2.2 TRIGONOMETRIC FOURIER SERIES

A periodic function $f(t)$ can be expressed in the form of trigonometric series as

$$f(t) = \frac{1}{2}a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \ldots$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \ldots \tag{2.1}$$

where $\omega_0 = 2\pi f = \frac{2\pi}{T}$, f is the frequency and a's and b's are the coefficients. The Fourier series exists only when the function $f(t)$ satisfies the following three conditions called ***Dirichlet's conditions.***

(i) $f(t)$ is well defined and single-valued, except possibly at a finite number of points, i.e. $f(t)$ has a finite average value over the period T.

(ii) $f(t)$ must possess only a finite number of discontinuities in the period T.

(iii) $f(t)$ must have a finite number of positive and negative maxima in the period T.

Equation 2.1 may be expressed by the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{2.2}$$

where a_n and b_n are the coefficients to be evaluated. Integrating Eq. 2.2 for a full period, we get

$$\int_{-T/2}^{T/2} f(t)\mathrm{d}t = \frac{1}{2}a_0 \int_{-T/2}^{T/2} \mathrm{d}t + \int_{-T/2}^{T/2} \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\mathrm{d}t$$

Integration of cosine or sine function for a complete period is zero.

Therefore, $\displaystyle\int_{-T/2}^{T/2} f(t)\mathrm{d}t = \frac{1}{2}a_0 T$

Hence, $$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\mathrm{d}t \tag{2.3}$$

or equivalently $\displaystyle a_0 = \frac{2}{T}\int_0^T f(t)\mathrm{d}t$

Multiplying both sides of Eq. 2.2 by $\cos m\omega_0 t$ and integrating, we have

$$\int_{-T/2}^{T/2} f(t)\cos m\omega_0 t\mathrm{d}t = \frac{1}{2}\int_{-T/2}^{T/2} a_0 \cos m\omega_0 t\mathrm{d}t +$$
$$\int_{-T/2}^{T/2} \sum_{n=1}^{\infty} a_n \cos n\omega_0 t \cos m\omega_0 t\mathrm{d}t + \int_{-T/2}^{T/2} \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \cos m\omega_0 t\mathrm{d}t$$

Here, $\displaystyle\frac{1}{2}\int_{-T/2}^{T/2} a_0 \cos m\omega_0 t\mathrm{d}t = 0$

$$\int_{-T/2}^{T/2} a_n \cos n\omega_0 t \cos m\omega_0 \mathrm{d}t = \frac{a_n}{2}\int_{-T/2}^{T/2} [\cos(m+n)\omega_0 t + \cos(m-n)\omega_0 t]\mathrm{d}t$$

$$= \begin{cases} 0, & \text{for } m \neq n \\ \frac{T}{2}a_n, & \text{for } m = n \end{cases}$$

$$\int_{-T/2}^{T/2} b_n \sin n\omega_0 t \cos m\omega_0 t\mathrm{d}t = \frac{b_n}{2}\int_{-T/2}^{T/2} [\sin(m+n)\omega_0 t - \sin(m-n)\omega_0 t]\mathrm{d}t$$

$$= 0$$

Therefore, $\displaystyle\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\mathrm{d}t = \frac{Ta_n}{2}, \quad \text{for } m = n$

Hence, $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\mathrm{d}t \tag{2.4}$$

or equivalently $\displaystyle a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\mathrm{d}t$

Similary, multiplying both sides of Eq. 2.2 by $\sin m\omega_0 t$ and integrating, we get

$$\int_{-T/2}^{T/2} f(t)\sin m\omega_0 t\mathrm{d}t = \frac{1}{2}\int_{-T/2}^{T/2} a_0 \sin m\omega_0 t\mathrm{d}t +$$

$$\int_{-T/2}^{T/2}\sum_{n=1}^{\infty} a_n \cos n\omega_0 t \sin m\omega_0 t\mathrm{d}t + \int_{-T/2}^{T/2}\sum_{n=1}^{\infty} b_n \sin n\omega_0 t \sin m\omega_0 t\mathrm{d}t$$

