Kraft's and McMillan's Inequalities

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Abstract

We study the existence of uniquely decodable or instantaneous r-ary codes for some given word-lengths. To do this, we prove and discuss the known Kraft and McMillan Inequalities by utilising graph theory. The approach is based on [JJ00].

Introduction

As uniquely decodable r-ary codes and instantaneous codes are important concepts, we want to know under which constraints these exist. We specifically look at r-ary codes with given word-lengths, as the inequalities we will later prove relate these concepts. After introducing a special kind of tree we show the relation between them and r-ary codes, which we use in the proof for Kraft's Inequality. Following the proofs we discuss the implications of these inequalities and give examples.

Consider a tree, which is defined to be an acyclic, connected and undirected graph. A tree $T = (V, E)$ has the set of vertices V and the set of edges E, which we denote by $V(T) := V, E(T) := E$. We call T a rooted tree, if we have a distinct vertex $v \in V(T)$, called the root of T and denoted as $root(T) = v$, and if we obtain T from a tree by replacing the edges with directed edges, such that we have a unique path from the root of T to each vertex of T. In this case each vertex $v \in V(T)$ has a height, denoted $height(v) = height_T(v)$, defined as the length of that unique path from $root(T)$ to v.

For the rest of this paper we will only consider rooted trees, in particular r-ary rooted trees of some height h, where $r, h \in \mathbb{N}$, which are rooted trees of finite height where each vertex of height less than h has exactly r children. We introduce subtrees and an ordering of vertices:

(1.1) Definition (Subtrees and Ordering).
Let T, T' be rooted Trees. We say T' is a rooted subtree of T, iff T' is a subgraph of T and write $T' \leq T$. We write $T' \leq_r T$ iff $r \in \mathbb{N}$ and T, T' are both r-ary. Now let $v, w \in V(T)$. We write $v \leq w$ iff the unique path from $root(T)$ to w visits v. Let $v \in T \setminus \{root(T)\}$. Set $V_v := \{u \in V(T) \mid v \leq u\}, E_v := \{(u, u') \in E(T) \mid u, u' \in V_v\}$ and let $sub(v) = sub_T(v) := (V_v, E_v)$ be the rooted subtree of T with v as its root. At last we define $T \setminus v := T \setminus sub(v)$ by usual graph difference to be the rooted subtree of T where all vertices and edges in $sub(v)$ are deleted from T.

In the definition for $T \setminus v$ we need $v \neq root(T)$, since otherwise $T \setminus v$ would be empty, and especially not rooted. Note also that if T is a r-ary rooted tree and $v \in V(T) \setminus \{root(T)\}$ we have $sub(v) \leq_r T$, but $(T \setminus v) \not\leq_r T$, although $(T \setminus v) \leq T$. From basic graph theory we get statements about counting vertices, which will be useful later:

(1.2) Remark (Number of vertices of some height in rooted r-ary Trees).

Let $h, r \in \mathbb{N}$ and T be a rooted r-ary tree of height h. Then T has exactly $r^{h'}$ vertices of height $h' \leq h$.

(1.3) Corollary (Number of Leaves of $T \setminus v$).

Let $h, r \in \mathbb{N}$, T be a rooted r-ary tree of height h, $T' \leq T$ and $v \in V(T') \setminus \{root(T')\}$ such that $sub_{T'}(v) \leq_r T$. If L is the number of leaves of T', then $T' \setminus v$ has $L - r^{h-height_T(v)}$ leaves.

Proof. Since $sub_{T'}(v)$ is r-ary and has height $h - height_T(v)$, we know $sub_{T'}(v)$ has $r^{h-height_T(v)}$ leaves by (1.2). Thus $T' \setminus v = T' \setminus sub_{T'}(v)$ has $L - r^{h-height_T(v)}$ leaves. $\qquad\square$

For us, intervals are over $\mathbb{N}_0$, so for $m, n \in \mathbb{N}_0$ we have $[m, n] := \{p \in \mathbb{N} \mid m \leq p \leq n\}$. Now we will come to the relationship between r-ary codes and r-ary rooted trees.

(1.4) Definition (r-ary Trees from r-ary Codes).

