一、利用行波法求解下述问题（15分）

$$\begin{cases} u_{tt}=u_{xx}+4(x+t) \\ u|_{t=x}=0,\quad u|_{t=-x}=3x^2 \end{cases}$$

解：由于 $a=1,\ b=0,\ c=-1$

$\Rightarrow$ 特征方程为 $\frac{dx}{dt}=\frac{b\pm\sqrt{b^2-ac}}{a}=\pm1 \Rightarrow \begin{matrix} x+t=C_1 \\ x-t=C_1 \end{matrix}$

令 $\xi=x+t$
$\eta=x-t$

$\Rightarrow u_t=u_\xi\xi_t+u_\eta\eta_t=u_\xi-u_\eta$

$u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi+u_\eta$

$u_{tt}=u_{\xi\xi}\xi_t+u_{\xi\eta}\eta_t-u_{\eta\xi}\xi_t-u_{\eta\eta}\eta_t=u_{\xi\xi}+u_{\eta\eta}-2u_{\xi\eta}$

$u_{xx}=u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x+u_{\eta\xi}\xi_x+u_{\eta\eta}\eta_x=u_{\xi\xi}+u_{\eta\eta}+2u_{\xi\eta}$

由 $u_{tt}=u_{xx}+4(x+t)$

$\Rightarrow -4u_{\xi\eta}=4\xi \Rightarrow u_{\xi\eta}=-\xi$

$u|_{t=x}=0 \Rightarrow u|_{\eta=0}=0.\quad u|_{t=-x}=3x^2 \Rightarrow u|_{\xi=0}=3\left(\frac{\xi+\eta}{2}\right)^2$

又由 $\frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\eta}\right)=-\xi$，把 η 当成参数（即固定 η）

$\Rightarrow \frac{\partial u}{\partial\eta}=-\frac{1}{2}\xi^2+C(\eta)$

$$\Rightarrow u(\xi,\eta) = -\frac{1}{2}\xi^2\eta + \int c(\eta)d\eta + C_1(\xi).$$

$$= -\frac{1}{2}\xi^2\eta + F(\eta) + G(\xi)$$

由条件 $u|_{\eta=0}=0$, $u|_{\xi=0} = 3\left(\frac{\xi+\eta}{2}\right)^2$

$$\Rightarrow \left.\begin{array}{l} F(0)+G(\xi)=0 \\ F(\eta)+G(0)=\frac{3}{4}\eta^2 \end{array}\right\} \Rightarrow \cancel{F(0)+}$$

$$\Rightarrow F(\eta) = \frac{3}{4}\eta^2 - G(0)$$

$$G(\xi) = G(0)$$

$$\Rightarrow F(\eta)+G(\xi) = \frac{3}{4}\eta^2$$

$$\Rightarrow u(\xi,\eta) = -\frac{1}{2}\xi^2\eta + \frac{3}{4}\eta^2$$

由 $\xi = x+t$, $\eta = x-t$

$$\Rightarrow u(x,t) = -\frac{1}{2}(x+t)^2(x-t) + \frac{3}{4}(x-t)^2$$

二、利用分离变量法求解下述定解问题（15分）

$$\begin{cases} u_t - 4u_{xx} = 0, & x\in(-\pi,\pi),\ t>0 \\ u(x,0) = x, \\ u(-\pi,t) = u(\pi,t),\ u_x(-\pi,t) = u_x(\pi,t) \end{cases}$$

解：设 $u(x,t) = X(x)T(t)$.