Here, $\displaystyle\frac{1}{2}\int_{-T/2}^{T/2} a_0 \sin m\omega_0 t\mathrm{d}t = 0$

$$\int_{-T/2}^{T/2} a_n \cos n\omega_0 t \sin m\omega_0 t\,\mathrm{d}t = 0$$

$$\int_{-T/2}^{T/2} b_n \sin n\,\omega_0 t \sin m\omega_0 t\,\mathrm{d}t = \begin{cases} 0, & \text{for } m \neq n \\ \frac{T}{2}b_n, & \text{for } m = n \end{cases}$$

Therefore, $\displaystyle\int_{-T/2}^{T/2} f(t)\sin m\omega_0 t\,\mathrm{d}t = \frac{T}{2}b_n, \quad \text{for } m = n$

Hence, $$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,\mathrm{d}t \tag{2.5}$$

or equivalently $\displaystyle b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,\mathrm{d}t$

The number $n = 1, 2, 3, \ldots$ gives the values of the harmonic frequencies

Symmetry Conditions

(i) If the function $f(t)$ is even, then $f(-t) = f(t)$. For example, $\cos t$, t^2, $t \sin t$, are all even. The cosine is an even function, since it may be expressed as the power series

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

The waveforms representing the even functions of t are shown in Fig. 2.2. Geometrically, the graph of an even function will be symmetrical with respect to the y-axis and only cosine terms are present (d.c. term optional). When $f(t)$ is even,

$$\int_{-a}^{a} f(t)\mathrm{d}t = 2\int_{0}^{a} f(t)\mathrm{d}t$$

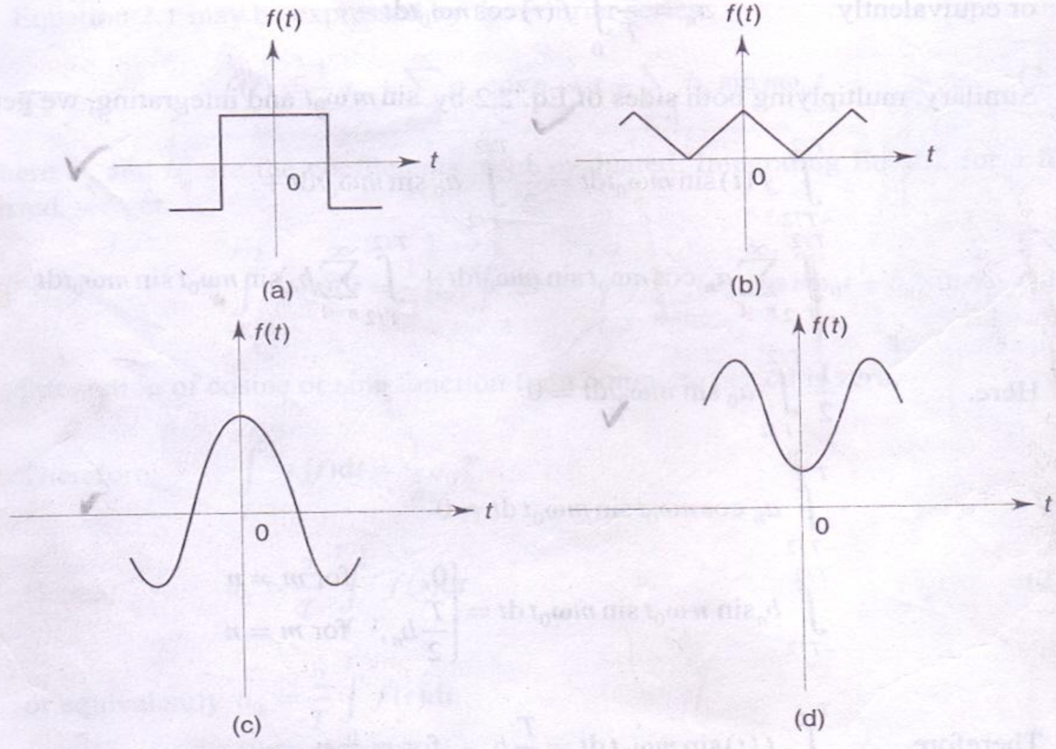


Fig. 2.2 *Waveforms Representing Even Functions*

The sum or product of two or more even function is an even function.

