LAPLACE TRANSFORM

$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$

Prove that $\mathcal{L}(1) = \frac{1}{s}$, $s>0$

Soln: $\mathcal{L}(1) = \int_0^\infty e^{-st}(1)\,dt$

$= \lim_{p\to\infty} \int_0^p e^{-st}\,dt$

$= \lim_{p\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^p$

$= \left(-\frac{1}{s}\right) \lim_{p\to\infty} \left[\frac{1}{e^{sp}} - 1\right]$

$= \frac{1}{s} - \lim_{p\to\infty} \frac{1}{e^{sp}}$

$= \frac{1}{s} - 0$ if $s>0$

$= \frac{1}{s}$ (proved)

Prove that, $\mathcal{L}(t) = \frac{1}{s^2}$

Soln: $\mathcal{L}(t) = \int_0^\infty e^{-st}(t)\,dt$

$= \lim_{p\to\infty} \int_0^p t e^{-st}\,dt$

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$$= \lim_{p\to\infty} \cancel{\left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]}$$

$$= \lim_{p\to\infty} \left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^p - \int_0^p 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= \lim_{p\to\infty} \left[t\left(\frac{e^{-st}}{-s}\right) - 1 \cdot \frac{e^{-st}}{s^2}\right]_0^p$$

$$= \lim_{p\to\infty} \left[\left\{ p\,\frac{e^{-sp}}{-s} - 0\right\} - \left\{\frac{1}{s^2}\cdot e^{-sp} - \frac{1}{s^2}\cdot 1\right\}\right]$$

$$= \lim_{p\to\infty} \left[\frac{pe^{-sp}}{-s} - \frac{e^{-sp}}{s^2} + \frac{1}{s^2}\right]$$

$$\Rightarrow \lim_{p\to\infty} \left[\frac{1}{s^2} - \frac{e^{-sp}}{s^2} - \frac{pe^{-sp}}{s}\right]$$

$$\Rightarrow \lim_{p\to\infty} \frac{1}{s^2} - 0 - 0 \quad \text{if } s>0$$

$$= \frac{1}{s^2} \text{ (proved)}$$

III Prove that $\mathcal{L}(t^2) = \frac{2}{s^3}$.

Soln: $\mathcal{L}(t^2) = \int_0^\infty e^{-st}(t^2)\, dt$

$$= \lim_{p\to\infty} \int_0^p t^2 e^{-st}\, dt$$

$$= \lim_{p\to\infty} \left[\left[t^2\left(\frac{e^{-st}}{-s}\right)\right]_0^p - \int_0^p (2t)\left(\frac{e^{-st}}{-s}\right) dt\right]$$

$$= \lim_{p\to\infty} \left[\frac{-p^2}{se^{sp}} + \frac{2}{s} \left\{ t\left(\frac{e^{-st}}{-s}\right)\Big|_0^p - \int_0^p 1.\left(\frac{e^{-st}}{-s}\right) dt \right] \right.$$

$$= \lim_{p\to\infty} \left[\frac{-p^2}{se^{sp}} - \frac{2}{s^2} \lim_{p\to\infty} \frac{p}{e^{sp}} + \frac{2}{s^2} \lim_{p\to\infty} \frac{e^{-st}}{-s} \Big|_0^p \right]$$

$$= \left[\lim_{p\to\infty} \frac{-p^2}{se^{sp}} - \frac{2}{s^2} \lim_{p\to\infty} \frac{p}{e^{sp}} - \frac{2}{s^3} \lim_{p\to\infty}\left(\frac{1}{e^{sp}} - 1\right) \right]$$

$$= -\frac{1}{s} \lim_{p\to\infty} \frac{-p^2}{e^{sp}} - \frac{2}{s^2} \lim_{p\to\infty} \frac{p}{e^{sp}} - \frac{2}{s^3} \lim_{p\to\infty} \frac{1}{e^{sp}} + \frac{2}{s^3}$$

