \begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 4 \\ 2 \end{pmatrix} \Leftrightarrow S_C : \begin{cases} x' = -x+8 \\ y' = -y+4 \end{cases}$

$$g = S_A \circ S_B \circ S_C = (x', y') \xrightarrow{S_C} (-x+8, -y+4) \xrightarrow{S_B}$$

$$\xrightarrow{S_B} (-(-x+8)+6, -(-y+4)+2) = (x-2, y-2) \xrightarrow{S_A}$$

$$\xrightarrow{S_A} (-(x-2)+2, -(y-2)) = (-x+4, -y+2) =$$

$$= -(x,y) + (4,2) = -(x,y) + 2(2,1)$$

Avem $g = S_A \circ S_B \circ S_C = -(x,y) + 2(2,1) = S_M$, unde $M(2,1)$

$$\boxed{g = (S_A \circ S_B \circ S_C)(x', y') = S_A(S_B(S_C(x', y')))}$$

extra: Găsiți o dreaptă invariantă în raport cu S_M, unde $M(2,1)$.

(TEORIE) Dacă $M \in d$, atunci $S_M(d) = d$.

Alegem $d: x - 2y = 0$, $M \in d$.

Verificare: $(d): x - 2y = 0 \Rightarrow x = 2y$

$$g(x,y) = g(2y,y) = (-2y+4, -y+2) = (2(-y+2), -y+2) \in d.$$

Ex 2. Fie $(d_1): x = 1$, $(d_2): x = 3$. Aflați $S_{d_2} \circ S_{d_1}$.

SOL: $(d_1): x - 1 = 0 \Rightarrow a = 1, b = 0, c = -1$

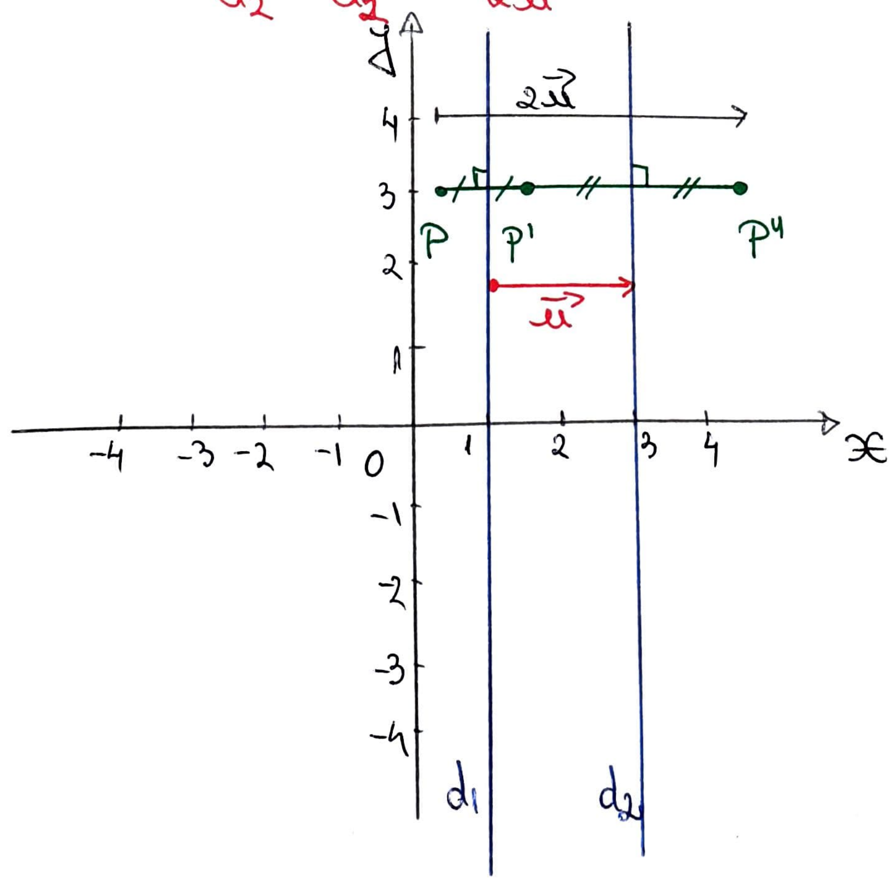
$$S_{d_1}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Leftrightarrow S_{d_1}: \begin{cases} x' = -x + 2 \\ y' = y \end{cases}$$

$$S_{d_2}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix} \Leftrightarrow S_{d_2}: \begin{cases} x' = -x + 6 \\ y' = y \end{cases}$$

$$S_{d_2} \circ S_{d_1} = (x,y) \xrightarrow{S_{d_1}} (-x+2, y) \xrightarrow{S_{d_2}} (-(-x+2)+6, y) =$$