(ii) If the function $f(t)$ is odd, then $f(-t) = -f(t)$ and only sine terms are present (d.c. term optional). For example, $\sin t$, t^3, $t \cos t$ are all odd. The waveforms shown in Fig. 2.3 represent odd functions of t. The graph of an odd function is symmetrical about the origin. If $f(t)$ is odd, $\int_{-a}^{a} f(t)\mathrm{d}t = 0$. The sum of two or more odd functions is an odd function and the product of two odd functions is an even function.

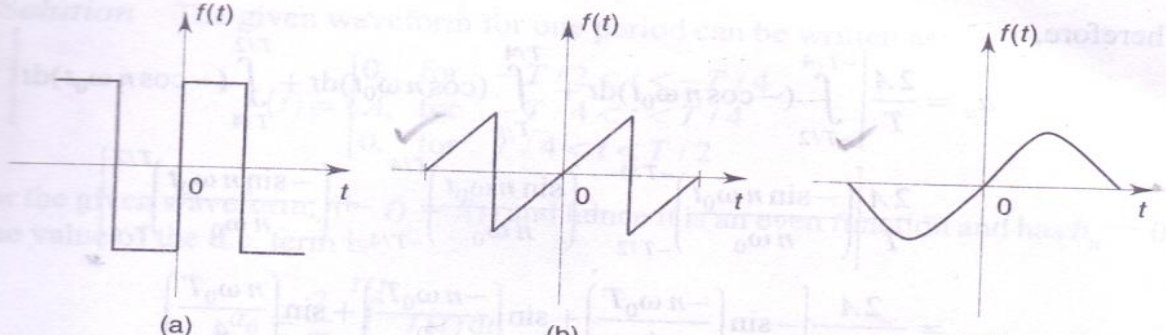
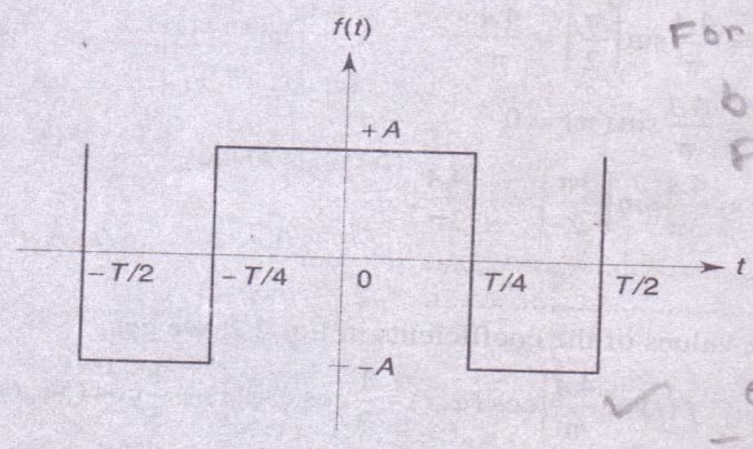


Fig. 2.3 *Waveforms Representing Odd Functions*

(iii) If $f(t + T/2) = f(t)$, only even harmonics are present.

(iv) If $f(t + T/2) = -f(t)$, only odd harmonics are present and hence the waveform has half-wave symmetry.

Example 2.1 *Obtain the Fourier components of the periodic square wave signal which is symmetrical with respect to the vertical axis at time $t = 0$, as shown in Fig. E2.1.*



For even func $b_n = 0$
For odd func $a_0 = 0$ $a_n = 0$

even x even = even
even x odd = odd

Fig. E2.1

Solution Since the given waveform is symmetrical about the horizontal axis, the average area is zero and hence the d.c. term $a_0 = 0$. In addition, $f(t) = f(-t)$ and so only cosine terms are present, i.e., $b_n = 0$.