$$= -0-0-0+\frac{2}{s^3} \quad [s>0]$$

$$= \frac{2}{s^3} \text{ (proved)}$$

$$* \; F(t^n) = \frac{n!}{s^{n+1}}$$

田 Prove that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

Soln:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} dt$$

$$= \lim_{p\to\infty} \int_0^p e^{at} e^{-st} dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-(s-a)t} dt$$

$$= \lim_{p\to\infty} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^p$$

$$= \frac{1}{-(s-a)} \lim_{p\to\infty} \left[\frac{1}{e^{(s-a)t}} \right]_0^p$$

$$= \frac{1}{-(s-a)} \lim_{p\to\infty} \left[\frac{1}{e^{(s-a)p}} - 1 \right]$$

$$= -\frac{\cancel{\lim}\ 1}{s-a} \lim_{p\to\infty} \frac{1}{e^{(s-a)p}} + \frac{1}{s-a}$$

$$= -0 + \frac{1}{s-a} \quad (s>a)$$

$$= \frac{1}{s-a} \text{ (proved)}$$

Prove that (a) $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$, $s>0$ (b) $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$

(a) Soln:

$$\mathcal{L}(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-st} \sin at \, dt$$

$$= \lim_{p\to\infty} \left[\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2} \right]_0^p$$

$$= \frac{1}{s^2+a^2} \lim_{p\to\infty} \left[\frac{s\sin ap - a\cos ap}{e^{sp}} + a \right]$$

$$= \frac{1}{s^2+a^2} [0+a]$$

$$= \frac{a}{s^2+a^2}$$

$$\int e^{\alpha t} \sin\beta t \, dt = \frac{e^{\alpha t}(\alpha \sin\beta t - \beta\cos(\beta t))}{\alpha^2+\beta^2}$$

Prove that $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

Soln:

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$$

$$\int e^{\alpha t}\cos\beta t\,dt = \frac{e^{\alpha t}(\alpha\cos\beta t+\beta\sin\beta t)}{\alpha^2+\beta^2}$$

$$= \lim_{p\to\infty}\int_0^p e^{-st}\cos at\,dt$$

$$= \lim_{p\to\infty}\left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right]_0^p$$

$$= \lim_{p\to\infty}\left[\frac{s}{s^2+a^2} - \frac{e^{-sp}(-s\cos ap + a\sin ap)}{s^2+a^2}\right]$$

$$= \frac{s}{s^2+a^2} - 0$$

$$= \frac{s}{s^2+a^2}$$

Prove that if $\mathcal{L}\{F(t)\} = f(s)$

∴ $\mathcal{L}\{F'(t)\} = s f(s) - F(0)$

Soln: we know,

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st}F'(t)\,dt$$

$$= \lim_{p\to\infty}\int_0^p e^{-st}F'(t)\,dt$$

$$= \lim_{p\to\infty}\left[\left[e^{-st}F(t)\right]_0^p - \int_0^p (-s)e^{-st}F(t)\,dt\right]$$

$$= \lim_{p\to\infty}\frac{F(p)}{e^{sp}} - F(0) + s\lim_{p\to\infty}\int_0^p e^{-st}F(t)\,dt$$

$= s\mathcal{L}\{F(t)\} - F(0)$

$= s\,f(s) - F(0)$

(proved)

Prove that, $\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

Soln: We know,

$$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0)$$
$$= s\,g(s) - G(0)$$

Let, $G(t) = F'(t)$

Then $\mathcal{L}\{F''(t)\} = s\mathcal{L}\{F'(t)\} - F'(0)$

$$= s[s\mathcal{L}\{F(t)\} - F(0)] - F'(0)$$
$$= s^2\mathcal{L}\{F(t)\} - sF(0) - F'(0)$$
$$= s^2 f(s) - sF(0) - F'(0)$$

(proved)

Prove that $\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) \cdots\cdots F^{(n-1)}(0)$

Soln: we first prove that,

$$\mathcal{L}\{F'(t)\} = s\,f(s) - F(0) \cdots\cdots (1)$$

Use mathematical induction, suppose,

$$\mathcal{L}\{F^{(n-1)}(t)\} = s^{n-1} f(s) - s^{n-2}F(0) - \cdots\cdots F^{(n-2)}(0) \cdots\cdots (2)$$