Let $q, r \in \mathbb{N}$, $[0, r-1]$ be the code-alphabet for some r-ary code $\mathcal{C}$ with word-lengths $l \in \mathbb{N}^q$. This choice of the code-alphabet can always be made since any other code-alphabet for $\mathcal{C}$ would stand in bijection to $[0, r-1]$. Set $h := \max\{l_i \mid i \in [1, q]\}$. Define $W := \bigcup_{i \in [0,h]} [0, r-1]^i$ to be the set of all words over $[0, r-1]$ of maximum length h. Thus $\mathcal{C} \subseteq W$. We construct a rooted r-ary tree $\mathcal{T}_r^h$ of height h indexed by W by setting $root(\mathcal{T}_r^h) := v_\varepsilon$, $V(\mathcal{T}_r^h) := \{v_w \mid w \in W\}$ and $E(\mathcal{T}_r^h) := \{(v_w, v_{w'}) \mid w, w' \in W, wx = w', x \in [0, r-1]\}$. Notice that with this we have $v_w \leq v_{w'} \iff w \sqsubseteq w'$, and $height(v_w) = |w|$.

(1.5) Examples. Note that $\mathcal{T}_r^h$ is uniquely determined by r and h. $\mathcal{T}_3^2$ is given by
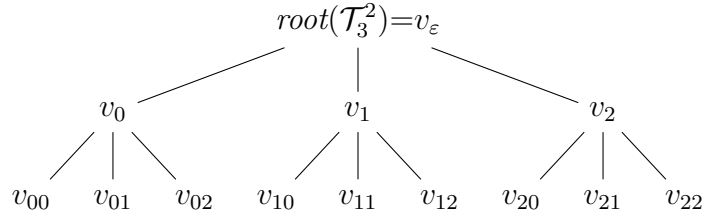
We have $height(v_\varepsilon) = 0$ and $height(v_{12}) = 2$. $v_0 \leq v_{02}$ holds, $v_0 \leq v_{10}$ does **not**. The subtrees $sub(v_1)$ and $\mathcal{T}_3^2 \setminus v_1$ are given by:
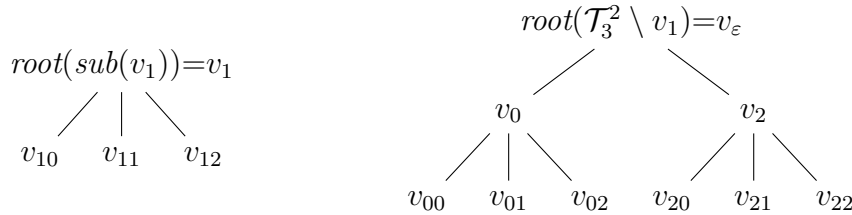
$sub(v_1)$ is a 3-ary rooted subtree of height 1, but $\mathcal{T}_3^2 \setminus v_1$ is only a rooted subtree of height 3, not r-ary for any $r \in \mathbb{N}$. We now have $height_{sub(v_1)}(v_1) = 0, height_{sub(v_1)}(v_{12}) = 1$, but still $height_{\mathcal{T}_3^2 \setminus v_1}(v) = height_{\mathcal{T}_3^2}(v)$ for $v \in V(\mathcal{T}_3^2 \setminus v_1)$.

(1.6) Theorem (Kraft's Inequality).

Let $q, r \in \mathbb{N}, l \in \mathbb{N}^q$. Then there is an instantaneous r-ary code $\mathcal{C}$ with word-lengths l iff

$$\sum_{k=1}^{q} \frac{1}{r^{l_k}} \leq 1 \tag{1}$$

Proof. If $q = 1$, then we always have an instantaneous code, and since $r \in \mathbb{N}$, (1) always holds as well. So assume w.l.o.g. that $q > 1$, $\forall i \in [1, q-1] : 0 < l_i \leq l_{i+1}$ and that the code-alphabet of $\mathcal{C}$ is $[0, r-1]$.

We first show that (1) implies the existence of an r-ary prefix-code, which by [JJ00] is instantaneous. Set $h := l_q$ to be the maximum length of the supposed code-words of $\mathcal{C}$. Thus we should have, like in (1.4), that $\mathcal{C} \subseteq \bigcup_{i \in [0,h]} [0, r-1]^i =: W$, where W is in bijection with $V(\mathcal{T}_r^h)$. So we construct the code-words w_i of the prefix-code $\mathcal{C}$, with $|w_i| = l_i$ for $i \in [1, q]$ via finite induction over i. The idea is to remove the subtrees rooted at v_{w_i} and then chose $v_{w_{i+1}}$ from the remaining vertices to uphold the prefix property of $\mathcal{C}$, since for all $j \in [1, i]$ we have $l_j \leq l_{i+1}$ and for all v_w with $w_j \sqsubseteq w$ we already removed v_w before chosing $v_{w_{i+1}}$.