$a_0 \neq 0$

Now, $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt$$ $a_0 =$

where $$f(t) = \begin{cases} -A, & \text{from } -T/2 < t < -T/4 \\ +A, & \text{from } -T/4 < t < +T/4 \\ -A, & \text{from } +T/4 < t < +T/2 \end{cases}$$

Therefore,

$$a_n = \frac{2A}{T}\left[\int_{-T/2}^{-T/4}(-\cos n\,\omega_0 t)\mathrm{d}t + \int_{-T/4}^{T/4}(\cos n\,\omega_0 t)\mathrm{d}t + \int_{T/4}^{T/2}(-\cos n\,\omega_0 t)\mathrm{d}t\right]$$

$$= \frac{2A}{T}\left[\left(\frac{-\sin n\,\omega_0 t}{n\,\omega_0}\right)_{-T/2}^{-T/4} + \left(\frac{\sin n\,\omega_0 t}{n\,\omega_0}\right)_{-T/4}^{T/4} + \left(\frac{-\sin n\,\omega_0 t}{n\,\omega_0}\right)_{T/4}^{T/2}\right]$$

$$= \frac{2A}{n\,\omega_0\,T}\left[-\sin\left(\frac{-n\,\omega_0 T}{4}\right) + \sin\left(\frac{-n\,\omega_0 T}{2}\right) + \sin\left(\frac{n\,\omega_0 T}{4}\right)\right.$$
$$\left. - \sin\left(\frac{-n\,\omega_0 T}{4}\right) - \sin\left(\frac{n\,\omega_0 T}{2}\right) + \sin\left(\frac{n\,\omega_0 T}{4}\right)\right]$$

$$= \frac{8A}{n\,\omega_0 T}\sin\left(\frac{n\,\omega_0 T}{4}\right) - \frac{4A}{n\,\omega_0 T}\sin\left(\frac{n\,\omega_0 T}{2}\right)$$

When $\omega_0 T = 2\pi$, the second term is zero for all integer of n.
Hence,

$$a_n = \frac{8A}{2n\pi}\sin\left(\frac{n\pi}{2}\right) = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$a_n = 0\,(\text{d.c. term})$$

$$a_1 = \frac{4A}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{4A}{\pi}$$

$$a_2 = \frac{4A}{\pi}\sin(\pi) = 0$$

$$a_3 = \frac{4A}{3\pi}\sin\left(\frac{3\pi}{2}\right) = -\frac{4A}{3\pi}$$

....

....

Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{4A}{\pi}\left|\cos(\varphi_0 t) - \frac{1}{3}\cos(3\varphi_0 t) + \frac{1}{5}\cos(5\varphi_0 t) - \cdots\right|$$

Example 2.2 *Obtain the Fourier Components of the periodic rectangular waveform shown in Fig. E2.2.*

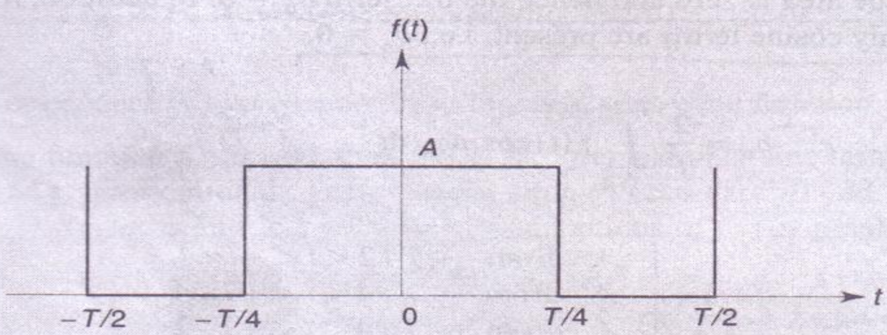


Fig. E2.2

Solution The given waveform for one period can be written as

$$f(t)=\begin{cases}0, & \text{for} \quad -T/2<t<-T/4\\ A, & \text{for} \quad -T/4<t<T/4\\ 0, & \text{for} \quad T/4<t<T/2\end{cases}$$

