MATH240 Assignment 2 - Real Analysis

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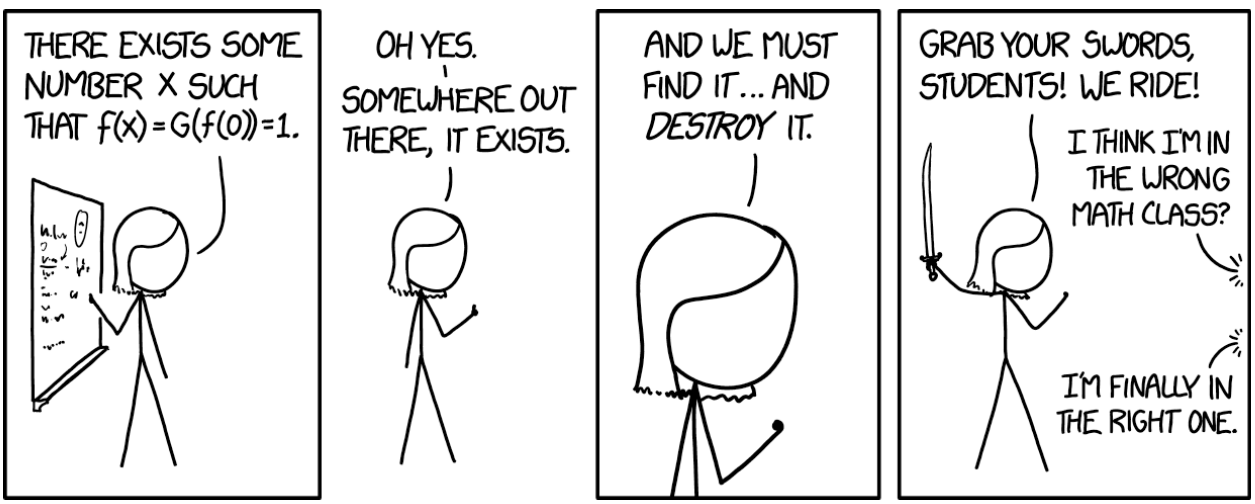

xkcd 1856: Existence Proof

Question 1

Let S be a non-empty subset of the real numbers which is bounded above. Let c be a real number and define T = {cx | x ∈ S}. Show that, if c > 0, then T is non empty, bounded above and that supT = csupS. Give an example of a set S as above, such that, with c = -1, the set T is still bounded above but supT ≠ csupS.

To show T is non-empty, observe $S \neq \emptyset$, so $T = \{cx \mid x \in S\} \neq \emptyset$.

Since S is bounded above, by definition there exists some upper bound U such that $x \leq U \; \forall \; x \in S$. So, since c is positive, $cx \leq cU \; \forall \; x \in S$. Therefore, cU is an upper bound for T.

Since S is non-empty and bounded above, we know $supS$ exists. By definition of $supS$, this is an upper bound for S, so from our previous result, we know $csupS$ is an upper bound for T. That is, $t \leq csupS \; \forall \; t \in T$. Additionally, since T is also non-empty and bounded above, $supT$ also exists. Hence, we know:

$$supT \leq csupS \tag{1}$$

Assume $supT < csupS$. We will find a contradiction.

Since $c > 0$:

$$supT < csupS$$
$$\Rightarrow \frac{supT}{c} < supS$$

By definition of $supS$, for any value $X < supS$, $\exists \; s \in S$ such that $s > X$. So, $\exists \; s' \in S$ such that

$$s' > \frac{supT}{c}$$
$$\Rightarrow cs' > supT$$

$cs' \in T$ by definition of T, so $cs' > supT$ is a contradiction.

$$\therefore supT \geq csupS \tag{2}$$

And so from (1) and (2), we know:

$$supT = csupS$$

As an example of $supT \neq supS$ when $c = -1$, take $S = \{-1, -2\}$, so $supS = -1$. With $c = -1$, $T = \{1, 2\}$ and $supT = 2$. But $csupS = 1 \neq 2$, so $csupS \neq supT$.

Question 2

(i) If (x_n) is a sequence such that $\lim|x_n| = 0$, prove that $\lim x_n = 0$ also.

By definition of the limit, $\lim |x_n| = 0 \Rightarrow \forall\ \mathcal{E} \geq 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$, $||x_n| - 0| \leq E$.

$$\begin{aligned} ||x_n| - 0| &\leq \mathcal{E} \\ \Rightarrow ||x_n|| &\leq \mathcal{E} \\ \Rightarrow |x_n| &\leq \mathcal{E} \\ \Rightarrow |x_n - 0| &\leq \mathcal{E} \\ \Rightarrow \lim x_n &= 0 \end{aligned}$$

Since the same definition applies to our end result.