代入方程得 $X(x)T'(t) - 4X''(x)T(t) = 0$

$$\Rightarrow \frac{T'(t)}{4T(t)} = \frac{X''(x)}{X(x)} = -\lambda \text{（常数）}$$

由 $u(-\pi,t) = u(\pi,t)$，$u_x(-\pi,t) = u_x(\pi,t)$

$\Rightarrow X(-\pi)T(t) = X(\pi)T(t)$，$X'(-\pi)T(t) = X'(\pi)T(t)$

$\Rightarrow X(-\pi) = X(\pi)$，$X'(-\pi) = X'(\pi)$

$$\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 \quad x \in (-\pi,\pi) \\ X(-\pi) = X(\pi) \\ X'(-\pi) = X'(\pi) \end{cases}$$

及 $T'(t) + 4\lambda T(t) = 0$

求解特征值问题：

当 $\lambda < 0$ 时，通解为 $X(x) = C_1 e^{-\sqrt{-\lambda}x} + C_2 e^{\sqrt{-\lambda}x}$

由 $X(-\pi) = X(\pi)$，$X'(-\pi) = X'(\pi)$ $\Rightarrow$

$$\begin{cases} C_1 e^{-\sqrt{-\lambda}\pi} + C_2 e^{\sqrt{-\lambda}\pi} = C_1 e^{\sqrt{-\lambda}\pi} + C_2 e^{-\sqrt{-\lambda}\pi} \\ -C_1\sqrt{-\lambda} e^{-\sqrt{-\lambda}\pi} + C_2\sqrt{-\lambda} e^{\sqrt{-\lambda}\pi} = -C_1\sqrt{-\lambda} e^{\sqrt{-\lambda}\pi} + C_2\sqrt{-\lambda} e^{-\sqrt{-\lambda}\pi} \end{cases}$$

$$\Rightarrow \begin{cases} C_1\left(e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi}\right) + C_2\left(e^{-\sqrt{-\lambda}\pi} - e^{\sqrt{-\lambda}\pi}\right) = 0 \\ C_1\sqrt{-\lambda}\left(e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi}\right) - C_2\sqrt{-\lambda}\left(e^{-\sqrt{-\lambda}\pi} - e^{\sqrt{-\lambda}\pi}\right) = 0 \end{cases}$$

对于 C_1, C_2 对应的系数 $\Rightarrow$ 只有零解 $C_1 = C_2 = 0$

当 $\lambda = 0$ 时，通解为 $X(x) = C_1 x + C_2$

由条件 $\Rightarrow \begin{cases} C_1(-\pi) + C_2 = C_1(\pi) + C_2 \\ C_2 = C_2 \end{cases} \Rightarrow C_1 = 0.$

$\Rightarrow \lambda=0$ 是特征值，对应的特征函数为

$X_0(x)=C_0$. 解 $T_0'(t) \cancel{+} =0 \Rightarrow T_0(t)=d_0$

当 $\lambda>0$ 时，通解为

$X(x)=C_1\sin\sqrt{\lambda}x + C_2\cos\sqrt{\lambda}x$.

由条件 $\Rightarrow \begin{cases} C_1\sin\sqrt{\lambda}\pi + C_2\cos\sqrt{\lambda}\pi = C_1\sin(-\sqrt{\lambda}\pi)+C_2\cos(-\sqrt{\lambda}\pi) \\ C_1\sqrt{\lambda}\cos\sqrt{\lambda}\pi \cancel{+} C_2\sqrt{\lambda}\sin\sqrt{\lambda}\pi = C_1\sqrt{\lambda}\cos(-\sqrt{\lambda}\pi) \cancel{+} C_2\sqrt{\lambda}\sin(-\sqrt{\lambda}\pi) \end{cases}$

$\Rightarrow \begin{cases} C_1\sin\sqrt{\lambda}\pi = 0 \\ C_2\sqrt{\lambda}\sin\sqrt{\lambda}\pi = 0 \end{cases} \Rightarrow \sin\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}=n,\ n=1,2,3\ldots$

$\Rightarrow \lambda_n = n^2,\ n=1,2,3\ldots$

对应的特征函数为

$X_n(x)=a_n\sin nx + b_n\cos nx,\quad n=1,2,3\ldots$

~~$\Rightarrow U_n(x,t)=X_n(x)T_n(t)$~~

解 $T_n'(t)+4\lambda_n T_n(t)=0 \Rightarrow T_n(t)=d_n e^{-4\lambda_n t} = d_n e^{-4n^2 t}$

$\Rightarrow U_n(x,t)=X_n(x)T_n(t)=A_n e^{-4n^2t}\sin nx + B_n e^{-4n^2t}\cos nx$.