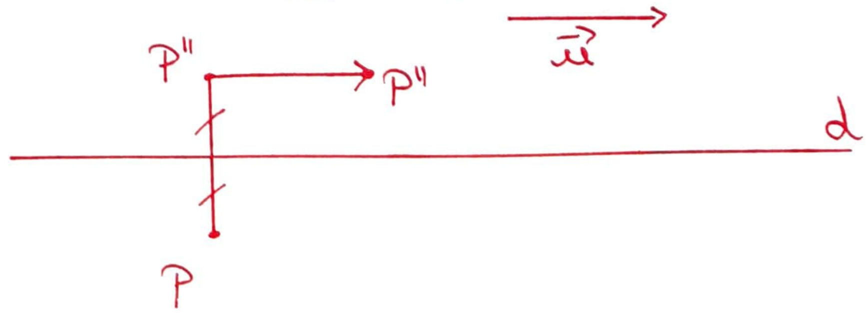
$$= (+x+4, y) = (x,y) + (4,0) = T_{2\vec{u}}$$

Din TEORIE: $S_{d_2} \circ S_{d_1} = T_{2\vec{u}} \Rightarrow 2\vec{u} = (4,0) \Rightarrow \boxed{\vec{u} = (2,0)}$



Ex 3 Fie $f = T_{\vec{u}} \circ S_d$ (glide reflection), $\vec{u} = (2,0)$, $d: y = 2$. Ecuația lui f ??

SOL Avem $T_{\vec{u}} \circ S_d(P)$, $\vec{u} \parallel d$



$(d): y - 2 = 0 \Rightarrow a = 0, b = 1, c = -2$

$$S_d : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \Leftrightarrow S_d : \begin{cases} x' = x \\ y' = -y + 4 \end{cases}$$

$$T_{\vec{u}} : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Leftrightarrow T_{\vec{u}} : \begin{cases} x' = x + 2 \\ y' = y \end{cases}$$

$$f = T_{\vec{u}} \circ S_d = (x, y) \xrightarrow{S_d} (x, -y+4) \xrightarrow{T_{\vec{u}}} (x+2, -y+4)$$

$$= (x, -y) + (2, 4).$$

Ex 4 a) Să se scrie ecuația lui $R_{O,\pi/3}$.

b) Să se determine $R_{O,\pi/3}(d) = d'$; $(d): x + y + 1 = 0$

c) Să se scrie ecuația lui $R_{M,\pi/3}$, $M(1,1)$

d) Să se determine $R_{M,\pi/3}(P) = P'$, $M(1,1)$, $P(3,3)$.

SOL $R_{N,\alpha} : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$N(x_0, y_0)$

a) $R_{O,\pi/3}: X' = A(\pi/3)X \iff \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$

$= \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \iff R_{O,\pi/2}: \begin{cases} x' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y \\ y' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y \end{cases}$

b) Din teorie știm că $A \cdot A^{T} = I_2$ (izometrii)

Așadar, avem că $A^{-1} = A^{T}$, căci $A \cdot A^{-1} = I_2$.

Noi avem: $X' = A(\pi/3)X \iff A^{-1}(\pi/3)X' = \underbrace{A^{-1}(\pi/3)A(\pi/3)}_{I_2}X$

$A^{-1}(\pi/3)X' = X$

Atunci avem:

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{cases} x = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \\ y = -\frac{\sqrt{3}}{2}x' + \frac{1}{2}y' \end{cases}$$

$d: x + y + 1 = 0$

$(d'): \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + \left(-\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right) + 1 = 0 \,\Big|\cdot 2$

$(d'): x' + \sqrt{3}y' + (-\sqrt{3}x') + y' + 2 = 0$

$(d'): (1-\sqrt{3})x' + (1+\sqrt{3})y' + 2 = 0.$

c) $R_{M,\pi/3}$, $M(1,1)$

$$R_{M,\pi/3}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$R_{M,\pi/3}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} x' = \frac{1}{2}(x-1) - \frac{\sqrt{3}}{2}(y-1) + 1 \\ y' = \frac{\sqrt{3}}{2}(x-1) + \frac{1}{2}(y-1) + 1 \end{cases}$$

d) $P(3,3)$

$$R_{M,\pi/2}(P) = P'\left(\frac{1}{2}(3-1) - \frac{\sqrt{3}}{2}(3-1)+1; \frac{\sqrt{3}}{2}(3-1) + \frac{1}{2}(3-1)+1\right) =$$

$$= P'\left(\frac{1}{\cancel{2}}\cdot \cancel{2} - \frac{\sqrt{3}}{\cancel{2}}\cdot \cancel{2}+1; \frac{\sqrt{3}}{\cancel{2}}\cdot \cancel{2} + \frac{1}{\cancel{2}}\cdot \cancel{2}+1\right) =$$

$$= P'(2-\sqrt{3}; 2+\sqrt{3}).$$

Ex 5. Fie punctele $A(1,0)$, $B = R_{0,2\pi/3}(A)$ și $C = R_{0,2\pi/3}(B)$.

