The Weibull process and reliability

Hanna Bekkare, Ines Besbes, Sara Rool

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1 Introduction

The Weibull distribution is a statistical technique used in reliability analysis and modeling failure times in various systems, such as to analyze tire wear over time. Its broad applicability stems from the ability to model different types of failure rates depending on the value of its shape parameter.

This process has been explored in the textbook *Processus stochastiques et fiabilité des systèmes* (C.Cocozza-Thivent, 1997).

The aim of this paper is to explore the properties and applications of the Weibull process presented in the textbook. In addition, we apply the Weibull process to a Boeing dataset, a leading aerospace manufacturer. The dataset includes information on the performance and failure rates of the air conditioning system in Boeing's aircrafts. By fitting Weibull models to these data, we seek to gain insight into the underlying failure mechanisms, evaluate the reliability of different components, and estimate their expected lifetimes.

The first part of this report provide an overview of the Weibull distribution, including its parameters and their interpretations. Secondly, we describe the methodology to fit Weibull models to real-world data, and detail the analysis of the Boeing dataset.

2 Analysis of Weibull intensity

The Weibull process is an inhomogeneous Poisson process that features independent events and a time-varying intensity given by

$$\lambda(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}, \quad \alpha > 0 \quad \text{and} \quad \beta > 0.$$

We denote $N(t)$ as the number of failures, which follows a $P(\int_0^t \lambda(x)dx)$ distribution. Furthermore, we define the cumulative distribution function as follows:

$$\Lambda(t) = \int_0^t \lambda(x)dx.$$

2.1 The impact of α on the intensity

We observe, in Appendix A.1, that α controls the scale of the intensity process, we call it the scale parameter. Essentially, α influences the "range" of the failure times. A larger α stretches the time scale, meaning the system is more robust and less likely to fail in the short term, with failure events occurring over a longer period. Conversely, a smaller α compresses the time scale, suggesting a less reliable system that fails sooner, with failure events occurring more rapidly. This could correspond to components that are expected to wear out faster or experience issues over a shorter time frame.

2.2 The impact of β on the intensity

The shape parameter β determines the distribution's form and the nature of the failure rate or hazard function, influencing how the probability of failure changes over time (see Figure 1).

- For $0 < \beta < 1$: The intensity function decreases over time. This corresponds to a process where the probability of failure is initially high but decreases as time progresses. It's the burn-in phase or early life failures.
- For $\beta = 1$: The intensity function is constant and equal to $\frac{1}{\alpha}$. The failure rate is constant over time, indicating no dependency on age or wear. Failures occur due to random, external factors unrelated to system aging. This corresponds to the operational phase, where the system operates reliably with random failures.
- For $1 < \beta < 2$: The intensity function increases with time, but at a decreasing rate. This represents the onset of wear and degradation in the system. This is the transition phase from normal operation to wear-out.
- For $\beta = 2$: The intensity function increases linearly with time, indicating that degradation occurs steadily as the system ages. This represents the steady wear-out phase, where failures occur more predictably due to aging.
- For $\beta > 2$: The intensity function increases rapidly with time. The failure rate increases rapidly with time, often due to cumulative stress, material exhaustion, or critical failure points being reached. This corresponds to the late wear-out phase, where systems are close to failure.

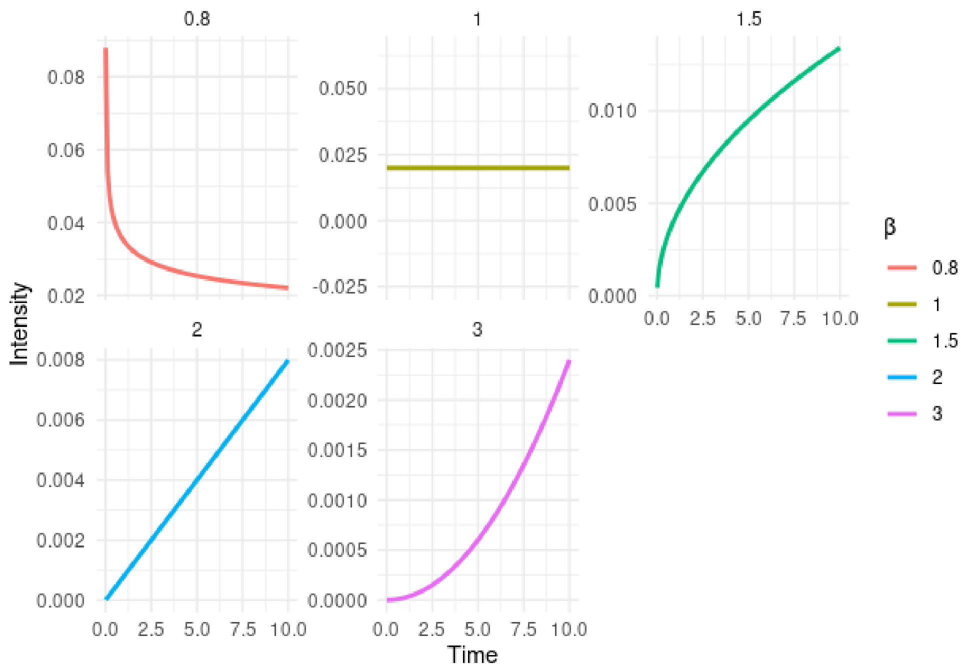


Figure 1: Intensity for different β with $\alpha = 50$.

3 Simulation of a Weibull process

Simulating a Weibull process is equivalent to simulating a non-homogeneous Poisson process with the Weibull intensity function. To do so, if we have an upper bound for the intensity function, we can use the thinning algorithm.

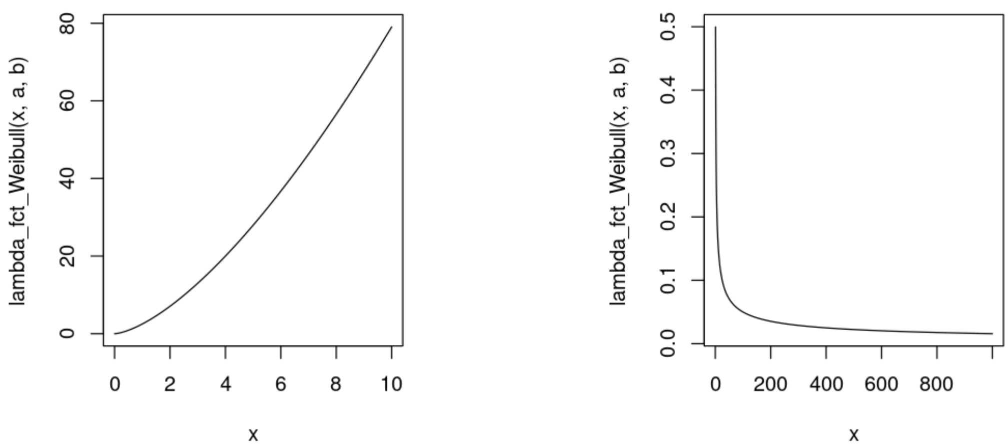
The intensity $\lambda(t)$ is maximal at:

- $t = 0^+$ for $\beta < 1$, where $\lambda(t)$ decreases as t increases.
- $t = Tmax$ (end of simulation interval) for $\beta > 1$, where $\lambda(t)$ increases with t.

The maximum intensity over the interval $[0, Tmax]$ is

$$\lambda_{\max} = \begin{cases} \lambda(Tmax) = \frac{\beta}{\alpha}\left(\frac{Tmax}{\alpha}\right)^{\beta-1}, & \text{if } \beta > 1, \\ \lambda(0^+) = \infty, & \text{if } \beta < 1. \end{cases}$$

If $\beta = 1$, we simulate a homogeneous Poisson process with rate $\lambda = \frac{1}{\alpha}$.



(a) Weibull intensity function with increasing failure rate ($\beta = 2.5$, $\alpha = 1$, $t \in [0, 10]$). The event rate grows as time increases.

(b) Weibull intensity function with decreasing failure rate ($\beta = 0.5$, $\alpha = 1$, $t \in [0, 1000]$). The event rate diminishes over time.

Figure 2: Comparison of inhomogeneous Poisson processes modeled using Weibull intensity functions for different shape parameters β. (a) shows an increasing failure rate with $\beta > 1$, while (b) illustrates a decreasing failure rate with $\beta < 1$.

