# Kraft's and McMillan's Inequalities

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**Abstract**

We study the existence of uniquely decodable or instantaneous $r$-ary codes for some given word-lengths. To do this, we prove and discuss the known Kraft and McMillan Inequalities by utilising graph theory. The approach is based on [JJ00].

# Introduction

As uniquely decodable $r$-ary codes and instantaneous codes are important concepts, we want to know under which constraints these exist. We specifically look at $r$-ary codes with given word-lengths, as the inequalities we will later prove relate these concepts. After introducing a certain rooted Tree we show the relation between them and $r$-ary codes, which we use in the proof for Kraft's Inequality. Following the proofs we discuss the implications of these inequalities and give an example. For a quick introduction to our graph-terminology:

Consider a tree, which is defined to be an acyclic, connected and undirected graph. A tree $T = (V, E)$ has the set of vertices $V$ and the set of edges $E$, which we denote by $V(T) := V, E(T) := E$. We call $T$ a rooted tree, if we have a distinct vertex $v \in V(T)$, called the root of $T$ and denoted as $\mathit{root}(T) = v$. Note that we then have a unique path from the root of $T$ to each vertex of $T$. In this case each vertex $v \in V(T)$ has a height, denoted $\mathit{height}(v) = \mathit{height}_T(v)$, defined as the length of that unique path from $\mathit{root}(T)$ to $v$.

For the rest of this paper we will only consider rooted trees, in particular $r$-ary rooted trees of some height $h$, where $r, h \in \mathbb{N}$, which are rooted trees of finite height $h$ where each vertex of height less than $h$ has exactly $r$ children. We introduce subtrees and an ordering of vertices:

**(1.1) Definition** (Subtrees and Ordering)**.**
Let $T, T'$ be rooted Trees. We say $T'$ is a rooted subtree of $T$, iff $T'$ is a subgraph of $T$ and write $T' \leq T$. We write $T' \leq_r T$ iff $r \in \mathbb{N}$ and $T, T'$ are both $r$-ary. Now let $v, w \in V(T)$. We write $v \leq w$ iff the unique path from $\mathit{root}(T)$ to $w$ visits $v$. Let $v \in T \setminus \{\mathit{root}(T)\}$. Set $V_v := \{u \in V(T) \mid v \leq u\}, E_v := \{(u, u') \in E(T) \mid u, u' \in V_v\}$ and let $\mathit{sub}(v) = \mathit{sub}_T(v) := (V_v, E_v)$ be the rooted subtree of $T$ with $v$ as its root. At last we define $T \setminus v := T \setminus \mathit{sub}(v)$ by usual graph difference to be the rooted subtree of $T$ where all vertices and edges in $\mathit{sub}(v)$ (all vertices reachable from $v$, including their associated edges) are deleted from $T$.

**TEXT**

**(1.2) Remark** (Number of vertices of some height in rooted r-ary Trees)**.**
Let $h, r \in \mathbb{N}$ and $T$ be a rooted $r$-ary tree of height $h$. Then $T$ has exactly $r^{h'}$ vertices of height $h' \leq h$.

**(1.3) Corollary** (Number of Leaves of $T \setminus v$)**.**
Let $h, r \in \mathbb{N}$, $T$ be a rooted $r$-ary tree of height $h$, $T' \leq T$ and $v \in V(T') \setminus \{root(T')\}$ such that $sub_{T'}(v) \leq_r T$. If $L$ is the number of leaves of $T'$, then $T' \setminus v$ has $L - r^{h - height_T(v)}$ leaves.

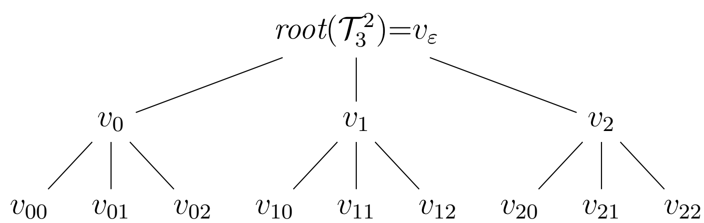
*Proof.* Since $sub_{T'}(v)$ is $r$-ary and has height $h - height_T(v)$, we know $sub_{T'}(v)$ has $r^{h-height_T(v)}$ leaves by (1.2). Thus $T' \setminus v = T' \setminus sub_{T'}(v)$ has $L - r^{h-height_T(v)}$ leaves. $\square$

For us, intervals are over $\mathbb{N}_0$, so for $m, n \in \mathbb{N}_0$ we have $[m, n] := \{p \in \mathbb{N}_0 \mid m \leq p \leq n\}$.
Now we will come to the relationship between $r$-ary codes and $r$-ary rooted trees.