$n=1,2,3,\ldots$

$U_0(x,t)=C_0\cdot d_0 \triangleq B_0$

则设 $U(x,t)=U_0(x,t)+\sum_{n=1}^{\infty}U_n(x,t)$

$=\sum_{n=0}^{\infty}U_n(x,t)=\sum_{n=0}^{\infty}\left(A_n e^{-4n^2t}\sin nx + B_n e^{-4n^2t}\cos nx\right)$

利用 $U(x,0)=x \Rightarrow \sum_{n=0}^{\infty}(A_n\sin nx + B_n\cos nx)=x$

由于 x 在 $(-\pi,\pi)$ 区间是奇函数

$\Rightarrow B_n = 0$

$$\text{又 } A_n = \frac{\int_{-\pi}^{\pi} \sin nx \cdot x\,dx}{\int_{-\pi}^{\pi} \sin^2 nx\,dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,d\left(\frac{\cos nx}{-n}\right)$$

$$= \frac{1}{\pi}\, x\frac{\cos nx}{-n}\Big|_{x=-\pi}^{x=\pi} + \frac{1}{n\pi}\underbrace{\int_{-\pi}^{\pi}\cos nx\,dx}_{0}$$

$$= -\frac{2\cos n\pi}{n} = -\frac{2(-1)^n}{n}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n}\sin nx \quad \#$$

三、利用分离变量法求解下述问题（15分）

$$\begin{cases} u_{xx}+u_{yy}=0, & x\in(0,\pi),\ y\in(0,\pi) \\ u(0,y)=y, & u(\pi,y)=0 \\ u(x,0)=\sin 2x, & u(x,\pi)=0 \end{cases}$$

解：令 $w(x,y)=\frac{(\pi-x)}{\pi}y$ （思路：构造关于 x 的线性函数使得边界齐次化，方程齐次化）

则 $w_{xx}+w_{yy}=0$

且 $w(0,y)=y$，$w(\pi,y)=0$。

令 $v(x,y)=u-w$ 则有

$$\begin{cases} v_{xx}+v_{yy}=0 \\ v(0,y)=0, \quad v(\pi,y)=0 \\ v(x,0)=\sin 2x, \quad v(x,\pi)=x-\pi. \end{cases}$$

设 $v(x,y)=X(x)Y(y)$. 代入方程

$X''(x)Y(y)+Y''(y)X(x)=0$

$\Rightarrow \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda$. (常数)

由于 $v(0,y)=v(\pi,y)=0$

$\Rightarrow X(0)Y(y)=X(\pi)Y(y)=0 \Rightarrow X(0)=X(\pi)=0$

$\Rightarrow \begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X(\pi)=0 \end{cases}$, $Y''(y)-\lambda Y(y)=0$

解特征值问题

当 $\lambda<0$ 时, $X(x)=C_1e^{-\sqrt{-\lambda}x}+C_2e^{\sqrt{-\lambda}x}$

由 $X(0)=X(\pi)=0 \Rightarrow C_1=C_2=0$

当 $\lambda=0$ 时. $X(x)=C_1x+C_2$

由 $X(0)=X(\pi)=0 \Rightarrow C_1=C_2=0$

当 $\lambda>0$ 时, 通解为 $X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$

由 $X(0)=X(\pi)=0 \Rightarrow \begin{cases} C_1=0 \\ C_1\cos\sqrt{\lambda}\pi+C_2\sin\sqrt{\lambda}\pi=0 \end{cases} \Rightarrow \begin{cases} \sin\sqrt{\lambda}\pi=0 \\ C_1=0 \end{cases}$

$\Rightarrow \sqrt{\lambda}=n$, $\Rightarrow \lambda_n=n^2$, $n=1,2,3,\cdots$

$\Rightarrow$ 特征函数 $X_n(x)=C_n\sin nx$.

