Waves on a String

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Abstract

An homogeneous and flexible string with a mass per unit length μ and a length l lied on two bridges where its two ends are fixed. A tension T was applied using masses hanging to a lever. The transversal motion was studied thanks to a detector which recorded the oscillations of the string. A frequency generator generated the oscillations via a driver coil placed near one of the bridge. The natural string frequency against the length and the tension was found what it enabled to obtain the relationship $v = \sqrt{\frac{T}{\mu}}$ where v is the velocity of the wave along the string. Then, by plucking the string and removing the driver, the system became damped and the resonance frequencies were changed. The damping constants were obtained for each harmonic by using the Fourier transform. These results were confirmed by the Fourier theory.

1 Introduction

The vibration phenomenon is illustrated by the waves. The fourier analysis enables to predict the characteristics of the transverse wave along a vibrating string with the both ends fixed. Let us see if the experiments are consistent with this. A driving force was first applied to the string thanks to a generator so as to study its permanent behaviour. Then, the string was plucked, the motion was consequently damped.

2 Theory

Let us consider a string with a mass per unit length μ and a length l.
The Newton's law is applied to a short segment -included between x and $x+dx$- of this continuous system by considering only the tension.

There is only a transversal displacement $y(x,t)$ which is given by the following relation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{1}$$

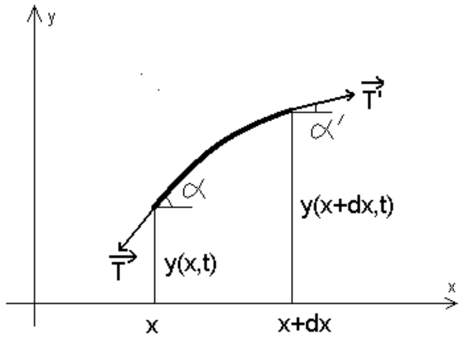

Figure 1: Tension on a segment of a string. [1]

where v is defined by $\sqrt{\frac{T}{\mu}}$.

The separation of variables, the initial and boundary conditions allow to solve this equation (1), therefore the transverse motion will have the following form:

$$y(x,t) = \sum_{1}^{\infty} A_n sin(\frac{n\pi x}{l}) cos(\frac{n\pi vt}{l}) \tag{2}$$

The resonance frequencies are thus defined by $f_n = \frac{w_n}{2\pi}$ -with $w_n = \frac{n\pi v}{l}$- i.e. $f_n = \frac{nv}{2l}$. Let us define the wavelength $\lambda_n = \frac{2l}{n}$. Consequently, the fundamental frequency is:

$$f = \frac{v}{2l} = \frac{v}{\lambda} \tag{3}$$

Moreover, $y(x,t)$ is initially the equation of the shape of the string which is assumed triangular when the string is plucked.

A_n is thus the "sine-Fourier coefficient" of the spatial function describing the shape of the string. It allows to calculate it. Hence,

$$A_n = \frac{2h sin(n\pi R)}{n^2 \pi^2 R(1-R)} \tag{4}$$

where Rl is the coordinate of the pluck The equation of the motion (1) is an approximation, actually the string is damped. It becomes:

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} + \frac{b}{\mu}\frac{\partial y}{\partial t} = 0 \tag{5}$$

$-b\frac{\partial y}{\partial t}$ refers to the damping force, afterwards b will be written β_n for a mode n . In fact, we assume that the friction coefficient b depends on the frequency. [4]

The form of the solution is deduced from the above study about the "ideal" case:

$$y(x,t) = \sum_{n=1}^{\infty} g_n(t) sin(\frac{n\pi x}{l}) \tag{6}$$

We substitute this expression into the differential equation(5):

$$g_n^{"} + \frac{\beta_n}{\mu} g_n^{'} + (\frac{n\pi v}{l})^2 g_n = 0 \tag{7}$$

The differential equation is solved for the case of light damping. We thus obtain the transversal displacement:

$$y(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta_n}{2\mu}t}(a_n cos(\Omega_n t) + b_n sin(\Omega_n t)) sin(n\pi x/l) \tag{8}$$