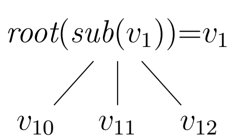
**(1.4) Definition** (r-ary Trees from r-ary Codes)**.**
Let $q, r \in \mathbb{N}, A := [0, r-1]$ be the code-alphabet for some $r$-ary code $\mathcal{C}$ with word-lengths $l \in \mathbb{N}^q$. This choice of the code-alphabet can always be made since any other code-alphabet for $\mathcal{C}$ would stand in bijection to $A$. Set $h := \max\{l_i \mid i \in [1, q]\}$. Define $W := \bigcup_{i \in [0,h]} A^i$ to be the set of all words over $A$ of maximum length $h$. Thus $\mathcal{C} \subseteq W$. We construct a rooted $r$-ary tree $\mathcal{T}_r^h$ of height $h$ indexed by $W$ by setting $root(\mathcal{T}_r^h) := v_\varepsilon$, $V(\mathcal{T}_r^h) := \{v_w \mid w \in W\}$ and $E(\mathcal{T}_r^h) := \{(v_w, v_{w'}) \mid w, w' \in W, wx = w', x \in A\}$.

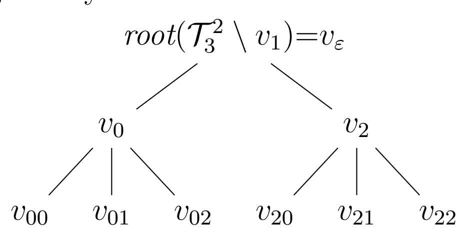
This gives us the correlation between tree-structure and prefix codes we talked about above: $v_w \leq v_{w'} \iff w \sqsubseteq w'$, and $height(v_w) = |w|$ (Without proof). We give an example:

**(1.5) Examples.** Note that $\mathcal{T}_r^h$ is uniquely determined by $r$ and $h$. $\mathcal{T}_3^2$ is given by




We have $height(v_\varepsilon) = 0$ and $height(v_{12}) = 2$. $v_0 \leq v_{02}$ holds, $v_0 \leq v_{10}$ does **not**.
The subtrees $sub(v_1)$ and $\mathcal{T}_3^2 \setminus v_1$ are given by:




$sub(v_1)$ is a 3-ary rooted subtree of height 1, but $\mathcal{T}_3^2 \setminus v_1$ is only a rooted subtree of height 3, not $r$-ary for any $r \in \mathbb{N}$. We now have $height_{sub(v_1)}(v_1) = 0$, $height_{sub(v_1)}(v_{12}) = 1$, but still $height_{\mathcal{T}_3^2 \setminus v_1}(v) = height_{\mathcal{T}_3^2}(v)$ for $v \in V(\mathcal{T}_3^2 \setminus v_1)$.

**(1.6) Theorem** (Kraft's Inequality)**.**
Let $q, r \in \mathbb{N}, l \in \mathbb{N}^q$. Then there is an instantaneous $r$-ary code $\mathcal{C}$ with word-lengths $l$ iff

$$\sum_{k=1}^{q} \frac{1}{r^{l_k}} \leq 1 \tag{1}$$

*Proof.* If $q = 1$, then we always have an instantaneous code, and since $r \in \mathbb{N}$, (1) always holds as well. So assume w.l.o.g. that $q > 1$, $\forall i \in [1, q-1] : 0 < l_i \leq l_{i+1}$ and that the code-alphabet of $\mathcal{C}$ is $[0, r-1]$.

We first show that (1) implies the existence of an $r$-ary prefix-code, which by [JJ00] is instantaneous. Set $h := l_q$ to be the maximum length of the supposed code-words of $\mathcal{C}$. Thus we should have, like in (1.4), that $\mathcal{C} \subseteq \bigcup_{i \in [0,h]} [0, r-1]^i =: W$, where $W$ is in bijection with $V(\mathcal{T}_r^h)$. So we construct the code-words $w_i$ of the prefix-code $\mathcal{C}$, with $|w_i| = l_i$ for $i \in [1, q]$ via finite induction over $i$. The idea is to remove the subtrees rooted at $v_{w_i}$ and then chose $v_{w_{i+1}}$ from the remaining vertices to uphold the prefix property of $\mathcal{C}$, since for all $j \in [1, i]$ we have $l_j \leq l_{i+1}$ and for all $v_w$ with $w_j \sqsubseteq w$ we already removed $v_w$ before chosing $v_{w_{i+1}}$.

