Lecture 19

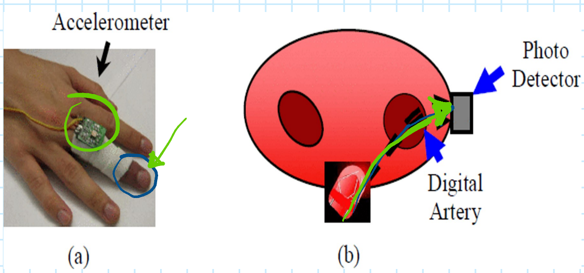
Monday, April 4, 2022 8:29 AM

Admin
1. HW 04 - due Next Monday.
2. Term projects.

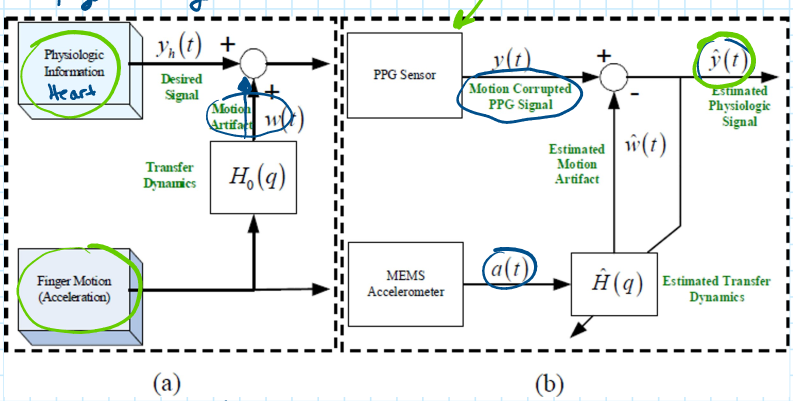
System Identification Theory: Frequency Domain Analysis

- Suggested Reading: Ljung - chap.3, 4
- Question: what are we doing in this class?
  - → Better understanding of a physical system
  - → Quantitative parameterization. ← Assumed some model structure.

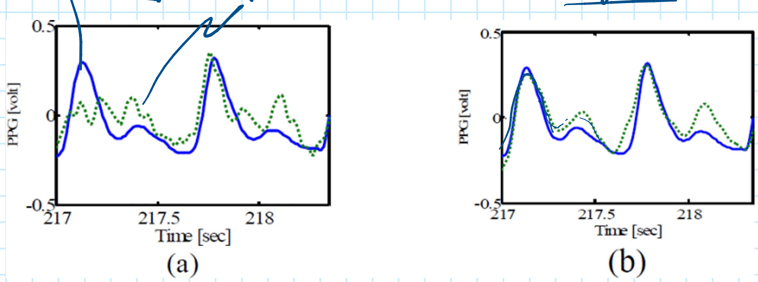
Example:



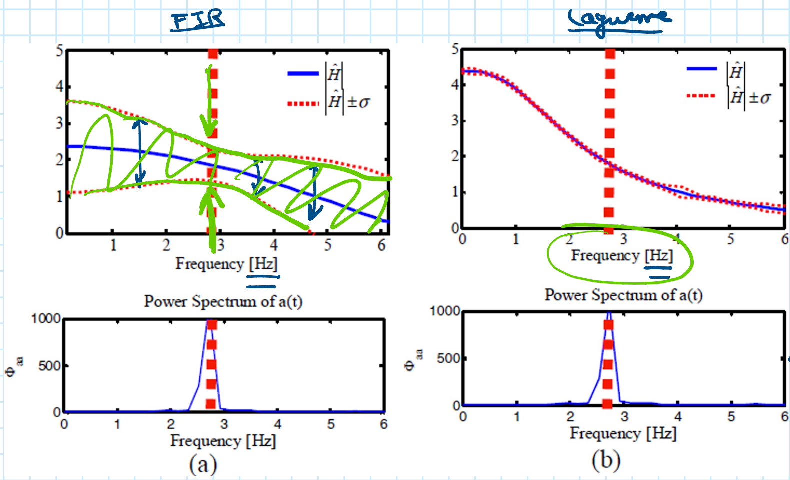
Physical System.