We introduce the damping constant $\alpha_n = \frac{\beta_n}{2\mu}$. The initial conditions give :

$$a_n = A_n \tag{9}$$

$$b_n = \frac{\beta_n A_n}{2\mu\Omega_n} \tag{10}$$

The transverse velocity is :

$$\frac{\delta y}{\delta t} = \sum_{n=1}^{\infty} A_n(-(\frac{b}{\mu\Omega_n})^2 - \Omega_n) e^{-\alpha_n t} sin(\Omega_n t) sin(\frac{n\pi x}{l}) \tag{11}$$

The damping force has several effects:

- The allowed frequency changed in comparison with the undamped case. The angular frequencies are Ω_n and they satisfy:

$$\Omega_n^2 = \omega_n^2 - \alpha_n^2 \tag{12}$$

- Moreover the amplitude of each harmonic is proportional to $A_n e^{-\alpha_n t}$. Indeed the amplitude of the displacement is maximum when its derivative is null i.e. $sin(\Omega_n t) = 0$ .By applying this condition in the expression of $y(x,t)$ (Eq.8), it gives an amplitude of the form $A_n sin(\frac{n\pi x}{l}) e^{-\alpha_n t}$. Therefore the envelope of each mode is a decreasing exponential. The coefficient A_n becomes null when $n = 1/R$ according to its expression, the Fourier development of $y(x,t)$ does not contain $(\frac{1}{R})$-th (in the case where $1/R$ is an integer) harmonics.

3 Experiments

The experiments were achieved with a sonometer. A string lied on two bridges, its length was changed by varying the space between the bridges. A tension T was applied thanks to masses hanged at a lever. To obtain vibrations, first a generator was used, then the string was plucked .

3.1 The Driven Oscillations

The experiments were only achieved with a string and masses, the varying parameters were thus the string mass and length, and the tension. The analysis of their effects on the wave velocity will allow to determine their link with the characteristics of the wave.

A frequency generator was linked to the string thanks to a driver coil (the detector must be far enough from it not to perturb the measurements), and the detector was linked to an oscilloscope which recorded the amplitude of the oscillations. This allowed to determine the fundamental frequency of the string against the tension. In the case of forced oscillations like this, the equation of motion (1) is modified :

$$\mu\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = F_0 cos(w_g t) \tag{13}$$

The right hand side refers to the force caused by the generator which provides a pulsation w_g.

3.1.1 Process of Measurements

A detector recorded the signal which was read on the computer and on an oscilloscope. The more the displacement was important the more the frequency generator provided a frequency close to the natural string frequency. Therefore, when the resonance was observed, i.e. the amplitude of the displacement was large and a sound was heard, it meant that the generator drove the string at one of these frequencies (the fundamental frequency or one of its harmonics). The lower frequency, for which the string vibrated strongly, was thus the fundamental one. The resonance frequency changed according to the set scale of the generator. So, in order to determine them accurately, Cool Edit and Final Project 98 were used. The first package recorded the vertical displacement $y(x, t)$ at a given x on an interval of about ten seconds. The second one yielded the Fourier decomposition of this function. The frequency of the first peak corresponded to the fundamental one.

3.1.2 Modes

When the generator provided a resonance frequency of the string, an analysis of the mode was achieved. After a reading of the frequency, the detector was moved along the string to determine the number of minima and maxima of the wave thanks to the observed amplitudes on the oscilloscope.

Frequency (Hz)	Positions of the nodes (cm)
$f_0 = 84.1 \pm 0.2$	$0/60$
234 ± 2	$0/20 \pm 0.2/40/60$
324 ± 2	$0/14.9 \pm 0.2/29.2 \pm 0.2/44.7 \pm 0.2/60$

Figure 2: $T = 49N$, $l = 60cm$. Position of the nodes according to the frequencies. The resonance is not only observed on a single frequency,but on a width of frequencies: this is what explains the error on the frequencies. The error on the position is only due to the precision of the ruler; indeed, in order to find the exact position of the node, we used a very high gain on the oscilloscope, so we were able to detect the small variations of the amplitude.