(ii) Give an example of a sequence (x_n) that does not converge but that $(|x_n|)$ converges.

Define (x_n) by $x_n = (-1)^n$, which does not converge.

$|x_n| = |(-1)^n| = |-1|^n = 1$ so $(|x_n|)$ is a constant sequence and converges to 1.

Question 3

Let $\sum x_n$ be a series that converges absolutely. Show that $\sum x_n^2$ converges.

Since $\sum |x_n|$ converges, we know that $\lim_{n\to\infty} |x_n| = 0$. So, by definition of the limit, we know $\exists\ N$ such that $\forall\ n \geq N$, $|x_n| < 1$.

We can remove a finite number of terms for the start of a series without changing its behaviour. So, we consider $\sum_{n=N}^{\infty} |x_n|$, so $|x_n| < 1$ for every term in the sequence.

Note that since $x_n^2 \geq 0$, $|x_n|^2 = x_n^2 = |x_n^2|$.

$|x_n| < 1$ so $|x_n|^2 = |x_n^2| < 1$.

$|x_n^2| < |x_n| \leq |x_n|\ \forall n \geq N$, so, since $\sum_{n=N}^{\infty} |x_n|$ is convergent, $\sum_{n=N}^{\infty} |x_n^2|$ is also convergent. Once again, N is just a finite number so removing the first N terms has not changed the behaviour of the series.

Therefore $\sum x_n^2$ is absolutely convergent, which implies it is convergent.

Question 4

Prove, from the definition of limit, that $\lim_{n\to\infty} x^2 + x = 2$.

The limit definition states:

$lim_{x\to p} f(x) = L \Rightarrow \forall\ \mathcal{E} > 0,\ \exists\ \delta > 0$ such that, $\forall x \in \mathbb{R},\ 0 < |x - p| < \delta$ we have x in the domain of f and $|f(x) - L| < \mathcal{E}$.

So, to show $lim_{x\to p}(x^2 + x - 2)$, we need to find $\delta > 0$ such that, $\forall\ x \in \mathbb{R},\ 0 < |x - 1| < \delta$, we have $|x^2 + x - 2| < \mathcal{E}\ \forall\ \mathcal{E}$.

Choose $\delta = min\left\{\frac{\mathcal{E}}{4}, 1\right\}$.

$$\begin{aligned}|x^2 + x - 2| &= |(x-1)(x-2)| \\ &= |x-1||x-2|\end{aligned}$$

From our definition, we already have $|x - 1| < \delta < \frac{\mathcal{E}}{4}$. So,

$$\begin{aligned}|x-2| &= |x-1+3| \\ &= |(x-1)+3| \\ &\leq |x-1| + |3|\end{aligned}$$

By the triangle inequality. Now using $|x - 1| < \delta$, with $\delta = min\left\{\frac{\mathcal{E}}{4}, 1\right\}$, we get

$$\begin{aligned}|x-2| &\leq |x-1| + 3 \\ &< 1 + 3 \\ &< 4\end{aligned}$$

Now we have

$$\begin{aligned}|x^2 + x - 2| &= |(x-1)||(x-2)| \\ &< \frac{\mathcal{E}}{4} \cdot 4 \\ &= \mathcal{E}\end{aligned}$$

So we have found a δ that satisfies the definition of the limit $\forall\ \mathcal{E}$.

Question 5

Let f be a continuous function on $(-1, 1)$ with $f(0) = 1$. Show that there exists $\mathcal{E} > 0$ such that for all x with $|x| < \mathcal{E}$ we have $f(x) > 2/3$.

Since $f(x)$ is continuous at 0, we know that:

$$\lim_{x \to 0} f(x) = 1$$

From the definition of the limit, we have some $\delta > 0$ such that $\forall\ x \in \mathbb{R}$, $0 < |x| < \delta$, we have $x \in D(f)$ and $|f(x) - 1| < \frac{1}{3}$.

$$\begin{aligned} |f(x) - 1| &< \frac{1}{3} \\ \Rightarrow \frac{2}{3} < f(x) &< \frac{4}{3} \end{aligned}$$

So, by setting $\mathcal{E} = \delta$, we have $\mathcal{E} > 0$ such that $f(x) > \frac{2}{3}\ \forall\ x$, $0 < |x| < \mathcal{E}$. $(\star)$

Since f is continuous with $f(0) = 1$, $|x| = 0 < \mathcal{E}$ works (since $f(|0|) = 1 > \frac{2}{3}$), so we can state that $(\star)$ holds $\forall\ x$ with $|x| < \mathcal{E}$ (including $|x| = 0$).