Then, $\mathcal{L}\{F^{(n)}(t)\} = \int_0^\infty e^{-st} F^{(n)}(t)\,dt$

$$= \lim_{p\to\infty} \int_0^p e^{-st} F^{(n)}(t)\,dt$$

$$= \lim_{p\to\infty} \left[e^{-st} F^{(n-1)}(t) \Big|_0^p - \int_0^p (-s) e^{-st} F^{(n-1)}(t)\,dt \right]$$

$$= \lim_{p\to\infty} \frac{F^{(n-1)}p}{e^{-sp}} - F^{(n-1)}(0) + s\int_0^p e^{-st} F^{(n-1)}(t)\,dt$$

$$= -F^{(n-1)}(0) + s\,\mathcal{L}\{F^{(n-1)}(t)\}$$

$$= s\,\mathcal{L}\{F^{(n-1)}(t)\} - F^{(n-1)}(0)$$

$$= s\left[s^{n-1} f(s) - s^{n-2}F(0) \cdots F^{(n-2)}(0)\right] - F^{(n-1)}(0)$$

$$= s^n f(s) - s^{n-1}F(0) - sF^{(n-2)}(0) - F^{(n-1)}(0) \text{ (proved)}$$

$\mathcal{L}\{F'(t)\} = s f(s) - F(0)$

$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

$\mathcal{L}\{F^{(n-1)}(t)\} = s^{n-1} f(s) - s^{n-2}F(0) - s^{n-3}F'(0) - \cdots - F^{(n-2)}(0).$

$\mathcal{L}\{F^{n}(t)\} = s^n f(s) - s^{n-1}F(0) - \cdots F^{(n-1)}(0).$

Laplace Transform of Integrals:

Prove that, if $\mathcal{L}\{F(t)\} = f(s)$
then, $\mathcal{L}\{\int_0^t F(u)du\} = f(s)/s$

Proof:

Let $G(t) = \int_0^t F(u)\, du$

Then $G'(t) = F(t)$ and $G(0) = 0$

Now taking Laplace on both sides,

$\mathcal{L}\{G'(t)\} = \mathcal{L}\{F(t)\}$

$\Rightarrow s\mathcal{L}\{G(t)\} - G(0) = f(s)$

$\Rightarrow s\,\mathcal{L}\{G(t)\} = f(s)$

$\Rightarrow \mathcal{L}\{G(t)\} = \frac{f(s)}{s}$

$\therefore \mathcal{L}\{\int_0^t F(u)\, du\} = f(s)/s$

(proved)

Multiplication by power of t.

* Prove that, if $\mathcal{L}\{F(t)\} = f(s)$ then
$\mathcal{L}\{t\,F(t)\} = -f'(s)$.

Proof: We know,
$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

taking derivative on both sides

$$f'(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st} F(t)\,dt\right)$$

$$= \int_0^\infty F(t)\,\frac{d}{ds}\,(e^{-st})\,dt$$

$$= \int_0^\infty F(t)\,(-t)\,e^{-st}\,dt$$

$$\cancel{= -\int_0^\infty F(t)(tF(t))\,dt} = -\int_0^\infty e^{-st}\{t\,F(t)\}\,dt$$

$$= -\mathcal{L}\{tF(t)\}$$

$$\therefore \mathcal{L}\{tF(t)\} = -f'(s).$$

(proved)

$$\mathcal{L}\{t^n F(t)\} = (-1)^n f^n(s).$$

Prove that, if $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

<u>Proof:</u> Let, $G(t) = \frac{F(t)}{t}$

$$\Rightarrow F(t) = t\,G(t)$$

taking laplace on both sides,

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{t\,G(t)\}$$

$$\Rightarrow f(s) = -g'(s)$$

taking derivative on both sides, with in limit ∞ to s

$$g(s) = -\int_\infty^s f(u)\,du$$

$$= \int_s^{\infty} f(u)\, du$$

$$\therefore \mathcal{L}\{G(t)\} = \int_s^{\infty} f(u)\, du$$

$$\Rightarrow \mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(u)\, du$$

(proved)

