C06 – SMT Solvers

Program Verification

FMI · Denisa Diaconescu · Spring 2025

Overview

What are SMT solvers?

First-order logic (101)

Some first-order theories

How SMT solvers works?

What are SMT solvers?

The SMT problem

The SMT problem:

Given a first-order logic formula, with symbols from (possibly several) theories, does it have a model?

Is the formula satisfiable? If so, how?

The SAT problem is a special case, in which

- the formula is quantifier-free, without function symbols or equality
- no theories are used

SAT solving algorithms are an important ingredient in SMT solvers

First-order theories

- Whereas the language of SAT solvers is Boolean logic, the language of SMT solvers is first-order logic.
- First-order theories allow us to capture structures which are used by programs (e.g., arrays, integers) and enable reasoning about them.
- Validity in first order logic (FOL) is undecidable!
  - Lambda calculus – Alonzo Church (1936)
  - Turing machines – Alan Turing (1937)
  - Recursive functions – Kurt Gödel (1934) and Stephen Kleene (1936)
- Validity in particular first-order theories is (sometimes) decidable.

SMT solvers

Combine propositional satisfiability search techniques with solvers for specific first-order theories:

- Linear arithmetic
- Bit vectors
- Arrays
- ...

Applications of SMT solvers

SMT solvers are used as core engines in many tools in

- program analysis and verification
- software engineering
- hardware verification
- ...
- symbolic execution
- concolic execution

SMT solvers

SMT solvers:

- Z3 (Microsoft)
- Yices
- MathSAT
- CVC4
- ...

First-order logic (101)

Language

- The language includes the Boolean operations of propositional logic, but instead of propositional variables, more complicated expressions are allowed.
- A first-order language must specify its signature: the set of constant, function, and predicate symbols that are allowed.
- Each predicate and function symbol has an associated arity: a natural number indicating how many arguments it takes.
    - Equality is a special predicate symbol of arity 2
    - Constant symbols can also be thought of as functions whose arity is 0

Language

Example (Propositional logic)

- Equality: no
- Predicate symbols: $x_1, x_2, \ldots$
- Constant symbols: none
- Function symbols: none

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Example (Elementary Number Theory)

- Equality: yes
- Predicate symbols: $<$
- Constant symbols: 0
- Function symbols: S (successor), $+$, $*$

Syntax

- Terms
  - Variables and constants are terms
  - For each function symbol f of arity n, and terms $t_1, \ldots, t_n$, $f(t_1, \ldots, t_n)$ is a term

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  - $P(t_1, \ldots, t_n)$ where P is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms.
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  - $t = t'$ if the logic is with equality where t, t' are terms.
- An atomic formula or its negation is called a literal.
- Formulas are built from literals using the Boolean operators and quantification. If α is a formula, then for every variables x
  - $\forall x.\ \alpha$ is a formula
  - $\exists x.\ \alpha$ is a formula

Semantics

Given a signature Σ and a set V of variables, a structure $\mathcal{M}$ of Σ consists of:

1. A nonempty set M called the domain of $\mathcal{M}$
2. For each constant c in Σ, an element $c^{\mathcal{M}} \in M$.
3. For each n-ary function symbol f in Σ, an n-ary function $f^{\mathcal{M}} : M^n \to M$.
4. For each n-ary predicate symbol P in Σ, an n-ary relation $P^{\mathcal{M}} \subseteq M^n$.

An interpretation I of the variables in V into a structure $\mathcal{M}$ maps each variable $v \in V$ to an element $I(v) \in M$.

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Is this sentence true in the structure?

Since 0 has this property, the sentence is true in this structure.

Term interpretation

Given

- a structure $\mathcal{M}$ of Σ, and
- an interpretation I of V into $\mathcal{M}$,

we can inductively extend the interpretation I to all terms over Σ with variables from V, as follows:

- $\bar{I}(v) = I(v)$ for any $v \in V$
- $\bar{I}(c) = c^{\mathcal{M}}$ for any constant symbol c in Σ
- $\bar{I}(f(t_1, \ldots, t_n)) = f^{\mathcal{M}}(\bar{I}(t_1), \ldots, \bar{I}(t_n))$, for any n-ary function symbol $f \in \Sigma$ and any terms $(t_i)_{1 \leq i \leq n}$

