ME 8873 – Statistical Model Estimation
Homework Set No. 1
Due February 2nd, 2022

Note: Datasets for this assignment are posted under the data locker folder on Canvas. For each problem involving coding, please use either R or MATLAB and attach your commented code at the end of the problem.

Problem 1: Consider the problem of determining if there is a significant difference between the means of two datasets, X1 and X2:

a) <u>Derive an expression</u> for standard error of the differences in means assuming normally distributed data. The data for the two variables can be found in the data locker as Data1-1. Use your analysis to determine if the groups are statistically significant and compare with a built-in t-test method in R or MATLAB. What makes your answer slightly different from the built-in function?

b) Use a second approach to determine if the groups are statistically significant by using the permutation resampling method.

Problem 2: Data 1-2 in the locker contains two scatter datasets relating Y to X1 and Y to X2. Compute linear regressions of each dataset and determine if the slope of each and intercept of each is significantly different from 0. Write your own code to conduct these analyses, and compare with the results from a regression package in MATLAB or R (e.g., lm in R)

Problem 3: Consider the following example of least squares estimation. Given

$$\begin{aligned} 3\theta_1 + 2\theta_2 &= 6 \\ \theta_1 + \theta_2 &= 1 \\ 4\theta_1 - 6\theta_2 &= -3 \\ 2\theta_1 + \theta_2 &= 4 \end{aligned}$$

Obtain the least squares estimate of parameters θ_1, θ_2. Suppose an additional measurement:

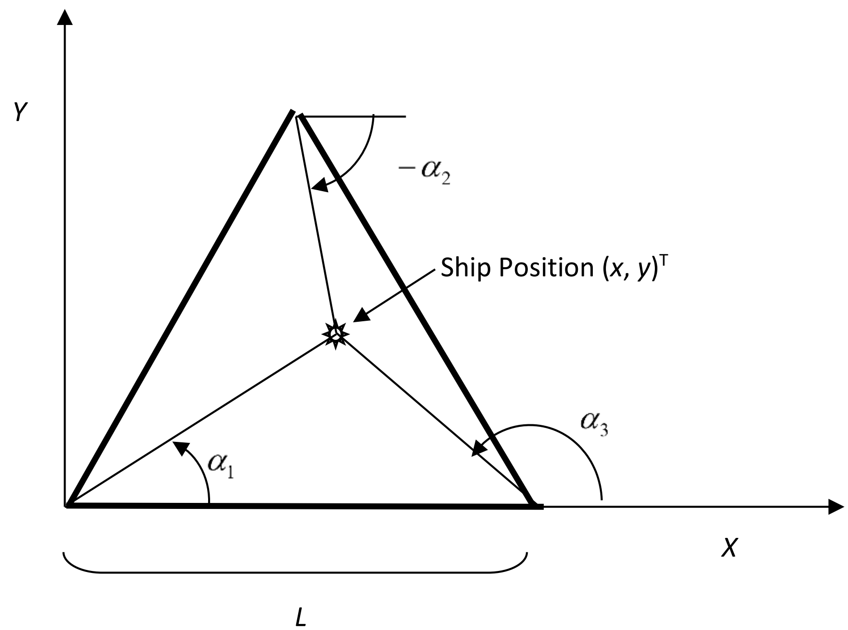
$$2\theta_1 + 2\theta_2 = 3$$

is obtained. Find the revised estimate by two methods:

Directly, by batch processing

Indirectly, by the recursive algorithm.

Problem 4: Shown below is a calibration range for determining the position of a ship. Three theodolite stations are placed at the three apexes of an equilateral triangle. Formulate the procedure for determining an optimal estimate of the ship's location, coordinates *x* and *y*, based on three angular measurements, $\alpha_1, \alpha_2, \alpha_3$. First, obtain model equations relating coordinates, *x* and *y,* and angular measurements, $\alpha_1, \alpha_2, \alpha_3$. Show that unknown parameters are ***linearly*** involved in the model equations. Obtain an optimal estimate. Write out the elements of the matrices involved.



Problem 5: Shown below is a two-link robot manipulator working in a horizontal plane, i.e. no gravity effect. Let q_1 and q_2 be joint angles for the first and the second joints, respectively. The torques generated by two motors are denoted by τ_1 and τ_2, as shown in the figure. The mass, moment of inertia (at the center of mass), and the location of the center of mass are denoted by m_i, I_i, and ℓ_{Ci}, respectively, for each link, *i* = 1, 2. It is known that the equations of motion are given by:

$$\tau_1 = H_{11}\ddot{q}_1 + H_{12}\ddot{q}_2 - h\dot{q}_2^2 - 2h\dot{q}_1\dot{q}_2$$

