

Examen Algebră liniară, Seria 11
17.01.2022

La rezolvarea problemelor veți folosi următorii parametri:

M = numărul de litere din primul vostru nume de familie

N = numărul de litere din primul vostru prenume.

De exemplu, pentru *Ionescu Ana-Maria* avem $M = 7$ și $N = 3$.

Scrieți pe prima pagină cu rezolvări valorile parametrilor voștri:

$M = \dots$, $N = \dots$.

Timp pentru rezolvarea problemelor și încărcarea soluțiilor în MoodleUB:

3 ore

(1) (2,25 pct.) În \mathbb{R}^4 considerăm vectorii

$$u_1 = \begin{pmatrix} 1 \\ M+2 \\ 2N+1 \\ N \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ N-1 \\ M+2 \\ N-1 \end{pmatrix}, u_3 = \begin{pmatrix} 7 \\ M+2N \\ 2N+2M+5 \\ 3N-2 \end{pmatrix}, u_4 = \begin{pmatrix} -1 \\ 2M-N+3 \\ 4N-M \\ N+1 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 4 \\ 3 \\ N+1 \end{pmatrix}$$

și subspațiul vectorial $V = \langle u_1, u_2, u_3, u_4 \rangle$.

(a) Aflați $\dim V$ și o bază \mathcal{B} pentru V .

(b) Completați \mathcal{B} la o bază pentru \mathbb{R}^4 .

(c) Aflați $pr_V(w)$ proiecția ortogonală a lui w pe V și $\|pr_V(w)\|$.

(d) Este w în V ? Argumentați.

(2) (1,5 pct.) Pentru orice $n \in \mathbb{N}, n \geq 2$ notăm

$$A_n = \begin{pmatrix} 2 & N & 0 & \dots & \dots & 0 \\ N & 2 & N & 0 & \dots & 0 \\ N & N & 2 & N & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ N & N & \dots & N & 2 & N \\ N & N & \dots & N & N & 2 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}) \text{ și } d_n = \det(A_n).$$

(a) Calculați d_2, d_3, d_4 .

(b) Stabiliți dacă matricea A_3 este inversabilă, iar în caz afirmativ găsiți-i inversa.

(c) Arătați că are loc relația $d_n = 2d_{n-1} + N^2(N-2) \cdot d_{n-3}$, pentru orice $n \geq 5$.

(3) (2,75 pct.) Fie aplicația liniară $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ dată de

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x - 2y - 4z \\ y \\ 12x - 4y - 7z \end{pmatrix}, \text{ pentru orice } x, y, z \in \mathbb{R}.$$

(a) Determinați subspațiul $\text{Ker } T \cap \text{Im } T$.

Subiectele continuă pe pagina următoare...

- (b) Verificați dacă $w = \begin{pmatrix} 3 \\ M-1 \\ N+1 \end{pmatrix}$ este vector propriu pentru T .
- (c) Aflați valorile proprii ale lui T .
- (d) Găsiți o bază \mathcal{B} în \mathbb{R}^3 astfel încât matricea lui T în baza \mathcal{B} să fie diagonală. Argumentați dacă poate fi \mathcal{B} aleasă să fie ortonormată.
- (4) (1,5 pct.) Fie v_1, v_2, v_3 vectori din \mathbb{R} -spațiul vectorial V . Arătați că dacă v_1, v_2, v_3 sunt liniar independenți, atunci și $Mv_1 + v_2, Mv_2 + v_3, Mv_3 + v_1$ sunt liniar independenți.
Este reciproca adevărată? Argumentați.
- (5) (1 pct.) Fie $A \in \mathcal{M}_{m,n}(\mathbb{R})$ o matrice de rang r . Arătați că există două matrici $B \in \mathcal{M}_{m,r}(\mathbb{R}), C \in \mathcal{M}_{r,n}(\mathbb{R})$ ambele de rang r astfel încât $A = BC$.

Se acordă 1 punct din oficiu.

Justificați toate răspunsurile date, arătând calculele efectuate.