Satisfaction

Given

- a structure $\mathcal{M}$ of Σ, and
- an interpretation I of V into $\mathcal{M}$,

we can inductively define a satisfaction relation for Σ-formulas with variables from V, as follows:

- $\mathcal{M}, I \models t_1 = t_2$ iff $\bar{I}(t_1) = \bar{I}(t_2)$
- $\mathcal{M}, I \models P(t_1, \ldots, t_n)$ iff $(\bar{I}(t_1), \ldots, \bar{I}(t_n)) \in P^{\mathcal{M}}$
- $\mathcal{M}, I \models \phi_1 \wedge \phi_2$ iff $\mathcal{M}, I \models \phi_1$ and $\mathcal{M}, I \models \phi_2$
- $\mathcal{M}, I \models \neg\phi$ iff $\mathcal{M}, I \not\models \phi$
- $\mathcal{M}, I \models \exists x.\phi$ iff there exists $a \in M$ such that $\mathcal{M}, I[x \leftarrow a] \models \phi$

Satisfiability and validity in a structure

Given

- a structure $\mathcal{M}$ of Σ, and
- a Σ formula ϕ with variables from V

we say that

- ϕ is satisfiable in $\mathcal{M}$ iff $\mathcal{M}, I \models \phi$ for some interpretation I
- ϕ is valid in $\mathcal{M}$ iff $\mathcal{M}, I \models \phi$ for any interpretation I

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Note that ϕ is valid iff $\neg\phi$ is not satisfiable. Indeed,

- ϕ is valid iff $\mathcal{M}, I \models \phi$ for any I
- iff there exists no I such that $\mathcal{M}, I \not\models \phi$
- iff there exists no I such that $\mathcal{M}, I \models \neg\phi$
- iff $\neg\phi$ is not satisfiable

Satisfiability, validity, and free variables

Given a formula ϕ and the set X of its free variables (i.e., not bounded by quantifiers)

- If I_1, I_2 interpretations such that $I_1|_X = I_2|_X$
- Then $\mathcal{M}, I_1 \models \phi$ iff $\mathcal{M}, I_2 \models \phi$

Hence, if a formula ϕ does not have free variables

- The satisfaction relation does not depend on interpretations
  We can therefore write $\mathcal{M} \models \phi$
- satisfiability and validity coincide for ϕ

Given a formula ϕ and the set $\{x_1, \ldots, x_n\}$ of its free variables,

- ϕ satisfiable in $\mathcal{M}$ iff $\mathcal{M} \models \exists x_1. \cdots \exists x_n.\phi$
- ϕ valid in $\mathcal{M}$ iff $\mathcal{M} \models \forall x_1. \cdots \forall x_n.\phi$

Satisfiability and validity in a theory

A theory is a set of sentences (no free variables).

Given a theory T, a formula φ is

- T-valid if φ is satisfied by all models of T for all interpretations of V.
- T-satisfiable if it is satisfied by some model of T for some interpretations of V.

Drinker's paradox

Example
$\exists x(D(x) \rightarrow \forall y D(y))$

There is someone in the pub such that
if he/she is drinking, then everyone in the pub is drinking.

How is this formula?

1. valid
2. unsatisfiable
3. satisfiable but not valid

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If everybody drinks, then anyone can be the witness for the validity of the formula.

If someone does not drink, then that particular non-drinking individual can be the witness for the validity of the formula.

SMT problem

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$$\boxed{\varphi \text{ is } T\text{-valid} \quad \text{iff} \quad \neg\varphi \text{ is } T\text{-unsatisfiable.}}$$

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SMT = satisfiability modulo theories

Satisfiability modulo theories

It is important to make a distinction between SMT and standard first-order satisfiability.

For example, is the following sentence satisfiable?

$$read(write(a, i, v), i) \neq v$$

Satisfiability modulo theories

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For example, is the following sentence satisfiable?

$$\mathit{read}(\mathit{write}(a, i, v), i) \neq v$$

If the set of allowable models is unrestricted, then the answer is yes.

However, if we only consider models that obey the axioms for *read* and *write* on arrays, then the answer is no.

