一.用行波法求解.

$$\begin{cases} u_{tt} - 2u_{xt} - 3u_{xx} = 0, & x \in (-\infty, +\infty),\ t > 0 \\ u|_{t=0} = x^2, & u_t|_{t=0} = 2x \end{cases}$$

解: 求特征方程 $\dfrac{dx}{dt} = \dfrac{-1 \pm \sqrt{(-1)^2+3}}{1} = -1 \pm 2$

$\Rightarrow x = -3t + C_1,\ x = t + C_2.$

$\Rightarrow x + 3t = C_1,\quad x - t = C_2$

令 $\xi = x + 3t,\quad \eta = x - t$

计算 u 关于 ξ, η 的方程.

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_t = u_\xi \xi_t + u_\eta \eta_t = 3u_\xi - u_\eta$

$u_{xt} = 3(u_{\xi\xi} + u_{\xi\eta}) - (u_{\eta\xi} + u_{\eta\eta})$

$\quad = 3u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta}$

$u_{tt} = 9u_{\xi\xi} + u_{\eta\eta} - 6u_{\xi\eta}$

$u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$

代入方程

$\Rightarrow (9u_{\xi\xi} + u_{\eta\eta} - 6u_{\xi\eta}) - 2(3u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta}) - 3(u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}) = 0$

$\Rightarrow -16u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0$

记 $u_\xi = f(\xi, \eta)$. $\Rightarrow \dfrac{\partial f}{\partial \eta}(\xi, \eta) = 0$.

(把 ξ 看成参数) $\Rightarrow f(\xi, \eta) = C(\xi)$

$\Rightarrow \dfrac{\partial u}{\partial \xi} = C(\xi)$. 把 η 看成参数 $\Rightarrow u(\xi, \eta) = \int C(\xi)\,d\xi + G(\eta)$

$\overset{记为}{=} F(\xi) + G(\eta)$

$\Rightarrow u(x,t) = F(x+3t) + G(x-t)$

由条件 $u|_{t=0} = x^2$, $u_t|_{t=0} = 2x$

$$\Rightarrow \left.\begin{aligned} F(x)+G(x) &= x^2 \\ 3F'(x) - G'(x) &= 2x \end{aligned}\right\} \Rightarrow \begin{cases} F(x)+G(x) = x^2 \\ 3F(x) - G(x) = x^2 + C \end{cases}$$

$$\Rightarrow \begin{cases} F(x) = \frac{x^2}{2} + \frac{C}{4} \\ G(x) = \frac{x^2}{2} - \frac{C}{4} \end{cases}$$

$$\Rightarrow u(x,t) = F(x+3t) + G(x-t) = \frac{(x+3t)^2}{2} + \frac{(x-t)^2}{2} \quad \#$$

二、用分离变量法求解

$$\begin{cases} r^2 u_{rr} + r u_r + u_{\theta\theta} = 0, & \theta \in (0,\pi),\ r > 1. \\ u|_{\theta=0} = u|_{\theta=\pi}, & u_\theta|_{\theta=0} = u_\theta|_{\theta=\pi} \\ u|_{r=1} = \cos(4\theta), & u \to 0\ (当\ r \to +\infty\ 时) \end{cases}$$

解: 设 $u(r,\theta) = R(r)\Phi(\theta)$ 代入方程

$\Rightarrow r^2 R''(r)\Phi(\theta) + rR'(r)\Phi(\theta) + R(r)\Phi''(\theta) = 0$

$$\Rightarrow -\frac{r^2 R''(r) + rR'(r)}{R(r)} = \frac{\Phi''(\theta)}{\Phi(\theta)} = -\lambda$$

由边界 $\Rightarrow \Phi(0) = \Phi(\pi)$, $\Phi'(0) = \Phi'(\pi)$

$\Rightarrow$ 特征值问题

$$\begin{cases} \Phi''(\theta)+\lambda\Phi(\theta)=0 \\ \Phi(0)=\Phi(\pi),\ \Phi'(0)=\Phi'(\pi) \end{cases}$$