4 Parameter estimation

4.1 Likelihood

The likelihood, parameterized by α and β, is given by

$$L(\alpha, \beta; T_1, T_2, \ldots, T_{N(T)}, t) = \left(\prod_{i=1}^{N(T)} \lambda(T_i)\right) \exp\left(-\int_0^t \lambda(s)\, ds\right),$$

where the intensity function is defined such that

$$\lambda(s; \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{s}{\alpha}\right)^{\beta-1}.$$

Substituting $\lambda(s; \alpha, \beta)$, we deduce

$$L(\alpha, \beta; T_1, T_2, \ldots, T_{N(T)}, t) = \left(\prod_{i=1}^{N(T)} \frac{\beta}{\alpha}\left(\frac{T_i}{\alpha}\right)^{\beta-1}\right) \exp\left(-\int_0^t \frac{\beta}{\alpha}\left(\frac{s}{\alpha}\right)^{\beta-1} ds\right).$$

The integral in the exponential term simplifies as follows

$$\int_0^t \frac{\beta}{\alpha}\left(\frac{s}{\alpha}\right)^{\beta-1} ds = \frac{\beta}{\alpha}\left[\frac{1}{\alpha^{\beta-1}}\frac{s^\beta}{\beta}\right]_0^t = \frac{\beta}{\alpha^\beta}\frac{t^\beta}{\beta} = \left(\frac{t}{\alpha}\right)^\beta.$$

Thus, the likelihood becomes

$$L(\alpha, \beta; T_1, T_2, \dots, T_{N(T)}, t) = \left(\prod_{i=1}^{N(T)} \frac{\beta}{\alpha} \left(\frac{T_i}{\alpha} \right)^{\beta - 1} \right) \exp \left(- \left(\frac{t}{\alpha} \right)^{\beta} \right).$$

4.2 Log-likelihood

Taking the logarithm of likelihood, we have

$$\log L(\alpha, \beta; T_1, T_2, \dots, T_{N(T)}, t) = \sum_{i=1}^{N(T)} \left[\log \left(\frac{\beta}{\alpha} \right) + (\beta - 1) \log \left(\frac{T_i}{\alpha} \right) \right] - \left(\frac{t}{\alpha} \right)^{\beta}.$$

Expanding the terms, we deduce

$$\log L(\alpha, \beta; T_1, T_2, \dots, T_{N(T)}, t) = N(T) \left[\log(\beta) - \log(\alpha) \right] + (\beta - 1) \sum_{i=1}^{N(T)} \left[\log(T_i) - \log(\alpha) \right] - \left(\frac{t}{\alpha} \right)^{\beta}.$$

Finally, reorganizing the terms yields

$$\log L(\alpha, \beta; T_1, T_2, \dots, T_{N(T)}, t) = N(T) \left[\log(\beta) - \beta \log(\alpha) \right] + (\beta - 1) \sum_{i=1}^{N(T)} \log(T_i) - \left(\frac{t}{\alpha} \right)^{\beta}.$$

4.3 Estimators of maximum likelihood $\hat{\alpha}$ and $\hat{\beta}$

Recall the log-likelihood formula

$$\mathcal{L}(\alpha, \beta) = N(t) \left[\log(\beta) - \beta \log(\alpha) \right] + (\beta - 1) \sum_{i=1}^{N(t)} \log(T_i) - \left(\frac{t}{\alpha} \right)^{\beta}.$$

Let's find $\hat{\alpha}, \hat{\beta}$ the criticals points of α, β.

- For α

$$\begin{aligned}
\frac{\partial \mathcal{L}(\alpha, \beta)}{\partial \alpha} = 0 &\iff \frac{-\beta N(t)}{\alpha} + \frac{\beta t^{\beta}}{\alpha^{\beta+1}} = 0 \\
&\iff \frac{\beta t^{\beta}}{\alpha^{\beta+1}} = \frac{\beta N(t)}{\alpha} \\
&\iff \log \left(\frac{\beta t^{\beta}}{\alpha^{\beta+1}} \right) = \log \left(\frac{\beta N(t)}{\alpha} \right), \text{ the terms are strictly positive} \\
&\iff \log(\beta) + \beta \log(t) - (\beta + 1) \log(\alpha) = \log(\beta) + \log(N(t)) - \log(\alpha) \\
&\iff \beta \log(t) - \beta \log(\alpha) = \log(N(t)) \\
&\iff \log(\alpha) = \log(t) - \frac{\log(N(t))}{\beta}
\end{aligned}$$

- For β

$$\begin{aligned}
\frac{\partial \mathcal{L}(\alpha, \beta)}{\partial \beta} = 0 &\iff \frac{N(t)}{\beta} - \log(\alpha) N(t) + \sum_{i=1}^{N(t)} \log(T_i) - \left(\frac{t}{\alpha} \right)^{\beta} \log \left(\frac{t}{\alpha} \right) = 0 \\
&\iff \frac{1}{\beta} = \log(\alpha) - \frac{1}{N(t)} \sum_{i=1}^{N(t)} \log(T_i) + \frac{1}{N(t)} \left(\frac{t}{\alpha} \right)^{\beta} \log \left(\frac{t}{\alpha} \right)
\end{aligned}$$

We then deduce $\log(\hat{\alpha}) = \log(t) - \frac{1}{\hat{\beta}}\log(N(t))$. We combine both formulas to find a reduced formula of $\hat{\beta}$. Detailed computation are in Appendix A.2.

$$\begin{aligned}\frac{1}{\hat{\beta}} &= \log(\hat{\alpha}) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}}\log\left(\frac{t}{\hat{\alpha}}\right) \\ &= \log(t) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i)\end{aligned}$$

Finally, we have $\log(\hat{\alpha}) = \log(t) - \frac{1}{\hat{\beta}}\log(N(t))$ and $\frac{1}{\hat{\beta}} = \log(t) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i)$. In Appendix A.3, we explain the computation of the hessian. We obtain

$$H[\mathcal{L},\hat{\alpha},\hat{\beta}] = \begin{pmatrix} -\frac{\hat{\beta}^2 N(t)}{\hat{\alpha}^2} & \frac{N(t)}{\hat{\alpha}}\log(N(t)) \\ \frac{N(t)}{\hat{\alpha}}\log(N(t)) & \frac{-N(t)}{\hat{\beta}^2}(1+\log(N(t))^2) \end{pmatrix}.$$

Then, we compute the trace and determinant.

$$Tr(H[\mathcal{L},\hat{\alpha},\hat{\beta}]) = -\frac{\beta^2 N(t)}{\hat{\alpha}^2} - \frac{N(t)}{\beta^2}(1+\log(N(t))^2) < 0$$

$$\begin{aligned}det(H[\mathcal{L},\hat{\alpha},\hat{\beta}]) &= -\frac{N(t)^2}{\hat{\alpha}^2}(-1-\log(N(t))^2) - \frac{N(t)^2}{\hat{\alpha}^2}\log(N(t))^2 \\ &= -\frac{N(t)^2}{\hat{\alpha}^2}[-1-\log(N(t))^2+\log(N(t))^2] \\ &= \frac{N(t)^2}{\hat{\alpha}^2} > 0\end{aligned}$$

The determinant of the hessian is the product of eigenvalues. It is negative, that means $\hat{\alpha}$ and $\hat{\beta}$ are both minima or maxima. In addition, we have a negative trace. Hence, $(\hat{\alpha},\hat{\beta})$ are local maxima of the log-likelihood. They are estimators of maximum likelihood of (α,β)

4.4 Numerical verification

With the results in Appendix A.4, we observe that we have an accurate estimation of β, but it is not the case for α. It is maybe because we approximate $\log(\alpha)$ instead of α directly and that we use $\hat{\beta}$ to compute $\hat{\alpha}$, so the error propagates. Also, the chosen values of α and β influence the number of failures, and so the accuracy of our estimators. The greater the α, the more robust the system, which gives us less data because there are fewer failures for the same time scale. So, if we have $\beta < 1$ this will concentrate our breakdowns at the start of life. Thus, the alliance of the two gives fewer breakdowns and therefore less data to estimate our parameters, so we lose accuracy.