Quiz time!

https://tinyurl.com/FMI-PV2023-Quiz7

Some first-order theories

First-order theories

A first-order theory T is defined by

- Signature Σ_T = set of constant, function, and predicate symbols
  - Have no meaning
- Axioms A_T = set of Σ_T-sentences (no free variables)
  - Provide meaning for the symbols of Σ_T

A fragment of a theory T is a syntactically restricted subset of formulas of the theory.

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Example
The quantifier-free fragment of a theory T (denoted $QFF\ T$) is the set of formulas without quantifiers that are valid in T.

A theory is decidable if for every formula in the theory we can automatically check whether the formula is valid or not.

Similarly for fragments of a theory.

First-order theories

- In principle, SMT can be applied to any theory T.
- In practice, when people talk about SMT, they are usually referring to a small set of specific theories.
- We will consider a few examples of theories which are of particular interest in program analysis and verification applications.

First-order theories

Theory	Description	Full Fragment	No Quantifiers
T_E	Equality	NO	YES
T_{PA}	Peano arithmetic	NO	NO
T_N	Presburger arithmetic	YES	YES
T_Z	Linear Integers	YES	YES
T_R	Reals (with *)	YES	YES
T_Q	Rationals (without *)	YES	YES
T_{RDS}	Recursive Data Structures	NO	YES
T_{RDS}^{+}	Acyclic Recursive Data Structures	YES	YES
T_A	Arrays	NO	YES
$T_A^{=}$	Arrays with extensionality	NO	YES

source: The Calculus of Computation, Manna and Bradley

Theory of Equality

Signature Σ_{E}:

- Any function, predicate, and constant
- The predicate $=$ which is interpreted (i.e., defined via axioms)

Axioms A_{E}:

- $\forall x.\ x = x$
- $\forall x, y.\ x = y \rightarrow y = x$
- $\forall x, y, z.\ x = y \land y = z \rightarrow x = z$
- $\forall x_1, \ldots, \forall x_n, y_1, \ldots, y_n.\ x_1 = y_1 \land \ldots \land x_n = y_n \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$

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T_E is undecidable

$QFF\ \ T_E$ is decidable

Example

$(a = b) \wedge (b = c) \rightarrow (g(f(a), b) = g(f(c), a))$

Exercise: Is this valid? Why?

Arithmetic: Natural numbers and Integers

Natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$

Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Three theories:

- Peano arithmetic T_{PA}
  - Natural numbers with addition ($+$), multiplication ($*$), equality ($=$)
- Presburger arithmetic T_{N}
  - Natural numbers with addition ($+$), equality ($=$)
- Theory of integers T_{Z}
  - Integers with addition ($+$), subtraction ($-$), comparison ($>$), equality ($=$), multiplication by constants

Theory of Peano Arithmetic

Signature Σ_E:

- Constants: $0, 1$
- Binary functions: $+, *$
- Predicate: $=$

Axioms A_{PA}:

- $\forall x.\ \neg(x + 1 = 0)$
- $\forall x.\ x + 0 = x$
- $\forall x.\ x * 0 = 0$
- $\forall x, y.\ x + 1 = y + 1 \to x = y$
- $\forall x, y.\ x + (y + 1) = (x + y) + 1$
- $\forall x, y.\ x * (y + 1) = (x * y) + x$
- For each formula $\varphi(x, \bar{y})$ in the language Σ_E,
  $\forall \bar{y}.\ \varphi(0, \bar{y}) \wedge (\forall x.\ \varphi(x, \bar{y}) \to \varphi(x + 1, \bar{y})) \to \forall x.\ \varphi(x, \bar{y})$ (induction)

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T_{PA} is undecidable

QFF T_{PA} is undecidable

Example
Is the formula

$3 * x + 2 = 2 * y$ in T_{PA}?

Theory of Peano Arithmetic

Example
Is the formula

$3 * x + 2 = 2 * y$ in T_{PA}?

