TP4 : Multigrid methods

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1 Introduction

In this TP, we will implement a V-cycle multigrid scheme with L + 1 grid levels. We keep the same problem setting as in TP3, but we use a different analytic solution. We search for a solution of a 1D Poisson problem.

$$-u'' = f \text{ in } \Omega = [0,1] \text{ and } u(0) = u(1) = 0,$$

The analytic solution is the following

$$u(x) = x^2(1-x)^2$$

and the right-hand site is

$$f = u'' = -2 + 12x - 12x^2$$

We are then able to compute the discretization error between the numerical and the exact solution of the PDE. This error depends on the discretization method and the mesh size h.

2 Multigrid in 1D

2.1 The grid

We will implement multigrid solvers on a hierachy of grid levels. Let therefore N be the number of elements on the finest grid of the interval $[0,1]$ with mesh size $h = \frac{1}{N}$. For a multigrid method with $L+1$ grid levels, we define the following grids

$$\begin{array}{llll}
\Omega_h & x_i = ih, & i = 0,\dots,N & \text{(finest grid)}, \\
\Omega_{2h} & x_i = 2^1 ih, & i = 0,\dots,\dfrac{N}{2^1} & \\
\vdots & \vdots & \vdots & \\
\Omega_{2^L h} & x_i = 2^L ih, & i = 0,\dots,\dfrac{N}{2^L} & \text{(coarsest grid)}.
\end{array}$$

2.2 Finite difference discretization

We use centered finite differences with

$$u_i'' \approx \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i, \quad i = 1, \dots, N-1$$

and obtain the linear system

$$A\vec{u} = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

Given the homogeneous Dirichlet boundary conditions, it is not necessary to include the values $u_0 = 0$ and $u_N = 0$ in the matrix. As a result, the system matrix on the fine grid has a dimension of $N - 1$.

3 V-cycle multigrid

3.1 Discrete L_2-error

We want to compute the discretization error between the exact solution u and its numerical approximation u_h on a given grid with mesh size h. To achieve this, we need a discrete version of the continuous L_2-error.

$$||u||_2 = \left(\int_\Omega u(x)^2 dx\right)^{1/2}$$

The discrete L_2 error is given by :

$$||u^h||_2^h = \left(h^d \sum_{i=1}^{N} u(x_i)^2 dx\right)^{1/2}$$

with d being the dimension of the domain.

Proof :
Let $g(x) = x, \Omega = [0,1], h = \frac{1}{N}$. Show that $||g||_2^2 = \frac{1}{3}$ & $||g^h||_2^2 \overset{h\to 0}{=} \frac{1}{3}$.

First, we compute $\|g\|_2^2$. We have:

$$\|g\|_2^2 = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

.
Now, we want the discrete approximation and its given by:

$$\begin{aligned}
\|g^h\|_2^2 &= h\sum_{i=1}^{N} g(x_i)^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{i}{N}\right)^2 \\
&= \frac{1}{N^3}\sum_{i=1}^{N} i^2.
\end{aligned}$$

With the hint we know that:

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}.$$

So,

$$\|g^h\|_2^2 = \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6N^2}.$$

We study the limit when $h \to 0$ ($N \to \infty$).

$$\lim_{N\to\infty} \frac{(N+1)(2N+1)}{6N^2} = \lim_{N\to\infty} \frac{2N^2+3N+1}{6N^2} = \lim_{N\to\infty} \frac{2+\frac{3}{N}+\frac{1}{N^2}}{6} = \frac{2}{6} = \frac{1}{3}.$$

So, we have:

$$\|g_h\|_2^2 \xrightarrow{h\to 0} \frac{1}{3} = \|g\|_2^2.$$

3.2 Computation of the discretization error

- What do you notice for the discretization error in errorL2?

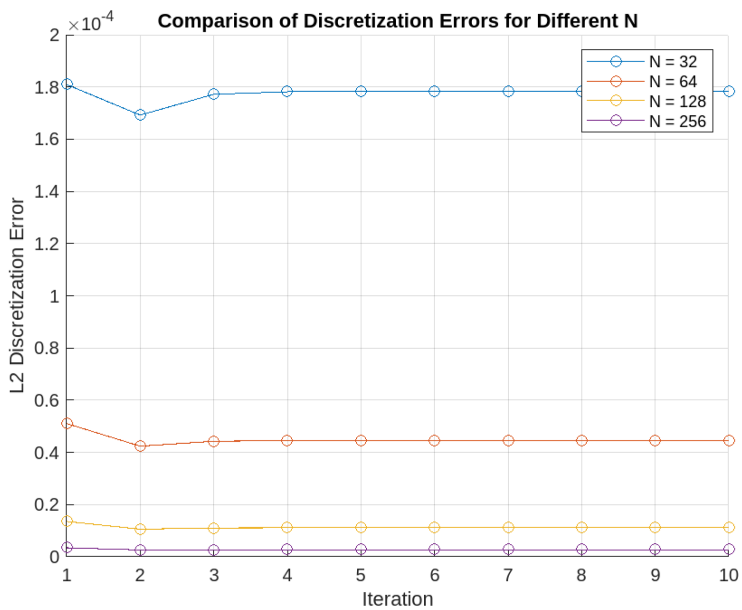

Figure 1: Comparison of discretization errors for different N

As the mesh is refined, the discretization error in the L_2-norm consistently decreases. This shows that the method is converging and that the numerical solution becomes more accurate with a finer grid.

- What do you notice for the residuals in res?

The residuals decrease rapidly over the V-cycle iterations. After just 3 to 4 iterations, the residuals reach very low values, indicating that the multigrid solver efficiently reduces high-frequency errors. This fast convergence confirms that the solver is well-tuned and effective for this problem.