解 $Y_n''(y)-n^2Y_n(y)=0 \Rightarrow Y_n(y)=a_ne^{ny}+b_ne^{-ny}$

$$\Rightarrow v_n(x,y) = X_n(x)Y_n(y) = A_n e^{ny}\sin nx + B_n e^{-ny}\sin nx.$$

令设 $v(x,y)=\sum_{n=1}^{\infty} v_n(x,y)$ 为方程的解.

由 $v(x,0)=\sin 2x$

$v(x,\pi)=x-\pi$

$$\Rightarrow \sum_{n=1}^{\infty}\left(A_n\sin nx + B_n \sin nx\right)=\sin 2x$$

$$\sum_{n=1}^{\infty}\left(A_n e^{n\pi}\sin nx + B_n e^{-n\pi}\sin nx\right)=x-\pi$$

$$\Rightarrow A_n+B_n = \frac{\int_0^{\pi}\sin 2x\cdot\sin nx\,dx}{\int_0^{\pi}\sin^2 nx\,dx}$$

$$A_n e^{n\pi}+B_n e^{-n\pi} = \frac{\int_0^{\pi}\sin nx\,(x-\pi)\,dx}{\int_0^{\pi}\sin^2 nx\,dx}$$

$$\Rightarrow A_n+B_n=\begin{cases}0, & n\neq 2\text{时}\\ 1, & n=2\text{时}\end{cases} \Rightarrow (A_n+B_n)e^{n\pi}=\begin{cases}0, & n\neq 2\text{时}\\ e^{n\pi}, & n=2\text{时}\end{cases}$$

$$A_n e^{n\pi}+B_n e^{-n\pi} = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{n}(-1)^{n+1}$$

$$\Rightarrow A_n=\begin{cases}\frac{2}{n}(-1)^{n+1}\frac{1}{e^{n\pi}-e^{-n\pi}}, & n\neq 2\text{时}\\ -\frac{e^{-2\pi}+1}{e^{2\pi}-e^{-2\pi}}, & n=2\text{时}\end{cases} \quad B_n=\begin{cases}\frac{2}{n}(-1)^{n}\frac{1}{e^{n\pi}-e^{-n\pi}}, & n\neq 2\text{时}\\ \frac{e^{2\pi}+1}{e^{2\pi}-e^{-2\pi}}, & n=2\text{时}\end{cases}$$

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$\Rightarrow v(x,y)=\sum_{n=1}^{\infty}\left(A_n e^{ny}\sin nx+B_n e^{-ny}\sin nx\right).$

$u(x,y)=v(x,y)+w(x,y)$ #

四、利用分离变量法求解下述混合问题（20分）.

$$\begin{cases} u_{tt}-u_{xx}=\cos 2\pi x, & 0<x<1,\ t>0 \\ u_x(0,t)=0,\ u_x(1,t)=0 \\ u(x,0)=x(x-2),\ u_t(x,0)=0 \end{cases}$$

解：同时把方程和边界齐次化.

解 $\begin{cases} w_{xx}=\cos 2\pi x \\ w_x(0)=0,\ w_x(1)=0. \end{cases}$ $\Rightarrow w(x)=\frac{1}{2\pi}\cos(2\pi x)$

$w=w(x)$

令 $v(x,t)=u(x,t)-w(x)$

$$\Rightarrow \begin{cases} v_{tt}-v_{xx}=0, & 0<x<1,\ t>0 \\ v_x(0,t)=0,\ v_x(1,t)=0 \\ v(x,0)=x(x-2)-\frac{1}{2\pi}\cos 2\pi x,\ v_t(x,0)=0. \end{cases}$$

