Petite Classe 8 - PHY550

November 30, 2022

Part I
Problems and exercices

1 Thruster description

The thruster is shown schematically in figure 1. It is made of a dielectric tube of radius $r_0 = 6$ cm and length $l = 10$ cm. The propellant is injected on the left-hand-side; here we consider Xenon (a noble gas). The plasma is generated by circulating an RF current in the 6-turn sinusoidal coil ($N = 6$). On the right-hand-side we have a pair of grids that can be DC biased to accelerate ions out of the thruster. The thrust is generated by the accelerated ions, and (marginally) by the neutrals leaking out of the thruster.

In the first part of the problem, we will first model the inductive plasma in a regime where the ionisation fraction remains small. In the second part, we use the Global Model software LPP0D to calculate useful quantities and compare it to simplified theories.

2 Model of a standard inductive discharge

Let us assume the following:

- The power remains moderate such that the ionization fraction remains low.
- The grids are closed and there is no flow; the pressure is fixed. Consequently, the Xenon atom density, noted n_g, is constant and related to the gas pressure via the ideal gas law, $p = n_g k_B T_g$ where p is the pressure and T_g the gas temperature
- The gas temperature is constant and $T_g = 300$ K.

In the above situation, the model presented in the lecture reduces to two variables, the electron density $n_e = n_i = n$ and the electron temperature T_e. The control parameters, at the operator disposal, are the neutral pressure p, and the RF current amplitude in the coil, I_{coil}.

2.1 Particle balance and electron temperature

1. Rewrite the two equations to be solved for the electron density and the electron temperature within the approximations listed above
2. Consider that the ionization rate has the form $K_{\text{iz}} = K_{\text{iz0}} \exp(-E_{\text{iz}}/k_B T_e)$, Show that the electron temperature is

$$T_e = -\frac{E_{\text{iz}}}{k_B}\left[\ln\left(\frac{u_B A_{\text{eff}}}{n_g K_{\text{iz0}} V}\right)\right]^{-1} \qquad (1)$$

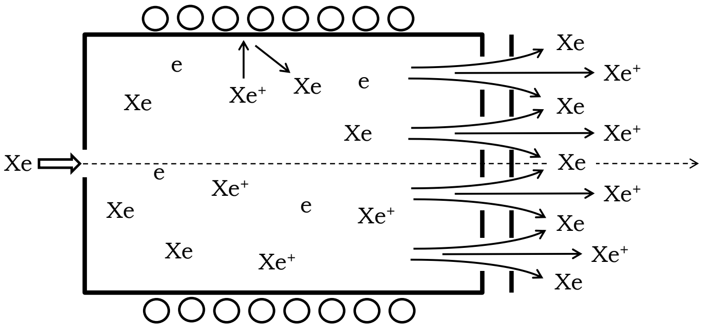

Figure 1: Schematic of the RF Ion gridded thruster powered by an inductive coil.

where k_B is the Boltzmann constant, and $A_{\rm eff}/V$ is the area-to-volume ratio of the thruster chamber.

2.2 Inductive coupling, power balance and electron temperature

We will now study the physics of inductive heating, for the system shown in figure 2. The current that circulates in the coil generates an induced electromagnetic field that penetrates the plasma volume. It can be shown that Maxwell's equation may be combined to obtain the Bessel equation for the complex amplitude $\widetilde{H}_z$, which gives the following (complex) fields

$$\begin{aligned} \widetilde{H}_{\rm z} &= H_{\rm z0}\frac{{\rm J}_0(kr)}{{\rm J}_0(kr_0)} \\ \widetilde{E}_\theta &= -\frac{{\rm i}kH_{\rm z0}}{\omega\varepsilon_0\varepsilon_{\rm p}}\frac{{\rm J}_1(kr)}{{\rm J}_0(kr_0)} \end{aligned}$$

where $k \equiv k_0\sqrt{\varepsilon_p}$ is the wave number in the plasma, $k_0 = \omega/c$, and

$$\varepsilon_p = 1 - \frac{\omega_{\rm pe}^2}{\omega\left(\omega - {\rm i}\nu_m\right)}$$

is the plasma dielectric constant. The magnitude of the fields depends on the coil current via the simple relation

$$H_{\rm z0} = \frac{NI_{\rm coil}}{l}$$

The power dissipated in the system composed of the coil loaded by the plasma (across the dielectric tube) can be calculated using the Poynting theorem and reads

$$\begin{aligned} \widetilde{P} &= -\frac{1}{2}\widetilde{E}_\theta(r_{\rm c})\widetilde{H}_{\rm z}(r_{\rm c})2\pi r_{\rm c}l \\ &= {\rm i}\frac{\pi N^2 I_{\rm coil}^2}{l}\left[\frac{kr_0{\rm J}_1(kr_0)}{\omega\varepsilon_0\varepsilon_{\rm p}{\rm J}_0(kr_0)} + \frac{1}{2}\omega\mu_0\left(r_{\rm c}^2 - r_0^2\right)\right] \end{aligned}$$

