Lecture 20

Wednesday, April 6, 2022 10:03 AM

Informative Datasets and Persistence of Excitation - Part I

- Reading : Wood & Asada 2007

<u>Last time:</u>

- Considerations of how to tell if a dataset is good enough.
- How accurate is the model?
- Power spectral analysis

<u>Theorem</u>

Let $H(z)$ be a transfer funcn. of BIBO process with a white noise input $e(t)$ w/ variance λ.

bounded input, bounded output.

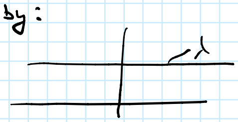
$$\underset{\text{stationary}}{e(t)} \longrightarrow \boxed{H(z)} \longrightarrow v(t)$$

$$E[e(t)\,e(t+\tau)] = \begin{cases} \lambda, & \tau = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$v(t) = H(z)\,e(t).$$

The power spectral density of $v(t)$ is given by:

$$\Phi_v(\omega) = \underset{\text{variance}}{\lambda}\,|H(e^{i\omega})|^2$$



- For a stationary signal, $\{w(t)\}$ with spectrum $\Phi_w(\omega)$

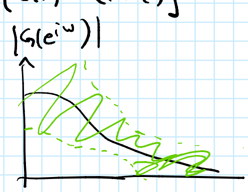
$$w(t) \longrightarrow \boxed{G(z)} \xrightarrow{s(t)} \quad \Rightarrow \quad s(t) = \underbrace{G(z)\,w(t)}$$

$\Rightarrow$ The power spectra are given by:

$$\underset{ss}{\Phi_s}(\omega) = |G(e^{i\omega})|^2\, \underset{ww}{\Phi_w}(\omega) \xleftarrow{\text{Fourier}} R_{ss}(\tau) = E[s(t)\,s(t+\tau)]$$

$$\Phi_{sw}(\omega) = G(e^{i\omega})\,\Phi_{ww}(\omega) \xleftarrow{\text{Fourier}} R_{sw}(\tau) = E[s(t)\,w(t+\tau)]$$

Can Find $G(e^{i\omega})$: $\quad G(e^{i\omega}) = \dfrac{\Phi_{sw}(\omega)}{\Phi_{ww}(\omega)}$



- Applying spectral analysis to System Identification/ parameter estimation.

$$y(t) = G(z)\,u(t) + \underbrace{H(z)\,e(t)}_{v(t)}$$

noise.

$$y(t) = G(z)u(t) + \underbrace{H(z)e(t)}_{v(t)}$$

noise

properties maybe changing w/ time.

__Two issues to address__

1) strictly speaking, the process is __not__ stationary: the input $u(t)$ drives the system.

$$\begin{array}{c} E[u(t)u(t+\tau)] \\ \| \\ R_u(\tau) \end{array} \underset{?}{\overset{\uparrow}{=}} \lim_{N\to\infty} \frac{1}{N} \sum u(t)u(t+\tau)$$

In real system, $R_u(\tau)$ cannot be defined in general.

$\to$ __But__ we need covariance to exist.

$\Rightarrow$ Extend the definition for existence of $R_u(\tau)$

Extension: Quasi-stationary (wide-sense stationary)

$$\Rightarrow R_u(\tau) = E[u(t)u(t+\tau)] \approx \underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{\infty} u(t)u(t+\tau)}_{\text{enables use of ergodicity.}}$$

- under this assumption, signal generated through stable filters $G(z)$

$$s(t) = G(z)e(t)$$

are also considered to be ergodic.

2) Informative datasets.

True system: $y(t) = G(z)u(t) + H(z)e(t)$

__predictor__: $$\hat{y}(t|t-1) = \underbrace{H^{-1}(z)G(z)}_{W_u(z)}u(t) + \underbrace{[1 - H^{-1}(z)]}_{W_y(z)}y(t)$$

$$= \underbrace{\begin{bmatrix} W_u(z) & W_y(z) \end{bmatrix}}_{W(z)} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = W(z)z(t)$$

__Definition 1__: Two models $W_1(z)$ and $W_2(z)$ are equal if frequency funcn.

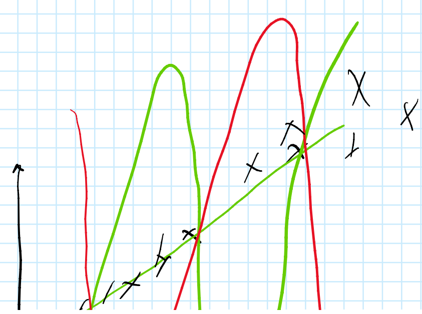
$$W_1(e^{i\omega}) = W_2(e^{i\omega}) \quad \text{for almost all } \omega$$

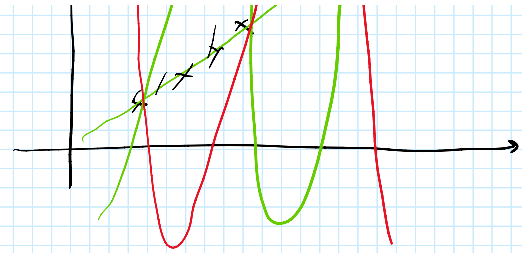
__Definition 2__: A quasi-stationary dataset z^∞ is informative with respect to the model structure, M, if for any two models in M:

$$\hat{y}_1(t, \theta_1) = W_1(z)z(t) \quad \& \quad \hat{y}_2(t, \theta_2) = W_2(z)z(t)$$

then the condition

$$E\left[\left(\hat{y}_1(t|\theta_1) - \hat{y}_2(t, \theta_2)\right)^2\right] = 0$$





$$E\left[\left(\hat{y}_1(t|\theta_1) - \hat{y}_2(t,\theta_2)\right)^2\right] = 0$$

implies

$W_1(e^{i\omega}) = W_2(e^{i\omega})$ for almost all ω.

-> Let's characterize the data:

$$\Phi_{zz}(\omega) = \begin{bmatrix} \Phi_{uu}(\omega) & \Phi_{uy}(\omega) \\ \Phi_{uy}(\omega) & \Phi_{yy}(\omega) \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

Theorem: A quasi-stationary dataset z^∞ is <u>informative</u> if the spectrum matrix $\Phi_{zz}(\omega)$ is strictly positive definite for almost all ω.

<u>proof</u> -> <u>Insert</u>

positive.

<u>Proof</u>:

$$\hat{y}_1(t|\theta_1) - \hat{y}_2(t,\theta_2) = \left[W_1(q) - W_2(q)\right]\cdot z(t) \leftarrow [u(t)\ y(t)]^T$$

$$E\left[\left(\hat{y}_1(t|\theta_1) - \hat{y}_2(t|\theta_2)\right)^2\right] = E\left\{\left[(W_1 - W_2)\, z(t)\right]^2\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[W_1(e^{i\omega}) - W_2(e^{i\omega})\right]^T \underbrace{\Phi_z(\omega)}_{\neq 0 \text{ for any } \omega.} \left[W_1(e^{i\omega}) - W_2(e^{i\omega})\right] d\omega$$

Since $\Phi_z(\omega)$ is strictly positive definite for almost all ω, the above integral becomes zero only when $W_1 - W_2$ is zero for almost all ω.

$\Rightarrow W_1(e^{i\omega}) = W_2(e^{i\omega})$.

- Consistency of prediction error estimate:

$\hat{\theta} \to \theta_0$ as $N \to \infty$

(θ_0: true parameters)

-> Ljung chap 8.

- For linear time invariant model structures based on minimizing squared error
  $\Rightarrow$ <u>estimates are consistent.</u>

<u>Next time</u>: persistence of excitation.