Let $i = 1$. Choose a code-word $w_1 \in [0, r-1]^{l_1}$ of length $l_1$. Since $w_1 \in W$ and $l_1 > 0$ we have $v_{w_1} \in V(\mathcal{T}_r^h) \setminus \{R(\mathcal{T}_r^h)\}$. Define $\mathcal{T}_0 := \mathcal{T}_r^h, \mathcal{T}_1 := \mathcal{T}_0 \setminus v_{w_1}$. We know from (1.3) that $\mathcal{T}_1$ has

$$r^h - r^{h - height(v_{w_1})} = r^h - r^{h-l_1} = r^h \left(1 - \sum_{k=1}^{1} \frac{1}{r^{l_k}}\right) > r^h \left(1 - \sum_{k=1}^{q} \frac{1}{r^{l_k}}\right) \overset{(1)}{\geq} 0$$

leaves. Now let $i \in [1, q-1]$ such that $\mathcal{C} := \{w_j \mid j \in [1, i]\}$ is a prefix-code with $|w_j| = l_j$ for $j \in [1, i]$ and such that $\mathcal{T}_i$ is a rooted subtree of $\mathcal{T}_r^h$ and has $r^h(1 - \sum_{k=1}^{i} \frac{1}{r^{l_k}}) > 0$ leaves. Then since $l_{i+1} \leq l_q = h$ we know that there must also be at least one vertex $v_w \in V(\mathcal{T}_i)$ with $height(v_w) = l_{i+1}$ since trees are connected and we have a leaf. So set $w_{i+1} := w$. If we had $w_j \sqsubseteq w_{i+1}$ for some $j \in [1, i]$, then also $v_{w_j} \leq v_{w_{i+1}}$, but then $v_{w_{i+1}} \notin V(\mathcal{T}_{j-1} \setminus v_{w_j}) = V(\mathcal{T}_j) \subseteq V(\mathcal{T}_i)$, a contradiction. Thus $\mathcal{C} := \{w_j \mid j \in [1, i+1]\}$ is still a prefix-code. If $i + 1 = q$ we are done, as we have constructed the desired prefix-code. Otherwise, we set $\mathcal{T}_{i+1} := \mathcal{T}_i \setminus w_{i+1}$ and we get for the number of leaves:

$$r^h \left(1 - \sum_{k=1}^{i} \frac{1}{r^{l_k}}\right) - r^{h-l_{i+1}} = r^h \left(1 - \sum_{k=1}^{i+1} \frac{1}{r^{l_k}}\right) > r^h \left(1 - \sum_{k=1}^{q} \frac{1}{r^{l_k}}\right) \overset{(1)}{\geq} 0$$

Thus we constructed the desired prefix-code $\mathcal{C}$ by finite induction.

Now we show the existence of an instantaneous $r$-ary code $\mathcal{C}$ with word-lengths $l$ implies (1). We know from [JJ00] that $\mathcal{C}$ is a prefix-code. Let $i \in [1, q], w_i \in \mathcal{C}, |w_i| = l_i$ and set

$$L_i := \{v_w \in V(\mathcal{T}_r^h) \mid w_i \sqsubseteq w \wedge |w| = h\} = \{v_w \in sub(v_{w_i}) \mid height_{\mathcal{T}_r^h}(v_w) = h\}$$

to be the set of leaves in $sub(v_{w_i})$. We know from (1.3) that $|L_i| = r^{h-l_i}$ for $i \in [1, q]$. Furthermore we know that for each $i \neq j \in [1, q]$ $L_i \cap L_j = \varnothing$:
Assume $i, j \in [1, q]$ and w.l.o.g. $i < j$. Let $v_w \in L_i \cap L_j$. Thus we get

$$v_{w_i} \leq v_w \wedge v_{w_j} \leq v_w \implies w_i \sqsubseteq w \wedge w_j \sqsubseteq w \overset{i<j}{\implies} w_i \sqsubseteq w_j$$

which is a contradiction to the fact that $\mathcal{C}$ is a prefix-code. So now, since $\mathcal{T}_r^h$ only has $r^h$ leafs, we get what we wanted to show:

$$r^h \geq |\bigcup_{i\in[1,q]} L_i| = \sum_{i=1}^q |L_i| = \sum_{i=1}^q r^{h-l_i} = r^h \sum_{i=1}^q \frac{1}{r^{l_i}} \quad \Longleftrightarrow \quad \sum_{i=1}^q \frac{1}{r^{l_i}} \leq 1$$