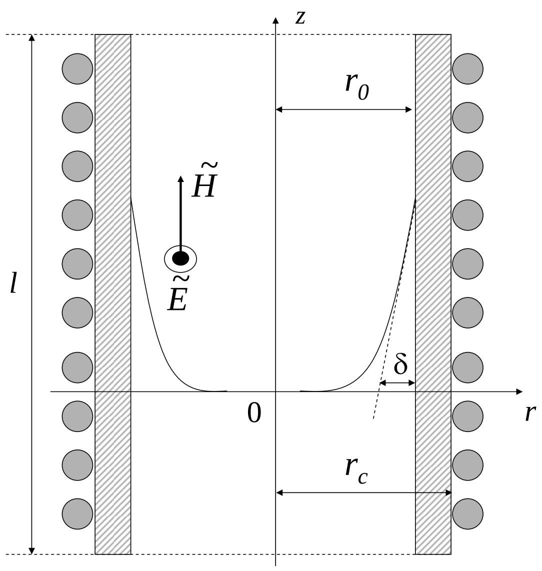

Figure 2: Schematic of inductive heating of the plasma.

If one defines the power such that

$$\widetilde{P} = \frac{1}{2} Z_{\text{ind}} I_{\text{coil}}^2$$

where Z_{ind} is the total complex impedance of the system, then it immediately follows that one can define dissipative and reactive components

$$\begin{aligned} R_{\text{ind}} &= \frac{2\text{Re}\left[\widetilde{P}\right]}{I_{\text{coil}}^2} \\ X_{\text{ind}} &= \frac{2\text{Im}\left[\widetilde{P}\right]}{I_{\text{coil}}^2} \end{aligned}$$

of the equivalent circuit shown in Figure 3.

1. Show that the coil inductance is

$$L_{\text{coil}} \approx \frac{\mu_0 \pi r_c^2 N^2}{l}$$

2. Show that

$$\begin{aligned} R_{\text{ind}} &= \frac{2\pi N^2}{l\omega\varepsilon_0} \text{Re}\left[\frac{\text{i}kr_0 \text{J}_1(kr_0)}{\varepsilon_\text{p} \text{J}_0(kr_0)}\right] \\ L_{\text{ind}} &= L_{\text{coil}}\left(1 - \frac{r_0^2}{r_\text{c}^2}\right) + \frac{2\pi N^2}{l\omega^2\varepsilon_0} \text{Im}\left[\frac{\text{i}kr_0 \text{J}_1(kr_0)}{\varepsilon_\text{p} \text{J}_0(kr_0)}\right] \end{aligned}$$

3. The thruster operates in a low pressure regime where $\nu_m \ll \omega$. Show that in this regime we have

$$kr_0 \approx \pm \frac{r_0}{\delta}\left(\frac{\nu_m}{2\omega} - \text{i}\right)$$

where $\delta \equiv c/\omega_{\text{pe}}$ is the collisionless plasma skin depth, i.e. the typical distance over which the fields decay in the plasma.

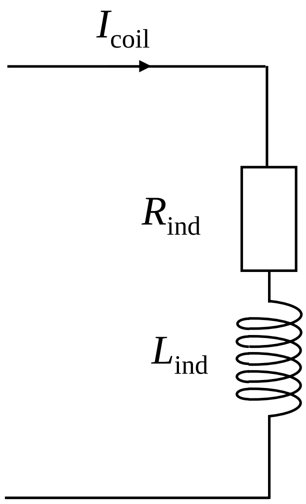

Figure 3: Equivalent circuit of the inductive plasma discharge.

4. The thruster also operates at fairly high plasma density in which the skin depth is small compared to the tube radius, i.e. $\delta \ll r_0$. Using the properties of Bessel functions given below, show that

$$L_{\text{ind}} \approx L_{\text{coil}}\left(1 - \frac{r_0^2}{r_c^2}\right)$$

5. In the same regime ($\nu_m \ll \omega$ and $\delta \ll r_0$) show that

$$R_{\text{ind}} \approx N^2 \frac{\pi r_0}{l\delta\sigma_m} \tag{2}$$

where $\sigma_m = n_e e^2/m_e\nu_m$ is the plasma conductivity.

6. Expressing the loss power as a function of the total energy spent for an electron-ion pair generation, $\varepsilon_T(T_e)$, and the absorbed power for the approximate expression of R_{ind} derived above, show that

$$n_{\text{e}} = \left[\frac{\pi r_0 N^2 \nu_{\text{m}} (m/\varepsilon_0)^{1/2}}{4u_{\text{B}}\left(h_{\text{l}}\pi r_0^2 + h_{\text{R}}\pi r_0 l\right) e\varepsilon_{\text{T}}(T_{\text{e}}) l\, c}\right]^{2/3} I_{\text{coil}}^{4/3} \tag{3}$$

3 Thruster model

We now jump to the thruster situation, in which the grid holes are open to let the neutrals and the ions exit the thruster. The gas enters at a mass flow rate Q_0. The full description of the model is given in Chabert 2012 and in the lecture notes chapter 8.