For the given waveform, $f(-t) = f(t)$ and hence it is an even function and has $b_n = 0$. The value of the d.c. term is

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\,\mathrm{d}t = \frac{2A}{T}\times\frac{T}{2} = A$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\,\omega_0 t\,\mathrm{d}t$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\cos n\,\omega_0 t\,\mathrm{d}t$$

$$= \frac{2A}{T}\left[\frac{\sin n\,\omega_0 t}{n\,\omega_0}\right]_{-T/4}^{T/4}$$

$$= \frac{4A}{n\,\omega_0 T}\sin(n\,\omega_0\,T/4)$$

When $\omega_0 T = 2\pi$, we have

$$a_n = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= 0, \qquad \text{for } n = 2, 4, 6, \ldots$$

$$= \frac{2A}{n\pi}, \qquad \text{for } n = 1, 5, 9, 13, \ldots$$

$$= -\frac{2A}{n\pi}, \qquad \text{for } n = 3, 7, 11, 15, \ldots$$

Substituting the values of the coefficients in Eq. 2.2, we obtain

$$f(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right)$$

Example 2.3 *Obtain the trigonometric Fourier series for the half-wave rectified sine wave shown in Fig. E2.3.*

Solution As the waveform shows no symmetry, the series may contain both sine and cosine terms. Here $f(t) = A\sin\omega_0 t$.

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t$$

where $$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$

$$= \frac{2}{T}\int_{-d/2}^{d/2} A\,\mathrm{d}t = \frac{2A}{T}[t]_{-d/2}^{d/2} = \frac{2Ad}{T}$$

Here, since the choice of $t = 0$ is at the centre of a pulse, b_n coefficient are zero.

Therefore, $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\,\omega_0 t\,dt = \frac{2}{T}\int_{-d/2}^{d/2} \cos n\,\omega_0\,t\,\mathrm{d}t$$

$$= \frac{2A}{T}\left[\frac{\sin n\,\omega_0 t}{n\,\omega_0}\right]_{-d/2}^{d/2} = \frac{2A}{n\,\omega_0 T}\left[\sin\left(\frac{n\,\omega_0 d}{2}\right) - \sin\left(\frac{-n\omega_0 d}{2}\right)\right]$$

$$= \frac{4A}{n\,\omega_0 T}\sin\frac{n\,\omega_0 d}{2}$$

Hence, $$f(t) = \frac{Ad}{T} + \frac{2Ad}{T}\sum_{n=1}^{\infty}\frac{\sin(n\omega_0 d/2)}{n\,\omega_0\,d/2}\cos n\omega_0 t$$

2.3 COMPLEX OR EXPONENTIAL FORM OF FOURIER SERIES

From Eq. 2.2, the trigonometric form of Fourier series is

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\,\omega_0 t + b_n \sin n\,\omega_0 t)$$

An alternative but convenient way of writing the periodic function $f(t)$ is in exponential form with complex quantities. Since

Eular's formula

$$\cos n\,\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin n\,\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Substituting these quantities in the expression for the Fourier series gives

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{(a_n - jb_n)e^{jn\omega_0 t}}{2}\right) + \left(\frac{(a_n + jb_n)e^{-jn\omega_0 t}}{-jb\ 2}\right)$$

Here, taking
$$c_n = \frac{1}{2}(a_n - jb_n) \tag{2.6}$$
$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$
$$c_0 = a_0/2$$

where, c_{2n} is the complex conjugate of c_n. Substituting expressions for the coefficients a_n and b_n from Eqs. 2.4 and 2.5 gives

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)[\cos n\omega_0 t - j\sin n\,\omega_0 t]\mathrm{d}t$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,e^{-jn\omega_0 t}\,\mathrm{d}t \tag{2.7}$$

and
$$c_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)[\cos n\,\omega_0 t + j\sin n\,\omega_0 t]\mathrm{d}t \tag{2.8}$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,e^{jn\omega_0 t}\,\mathrm{d}t$$

with
$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t} \tag{2.9}$$

where the values of n are negative in the last term and are included under the Σ sign. Also, c_0 may be included under the Σ sign by using the value of $n = 0$.