5 Reliability estimation

We aim to estimate the reliability of equipment at a given time t based on observations over the interval $[0,t]$. We can quantify this reliability by the number of failures that occurs within this interval. We denote $N(t)$ as the number of failures, which follows a $P(\Lambda(t))$ distribution with expectation $\Lambda(t)$. However, we know that

$$\Lambda(t) = \int_0^t \lambda(s)\,ds = \int_0^t \frac{\beta}{\alpha}\left(\frac{s}{\alpha}\right)^{\beta-1}ds = \frac{\beta}{\alpha^\beta}\left[\frac{s^\beta}{\beta}\right]_0^t = \left(\frac{t}{\alpha}\right)^\beta.$$

We estimate $\Lambda(t)$ by

$$\widehat{\Lambda}(t) = \left(\frac{t}{\alpha}\right)^{\widehat{\beta}} = e^{\widehat{\beta}\log(\frac{t}{\alpha})} = e^{\widehat{\beta}\frac{1}{\widehat{\beta}}log(N(t))} = N(t).$$

We can also estimate the reliability at a time $t' \neq t$ to:

- **Anticipate future breakdowns :** If $t' > t$, we can estimate how many additional failures may occur in the interval $[t, t']$, which is crucial for preventive maintenance, resource planning or system improvement.
- **Analyze past reliability :** If $t' < t$, this allows us to reconstruct or verifying past system behavior to identify anomalies or refine models.

In the case $t' > t$,

$$\frac{\Lambda(t')}{\Lambda(t)} = \left(\frac{t'}{t}\right)^{\beta}.$$

We can therefore estimate future failures as

$$\Lambda(t') = \left(\frac{t'}{t}\right)^{\beta} \Lambda(t) = \left(\frac{t'}{t}\right)^{\beta} N(t).$$

6 Convergence of the estimators

We aim to study the convergence of the two Maximum Likelihood Estimators (MLE) of α and β. We simulate data for different values of T_{max}, the maximum observation time, and analyze how the MLEs stabilize around the true parameter values.

6.1 Convergence of β

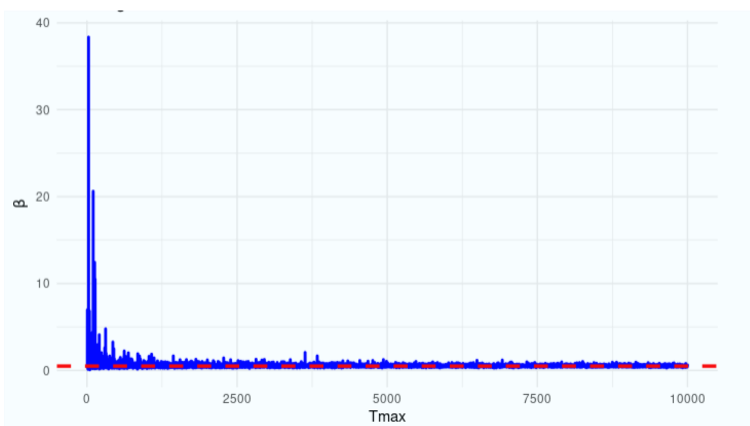


Figure 3: Convergence of β as a function of T_{max}

Figure 3 shows the evolution of $\hat{\beta}$ as a function of T_{max}. The horizontal dashed line represents the true value of $\beta = 0.5$. This visualization shows that $\hat{\beta}$ stabilizes near the true value as T_{max} increases. The estimator of β is consistent, this result confirms it is the correct estimator.

6.2 Convergence of α

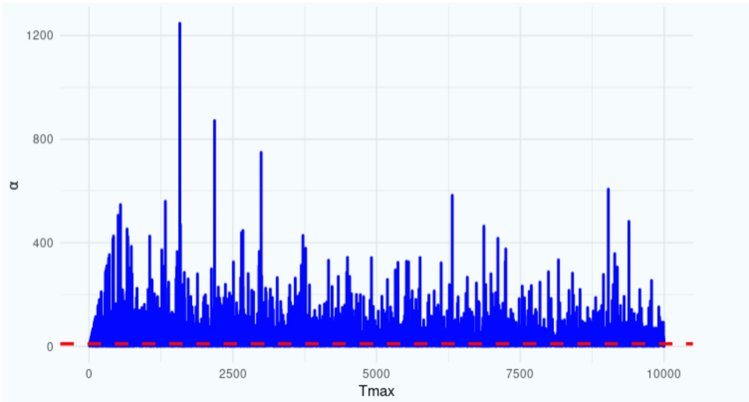


Figure 4: Convergence of α as a function of T_{max}

We study the convergence of the estimator of α, $\hat{\alpha}$, the same ways as $\hat{\beta}$. However, we do not observe the same results in Figure 4, $\hat{\alpha}$ does not converge towards α over time. This means that the estimator we computed is not consistent.

7 Construction of a confidence interval for β

- **Estimator of** β: β is estimated using its maximum likelihood estimator $\hat{\beta}$.
- **Pivotal statistic**

From Proposition 3.5 in the textbook *Processus stochastiques et fiabilité des systèmes* (C.Cocozza-Thivent, 1997), we have

$$\frac{2n\beta}{\hat{\beta}} \mid \{N_t = n\} \sim \chi^2(2n).$$

β is the only unknown parameter, and the distribution is known. It corresponds to the pivotal statistic.

- **Confidence interval at level** $1 - \epsilon$

Let $x_{d,\eta}$ denote the η-quantile of a $\chi^2(d)$ distribution. Thus

$$\mathbb{P}(x_{2n,\eta/2} \leq \frac{2n\beta}{\hat{\beta}} \leq x_{2n,1-\eta/2}) = 1 - \eta,$$

i.e.

$$\mathbb{P}(\frac{\hat{\beta}x_{2n,\eta/2}}{2n} \leq \beta \leq \frac{\hat{\beta}x_{2n,1-\eta/2}}{2n}) = 1 - \eta.$$

Finally, we deduce $CI_{1-\eta}(\beta) = [\frac{\hat{\beta}x_{2n,\eta/2}}{2n}, \frac{\hat{\beta}x_{2n,1-\eta/2}}{2n}]$.

8 Construction of an asymptotic confidence interval for β

- **Estimator of** β: β is estimated using its maximum likelihood estimator $\hat{\beta}$.
- **Pivotal statistic**

From Proposition 3.11 in the textbook *Processus stochastiques et fiabilité des systèmes* (C.Cocozza-Thivent, 1997). , we know that

$$\sqrt{\Lambda(t)} \cdot (\frac{\beta}{\hat{\beta}} - 1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

This result is demonstrated in Appendix A.5. In addition, we have

$$\sqrt{\Lambda(t)} \cdot (\frac{\beta}{\hat{\beta}} - 1) = \sqrt{N(t)} \cdot (\frac{\beta}{\hat{\beta}} - 1).$$

In fact, we know that $\sqrt{\Lambda(t)} \cdot (\frac{\beta}{\hat{\beta}} - 1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$, and $\sqrt{\frac{N(t)}{\Lambda(t)}} \xrightarrow{\mathbb{P}} 1$. So, by Slutsky's theorem, we deduce that

$$\sqrt{\frac{N(t)}{\Lambda(t)}} \cdot \sqrt{\Lambda(t)} \cdot (\frac{\beta}{\hat{\beta}} - 1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1),$$

i.e.

$$\sqrt{N(t)} \cdot (\frac{\beta}{\hat{\beta}} - 1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

This is the pivotal statistic.

- **Confidence interval at level** $1 - \epsilon$

Let z_ϵ denote the ϵ-quantile of the asymptotic $\mathcal{N}(0,1)$ distribution. Then, by definition of convergence in distribution,

$$\mathbb{P}(-z_{1-\frac{\epsilon}{2}} \leq \sqrt{N(t)} \cdot (\frac{\beta}{\hat{\beta}} - 1) \leq z_{1-\frac{\epsilon}{2}}) \underset{t\to\infty}{\longrightarrow} \mathbb{P}(-z_{1-\frac{\epsilon}{2}} \leq Z \leq z_{1-\frac{\epsilon}{2}}) = 1 - \epsilon.$$

By isolating the parameter β in the above probability, we deduce that

$$\mathbb{P}(\hat{\beta} \cdot (1 - \frac{z_{1-\frac{\epsilon}{2}}}{\sqrt{N(t)}}) \leq \beta \leq \hat{\beta} \cdot (1 + \frac{z_{1-\frac{\epsilon}{2}}}{\sqrt{N(t)}}) \underset{t\to\infty}{\longrightarrow} 1 - \epsilon.$$

Finally, we obtain $CI^a_{1-\epsilon}(\beta) = [\hat{\beta} \cdot (1 - \frac{z_{1-\frac{\epsilon}{2}}}{\sqrt{N(t)}}); \hat{\beta} \cdot (1 + \frac{z_{1-\frac{\epsilon}{2}}}{\sqrt{N(t)}})]$.

