Lecture 12

Monday, February 21, 2022 9:46 AM

Kalman Filtering - Part II

Admin

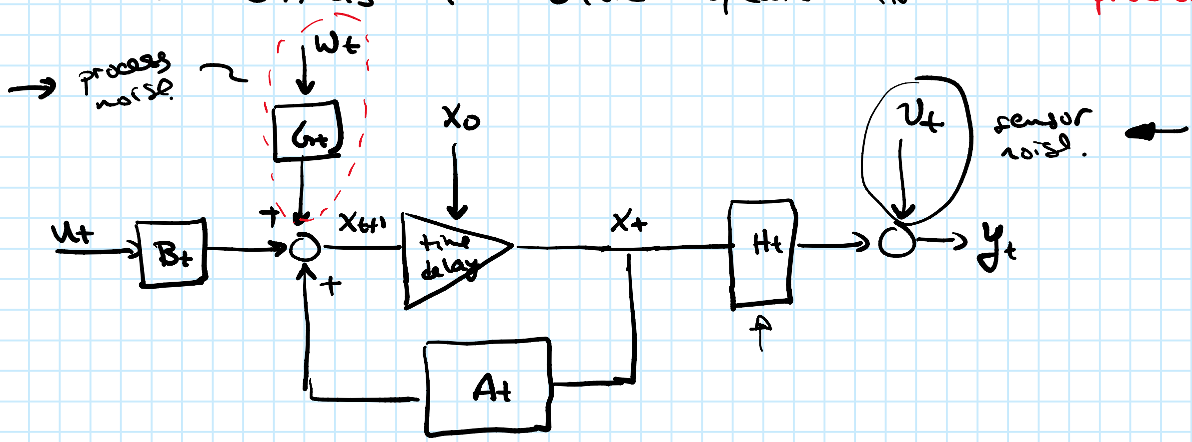
1. HW set 02 due weds next week.
2. Hotelling T^2 test problem 1: optional.
3. weds, Mar 2nd: class cancelled.

Objectives

1. Recap of Kalman Filter, discussion of Random elements.
2. Discrete Kalman Filter as an optimal linear filter.

What is a Kalman Filter?

- An algorithm that predicts the next "state" of a time-varying, linear dynamic system using noisy measurements.
- It is a recursive "Prediction-Error-Correction" method that takes the weighted average of previous system infor., places greater importance on high-certainty data.
- Luenberger's state observer is meant for deterministic systems, but in reality most systems are susceptible to noise that is stochastic in nature.
  - → Noise affects the sensors, ← sensor noise
  - → Noise affects the state update itself ← process noise.



The system equations can be written in the following form.

$$x_{t+1} = A_t x_t + B_t u_t + G_t w_t$$

This is deterministic & predictable, so can be eliminated.

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$$\boxed{\begin{aligned} x_{t+1} &= A_t x_t + G_t w_t \\ y_t &= H_t x_t + v_t \end{aligned}}$$

$\Leftarrow$ Kalman filter framework.

where:

x_t : state vector $\in \mathbb{R}^{n\times 1}$

u_t : input vector $\in \mathbb{R}^{r\times 1}$

A_t : state transition matrix $\in \mathbb{R}^{n\times n}$

B_t : input matrix of proper dims.

y_t : output vector $\in \mathbb{R}^{\ell\times 1}$

H_t : observation matrix $\in \mathbb{R}^{\ell\times n}$

G_t : process noise matrix $\in \mathbb{R}^{n\times n}$

v_t : measurement noise $\in \mathbb{R}^{\ell\times 1}$

w_t : process noise $\in \mathbb{R}^{n\times 1}$

- If we consider both forms of noise as multivariable random processes, we ~~can~~ must. make the following assumption.