The results are consistent with what we expected: we obtained a fundamental frequency and its multiples (approximately). According to the theory, we should have (n+1) nodes for the n-th harmonics: this is exactly what we found. Moreover, these nodes were found, as expected by the theory, at the positions $x = \frac{lp}{n}$ (where p is an integer).

3.2 Variation of the Tension

This experiment was repeated for several tensions, it yielded the link between the frequency and the tension. Knowing the relation between the frequency and the velocity (3), the variations of the velocity against the tension were thus deduced.

According to the figure (3), the velocity varies proportionally with the squared root of the tension. Indeed the plot of $< \frac{v}{\sqrt{T}} > \sqrt{T}$ (where $< \frac{v}{\sqrt{T}} >= 29.30m^{1/2}.kg^{-1/2} \pm 0.07m^{1/2}.kg^{-1/2}$ is the average of $\frac{T}{\sqrt{v}}$) approximates pretty well the data, the largest difference between the fit and the data is $2.12m.s^{-1}$, i.e 3 percent of the smallest velocity.

$$\mu = \frac{1}{29.30^2} = 0.001165kg.m^{-1} \pm 6.10^{-6}kg.m^{-1} \tag{14}$$

The real mass per length unit is given by: $\mu = d * \pi * (D/2)^2 =$ (d and D respectively are the density ($d = 7.9g.cm^{-3}$) and the diameter of the string ($D = 0.8mm \pm 0.2mm$). So

$$\mu = 0.00397097kg.m^{-1} \pm 0.002kg.m^{-1} \tag{15}$$

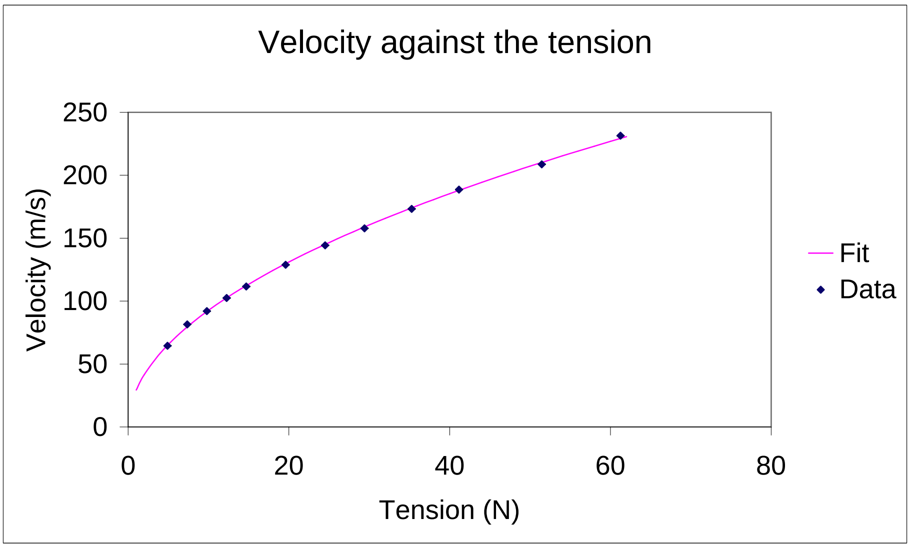

Figure 3: Velocity against the tension. The errors on the frequencies are small (that is why they are visible with difficulty) because they are calculated by only considering the reading error on the computer. Let us notice that there are supplementary errors due to the determination of the resonance but they are not calculable.

3.3 Variation of the Length

Then, at a given tension $T = 12.25N$, the length was changed several times, by varying the distance between the bridges from $30cm$ to $60cm$, to determine the effect of the length on the natural frequency of the vibrating system. The link between the velocity and the length was determined (figure 4).

The velocity is considered independent of the string length. Actually, it slightly decreases as the length increases according to the figure 4. Indeed, the amplitude of the transverse displacement becomes consequent with the decreasing of the length string, the angle θ is thus not so small enough to apply an approximation, what implies that the tension is really not steady along the x-axis. Moreover, when the vibrations were too intense, the string touched the detector, the measurements were consequently perturbed. The average of the velocity is $104.34m/s \pm 0.07m/s$, the data are equal to $104.34m/s \pm 2.40m/s$, the velocity is thus assimilated to a constant.