Therefore,
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{2.10}$$

It is clear from the result given in Eq. 2.10 that the periodic function $f(t)$ may be expressed mathematically by an infinite set of positive and negative frequency components. The negative frequencies have not only mathematical significance, but also physical significance, since a positive frequency may be associated with an anti-clockwise rotation and a negative frequency with a clockwise rotation.

The complex Fourier series furnishes a method of decomposing a signal in terms of a sum of elementary signals of the form $\{e^{jn\omega_0 t}\}$. This representation may be used for signals $f(t)$ that are

(i) Periodic, $f(t) = f(t + T)$, in which case the representation is valid on $(-\infty, \infty)$

(ii) Aperiodic, in which case the representation is valid on a finite interval (t_1, t_2). The periodic extension of $f(t)$ is obtained outside of (t_1, t_2).

Note that similar to the evaluation of integrals a_n and b_n, the limits of integration in Eq. 2.7 may be the end points of any convenient full period and not essentially 0 to T or 0 to 2π. For $f(t)$ to be real, $C_{-n} = C_n^*$, so that only positive value on n are considered in Eq. 2.7. Also, we have

$$a_n = 2\,\mathrm{Re}\,[c_n] \quad \text{and} \quad b_n = -2\,\mathrm{Im}\,[c_n] \tag{2.11}$$

For an even waveform, the trigonometric Fourier series has only cosine terms and hence, by Eq. 2.6, the exponential Fourier series coefficients will be pure real numbers. Similarly, for an odd waveform, the trigonometric Fourier series contains only sine terms and hence the exponential Fourier series coefficients will be pure imaginary.

Example 2.6

(a) *Find the trigonometric Fourier series of the waveform shown in Fig. E2.6 and*
(b) *Determine the exponential Fourier series and hence find a_n and b_n of the trigonometric series and compare the results.*

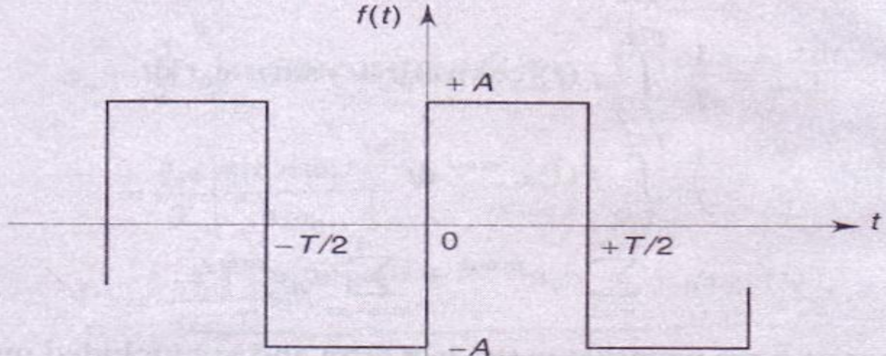


Fig. E2.6

Solution The function of the given waveform for one period can be written as

$$f(t)=\begin{cases}-A, & \text{for} -T/2<t<0\\ +A, & 0<t<T/2\end{cases}$$

As the waveform is symmetrical about the origin, the function of the waveform is odd and hence $a_0 = a_n = 0$, and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\, \mathrm{d}t$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0}(-A\sin n\omega_0 t)\,\mathrm{d}t + \int_{0}^{T/2} A\sin n\omega_0 t\,\mathrm{d}t\right]$$