与 $r^2R''(r)+rR'(r)-\lambda R(r)=0$ ③

解特征值问题，

当 $\lambda<0$ 时，~~通~~通解为

$$\Phi(\theta)=C_1e^{-\sqrt{-\lambda}\theta}+C_2e^{\sqrt{-\lambda}\theta}$$

要使得 $\Phi(0)=\Phi(\pi)$, $\Phi'(0)=\Phi'(\pi)$ $\Rightarrow C_1=C_2=0$

当 $\lambda=0$ 时，通解为

$\Phi(\theta)=C_1+C_2\theta$，由边值条件 $\Rightarrow C_2=0$

$\Rightarrow \Phi(\theta)=C$

当 $\lambda>0$ 时，通解为

$$\Phi(\theta)=C_1\cos\sqrt{\lambda}\theta+C_2\sin\sqrt{\lambda}\theta$$

由边值 $\Rightarrow C_1=C_1\cos\sqrt{\lambda}\pi+C_2\sin\sqrt{\lambda}\pi$

$C_2\sqrt{\lambda}=-C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi+C_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi$

$$\Rightarrow \begin{matrix} C_1=C_1\cos\sqrt{\lambda}\pi+C_2\sin\sqrt{\lambda}\pi \\ C_2=-C_1\sin\sqrt{\lambda}\pi+C_2\cos\sqrt{\lambda}\pi \end{matrix} \Rightarrow \begin{cases} (\cos\sqrt{\lambda}\pi-1)C_1+\sin\sqrt{\lambda}\pi\, C_2=0 \\ -\sin\sqrt{\lambda}\pi\, C_1+(\cos\sqrt{\lambda}\pi-1)C_2=0 \end{cases}$$

要使得~~满足~~ C_1, C_2 不同时为零，则必有

$$\begin{vmatrix} \cos\sqrt{\lambda}\pi-1 & \sin\sqrt{\lambda}\pi \\ -\sin\sqrt{\lambda}\pi & \cos\sqrt{\lambda}\pi-1 \end{vmatrix}=0 \quad \text{即} \ (\cos\sqrt{\lambda}\pi-1)^2+\sin^2\sqrt{\lambda}\pi=0$$

$$\Rightarrow 2-2\cos\sqrt{\lambda}\pi=0$$

$$\Rightarrow \sqrt{\lambda}\pi=2n\pi \Rightarrow \lambda=(2n)^2,\ n=1,2,3,\cdots$$

~~特别地，由 $C_1=C_1\cos\sqrt{\lambda}\pi+C_2\sin\sqrt{\lambda}\pi$~~

$\Rightarrow$ $\Rightarrow$ 特征函数为 $\Phi_n(\theta)=C_n\cos 2n\theta+d_n\sin 2n\theta$, $n=1,2,3,\cdots$

结合 $\lambda=0$ 的情形.

$\Rightarrow$ 特征值为 $\lambda_n=(2n)^2,\ n=0,1,2,3,\cdots$

特征函数为 $\Phi_n(\theta)=C_n\cos(2n\theta)+d_n\sin(2n\theta),\quad n=0,1,2,3\cdots$

解 $r^2R''(r)+rR'(r)-\lambda_nR(r)=0$

即 $r^2R''(r)+rR'(r)-(2n)^2R(r)=0$

令 $r=e^t$, 即 $t=\ln r$.

$\Rightarrow R'(r)=\frac{dR}{dt}\cdot\frac{1}{r}$

$R''(r)=\frac{d^2R}{dt^2}\cdot\frac{1}{r^2}+\frac{-1}{r^2}\frac{dR}{dt}$

$\Rightarrow rR'(r)=\frac{dR}{dt}$

$r^2R''(r)=\frac{d^2R}{dt^2}-\frac{dR}{dt}$

$\Rightarrow \frac{d^2R}{dt^2}-(2n)^2R(t)=0 \Rightarrow R_n(t)=a_ne^{2nt}+b_ne^{-2nt},\quad n=1,2,3\cdots$

$R_0(t)=a_0+b_0t$.

由 $r=e^t$

$\Rightarrow R_n(r)=a_nr^{2n}+b_nr^{-2n},\quad n=1,2,3,\cdots$

$R_0(r)=a_0+b_0\ln r,\quad n=0.$

由于 $\lim\limits_{r\to\infty}u=0 \Rightarrow a_n=0,\quad n=0,1,2,3,\cdots$

$b_0=0.$

$\Rightarrow R_n(r)=b_nr^{-2n},\ n=1,2,3,\cdots$

$R_0(r)=0$

$\Rightarrow u_n(r,\theta)=R_n(r)\Phi_n(\theta)=\left(A_n\cos(2n\theta)+B_n\sin(2n\theta)\right)r^{-2n},\ n=1,2,3,\cdots$

$u_0(r,\theta)=0$

令 $u(r,\theta)=\sum_{n=1}^{\infty}u_n(r,\theta)=\sum_{n=1}^{\infty}\left(A_n\cos(2n\theta)+B_n\sin(2n\theta)\right)r^{-2n}$, (5)

$\Rightarrow$ 由 $u|_{r=1}=\cos(4\theta)$,

$\Rightarrow \sum_{n=1}^{\infty}\left(A_n\cos(2n\theta)+B_n\sin(2n\theta)\right)=\cos(4\theta)$.