Let $i = 1$. Choose a code-word $w_1 \in [0, r-1]^{l_1}$ of length l_1. Since $w_1 \in W$ and $l_1 > 0$ we have $v_{w_1} \in V(\mathcal{T}_r^h) \setminus \{R(\mathcal{T}_r^h)\}$. Define $\mathcal{T}_0 := \mathcal{T}_r^h, \mathcal{T}_1 := \mathcal{T}_0 \setminus v_{w_1}$. We know from (1.3) that $\mathcal{T}_1$ has

$$r^h - r^{h-height(v_{w_1})} = r^h - r^{h-l_1} = r^h \left(1 - \sum_{k=1}^{1} \frac{1}{r^{l_k}} \right) > r^h \left(1 - \sum_{k=1}^{q} \frac{1}{r^{l_k}} \right) \overset{(1)}{\geq} 0$$

leaves. Now let $i \in [1, q-1]$ such that $\mathcal{C} := \{w_j \mid j \in [1, i]\}$ is a prefix-code with $|w_j| = l_j$ for $j \in [1, i]$ and such that $\mathcal{T}_i$ is a rooted subtree of $\mathcal{T}_r^h$ and has $r^h(1 - \sum_{k=1}^{i} \frac{1}{r^{l_k}}) > 0$ leaves. Then since $l_{i+1} \leq l_q = h$ we know that there must also be at least one vertex $v_w \in V(\mathcal{T}_i)$ with $height(v_w) = l_{i+1}$ since trees are connected and we have a leaf. So set $w_{i+1} := w$. If we had $w_j \sqsubseteq w_{i+1}$ for some $j \in [1, i]$, then also $v_{w_j} \leq v_{w_{i+1}}$, but then $v_{w_{i+1}} \notin V(\mathcal{T}_{j-1} \setminus v_{w_j}) = V(\mathcal{T}_j) \subseteq V(\mathcal{T}_i)$, a contradiction. Thus $\mathcal{C} := \{w_j \mid j \in [1, i+1]\}$ is still a prefix-code. If $i + 1 = q$ we are done, as we have constructed the desired prefix-code. Otherwise, we set $\mathcal{T}_{i+1} := \mathcal{T}_i \setminus w_{i+1}$ and we get for the number of leaves:

$$r^h \left(1 - \sum_{k=1}^{i} \frac{1}{r^{l_k}} \right) - r^{h-l_{i+1}} = r^h \left(1 - \sum_{k=1}^{i+1} \frac{1}{r^{l_k}} \right) > r^h \left(1 - \sum_{k=1}^{q} \frac{1}{r^{l_k}} \right) \overset{(1)}{\geq} 0$$

Thus we constructed the desired prefix-code $\mathcal{C}$ by finite induction.

Now we show the existence of an instantaneous r-ary code $\mathcal{C}$ with word-lengths l implies (1). We know from [JJ00] that $\mathcal{C}$ is a prefix-code. Let

$$L_i := \{v_w \in V(\mathcal{T}_r^h) \mid w_i \sqsubseteq w \wedge |w| = h\} = \{v_w \in sub(v_{w_i}) \mid height_{\mathcal{T}_r^h}(v_w) = h\}$$

be the set of leaves in $sub(v_{w_i})$, where $w_i \in \mathcal{C}$ with $|w_i| = l_i$ for $i \in [1, q]$. We know from (1.3) that $|L_i| = r^{h-l_i}$ for $i \in [1, q]$. Furthermore we know that for each $i \neq j \in [1, q]$ $L_i \cap L_j = \varnothing$: Assume $i, j \in [1, q]$ and w.l.o.g. $i < j$. Let $v_w \in L_i \cap L_j$. Thus we get

$$v_{w_i} \leq v_w \wedge v_{w_j} \leq v_w \implies w_i \sqsubseteq w \wedge w_j \sqsubseteq w \implies w_i \sqsubseteq w_j$$

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which is a contradiction to the fact that $\mathcal{C}$ is a prefix-code. So now, since $\mathcal{T}_r^h$ only has r^h leafs, we get what we wanted to show:

$$r^h \geq |\bigcup_{i \in [1,q]} L_i| = \sum_{i=1}^q |L_i| = \sum_{i=1}^q r^{h-l_i} = r^h \sum_{i=1}^q \frac{1}{r^{l_i}} \iff \sum_{i=1}^q \frac{1}{r^{l_i}} \leq 1$$

$\square$

One could assume, that because being instantaneous implies being uniquely decodable, the constraints for being the latter are weaker. Suprisingly, this is not the case:

(1.7) Theorem (McMillan's Inequality)**.**
Let $q, r \in \mathbb{N}, l \in \mathbb{N}^q$. Then there is an uniquely decodable r-ary code $\mathcal{C}$ iff

$$\sum_{i=1}^q \frac{1}{r^{l_i}} \leq 1 \tag{1}$$

Proof. If we assume (1), then by Kraft's inequality we know that $\mathcal{C}$ is instantaneous, which by [JJ00] implies unique decodability.