9 Construction of an asymptotic confidence interval for α

- **Estimator of** α: α is estimated using its maximum likelihood estimator $\hat{\alpha}$
- **Pivotal statistic**

Let

$$\frac{\sqrt{N(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha)) = \sqrt{\frac{N(t)}{\Lambda(t)}} \frac{\sqrt{\Lambda(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha)).$$

We know that $\frac{\sqrt{\Lambda(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha)) \underset{t\to+\infty}{\xrightarrow{\mathcal{L}}} \mathcal{N}(0,1)$ (the proof is in Appendix A.6) and $\sqrt{\frac{N(t)}{\Lambda(t)}} \underset{t\to+\infty}{\xrightarrow{\mathbb{P}}} 1$. So, by Slutsky's theorem

$$\frac{\sqrt{N(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha)) \underset{t\to+\infty}{\xrightarrow{\mathcal{L}}} \mathcal{N}(0,1).$$

- **Confidence interval at level** $1 - \epsilon$

We search an asymptotic interval for α at level $1 - \epsilon$. We search $c > 0$ such that

$$\mathbb{P}\left(\left|\frac{\sqrt{N(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha))\right| \leq c\right) \underset{t\to+\infty}{\longrightarrow} \mathbb{P}(X \leq c) = 1 - \epsilon \text{ with } X \sim \mathcal{N}(0,1).$$

So $c = z_{1-\frac{\epsilon}{2}}$, with $z_{1-\frac{\epsilon}{2}}$ being the $\left(1-\frac{\epsilon}{2}\right)$-quantile of a $\mathcal{N}(0,1)$. We then have

$$
\begin{aligned}
&\mathbb{P}\left(\left|\frac{\sqrt{N(t)}}{\log(t)}(\log(\hat{\alpha})-\log(\alpha))\right| \leq z_{1-\frac{\epsilon}{2}}\right) \underset{t\to+\infty}{\longrightarrow} 1-\epsilon \\
\iff\ &\mathbb{P}\left(|\log(\hat{\alpha})-\log(\alpha)| \leq \frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}}\right) \underset{t\to+\infty}{\longrightarrow} 1-\epsilon \\
\iff\ &\mathbb{P}\left(-\frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}} \leq \log(\hat{\alpha})-\log(\alpha) \leq \frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}}\right) \underset{t\to+\infty}{\longrightarrow} 1-\epsilon \\
\iff\ &\mathbb{P}\left(\log(\hat{\alpha})-\frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}} \leq \log(\alpha) \leq \log(\hat{\alpha})+\frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}}\right) \underset{t\to+\infty}{\longrightarrow} 1-\epsilon \\
\iff\ &\mathbb{P}\left(\log(\alpha) \in \left[\log(\hat{\alpha}) \pm \frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}}\right]\right) \underset{t\to+\infty}{\longrightarrow} 1-\epsilon \\
\iff\ &\mathbb{P}\left(\alpha \in \left[\hat{\alpha}\exp\left(\pm\frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}}\right)\right]\right) \underset{t\to+\infty}{\longrightarrow} 1-\epsilon
\end{aligned}
$$

Finally, we have

$$
CI^{a}_{1-\epsilon}(\alpha) = \left[\hat{\alpha}\exp\left(-\frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}}\right), \hat{\alpha}\exp\left(+\frac{\log(t)}{\sqrt{N(t)}} z_{1-\frac{\epsilon}{2}}\right)\right].
$$