$\Rightarrow B_n=0,\ n=1,2,3,\cdots$

$A_2=1$, $A_n=0$, 当 $n\neq2$ 时

$\Rightarrow u(r,\theta)=\cos(4\theta)r^{-2n}$ #

三、求常数 A 使得半无界初边值问题

$$\begin{cases} u_{tt}-u_{xx}=0, & x>0,\ t>0 \\ u_x|_{x=0}=At \\ u|_{t=0}=x^2, & u_t|_{t=0}=0 \end{cases}$$

的解 $u(t,x)$ 在点 $(t,x)=(2,1)$ 处的取值为零.

解: 把边界齐次化.

令 $w(t,x)=Atx$.

令 $U(t,x)=u-w=u-Atx$

$$\Rightarrow \begin{cases} U_{tt}-U_{xx}=0, & x>0 \\ U_x|_{x=0}=0, & t>0 \\ U|_{t=0}=x^2, & U_t|_{t=0}=-Ax. \end{cases}$$

作偶延拓.

令 $W(t,x)=\begin{cases} U(t,x), & x>0 \\ U(t,-x), & x<0, \end{cases}$

$$\Phi(x)=\begin{cases} x^2, & x>0 \\ x^2, & x<0 \end{cases} \qquad \Psi(x)=\begin{cases} -Ax, & x>0 \\ Ax, & x<0 \end{cases}$$

则 W 满足

$$\begin{cases} W_{tt}-W_{xx}=0 & x\in(-\infty,+\infty),\ t>0 \\ W|_{t=0}=\Phi(x), \quad W_t|_{t=0}=\Psi(x), \end{cases}$$

由达朗贝尔公式

$$\Rightarrow W(t,x)=\frac{1}{2}\left(\Phi(x+t)+\Phi(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t}\Psi(\xi)\,d\xi$$

$$=\frac{1}{2}\left((x+t)^2+(x-t)^2\right)+\cancel{\frac{1}{2}\int_0^{x+t}(-A\xi)\,d\xi+\frac{1}{2}\int_{x-t}^{0}(A\xi)}$$

$$+\frac{1}{2}\int_{x-t}^{x+t}\Psi(\xi)\,d\xi$$

$$=(x^2+t^2)+\frac{1}{2}\int_{x-t}^{x+t}\Psi(\xi)\,d\xi$$

当 $x-t>0$ 时.

$$\int_{x-t}^{x+t}\Psi(\xi)\,d\xi=\int_{x-t}^{x+t}-A\xi\,d\xi=-\frac{A}{2}\left((x+t)^2-(x-t)^2\right)$$

$$=-2Axt$$

当 $x-t<0$ 时

$$\int_{x-t}^{x+t}\Psi(\xi)\,d\xi=\int_0^{x+t}-A\xi\,d\xi+\int_{x-t}^{0}A\xi\,d\xi$$

$$=-\frac{A}{2}(x+t)^2-\frac{1}{2}A(x-t)^2=\cancel{-\frac{A}{2}}-A(x^2+t^2)$$

$\Rightarrow$

当 $x>0$，$x-t>0$ 时

$$U(t,x)=W(t,x)=(x^2+t^2)-Axt$$

当 $x>0$，$x-t<0$ 时

$$U(t,x)=W(t,x)=(x^2+t^2)-\frac{A}{2}(x^2+t^2)=\left(1-\frac{A}{2}\right)(x^2+t^2)$$

$$\Rightarrow u(t,x)=U+Atx$$

$$=\begin{cases} x^2+t^2, & x>t, \\ \left(1-\frac{A}{2}\right)(x^2+t^2)+Atx, & x<t \end{cases}$$

此时 $u(2,1)=\left(1-\frac{A}{2}\right)(1+4)+2A$

$$=5-\frac{1}{2}A=0$$

$\Rightarrow A=10$. #

四. 用分离变量法求解：

$$\begin{cases} u_t-u_{xx}+2tu=t, & x\in(0,2),\ t>0 \\ u_x|_{x=0}=u_x|_{x=2}=0 \\ u|_{t=0}=3\cos(2\pi x) \end{cases}$$