  - The noise sources must have zero means

  $$E[v_t] = 0, \quad E[w_t] = 0.$$

  - The noise sources are uncorrelated:

  $$\mathrm{Cov}(w, v) = E[w_t \cdot v_t^T] = 0.$$

  - The noise sources have some covariance:

  $$\mathrm{Cov}[v_t] = E[v_t \cdot v_t^T] = R_t \in \mathbb{R}^{\ell\times\ell}$$
  $$\mathrm{Cov}[w_t] = E[w_t \cdot w_t^T] = Q_t \in \mathbb{R}^{n\times n}.$$

- ultimately, the goal of the Kalman Filter is to find an optimal estimate of the state vector $\hat{x}_t$ that minimizes the mean squared error:

$$J_t = E\left[(\hat{x}_t - x_t)^T (\hat{x}_t - x_t)\right]$$

The Discrete Kalman Filter as an optimal linear filter

- The discrete Kalman Filter provides a recursive soln. to minimize the mean squared error problem above, by:
  1. taking a prior estimate of the system's current state, called $\hat{x}_t'$
  2. updating it with new measurement data:

"update"

accounts for noise

new data

$$\underbrace{\hat{x}_t}_{\text{"a posteriori estimate"}} = \underbrace{\hat{x}_t'}_{\text{"a priori estimate"}} + \underbrace{K_t}_{\text{Kalman Gain}} \underbrace{(y_t - H_t \hat{x}_t')}_{\text{"Innovation"}}$$

↳ "Measurement Residual," "i_t"

- The Kalman Filter consists of two steps:
  1) "prediction" or "propogation"
  2) "update" or "correction"

$e_t = \hat{x}_t - x_t$

Step 1: (a priori)

- ↳ predicted state estimate $\hat{x}_t' = A_{t-1} \hat{x}_{t-1}$ — "a priori" estimate; previous state
- ↳ predict error covariance: $P_t' = A_t P_{t-1} A_t^T + Q_t$ — previous error cov.; process noise

$P_t = E[e_t e_t^T]$

Step 2 Correction ("a posteriori")

- ↳ Measurement residual : $y_t - H_t \hat{x}_t' = i_t$
- ↳ Kalman Gain : $K_t = P_t' H_t^T [R + H P_t' H^T]^{-1}$ — measurement noise
- ↳ update state estimate: $\hat{x}_t = \hat{x}_t' + K_t i_t$
- ↳ update error covariance: $P_t = (I - K_t H_t) P_t'$