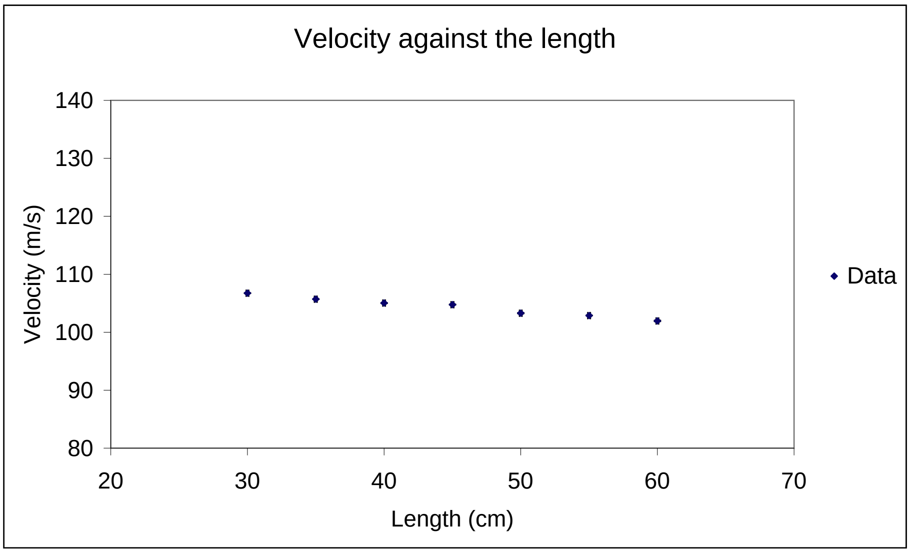

Figure 4: Velocity against the length. No fit is plotted because the velocity is clearly constant.

3.4 Characteristics of the Wave

The length does not have effect on the velocity. Therefore, the proportionality coefficient between the square root of the tension and the velocity only depends on the mass per length unit, the only unchanged parameter throughout the experiments. The dimension of this coefficient is such that we can say $v = \sqrt{\frac{T}{\mu}}$. What was yielded by the experimental data: It indeed confirmed the following hypothesis:

$$v = \frac{\sqrt{T}}{\mu} \tag{16}$$

The system is thus non-dispersive.

4 Plucked String

The string was plucked at several positions, for each one of them, the detector recorded the transversal displacement for about 6 seconds at two different values of x.

4.1 Damping Constants

In order to obtain the relation between the frequency and the rate decay, a Fourier analysis was performed with Final Project 98 for each set of data:

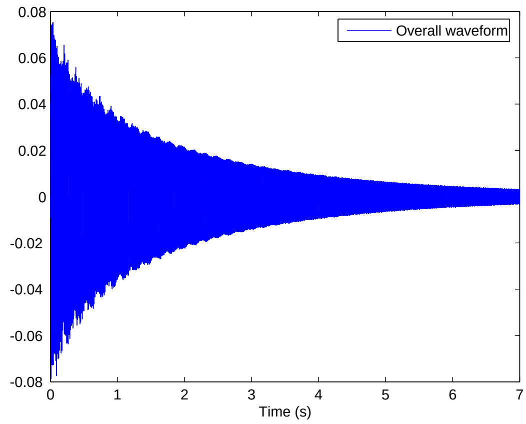

Figure 5: Overall waveform in the case of the pluck and the detector both at the mid-point. The oscillations are damped and the envelope is, as predicted, exponential.

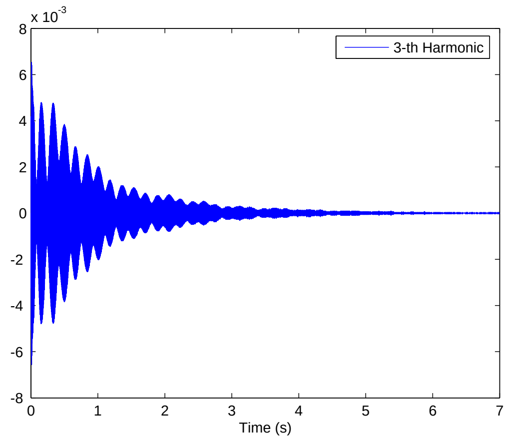

Figure 6: Third harmonics in the case of the pluck and the detector both at the mid-point. The envelope is not as perfect as the previous one, we have oscillations. This may be due to the fact that we selected a range of frequency for a peak, and as there were interferences between these peaks, the whole peak was not selected.