Să se determine $R_{0,\pi/2}(\Delta ABC) = \Delta A'B'C'$.

Sol:

$$R_{0,2\pi/3}: X' = A\left(\frac{2\pi}{3}\right)X \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x' = -\frac{1}{2}x - \frac{\sqrt{3}}{2}y \\ y' = \frac{\sqrt{3}}{2}x - \frac{1}{2}y \end{cases}$$

$$R_{0,2\pi/3}(A) = B = \left(-\frac{1}{2}\cdot 1 - \frac{\sqrt{3}}{2}\cdot 0; \frac{\sqrt{3}}{2}\cdot 1 - \frac{1}{2}\cdot 0\right) = \left(\frac{-1}{2}; \frac{\sqrt{3}}{2}\right)$$

$$R_{0,2\pi/3}(B) = C = \left(\frac{-1}{2}\cdot\left(\frac{-1}{2}\right) - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}; \frac{\sqrt{3}}{2}\cdot\left(\frac{-1}{2}\right) - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) =$$

$$= \left(\frac{1}{4} - \frac{3}{4}; -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) = \left(-\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$$

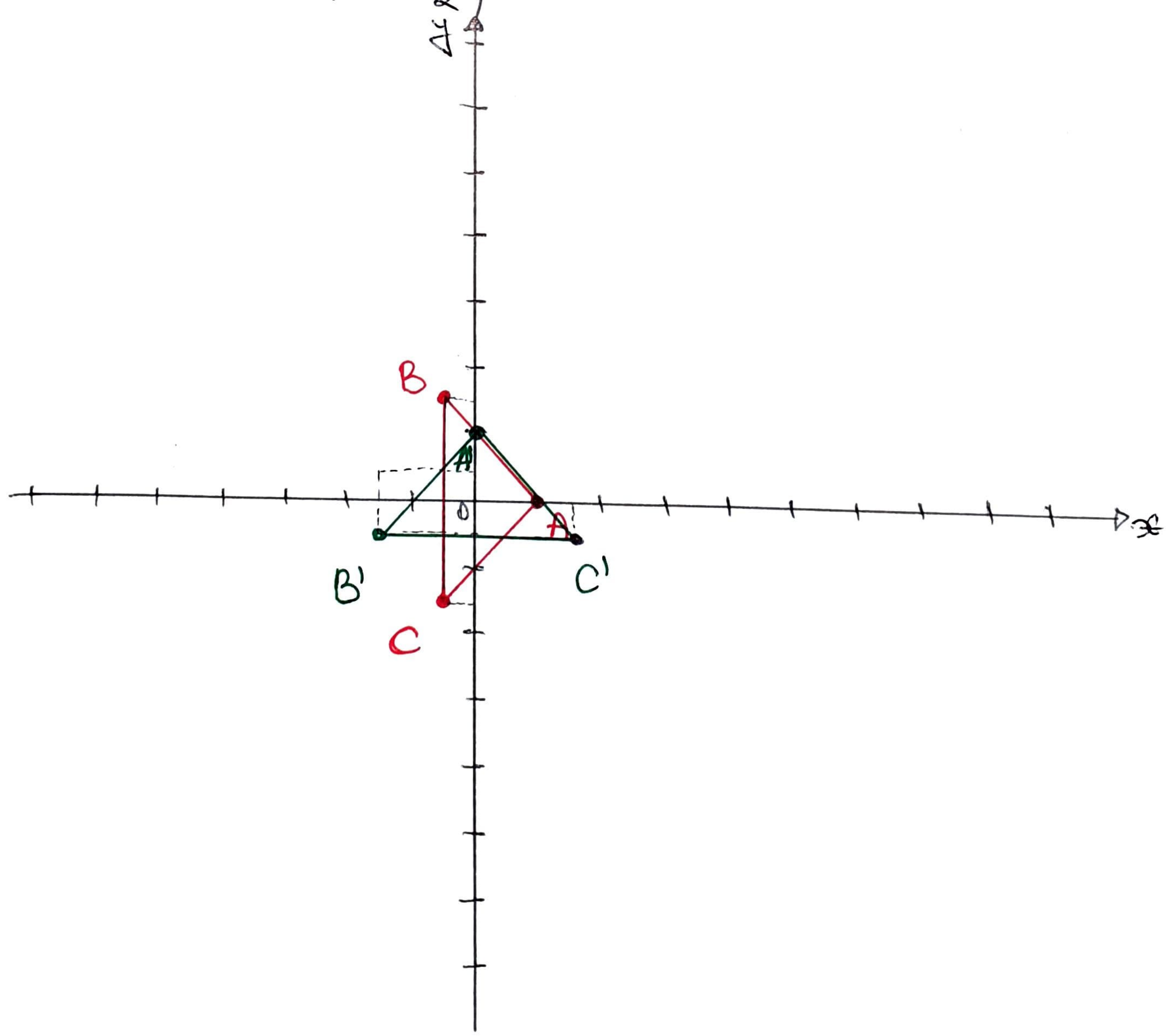
$A(1,0)\ ,\ B\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)\ ,\ C\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