Now assume that $\mathcal{C}$ is a uniquely decodable r-ary code with word-lengths l.
Let $K := \sum_{i=1}^q \frac{1}{r^{l_i}}$ and $n \in \mathbb{N}$. Then we have

$$K^n = \left(\sum_{i=1}^q \frac{1}{r^{l_i}} \right)^n = \sum_{i \in [1,q]^n} \prod_{k=1}^n \frac{1}{r^{l_{i_k}}} = \sum_{i \in [1,q]^n} r^{-\sum_{k=1}^n l_{i_k}} \tag{2}$$

where the $i \in [1,q]^n$ represents n choices of q possible summands (with repitition).

Now there are many different $i \in [1,q]^n$ which have the same sum $\sum_{k=1}^n l_{i_k}$ (consider permutations for example). Set $M := \max\{l_k \mid k \in [1,q]\}, m := \min\{l_k \mid k \in [1,q]\}$. Then we get $mn \leq \sum_{k=1}^n l_{i_k} \leq Mn$ for all $i \in [1,q]^n$ (3). We define for $j \in [mn, Mn], p \in [1, j]$:

$$N_{j,p} := \{w_{i_1} w_{i_2} \cdots w_{i_p} \mid i \in [1,q]^n \wedge |w_{i_1} \cdots w_{i_n}| = j\}$$

So $t \in N_{p,j}$ is a code-sequence of length j, consisting of p code-words in $\mathcal{C}$.
But since $\mathcal{C}$ is uniquely decodable, we know that $\forall t \in N_{j,p} : \exists! \ i \in [1,q]^n : t = w_{i_1} \cdots w_{i_n}$, meaning there is only one way to construct $t \in N_{p,k}$ from p code-words of $\mathcal{C}$.
This implies that

$$|\{i \in [1,q]^n \mid \sum_{k=1}^n l_{i_k} = j\}| = |\{i \in [1,q]^n \mid \sum_{k=1}^n |w_{i_k}| = j\}| = |N_{j,p}| \tag{4}$$

Furthermore, since $N_{j,p} \subseteq [0, r-1]^j$, we have $|N_{j,p}|$. Thus, from (2), (3), (4) we get

$$K^n = \sum_{j=mn}^{Mn} \frac{|N_{j,n}|}{r^j} \leq \sum_{j=mn}^{Mn} 1 = (l-m)n + 1 \implies \frac{K^n}{n} \leq (M-m) + \frac{1}{n}$$

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Now M, m, K are fixed, while n may be arbitrarily large. From Analysis we know that as $n \to \infty$, the only way that $\frac{K^n}{n}$ stays bounded is if $K \leq 1$. Thus we get the desired result:

$$\sum_{i=1}^{q} \frac{1}{r^{l_i}} = K \leq 1$$

$\square$

(1.8) Corollary.

Let $r, q \in \mathbb{N}, l \in \mathbb{N}^q$. Then by the above inequalities we get that there exists an instantaneous r-ary code with word-lengths l iff there exists an uniquely decodable r-ary code with word-lenghts l.

