MATH321 Tutorial 1

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Question 1

Let d be an integer. Show that $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$ is an integral domain.

First, observe $\mathbb{Z}[\sqrt{d}]$ is a ring because it is a subset of $\mathbb{C}$ that is non-empty (as $a, b \in \mathbb{Z} \neq \phi$), and $\forall a_1, a_2, b_1, b_2 \in \mathbb{Z}[\sqrt{d}]$ it is:

1) Closed under subtraction:

$$(a_1 + b_1\sqrt{d}) - (a_2 + b_2\sqrt{d}) = (a_1 - a_2) + (b_1 - b_2)\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$$

2) Closed under multiplication:

$$(a_1 + b_1\sqrt{d})(a_2 + b_2\sqrt{d}) = (a_1a_2 + b_1b_2d) + (a_1b_2 + b_1a_2)\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$$

[Note: ab, $a - b \in \mathbb{Z}\ \forall\ a, b \in \mathbb{Z}$]

Now, with $a = 1$, $b = 0$ we have $e = 1$ in $\mathbb{Z}[\sqrt{d}]$ such that $\forall\ a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$:

$$e(a + b\sqrt{d}) = (a + b\sqrt{d})e = (a + b\sqrt{d})$$

So, $\mathbb{Z}[\sqrt{d}]$ has a unity. Also, since we have shown $\mathbb{Z}[\sqrt{d}]$ is a subring of $\mathbb{C}$ and $\mathbb{C}$ does not have any zero divisors, $\mathbb{Z}[\sqrt{d}]$ also can not have any zero divisors.

Therefore, as $\mathbb{Z}[\sqrt{d}]$ is a ring with unity and no zero divisors, it is an integral domain.

Question 2

Find all units and zero-divisors in the ring $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Since addition and multiplication in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ are performed pointwise, we need to find units and zero divisors in $\mathbb{Z}_4$ and $\mathbb{Z}_3$ and pair them up.

The units of $\mathbb{Z}_4$ and $\mathbb{Z}_3$ are precisely those elements relatively prime to 4 and 3 respectively.

$\mathbb{Z}_4$ units = {1,3}
$\mathbb{Z}_3$ units = {1,2}

So by considering all possible pairs, we get the units of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ are (1,1), (1,2), (3,1) and (3,2).

Similarly, by constructing Cayley tables we can observe that $\mathbb{Z}_4$ has a zero-divisor of 2, and $\mathbb{Z}_3$ has no zero-divisors. Thus, the only zero-divisor of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ is (2,0).

Question 3

Prove or disprove that

(a) $R = \left\{ \begin{bmatrix} 0 & a \\ b & a \end{bmatrix} \mid a, b, c \in \mathbb{Q} \right\}$

(b) $S_d = \left\{ \begin{bmatrix} a & bd \\ b & c \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

is a subring of the matrix ring $M_2(\mathbb{R})$, where d is an arbitrary (but fixed) integer.

(a) This fails the subring test:

Let $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{Q}$.

$$\begin{bmatrix} 0 & a_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 0 & a_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 b_2 & a_1 c_1 \\ c_1 b_2 & b_1 a_2 + c_1 c_2 \end{bmatrix} \notin R.$$

So R is not a subring of $M_2(\mathbb{R})$ as it is not closed under multiplication.

(b) Again we apply the subring test. S_d is clearly non-empty, so take $a_1, a_2, b_1, b_2 \in \mathbb{R}, d \in \mathbb{Z}$.

$$\begin{bmatrix} a_1 & b_1 d \\ b_1 & a_1 \end{bmatrix} - \begin{bmatrix} a_2 & b_2 d \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 & b_1 d - b_2 d \\ b_1 - b_2 & a_1 - a_2 \end{bmatrix} \in S_d$$

So S_d is closed under subtraction.

$$\begin{bmatrix} a_1 & b_1 d \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 d \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 + b_1 d b_2 & a_1 b_2 d + b_1 d a_2 \\ b_1 a_2 + a_1 b_2 & a_1 a_2 + b_1 b_2 d \end{bmatrix} = \begin{bmatrix} a_1 a_2 + b_1 b_2 d & (a_1 b_2 + b_1 a_2) d \\ a_1 b_2 + b_1 a_2 & a_1 a_2 + b_1 b_2 d \end{bmatrix} \in S_d$$

(By taking $a = a_1 a_2 + b_1 b_2 d, b = a_1 b_2 + b_1 a_2$)

So S_d is closed under multiplication as well, and is therefore a subring of $M_2(\mathbb{R})$.

Question 4

Find units and zero divisors in the RING examples in Q3. Do you find them all? Does the answer depend on the value of d?

The units of S_d are the invertible matrices of the form

$$\begin{bmatrix} a & bd \\ b & a \end{bmatrix}$$

These are invertible when

$$\begin{aligned} a^2 - b^2 d &\neq 0 \\ a^2 &\neq b^2 d \\ a &\neq \pm b\sqrt{d} \end{aligned}$$

Zero-divisors are any elements

$$\begin{bmatrix} a_1 & b_1 d \\ b_1 & a_1 \end{bmatrix}$$

Such that for some $a_2, b_2 \in \mathbb{R}$

$$\begin{bmatrix} a_1 & b_1 d \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 d \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

I'm too lazy to solve simultaneous equations or mess around with null spaces, so instead I will take a shortcut by observing that zero-divisors can never be units. So, by the working earlier used to classify units, we know that all zero divisors must have $a = \pm b\sqrt{(}d)$. Note this is a sufficient but not necessary condition for the zero divisors, so we are not done. We have however, eliminated some variables.

By substituting $a_1 = \pm b_1\sqrt{(}d)$ and $a_2 = \pm b_2\sqrt{(}d)$ into the product calculated in Question 3 and solving each entry for 0, we find that for any matrix of the form

$$\begin{bmatrix} \pm b_1\sqrt{(}d) & b_1 d \\ b_1 & \pm b_1\sqrt{(}d) \end{bmatrix}$$

We can calculate $a_2 = \mp b_2\sqrt{(}d)$ and find another matrix of the form

$$\begin{bmatrix} \mp b_2\sqrt{(}d) & b_2 d \\ b_2 & \mp b_2\sqrt{(}d) \end{bmatrix}$$

And their product will be 0 as shown.

$$\begin{bmatrix} \pm b_1\sqrt{(}d) & b_1 d \\ b_1 & \pm b_1\sqrt{(}d) \end{bmatrix} \begin{bmatrix} \mp b_2\sqrt{(}d) & b_2 d \\ b_2 & \mp b_2\sqrt{(}d) \end{bmatrix} = \begin{bmatrix} -b_1 b_2 d + b_1 b_2 d & \mp b_1 b_2 d\sqrt{d} \pm b_1 b_2 d\sqrt{d} \\ \mp b_1 b_2\sqrt{d} \pm b_1 b_2\sqrt{d} & b_1 b_2 d - b_1 b_2 d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The condition on d is that S_d is only defined over the reals, so $d \geq 0$. If $d < 0$ then every element in S_d is invertible, and there are no zero-divisors.

Interestingly, despite me saying not being a unit was only a necessary condition for being a zero-divisor, in this ring (ignoring the zero matrix), it turned out to be sufficient. I have not convinced myself of a reason why this is the case - the ring is commutative, but not finite so the result shown in class doesn't apply. I feel like I'm missing something.