解：由于 $v(x,t)=\frac{1}{2}$，满足 $v_t-v_{xx}+2tv=t$

且 $v_x|_{x=0}=v_x|_{x=2}=0$

令 $w(x,t)=u-v$ $\Rightarrow$ $\begin{cases} w_t-w_{xx}+2tw=0, & x\in(0,2),\ t>0 \\ w_x|_{x=0}=w_x|_{x=2}=0 \\ w|_{t=0}=3\cos(2\pi x)-\frac{1}{2}. \end{cases}$

⑧

设 $w = X(x)T(t)$，代入方程

$\Rightarrow T'(t)X(x) - X''(x)T(t) + 2tX(x)T(t) = 0$

$\Rightarrow \dfrac{T'(t)}{T(t)} - \dfrac{X''(x)}{X(x)} + 2t = 0$

$\Rightarrow \dfrac{T'(t)}{T(t)} + 2t = \dfrac{X''(x)}{X(x)} = -\lambda.$

由 $w_x|_{x=0} = w_x|_{x=2} = 0 \Rightarrow X'(0) = X'(2) = 0$

$\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(2) = 0. \end{cases}$ 与 $T'(t) + (2t+\lambda)T(t) = 0$

解特征值问题

由定理 $\Rightarrow \lambda \geq 0$，当 $\lambda = 0$ 时，$X_0(x) = C_0$

当 $\lambda > 0$ 时，~~解~~ 通解 $X(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$\Rightarrow X'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\Rightarrow \begin{cases} C_2\sqrt{\lambda} = 0 \\ -C_1\sqrt{\lambda}\sin\sqrt{\lambda}2 = 0 \end{cases} \Rightarrow 2\sqrt{\lambda} = n\pi \Rightarrow \lambda = \left(\dfrac{n\pi}{2}\right)^2,\ n = 1, 2, 3, \ldots$

$\Rightarrow$ 特征值为 $\lambda_0 = 0,\ \lambda_n = \left(\dfrac{n\pi}{2}\right)^2,\ n = 1, 2, 3, \ldots$

特征函数 $X_0(x) = C_0,\ X_n(x) = C_n\cos\dfrac{n\pi}{2}x.$

解 $T'(t) + (2t+\lambda_n)T(t) = 0 \Leftrightarrow \dfrac{T'(t)}{T(t)} = -(2t+\lambda_n)$

$\Leftrightarrow (\ln T(t))' = -(t^2+\lambda_n t)'$

$\Rightarrow \ln T_n(t) = -(t^2+\lambda_n t) + C$

$\Rightarrow T_n(t) = A_n e^{-(t^2+\lambda_n t)}$，其中 $A_n = e^C$ 常数

$$\Rightarrow W_n(x,t) = X_n(x)T_n(t) = B_n e^{-(t^2+(\frac{n\pi}{2})^2 t)} \cos\frac{n\pi}{2}x, \quad n=0,1,2,3,\cdots$$

考虑 $W(x,t)=\sum_{n=0}^{\infty} W_n(x,t) = \sum_{n=0}^{\infty} B_n e^{-(t^2+(\frac{n\pi}{2})^2 t)} \cos\frac{n\pi}{2}x$

由 $W|_{t=0} = 3\cos(2\pi x) - \frac{1}{2}$

$$\Rightarrow \sum_{n=0}^{\infty} B_n \cos\frac{n\pi}{2}x = 3\cos(2\pi x) - \frac{1}{2}$$

$\Rightarrow B_0 = -\frac{1}{2}$，$B_4 = 3$，$B_n = 0$，当 $n\neq 0, n\neq 4$ 时

$$\Rightarrow W(x,t) = -\frac{1}{2}e^{-t^2} + 3e^{-(t^2+4\pi^2 t)}\cos(2\pi x)$$

$$\Rightarrow u(x,t) = w + v = \frac{1}{2} - \frac{1}{2}e^{-t^2} + 3e^{-(t^2+4\pi^2 t)}\cos(2\pi x).$$

五：(1) 证明

$$\mathcal{F}[xf(x)](\lambda) = i\frac{d}{d\lambda}\mathcal{F}[f(x)](\lambda)$$

$$\mathcal{F}^{-1}[f(x-a)](\lambda) = e^{i\lambda a}\mathcal{F}^{-1}[f(x)](\lambda)$$

(2) 已知 $\mathcal{F}[e^{-x^2}](\lambda) = \sqrt{\pi}e^{-\frac{1}{4}\lambda^2}$. 记 $\mathcal{F}[u(t,x)](\lambda) = v(t,\lambda)$.