Initial Value Theorem on Laplace:

Prove the initial-value theorem: $\lim_{t\to 0} F(t) = \lim_{s\to\infty} s f(s)$

Soln: We know,

$$\mathcal{L}\{F'(t)\} = \int_0^{\infty} e^{-at} F'(t)\, dt = s f(s) - F(0) \quad \cdots (1)$$

But if $F(t)$ is sectionally continuous and of exponential order, we have

$$\lim_{s\to\infty} \int_0^{\infty} e^{-at} F'(t)\, dt = 0 \quad \cdots (2)$$

Then taking the limit as $s \to \infty$ in (1), assuming $F(t)$ continuous at $t=0$, we find that:

$$0 = \lim_{s\to\infty} s f(s) - F(0)$$

$$\text{or } \lim_{s\to\infty} s f(s) = F(0)$$

$$= \lim_{t\to 0} F(t)$$

(proved)

Prove the final value theorem : $\lim_{t\to\infty} F(t) = \lim_{s\to 0} s f(s)$

Proof: we know,

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt = s f(s) - F(0) \quad \cdots (1)$$

The limit of the left hand side as $s\to 0$ is

$$\lim_{s\to 0} \int_0^\infty e^{-st} F'(t) dt = \int_0^\infty F'(t)\, dt$$

$$= \lim_{p\to\infty} \int_0^p F'(t) dt$$

$$= \lim_{p\to\infty} \{F(p) - F(0)\}$$

$$= \lim_{t\to\infty} F(t) - F(0)$$

The limit of the right hand side as $s\to 0$ is in (1)

$$\lim_{s\to 0} s f(s) - F(0)$$

Thus $\lim_{t\to\infty} F(t) - F(0) = \lim_{s\to 0} s f(s) - F(0)$

$$\Rightarrow \lim_{t\to\infty} F(t) = \lim_{s\to 0} s f(s).$$

(proved)

APPLICATION OF INTEGRATION:

1. $\int_0^\infty te^{-st}\cos at\,dt$

Soln: $\int_0^\infty te^{-st}\cos at\,dt$

$\Rightarrow \mathcal{L}\{t\cos at\}$

$= -\frac{d}{ds}\{\mathcal{L}\{\cos at\}\}$

$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$

$= -\frac{(s^2+a^2)\cdot 1 - s(2s)}{(s^2+a^2)^2}$

$= \frac{-(a^2-s^2)}{s^2+a^2} = \frac{s^2-a^2}{s^2+a^2}$

2. $\int_0^\infty \left(\frac{e^{-3t}-e^{6t}}{t}\right)dt$

$= \lim_{s\to 0}\left[\int_0^\infty e^{-st}\left(\frac{e^{-3t}-e^{6t}}{t}\right)dt\right]$

$= \lim_{s\to 0} \mathcal{L}\left\{\frac{F(t)}{t}\right\}$

$\mathcal{L}\{F(t)\} = \mathcal{L}\{e^{-3t}-e^{-6t}\}$

$= \mathcal{L}\{e^{-3t}\} - \mathcal{L}\{e^{-6t}\}$

$= \frac{1}{s+3} - \frac{1}{s+6}$

Now, $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(u)\, du$

$$= \int_s^{\infty}\left(\frac{1}{s+3} - \frac{1}{s+6}\right) du$$

$$= \lim_{p\to\infty} \Big[\ln(u+3) - \ln(u+6)\Big]_s^p$$

$$= \lim_{p\to\infty} \left[\ln \frac{u+3}{u+6}\right]_s^p$$

$$= \lim_{p\to\infty} \ln\left(\frac{p+3}{p+6}\right) - \ln\left(\frac{s+3}{s+6}\right)$$

$$= \ln \frac{s+6}{s+3}$$

Therefore the required result is $\lim_{s\to 0} \ln\left(\frac{s+6}{s+3}\right)$

$$= \ln \frac{6}{3} = \ln 2$$

Prove that: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

Proof:

we know,

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st}\, F(t)\, dt$$

Here, $F(t) = t^n$

$$\therefore \mathcal{L}\{t^n\} = \int_0^{\infty} e^{-st}\, t^n\, dt \text{ ----- } (1)$$

putting $st = y$

$\Rightarrow s\,dt = dy$

$\Rightarrow dt \Rightarrow \frac{1}{s}\,dy$

$st = y$

$\Rightarrow s^n t^n = y^n$

$\Rightarrow t^n = \frac{1}{s^n}\,y^n$

when $t = 0$, $y = 0$

$t = \infty$, $y = \infty$

From equation (1) we have,

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-y}\frac{1}{s^n}\,y^n\,\frac{1}{s}\,dy$$

$$= \lim_{p \to \infty}\int_0^p e^{-y}\frac{1}{s^n}\,y^n\,\frac{1}{s}\,dy$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-y}\,y^n\,dy$$

$$= \frac{1}{s^{n+1}}\,\Gamma(n+1)$$

$$= \frac{n!}{s^{n+1}}$$

By using gama function

$$\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x}\,x^n\,dx$$

$$\therefore \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

[7] Prove that (a) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$ (b) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

Proof: (a) $\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$$= \int_0^\infty e^{-at}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$

$= \frac{1}{2}\int_0^\infty e^{-st} e^{at} dt - \frac{1}{2}\int_0^\infty e^{-st} e^{-at} dt$

$= \frac{1}{2} \mathcal{L}\{e^{at}\} - \frac{1}{2} \mathcal{L}\{e^{-at}\}$

$= \frac{1}{2} \left\{ \frac{1}{s-a} - \frac{1}{s+a} \right\}$

$= \frac{a}{s^2-a^2}, s>|a|$

(b) We know,

$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{ \frac{e^{at}+e^{-at}}{2} \right\}$

$= \int_0^\infty e^{-st} \left(\frac{e^{at}+e^{-at}}{2} \right) dt$

$= \frac{1}{2}\int_0^\infty e^{-st} e^{at} dt + \frac{1}{2}\int_0^\infty e^{-st} e^{-at} dt$

$= \frac{1}{2} \mathcal{L}\{e^{at}\} + \frac{1}{2} \mathcal{L}\{e^{-at}\}$

$= \frac{1}{2} \left\{ \frac{1}{s-a} + \frac{1}{s+a} \right\}$

$= \frac{1}{2} \times \frac{s+a+s-a}{s^2-a^2}$

$= \frac{1}{2} \times \frac{2s}{s^2-a^2}$

$= \frac{s}{s^2-a^2}$ for $s>|a|$

Ex Prove that $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$.

Proof: we have,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = f(s)$$

$$\therefore \mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st}\{e^{at}F(t)\}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}F(t)\,dt$$

$$= f(s-a) \text{ (proved)}$$

Ex Show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Soln:

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \lim_{s\to 0}\int_0^\infty e^{-st}\left(\frac{\sin t}{t}\right)dt$$

$$= \lim_{s\to 0}\mathcal{L}\left\{\frac{\sin t}{t}\right\}$$

$$= \lim_{s\to 0}\mathcal{L}\left\{\frac{F(t)}{t}\right\}$$

$$\therefore \mathcal{L}\{F(t)\} = \mathcal{L}\{\sin t\}$$

$$= \frac{1}{s^2+1}$$

$$\therefore \mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_0^\infty f(u)\,du$$

$$\cancel{= \int_s^\infty \frac{1}{u^2+1}\,du} \quad \int_0^\infty \frac{1}{u^2+1}\,du$$

$$= \cancel{\lim_{s\to 0}\int_0^{\infty}\frac{1}{s^2+1}\,du}$$

$$= \tan^{-1} u \Big|_0^{\infty}$$

$$= \frac{\pi}{2}$$ (showed)