$R_{O,\pi/2}(\Delta ABC) = \Delta A'B'C'$

$R_{O,\pi/2}:\ X' = A\left(\frac{\pi}{2}\right)X \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x' = -y \\ y' = x \end{cases}$

$$\begin{cases} R_{O,\pi/2}(A) = A'(0,1) \\ R_{O,\pi/2}(B) = B'\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \\ R_{O,\pi/2}(C) = C'\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \end{cases}$$



Ex 7. Fie $f: \mathcal{E}_2 \to \mathcal{E}_2$ o transformare geometrică de ecuație $X' = AX + X_0$.

$$A = \begin{pmatrix} 7/25 & -24/25 \\ 24/25 & 7/25 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

a) Arătați că f este o izometrie.

b) Precizați speța, mulțimea de puncte fixe și tipul.

SOL

a) f este izometrie dacă A este matrice ortogonală, adică $A \cdot A^T = A^T \cdot A = I_2$.

$$A^T = \begin{pmatrix} 7/25 & +24/25 \\ -24/25 & 7/25 \end{pmatrix} \Rightarrow A \cdot A^T = \begin{pmatrix} \frac{7}{25} & \frac{-24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{pmatrix} \begin{pmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{-24}{25} & \frac{7}{25} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{49}{25^2} + \frac{576}{25^2} & \frac{7 \cdot 24}{25^2} - \frac{24 \cdot 7}{25^2} \\ \frac{24 \cdot 7}{25^2} - \frac{24 \cdot 7}{25^2} & \frac{576}{25^2} + \frac{49}{25^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Deci, $A \cdot A^T = I_2 \Rightarrow f$ izometrie.

b)

$$\det A = \begin{vmatrix} \frac{7}{25} & -\frac{24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{vmatrix} = \frac{49}{625} + \frac{576}{625} = \frac{625}{625} = 1 \Rightarrow \text{speța } 1.$$

Puncte fixe: $X = AX + X_0 \Leftrightarrow (I_2 - A)X = X_0$

$$\underbrace{\begin{pmatrix} \frac{18}{25} & \frac{24}{25} \\ -\frac{24}{25} & \frac{18}{25} \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\det B = \begin{vmatrix} 18/25 & 24/25 \\ -24/25 & 18/25 \end{vmatrix} = \frac{324 + 576}{625} = \frac{900^{(25}}{625} = \frac{36}{25} \neq 0 \Rightarrow$$

$\Rightarrow$ sistemul are soluție unică $\Rightarrow$ are un singur pct. fix

$\Downarrow$

f este o rotație.

$$\Delta_x = \begin{vmatrix} 2 & \frac{24}{25} \\ 0 & \frac{18}{25} \end{vmatrix} = \frac{36}{25} \Rightarrow x = \frac{\Delta_x}{\Delta} = \frac{\cancel{36}}{\cancel{25}} \cdot \frac{\cancel{25}}{\cancel{36}} = 1$$

$$\Delta_y = \begin{vmatrix} \frac{18}{25} & 2 \\ -\frac{24}{25} & 0 \end{vmatrix} = \frac{48}{25} \Rightarrow y = \frac{\Delta_y}{\Delta} = \frac{\overset{4}{\cancel{48}}}{\cancel{25}} \cdot \frac{\cancel{25}}{\underset{3}{\cancel{36}}} = \frac{4}{3}$$

$\Rightarrow$

$\Rightarrow M\left(1, \frac{4}{3}\right)$ punct fix.