Lecture 13

Wednesday, February 23, 2022 8:57 AM

Modeling of Linear Dynamical Systems - Part I

Admin

① OH → tomorrow 3–4pm
↳ email me if you need a different time.

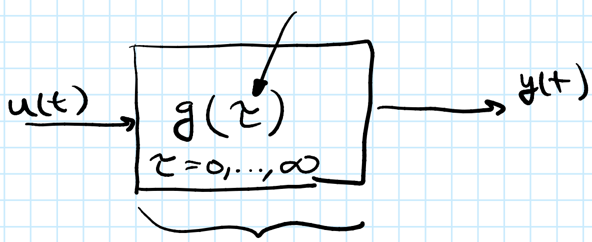
② Graded HW01 back tomorrow.

Objectives

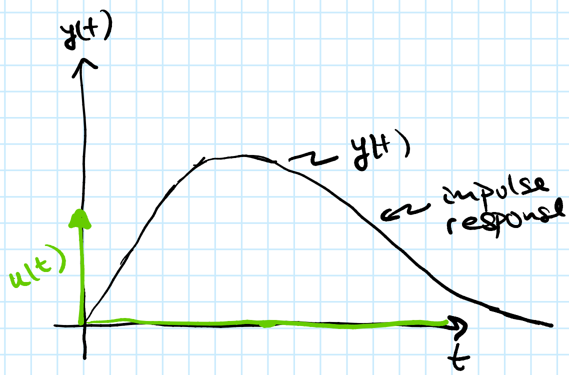
① Define "transfer funcn." modeling for linear dynamical systems
② Define "black-box" modeling structures for dynamic systems

Relevent Reading: Ljung chaps. 2–4.

- Consider a linear time invariant system.



Impulse Response completely characterizes the system.



Continuous time convolution

$$y(t) = \int_0^{\infty} g(\tau)\underbrace{u(t-\tau)}_{\text{Impulse}}\, d\tau$$

$$u(t) = \begin{cases} \delta & @\ t=0 \\ 0 & \text{otherwise.} \end{cases}$$

- Given Impulse response $\{g(\tau) \mid \tau = 0, \dots, \infty\}$ and input $\{u(s) \mid s \le t\}$
$\Rightarrow$ can define output $\{y(s) \mid s \le t\}$

- In discrete time

$$y(t) = \sum_{k=1}^{\infty} g(k)\, u(t-k)$$

- We frequently describe the time shift using an algebraic time shift operator: q

$$q\, u(t) = u(t+1)$$

→ Backward shift operator : q^{-1}

$q\, u(t) = u(t+1)$

$\to$ Backward shift operator : q^{-1}

$q^{-1} u(t) = u(t-1)$

$$\Rightarrow y(t) = \sum_{k=1}^{\infty} g(k)\left[q^{-k} u(t)\right]$$

$$= \underbrace{\left\{ \sum_{k=1}^{\infty} \left[g(k)\, q^{-k} \right] \right\}}_{\text{"Transfer funcn.": } G(q)} u(t)$$

$\Rightarrow \quad y(t) = G(q)\, u(t)$

<u>Example</u> : Finding a transfer func:

$y(t) = u(t) + c\,u(t-1)$

$\quad = u(t) + c q^{-1} u(t)$

$\quad = [1 + c q^{-1}]\, u(t)$

$\Rightarrow G(q) = G(q^{-1}) = 1 + c q^{-1}$

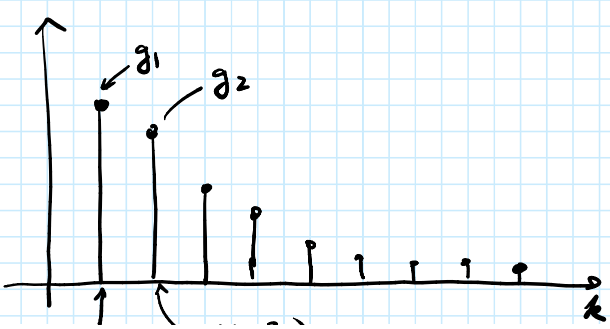
<u>Note</u>: Related to z ransform : $G(z) = 1 + cz^{-1} = \frac{z+c}{z}$.