□

One could assume, that because being instantaneous implies being uniquely decodable, the constraints for being the latter are weaker. Suprisingly, this is not the case:

**(1.7) Theorem** (McMillan's Inequality)**.**
Let $q, r \in \mathbb{N}, l \in \mathbb{N}^q$. Then there is an uniquely decodable $r$-ary code $\mathcal{C}$ iff

$$\sum_{i=1}^q \frac{1}{r^{l_i}} \leq 1 \tag{1}$$

*Proof.* If we assume (1), then by Kraft's inequality we know that $\mathcal{C}$ is instantaneous, which by [JJ00] implies unique decodability.

Now assume that $\mathcal{C}$ is a uniquely decodable $r$-ary code with word-lengths $l$.
Let $K := \sum_{i=1}^q \frac{1}{r^{l_i}}$ and $n \in \mathbb{N}$. Then we have

$$K^n = \left(\sum_{i=1}^q \frac{1}{r^{l_i}}\right)^n = \sum_{i\in[1,q]^n} \prod_{k=1}^n \frac{1}{r^{l_{i_k}}} = \sum_{i\in[1,q]^n} r^{-\sum_{k=1}^n l_{i_k}} \tag{2}$$

where the $i \in [1,q]^n$ represents $n$ choices of $q$ possible summands (with repitition).

Now there are many different $i \in [1,q]^n$ which have the same sum $\sum_{k=1}^n l_{i_k}$ (consider permutations for example). Set $M := \max\{l_k \mid k \in [1,q]\}, m := \min\{l_k \mid k \in [1,q]\}$. Then we get $mn \leq \sum_{k=1}^n l_{i_k} \leq Mn$ for all $i \in [1,q]^n$ (3). We define for $j \in [mn, Mn], p \in [1,j]$:

$$N_{j,p} := \{w_{i_1} w_{i_2} \cdots w_{i_p} \mid i \in [1,q]^n \wedge |w_{i_1} \cdots w_{i_n}| = j\}$$

So $t \in N_{p,j}$ is a code-sequence of length $j$, consisting of $p$ code-words in $\mathcal{C}$.
But since $\mathcal{C}$ is uniquely decodable, we know that $\forall t \in N_{j,p} : \exists!\ i \in [1,q]^n : t = w_{i_1} \cdots w_{i_n}$, meaning there is one and only one way to construct $t \in N_{p,k}$ from $p$ code-words of $\mathcal{C}$.
This implies that

$$|\{i \in [1,q]^n \mid \sum_{k=1}^n l_{i_k} = j\}| = |\{i \in [1,q]^n \mid \sum_{k=1}^n |w_{i_k}| = j\}| = |N_{j,p}| \tag{4}$$

Furthermore, since $N_{j,p} \subseteq [0, r-1]^j$, we have $|N_{j,p}| \leq r^j$. Thus, from (2), (3), (4) we get

$$K^n = \sum_{j=mn}^{Mn} \frac{|N_{j,n}|}{r^j} \leq \sum_{j=mn}^{Mn} 1 = (l-m)n + 1 \implies \frac{K^n}{n} \leq (M-m) + \frac{1}{n}$$

Now $M, m, K$ are fixed, while $n$ may be arbitrarily large. From Analysis we know that as $n \to \infty$, the only way that $\frac{K^n}{n}$ stays bounded is if $K \leq 1$. Thus we get the desired result:

$$\sum_{i=1}^{q} \frac{1}{r^{l_i}} = K \leq 1$$

$\square$

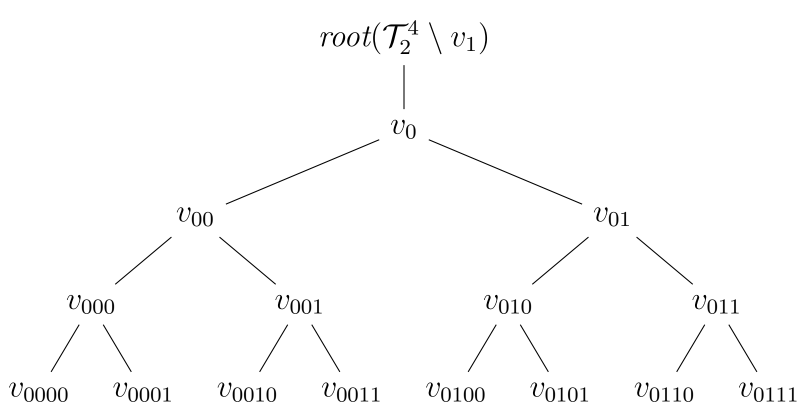
**(1.8) Corollary.**
Let $r, q \in \mathbb{N}, l \in \mathbb{N}^q$. Then by the above inequalities we get that there exists an instantaneous $r$-ary code with word-lengths $l$ iff there exists an uniquely decodable $r$-ary code with word-lenghts $l$.