- The frequency spectrum of the waveform was plotted.
- The fundamental and its harmonics were selected. This was a part that induced errors on the results. Indeed, a resonance peak is not a single frequency, but has a certain width. As this width was not too small, there were frequencies that had both the contribution of two successive peaks. We had to choose the peak they belonged to, knowing it was an approximation.
- For each one of these peaks, the Fourier inverse was determined. We obtained different kinds of waveforms (as seen on figures 5 and 6). Even if some of them yielded directly the damping constant (figure 5), others had to be treated first (figure 6).

From each Fourier inverse: Thanks to Matlab, an average of the data was achieved on each interval of $0.2s$. The resulting plot was a decreasing exponential without oscillations. It was then fitted by an exponential function that allowed us to determine the damping constant.

We then try to determine the variations of the damping constant with the position of the pluck, the frequency and the position of the detector.

Pluck	Detector	Overall Damping Constant ($s^{-1} \pm 0.1$)
1/2	1/2	0.41
1/2	2/3	0.36
1/3	1/2	0.39
1/3	2/3	0.37
1/4	1/2	0.39
1/4	2/3	0.36

Figure 7: Overall damping constant according to the place where the string is plucked and the position of the detector. The overall damping constant is approximately constant, an average gives $\alpha = 0.38s^{-1}$.

Given the error, we can say there is no dependence of the damping constant in the positions of the pluck and detector: The first verification we can infer from this table is that the damping does not depend on the coordinate at which the string was plucked.

We see in figures 8, 9 and 10 that the damping constant increases with the frequency, i.e. the highest frequencies are more damped than the lowest ones. It means that the fundamental remains the most important frequency throughout the time. There is a systematic error on all the frequencies given: indeed, the Fourier transform gave values with a precision of $0.08Hz$, which is negligible compared to the other errors.

We can now try to derive a general relation between the damping constant and the frequency. In order to do this, we have to select only some damping constants. We saw at the end of part 2 that when n is a multiple of $1/R$, the

Frequency (Hz)	Damping (s^{-1})
85.46	0.41
171.00	0.84
262.27	0.83
349.70	1.20
440.42	1.20
	1.70

Frequency (Hz)	Damping (s^{-1})
86.13	0.36
172.43	0.67
258.73	0.84
355.63	0.99
441.77	1.00
528.07	1.50

Figure 8: Damping constant for the string plucked at the mid-point. On the left: with the detector at the mid-point. On the right: with the detector moved by $10cm$.

Frequency (Hz)	Damping (s^{-1})
85.88	0.39
171.76	0.77
263.36	0.72
360.46	1.00
442.19	1.00
528.07	1.60

Frequency (Hz)	Damping (s^{-1})
86.22	0.37
172.52	0.59
260.42	0.94
352.35	0.85
441.94	1.00
528.24	1.50

Figure 9: Damping constant for the string plucked at the mid-point. On the left: with the detector at the mid-point. On the right: with the detector moved by $10cm$.

Frequency (Hz)	Damping (s^{-1})
85.80	0.38
171.59	0.76
263.19	0.78
348.99	1.10
441.94	1.10
527.73	1.50

Frequency (Hz)	Damping (s^{-1})
86.30	0.36
172.60	0.63
258.90	0.86
355.89	0.96
442.19	1.00
528.49	1.50

Figure 10: Damping constant for the string plucked at the mid-point. On the left: with the detector at the mid-point. On the right: with the detector moved by $10cm$.

damping constant has no meaning since the harmonics does not exist. Consequently, when the string is plucked at its half for example, the only relevant damping constants are for the first, third and fifth harmonics. This can also be seen with the amplitudes of the waveforms: we indeed found very small amplitudes for the "non-existing" modes compared to those for the "existing" modes.

