AMS X01 Project Report

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Introduction to periodic homogenization

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Mars 2022

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1 Introduction

This report highlights an application of the finite element method to the special case of elliptical problems. The first part concerns the study of a partial differential equation in a stationary regime with Neumann conditions at the edges. The second part will be very similar, except that the edge conditions will be homogeneous Dirichlet. Finally, we will adapt the technique to deal with the case of a periodic solution in a lattice of size $[0, L]^2$.

2 Elliptical problem with Neumann conditions

$\forall v \in H^1(\Omega)$,

$\int_\Omega uv \,\mathrm{d}x + \int_\Omega \vec{\nabla} v \cdot A(x,y) \cdot \vec{\nabla} u \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x$.

We therefore pose a: $(u, v) \longmapsto \int_\Omega uv \,\mathrm{d}x + \int_\Omega \nabla v A(x,y) \nabla u \,\mathrm{d}x$ and $l : v \longmapsto \int_\Omega fv \,\mathrm{d}x$. The continuity of l in $H^1(\Omega)$ is obvious, as is that of a given the tensor majorization A: $forall(x, y) \in \Omega \quad \forall(i, j) \quad |A_{i,j}| \leq C$. The coercivity of a in $H^1(\Omega)$ is also immediate given the positivity property of A : $\forall u \in H^1(\Omega), \quad a(u, u) \geq C^{'}|u\|_{H^1(\Omega)}, \quad C^{'} = min(1, c)\|u\|_{H^1(\Omega)}$, c being the coercivity constant of A. From the Lax-Milgram theorem, we can deduce the existence and uniqueness of u $\in H^1(\Omega)$, the solution to the variational problem. The problem is therefore well posed.

For the computer solution, we discretise the spaces using a conformal method, i.e. we take $V_h \in H^1(\Omega)$ and such that $dim(V_h) = N \leq \infty$. We therefore construct an approximation to the variational problem, by decomposing the solution u_h in the basis of V_h, i.e. $u_h = \sum_{i=1}^{N} u_h(M_I) w_I$, where M_I are the nodes of the mesh and w_I are the Lagrange polynomials of degree 1 such that $w_I(M_J) = \delta_{I,J}$. By noting

$$athbf M_{I,J} = \int_\Omega w_I w_J \, dx,$$

and

$$athbf K_{I,J} = \int_\Omega \nabla w_I \nabla w_J \, dx,$$

we have

$$(\mathbf{K} + \mathbf{M})\vec{U} = \vec{L}.$$

We find that the integrals can be calculated exactly in the case of Lagrange finite elements of order . A quick calculation gives :

$$\int_{T_l} w_I w_J dx = \begin{cases} |D|/24 & \text{si} I = J, \\ |D|/12 & \text{si} I = J, \end{cases} \tag{1}$$

and

$$\int_{T_l} \nabla w_I \nabla w_J dx =$$

$$\frac{(y_{(I+1)\mathrm{mod}3,(I+2)\mathrm{mod}3} y_{(J+1)\mathrm{mod}3,(J+2)\mathrm{mod}3} - x_{(I+1)\mathrm{mod}3,(I+2)\mathrm{mod}3} x_{(J+1)\mathrm{mod}3,(J+2)\mathrm{mod}3})}{|2D|}.$$

We have $\vec{L}_I = \int_\Omega f w_I(x) dx$. By replacing f by $\pi_h f = \sum_{i=1}^{N} f(M_I) w_I$, we recognise the mass matrix. $\vec{L} \simeq \mathbf{M}\vec{F_h}$, where $\vec{F_h} \in \mathbf{R}^N$ is the column vector containing the value of f on the nodes M_I.

Figures 1 and 2 show the relative errors as a function of step size, in logarithmic scale. Remember that we have chosen $A = 1$ and $u(x, y) = \cos(\pi x)\cos(2\pi y)$.

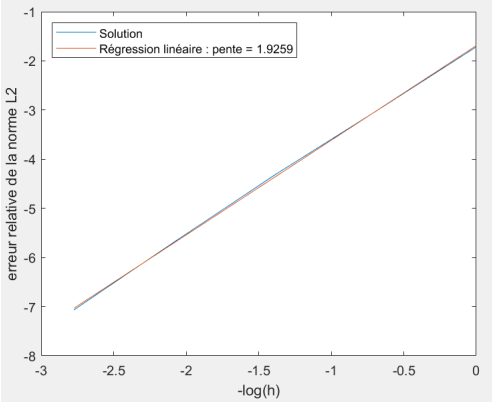

Figure 1: $||u - u_h||_{L^2(\Omega)}$ as a function of h.

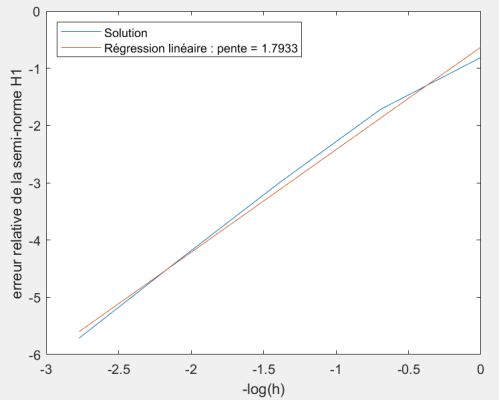

Figure 2: $|u - u_h|_{H_0^1(\Omega)}$ as a function of h.