设 u 满足

$$(*)\begin{cases} u_{tt} - \frac{1}{t}u_t + t^2x^2u = 0, & x\in(-\infty,+\infty),\ t>1. \\ u|_{t=1} = e^{-x^2}, \quad u_t|_{t=1} = 0 \end{cases}$$

试求 $v(t,\lambda)$ 的方程和条件.

(3) 试求方程 $(*)$ 的解 $u(t,x)$.

解:

(1). 由于 $\mathcal{F}[f(x)](\lambda)=\int_{-\infty}^{+\infty} f(x)e^{-i\lambda x}dx$

$$\Rightarrow \frac{d}{d\lambda}\mathcal{F}[f(x)](\lambda)=\int_{-\infty}^{+\infty}(-ix)f(x)e^{-i\lambda x}dx=-i\int_{-\infty}^{+\infty}xf(x)e^{-i\lambda x}dx$$

$$=-i\mathcal{F}[xf(x)](\lambda)$$

$$\Rightarrow i\frac{d}{d\lambda}\mathcal{F}[f(x)](\lambda)=\mathcal{F}[xf(x)](\lambda)$$

(2)

$$\mathcal{F}^{-1}[f(x-a)](\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(x-a)e^{i\lambda x}dx$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\xi)e^{i\lambda(\xi+a)}d\xi$$

$$=e^{i\lambda a}\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\xi)e^{i\lambda\xi}d\xi=e^{i\lambda a}\mathcal{F}^{-1}[f(x)](\lambda)$$

(2). 由于 (1)中的结论 $\mathcal{F}[x^2u(t,x)](\lambda)=i\frac{d}{d\lambda}\mathcal{F}[xu(t,x)](\lambda)$

$$=i\frac{d}{d\lambda}\left(i\frac{d}{d\lambda}\mathcal{F}[u(t,x)](\lambda)\right)$$

$$=-\frac{d^2}{d\lambda^2}\mathcal{F}[u(t,x)](\lambda).$$

$\Rightarrow v$ 满足 $\cancel{v_{tt}-\frac{1}{t}v_t - t^2 v_{\lambda\lambda}}$

$$\begin{cases} v_{tt}-\frac{1}{t}v_t-t^2v_{\lambda\lambda}=0, & \lambda\in(-\infty,+\infty),\ t>1 \\ v|_{t=1}=\sqrt{\pi}e^{-\frac{1}{4}\lambda^2}, & v_t|_{t=1}=0 \end{cases}$$

(3) 求解 v 的方程（用行波法）

解特征方程. $\frac{d\lambda}{dt}=\frac{\pm\sqrt{t^2}}{1}=\pm t$.

$\Rightarrow \lambda=\frac{1}{2}t^2+c_1 \quad \lambda=-\frac{1}{2}t^2+c_2$

$\Rightarrow \lambda-\frac{1}{2}t^2=c_1, \quad \lambda+\frac{1}{2}t^2=c_2$.

令 $\xi(\lambda,t)=\lambda-\frac{1}{2}t^2, \quad \eta(\lambda,t)=\lambda+\frac{1}{2}t^2$

求出 v 关于 ξ,η 的方程.

$v_\lambda=v_\xi\xi_\lambda+v_\eta\eta_\lambda=v_\xi+v_\eta$

$v_{\lambda\lambda}=v_{\xi\xi}+v_{\eta\eta}+2v_{\xi\eta}$.

$v_t=v_\xi\xi_t+v_\eta\eta_t=-tv_\xi+tv_\eta$

$v_{tt}=-v_\xi+v_\eta+t^2v_{\xi\xi}+t^2v_{\eta\eta}-2t^2v_{\xi\eta}$

$\Rightarrow v_{tt}-\frac{1}{t}v_t-t^2v_{\lambda\lambda}$

$=t^2v_{\eta\eta}+t^2v_{\xi\xi}-2t^2v_{\xi\eta}-v_\xi+v_\eta-v_\eta+v_\xi-t^2(v_{\xi\xi}+v_{\eta\eta}+2v_{\xi\eta})$

$=-4t^2v_{\xi\eta}=0$

$\Rightarrow v_{\xi\eta}=0$

类似行波法（或第一题的做法）$\Rightarrow v(\xi,\eta)=F(\xi)+G(\eta)$

$\Rightarrow v(\lambda,t)=F(\lambda-\frac{1}{2}t^2)+G(\lambda+\frac{1}{2}t^2)$

由条件 $\Rightarrow$ $F(\lambda-\frac{1}{2})+G(\lambda+\frac{1}{2})=\sqrt{\pi}e^{-\frac{1}{4}\lambda^2}$.