Prove that, $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$

<u>Proof:</u> $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \mathcal{L}\{\sin t\}\,du$$

$$= \int_s^{\infty} \frac{1}{u^2+1}\,du$$

$$= \lim_{p\to\infty}\left[\tan^{-1} u\right]_s^p$$

$$= \lim_{p\to\infty}\left(\tan^{-1} p - \tan^{-1} s\right)$$

$$= \lim_{p\to\infty}\left\{\tan^{-1}\left(\frac{p-s}{1+ps}\right)\right\}$$

$$= \lim_{p\to\infty}\left\{\tan^{-1}\left(\frac{1-s/p}{1/p+s}\right)\right\}$$

$$= \tan^{-1}\left(\frac{1}{s}\right)$$

$$\therefore \mathcal{L}\left\{\int_0^t \left(\frac{\sin u}{u}\right) du\right\} = \frac{f(s)}{s}$$

$$= \frac{\mathcal{L}\left\{\frac{\sin t}{t}\right\}}{s}$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s}$$

(proved)

Inverse Laplace Transform:

If the Laplace transform of a function $F(t)$ is $f(s)$ i.e. if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace transform of $f(s)$ and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator.

Example: Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}$

Soln: $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}$

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$\Rightarrow 1 = A(s-2) + B(s-1)$$

put, $s = 1$

$$1 = A(1-2) + B\cdot 0$$

$$\Rightarrow A = -1$$

put $s = 2$

$1 = A\cdot 0 + B(2-1)$

$\Rightarrow B = 1$

$\therefore \frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}$

$\therefore -\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$

$= -e^{t} + e^{2t}$

$= \cancel{F(t)} = e^{2t} - e^{t}$

Evaluate $\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+18}\right\}$

Soln: $\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+18}\right\} = \mathcal{L}^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$

$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$

$= 4e^{-4t} - 4t\,e^{-4t}$

$= 4e^{-4t}(1-t)$ (Ans)

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Soln: $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

we have, $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$

Then by convolution theorem,

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t (ue^{-u})(t-u)\,du$$

$$= \int_0^t (ut-u^2)\,e^{-u}\,du$$

$$= (ut-\cancel{2}u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\Big|_0^t$$

$$= \{(t^2-t^2)(-e^{-t}) - (t-2t)(e^{-t}) + (-2)(-e^{-t})\}$$

$$\cancel{= 0 + t \quad 1 \quad 2\cdot 1} - \{0 - t + 2\}$$

$$= 0 - (-t)e^{-t} + 2e^{-t} + t - 2$$

$$= te^{-t} + 2e^{-t} + t - 2 \text{ (Ans)}$$

Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Soln: We have $\frac{d}{ds}\left\{\frac{1}{s^2+a^2}\right\} = \frac{-2s}{(s^2+a^2)^2}$

Thus, $\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)$

Then since $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{1}{2}t\left(\frac{\sin at}{a}\right)$$

$$= \frac{t\sin at}{2a}.$$

Convolution:

Let $f(t)$ and $G(t)$ be two functions of a class A, then the convolution of $F(t)$ and $G(t)$ is denoted by $F*G$ and is defined as follows:

$$F*G=\int_0^t F(u)\,G(t-u)\,du$$

Convolution Theorem or Convolution Property:

Statement: If $\mathcal{L}^{-1}\{f(s)\}=F(t)$

and $\mathcal{L}^{-1}\{g(s)\}=G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\,g(s)\}=\int_0^t F(u)\,G(t-u)\,du$$

$$=F*G$$

or $f(s)\,g(s)=\mathcal{L}\left\{\int_0^t F(u)\,G(t-u)\,du\right\}$

Proof: Here the required result follows if we can prove that,

$$\mathcal{L}\left\{\int_0^t F(u)\,G(t-u)\,du\right\}=f(s)\,g(s) \quad \text{------} \; (1)$$

where, $f(s)=\mathcal{L}\{F(t)\}$ and $g(s)=\mathcal{L}\{G(t)\}$

Now, $\mathcal{L}\left\{\int_0^t F(u)\,G(t-u)\,du\right\}=\int_0^\infty e^{-st}\left\{\int_{u=0}^t F(u)\,G(t-u)\,du\right\}dt$

$$=\int_{t=0}^\infty\int_{u=0}^t e^{-st}\,F(u)\,G(t-u)\,du\,dt$$

$$= \lim_{T\to\infty} S_T$$

where,

$$S_T = \int_{t=0}^{T}\int_{u=0}^{t} e^{-st} F(u)\, G(t-u)\, du\,dt \quad \text{------ (ii)}$$

