# Lecture 23

Wednesday, April 20, 2022 9:35 AM

**The Art of System Identification: Part I - Experimental Design**

Admin

1. HW04 → due.
2. HW05 → due Friday.
3. HW07 → Monday. →
4. projects.

## Review of Theories for Experimental Design

- Key requirements for system Identification (parameter estimation):
  - → Consistent (unbiased estimate): $\hat{\theta}_N \to \theta_0$
  - → This depends on system being Ergodic

## Major Theoretic Results

$1 + g_1 z^{-1} + \dots$

i) Informative data sets

Is dataset ~~zoo~~ informative enough to distinguish any two different models, $W_1(z)$ & $W_2(z)$?

→ from same model set $M(\theta)$

$$E\left[\left(\hat{y}_1(t|\theta_1) - \hat{y}_2(t|\theta_2)\right)^2\right] = 0 \quad W_1(\overset{e^{iw}}{z}) \equiv W_2(z)$$

⇒ The dataset is Informative (for any linear time invariant model)

If the spectrum matrix

$$\Phi_z(w) = \begin{bmatrix} \Phi_u(w) & \Phi_{uy}(w) \\ \Phi_{uy}(w) & \Phi_y(w) \end{bmatrix}$$

is positive definite for almost all $w$.

$u(t) \to \boxed{M(\theta)} \to y(t)$

ii) persistence of excitation.

Any $(n-1)^{st}$ order moving average filter cannot filter the

(i) persistence of excitation.

Any $(n-1)^{st}$ order moving average filter cannot filter the input to zero if the input $u(t)$ is persistently exciting of order $n$.

$$u(t) \longrightarrow \boxed{M_n(q)} \longrightarrow y(t)$$

$$M_n(q) = m_1 q^{-1} + m_2 q^{-2} + \cdots + m_n q^{-n}$$

$\Rightarrow$ persistently exciting of order $n$ if

$$\underbrace{\bar{R}_N}_{\text{auto-covariance}} = \underbrace{\begin{bmatrix} R_u(0) & \cdots & R_u(n-1) \\ \vdots & & \vdots \\ R_u(n-1) & & R_u(0) \end{bmatrix}}_{n \times n}$$

is non-singular.

$$\Rightarrow |\Delta M(e^{i\omega})|^2 \Phi_u(\omega) = 0 \overset{\text{implies}}{\Longrightarrow} \Delta M(e^{i\omega}) = 0.$$

- can extended.

$$G(q) = \frac{\overbrace{q^{-n_k}(b_1 + b_2 q^{-1} + \cdots + b_{n_b} q^{-n_b+1})}^{B(q)}}{\underbrace{1 + f_1 q^{-1} + \cdots + f_{n_f} q^{-n_f}}_{F(q)}}$$

$$\Rightarrow \Delta G(q) = \frac{B_1}{F_1} - \frac{B_2}{F_2} = \frac{B_1 F_2 - B_2 F_1}{F_1 F_2}$$

$B_1 F_2 - B_2 F_1$: polynomial of order $n_b + n_f$

$\Rightarrow$ persistence of excitation if $\bar{R}_{n_b+n_f}$ is non-singular.

$$\Rightarrow |B_1(e^{i\omega})F_2(e^{i\omega}) - B_2(e^{i\omega})F_1(e^{i\omega})|^2 \Phi_u(\omega) = 0$$

$\Rightarrow$ Implies $\Delta G(q) = 0$.

$\Rightarrow$ The sequence $u(t)$ that is persistently exciting of order $n_b + n_f$ is informative enough for distinguish two transfer

=) the sequence $u(t)$ that is persistently exciting of order $n_b + n_f$ is informative enough to distinguish two transfer funcs. of above form.

iii) signal to noise ratio.

The prediction error method determines the parameter vector $\theta$ such that a frequency-weighted squared error is minimized. w/ Fixed noise model

$$H(q, \theta) = H(q)$$

then

$$\hat{\theta} = \arg\min_{\theta} \int_{-\pi}^{\pi} \Big| \underbrace{G_0(e^{i\omega})}_{\text{true system}} - \underbrace{G(e^{i\omega})}_{\text{model}} \Big|^2 \cdot \underbrace{\frac{\Phi_{uu}(\omega)}{|H(e^{i\omega})|^2 \lambda}}_{\text{signal-to-noise ratio}} \, d\omega.$$

If $H = 1$

$$\operatorname{var}(\hat{G}(e^{i\omega})) \sim \frac{n}{N} \frac{\cancel{\Phi_{ee}(\omega)} \to \lambda}{\Phi_{uu}(\omega)}$$

$n$: # of parameters

$N$: # of data points

iv) Asymptotic variance

consider the prediction error model with

$$V_N(\theta, Z^N) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} \varepsilon^2(t, \theta)$$

If the true model is involved in the model set: $\theta_0 \in M$, variance in parameter estimation error approach the folling asymptotic result:

$$\operatorname{Cov}(\hat{\theta}_N) \sim \frac{\lambda_0}{N} \Big[ \underbrace{E\big[ \psi(t, \theta_0) \, \psi^T(t, \theta_0) \big]}^{-1} \Big] \quad (\text{as } N \to \infty$$

covariance matrix

- depends on
  - $N$: number of data,
  - $\lambda_0$: Noise level
  - Sensitivity : $\psi(t,\theta) = \frac{\partial}{\partial \theta} \hat{y}(t|\theta) \Big|_{\theta=\theta_0}$

If $\hat{y}(t,\theta) = \varphi^T(t)\theta$,
the $E[\ \ ]$ = covariance matrix.