Table 4.1 Properties of CTFS

Properties	Continuous-time Fourier series $x(t) \xleftrightarrow{\text{CTFS}} a_k$, $y(t) \xleftrightarrow{\text{CTFS}} b_k$
Linearity	$A\,x(t)+B\,y(t) \xleftrightarrow{\text{CTFS}} Aa_k + Bb_k$
Time shifting	$x(t-t_0) \xleftrightarrow{\text{CTFS}} e^{-jk\omega_0 t_0}\ a_k$
Time reversal	$x(-t) \xleftrightarrow{\text{CTFS}} a_{-k}$
Time scaling	$x(\beta t) \xleftrightarrow{\text{CTFS}} \frac{1}{\lvert\beta\rvert} a_{(k/\beta)}$
Multiplication	$x(t)\times y(t) \xleftrightarrow{\text{CTFS}} \sum_{l} a_l\ a_{k-l}$
Conjugation	$x^*(t) \xleftrightarrow{\text{CTFS}} a^*_{-k}$
Parseval's theorem	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt \xleftrightarrow{\text{CTFS}} \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2.$
Differentiation	$\frac{dx(t)}{dt} \xleftrightarrow{\text{CTFS}} jk\left(\frac{2\pi}{T}\right) a_k$
Integration	$\int_{-\infty}^{t} x(t) \xleftrightarrow{\text{CTFS}} \frac{1}{jk\left(\frac{2\pi}{T}\right)} a_k$

4.6 FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNAL

A discrete-time signal $x(n)$ is periodic with period N if

$$x(n) = x(n+N) \tag{4.41}$$

where N = smallest positive integer.

The fundamental period, N is the smallest positive integer for which equation (4.41) holds good.

The fundamental frequency,

$$\omega_0 = \frac{2\pi}{N} \tag{4.42}$$

Discrete-time complex exponential signals are represented as

$$\theta_k(n) = e^{jk\omega_0 n} \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\theta_k(n) = e^{jk\left(\frac{2\pi}{N}\right)n} \tag{4.43}$$

The fundamental frequency of the discrete-time complex exponential signal is N.

Any periodic signal can be represented as the linear combination of the sequence $\theta_k(n)$, i.e.

$$x(n) = \sum_k a_k \theta_k(n)$$

$$x(n) = \sum_k a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{4.44}$$

The summation on k, as k varies over a range of N successive integers is represented by $k = \langle N \rangle$, i.e.

$$x(n) = \sum_{k=\langle N \rangle} a_k \theta_k(n)$$

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \quad k = 0, 1, 2, \ldots, (N-1) \tag{4.45}$$

The equation (4.45) is referred as the discrete-time Fourier series and the coefficient a_k is called the discrete-time Fourier coefficient.

4.6.1 To Calculate Discrete-time Fourier Series Coefficient

Let us consider equation (4.45)

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

For $n = 0$,
$$x(0) = \sum_{k=\langle N \rangle} a_k$$

For $n = 1$,
$$x(1) = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)}$$

For $n = 2$,
$$x(2) = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)2}$$

For $n = (N-1)$,
$$x(N-1) = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)(N-1)} \tag{4.46}$$

The equation (4.46) represents a set of N linear equations that consists of unknown Fourier series coefficients a_k as k ranges over a set of N integers. The Fourier series coefficients can be obtained by solving the set of N equations. The equations are linearly independent and hence can be solved.

Multiply by the term $e^{-jl\left(\frac{2\pi}{N}\right)n}$ on either side of equation (4.45)

$$x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} e^{-jl\left(\frac{2\pi}{N}\right)n} \tag{4.47}$$

Summing equation (4.47) over N terms

$$\sum_{x=<N>} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} = \sum_{x=<N>}\left[\sum_{k=<N>} a_k e^{j(k-l)\frac{2\pi}{N}n}\right] \tag{4.48}$$

Interchanging the summation of equation (4.48),

$$\sum_{x=<N>} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k\left[\sum_{n=<N>} e^{j(k-l)\frac{2\pi}{N}n}\right] \tag{4.49}$$

Since the length of the series k and l are restricted to the same interval N, this can be represented as a finite geometric series whose sum depends on whether $k = l$ or $k \neq l$.