Examen algebra liniară

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$$M=5$$

$$N=4$$

$$(1) u_1 = \begin{pmatrix} 1 \\ 7 \\ 9 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ 3 \\ 7 \\ 3 \end{pmatrix}, u_3 = \begin{pmatrix} 4 \\ 13 \\ 23 \\ 10 \end{pmatrix}, u_4 = \begin{pmatrix} -1 \\ 9 \\ 11 \\ 5 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$V = \langle u_1, u_2, u_3, u_4 \rangle$$

(a) Verifică liniar independența lui u_1, u_2, u_3, u_4

$$\begin{pmatrix} 1 & 3 & 7 & -1 \\ 7 & 3 & 13 & 9 \\ 9 & 7 & 23 & 11 \\ 1 & 3 & 10 & 5 \end{pmatrix} \xrightarrow{\substack{L'_2 = L_2 - 7L_1 \\ L'_3 = L_3 - 9L_1 \\ L'_4 = L_4 - L_1}} \begin{pmatrix} 1 & 3 & 7 & -1 \\ 0 & -18 & -36 & 16 \\ 0 & -20 & -40 & 20 \\ 0 & -9 & -18 & 9 \end{pmatrix} \xrightarrow{\substack{L'_3 = -\frac{1}{20}L_3 \\ L_2 \leftrightarrow L_3}}$$

$$\begin{pmatrix} 1 & 3 & 7 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & -18 & -36 & 16 \\ 0 & -9 & -18 & 9 \end{pmatrix} \xrightarrow{\substack{L'_1 = L_1 - 3L_2 \\ L'_3 = L_3 + 18L_2 \\ L'_4 = L_4 + 9L_2}} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{L'_3 = -\frac{1}{2}L_3}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\substack{L'_1 = L_1 - 2L_3 \\ L'_2 = L_2 + L_3}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{am 3 pivoli} \Rightarrow$$

$\Rightarrow u_1, u_2, u_4$ sunt liniar independenți $\Rightarrow u_3$ e o combinație

liniară de $u_1, u_2, u_4 \Rightarrow u_3 \in \langle u_1, u_2, u_4 \rangle$

$B = \{u_1, u_2, u_4\}$ bază pt. $V \Rightarrow \dim B = 3$

(b) Verific dacă B poate fi completată cu e_1 :

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 5 \\ 7 & 3 & 9 & 0 & 4 \\ 9 & 7 & 11 & 0 & 3 \\ 4 & 3 & 5 & 0 & 5 \end{array} \right) \begin{array}{l} L'_2 = L_2 - 7L_1 \\ L'_3 = L_3 - 9L_1 \\ L'_4 = L_4 - 4L_1 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 5 \\ 0 & -18 & 16 & -7 & -31 \\ 0 & -20 & 20 & -9 & -42 \\ 0 & -9 & 9 & -4 & -15 \end{array} \right) \xrightarrow{L'_2 = -\frac{1}{18}L_2}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 5 \\ 0 & 1 & -8/9 & 7/18 & 31/18 \\ 0 & -20 & 20 & -9 & -42 \\ 0 & -9 & 9 & -4 & -15 \end{array} \right) \begin{array}{l} L'_1 = L_1 - 3L_2 \\ L'_3 = L_3 + 20L_2 \\ L'_4 = L_4 + 9L_2 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 5/3 & -1/6 & -1/6 \\ 0 & 1 & -8/9 & 7/18 & 31/18 \\ 0 & 0 & 20/9 & -11/9 & -11/9 \\ 0 & 0 & 1 & -1/2 & -1/2 \end{array} \right) \xrightarrow{L'_4 = L_4 - \frac{9}{20}L_3}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 5/3 & -1/6 & -1/6 \\ 0 & 1 & -8/9 & 7/18 & 31/18 \\ 0 & 0 & 20/9 & -11/9 & -11/9 \\ 0 & 0 & 0 & 1/20 & 1/20 \end{array} \right) \Rightarrow \text{am pivot pe fiecare coloană}$$

$\Rightarrow u_1, u_2, u_3, e_1$ sunt liniar independenți

$\Rightarrow B = \{u_1, u_2, u_3, e_1\}$ bază pt. \mathbb{R}^4

(d) $w \in V \Leftrightarrow \exists a, b, c, d \in \mathbb{R} \text{ a. d. } w = au_1 + bu_2 + cu_3 + du_4$