10 Numerical verification of confidence intervals for α and β

The following figures (Figure 5, Figure 6, Figure 7) present numerical verification of confidence intervals for α and β.

```
## Asymptotic interval for alpha = 1 with beta = 2.5 : [0.99695, 1.00177]
## Asymptotic interval for alpha = 1 with beta = 0.8 : [0.71033, 4.02072]
## Asymptotic interval for alpha = 30 with beta = 2.5  : [25.66577, 36.08883]
## Asymptotic interval for alpha = 30 with beta = 0.8 : [2.82422, 2459.48788]
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Figure 5: Asymptotic confidence interval results for α

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Asymptotic interval for beta = 2.5 with alpha=1: [2.49894, 2.50068]
Asymptotic interval for beta = 0.8 with alpha=1: [0.74707, 0.96246]
Asymptotic interval for beta = 2.5 with alpha=30: [2.51557, 2.64290]
Asymptotic interval for beta = 0.8 with alpha=30: [0.50266, 1.36586]
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Figure 6: Asymptotic confidence interval results for β

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Confidence interval for beta = 2.5 with alpha=1: [2.49894, 2.50068]
Confidence interval for beta = 0.8 with alpha=1: [0.75045, 0.96246]
Confidence interval for beta = 2.5 with alpha=30: [2.51595, 2.64290]
Confidence interval for beta = 0.8 with alpha=30: [0.55370, 1.36586]
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Figure 7: Confidence interval results for β

We notice that for $\beta > 1$, the intervals tend to be more precise. $\beta > 1$ indicates an increasing intensity over time, generally leading to more precise confidence intervals if the data adequately covers the time range. Conversely, a $\beta < 1$ corresponds to a decreasing intensity, resulting in wider confidence intervals due to the difficulty in estimating the intensity at longer times. In addition, we notice that the size of the interval increases when α increases. In fact, the scale parameter α stretches or compresses the time range of the intensity: a larger α allows for the analysis of longer times but may increase uncertainty if the data is concentrated in shorter times. This is consistent with the results of the estimators, which were not precise in the case of large α and $\beta < 1$.

11 Analysis of Boeing data

In the first part of this report, we explored the theoretical aspects of Weibull processes. Now, we apply this model to real-world data. The data (available in Appendix A.7) is sourced from the article *Theoretical Explanation of Observed Decreasing Failure Rate* (Proschan, 1963). This dataset represents the occurence times of successive failures of the air conditioning system of 15,720 aircrafts.

11.1 General analysis

For the remainder of the analysis, we focus on three aircraft (B09, B12, B13), as they contain the most data. Figure 8 represents the cumulative number of failures over time for our three aircrafts.

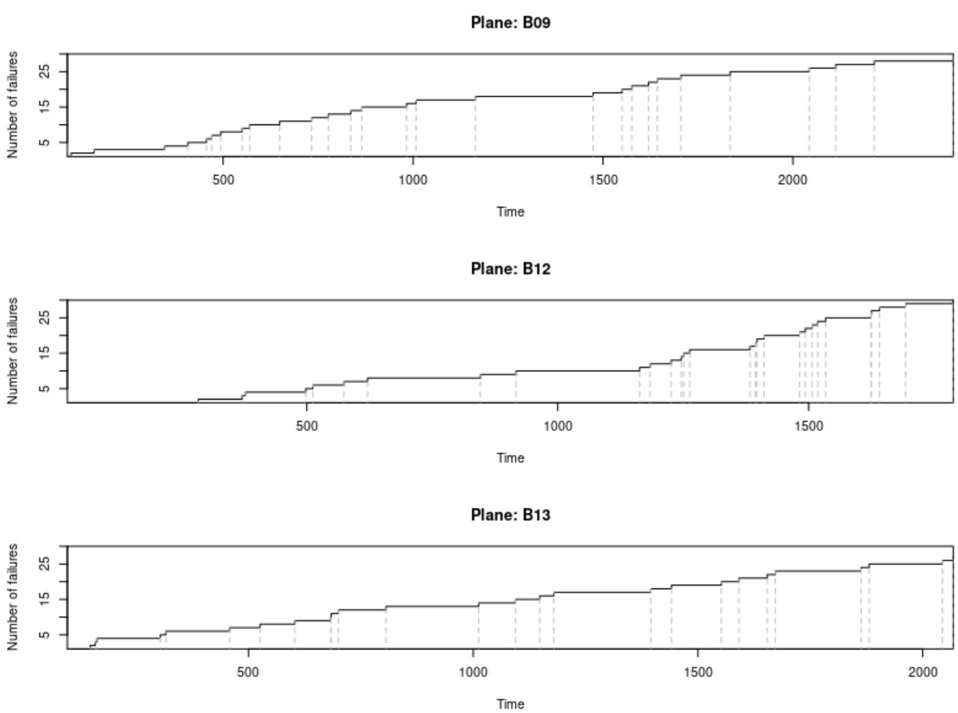

Figure 8: Cumulative number of failures for each aircraft

For B09, the failures initially occur more frequently and then gradually decrease over time. This may indicate a decreasing failure rate as the air conditioning system stabilizes after early-phase failures. In contrast, B12 shows a more irregular pattern, with periods of stability followed by bursts of failures, particularly toward the end of the timeline. This suggests an aging system or accumulation of wear that leads to a higher failure rate in later stages. And B13 shows a regular pattern, which may be a sign of a system in use.

We assume that the data can be modeled using Weibull processes. To do this, we estimate the parameters α and β for each aircraft, with the results provided in Appendix A.9. Using these estimated parameters, we then simulate the Weibull intensities for both aircrafts. The results are in Figure 9.

For B09 ($\alpha = 57.62, \beta = 0.9$), the intensity starts low and gradually decreases over time. $\beta < 1$ indicates a decreasing failure rate, typical of systems prone to early failures. This is possibly due to initial defects or wear-out phases. The relatively small α suggests that these failures occur over a shorter time scale. For B12 ($\alpha = 186.85, \beta = 1.51$), the intensity steadily increases, suggesting a rising failure rate, $\beta > 1$. This behavior reflects system degradation or wear-out over time. The large value of α shows that these failures occur over a longer time scale, suggesting progressive deterioration. For B13 ($\alpha = 72.94, \beta = 0.99$), the intensity starts higher but decreases slightly over time. Since $\beta \approx 1$, the failure rate is nearly constant. Moderate α implies that failures are distributed over an intermediate time scale.