We used all these relevant values to find the variations with the frequency (figure 11):

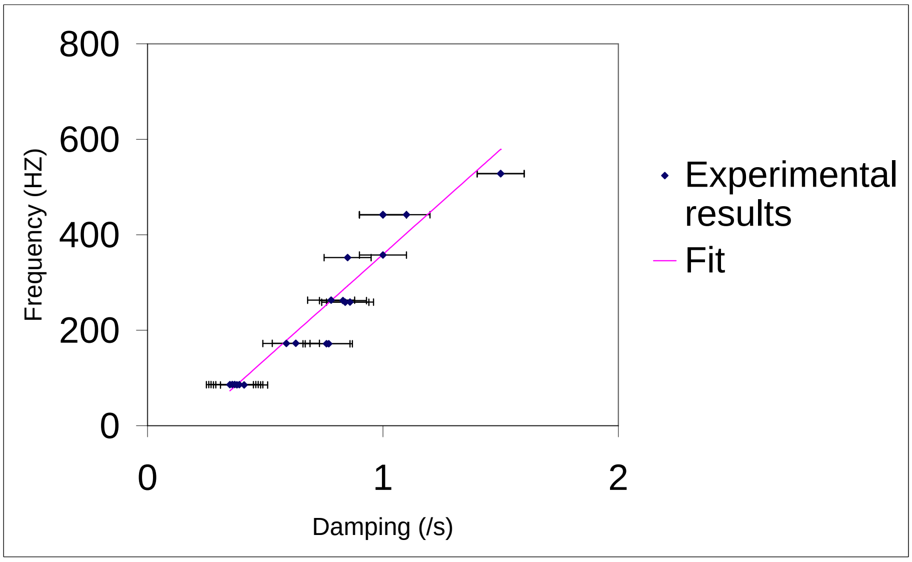

Figure 11: Frequency against the damping. The frequency varies quasi-linearly with the damping: the fit of the data gives $f = 440\alpha - 81Hz$.

4.2 Errors

The errors on the frequencies and damping constants are difficult to estimate, even though their origins are well known. The choices of the limits between two successive peaks in the Fourier transform as said previously induced errors. The fit used to determine the damping constants induced an error since the waveforms were not purely exponential. In fact, it was performed after averaging the amplitudes.
We estimated the error on the damping constants as $\Delta\alpha_n = 0.1$ after comparing the different results of the fit depending on the place where the string was plucked; indeed, the damping should not depend on the plucking coordinate.

5 Conclusion

In the undamped case, we obtained results that confirms the theory: the frequency of the harmonics are multiples of the fundamental one, and the n-th harmonics has (n+1) nodes, as expected. The relation $v = \sqrt{\frac{T}{\mu}}$ is satisfied, even if the value of μ we found is quite far from the one expected.
In the damped case, the model chosen to interpret the vibrating string was not exactly consistent with our results, it can be explained by the presence of extra terms in the equation, such as a fourth-order time-derivative of the vertical displacement. The method used allowed to find a linear relation between the frequency ant the damping thanks to a fit: $f = 440\alpha - 81Hz$. Moreover, by taking a travelling wave as a solution for our differential equation, the damping was given by a more accurate formula:
$\alpha_n = \frac{v_n}{\lambda_{n,attenuation}} = \frac{\Omega_n\sqrt{-1+\sqrt{1+(\frac{\beta_n}{\mu\Omega_n})^2}}}{\sqrt{1+\sqrt{1+(\frac{\beta_n}{\mu\Omega_n})^2}}}$.
We could have then found more precise values for the variations of the coefficient β_n.