$$= \frac{2A}{T}\left[\left[\frac{\cos n\omega_0 t}{n\omega_0}\right]_{-T/2}^{0} + \left[\frac{-\cos n\omega_0 t}{n\omega_0}\right]_{0}^{T/2}\right]$$

$$= \frac{2A}{n\omega_0 T}\{[1-\cos(n\omega_0 T/2)]+[1-\cos(n\omega_0 T/2)]\}$$

$$= \frac{4A}{n\omega_0 T}[1-\cos(n\omega_0 T/2)]$$

For an even waveform, the trigonometric Fourier series has only cosine terms and hence, by Eq. 2.6, the exponential Fourier series coefficients will be pure real numbers. Similarly, for an odd waveform, the trigonometric Fourier series contains only sine terms and hence the exponential Fourier series coefficients will be pure imaginary.

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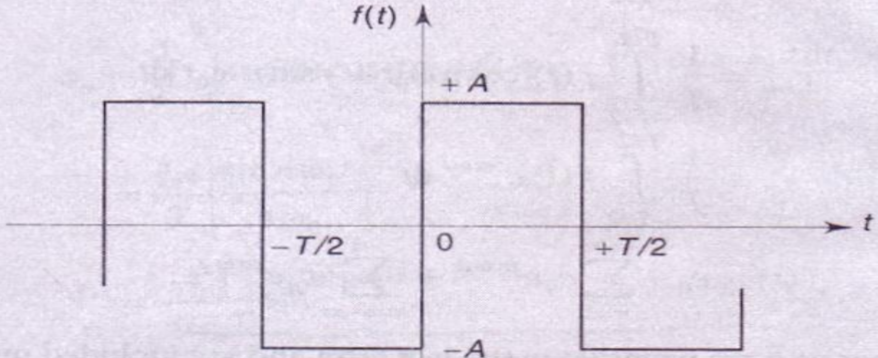

Fig. E2.6

Solution The function of the given waveform for one period can be written as

$$f(t)=\begin{cases}-A, & \text{for} -T/2<t<0\\ +A, & 0<t<T/2\end{cases}$$

As the waveform is symmetrical about the origin, the function of the waveform is odd and hence $a_0 = a_n = 0$, and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,\mathrm{d}t$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0}(-A\sin n\omega_0 t)\,\mathrm{d}t + \int_{0}^{T/2} A\sin n\omega_0 t\,\mathrm{d}t\right]$$

$$= \frac{2A}{T}\left\{\left[\frac{\cos n\omega_0 t}{n\omega_0}\right]_{-T/2}^{0} + \left[\frac{-\cos n\omega_0 t}{n\omega_0}\right]_{0}^{T/2}\right\}$$

$$= \frac{2A}{n\omega_0 T}\{[1-\cos(n\omega_0 T/2)]+[1-\cos(n\omega_0 T/2)]\}$$

$$= \frac{4A}{n\omega_0 T}[1-\cos(n\omega_0 T/2)]$$

When $\omega_0 = \dfrac{2\pi}{T}$, we have

$$b_n = \frac{4A}{n \cdot 2\pi}\left[1 - \cos n\left(\frac{2\pi}{T} \cdot T/2\right)\right]$$

$$= \frac{2A}{n\pi}[1 - \cos n\pi]$$

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4A}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Substituting the values of the coefficients in Eq. 2.2, we obtain

$$f(t) = \frac{4A}{\pi}\left[\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right], \quad \text{where} \quad \omega_0 = \frac{2\pi}{T}$$

(b) To determine exponential Fourier series

Here $c_0 = \left|\dfrac{1}{2}a_0\right| = 0$

To evaluate c_n

Since the wave is odd, c_n consists of pure imaginary coefficients. From Eq. 2.7, we have