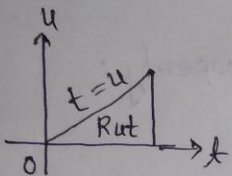


Fig: 1

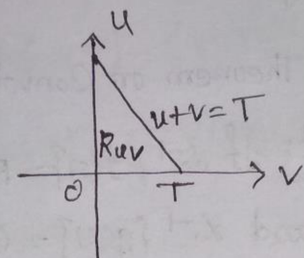


Fig: 2

The region in the ut-plane over which the integration (2) is defined is shown in Fig-1, Let $t-u=v$ or $t=u+v$, the region R_{uv} shown in the Fig-2 in the UV-plane. Then be the tranformation of multiple integrals, we have

$$S_T = \iint_{R_{ut}} e^{-st} F(u)\, G(t-u)\, du\,dt$$

$$= \iint_{R_{uv}} e^{-s(u+v)} F(u)\, G(v)\, \frac{\delta(u,t)}{\delta(u,v)}\, du\,dt \quad \text{---- (iii)}$$

where the Jacobian of the transformation is,

$$\frac{\delta(u,t)}{\delta(u,v)} = \begin{vmatrix} \frac{\delta u}{\delta u} & \frac{\delta u}{\delta v} \\ \frac{\delta t}{\delta u} & \frac{\delta t}{\delta v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

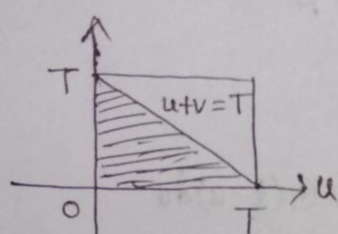
Thus from equation (iii) we have,

$$S_T = \int_{v=0}^{T} \int_{u=0}^{T-u} e^{-s(u+v)} F(u)\, G(v)\, du\, dv \text{ ----- (iv)}$$

Let us define a new function

$$K(u,v) = \begin{cases} e^{-s(u+v)} \; F(u) \; G(v) & , u+v \leq T \\ 0 & u+v > T \end{cases} \text{ ---- (v)}$$

This function is defined over a square ~~of~~ of Fig-3



But as indicated in equation (v), is zero over the unsaded portion. In terms of new function we can write,

$$S_T = \int_{v=0}^{T} \int_{u=0}^{T} e^{-s(u+v)} F(u)\, G(v)\, du\, dv$$

From equation (ii) we have,

$$\mathcal{L}\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = \lim_{T \to \infty} S_T$$

$$= \lim_{T\to\infty} \left[\int_{v=0}^{T} \int_{u=0}^{T} e^{-s(u+v)} F(u)\, G(v)\, du\, dv \right]$$

$$= \int_{v=0}^{\infty} \int_{u=0}^{\infty} e^{-s(u+v)} F(u)\, G(v)\, du\, dv$$

$$= \left[\int_{u=0}^{\infty} e^{-su} F(u)\, du \right] \left[\int_{v=0}^{\infty} e^{-sv} G(v)\, dv \right]$$

$$= \mathcal{L}\{F(u)\}\, \mathcal{L}\{G(v)\}$$

$$= f(s)\, g(s)$$ (proved)

■ Prove that $F * G = G * F$

Proof: By defination,

$$F * G = \int_0^t F(u)\, G(t-u)\, du$$

let , $t - u = v$

or $t = u + v$

$$F * G = \int_0^t F(t-v)\, G(v)\, dv$$

$$= \int_0^t G(v)\, F(t-v)\, dv$$

$$= G * F$$ (proved)