$$\left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 7 & 3 & 13 & 9 & 4 \\ 9 & 7 & 23 & 11 & 3 \\ 4 & 3 & 10 & 5 & 5 \end{array} \right) \begin{array}{l} L'_2 = L_2 - 7L_1 \\ L'_3 = L_3 - 9L_1 \\ L'_4 = L_4 - 4L_1 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 0 & -18 & -36 & 16 & -31 \\ 0 & -20 & -40 & 20 & -42 \\ 0 & -9 & -18 & 9 & -15 \end{array} \right) \begin{array}{l} L'_3 = -\frac{1}{20}L_3 \\ L_3 \leftrightarrow L_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 7 & -1 & 5 \\ 0 & 1 & 2 & -1 & 21/10 \\ 0 & -18 & -36 & 16 & -31 \\ 0 & -9 & -18 & 9 & -15 \end{array} \right) \begin{array}{l} L'_1 = L_1 - 3L_2 \\ L'_3 = L_3 + 18L_2 \\ L'_4 = L_4 + 9L_2 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -4/5 \\ 0 & 1 & 2 & -1 & 21/10 \\ 0 & 0 & 0 & -2 & 34/5 \\ 0 & 0 & 0 & 0 & -15 + \frac{9 \cdot 21}{10} \end{array} \right)$$

ultima ecuație devine $0 \cdot a + 0 \cdot b + 0 \cdot c + 0 \cdot d = -15 + \frac{9 \cdot 21}{10}$ Fals

$\Rightarrow \nexists a, b, c, d \in \mathbb{R} \text{ a. d. } w = au_1 + bu_2 + cu_3 + du_4 \Leftrightarrow$

$\Leftrightarrow w \notin V$

(2)

$$(2) A_n = \begin{pmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{pmatrix}$$

$$\det(A_n) = d_n$$

$$(a) d_2 = \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = -12$$

$$d_3 = \begin{vmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{vmatrix} = 8 + 64 - 32 - 32 = 8$$

$$d_4 = \begin{vmatrix} 2 & 4 & 0 & 0 \\ 4 & 2 & 4 & 0 \\ 4 & 4 & 2 & 4 \\ 4 & 4 & 4 & 2 \end{vmatrix} = 2 \cdot (-1)^{1+1} \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + 4 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 4 \\ 4 & 4 \end{vmatrix} \\ = 2 \cdot d_2 - 4 \cdot (16 + 64 - 64 - 32)$$

$$= 2 \cdot 16 + 4 \cdot 16$$

$$= 5 \cdot 16 = 80$$

$$(b) \left(\begin{array}{cccc|cccc} 2 & 4 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & 2 & 0 & 0 & 0 & \dots & 1 \end{array} \right) \xrightarrow{\substack{L_1' = \frac{1}{2}L_1 \\ L_2' = L_2 - 4L_1 \\ L_3' = L_3 - 4L_1 \\ \vdots \\ L_n' = L_n - 4L_1}} \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & \dots & 0 & 1/2 & 0 & 0 & \dots & 0 \\ 0 & -6 & 4 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & -4 & -4 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -4 & 4 & \dots & 4 & -2 & 0 & 0 & \dots & 2 \end{array} \right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & \dots & 0 & 1/2 & 0 & 0 & \dots & 0 \\ 0 & -6 & 4 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & -4 & -4 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -4 & 4 & \dots & 4 & -2 & 0 & 0 & \dots & 2 \end{array} \right)$$

$$\xrightarrow{\substack{L_2' = L_2 - 4L_1 \\ \vdots \\ L_n' = L_n - 4L_1}}$$

$$(c) d_n = \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{\substack{\text{det } 0 \\ n \text{ coloane}}} = 2 \cdot (-1)^{1+1} \cdot \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} +$$

$$+ 4 \cdot (-1)^{1+2} \cdot \underbrace{\begin{vmatrix} 4 & 4 & 0 & 0 & \dots & 0 \\ 4 & 2 & 4 & 0 & \dots & 0 \\ 4 & 4 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} = C_1 = C_1 - C_2$$

$$= 2 \cdot d_{n-1} - 4 \cdot \underbrace{\begin{vmatrix} 0 & 4 & 0 & \dots & 0 \\ 2 & 2 & 4 & \dots & 0 \\ 0 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-1 \text{ coloane}} = 2d_{n-1} - 4 \cdot 4 \cdot (-1)^{1+2}$$

$$\underbrace{\begin{vmatrix} 2 & 4 & 0 & 0 & \dots & 0 \\ 0 & 2 & 4 & 0 & \dots & 0 \\ 0 & 4 & 2 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-2 \text{ coloane}} = 2d_{n-1} + 16 \cdot 2 \cdot (-1)^{1+1} \cdot \underbrace{\begin{vmatrix} 2 & 4 & 0 & \dots & 0 \\ 4 & 2 & 4 & \dots & 0 \\ 4 & 4 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & 2 \end{vmatrix}}_{n-3 \text{ coloane}} =$$

$$= 2d_{n-1} + 32d_{n-3} \Leftrightarrow d_n = 2d_{n-1} + 4^2(4-2)d_{n-3}, \quad \forall n \geq 5$$