A quick calculation gives $||u||_{L^2(\Omega)} = \frac{\pi}{\sqrt{2}}$ and $|u|_1 = \sqrt{\frac{3}{2}}\pi$. We find a step size of about 2.25 for the $L^2(\Omega)$ norm and a step size of 1.8 for the $|\cdot|_1$ semi-norm. This is consistent with the theory, since the speed of convergence should be in h^2 for the $L^2(\omega)$ norm and in h for the **au minimum** semi-norm. The slope is therefore perfectly allowed to take these values.

In the case where A depends on x and y (but remains a scalar), the errors found are shown in Figures 3 and 4.

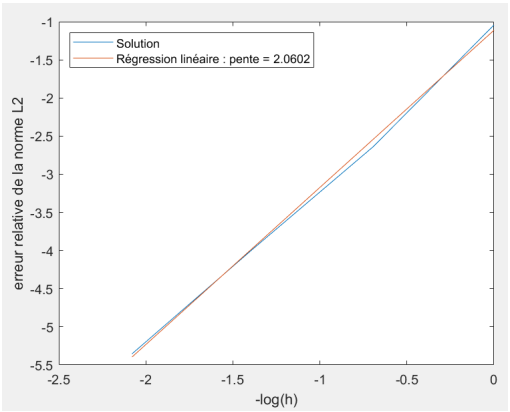

Figure 3: $||u - u_h||_{L^2(\Omega)}$ with A non constant

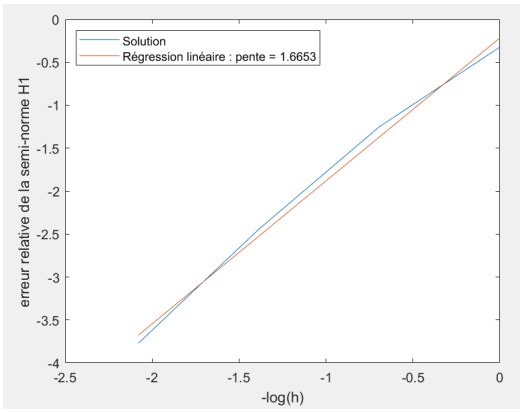

Figure 4: $|u - u_h|_{H_0^1(\Omega)}$ with A non constant

Similarly, we find a step size of 2.0602 for the $L^2(\Omega)$ norm and a step size of 1.6653 for the $H^1(\Omega)$ semi-norm. The presence of the $\mathbf{A}$ tensor only slightly affects the speed of convergence as it is uniformly bounded on the domain.

I represent the solution in Figures 5 and 6 for the source term $f(x, y) = \cos(\cos(x))\sin(\sin(y))$. I notice that the solution remains almost strictly identical, which is why I decided to display only 2: when $A = \sin(2\pi x)\sin(2\pi y) + 2$ and $A = \sin(32\pi x)\sin(32\pi y) + 2$. The figure should have shown slight oscillations around the 'slower' solution proposed in figure 5. My choice of solution representation does not highlight this.

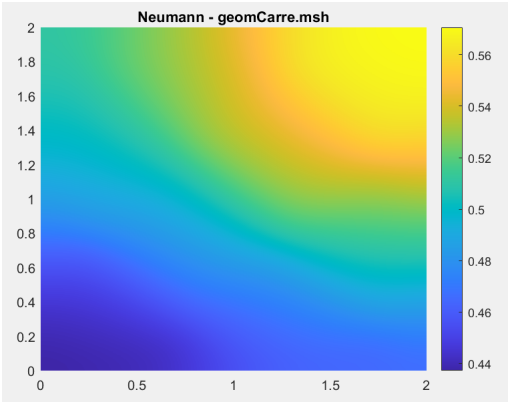

Figure 5: Solution dans le cas $f = 1$.

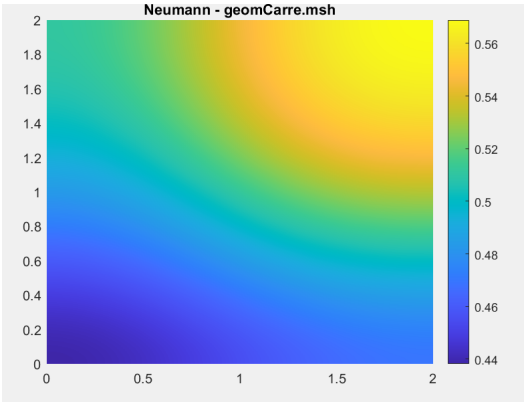

Figure 6: Solution dans le cas $f = 16$.