So now we have necessary and sufficient conditions for when r-ary Codes of some given length exist. These Inequalities may be interpreted as the following: One usually wants a code to be instantaneous, have as small word-lengths as possible and a code-alphabet as small as possible. These 3 properties are related through the Inequalites we proved. In particular it is not possible to construct an instantaneous or uniquely decodable r-ary Code with arbitrarily small word-lengths for some fixed r. There exists a lower bound given by these Inequalities.
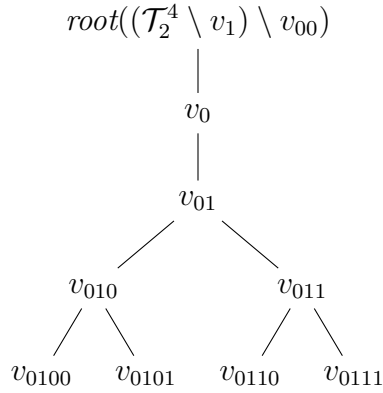
(1.9) Remark. Note that we know that if $q, r \in \mathbb{N}, l \in \mathbb{N}^q$ satisfy Kraft's Inequality, there **exists** an instantaneous r-ary Code. This does in no way imply that every code with codewords of these lengths is instantaneous. Consider for example $r = 2, q = 3, l = (1, 2, 3)$. Then we have $\sum_{k=1}^{q} \frac{1}{r^{l_k}} = \frac{7}{8} \leq 1$, but the 2-ary code $\{0, 01, 011\}$ is obviously not a prefix-code $\mathcal{C}$ and thus not instantaneous. Similarly, by (1.8) we know that if we have some uniquely decodable code, there **exists** an instantaneous code with the same word lengths, not that $\mathcal{C}$ is instantaneous. For this, consider the code $\{0, 01, 11\}$, which is uniquely decodable, but not instantaneous since $0 \sqsubseteq 01$.

As the proof for Kraft's Inequality is constructive, we conclude with an example of constructing an instantaneous code for given constraints satisfying the Inequality:
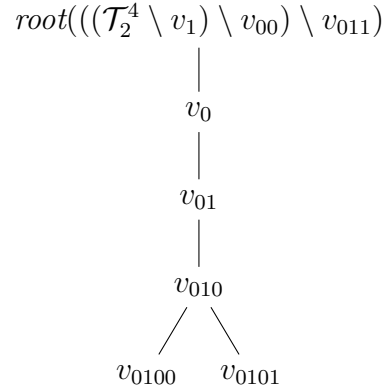
(1.10) Example. Let $r = 2, q = 4, l = (1, 2, 3, 4)$, which satisfy the Kraft Inequality. We may chose $w_1 \in [0, r-1]^{l_1} = [0, 1]$ so set $w_1 := 1$. Now consider $\mathcal{T}_2^{\max l} \setminus v_{w_1} = \mathcal{T}_2^4 \setminus v_1$:

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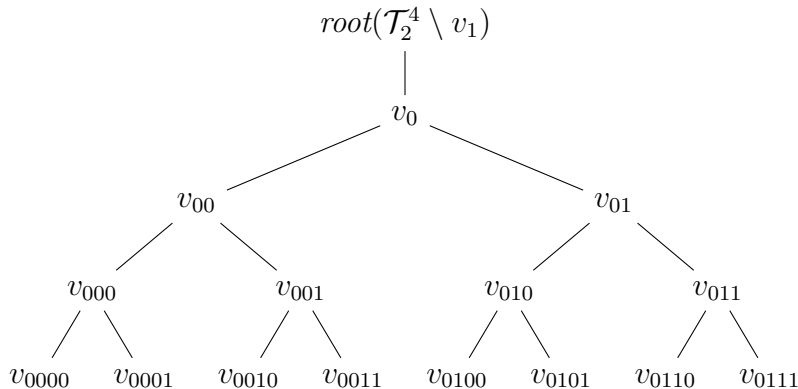
Now we chose one of the vertices at the height $l_2 = 2$, lets say v_{00}, and thus set $w_2 := 00$. Again consider $(\mathcal{T}_2^4 \setminus v_1) \setminus v_{00}$:

$$root((\mathcal{T}_2^4 \setminus v_1) \setminus v_{00})$$

$$v_0$$

$$v_{01}$$

$$v_{010} \qquad v_{011}$$

$$v_{0100} \quad v_{0101} \quad v_{0110} \quad v_{0111}$$

For $l_3 = 3$ we chose v_{011}, $w := 011$ and see $((\mathcal{T}_2^4 \setminus v_1) \setminus v_{00}) \setminus v_{011}$ is given by:

$$root(((\mathcal{T}_2^4 \setminus v_1) \setminus v_{00}) \setminus v_{011})$$

$$v_0$$

$$v_{01}$$

$$v_{010}$$

$$v_{0100} \qquad v_{0101}$$

For $l_4 = 4$ we have 2 choices, the leaves of $((\mathcal{T}_2^4 \setminus v_1) \setminus v_{00}) \setminus v_{011}$, left and set $w_4 := 0100$. Now we have constructed the r-ary prefix-code $\mathcal{C} := \{1, 00, 011, 0100\}$ with word-lengths l, which we know is instantaneous.

References

[JJ00] Gareth A. Jones and J. Mary Jones. *Information and Coding Theory.* 2000.