令 $v(x,t)=X(x)T(t)$. 代入 $v_{tt}-v_{xx}=0$

$\Rightarrow T''(t)X(x)-X''(x)T(t)=0.$

$\Rightarrow \frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda$（常数）

由 $v_x(0,t)=v_x(1,t)=0 \Rightarrow X'(0)T(t)=X'(1)T(t)=0$

$\Rightarrow X'(0)=X'(1)=0$

求解特征值问题

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X'(1)=0, \end{cases}$$

$T''(t)+\lambda T(t)=0$

当 $\lambda<0$ 时. ⇒ 只有零解

当 $\lambda=0$ 时. 通解 $X(x)=C_1x+C_2$.

由 $X'(0)=X'(1)=0 \Rightarrow C_1=0$.

即 $\lambda=0$ 为特征值 且 $X_0(x)=C_0$. 为特征函数

当 $\lambda>0$ 时 通解 $X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$

由 $X'(0)=X'(1)=0$

$$\Rightarrow \begin{cases} C_2\sqrt{\lambda}=0 \\ -C_1\sqrt{\lambda}\sin\sqrt{\lambda}+C_2\sqrt{\lambda}\cos\sqrt{\lambda}=0 \end{cases}$$

$$\Rightarrow \begin{cases} C_2=0 \\ \sin\sqrt{\lambda}=0 \end{cases} \Rightarrow \begin{cases} C_2=0 \\ \lambda=(n\pi)^2, \quad n=1,2,3,\cdots \end{cases}$$

$\Rightarrow X_n(x)=C_n\cos(n\pi x)$. $n=1,2,3,\cdots$

解 $T_n''(t)+(n\pi)^2T_n(t)=0 \Rightarrow T_n(t)=a_n\cos n\pi t+b_n\sin n\pi t$
$n=1,2,3,\cdots$

解 $T_0''(t)=0 \Rightarrow T_0(t)=a_0t+b_0$.

$\Rightarrow V_n(x,t)=X_n(x)T_n(t)=\left(A_n\cos(n\pi t)+B_n\sin(n\pi t)\right)\cos(n\pi x)$, $n=1,2,3,\cdots$

$V_0(x,t)=X_0(x)T_0(t)=B_0t+A_0$

设 $v(x,t)=v_0(x,t)+\sum_{n=1}^{\infty}v_n(x,t)$ 为方程的解

由 $v(x,0)=x(x-2)-\frac{1}{2\pi}\cos 2\pi x$,

$v_t(x,0)=0$

$$\Rightarrow A_0+\sum_{n=1}^{\infty}A_n\cos(n\pi x)=x(x-2)-\frac{1}{2\pi}\cos 2\pi x$$

$$B_0+\sum_{n=1}^{\infty}\left(-B_n(n\pi)\right)\cos(n\pi x)=0$$

$\Rightarrow$ ~~$A_n=0$~~, $B_n=0$

$$A_0=\int_0^1\left(x(x-2)-\frac{1}{2\pi}\cos 2\pi x\right)dx=\int_0^1 x(x-2)\,dx=\left(\frac{1}{3}x^3-x^2\right)\Big|_{x=0}^{x=1}=-\frac{2}{3}$$

$$A_n=\frac{\int_0^1\left(x(x-2)-\frac{1}{2\pi}\cos 2\pi x\right)\cos n\pi x\,dx}{\int_0^1\cos^2 n\pi x\,dx},\quad n=1,2,3,\ldots$$

$=$

$\Rightarrow u(x,t)=v(x,t)+w(x)$ 为方程的解

其中 $v(x,t)=v_0(x,t)+\sum_{n=1}^{\infty}v_n(x,t)$.

