University of Canterbury, School of Mathematics and Statistics

MATH240 Assignment 2 - Real Analysis

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October 8, 2019

Question 1

Let S be a non-empty subset of the real numbers which is bounded above. Let c be a real number and define T = {cx | x ∈ S}. Show that, if c > 0, then T is non empty, bounded above and that supT = csupS. Give an example of a set S as above, such that, with c = -1, the set T is still bounded above but supT ≠ csupS.

To show T is non-empty, observe $S \neq \emptyset$, so $T = \{cx \mid x \in S\} \neq \emptyset$.

Since S is bounded above, by definition there exists some upper bound U such that $x \leq U \ \forall \ x \in S$. So, by transitivity, $cx \leq cU \ \forall \ x \in S$. Therefore, cU is an upper bound for T.

Since S is non-empty and bounded above, we know $supS$ exists. By definition of $supS$, $\forall \ s \in S$, $s \leq supS$. So, since c is positive, $cs \leq csupS$. This means that $t \leq csupS \ \forall \ t \in t$ and therefore $csupS$ is an upper bound for T.

Since T is also non-empty and bounded above, we know $supT$ also exists. Assume $supT < csupS$. We will find a contradiction.

Since $c > 0$:

$$\begin{aligned} supT &< csupS \\ \Rightarrow \frac{supT}{c} &< supS \end{aligned}$$

By definition of $supS$, for any value $X < supS$, $\exists \ s \in S$ such that $s > X$. So, $\exists \ s' \in S$ such that

$$\begin{aligned} s' &> \frac{supT}{c} \\ \Rightarrow cs' &> supT \end{aligned}$$

$cs' \in T$ by definition of T, so $cs' > supT$ is a contradiction.

$$\therefore supT \geq csupS \tag{1}$$

But $csupS$ is an upper bound for T.

$$\therefore supT \leq csupS \tag{2}$$

And so from (1) and (2), we know:

$$supT = csupS$$

As an example of $supT \neq supS$ when $c = -1$, take $S = \{-1, -2\}$, so $supS = -1$. With $c = -1$, $T = \{1, 2\}$ and $supT = 2$. But $csupS = 1 \neq 2$, so $csupS \neq supT$.

This is true for any bounded S with $|S| > 1$.

Question 2

(i) If (x_n) is a sequence such that $\lim |x_n| = 0$, prove that $\lim x_n = 0$ also.

By definition of the limit, $\lim |x_n| = 0 \Rightarrow \forall\ \mathcal{E} \geq 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$,

$$\begin{aligned} ||x_n| - 0| &\leq \mathcal{E} \\ \Rightarrow ||x_n|| &\leq \mathcal{E} \\ \Rightarrow |x_n| &\leq \mathcal{E} \\ \Rightarrow |x_n - 0| &\leq \mathcal{E} \\ \Rightarrow \lim x_n &= 0 \end{aligned}$$

(ii) Give an example of a sequence (x_n) that does not converge but that $(|x_n|)$ converges.

Define (x_n) by $x_n\ = (-1)^n$ which does not converge.

$|x_n| = |(-1)^n| = |-1|^n = 1$ so $(|x_n|)$ converges to 1.

Question 3

Let $\sum x_n$ be a series that converges absolutely. Show that $\sum x_n^2$ converges.

Since $\sum |x_n|$ converges, we know that $\lim_{n\to\infty} |x_n| = 0$. So, by definition of the limit, we know $\exists\ N$ such that $\forall\ n \geq N$, $|x_n| \leq 1$.

We can remove a finite number of terms for the start of a series without change its behaviour. So, we consider $\sum_{n=N}^{\infty} |x_n|$, so $|x_n| \leq 1$ for every term in the sequence.

Note that since $x_n^2 \geq 0$, $|x_n|^2 = x_n^2 = |x_n^2|$.

$|x_n| \leq 1$ so $|x_n|^2 = |x_n^2| \leq 1$.

$|x_n^2| \leq |x_n| \forall n \geq N$, so, since $\sum_{n=N}^{\infty} |x_n|$ is convergent, $\sum_{n=N}^{\infty} |x_n^2|$ is also convergent. Once again, N is just a finite number so it does not change the behaviour of the series.

Therefore x_n^2 is absolutely convergent, which implies it is convergent.

Question 4

Write question 4 here

Solve question 4 here.

Question 5

Write question 5 here

Solve question 5 here.