<u>Parameterizing System Dynamics</u>

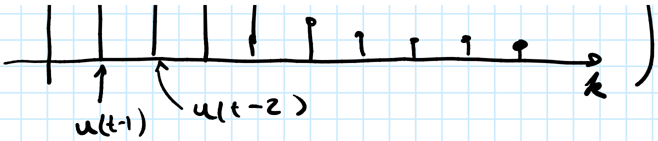
- A linear, time invariant system is <u>entirely</u> characterized by its impulse response.

$u(t) \longrightarrow \boxed{G(q)} \longrightarrow y(t)$

$$\underbrace{G(q) = \sum_{k=1}^{\infty} g(k)\, q^{-k}.}$$



$g_1, g_2, \ldots g_{30} \ldots$ too many params.



$\Rightarrow$ Can we represent the system using fewer parameters?

consider $g(k) = a^{k-1}, \quad k = 1, 2, 3 \quad ; \quad a < 1$

$$G(q) = \sum_{k=1}^{\infty} a^{k-1} q^{-k}$$

Multiply by $\frac{a}{q}$:

$$\frac{a}{q} G(q) = \sum_{k=1}^{\infty} a^{k} q^{-k-1} = \sum_{k=2}^{\infty} a^{k-1} q^{-k} \Rightarrow \left[\frac{1}{q} + \sum_{k=2}^{\infty} a^{k-1} q^{-k} \right] - \frac{1}{q}$$

$$= \underbrace{\sum_{k=1}^{\infty} a^{k-1} q^{-k}}_{G(q)} - \frac{1}{q}$$

$$\Rightarrow \frac{a}{q} G(q) = G(q) - \frac{1}{q}$$

$$\Rightarrow \left[1 - \frac{a}{q} \right] G(q) = \frac{1}{q}$$

$$\Rightarrow G(q) = \frac{q^{-1}}{1 - a q^{-1}} = \frac{1}{q - a}$$

$\Rightarrow$ $G(q)$ is represented by a single parameter (1 pole) by using a proper representation.

<u>Model</u>

- Maybe determined based on apriori knowledge.
  - $\rightarrow$ system physics.
- Maybe unknown, need to be estimated:

$$\theta = (\theta_1, \ldots \theta_d) \in \mathbb{R}^d$$

(Adjustable)