$v_0(x,t)=B_0t+A_0$

$v_n(x,t)=\left(A_n\cos(n\pi t)+B_n\sin(n\pi t)\right)\cos n\pi x,\quad n=1,2,3,\ldots$

五.(1) 计算函数 $f(x)=\begin{cases}0 & |x|>1\\ 1-|x|, & |x|\leq 1\end{cases}$ 的 Fourier 变换

(2) 已知函数 e^{-x^2} 的 Fourier 变换是 $\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$，利用 Fourier 变换，求解下述问题 (a, b, c 为常数且 $a>0$)

$$\begin{cases} u_t - a^2 u_{xx} - b u_x - c u = f(x,t), & x\in\mathbb{R},\ t>0 \\ u(x,0)=0 \end{cases}$$

解：(1)：

$$F[f(x)](\lambda) = \int_{-\infty}^{+\infty} f(x) e^{-i\lambda x}\,dx$$

$$= \int_{-1}^{1}(1-|x|)e^{-i\lambda x}\,dx = \int_0^1 (1-x)e^{-i\lambda x}\,dx + \int_{-1}^{0}(1+x)e^{-i\lambda x}\,dx$$

$$= (1-x)\frac{e^{-i\lambda x}}{-i\lambda}\Big|_{x=0}^{x=1} + (1+x)\frac{e^{-i\lambda x}}{-i\lambda}\Big|_{x=-1}^{0}$$

$$+ \int_0^1 -\frac{e^{-i\lambda x}}{i\lambda}\,dx + \int_{-1}^{0}\frac{e^{-i\lambda x}}{i\lambda}\,dx$$

$$= \frac{1}{i\lambda} - \frac{1}{i\lambda} + \frac{e^{-i\lambda x}}{(i\lambda)^2}\Big|_{x=0}^{x=1} - \frac{e^{-i\lambda x}}{(i\lambda)^2}\Big|_{x=-1}^{0}$$

$$= \frac{e^{-i\lambda}}{(i\lambda)^2} + \frac{e^{i\lambda}}{(i\lambda)^2} \cancel{+} - \frac{2}{(i\lambda)^2} = -\frac{2\cos\lambda}{\lambda^2} + \frac{2}{\lambda^2}$$

(2) 记 $\hat{u}(\lambda,t)=F[u(x,t)](\lambda)$，$\hat{f}(\lambda,t)=F[f(x,t)](\lambda)$

对方程两边作 Fourier 变换

$$\Rightarrow \begin{cases} \frac{d}{dt}\hat{u}(\lambda,t) - a^2(i\lambda)^2\hat{u} - b(i\lambda)\hat{u} - c\hat{u} = \hat{f} \\ \hat{u}(\lambda,0)=0 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\hat{u}(\lambda,t) + (a^2\lambda^2 - ib\lambda - c)\hat{u} = \hat{f} \\ \hat{u}(\lambda,0) = 0 \end{cases}$$

由齐次化原理. 只要求

$$\begin{cases} \dfrac{d}{dt}w + (a^2\lambda^2 - ib\lambda - c)w = 0 \\ w|_{t=0} = \hat{f}(\lambda,\tau) \end{cases}$$

$$\Rightarrow w(\lambda,t,\tau) = \hat{f}(\lambda,\tau)\, e^{-(a^2\lambda^2 - ib\lambda - c)t}$$

$$\Rightarrow \hat{u}(\lambda,t) = \int_0^t \hat{f}(\lambda,\tau)\, e^{-(a^2\lambda^2 - ib\lambda - c)(t-\tau)}\, d\tau$$

$$\Rightarrow u(x,t) = \mathcal{F}^{-1}[\hat{u}(\lambda,t)](x) = \int_0^t \mathcal{F}^{-1}\left[\hat{f}(\lambda,\tau)\, e^{-(a^2\lambda^2 - ib\lambda - c)(t-\tau)}\right](x)\, d\tau$$