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$$(2)(b) \quad A_3 = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{pmatrix}$$

$\det A_3 = d_3 = 8 \neq 0 \Rightarrow A_3$ - inversabilă

$$A_3 = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \Rightarrow A_3^* = \begin{pmatrix} -12 & -8 & 16 \\ 8 & 4 & -8 \\ 8 & 8 & -12 \end{pmatrix}$$

$$A_3^{-1} = \frac{1}{\det A_3} \cdot A_3^* = \begin{pmatrix} -3/2 & -1 & 2 \\ 1 & 1/2 & -1 \\ 1 & 1 & -3/2 \end{pmatrix}$$

$$(3) T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x - 2y - 4z \\ 7 \\ 12x - 4y - 7z \end{pmatrix} = \begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(a) \text{Ker } T = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \mid T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\Leftrightarrow \begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{L_1 \leftrightarrow L_3} \begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix} \xrightarrow{L_1 = \frac{1}{7} L_1} \begin{pmatrix} 1 & -2/7 & -4/7 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix}$$

$$\xrightarrow{L_3 = L_3 - 12L_1} \begin{pmatrix} 1 & -2/7 & -4/7 \\ 0 & 1 & 0 \\ 0 & -4/7 & -1/7 \end{pmatrix} \xrightarrow{\begin{matrix} L_1 = L_1 + \frac{2}{7} L_2 \\ L_3 = L_3 + \frac{4}{7} L_2 \end{matrix}}$$

$$\begin{pmatrix} 1 & 0 & -4/7 \\ 0 & 1 & 0 \\ 0 & 0 & -1/7 \end{pmatrix} \xrightarrow{L_3 = -7L_3} \begin{pmatrix} 1 & 0 & -4/7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{L_1 = L_1 + \frac{4}{7} L_3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow a = b = c = 0 \Rightarrow \text{Ker } T = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \dim(\text{Ker}(T)) = 0$$

$\Rightarrow T$ -injective

$$\dim T \text{ rang - defect} \Rightarrow \dim \mathbb{R}^3 = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$$

$$\Rightarrow \dim(\text{Im}(T)) = 3 \Rightarrow T \text{ surjective} \Rightarrow \text{Im } T = \mathbb{R}^3$$

$$\text{Im } T = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \mid \exists \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ a. v. } T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow x \cdot C_1(A) + y \cdot C_2(A) + z \cdot C_3(A) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \langle C_1(A), C_2(A), C_3(A) \rangle$$

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$\dim \text{Ker } T$ am arătat că $C_1(A), C_2(A), C_3(A)$ sunt
linii independente: $\Rightarrow \langle C_1(A), C_2(A), C_3(A) \rangle$ bază pt. $\text{Im } T$
 $\Rightarrow \dim(\text{Im } T) = 3$; $\text{Ker } T = \{0_{\mathbb{R}^3}\}$; $\text{Im } T = \mathbb{R}^3$

$$\text{Ker } T \cap \text{Im } T = \{x \in \text{Ker } T \mid x \in \text{Im } T\}$$

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x = a \cdot C_1(A) + b \cdot C_2(A) + c \cdot C_3(A) \mid \Rightarrow a = b = c = 0$$

$$\Rightarrow \text{Ker } T \cap \text{Im } T = \text{Ker } T = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

~~$$(b) w = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$~~

$$(c) P_A(x) = \det(x \cdot I_3 - A)$$

$$\det \left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 7 & -2 & -4 \\ 0 & 1 & 0 \\ 12 & -4 & -7 \end{pmatrix} \right) = \det \begin{pmatrix} x-7 & 2 & 4 \\ 0 & x-1 & 0 \\ -12 & 4 & x+7 \end{pmatrix} =$$

$$= \begin{vmatrix} x-7 & 2 & 4 \\ 0 & x-1 & 0 \\ -12 & 4 & x+7 \end{vmatrix} = (x-1)(x^2-49) + 48(x-1) =$$

$$= (x-1)(x^2-49+48)$$

$$= (x-1)(x^2-1)$$

$$= (x-1)^2(x+1)$$

\Rightarrow valabile propri sunt $\lambda_1 = 1$, $\lambda_2 = -1$

$$(b) \lambda_1 = 1 \Rightarrow (A - 1 \cdot I_3) v_1 = 0_{\mathbb{R}^3}$$

$$\begin{pmatrix} 6 & -2 & -4 \\ 0 & 0 & 0 \\ 12 & -4 & -8 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & -4 \\ 12 & -4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow \frac{1}{6} L_1} \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} \\ 12 & -4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_2 \leftarrow L_2 - 12 L_1}$$