3.1 Study as a function of the coil current or the RF power

1. Show that in this case and before switching on the plasmas, the neutral gas density n_{g0} and the neutral gas pressure p_0 are

$$n_{\text{g0}} = \frac{4Q_0}{v_{\text{g0}} A_{\text{open,g}}} \tag{4}$$

$$p_0 = n_{\text{g0}} k_B T_{\text{g0}} \tag{5}$$

where $T_{\rm g0}$ is the gas temperature without plasma and

$$
\begin{aligned}
v_{\rm g0} &= \left(\frac{8k_B T_{\rm g0}}{\pi M}\right)^{1/2} \\
A_{\rm open,g} &= \beta_g \pi R^2
\end{aligned}
$$

are the averaged gas velocity and the open area for neutrals leaking out of the thruster, respectively.

2. Let us now consider a gas flow rate of $Q_{\rm sccm} = 29$ sccm and a grid transparency for neutrals of $\beta_g = 0.3$. Use the numerical code to plot n_g, n_e, T_g, T_e and p (the neutral gas pressure) as a function of the coil current, varying $I_{\rm coil}$ from 2.5 to 25 Amps.

3. Compare on dedicated figures the above numerical solution (that includes neutral dynamics) to the analytical model where the neutral dynamics was not included, choosing $T_{\rm g0} = 400$ K, $n_{\rm g0}$ calculated from Eq. 4, n_e calculated from Eq. 3, T_e calculated from Eq. 1, and p_0 calculated from Eq. 5. For this calculation use

$$
\begin{aligned}
u_B &= 3000 \text{ m/s} \\
h_R &= h_L = 0.5 \\
E_{\rm iz} &= 12.1\text{eV} \\
K_{\rm iz0} &= 10^{-13} \text{ m}^3/\text{s} \\
K_{\rm el} &= 2 \times 10^{-13} \text{ m}^3/\text{s} \\
\nu_m &= n_{\rm g0} K_{\rm el}
\end{aligned}
$$

Comment on the discrepancies (in particular recall that in the analytical model the plasma resistance was approximated in the high electron density regime).

4. Use the numerical model to plot the ion current density J as a function of the RF power. Identify the Child-Langmuir limit and deduce the maximum RF power at which the thruster shoud operate. At this limit, evaluate the thrust.

5. Use the numerical model to plot the thrust-to-power ratio as a function of the RF power and compare it to the theoretical formula given in the lecture. Discuss the discrepancy between the numerical solution and the theory.

6. Use the numerical model to plot the inductive power transfer efficiency, defined as

$$
\zeta = \frac{R_{\rm ind}}{R_{\rm ind} + R_{\rm coil}}
$$

as a function of the RF power. Compare it to the theoretical formula obtained using Eq. 2. For this calculation use $R_{\rm coil} = 2\ \Omega$ and use the electron and gas densities calculated from the coil (not the approximate formulas).

3.2 Study as a function of the mass flow rate

From this point, we shall now use the model to study the thruster efficiency as the gas (mass) flow rate is varied. For each mass flow rate, the power will be pushed to the maximum value required to reach the Child-Langmuir limit for the ion current density.

1. Use the numerical model to plot the mass utilization efficiency η, electrical power efficiency γ, and the total thruster efficiency $\eta_{\rm thruster} = \eta\gamma$ as a function of Q_m.

2. Run the simulation code again for $L = 0.03$ m instead of $L = 0.1$ m and plot the total efficiency as a function of Q_m. Discuss/explain the results.

3. Run the simulation code for $L = 10$ m and $L = 0.03$ m for argon, krypton and xenon and identify the maximum efficiency for the three propellant. Plot the thrust-to-power ratio for the same conditions ($L = 0.01$ m and $L = 0.03$ m for argon, krypton and xenon).

4. Plot the thrust as a function of the mass flow rate for the three propellants.

5. Comment all the above results and use them to evaluate the best propellant for plasma propulsion.

4 Useful relations an limits for Bessel functions with complex arguments

This problem requires an appreciation of Bessel functions of complex arguments. Some of their useful properties are summarized here. If x is a real number, then

$$\begin{aligned} \mathrm{J}_0(\mathrm{i}x) &= \mathrm{I}_0(x) \\ \mathrm{J}_1(\mathrm{i}x) &= \mathrm{i}\mathrm{I}_1(x) \end{aligned}$$

where $\mathrm{I}_0(x)$ and $\mathrm{I}_1(x)$ are modified Bessel functions, which increase exponentially when x is large. In addition, note that

$$\frac{\mathrm{I}_1(x)}{\mathrm{I}_0(x)} \rightarrow 1$$

when $x \rightarrow \infty$. In the other limit when $x \rightarrow 0$, the modified Bessel functions may, at the lower order, be approximated by:

$$\begin{aligned} \mathrm{I}_0(x) &\approx 1 \\ \mathrm{I}_1(x) &\approx \frac{x}{2} \end{aligned}$$

Unfortunately, when the argument has both real and imaginary parts such simplifications do not arise.