Recursive Calculation procedure for Discrete Kalman Filter

Initial Error Covariance

P_0 @ $t=1$

Initial State

Estimate $\hat{x}_0$ Measurement y_t

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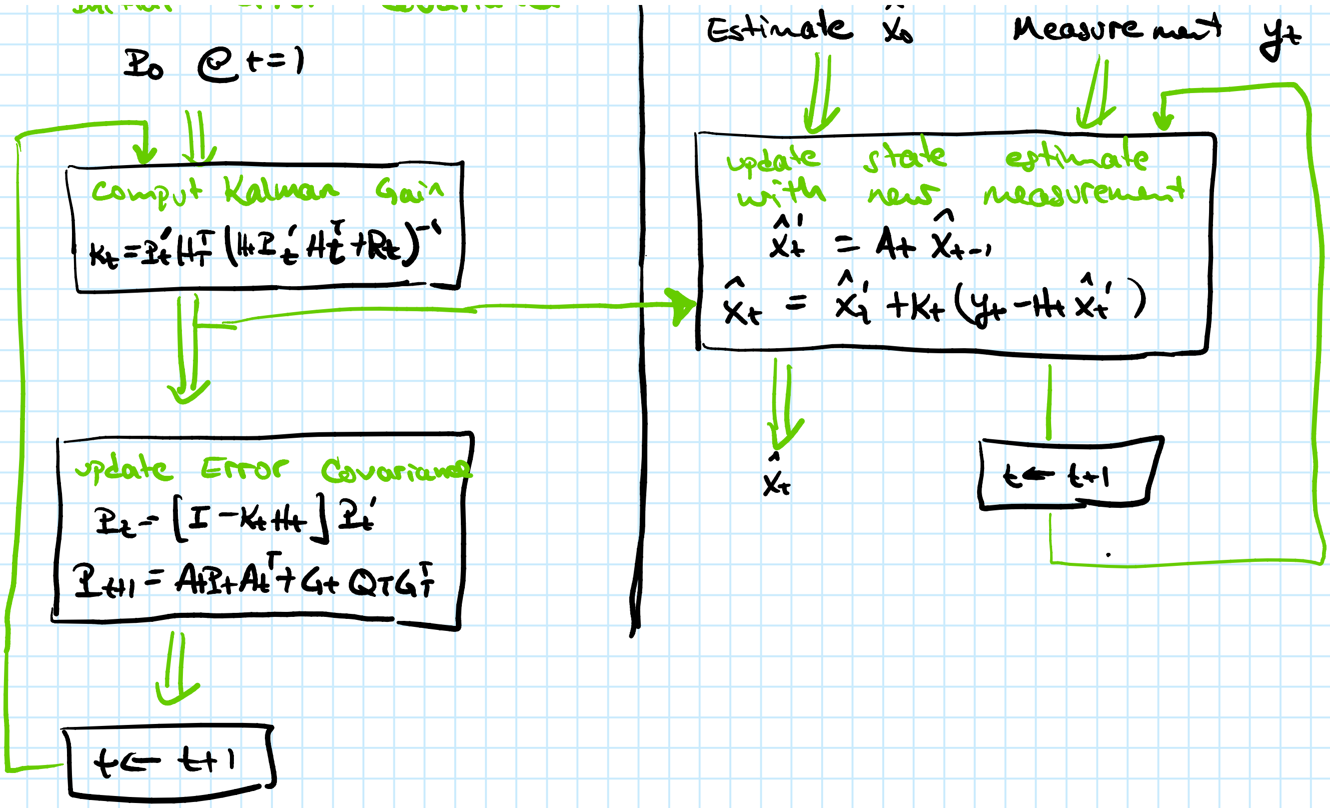

Addendum: Derivation of Kalman Filter

• Derivation of the Kalman Filter

The predicted state estimate $\hat{x}'_t$ and measurement residual i_t can easil[y] be calculated without explicit use of system error and noise distributio[n] during the algorithm's first iteration.

– Expected State Transition:

$$\hat{x}'_t = A_{t-1}\hat{x}_{t-1} + G_{t-1}w_{t-1} \xRightarrow{\text{same as}} \hat{x}'_t = E[A_{t-1}\hat{x}_{t-1} + G_{t-1}w_{t}$$

Since only the stochastic terms have expected values,

$$\hat{x}'_t = A_{t-1}\hat{x}_{t-1} + G_{t-1}\cancel{E[w_{t-1}]}$$ this is zero b.c. process noise has zero mean...

$$\Downarrow$$

$$\hat{x}'_t = A_{t-1}\hat{x}_{t-1}$$ (a priori predicted state estimate)

– Estimated Output:

$$\hat{y}_t = H_t\hat{x}'_t + \cancel{V_t} \Longrightarrow E[V_t] = 0, \text{ so}$$

$$\hat{y}_t = H_t\hat{x}'_t$$ (measurement prediction)

The other calculations in the Kalman Filter rely on observations of the system's error covariance to update parameters.

Matrix P_t is the error covariance of the prediction at time t:

$$P_t = E[e_t e_t^T] = E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^T] \qquad (10)$$

To determine the Kalman Gain K_t for the optimal linear filter, we must find the form that minimizes P_t.