$$\tau_2 = H_{22}\ddot{q}_2 + H_{21}\ddot{q}_1 + h\dot{q}_1^2$$

where the coefficients are given by

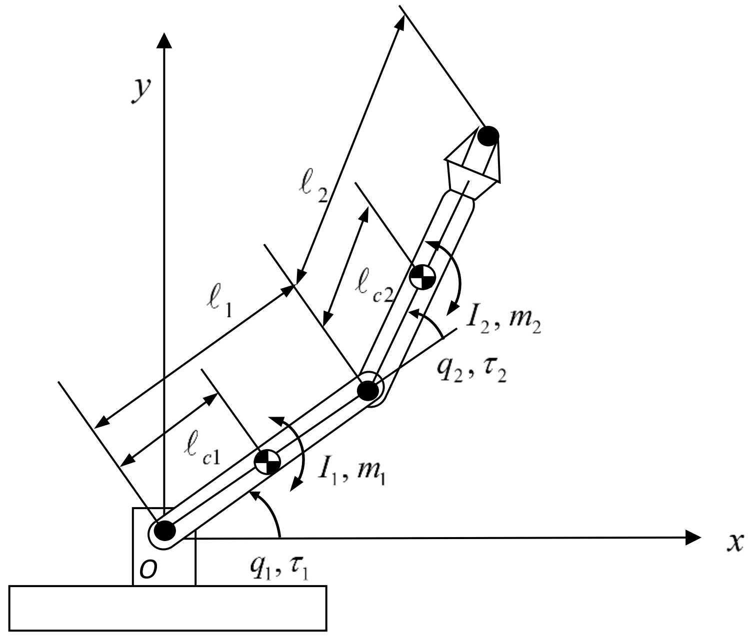
$$H_{11} = m_1\ell_{c1}^2 + I_1 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_2$$

$$H_{22} = m_2\ell_{c2}^2 + I_2$$

$$H_{12} = H_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2$$

$$h = m_2\ell_1\ell_{c2}\sin q_2$$

We want to estimate the mass parameters, m_i, I_i, and ℓ_{Ci}, based on experiment. We can measure joint torques, $\tau_1(t)$, $\tau_2(t)$, and joint angles $q_1(t)$, $q_2(t)$ as well as their derivatives, $\dot{q}_1(t)$, $\dot{q}_2(t)$, for time *t* = 1, 2, 3,.., *N*. Furthermore, we can obtain joint accelerations from the data: $\ddot{q}_1(t)$, $\ddot{q}_2(t)$. Using these data we want to identify the mass parameters based on the RLS algorithm. The difficulty is that the parameters are *not* linearly involved in the above dynamic equations.



Interestingly, however, if we use a new set of parameters, we can reduce the above dynamic equations into our favorite form, where unknown parameters are linearly involved. Consider the following new parameters:

$$\theta = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}^T$$
$$b_1 = m_2, b_2 = m_2 \ell_{C2}, b_3 = I_1 + m_1 \ell_{C1}^2, b_4 = I_2 + m_2 \ell_{C2}^2$$

The trick is to use the this new set of parameters instead of the original set of parameters, m_i, I_i, and ℓ_{Ci}, so that the standard RLS algorithm can be applied. Show that with the new parameters $\theta = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}^T$, the original dynamic equations can be represented as

$$\tau_1 = \varphi_1^T \theta, \tau_2 = \varphi_2^T \theta$$

where both φ_1 and φ_2 are observation vectors consisting of measured or known values.

Obtain observation vectors φ_1 and φ_2 as functions of $q_1(t)$, $q_2(t)$, $\dot{q}_1(t)$, $\dot{q}_2(t)$, and $\ddot{q}_1(t)$, $\ddot{q}_2(t)$. Assume the link length ℓ_1 is known. Also write out equations needed for coding the RLS algorithm for this problem. (No need to actually implement it.)

Problem 6: Consider the recursive least-squares algorithm with exponential weighting:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \frac{P_{t-1}\varphi(t)}{\alpha + \varphi^T(t)P_{t-1}\varphi(t)}\{y(t) - \varphi^T(t)\hat{\theta}(t-1)\}$$

$$P_t = \frac{1}{\alpha}[P_{t-1} - \frac{P_{t-1}\varphi(t)\varphi^T(t)P_{t-1}}{\alpha + \varphi^T(t)P_{t-1}\varphi(t)}]$$

where $0 < \alpha < 1$.

a). Implement the above algorithm using MATLAB. (MATLAB already has this algorithm in its library, but you should code it on your own.) Test out your code with Data 1-6a in the data locker. Plot the estimated parameter values against time *t*. Repeat the computation for different values of forgetting factor α, and discuss the effect of α. Note that the data in the course locker are for a third order FIR model: *m* = 3. The output is obtained from *t* = 3.

b). Use Data 1-6b also in the course locker, and plot the parameter values. Examine the P matrix, and discuss the results. Note also that the model order is *m* = 3.

The question below is a challenging problem, which requires a bit more theoretical understanding. If you can solve it, that will be great. You will get extra credit for the work. Do not feel bad if you cannot complete it.

c). The data set in Part b) of the above question is a special case where $\{\varphi(t),\, t = 1, 2\cdots\}$ lies in a hyperplane of dimension lower than that of parameter vector θ. Theoretically show that P_t diverges. Discuss the implication of this observation.