In summary, B12 shows a clear wear pattern and may require more maintenance due to its increasing failure rate. B09 and B13 both stabilize over time, with B09 exhibiting early failures over a shorter period, while B13 remains relatively stable. The differences in α highlight the varying time scales over which these systems fail.

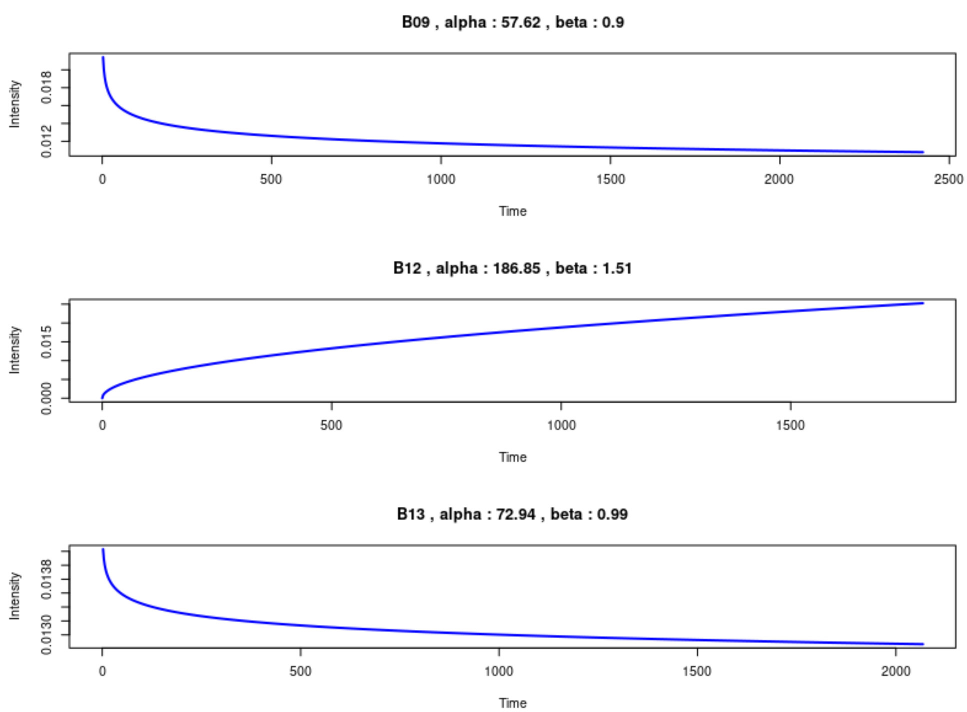

Figure 9: Weibull intensity with the estimated parameters

Furthermore, the confidence intervals for α and β, available in Appendix A.10, are very wide. Thus, we can assume a large uncertainty in our estimators, maybe due to the amount of data we have. We also present the estimated number of failures (in Appendix A.11) and compare it to the figures derived from the data, which validates our modeling choice.

11.2 Analysis before and after repair

In Section 11.1, we did not take into account that aircrafts may have undergone repairs. Now, we look at the breakdowns before and after repair and observe if there are any changes. The data are in Appendix A.8.

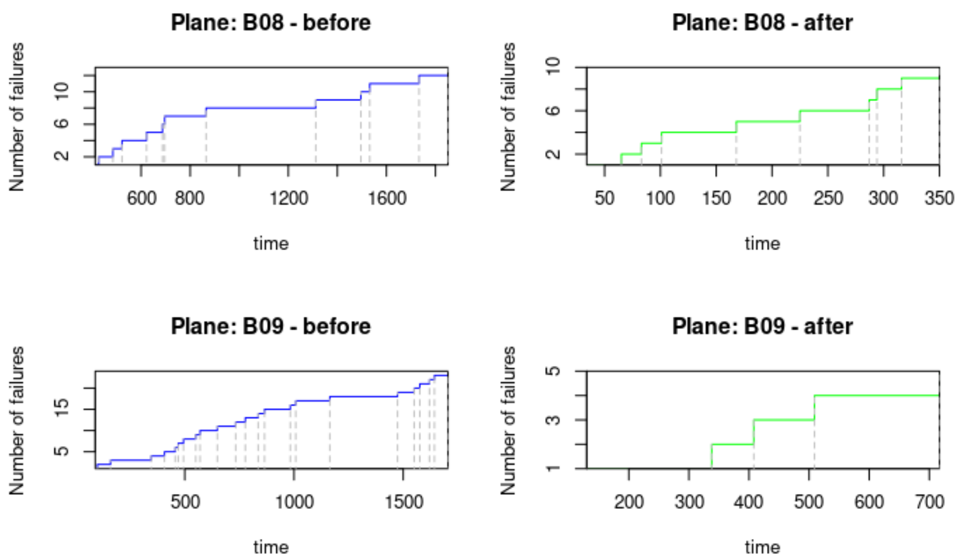

Figure 10: Cumulative number of failures evolution for each aircraft before and after a major repair

Figure 10 shows the cumulative number of failures over time for aircrafts B08 and B09, before and after repairs.

For aircraft B08, before repairs, the total number of failures gradually increases over a long period, with occasional failures. There are periods without failures, followed by times with more frequent failures. After repairs, the number of failures increases rapidly in a shorter time. The failures are more concentrated with shorter intervals between them. This suggests that the repair did not improve reliability. Instead, it led to more frequent failures in a shorter period.

For aircraft B09, before repair, failures increase steadily over a long period, with fairly regular intervals between them. This shows a system with a nearly constant failure rate. After repair, the number of failures increases much more slowly, with only a few failures over a longer period. The intervals between failures become longer, suggesting that the repair effectively reduced the frequency of failures and temporarily stabilized the system.

By comparing both aircraft, we conclude that the repair of B09 was more effective. B08 experienced a shorter operational period with more frequent failures after repair, while B09 showed a significant reduction in failures and longer intervals between them.

As before, we estimate the parameters (α, β). The R output can be found in Appendix A.12.

In Figure 11, before repair, the Weibull intensity function for B08 ($\alpha = 247.17, \beta = 1.27$) shows a steadily increasing failure intensity over time. It reflects a system experiencing wear and degradation. The high-scale parameter α suggests that failures are distributed over a long period. The shape parameter β confirms an increasing failure rate, typical of a system aging over time. After repair, the intensity function changes significantly ($\alpha = 51.2, \beta = 1.2$). The scale parameter decreases, indicating that failures occur over a much shorter time frame, and the initial failure intensity is higher. This suggests that the repair may not have fully addressed the underlying degradation.

Before repair of B09, the Weibull intensity function ($\alpha = 80.06, \beta = 1.04$) indicates a nearly constant failure rate, as the shape parameter is close to 1. The scale parameter suggests that failures are spaced over time, and the overall system shows stable behavior with minimal degradation. After repair, the intensity function also changes significantly ($\alpha = 242.65, \beta = 1.49$). The scale parameter increases, meaning that failures are distributed over a longer period of time. However, the shape parameter indicates a steep increase in failure intensity over time. This suggests that while the repair temporarily stabilized the system, it did not prevent accelerated degradation. The system shows signs of wear-out and failures become more frequent over time.

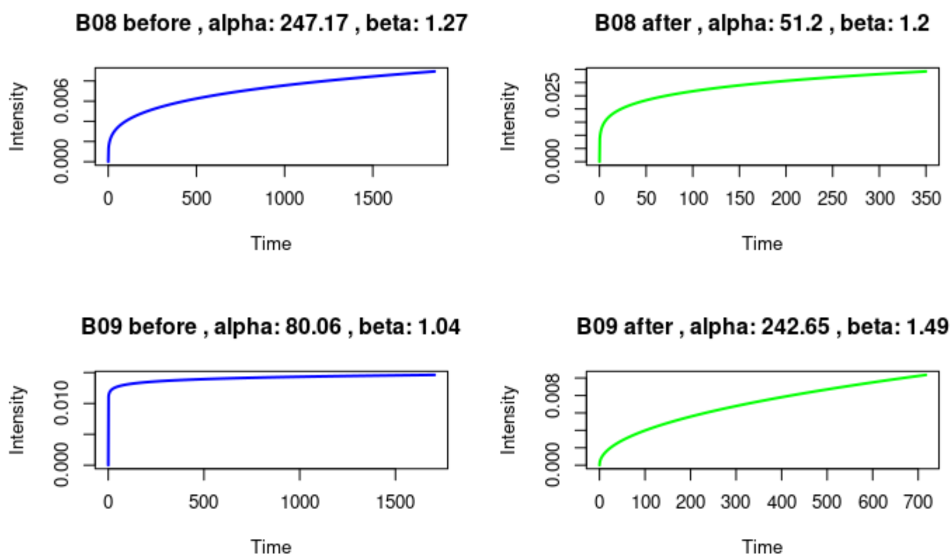

Figure 11: Weibull intensity with the estimated parameters, before and after a major repair

As before, we display the confidence intervals of α and β (in Appendix A.13) and the estimated number of failures (in Appendix A.14). We cannot really conclude on the quality of the results given the amount of data.

12 Conclusion

In this project, we have studied the Weibull distribution from a theoretical point of view and used it to analyze the air conditioning failures of Boeing aircrafts.

We estimated the parameters of the intensity function of the Weibull process. Moreover, we constructed confidence intervals for the parameters α and β. We noticed that the choice of these parameters greatly impacts the span of confidence intervals. Then we applied the model to real-world data to determine whether we can predict the number of breakdowns and recognize the different phases of the air conditioning system's lifecycle.