Yes! It can be written as

$(1 + 1 + 1) * x + 1 + 1 = (1 + 1) * y$

Theory of Presburger Arithmetic

Signature Σ_{N}:

- Constants: $0, 1$
- Binary functions: $+$ NO multiplication!
- Predicate: $=$

Axioms A_{N}:

- $\forall x.\ \neg(x + 1 = 0)$
- $\forall x.\ x + 0 = x$
- $\forall x, y.\ x + 1 = y + 1 \rightarrow x = y$
- $\forall x, y.\ x + (y + 1) = (x + y) + 1$
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T_{N} is decidable
QFF T_{N} is decidable

Theory of Integers

Signature $\Sigma_{\mathbb{Z}}$:

- Constants: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
- Unary function: $\ldots, -3*, -2*, 2*, 3*, \ldots$
  (intended meaning $2 * x$ is $x + x$, $-3 * x$ is $-x - x - x$)
- Binary functions: $+$ and $-$
- Predicate: $=$ and $>$

$T_{\mathbb{Z}}$ is decidable

QFF $T_{\mathbb{Z}}$ is decidable

Theory of Arrays

Signature Σ_A:

- Functions:
  - *read*$(\_, \_)$
    - written for simplicity as $\_[\_]$
    - e.g. $a[i]$
  - *write*$(\_, \_, \_)$
    - e.g. $\mathit{write}(a, v, i)$ denotes the array a' where $a'[v] = i$ and all other entries are the same as a
- Predicate: $=$

Axioms A_A:

- Same as A_E
- $\forall a, i, j.\ i = j \rightarrow a[i] = a[j]$
- $\forall a, v, i, j.\ i = j \rightarrow \mathit{write}(a, i, v)[j] = v$
- $\forall a, v, i, j.\ i \neq j \rightarrow \mathit{write}(a, i, v)[j] = a[j]$

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How SMT solvers works?

Satisfiability modulo theories

There are two main approaches for SMT solvers:

- The eager approach
  - Tries to find ways of encoding an entire SMT problem into SAT.
  - There are a variety of techniques
  - For some theories, this works quite well.

Satisfiability modulo theories

There are two main approaches for SMT solvers:

- The eager approach
  - Tries to find ways of encoding an entire SMT problem into SAT.
  - There are a variety of techniques
  - For some theories, this works quite well.
- The lazy approach
  - Tries to combine SAT and theory reasoning.
  - The basis for most modern SMT solvers.

SMT: The Big Questions

1. How to solve conjunctions of literals in a theory?
    - Use a Theory solver
2. How to combine a theory solver and a SAT solver to reason about arbitrary formulas?
    - The DPLL(T) framework
3. How to combine theory solvers for several theories?
    - The Nelson-Oppen method and its variants

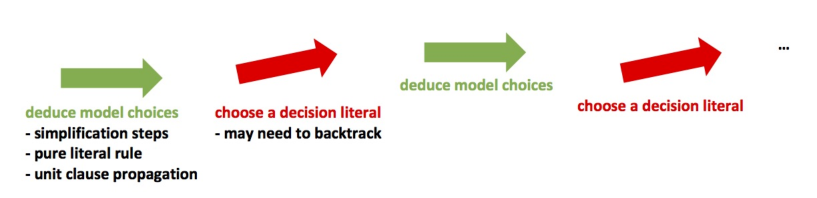
Theory solver

- Given a theory T, a Theory solver for T takes as input a set (interpreted as an implicit conjunction) φ of literals and determines whether φ is T-satisfiable.
  - φ is T-satisfiable if there is some model $\mathcal{M}$ of T such that φ holds in $\mathcal{M}$.
- In order to integrate a Theory solver into a modern SMT solver, it is helpful if the Theory solver can do more than just check satisfiability.

Propositional Abstraction

- An atom is a formula without propositional connectives or quantifiers
  - depending on the signature $f(a) = b, m * n \leq 42$ could be atoms; 42 is not
  - a propositional atom is an uninterpreted constant symbol of sort Bool
- A (first-order) literal is an atom or its negation
- For a given signature Σ, we define a signature Σ^p containing only:
  - the propositional Σ-atoms
  - a fresh propositional atom for each non-propositional Σ-atom
- We then fix an injective mapping from the non-propositional Σ-atoms to the Σ^p-atoms.
- For a Σ-formula φ, the formula φ^p is the propositional abstraction of φ, given by replacing all non-propositional Σ-atoms in φ with their image under this mapping.
- An Σ-formula φ is propositional unsatisfiable if $\varphi^p \models \bot$.
- An Σ-formula φ propositionally entails an Σ-formula ψ if $\varphi^p \models \psi^p$.
  - Note that $\varphi^p \models \psi^p$ implies $\varphi \models \psi$, but not necessarily vice-versa.