(7)

$$\begin{pmatrix} 1 & -1/3 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 - \frac{1}{3}y_1 - \frac{2}{3}z_1 = 0$$

x_1 - nec. principală
 y_1, z_1 nec. secundare; $y_1 = s, z_1 = t, s, t \in \mathbb{R}$

$$\Rightarrow x_1 = \frac{1}{3}s + \frac{2}{3}t$$

$$x_1 = \frac{1}{3}s + \frac{2}{3}t$$

$$v_1 = \left\{ s \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2/3 \\ 0 \\ 1 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$$

$$x_2 = -1 \Rightarrow (A + iI_3)v_2 = 0_{\mathbb{R}^3}; v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -2 & -4 \\ 0 & 2 & 0 \\ 12 & -4 & -6 \end{pmatrix} \xrightarrow{L'_1 \leftrightarrow L_1} \begin{pmatrix} 1 & -1/4 & -1/2 \\ 0 & 2 & 0 \\ 12 & -4 & -6 \end{pmatrix} \xrightarrow{L'_3 \leftrightarrow L_3 - 12L_1}$$

$$\begin{pmatrix} 1 & -1/4 & -1/2 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{L'_2 \leftrightarrow L_2} \begin{pmatrix} 1 & -1/4 & -1/2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{\begin{matrix} L'_1 \leftrightarrow L_1 + \frac{1}{4}L_2 \\ L'_3 \leftrightarrow L_3 + L_2 \end{matrix}}$$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow y_2 = 0 \Rightarrow x_2 - \frac{1}{2}z_2 = 0$$

$$z_2 = \alpha, \alpha \in \mathbb{R} \Rightarrow x_2 = \frac{1}{2}\alpha$$

$$v_2 = \left\{ \alpha \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$$

$$w = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \notin v_2$$

$$\text{pt. } s=4, t=5 \Rightarrow v_1 = \begin{pmatrix} 14/3 \\ 4 \\ 5 \end{pmatrix} \Rightarrow w \notin v_1 \Rightarrow$$

w nu este vector propriu
 pt. T

(4) v_1, v_2, v_3 - linear independenti \Leftrightarrow

$$\Leftrightarrow \nexists a, b, c \in \mathbb{R} \text{ a.v. } av_1 + bv_2 + cv_3 = 0_V$$

$$5v_1 + v_2, 5v_2 + v_3, 5v_3 + v_1 - \text{lin. indep.} \Leftrightarrow$$

$$\Leftrightarrow \nexists x, y, z \in \mathbb{R} \text{ a.v.}$$

$$5xv_1 + xv_2 + 5yv_2 + yv_3 + 5zv_3 + zv_1 = 0_V$$

$$v_1(5x+z) + v_2(x+5y) + v_3(y+5z) = 0_V$$

$$\begin{cases} 5x + 0 + z = 0 \\ x + 5y + 0 = 0 \\ 0 + y + 5z = 0 \end{cases}$$

$$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} 1 & 5 & 0 \\ 5 & 0 & 1 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{L_2' = L_2 - 5L_1} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -25 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\xrightarrow{L_2 \leftrightarrow L_3} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & -25 & 1 \end{pmatrix} \xrightarrow{\begin{matrix} L_1' = L_1 - 5L_2 \\ L_3' = L_3 + 25L_2 \end{matrix}} \begin{pmatrix} 1 & 0 & -25 \\ 0 & 1 & 5 \\ 0 & 0 & 126 \end{pmatrix} \xrightarrow{L_3' \cdot \frac{1}{126} L_3}$$

$$\begin{pmatrix} 1 & 0 & -25 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{matrix} L_1' = L_1 + 25L_3 \\ L_2' = L_2 - 5L_3 \end{matrix}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow x = y = z = 0$$

$$\Rightarrow \nexists x, y, z \in \mathbb{R} \text{ a.v. } v_1(5x+z) + v_2(x+5y) + v_3(y+5z) = 0_V$$