<u>Families of Transfer Func. Models</u>

- ARX (Auto Regressive with eXogenous inputs) Model structure

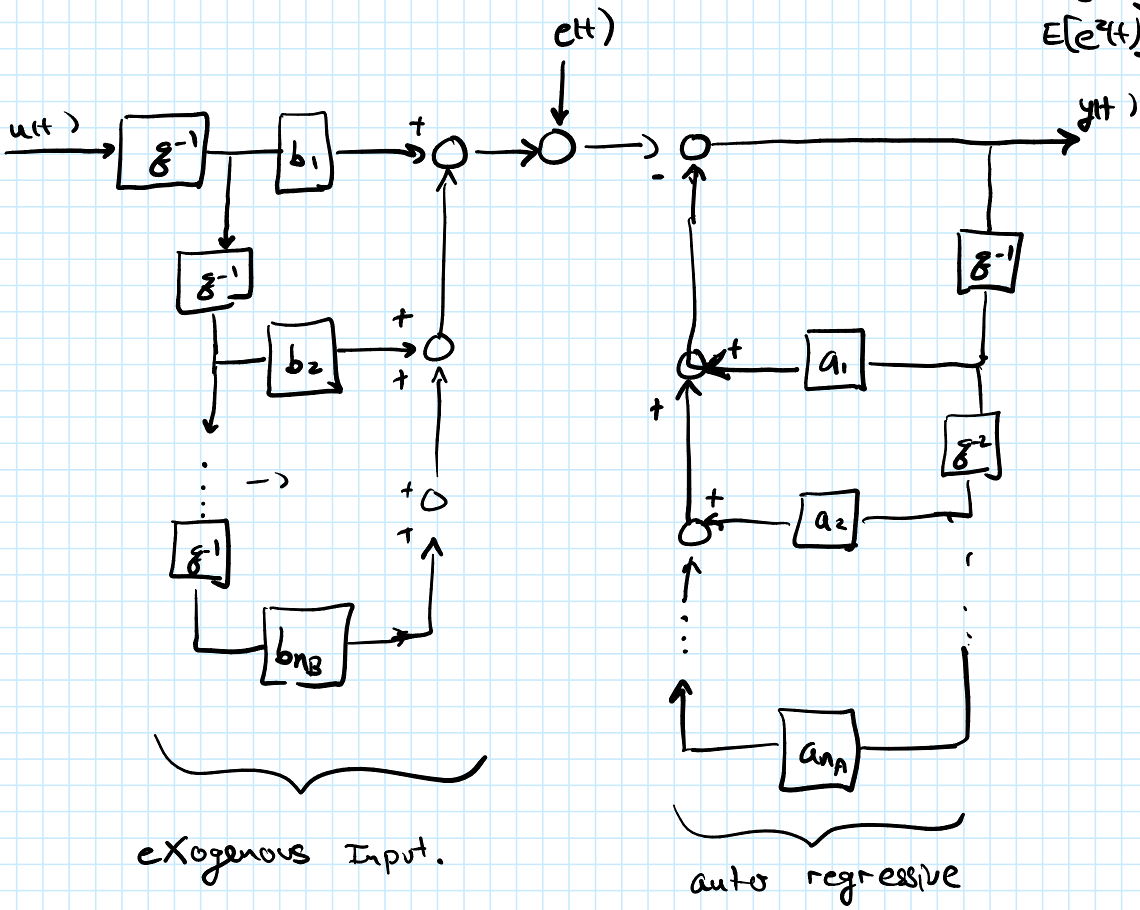
Families of Transfer Func. Models

- ARX (Auto Regressive with eXogenous inputs) Model structure

$$y(t) + a_1 y(t-1) + \cdots + a_{n_A} y(t-n_a) = b_1 \underbrace{u(t-1)} + \cdots + b_{n_b} u(t-n_b) + e(t)$$

• why not u(t)?
→ Causal Model.

Gaussian (Uncorrelated)
$E[e(t)] = 0$
$E[e^2(t)] = \lambda$



- Adjustable parameters: $\theta = [a_1, \dots a_{n_a}, b_1, \dots b_{n_b}]^T$

Define $A(z) = 1 + a_1 z^{-1} + \cdots a_{n_A} z^{-n_A}$

$B(z) = b_1 z^{-1} + \cdots + b_{n_B} z^{-n_B}$

Then the ARX structure becomes:

$$A(z) y(t) = B(z) u(t) + e(t)$$

- If $n_A = 0$, then $y(t) = B(z) u(t) + e(t)$

⇒ Finite Impulse Response (FIR) model.

• If $n_A = 0$, then $y(t) = B(q)u(t) + e(t)$

$\Rightarrow$ Finite Impulse Response (FIR) model.

ARMAX (Auto Regressive Moving Average w/ eXogenous Inputs) Model

$$\Rightarrow y(t) + a_1 y(t-1) + \cdots + a_{n_A} y(t-n_A) = b_1 u(t-1) + \cdots + b_{n_b} u(t-n_b) + e(t) + c_1 e(t-1) + \cdots + c_{n_c} e(t-n_c)$$

Use $C(q) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}$, then

$$\frac{A(q)}{A(q)} y(t) = \frac{B(q)}{A(q)} u(t) + \frac{C(q)}{A(q)} e(t)$$

Then,

$$G(q,\theta) = \frac{B(q)}{A(q)} \ ; \quad H(q,\theta) = \frac{C(q)}{A(q)}$$

• This structure consists of
- The moving average part (MA) : $C(q)$
- The autoregressive part : $A(q)$
- The exogenous inputs (X) : $B(q)$