Recall DPLL/CDCL Algorithms

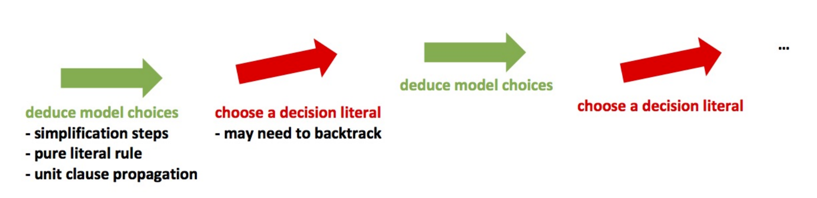


...until...

- conflict reached
  - backtrack - try flipping a decision literal
  - (if CDCL) learn new clause, back-jump
- model found
  - return the model

Adapting DPLL to DPLL(T)

Run DPLL on the propositional abstraction φ^p of the T-input formula φ



...until...

- conflict reached: backtrack/jump, learn clauses as usual
- model found (represented by a set Γ of literals)
    - It is not necessarily a T-model!
    - Ask theory solver: is Γ T-satisfiable?
        - If yes, we are done.
        - If no, backtrack in the original search.
        - (CDCL) get a T-unsatisfiable subset for clause learning/back-jumping

Adapting DPLL to DPLL(T)

Example

$$\underbrace{g(a) = c}_{1} \wedge (\underbrace{f(g(a)) \neq f(c)}_{\neg 2} \vee \underbrace{g(a) = d}_{3}) \wedge \underbrace{c \neq d}_{\neg 4}$$

Adapting DPLL to DPLL(T)

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- Call SAT solver with input $[1, \neg 2 \vee 3, \neg 4, \neg 1 \vee 2, \neg 1 \vee \neg 3 \vee 4]$
- SAT solver detects unsat

Theory combination

- Given a theory T, a Theory solver for T takes as input a set (interpreted as an implicit conjunction) φ of literals and determines whether φ is T-satisfiable.
- We are often interested in using two or more theories at the same time.
- Can we combine two theory solvers to get a theory solver for the combined theory?

Theory combination

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- We are often interested in using two or more theories at the same time.
- Can we combine two theory solvers to get a theory solver for the combined theory?

Example
The following formula uses both T_{E} and T_{Z}

$$\varphi := 1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$$

The Nelson-Oppen Method

A very general method for combining theory solvers is the Nelson-Oppen method.

This method is applicable when:

1. The signatures Σ_i are disjoint.
2. The theories T_i are stably-infinite.
   - A Σ-theory T is stably-infinite if every T-satisfiable quantifier-free Σ-formula in satisfiable in an infinite model.
3. The formulas to be tested for satisfiability are conjunctions of quantifier-free literals.

Extensions exist that can relax each of these restrictions in some cases.

The Nelson-Oppen Method

Some definitions:

- A member of Σ_i is an *i-symbol*.
- A term t is an *i-term* if it starts with an i-symbol.
- An *atomic i-formula* is
  - an application of an i-predicate,
  - an equation whose lhs and rhs are an i-terms, or
  - an equation whose lhs is a variable and whose rhs in an i-term
- An *i-literal* is an atomic i-formula or the negation of one.
- An occurrence of a term t in either an i-term or an i-literal is *i-alien* if it is a j-term with $i \neq j$ and all of its super-terms (if any) are i-terms.
- An expression is *pure* if it contains only variables and i-symbols for some i.

Conversion to separate form

Given a conjunction of literals φ, we want to convert it into a separate form: a T-equisatisfiable conjunction of literals $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$ where each φ_i is a Σ_i-formula.

We have the following algorithm:

1. Let ψ be some literal in φ.
2. If ψ is a pure i-literal, for some i, remove ψ from φ and add ψ to φ_i.
   If φ is empty then stop; otherwise goto step 1.
3. Otherwise, ψ is an i-literal for some i.
   Let t be a term occurring i-alien in ψ.
   Replace t in φ with a new variable z and add $z = t$ to φ.
   Goto step 1.