Physiologic signal. FIR Reconstruction Laguerre



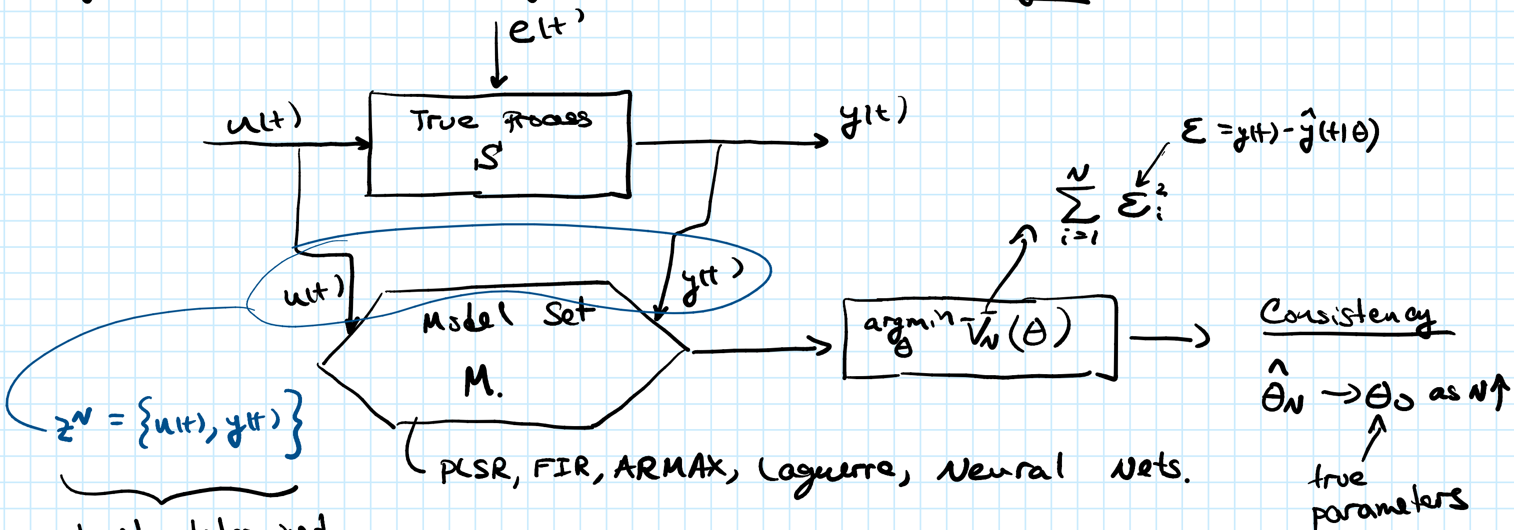
$H_{laguerre}$ is better than H_{FIR}



define confidence in the model.

← Input $\Phi_{aa}(\omega)$

High-level view of System Identification Theory



Dataset determined by experimental design.

Key Questions

Q1: Is a given dataset informative enough to uniquely determin a model ~~from a~~ given model set, M?

→ Does Z^N contain sufficient information to distinguish if one of two models in M is better?

Q2: Is minimization of $V_N(\theta)$ good enough to obtain true (unbiased) model?

→ what if true model is not in model set?

→ how is model fitting influenced by noise characteristics?

Q3: How accurate is the model?

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-> How much variance is expected?
-> How much data is needed?

How do we design experiment

Key Results:

- Informative dataset & persistent excitation condition.
- Consistent (unbiased) estimate.
- signal - to - noise ratio.
- Asymptotic variance.
- Input design: pseudo random binary signal.
- system order estimate: Model selection process.

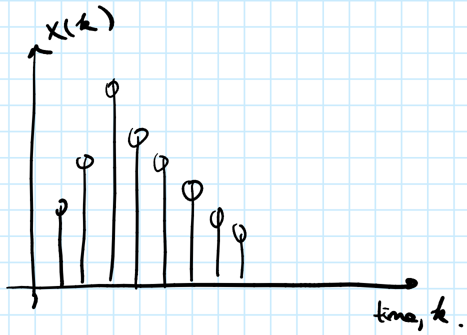
Mathematical Tools

- Discrete Fourier Transform, spectral analysis
- Central limit theorems
- Random processes: stationary process & ergodicity.

Frequency domain analysis

- Discrete Fourier Transform & power spectrum.

Discrete Fourier Transform of a sampled data system: $x(k)$



$$X(\omega) = \sum_{k=-\infty}^{\infty} x(k) e^{-i\omega k}.$$

- Inverse transform:

$$x(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{i\omega k} \, d\omega$$

Power Spectrum

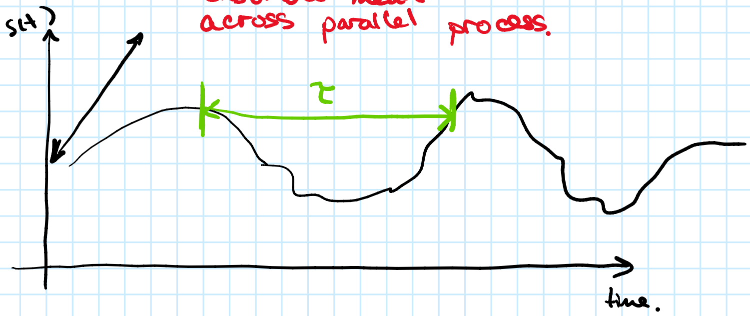
- Consider a deterministic, bounded $\{s(t)\}$ for which

$$R_s(\tau) \approx \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} (s(t)\, s(t+\tau)) \quad \leftarrow \text{covariance funcn.}$$

$$= \underbrace{\boxed{E[s(t)\, s(t+\tau)]}}_{\text{ensemble mean}}$$

if equal, the process is ergodic

$$= \underbrace{E[s(t)\, s(t+\tau)]}_{\text{ensemble mean across parallel process.}}$$



$E[s(t) \cdot s(t+\tau)]$

- The power spectrum of $\{s(t)\}$ is defined as the Fourier transform of auto covariance funcn, $R_s(\tau)$:

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)\, e^{-i\omega\tau}$$

- Inverse

$$R_s(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_s(\omega)\, e^{i\omega\tau}\, d\omega$$

$\sqrt{-1}$ frequency

White (Gaussian Noise)

- We define $\{e(t)\}$ as a sequence of indep. random variables:
  - zero mean.
  - Covariance: $E[e(t)\, e(t+\tau)] = \begin{cases} \lambda, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$

$$R_e(\tau) = \overbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} e(t)\, e(t+\tau)}^{\text{average over time.}} \Longleftrightarrow \overbrace{E[e(t)\, e(t+\tau)]}^{\text{average over ensemble}} = \lambda\,\delta(\tau)$$

dirac delta

same?

must have ergodicity.

- The left side is a time average
- The right side is an ensemble

If same, process is called ergodic.

- We assume ergodicity for most processes.

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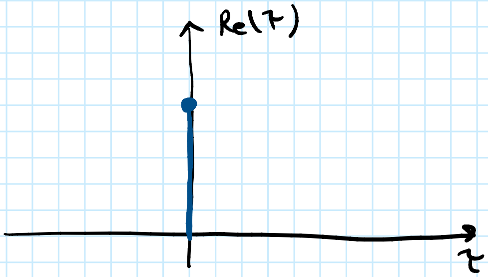
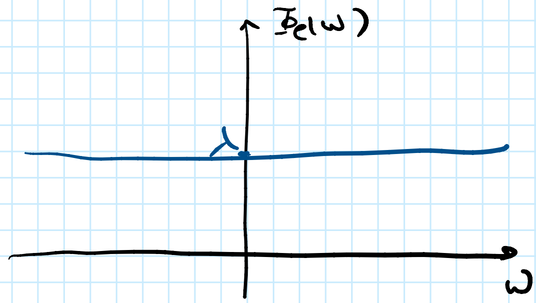
- For the above, power spectrum is given by:

$$\Phi_e(\omega) = \sum_{\tau=-\infty}^{\infty} R_e(\tau)\, e^{-i\tau\omega} = \sum_{\tau=-\infty}^{\infty} \lambda\,\delta(\tau)\, e^{-i\tau\omega} = \lambda = \Phi_e(\omega)$$

<u>Auto-covariance</u>



Power spectrum

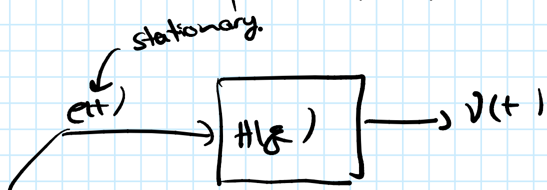


- A correlated (colored) signal can be created by white noise going throug a dynamic process.

<u>Theorem</u>:

bounded input - bounded output.

Let $H(z)$ be a transfer func. of a BIBO process with a white noise input $e(t)$ w/ variance λ:



$$E[e(t)e(t+\tau)] = \begin{cases} \lambda, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$$

$$v(t) = H(z)\, e(t)$$

<u>say</u>

$$H(z) = b_1 z^{-1} + b_2 z^{-2} + \cdots$$

The power spectrum of $v(t)$ is given by:

$$\Phi_v(\omega) = \lambda \left| H(e^{i\omega}) \right|^2$$

<u>Proof.</u>

Proof $v(k)=\sum_{l=0} h(l)e(k-l)$

$=\qquad v(k-\tau)=\sum_{m=0}^{\infty} h(m)e(k-\tau-m)$

combining these two

(15)

$$R_v(\tau)=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} v(k)v(k-\tau)$$

$$=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{l=0}^{\infty} h(l)e(k-l)\sum_{m=0}^{\infty} h(m)e(k-\tau-m)$$

$$R_v(\tau)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} h(l)h(m)\underbrace{\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} e(k-l)e(k-\tau-m)}_{\parallel}$$

$$\lambda\delta(l-\tau-m)=\begin{cases}\lambda: & l=\tau+m\\ 0: & otherwise\end{cases}$$

$$\therefore\quad R_v(\tau)=\lambda\sum_{l=\max(0,\tau)}^{\infty} h(l)h(l-\tau)\quad:\quad h(l)=0\quad for\quad l<0$$

The power spectrum is then given by

(16)

$$\Phi_v(\omega)=\sum_{\tau=-\infty}^{\infty} R_v(\tau)e^{-i\omega\tau}$$

$$=\lambda\sum_{\tau=-\infty}^{\infty}\sum_{l=\max(0,\tau)}^{\infty} h(l)e^{-i\omega l}h(l-\tau)e^{i\omega(l-\tau)}$$

$$\text{Re}\,placing\quad l-\tau\quad by\quad s$$

$$=\lambda\sum_{l=0}^{\infty} h(l)e^{-il\omega}\sum_{s=0}^{\infty} h(s)e^{i\omega s}$$

$$=\lambda H\left(e^{i\omega}\right)\cdot H\left(e^{-i\omega}\right)=\lambda\left|H\left(e^{i\omega}\right)\right|^2$$