3 Elliptical problem with Dirichlet conditions

The condition becomes natural because it appears in the functional space in which we are going to look for the solution: $H_0^1(\Omega)$. This space is also given the norm of $H^1(\Omega)$. The edge term will therefore appear when we integrate by parts using Green's formula. However, as the space of the test functions is the same as the space in which we will be looking for the solution, the edge term will still cancel out: $v|_{\partial\Omega} = 0$. The variational formula is therefore strictly identical to that in Q1), we just change $H^1(\Omega)$ to $H_0^1(\Omega)$.
The mass and stiffness matrix is the same as before, only it's smaller because it only contains vertices on the interior nodes. If N_0 is the number of interior nodes, we take the sum in the case Neumann is only up to N_0 and not N.
I constructed the **P** matrix as follows: I create a sparse matrix of size $N_0 \times N$. I go through the nodes, and when I find the ith internal node numbered x, I put a 1 in row i and column x.
I have chosen to take the tensor $A(x, y) = \sin(2\pi x)+2$ and $u(x, y) = \sin(\pi x)\sin(2\pi y)$. We can check that $u|_(\partial\Omega) = 0$. Figures 7 and 8 show the errors found in this case, with their associated linear regressions. We find a step size of 1.8772 for the L^2 norm, which is below the permitted limit. Calculation errors may have accumulated or the step size may not have been small enough. On the other hand, for the $H^1(\Omega)$ semi-norm, we find 1.7814, which is much better than the minimum unit step announced by the theory for the P_1 Lagrange elements.

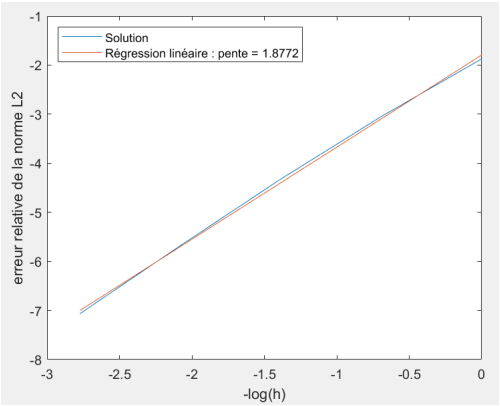

Figure 7: $||u - u_h||_{L^2(\Omega)}$ en fonction de h.

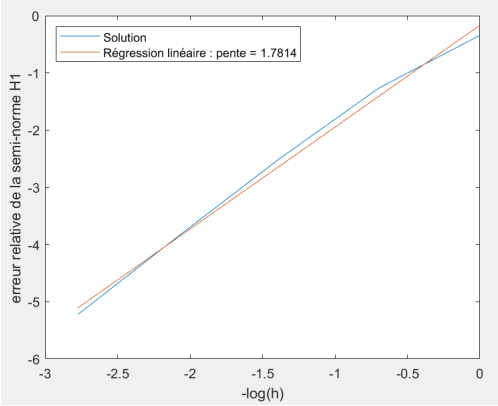

Figure 8: $|u - u_h|_{H_0^1(\Omega)}$ en fonction de h.

4 Elliptical problem with periodic conditions

Once again, the variational formula itself does not change, only the function space. The conditions on the gradients $A\nabla u$ on each of the edges will allow the edge term to disappear naturally: $\int_{\partial\Omega} A\nabla u \cdot \vec{n} v d\Gamma$, where $\vec{n}$ is the unit vector leaving normal to the edge. The edge conditions on u will be taken into account in the functional space: $H_\sharp^1(\Omega) =$

Figures 10 and 11 show the error plots. Once again, the steps are slightly better than those predicted by the theory, particularly for the H^1 semi-norm.

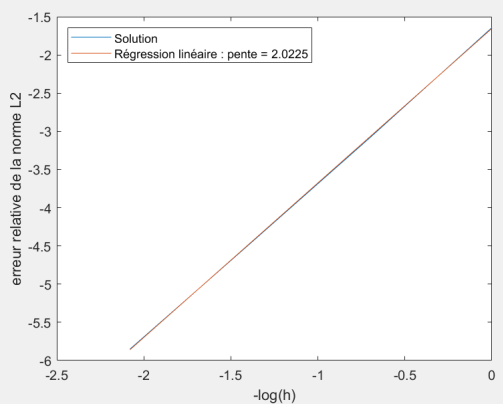

Figure 10: $||u - u_h||_{L^2(\Omega)}$ as a function of h.

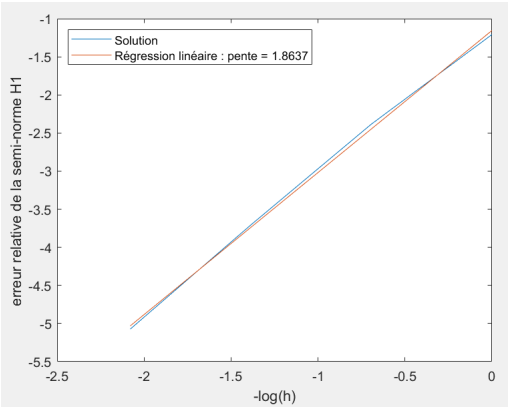

Figure 11: $|u - u_h|_{H_0^1(\Omega)}$ as a function of h.

The techniques implemented to take account of the natural boundary conditions proved very effective, with relative errors evolving in accordance with finite element theory. The only failure was the lack of difference between the solutions calculated for the different frequencies of the A tensor.