$$\sum_{n=0}^{N-1} e^{j(k-l)\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & k=l \\ \dfrac{1-e^{-j(k-l)2\pi}}{1-e^{-j(k-l)\left(\frac{2\pi}{N}\right)}}, & k \neq l \end{cases} \tag{4.50}$$

$\sum_{K=0}^{N} a^K = \frac{1-\left(\frac{1}{a}\right)^N}{1-\left(\frac{1}{a}\right)}$

Since $e^{-j(k-l)2\pi} = 1$, the equation (4.50) reduces to

$$\sum_{n=0}^{N-1} e^{j(k-l)\frac{2\pi}{N}n} = \begin{cases} N, & k=l \\ 0, & k \neq l \end{cases} \tag{4.51}$$

The equation (4.51) shows that the signals (complex sinusoids) with frequencies separated by an integer multiples of fundamental frequency are orthogonal.

Substituting equation (4.51) in (4.49),

$$\sum_{n=<N>} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} = Na_k$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} \tag{4.52}$$

The discrete-time Fourier series pair is given by

$$x(n) = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{4.53}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{4.54}$$

This relation is denoted by $x(n) \xleftrightarrow{\text{DTFS}} a_k$

SOLVED PROBLEMS

Problem 4.10 Find the discrete-time Fourier series coefficients for

$$x(n)=1+\sin\left(\frac{2\pi}{N}\right)n+\cos\left(\frac{2\pi}{N}\right)n+2\cos\left(\frac{6\pi}{N}\right)n+3\sin\left(\frac{8\pi}{N}n+\frac{\pi}{3}\right)$$

Solution

$$x(n)=1+\sin\left(\frac{2\pi}{N}\right)n+\cos\left(\frac{2\pi}{N}\right)n+2\cos\left(\frac{6\pi}{N}\right)n+3\sin\left(\frac{8\pi}{N}n+\frac{\pi}{3}\right)$$

The signal is periodic with period N.

Expanding the given signal as a sum of two exponentials,

$$x(n)=1+\left[\frac{e^{j\frac{2\pi}{N}n}-e^{-j\frac{2\pi}{N}n}}{2j}\right]+\left[\frac{e^{j\frac{2\pi}{N}n}+e^{-j\frac{2\pi}{N}n}}{2}\right]+2\left[\frac{e^{j\frac{6\pi}{N}n}+e^{-j\frac{6\pi}{N}n}}{2}\right]+3\left[\frac{e^{j\left(\frac{8\pi}{N}n+\frac{\pi}{3}\right)}-e^{-j\left(\frac{8\pi}{N}n+\frac{\pi}{3}\right)}}{2j}\right]$$

$$x(n)=1+\frac{1}{2j}e^{j\left(\frac{2\pi}{N}\right)n}-\frac{1}{2j}e^{-j\left(\frac{2\pi}{N}\right)n}+\frac{1}{2j}e^{j\left(\frac{2\pi}{N}\right)n}+\frac{1}{2j}e^{-j\left(\frac{2\pi}{N}\right)n}+e^{j\left(\frac{6\pi}{N}\right)n}+e^{-j\left(\frac{6\pi}{N}\right)n}$$
$$+\frac{3}{2j}e^{j\frac{\pi}{3}}e^{j\left(\frac{8\pi}{N}\right)n}-\frac{3}{2j}e^{-j\frac{\pi}{3}}e^{-j\left(\frac{8\pi}{N}\right)n}$$

$$x(n)=1+\frac{1}{2}(1-j)e^{j\left(\frac{2\pi}{N}\right)n}+(1+j)e^{-j\left(\frac{2\pi}{N}\right)n}+e^{j\left(\frac{6\pi}{N}\right)n}+e^{-j\left(\frac{6\pi}{N}\right)n}+\frac{3}{2j}e^{j\frac{\pi}{3}}e^{j\left(\frac{8\pi}{N}\right)n}-\frac{3}{2j}e^{-j\frac{\pi}{3}}e^{-j\left(\frac{8\pi}{N}\right)n}$$