由于 $\mathcal{F}\left[e^{-(a^2x^2 - ibx - c)(t-\tau)}\right](\lambda)$

$$= e^{c(t-\tau)}\, \mathcal{F}\left[e^{-(a^2x^2 - ibx)(t-\tau)}\right](\lambda)$$

$$= e^{c(t-\tau)}\, \mathcal{F}\left[e^{-a^2(t-\tau)x^2}\right](\lambda - b(t-\tau))$$

$$= e^{c(t-\tau)}\, \frac{\sqrt{\pi}}{\sqrt{a^2(t-\tau)}}\, e^{-\frac{(\lambda - b(t-\tau))^2}{4a^2(t-\tau)}}$$

$$\Rightarrow \mathcal{F}^{-1}\left[e^{-(a^2\lambda^2 - ib\lambda - c)(t-\tau)}\right](x) = \frac{e^{c(t-\tau)}}{2\sqrt{a^2(t-\tau)\pi}}\, e^{-\frac{(x+b(t-\tau))^2}{4a^2(t-\tau)}}$$

$$\Rightarrow \mathcal{F}^{-1}\left[\hat{f}(\lambda,\tau)\, e^{-(a^2\lambda^2 - ib\lambda - c)(t-\tau)}\right](x)\,d\tau$$

$$= \left(f(y,\tau) * \frac{e^{c(t-\tau)}}{2\sqrt{a^2(t-\tau)\pi}}\, e^{-\frac{(y+b(t-\tau))^2}{4a^2(t-\tau)}}\right)(x)$$

$$= \int_{-\infty}^{+\infty} f(\xi,\tau)\, e^{-\frac{(x-\xi+b(t-\tau))^2}{4a^2(t-\tau)}}\, \frac{e^{c(t-\tau)}}{2\sqrt{a^2(t-\tau)\pi}}\, d\xi$$

$$\Rightarrow u(x,t) = \int_0^t \int_{-\infty}^{+\infty} f(\xi,\tau)\, e^{-\frac{(x-\xi+b(t-\tau))^2}{4a^2(t-\tau)}}\, \frac{e^{c(t-\tau)}}{2\sqrt{a^2(t-\tau)\pi}}\, d\xi\, d\tau$$

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六、先化简方程再求解下述半无界问题（15分）

$$\begin{cases} u_{tt} - u_{xx} + 2u_t - 2u_x = 0, & x>0,\ t>0 \\ u(x,0) = e^{-x}\sin x, & u_t(x,0) = 1. \\ u(0,t) = te^{-t} \end{cases}$$

解：~~方程、边界同时齐次化，~~

~~求解 $w=w(t)$ 使得~~ $\begin{cases} \cancel{w_{tt}+2w_t=0} \\ \cancel{w} \end{cases}$

先求整体解 $\tilde{u}$，$x>0$，$t>0$，使得 $\tilde{u}(0,t)=te^{-t}$.

再考虑 $v(x,t) = u-\tilde{u}$，利用延拓法求 v.

下

特征方程 $\frac{dx}{dt} = \pm 1 \Rightarrow \begin{array}{l} x+t = c_1 \\ x-t = c_2. \end{array}$

令 $\xi = x+t$
$\quad \eta = x-t$

$$\Rightarrow u_t = u_\xi \xi_t + u_\eta \eta_t = u_\xi - u_\eta \Rightarrow u_{tt} = u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \qquad u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$$

由 $u_{tt} - u_{xx} + 2(u_t - u_x) = 0$

$$\Rightarrow -4u_{\xi\eta} - 4u_\eta = 0 \Rightarrow u_{\xi\eta} + u_\eta = 0$$

$$\Rightarrow \frac{\partial}{\partial \xi}(u_\eta) + u_\eta = 0 \Rightarrow \frac{\partial}{\partial \xi}(e^{\xi} u_\eta) = 0$$

$$\Rightarrow e^{\xi} u_\eta = C(\eta) \Rightarrow u_\eta = e^{-\xi} C(\eta)$$

$$\Rightarrow u = \int e^{-\xi} C(\eta)\, d\eta + C_1(\xi)$$

$$= e^{-\xi} F(\eta) + G(\xi)$$

$\Rightarrow u(x,t) = e^{-(x+t)} F(x-t) + G(x+t)$

由 $u(0,t) = te^{-t} \Rightarrow e^{-t} F(-t) + G(t) = te^{-t}$

取 $G = 0$. $F(x) = -x$.