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6 Appendix

- In the undamped case, a tension stretched the string. $\overrightarrow{T}$ and $\overrightarrow{T'}$ are applied on the short element of the string.Let $\overrightarrow{T(x)}$ denote the tension in x from the right part on the left part.The total force is thus:

$$\overrightarrow{T} + \overrightarrow{T'} = \overrightarrow{T(x+dx)} - \overrightarrow{T(x)} = \frac{\partial \overrightarrow{T}}{\partial x} dx \tag{17}$$

The two components of $\overrightarrow{T}$ are $Tsin\theta$ along the y-axis and $Tcos\theta$ along the x-axis. By applying the Newton's law and the approximation of the small oscillations, we obtain:

$$\mu dx\frac{\delta^2 y}{\delta t^2} = \frac{\partial Tsin\theta}{\partial x}dx \tag{18}$$

$$0 = \frac{\partial T}{\partial x} \tag{19}$$

$sin\theta \sim tan\theta \Longrightarrow sin\theta \sim \frac{\delta y}{\delta x}$
Therefore:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} \tag{20}$$

We put $v = \sqrt{\frac{T}{\mu}}$
By separating the variables and by using the boundary and initial conditions ($y(0,t) = 0$, $y(l,t) = 0$, $\frac{\partial y}{\partial t} = 0$), we find:

$$y(x,t) = \sum_{n=1}^{\infty} A_n cos(\frac{n\pi vt}{l})sin(\frac{n\pi x}{l}) \tag{21}$$

At $t = 0$, the shape of the string is $f(x)$, hence $y(x,0) = f(x) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{l})$.
A_n is the spatial Fourier coefficient of $f(x)$.
Let's assume $f(x)$ is triangular:
$0 \leq x \leq Rl$:

$$f(x) = \frac{hx}{l} \tag{22}$$

$Rl \leq x \leq l$:

$$f(x) = \frac{h(l-x)}{(1-R)l} \tag{23}$$

The calculation yields:

$$A_n = \frac{2hsin(n\pi R)}{n^2\pi^2 R(1-R)} \tag{24}$$

- In the damped case, the transversal displacement satisfies:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} + \frac{b}{\mu}\frac{\partial y}{\partial t} = 0 \tag{25}$$

The form of the solution is deduced from the above study about the "ideal" case:

$$y(x,t) = \sum_{n=1}^{\infty} g_n(t) sin(\frac{n\pi x}{l}) \tag{26}$$

This expression is placed in the differential equation():

$$g_n^{"} + \frac{\beta}{nb\mu g_n' + (\frac{n\pi v}{l})^2 g_n = 0(27)}$$

The resolution gives :

$$y(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta_n}{2\mu}t}(a_n cos(\Omega_n t) + b_n sin(\Omega_n t)) sin(n\pi x/l) \tag{28}$$

with $\Omega_n = \sqrt{(}(\frac{n\pi v}{l})^2 - (\frac{\beta_n}{2\mu})^2)$

- At $t = 0s$, $y(x,0) = f(x)$ where $f(x)$ is the shape of the string when it is plucked.

$$y(x,0) = \sum_{n=1}^{\infty} a_n sin(\frac{n\pi x}{l}) \tag{29}$$

  It yields $a_n = A_n$
- The initial velocity is null:

$$\frac{\delta y(x,0)}{\delta t} = 0 = \sum_{n=1}^{\infty}(\frac{-bA_n}{2\mu} + b_n\Omega_n) sin(\frac{n\pi x}{l}) \tag{30}$$

$$b_n\Omega_n = \frac{bA_n}{2\mu} \tag{31}$$

The transverse displacement and velocity are thus:

$$y(x,t) = \sum_{n=1}^{\infty} e^{(-\frac{\beta_n t}{2\mu})}(a_n cos(\Omega_n t) + \frac{ba_n}{2\mu\Omega_n} sin(\Omega_n t)) sin(\frac{n\pi x}{l}) \tag{32}$$

$$\frac{\delta y(x,t)}{\delta t} = \sum_{n=1}^{\infty} A_n(-(\frac{b}{\mu\Omega_n})^2 - \Omega_n) e^{-\alpha_n t} sin(\Omega_n t) sin(\frac{n\pi x}{l}) \tag{33}$$

The displacement is maximum when its derivative is null i.e $sin(\Omega_n t) = 0$.By applying this condition in the expression of $y(x,t)$, it gives an amplitude of the form $A_n sin(\frac{n\pi x}{l}) e^{-\alpha t} = A_0 e^{-\alpha t}$ (A_0 is the initial amplitude) for each harmonic.
The function $A_{0n} e^{-\alpha_n t}$ is the envelope of each harmonic.