Rearranging the sequence,

$$x(n)=\frac{-3}{2j}e^{-j\frac{\pi}{3}}e^{-j4\left(\frac{2\pi}{N}\right)n}+e^{-j3\left(\frac{2\pi}{N}\right)n}+(1+j)e^{-j\frac{2\pi}{N}n}+1+(1-j)e^{j\left(\frac{2\pi}{N}\right)n}+e^{j3\left(\frac{2\pi}{N}\right)n}+\frac{3}{2j}e^{+j\frac{\pi}{3}}e^{j4\left(\frac{2\pi}{N}\right)n}$$

$$a_0=1$$
$$a_{-4}=\frac{-3}{2j}e^{-j\frac{\pi}{3}};\ a_4=\frac{3}{2j}e^{j\frac{\pi}{3}}$$
$$a_{-3}=-1;\ a_3=+1$$
$$a_{-2}=a_2=0$$
$$a_{-1}=\frac{1}{2}(1+j);\ a_1=\frac{1}{2}(1-j)$$

2.6 FOURIER TRANSFORM

The plot of amplitudes at different frequency components for a periodic wave is known as discrete (line) frequency spectrum because amplitude values have significance only at discrete values of $n\,\omega_0$ where $\omega_0 = 2\pi/T$ is the separation between two adjacent (consecutive) harmonic components. If the repetition period T increases, ω_0 decreases.

Hence, when the repetition period T becomes infinity, i.e. $T \to \infty$, the wave $f(t)$ will become non-periodic, the separation between two adjacent harmonic components will be zero, i.e. $\omega_0 = 0$. Therefore, the discrete spectrum will become a continuous spectrum. When $T \to \infty$, the adjacent pulses virtually never occur and the pulse train reduces to a single isolated pulse. The exponential form of the Fourier series given in Eq. 2.10 can be extended to aperiodic waveforms such as single pulses or single transients by making a few changes.

Assuming $f(t)$ is initially periodic, from Eq. 2.10, we have

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where
$$c_n = 1/T \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

In the limit, for a single pulse, we have

$$T \to \infty,\ \omega_0 = 2\pi/T \to d\omega \text{ (a small quantity)}$$

or
$$1/T = \omega_0/2\pi \to d\omega/2\pi$$

Furthermore, the n^{th} harmonic in the Fourier series is $n\,\omega_0 \to nd\,\omega$. Here n must tend to infinity as ω_0 approaches zero, so that the product is finite, i.e. $n\,\omega_0 \to \omega$.

In the limit, the Σ sign leads to an integral and we have

$$c_n = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

and,
$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \right] e^{j\omega t}$$

When evaluated, the quantity in bracket is a function of frequency only and is denoted as $F(j\omega)$ where

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \tag{2.16}$$

It is called the *Fourier transform* of $f(t)$.
Substituting for $f(t)$ above. We obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega$$

or equivalently,

$$f(t) = \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\mathrm{d}f \tag{2.17}$$

$\frac{d\omega}{2\pi} = df$

which is called the *inverse* Fourier *transform*. Now the time function $f(t)$ represents the expression for a single pulse or transient only. Equations 2.16 and 2.17 constitute a Fourier *transform pair*.

From Eqs. 2.16 and 2.17, it is apparent that the Fourier transform and inverse Fourier transform are similar, except for sign change on the exponential component.

2.6.1 Energy Spectrum for a Non-Periodic Function

For a non-periodic energy signal, such as a single pulse, the total energy in $(-\infty, \infty)$ is finite, whereas the average power, i.e. energy per unit time is zero because $\frac{1}{T}$ tends to zero as T tends to infinity. Hence, the total energy associated with $f(t)$ is given by

$$E = \int_{-\infty}^{\infty} f^2(t)\mathrm{d}t$$

Since, $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$, we obtain

$$E = \int_{-\infty}^{\infty} f(t)\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\mathrm{d}\omega\ \mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\left[\int_{-\infty}^{\infty} f(t)e^{j\omega t}\mathrm{d}t\right]\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F(-j\omega)\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F^*(j\omega)\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\ \mathrm{d}\omega$$

$$= \int_{-\infty}^{\infty} |F(f)|^2\ \mathrm{d}f, \quad \text{joules}$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \int_{-\infty}^{\infty} |F(f)|^2\ \mathrm{d}f$$

This result is called ***Rayleigh's energy theorem*** or ***Parseval's theorem*** for Fourier transform. The quantity $|F(f)|^2$ is referred to as the *energy spectral density*, $S(f)$, which is equal to the energy per unit frequency.

The integration in Eq. 2.18 is carried out over positive and negative frequencies. If $f(t)$ is real, then $|F(j\omega)| = |F(-j\omega)|$, then the Eq. 2.18 becomes,