The results confirm our initial interpretations about the failure patterns. The three aircrafts we studied describe the three phases of a system's lifecycle: B09 the burn-in phase, B13 the operational phase, and B12 the wear-out phase.

In the experimental section, we observed that the results were not always convincing, likely due to the limited availability of Boeing data.

For future studies, it would be interesting to gather more data to have representative results.

13 References

- Christiane Cocozza-Thivent. Processus stochastiques et fiabilité des systèmes, volume 28. Springer Science & Business Media, 1997.
- Frank Proschan. Theoretical explanation of observed decreasing failure rate. Technometrics, 5(3):375–383, 1963.
- Mélisande Albert. Poisson Processes and Applications to Reliability Theory and Actuarial Science, 2024.

A Appendix

A.1 The impact of α on Weibull intensity

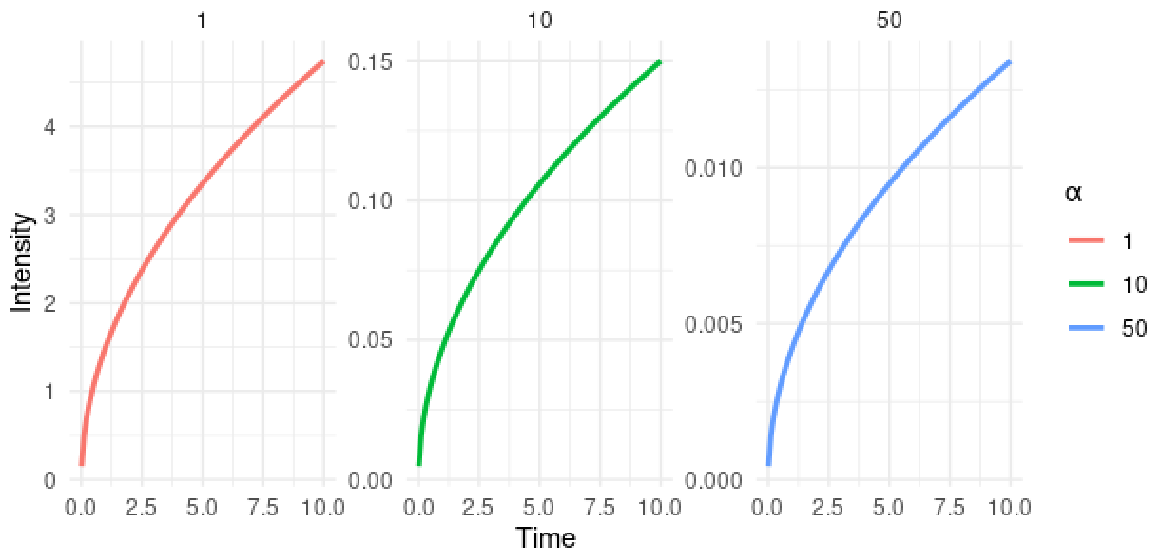

Figure 12: Intensity for different α with $\beta = 1.5$.

A.2 Detailed justification of $\hat{\beta}$ in part 3.3

$$
\begin{aligned}
\frac{1}{\hat{\beta}} &= \log(\hat{\alpha}) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}}\log\left(\frac{t}{\hat{\alpha}}\right) \\
&= \log(\hat{\alpha}) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}}(\log(t)-\log(\hat{\alpha})) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}}(\log(t)-\log(t)+\frac{1}{\hat{\beta}}\log(N(t))) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}}\frac{1}{\hat{\beta}}\log(N(t)) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\frac{1}{\hat{\beta}}\log(N(t))\left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}} \\
&= \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\frac{1}{\hat{\beta}}\log(N(t))\exp(\hat{\beta}(\log(t)-\log(\hat{\alpha}))) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\frac{1}{\hat{\beta}}\log(N(t))\exp(\hat{\beta}\frac{1}{\hat{\beta}}\log(N(t))) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i) + \frac{1}{N(t)}\frac{1}{\hat{\beta}}\log(N(t))N(t) \\
&= \log(t) - \frac{1}{N(t)}\sum_{i=1}^{N(t)}\log(T_i)
\end{aligned}
$$

A.3 Detailed justification of the Hessian in part 3.3

Let's compute the Hessian to check that we have local maxima.

$$
\begin{aligned}
H[\mathcal{L},\alpha,\beta] &= \begin{pmatrix} \frac{\beta N(t)}{\alpha^2} - \frac{\beta(\beta+1)t^\beta}{\alpha^{\beta+2}} & \frac{-N(t)}{\alpha} + \frac{t^\beta}{\alpha^{\beta+1}} + \frac{\beta\log(t)t^\beta}{\alpha^{\beta+1}} - \frac{\beta t^\beta\log(\alpha)}{\alpha^{\beta+1}} \\ \frac{-N(t)}{\alpha} + \frac{t^\beta}{\alpha^{\beta+1}} + \frac{\beta\log(t)t^\beta}{\alpha^{\beta+1}} - \frac{\beta t^\beta\log(\alpha)}{\alpha^{\beta+1}} & \frac{-N(t)}{\beta^2} - \log\left(\frac{t}{\alpha}\right)^2\left(\frac{t}{\alpha}\right)^\beta \end{pmatrix} \\
&= \begin{pmatrix} \frac{\beta N(t)}{\alpha^2} - \frac{\beta(\beta+1)t^\beta}{\alpha^{\beta+2}} & \frac{-N(t)}{\alpha} + \frac{t^\beta}{\alpha^{\beta+1}} + \frac{\beta t^\beta}{\alpha^{\beta+1}}\log\left(\frac{t}{\alpha}\right) \\ \frac{-N(t)}{\alpha} + \frac{t^\beta}{\alpha^{\beta+1}} + \frac{\beta t^\beta}{\alpha^{\beta+1}}\log\left(\frac{t}{\alpha}\right) & \frac{-N(t)}{\beta^2} - \log\left(\frac{t}{\alpha}\right)^2\left(\frac{t}{\alpha}\right)^\beta \end{pmatrix}
\end{aligned}
$$

From the $\hat{\alpha}$ formula, we can deduce two equality. The first one

$$
\begin{aligned}
\log(\hat{\alpha}) = \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) &\iff \hat{\beta}\log(\hat{\alpha}) = \hat{\beta}\log(t) - \log(N(t)) \\
&\iff \log(\hat{\alpha}^{\hat{\beta}}) = \log(t^{\hat{\beta}}) - \log(N(t)) \\
&\iff \log(N(t)) = \log\left(\frac{t^{\hat{\beta}}}{\hat{\alpha}^{\hat{\beta}}}\right) \\
&\iff N(t) = \frac{t^{\hat{\beta}}}{\hat{\alpha}^{\hat{\beta}}} \\
&\iff \frac{\hat{\beta}N(t)}{\hat{\alpha}} = \frac{\hat{\beta}t^{\hat{\beta}}}{\hat{\alpha}^{\hat{\beta}+1}}.
\end{aligned}
$$

And the second

$$\log(\hat{\alpha}) = \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) \iff \log\left(\frac{t}{\hat{\alpha}}\right) = \frac{1}{\hat{\beta}}\log(N(t)).$$

So, we can simplify the Hessian

$$\begin{aligned} H[\mathcal{L}, \hat{\alpha}, \hat{\beta}] &= \begin{pmatrix} -\frac{\hat{\beta}^2 t^{\hat{\beta}}}{\hat{\alpha}^{\hat{\beta}+2}} & \frac{\hat{\beta} t^{\hat{\beta}}}{\hat{\alpha}^{\hat{\beta}+1}} \log\left(\frac{t}{\hat{\alpha}}\right) \\ \frac{\hat{\beta} t^{\hat{\beta}}}{\hat{\alpha}^{\hat{\beta}+1}} \log\left(\frac{t}{\hat{\alpha}}\right) & \frac{-N(t)}{\hat{\beta}^2} - \frac{1}{\hat{\beta}^2}\log(N(t))^2\left(\frac{t}{\hat{\alpha}}\right)^{\hat{\beta}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\hat{\beta}^2 N(t)}{\hat{\alpha}^2} & \frac{N(t)}{\hat{\alpha}}\log(N(t)) \\ \frac{N(t)}{\hat{\alpha}}\log(N(t)) & \frac{-N(t)}{\hat{\beta}^2}(1+\log(N(t))^2) \end{pmatrix}. \end{aligned}$$

A.4 Numerical verification for the estimators of α and β