So now we have necessary conditions for when instantaneous $r$-ary Codes of some given length exist for some $r \in \mathbb{N}$. When searching / constructing a code, one usually wants a it to be instantaneous and have its word lengths and code-alphabet as small as possible. These properties are related through the Inequalites we proved. In particular it is not possible to construct an instantaneous or uniquely decodable $r$-ary Code with arbitrarily small word-lengths for some fixed $r$, neither for an arbitrarily small $r$, given fixed word-lengths;
There exists a lower bound given by these Inequalities.

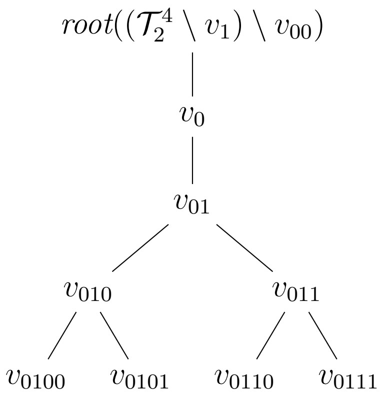
**(1.9) Remark.** Note that we know that if $q, r \in \mathbb{N}, l \in \mathbb{N}^q$ satisfy Kraft's Inequality, there **exists** an instantaneous $r$-ary Code. This does in no way imply that every code with code-words of these lengths is instantaneous. Consider for example $r = 2, q = 3, l = (1, 2, 3)$. Then we have $\sum_{k=1}^{q} \frac{1}{r^{l_k}} = \frac{7}{8} \leq 1$, but the 2-ary code $\{0, 01, 011\}$ is obviously not a prefix-code $\mathcal{C}$ and thus not instantaneous. Similarly, by (1.8) we know that if we have some uniquely decodable code, there **exists** an instantaneous code with the same word lengths, not that $\mathcal{C}$ is instantaneous. For this, consider the code $\{0, 01, 11\}$, which is uniquely decodable, but not instantaneous since $0 \sqsubseteq 01$.

As the proof for Kraft's Inequality is constructive, we conclude with an example of constructing an instantaneous code for given constraints satisfying Kraft's Inequality:

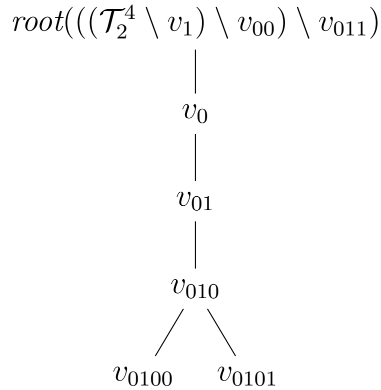
**(1.10) Example.** Let $r = 2, q = 4, l = (1, 2, 3, 4)$, which satisfy the Kraft Inequality. We may chose $w_1 \in [0, r-1]^{l_1} = [0, 1]$ so set $w_1 := 1$. Now consider $\mathcal{T}_2^{\max l} \setminus v_{w_1} = \mathcal{T}_2^4 \setminus v_1$:



Now we chose one of the vertices at the height $l_2 = 2$, lets say $v_{00}$, and thus set $w_2 := 00$. Again consider $(\mathcal{T}_2^4 \setminus v_1) \setminus v_{00}$:




For $l_3 = 3$ we chose $v_{011}$, $w := 011$ and see $((\mathcal{T}_2^4 \setminus v_1) \setminus v_{00}) \setminus v_{011}$ is given by:




For $l_4 = 4$ we have 2 choices, the leaves of $((\mathcal{T}_2^4 \setminus v_1) \setminus v_{00}) \setminus v_{011}$, left and set $w_4 := 0100$. Now we have constructed the $r$-ary prefix-code $\mathcal{C} := \{1, 00, 011, 0100\}$ with word-lengths $l$, which we know is instantaneous.

# References

[JJ00] Gareth A. Jones and J. Mary Jones. *Information and Coding Theory*. 2000.