– First, plug the observation estimate equation into the state estimate update equation to get:

$$\hat{x}_t = \hat{x}'_t + K_t(H_t x_t + V_t - H\hat{x}'_t) \qquad (11)$$

- Then plug (11) into (10) to get:

$$P_t = E\Big[\big[(I-K_tH_t)(x_t-\hat{x}_t')-K_tV_t\big]\big[(I-K_tH_t)(x_t-\hat{x}_t')-K_tV_t\big]^T\Big] \quad (12)$$

- Knowing that the error of the prior state estimate, $(x_t-\hat{x}_t')$, is not correlated with measurement noise V_t, we can write:

$$P_t = (I-K_tH_t)E\big[(x_t-\hat{x}_t')(x_t-\hat{x}_t')^T\big](I-K_tH) + K_tE\big[V_tV_t^T\big]K_t^T \quad (13)$$

- Plugging in our symbols for the error and noise covariances we get:

$$P_t = (I-K_tH_t)P_t'(I-K_tH)^T + K_tR_tK_t^T \quad (14)$$

(Error Covariance Update)

* Note: The diagonal elements of the error covariance matrix contain the mean square errors:

$$P_t = \begin{bmatrix} \underline{E[e_{t-1}e_{t-1}^T]} & E[e_te_{t-1}^T] & E[e_{t+1}e_{t-1}^T] \\ E[e_{t-1}e_t^T] & \underline{E[e_te_t^T]} & E[e_{t+1}e_t^T] \\ E[e_{t-1}e_{t+1}^T] & E[e_te_{t+1}^T] & \underline{E[e_{t+1}e_{t+1}^T]} \end{bmatrix}$$

Diagonal

- The trace of P_t, $T[P_t]$ sums these square errors, so we want to minimize that..

- Expanding (14), we get:

$$P_t = P_t' - K_tH_tP_t' - P_t'H_t^TK_t^T + K_t(H_tP_t'H^T+R_t)K_t^T \quad (15)$$

- Since the trace of a matrix is equal to the trace of its transpose, we can rewrite (15) as:

$$T[P_K] = T[P_K'] - 2T[K_tH_tP_t'] + T[K_t(H_tP_t'H_t^T+R)K_t^T] \quad (16)$$

- Differentiating (16) with respect to K_t, we get:

$$\frac{\partial T[P_t]}{\partial K_t} = -2(H_t P_t')^T + 2K_t(H_t P_t' H_t^T + R) \qquad (17)$$

- Setting this to zero and rearranging terms:

$$(H_t P_t')^T = K_t(H_t P_t' H_t^T + R_t)$$

- Now, solving for K_t, we get:

$$\boxed{K_t = P_t' H_t^T (H_t P_t' H_t^T + R_t)^{-1}} \qquad (18)$$

(the Kalman Gain equation)

* This is the optimal gain that minimizes error covariance and, by extension, mean-square error.

- Plugging the Kalman Gain (18) into (15) we get:

$$\begin{aligned} P_t &= P_t' - P_t' H_t^T (H_t P_t' H_t^T + R_t)^{-1} H_t P_t' \\ &= P_t' - K_t H_t P_t' \end{aligned}$$

$$\boxed{P_t = (I - K_t H_t) P_t'} \qquad (19)$$

(Error Covariance Update)

- Finally, we need to find an equation to project the error covariance matrix to the next time step $t+1$. We do this by using the equation for prior error.

$$\begin{aligned} e_{t+1}' &= x_{t+1} + \hat{x}_{t+1}' \\ &= (A_t x_t + w_t) - A_t \hat{x}_t' \\ &= A_t e_t + w_t \end{aligned}$$

Extending this to (10) for P_t we get:

$$P_{t+1}' = E[e_{t+1}' e_{t+1}'^T] = E[(A_t e_t + w_t)(A_t e_t + w_t)^T]$$

Since w_t and e_t have no cross-correlation, we can write:

$$P'_{t+1} = E[e'_{t+1} e'^{T}_{t+1}]$$

$$= E[A_t e_t (A_t e_t)^T] + E[w_t w_t^T]$$

$$\boxed{P'_{t+1} = A_t P_t A_t^T + Q_t} \quad (20)$$

Second term can also be written as: $G_t Q_t G_t^T$

(Predicted Error Covariance)