S_1/ 1. Cf. Th. lui Cayley, există o scufundare a grupului S_3 în grupul S_6. Găsiți imaginea transpoziției $(1,2) \in S_3$ prin această scufundare.

Sol.: S_3 are $3!$ elemente (în general S_n are $n!$ elem.)

$1 = id$ $\quad$ $4 = (2,3)$

$2 = (1,2)$ $\quad$ $5 = (1,2,3)$

$3 = (1,3)$ $\quad$ $6 = (1,3,2)$

1	2	3	4	5	6
1	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
(1,2)	id	(1,3,2)	(1,2,3)	(2,3)	(1,3)
2	1	6	5	4	3

$$(1,2)(1,2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = id$$

$$(1,2)(1,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1,3,2)$$

$$(1,2)(2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1,2,3)$$

$$(1,2)(1,2,3) = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$$

$$= \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \cancel{(3,2)}\,(2,3)$$

$$\begin{pmatrix}1&2&3&4&5&6\\2&1&6&5&4&3\end{pmatrix} \in S_6$$

$$(1,2) \in S_3$$

2. Demonstrați identitatea: $(k,k+1) = (1,k,\dots,n)^{k-1}(1,2)(1,2,\dots,n)^{-(k-1)}$

Sol.: Not. $\sigma = (1,2,\dots,n)$

ip. se rescrie: $(k,k+1) = \sigma^{k-1}\cdot(1,2)\sigma^{-(k-1)}$

echiv: $(k \leftarrow k+1)$: $(k+1,k+2) = \sigma^{k}(1,2)\sigma^{-k}$

Inducție: $k=0$: $(1,2)=(1,2)$

$k=1$: $\sigma(1,2)\sigma^{-1} = (2,3)$

$n=3$

$$\sigma = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix}2&3&1\\1&2&3\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$$

$$\sigma(1,2)\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2,3)$$

$$\sigma(2,3)\sigma^{-1} = (3,4)$$

$$\vdots$$

$$\sigma(k,k+1)\sigma^{-1} = (k+1,k+2)$$

$k \geq 2$:
$$\begin{aligned}\sigma^{k}(1,2)\sigma^{-k} &= \sigma^{k-1}\left(\sigma(1,2)\sigma^{-1}\right)\sigma^{-(k-1)} \\ &= \sigma^{k-1}(2,3)\sigma^{-(k-1)} \\ &= \sigma^{k-2}\left(\sigma(2,3)\sigma^{-1}\right)\sigma^{-(k-2)} \\ &= \sigma^{k-2}(3,4)\sigma^{-(k-2)} \\ &\vdots \\ &= \sigma(k,k+1)\sigma^{-1} \\ &= (k+1,k+2)\end{aligned}$$

3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix} = \sigma = (1,3,6,4)(2,5)$

Scrieți σ în produs de cicli disjuncți.

Obs. Orice permutare poate fi descompusă în produs de cicli disjuncți.

4. Arătați că orice permutare poate fi descompusă într-un produs de transpoziții folosind următorul ex.:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix} \qquad (i, \sigma(i)) \quad i \neq \sigma(i)$$

Dem.: $(1,3)\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 3 & 2 & 4 \end{pmatrix} = \sigma_1$

$(2,5)\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 6 & 3 & 5 & 4 \end{pmatrix} = \sigma_2$

$(3,6)\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix} = \sigma_3$

$(4,6)\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = id$

$id = (4,6)(3,6)(2,5)(1,3)\sigma$

$\Rightarrow \sigma = (1,3)(2,5)(3,6)(4,6)$

5. Calc. elem. generate de perm. ciclică $\underset{\sigma}{\overset{\shortparallel}{(1,2,3,4,5,6)}}$

Sol.: $(1,2,3,4,5,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$

$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix} = (1,3,5)(2,4,6)$

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} = (1,4)(2,5)(3,6)$$

$$\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} = (1,5,3)(2,6,4)$$

$$\sigma^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (1,6,5,4,3,2)$$

$$\sigma^6 = id$$

6. Arătați că toți ciclii generați de $(1,2,3,4,5)$ au lungimea 5.

$\sigma = (1,2,3,4,5)$

$\sigma^2 = (1,3,5,2,4)$

$\sigma^3 = (1,4,2,5,3)$

$\sigma^4 = (1,5,4,3,2)$

Obs.: Toate permutările de lungime p (p prim) se scriu ca cicli de lungime p.

7. Fie $\sigma \in S_n$ și $(a_1, \dots, a_k)$ un ciclu. Arătați că $\sigma(a_1, \dots, a_k)\sigma^{-1} = (\sigma(a_1), \dots, \sigma(a_k))$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$$

Sol.: Considerăm $x \neq \overline{a_1, a_k}$

$$\sigma = \begin{pmatrix} a_1 & a_2 & \dots & a_k & x \\ \sigma(a_1) & \sigma(a_2) & \dots & \sigma(a_k) & \sigma(x) \end{pmatrix}$$

$$\sigma(a_1, \dots, a_k) \begin{pmatrix} \sigma(a_1) & \sigma(a_2) & \dots & \sigma(a_k) & \sigma(x) \\ a_1 & a_2 & \dots & a_k & x \end{pmatrix} =$$

$$= \sigma \begin{pmatrix} \sigma(a_1) & \sigma(a_2) & \dots & \sigma(a_k) & \sigma(x) \\ a_2 & a_3 & \dots & a_1 & x \end{pmatrix}$$

$$= \begin{pmatrix} \sigma(a_1) & \sigma(a_2) & \dots & \sigma(a_k) & \sigma(x) \\ \sigma(a_2) & \sigma(a_3) & \dots & \sigma(a_1) & \sigma(x) \end{pmatrix}$$

$$= (\sigma(a_1), \sigma(a_2), \sigma(a_3) \dots, \sigma(a_k))$$

8. Descrieți grupul unităților lui $\mathbb{Z}_{12}$.

$U_{12} = \{1, 5, 7, 11\}$

$\mathbb{Z}_p$, p prim $U_p = \{1, 2, 3, \dots, p-1\}$

9. Scrieți elementele mulțimii generate de 7 $(\langle 7 \rangle)$ în U_{18}.

$U_{18} = \{1, 5, 7, 11, 13, 17\}$

$7^0 \overset{18}{\equiv} 1$ $\quad$ $7^3 \overset{18}{\equiv} 1$

$7^1 \overset{18}{\equiv} 7$

$7^2 \overset{18}{\equiv} 13$ $\quad \Rightarrow \langle 7 \rangle = \{1, 7, 13\}$