ME8873 – Statistical Model Estimation
Homework Set No. 3
Due March 16th, 2022

Note: Datasets for this assignment are posted under the data locker folder on Canvas. For each problem involving coding, please use either R or MATLAB and attach your commented code at the end of the problem.

Problem 1

Consider a linear time-invariant, single-output model given by

$$y(t) = \frac{bq^{-1}}{1+aq^{-1}}u(t) + \frac{1}{(1+aq^{-1})(1+cq^{-1})}e(t)$$

where $e(t)$ is an uncorrelated random variable, $y(t)$ the output, and $u(t)$ the deterministic input. Answer the following questions.

a). Obtain the one-step-ahead predictor of $y(t)$. Write out predictor $\hat{y}(t\,|\,t-1)$ as a function of $u(t-1)$, $u(t-2),\cdots, y(t-1),\ y(t-2),\cdots$.

b). Assuming that parameter *c* is known: $c = 0.5$, rewrite the predictor in linear regression form: $\hat{y}(t\,|\,\theta) = \varphi^T(t)\cdot\theta + \mu(t)$, where θ is an unknown parameter vector, $\theta = (a, b)^T$. Obtain the regressor vector $\varphi(t)$ and scalar time function $\mu(t)$.

Problem 2

Figure 1 shows the cardiovascular network of a human. A high pressure blood flow generated at the left ventricle is distributed through the arterial network. One of the clinically important measures of cardiac function is Cardiac Output, determined from the aortic blood flow, *u*(*t*), as shown in Figure 3. Currently Cardiac Output is directly measured with a catheter inserted into the heart. Since catheterization is a dangerous and costly procedure, it is desirable to estimate the cardiac output from a peripheral pressure using a non-invasive sensor, as shown in Figure 2. This requires the identification of the arterial dynamics relating the peripheral sensor output *y*(*t*) to the input, i.e. the cardiac output, *u*(*t*). Inverting this transfer function *H*(*q*) yields the cardiac output, $u(t) = H^{-1}(q)\,y(t)$.

Non-invasive estimate of cardiac output $u(t)$ using a peripheral pressure sensor $y(t)$:

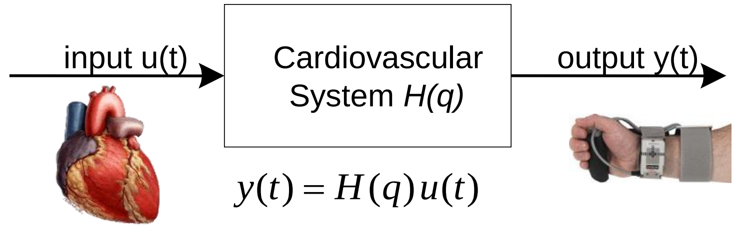

Fig. 2 Input-output

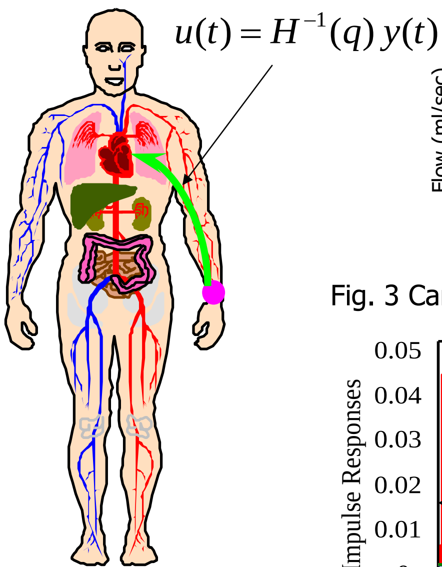

Fig. 1 Cardiovascular network

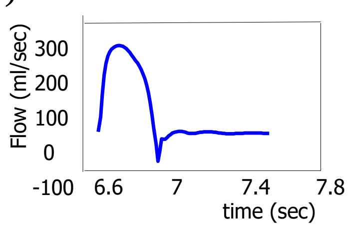

Fig. 3 Cardiac output u(t), aortic flow

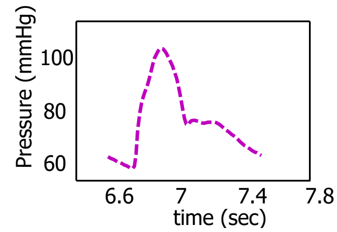

Fig. 4 Radial pressure $y(t)$

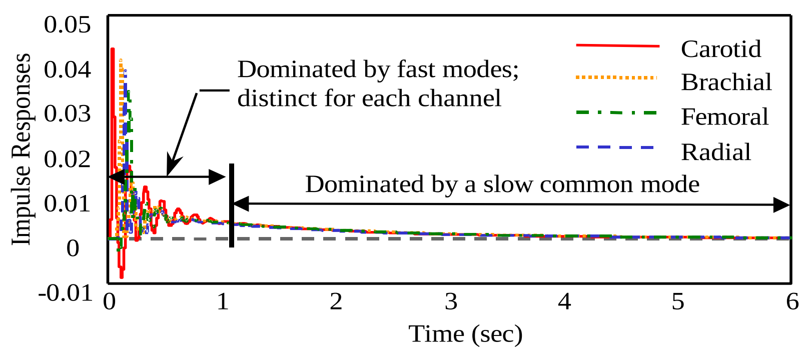

Fig. 5 Impulse response: Aortic flow input to peripheral pressure outputs

Based on experimental input-output data, $\{u(t),\ y(t);\ t = 1, 2, \cdots\}$, this transfer function can be determined. A FIR model is uniquely featured, in that it provides a consistent estimate when determined with a standard LSE algorithm, even though the noise is colored. Unfortunately, however, the impulse response of the arterial dynamics includes a slow decaying mode. Figure 5 shows typical impulse responses from the aortic flow to pressure outputs measured at the carotid, brachial, femoral, and radial arteries. All of the impulse responses show a slow decaying mode. Therefore, an FIR model of the arterial dynamics tends to be of high order, having many parameters to estimate. Answer the following questions.

a). The input data, $u(t)$, are repetitive waveforms, as shown in Figure 3. Therefore the “richness” of the input signal may be limited. Download an input data (data3-2) file from the course data locker and examine the eigenvalues of the matrix, $\Phi\Phi^T$, where

$$\Phi^T = \begin{pmatrix} u(m) & u(m-1) & \cdots & u(1) \\ u(m+1) & u(m) & \cdots & u(2) \\ \vdots & \vdots & \ddots & \vdots \\ u(m+N-1) & \cdots & \cdots & u(N) \end{pmatrix} \in R^{Nxm}$$

Note that m is the order of the FIR model and that N is the number of usable data sets for identification. Discuss the results with respect to the number of non-zero eigenvalues (including almost zero eigenvalues), the rank of the matrix, and the number of FIR model parameters that can be identified with the input data.

b). Use the Laguerre series expansion for compressing the input data and obtain a compact FIR model. You need to find a proper combination of Laguerre pole a and the reduced FIR model order n through trial-and-error LSE computation. Discuss the results with respect to the speed of conversion, i.e. the reduced FIR model order, v.s. the Laguerre pole as well as the accuracy of the reduced FIR model.

Problem 3

An important step in training a radial-basis-function (RBF) network is to determine the center location and the dilation parameter of each radial basis function so that a limited number of RBF functions may effectively approximate a nonlinear map. Shown below is an example of optimal allocation of RBF functions for voice data processing. Twenty RBF functions are placed optimally for covering approximately 300 data points in 2-dimensional input space. The dilation parameter, shown by the radius of each circle, is determined based on the variance of the data classified into the same RBF function.

A similar data set has been uploaded to the course data locker. You are requested to classify these data for the purpose of tuning a RBF network.

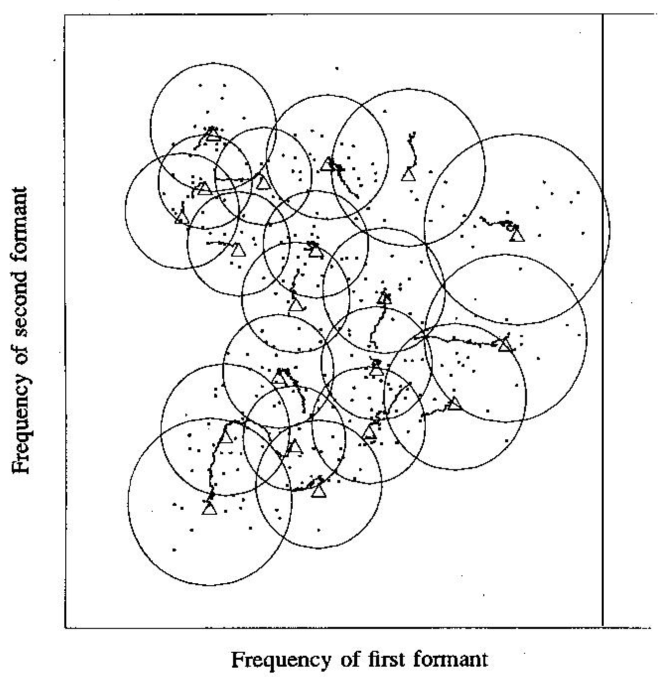

Figure 3 Two-dimensional classification example (vector quantization)
[J. Moody and C. Darken, 1989]

a). Implement the Generalized Lloyd Algorithm discussed in class for classifying N points of 2-dimensional input data into m clusters, i.e. m RBF functions. Download the data (data 3-3 from course locker) and test your program with the data. Set m = 9, create an equally-spaced 3-by-3 grid in the 2-dimensional space, and place the center points of the nine RBF functions initially at those grid points. After optimizing the center locations, compute the dilation parameter for each cluster. Plot the results in the same way as the above example.

b). Using the Least Square Estimate algorithm, obtain the scales of the RBF functions to approximate the downloaded training data.

c). To evaluate the validity of the tuned RBF network, another set of data has been uploaded to the data locker. These data are not used for training the RBF network but are used for evaluating the accuracy of the trained network. Using this data file, evaluate the mean squared error of the RBF network.

d). Repeat Parts a) through c) for m = 25. If time permits, try out a much larger number of RBF functions, say m = 100. Discuss pros and cons of using many RBF functions.