

## Lecture 23

Wednesday, April 20, 2022 9:35 AM

### The Art of System Identification: Part I - Experimental Design

#### Admin

- ① HW Ø4 → due.
- ② HW Ø5 → due Friday.
- ③ HW Ø7 → Monday.  
→
- ④ projects.

#### Review of Theories for experimental design

- Key requirements for system identification (parameter estimation)
  - Consistent (unbiased estimate) :  $\hat{\Theta}_N \rightarrow \Theta_0$
  - This depends on system being Ergodic

#### Major Theoretic Results

##### i) Informative datasets

Is dataset  $x$  informative enough to distinguish any two different models,  $w_1(\xi) \in W_1(\xi)$  &  $w_2(\xi) \in W_2(\xi)$ ?

→ from same model set  $M(\theta)$

$$E[(\hat{y}_1(t|\theta_1) - \hat{y}_2(t|\theta_2))^2] = 0 \quad w_1(\xi) \neq w_2(\xi)$$

⇒ The dataset is informative (for any linear time invariant model)

If the spectrum matrix

$$\underline{\underline{S}}_2(\omega) = \begin{bmatrix} \underline{\underline{D}}_{11}(\omega) & \underline{\underline{D}}_{12}(\omega) \\ \underline{\underline{D}}_{21}(\omega) & \underline{\underline{D}}_{22}(\omega) \end{bmatrix}$$

is positive definite for almost all  $\omega$ .

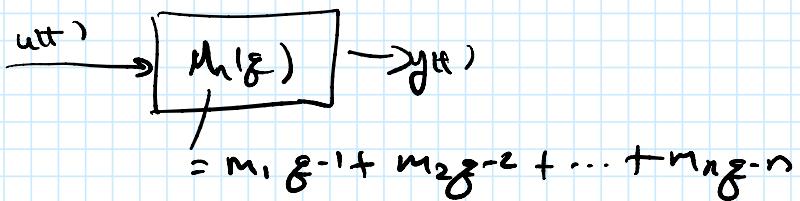
$$u(t) \rightarrow \boxed{M(\theta)} \rightarrow y(t)$$

##### ii) persistence of excitation.

An  $n-1$ st order linear overall filter cannot filter the

(i) persistence of excitation.

Any  $(n-1)^{\text{st}}$  order moving average filter cannot filter the input to zero if the input  $u(t)$  is persistently exciting of order  $n$ .



$\Rightarrow$  persistently exciting of order  $n$  if

$$\bar{R}_W = \begin{bmatrix} R_W(0) & \dots & R_W(n-1) \\ \vdots & & \vdots \\ R_W(n-1) & & R_W(n) \end{bmatrix}_{n \times n}$$

auto-covariance

is non-singular.

$$\Rightarrow |\Delta M(e^{i\omega})|^2 \bar{R}_W(\omega) = 0 \stackrel{\text{implies}}{\Rightarrow} \Delta M(e^{i\omega}) = 0.$$

- can extend.

$$G(z) = \frac{B(z)}{F(z)} = \frac{z^{-n_B}(b_1 + b_2 z^{-1} + \dots + b_{n_B} z^{-n_B+1})}{1 + f_1 z^{-1} + \dots + f_{n_F} z^{-n_F}}$$

$\Rightarrow$  persistence of excitation if  
 $\bar{R}_W$  is non-singular.

$$\Rightarrow \Delta G(z) = \frac{B_1}{F_1} - \frac{B_2}{F_2} = \frac{B_1 F_2 - B_2 F_1}{F_1 F_2}$$

$$\Rightarrow |B_1(e^{i\omega})F_2(e^{i\omega}) - B_2(e^{i\omega})F_1(e^{i\omega})|^2 \bar{R}_W(\omega) = 0$$

$$\Rightarrow \text{Implies } \Delta G(z) = 0.$$

$\Rightarrow$  The sequence  $u(t)$  that is persistently exciting of order  $n_B + n_F$  is informative enough to distinguish two transfer

$\Rightarrow$  the sequence  $w(t)$  that is persistently exciting of order  $n_H + n_F$  is informative enough to distinguish two transfer funs. of above form.

### iii) Signal to noise ratio.

The prediction error method determines the parameter vector  $\theta$  such that a frequency-weighted squared error is minimized w/ fixed noise model

$$H(g, \theta) = H(g)$$

then

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{-\pi}^{\pi} \left| \underbrace{G_0(e^{j\omega})}_{\text{true system}} - \underbrace{G(e^{j\omega})}_{\text{model}} \right|^2 \cdot \underbrace{\frac{E_{\text{out}}(\omega)}{|H(e^{j\omega})|^2} \lambda}_{\text{Signal-to-noise ratio.}} d\omega.$$

If  $H = I$

$$\operatorname{var}(\hat{g}(e^{j\omega})) \sim \frac{1}{N} \underbrace{\frac{\cancel{S_{\text{out}}(\omega)}}{E_{\text{out}}(\omega)}}_{\# \text{ of parameters}}$$

$\cancel{S_{\text{out}}(\omega)}$

# of data points

### iv) Asymptotic variance

consider the prediction error model with

$$V_N(\theta, z^N) = \frac{1}{N} \sum_{t=1}^N \frac{1}{2} \varepsilon^2(t, \theta)$$

If the true model is involved in the model set:  $\theta_0 \in M$ , variance in parameter estimation error approach the following asymptotic result:

$$\operatorname{Cov}(\hat{\theta}_N) \sim \frac{\lambda_0}{N} \underbrace{\left[ E[\gamma(t, \theta_0) \gamma^T(t, \theta_0)] \right]^{-1}}_{\text{as } N \rightarrow \infty}$$

- depends on

- $N$ : number of data,

- $\lambda_0$ : noise level

- sensitivity:  $\gamma(t, \theta) = \frac{1}{\partial \theta} \tilde{y}(t + \theta) \Big|_{\theta=0}$

covariance matrix

If  $\tilde{y}(t, \theta) = \varphi^T(t) \theta$ ,  
the  $E[\cdot]$  = covariance matrix.