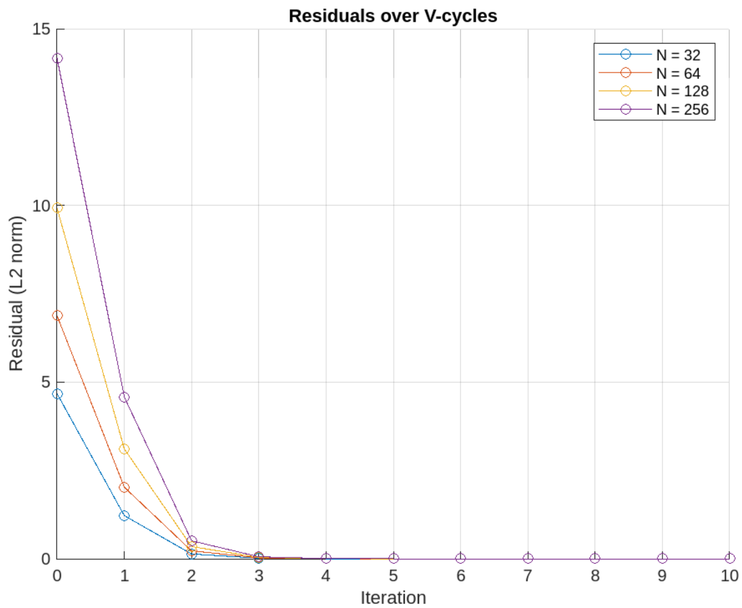

Figure 2: Residuals over V-cycles for different N

- Write down the converged discretization error for the four tests. What is the ratio between two following errors? What does this tell you?

N	L2 Error	Ratio to previous
32	1.7830×10^{-4}	–
64	4.4574×10^{-5}	4.00
128	1.1143×10^{-5}	4.00
256	2.7859×10^{-6}	4.00

Table 1: Converged L2 errors and ratios

The ratio between successive errors is 4.00, which is consistent across all mesh refinements. This confirms that the method achieves second-order accuracy ($\mathcal{O}(h^2)$), meaning the error is proportional to the square of the mesh size.

3.3 From 2-grid to multigrid

- Which part of the above algorithm has to be modified?

To implement a multigrid method with $L + 1$ levels instead of just two, the `V_cycle.m` function needs to be made recursive. Specifically, the part where the algorithm moves to the coarser level should now call V_cycle recursively on the next coarser grid, unless the coarsest level has been reached.

- What is a reason to use more than two levels?

Using more than two levels helps the algorithm solve the problem faster. A two-level method is good at removing high-frequency errors, but it's not as good at removing low-frequency errors. By adding more levels, we solve these smooth errors on coarser grids where they look like high-frequency errors again. This makes the method more efficient and helps it converge in fewer steps.

- What is the maximal level L that can be used?

The maximum number of levels is determined by how many times the grid can be halved while keeping a valid discretization. With $N = 256$, the maximum number of levels is 8.

$$L_{\text{max}} = \log_2(N) = \log_2(256) = 8$$

- Verify that you obtain the same discretization error for varying L as for the two-level method.

Level L	**Converged** L_2 **Error**
1	2.7859×10^{-6}
2	2.7859×10^{-6}
3	2.7859×10^{-6}
4	2.7859×10^{-6}
5	2.7859×10^{-6}
6	2.7858×10^{-6}
7	2.7856×10^{-6}
8	2.7841×10^{-6}

Table 2: Converged L_2 errors for different levels L (with $N = 256$)

The converged L_2 error remains nearly identical across all values of L, from 1 to 8. This confirms that the multigrid method, regardless of the number of levels used, produces the same discretization error, which is determined by the grid size N, not by the algorithm's structure. Therefore, all levels converge to the same numerical accuracy.

- What do you observe for the error reduction?

From the plot, we notice that increasing the number of levels beyond a certain point ($L > 3$) does not improve the convergence speed - in fact, it slightly slows it down. This is likely due to the fact that with deeper multigrid hierarchies, the coarser grids may not effectively contribute to error smoothing. Thus, using too many levels can become counterproductive, especially if the coarsest levels are too small to be useful.

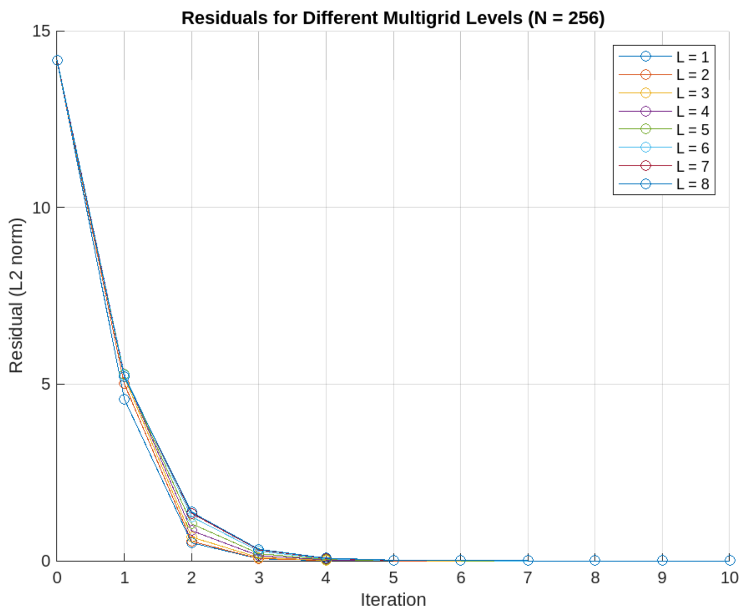

Figure 3: Residuals over V-cycles for different L with $N = 256$