Conversion to separate form

Example
Consider the following $\Sigma_{\mathrm{E}} \cup \Sigma_{\mathrm{Z}}$:

$$\varphi := 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ f(x) \neq f(1) \ \wedge \ f(x) \neq f(2)$$

We convert φ to a separate form:

- $\varphi_{\mathrm{E}} :=?$
- $\varphi_{\mathrm{Z}} :=?$

Conversion to separate form

Example
Consider the following $\Sigma_{\mathrm{E}} \cup \Sigma_{\mathrm{Z}}$:

$$\varphi = 1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$$

We convert φ to a separate form:

- $\varphi_{\mathrm{E}} :=?$
- $\varphi_{\mathrm{Z}} := 1 \leq x$

Conversion to separate form

Example
Consider the following $\Sigma_{\mathrm{E}} \cup \Sigma_{\mathrm{Z}}$:

$$\varphi = \cancel{1 \leq x \wedge} \; x \leq 2 \; \wedge \; f(x) \neq f(1) \; \wedge \; f(x) \neq f(2)$$

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- $\varphi_{\mathrm{E}} := ?$
- $\varphi_{\mathrm{Z}} := 1 \leq x \wedge x \leq 2$

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Example
Consider the following $\Sigma_{\mathrm{E}} \cup \Sigma_{\mathrm{Z}}$:

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Example

Consider the following $\Sigma_{\mathrm{E}} \cup \Sigma_{\mathrm{Z}}$:

$$\varphi = \cancel{1 \leq x \wedge x \leq 2 \wedge} f(x) \neq f(y) \wedge f(x) \neq f(2) \wedge y = 1$$

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- $\varphi_{\mathrm{E}} :=?$
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The Nelson-Oppen Method

- As each φ_i is a Σ_i-formula, we can run a Theory solver Sat_i for each φ_i.
- If any Sat_i reports that φ_i is unsatisfiable, then φ is unsatisfiable.
- The converse is not true in general!
- We need a way for the decision procedures to communicate with each other about shared variables.
- If S is a set of terms and $\sim$ is an equivalence relation on S, then the arrangement of S induced by $\sim$ is

$$Ar_{\sim} = \{x = y \mid x \sim y\} \cup \{x \neq y \mid x \not\sim y\}$$

The Nelson-Oppen Method

Suppose that T_1 and T_2 are theories with disjoint signatures Σ_1 and Σ_2.

Let $T = \bigcup T_i$ and $\Sigma = \bigcup \Sigma_i$.

Given a Σ-formula φ and decision procedures Sat_1 and Sat_2 for T_1 and T_2, we wish to determine if φ is T-satisfiable.

The non-deterministic Nelson-Oppen algorithm:

1. Convert φ to its separate form $\varphi_1 \wedge \varphi_2$.
2. Let S be the set of variables shared between φ_1 and φ_2.
   Guess an equivalence relation $\sim$ on S.
3. Run Sat_1 on $\varphi_1 \cup Ar_\sim$.
4. Run Sat_2 on $\varphi_2 \cup Ar_\sim$.

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If there exists an equivalence relation $\sim$ such that both Sat_1 and Sat_2 succeed, then φ is T-satisfiable.

If no such equivalence relation exists, then φ is T-unsatisfiable.

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If there exists an equivalence relation $\sim$ such that both Sat_1 and Sat_2 succeed, then φ is T-satisfiable.

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The generalization to more than two theories is straightforward.

The Nelson-Oppen Method

Example
Consider the following $\Sigma_{\mathrm{E}} \cup \Sigma_{\mathrm{Z}}$:

$$\varphi := 1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$$

We first convert φ to a separate form:

- $\varphi_{\mathrm{E}} := f(x) \neq f(y) \wedge f(x) \neq f(z)$
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The shared variables are $\{x, y, z\}$.

There are 5 possible arrangements based on equivalence classes of x, y, and z *(see Bell number)*.

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- $\varphi := 1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$
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1. $\{x = y, x = z, y = z\}$ inconsistent with T_{E}
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5. $\{x \neq y, x \neq z, y \neq z\}$ inconsistent with T_{Z}

Conclusion: φ is $T_{\mathrm{E}} \cup T_{\mathrm{Z}}$-unsatisfiable!

SMT solvers

Recall the ingredients:

- Theory solvers for different theories
- Combine a Theory solver and a SAT solver
- Combine Theory solvers for different theories

References

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