$\Rightarrow \tilde{u}(x,t) = e^{-(x+t)}(t-x)$ 满足 $\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} + 2\tilde{u}_t - 2\tilde{u}_x = 0 \\ \tilde{u}|_{x=0} = te^{-t} \end{cases}$

令 $w(x,t) = u(x,t) - \tilde{u}(x,t)$

$$\Rightarrow \begin{cases} w_{tt} - w_{xx} + 2w_t - 2w_x = 0, & x>0,\ t>0 \\ w(x,0) = e^{-x}\sin x + xe^{-x},\ w_t(x,0) = 1 - e^{-x} - xe^{-x} \\ w(0,t) = 0. \end{cases}$$

考虑奇延拓. w_t(奇) $\Phi(x) = \begin{cases} e^{-x}\sin x + xe^{-x}, & x\geq 0 \\ e^{x}\sin x + xe^{x}, & x<0 \end{cases}$

$$\Psi(x) = \begin{cases} 1 - e^{-x} - xe^{-x}, & x\geq 0 \\ -1 + e^{x} - xe^{x}, & x\leq 0 \end{cases}$$

解 $\begin{cases} W_{tt} - W_{xx} + 2W_t - 2W_x = 0, & x\in\mathbb{R},\ t>0 \\ W(x,0) = \Phi(x),\ W_t(x,0) = \Psi(x) \end{cases}$

$$\Rightarrow W(x,t) = e^{-(x+t)}F(x-t) + G(x+t)$$

$$\Rightarrow \left.\begin{aligned} & e^{-x}F(x) + G(x) = \Phi(x) \\ & -e^{-x}F(x) - e^{-x}F'(x) + G'(x) = \Psi(x) \end{aligned}\right\}$$

$$\Phi'(x) - \Psi(x) = 2e^{-x}F'(x) \Rightarrow F'(x) = \frac{1}{2}e^{x}\left(\Phi'(x) - \Psi(x)\right)$$

$$F(x) = \frac{1}{2}\int_0^x e^{y}\left(\Phi'(y) - \Psi(y)\right)dy + C_0$$

$$G(x) = \Phi(x) - e^{-x}F(x) = \Phi(x) - \frac{1}{2}e^{-x}\int_0^x e^{y}\left(\Phi'(y) - \Psi(y)\right)dy - C_0e^{-x}$$

$$\Rightarrow W(x,t) = e^{-(x+t)} F(x-t) + G(x+t)$$

$$= \frac{1}{2} e^{-(x+t)} \int_0^{x-t} e^{y} \left(\Phi'(y) - \Psi(y) \right) dy$$

$$+ C_0 e^{-(x+t)} + \Phi(x+t) - C_0 e^{-(x+t)}$$

$$- \frac{1}{2} e^{-(x+t)} \int_0^{x+t} e^{y} \left(\Phi'(y) - \Psi(y) \right) dy$$

$$= \Phi(x+t) - \frac{1}{2} e^{-(x+t)} \int_{x-t}^{x+t} e^{y} \left(\Phi'(y) - \Psi(y) \right) dy$$

在区域 $x>0, t>0$ 上.

知 $\Phi(x+t) = e^{-(x+t)} \sin(x+t) + (x+t) e^{-(x+t)}$

当 $x-t>0$ 时

$$\int_{x-t}^{x+t} e^{y} \left(\Phi'(y) - \Psi(y) \right) dy = \int_{x-t}^{x+t} e^{y} \Big($$

当 $x-t<0$ 时, ...

$\Rightarrow u(x,t) = W(x,t) + \tilde{u}(x,t).$ #