$\left[F'(\lambda-\frac{1}{2}t^2)(-t)+G'(\lambda+\frac{1}{2}t^2)t\right]\Big|_{t=1}=0$

$$\Rightarrow \begin{cases} F(\lambda-\frac{1}{2})+G(\lambda+\frac{1}{2})=\sqrt{\pi}\,e^{-\frac{1}{4}\lambda^2} & ① \\ -F'(\lambda-\frac{1}{2})+G'(\lambda+\frac{1}{2})=0 & ② \end{cases}$$

①对λ求导

$$\Rightarrow F'(\lambda-\frac{1}{2})+G'(\lambda+\frac{1}{2})=\sqrt{\pi}\left(-\frac{\lambda}{2}\right)e^{-\frac{1}{4}\lambda^2} \quad ③$$

由②③

$$\Rightarrow G'(\lambda+\frac{1}{2})=-\frac{\sqrt{\pi}\lambda}{4}e^{-\frac{1}{4}\lambda^2}$$

$$F'(\lambda-\frac{1}{2})=-\frac{\sqrt{\pi}}{4}\lambda e^{-\frac{1}{4}\lambda^2}$$

由于

$$\left(G(\lambda+\frac{1}{2})\right)'=G'(x)\Big|_{x=\lambda+\frac{1}{2}}$$

$$\Rightarrow G(\lambda+\frac{1}{2})=\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}\lambda^2}+C_0$$

由①

$$\Rightarrow F(\lambda-\frac{1}{2})=\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}\lambda^2}-C_0$$

$$\Rightarrow G(x)=\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}(x-\frac{1}{2})^2}+C_0,\quad F(x)=\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}(x+\frac{1}{2})^2}-C_0$$

$$\Rightarrow v(t,\lambda)=F(\lambda-\frac{1}{2}t^2)+G(\lambda+\frac{1}{2}t^2)$$

$$=\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}(\lambda-\frac{1}{2}t^2+\frac{1}{2})^2}+\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}(\lambda+\frac{1}{2}t^2-\frac{1}{2})^2}$$

作Fourier逆变换

由(1)中的公式

$$u(t,x)=\mathcal{F}^{-1}[v(t,\lambda)](x)=\frac{e^{i(\frac{1}{2}t^2-\frac{1}{2})x}}{2}\mathcal{F}^{-1}[\sqrt{\pi}e^{-\frac{1}{4}\lambda^2}]+\frac{1}{2}e^{i(-\frac{1}{2}t^2+\frac{1}{2})x}\mathcal{F}^{-1}[\sqrt{\pi}e^{-\frac{1}{4}\lambda^2}]$$

而明 $\mathcal{F}[e^{-x^2}](\lambda)=\sqrt{\pi}e^{-\frac{1}{4}\lambda^2}$

$$\Rightarrow \mathcal{F}^{-1}[\sqrt{\pi}e^{-\frac{1}{4}\lambda^2}](x)=e^{-x^2}$$

$$\Rightarrow u(t,x)=\frac{1}{2}e^{i(\frac{1}{2}t^2-\frac{1}{2})x}e^{-x^2}+\frac{1}{2}e^{i(-\frac{1}{2}t^2+\frac{1}{2})x}e^{-x^2}$$

$$=\frac{1}{2}e^{-x^2}\left(\cos\left(\tfrac{1}{2}t^2-\tfrac{1}{2}\right)x+i\sin\left(\tfrac{1}{2}t^2-\tfrac{1}{2}\right)x\right)$$

$$+\frac{1}{2}e^{-x^2}\left(\cos\left(\tfrac{1}{2}t^2-\tfrac{1}{2}\right)x-i\sin\left(\tfrac{1}{2}t^2-\tfrac{1}{2}\right)x\right)$$

$$=\cancel{\tfrac{1}{2}}\, e^{-x^2}\cos\left(\tfrac{1}{2}t^2-\tfrac{1}{2}\right)x. \quad \#$$