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} (-A)\, e^{-jn\omega_0 t}\, dt + \int_0^{T/2} A e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{A}{T}\left\{\left[(-1)\frac{1}{(-jn\omega_0)}e^{-jn\omega_0 t}\right]_{-T/2}^{0} + \left[\frac{1}{(-jn\omega_0)}e^{-jn\omega_0 t}\right]_0^{T/2}\right\}$$

$$= \frac{A}{T} \cdot \frac{1}{(-jn\omega_0)}\{-e^0 + e^{jn\omega_0 (T/2)} + e^{-jn\omega_0 (T/2)} - e^0\}$$

When $\omega_0 = \dfrac{2\pi}{T}$, we get

$$c_n = \frac{A}{T} \cdot \frac{T}{-jn2\pi}\{-e^0 + e^{jn(2\pi/T)(T/2)} + e^{-jn(2\pi/2)(t/2)} - e^0\}$$

$$= \frac{A}{(-j2\pi n)}\{-e^0 + e^{jn\pi} + e^{-jn\pi} - e^0\} = j\frac{A}{n\pi}(e^{jn\pi} - 1)$$

Here, $e^{jn\pi} = +1$ for even n and $e^{jn\pi} = -1$ for odd n.

Therefore, $c_n = -j\left(\dfrac{2A}{n\pi}\right)$ for odd n only.

Hence, the exponential Fourier series is

$$f(t) = \ldots + j\frac{2A}{3\pi}e^{-j3\omega_0 t} + j\frac{2A}{\pi}e^{-j\omega_0 t} - j\frac{2A}{\pi}e^{j\omega_0 t} - j\frac{2A}{3\pi}e^{j3\omega_0 t}$$

By using Eq. 2.11, the trigonometric Fourier series coefficients, a_n and b_n can be evaluated as

$$a_n = 2\,\text{Re}\,[c_n] = 2\,|\,c_n\,| = 0 \text{ and } b_n = -2\,\text{Im}\,[c_n] = \frac{4A}{n\pi} \text{ for odd } n \text{ only.}$$

These coefficients are the same as the coefficients obtained in the trigonometric Fourier series.

Example 2.7

(a) *Find the trigonometric Fourier series of the waveform shown in Fig. E2.7 and*

(b) *Determine the exponential Fourier series and hence find a_n and b_n of the trigonometric series and compare the results.*

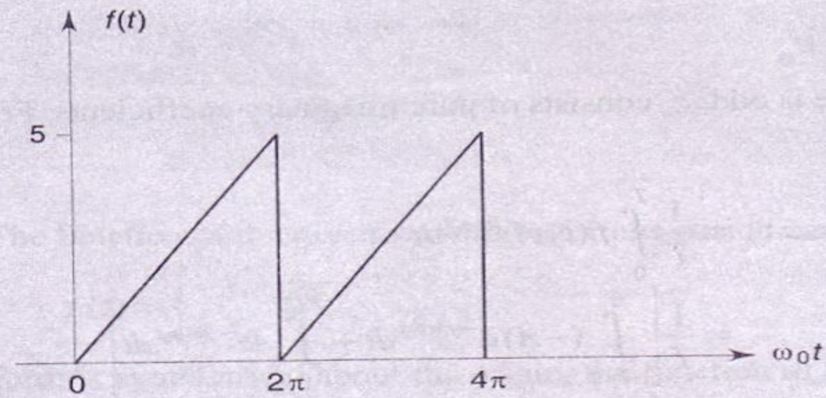

Fig. E2.7

Solution (a) As the waveform is periodic with period 2π in $\omega_0 t$ and continuous for $0 < \omega_0 t < 2\pi$, with discontinuities at $\omega_0 t = n(2\pi)$, where $n = 0, 1, 2, \ldots$, the Dirichlet conditions are satisfied.

To find f(t) for the given waveform of region $0 < \omega_0 t < 2\pi$

The equation of the straight line is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

Substituting $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (2\pi, 5)$, we get

$$\frac{f(t) - 0}{\omega_0 t - 0} = \frac{0 - 5}{0 - 2\pi}$$