```
## True values : (1, 2.5), estimated values : (0.99888, 2.49959)
```

```
## True values : (1, 0.8), estimated values : (1.52949, 0.84605)
```

```
## True values : (30, 2.5), estimated values : (31.01203, 2.51917)
```

```
## True values : (30, 0.8), estimated values : (15.91262, 0.57911)
```

Figure 13: Estimation of α and β

A.5 Proof of the pivotal statistic of β

To do so, it suffices to show that

$$Z_t = \frac{N(t)}{\Lambda(t)} \cdot \sqrt{\Lambda(t)} \cdot \left(\frac{\beta}{\hat{\beta}} - 1\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

Using Proposition 3.5 and Proposition 1.11 of the book, we deduce

$$\begin{aligned} \mathbb{E}\left(\exp(iuZ_t) \,\middle|\, \{N_t = n\}\right) &= \mathbb{E}\left(\exp\left(iu\left(\frac{N(t)}{\Lambda(t)} \cdot \sqrt{\Lambda(t)} \cdot \left(\frac{\beta}{\hat{\beta}} - 1\right)\right)\right) \,\middle|\, \{N_t = n\}\right) \\ &= \mathbb{E}\left(\exp\left(iu\left(\frac{n}{\sqrt{\Lambda(t)}} \cdot \left(\frac{\beta}{\hat{\beta}} - 1\right)\right)\right)\right) \\ &= \mathbb{E}\left(\exp\left(\frac{iun\beta}{\sqrt{\Lambda(t)}\hat{\beta}}\right) \cdot \exp\left(-\frac{iun}{\sqrt{\Lambda(t)}}\right)\right) \\ &= \exp\left(-\frac{iun}{\sqrt{\Lambda(t)}}\right) \cdot \mathbb{E}\left(\exp\left(\frac{iun\beta}{\sqrt{\Lambda(t)}\hat{\beta}}\right)\right) \\ &= \exp\left(-\frac{iun}{\sqrt{\Lambda(t)}}\right) \cdot \frac{1}{(1+\frac{iu}{\Lambda(t)})^n}. \end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}(e^{iuZ_t}) &= \sum_{n\geq 0} \frac{e^{-iun\sqrt{\Lambda(t)}}}{(1-iu/\sqrt{\Lambda(t)})^n} \frac{e^{-\Lambda(t)}\Lambda(t)^n}{n!} \\
&= \exp\left(-\Lambda(t) + \frac{\Lambda(t)}{1-\frac{iu}{\sqrt{\Lambda(t)}}} \cdot \exp\left(\frac{-iu}{\sqrt{\Lambda(t)}}\right)\right) \\
&= \exp\left(\Phi(t)\right).
\end{aligned}$$

We have

$$\Phi(t) = \frac{\Lambda(t)}{1-\frac{iu}{\sqrt{\Lambda(t)}}} \cdot \left(\exp\left(\frac{-iu}{\sqrt{\Lambda(t)}}\right) - 1 + \frac{iu}{\sqrt{\Lambda(t)}}\right).$$

We perform a Taylor expansion of $\exp\left(\frac{-iu}{\sqrt{\Lambda(t)}}\right)$ around $u=0$, assuming that $\frac{-iu}{\sqrt{\Lambda(t)}}$ is small when $\Lambda(t)$ is large. Plus, we know that $\Lambda(t)\to\infty$ as $\Lambda(t)$ is typically the cumulative intensity function, which increases as $t\to\infty$. This implies

$$\exp\left(\frac{-iu}{\sqrt{\Lambda(t)}}\right) \approx 1 - \frac{iu}{\sqrt{\Lambda(t)}} - \frac{u^2}{2\Lambda(t)} + \mathcal{O}\left(\frac{u^3}{\Lambda(t)^{3/2}}\right), \text{ as } \frac{-iu}{\sqrt{\Lambda(t)}} \to 0.$$

Therefore, we deduce

$$\begin{aligned}
\Phi(t) &= \frac{\Lambda(t)}{1-\frac{iu}{\sqrt{\Lambda(t)}}} \cdot \left(1 - \frac{iu}{\sqrt{\Lambda(t)}} - \frac{u^2}{2\cdot\Lambda(t)} - 1 + \frac{iu}{\sqrt{\Lambda(t)}}\right) \\
&= \frac{-u^2}{2\cdot(1-\frac{iu}{\Lambda(t)})} \\
&\underset{t\to\infty}{\approx} \frac{-u^2}{2}.
\end{aligned}$$

Hence

$$\mathbb{E}(e^{iuZ_t}) = \exp\left(-\frac{u^2}{2}\right).$$

This matches the characteristic function of a standard normal distribution, so $Z_t \xrightarrow[t\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$.

A.6 Proof of the pivotal statistic of α

Let's prove that

$$\frac{\sqrt{\Lambda(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha)) \xrightarrow[t\to+\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

We know that

$$\begin{aligned}
\log(\hat{\alpha}) &= \log(t) - \frac{1}{\hat{\beta}}\log(N(t)) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log\left(\frac{N(t)}{\Lambda(t)}\right) - \frac{1}{\hat{\beta}}\log(\Lambda(t)) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log\left(\frac{N(t)}{\Lambda(t)}\right) - \frac{1}{\hat{\beta}}\log\left(\left(\frac{t}{\alpha}\right)^{\beta}\right) \\
&= \log(t) - \frac{1}{\hat{\beta}}\log\left(\frac{N(t)}{\Lambda(t)}\right) - \frac{\beta}{\hat{\beta}}\log\left(\frac{t}{\alpha}\right) \\
&= \frac{\beta}{\hat{\beta}}\log(\alpha) - \frac{1}{\hat{\beta}}\log\left(\frac{N(t)}{\Lambda(t)}\right) - \left(\frac{\beta}{\hat{\beta}} - 1\right)\log(t).
\end{aligned}$$

Then, if we replace $\log(\hat{\alpha})$ by the previous formula, we have

$$\begin{aligned}
\frac{\sqrt{\Lambda(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha)) &= \frac{\sqrt{\Lambda(t)}}{\log(t)}\left(\frac{\beta}{\hat{\beta}}\log(\alpha) - \frac{1}{\hat{\beta}}\log\left(\frac{N(t)}{\Lambda(t)}\right) - \left(\frac{\beta}{\hat{\beta}} - 1\right)\log(t) - \log(\alpha)\right) \\
&= \frac{\log(\alpha)}{\log(t)}\sqrt{\Lambda(t)}\left(\frac{\beta}{\hat{\beta}} - 1\right) - \frac{1}{\hat{\beta}\log(t)}\sqrt{\Lambda(t)}\log\left(\frac{N(t)}{\Lambda(t)}\right) - \sqrt{\Lambda(t)}\left(\frac{\beta}{\hat{\beta}} - 1\right).
\end{aligned}$$

Now, we study the convergence of each terms.

- For $-\frac{1}{\hat{\beta}\log(t)}\sqrt{\Lambda(t)}\log\left(\frac{N(t)}{\Lambda(t)}\right)$: First, we know that, $\sqrt{\Lambda(t)}\left(\frac{N(t)}{\Lambda(t)} - 1\right) \xrightarrow[t\to+\infty]{\mathcal{L}} \mathcal{N}(0,1)$ and $\frac{N(t)}{\Lambda(t)} \xrightarrow[t\to+\infty]{\mathbb{P}} 1$. So, by using proposition 3.9, with $X(t) = \frac{N(t)}{\Lambda(t)}$, $m = 1$, $c(t) = \sqrt{\Lambda(t)}$, $\Gamma = 1$ and $f = \log$, we have,

$$\sqrt{\Lambda(t)}\log\left(\frac{N(t)}{\Lambda(t)}\right) = \sqrt{\Lambda(t)}\left(\log(\frac{N(t)}{\Lambda(t)}) - \log(1)\right) \xrightarrow[t\to+\infty]{\mathcal{L}} \mathcal{N}(0,K)\text{with } K = Jf(m)\Gamma Jf(m)^T = 1$$

We also know that, $-\frac{1}{\hat{\beta}\log(t)} \xrightarrow[t\to+\infty]{\mathbb{P}} 0$. So, by proposition 3.8, we have $-\frac{1}{\hat{\beta}\log(t)}\sqrt{\Lambda(t)}\log\left(\frac{N(t)}{\Lambda(t)}\right) \xrightarrow[t\to+\infty]{\mathbb{P}} 0$.

- For $\frac{\log(\alpha)}{\log(t)}\sqrt{\Lambda(t)}\left(\frac{\beta}{\hat{\beta}} - 1\right)$: We know by previous results that $\sqrt{\Lambda(t)}\left(\frac{\beta}{\hat{\beta}} - 1\right) \xrightarrow[t\to+\infty]{\mathcal{L}} \mathcal{N}(0,1)$ and $\frac{\log(\alpha)}{\log(t)} \xrightarrow[t\to+\infty]{\mathbb{P}} 0$. So, by proposition 3.8, $\frac{\log(\alpha)}{\log(t)}\sqrt{\Lambda(t)}\left(\frac{\beta}{\hat{\beta}} - 1\right) \xrightarrow[t\to+\infty]{\mathbb{P}} 0$.

- For $-\sqrt{\Lambda(t)}(\frac{\beta}{\hat{\beta}} - 1)$: Analogously, $-\sqrt{\Lambda(t)}(\frac{\beta}{\hat{\beta}} - 1) \xrightarrow[t\to+\infty]{\mathcal{L}} -X$ with $X \sim \mathcal{N}(0,1)$. But the law of $-X$ is equal to the law of X.

Finally, we have

$$\frac{\sqrt{\Lambda(t)}}{\log(t)}(\log(\hat{\alpha}) - \log(\alpha)) \xrightarrow[t\to+\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

A.7 Boeing data

B07	B08	B09	B10	B11	B12	B13	B14	B15	B16	B17	B44	B45
194	413	90	74	55	23	97	50	359	50	130	487	102
15	14	10	57	320	261	51	44	9	254	493	18	209
41	58	60	48	56	87	11	102	12	5		100	14
29	37	186	29	104	7	4	72	270	283		7	57
33	100	61	502	220	120	141	22	603	35		98	54
181	65	49	12	239	14	12	39	3	12		5	32
	9	14	70	47	62	142	3	104			85	67
	169	24	21	246	47	68	15	2			91	59
	447	56	29	176	225	77	197	438			43	134
	184	20	386	182	71	80	188				230	152
	36	79	59	33	246	1	79				3	27
	201	84	27		21	16	88				130	14
	118	44		15	42	106	46					230
	34	59	153	104	20	206	5					66
	31	29	26	35	5	82	5					61
	18	118	326		12	54	36					34
	18	25			120	31	22					
	67	156			11	216	139					
	57	310			3	46	210					
	62	76			14	111	97					
	7	26			71	39	30					
	22	44			11	63	23					
	34	23			14	18	13					
		62			11	191	14					
		130			16	18						
		208			90	163						
		70			1	24						
		101			16							
		208			52							
					95							

Figure 14: Boeing data

A.8 Boeing data before and after repair

B08_before	B08_after	B09_before	B09_after
413	34	90	130
14	31	10	208
58	18	60	70
37	18	186	101
100	67	61	208
65	57	49	
9	62	14	
169	7	24	
447	22	56	
184	34	20	
36		79	
201		84	
118		44	
		59	
		29	
		118	
		25	
		156	
		310	
		76	
		26	
		44	
		23	
		62	

Figure 15: Boeing data before and after a major repair

A.9 Estimation of α and β of Boeing data

```
## $B09_T
## [1] 57.6238078  0.9007287
##
## $B12_T
## [1] 186.847858   1.505915
##
## $B13_T
## [1] 72.9409859  0.9853948
```

Figure 16: Estimation of α and β for each aircraft

A.10 Confidence intervals of α and β of Boeing data

```
## $B09_T
## [1]   3.379955 982.410574
##
## $B12_T
## [1]   12.81341 2724.65492
##
## $B13_T
## [1]    4.096017 1298.917369
```

Figure 17: Asymptotic confidence intervals of α for each aircraft

```
$B09_T
[1] 0.6032321 1.2569139

$B12_T
[1] 1.016035 2.090654

$B13_T
[1] 0.6493814 1.3903565
```

Figure 18: Confidence intervals of β for each aircraft

A.11 Estimated number of failures of Boeing data

```
## $B09_T
## [1] 29
##
## $B12_T
## [1] 30
##
## $B13_T
## [1] 27
```

Figure 19: Estimated number of failures of Boeing data

A.12 Estimation of α and β of Boeing data before and after repair

```
## $B08_before_T
## [1] 247.170766   1.273938
##
## $B08_after_T
## [1] 51.201886  1.197917
##
## $B09_before_T
## [1] 80.060185  1.039075
##
## $B09_after_T
## [1] 242.645060   1.485439
```

Figure 20: Numerical results of the estimations of α and β, before and after a major repair

A.13 Confidence intervals of α and β of Boeing data before and after repair

```
## $B08_before_T
## [1]     4.138542 14762.055297
##
## $B08_after_T
## [1]    1.356704 1932.354001
##
## $B09_before_T
## [1]    4.07832 1571.63563
##
## $B09_after_T
## [1] 7.621572e-01 7.724998e+04
```

Figure 21: Asymptotic confidence interval for α before and after a major repair

```
$B08_before_T
[1] 0.6783184 2.0541355

$B08_after_T
[1] 0.574448 2.046618

$B09_before_T
[1] 0.6657549 1.4941590

$B09_after_T
[1] 0.4823181 3.0426515
```

Figure 22: Confidence interval for β before and after a major repair

A.14 Estimated number of failures of Boeing data before and after repair

```
## $B08_before_T
## [1] 13
##
## $B08_after_T
## [1] 10
##
## $B09_before_T
## [1] 24
##
## $B09_after_T
## [1] 5
```

Figure 23: Number of failures per aircraft before and after a major repair