Week 4 Assignment

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1 Question 1

Given specification:

1. The DC gain is 1.
2. The gain at 1 MHz is at least 0.95.
3. The gain at 2 MHz is at most 10^{-1}

1.a Finding filter parameters

1.a.1 Minimum order needed

For specification 3:
Let cutoff frequency be f_c, hence:

$$\left[\frac{1}{\sqrt{1+(f/f_c)^{2n}}}\right]_{f=2\times10^6} = 10^{-1}$$

$$\frac{1}{\sqrt{1+(2\times10^6/f_c)^{2n}}} = 10^{-1}$$

$$\left(2\times10^6/f_c\right)^{2n} = 10^2-1$$

$$= 99$$

$$2n\left[\log(2\times10^6)-\log f_c\right] = \log 99 \tag{1}$$

For specification 2:

$$\left[\frac{1}{\sqrt{1+(f/f_c)^{2n}}}\right]_{f=10^6} = 0.95$$

$$2n\left[\log 10^6-\log f_c\right] = \log(1/0.95^2-1) \tag{2}$$

Subtracting (2) from (1):

$$2n\log 2 = \log 99 - \log(1/0.95^2-1)$$

$$n = \frac{\log 99 - \log(1/0.95^2-1)}{2\log 2} = 4.9$$

$$\boxed{\text{So Order of filter is } n = 5}$$

1.a.2 3dB bandwidth

Substituting n=5 in equation (1):

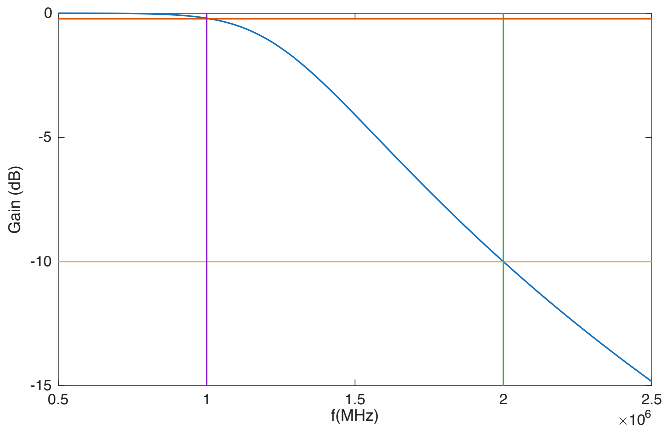
$$10\left[\log(2\times10^6)-\log f_c\right] = \log 99$$

$$\log f_c = -\frac{\log 99}{10} + \log(2\times10^6)$$

$$\therefore f_c = 1.263\text{MHz}$$

$$\boxed{\text{So 3dB bandwidth is 1.263 MHz}}$$

1.a.3 Magnitude response plot



Here orange line shows that passband criteria has met ($|H(j\omega)| \geq -0.22dB$).

1.b New order for tolerance in 3dB bandwidth

If f_c varies by $\pm5\%$ then the range of f_c is from 1.199 MHz to 1.326 MHz
For f_c = 1.199 MHz:
stop band:

$$\frac{1}{\sqrt{1+\left(\frac{2\times10^6}{1.199\times10^6}\right)^{2n}}} = 10^{-1}$$

$$n = \frac{\log 99}{2\log(2/1.199)}$$

$$n = 4.49$$

pass band:

$$\frac{1}{\sqrt{1 + \left(\frac{1\times10^6}{1.199\times10^6}\right)^{2n}}} = 0.95$$

$$n = \frac{\log(1/0.95^2 - 1)}{-2\log(1.199)}$$

$$\boxed{n = 6.13}$$

For f_c = 1.326 MHz:
stop band:

$$\frac{1}{\sqrt{1 + \left(\frac{2\times10^6}{1.326\times10^6}\right)^{2n}}} = 10^{-1}$$

$$n = \frac{\log 99}{2\log(2/1.326)}$$

$$n = 5.59$$

pass band:

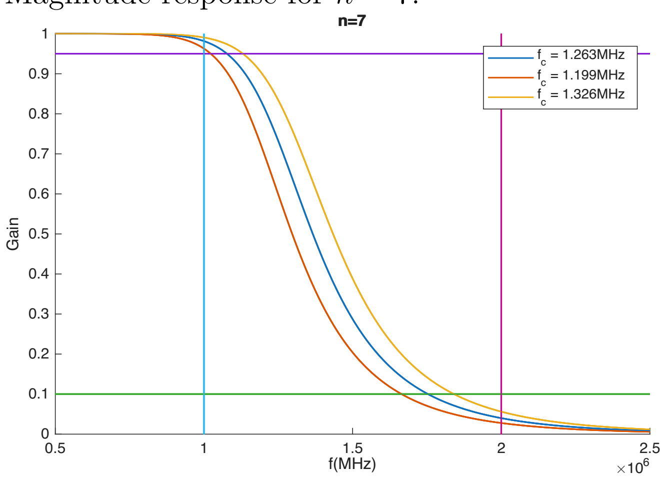
$$\frac{1}{\sqrt{1 + \left(\frac{1\times10^6}{1.326\times10^6}\right)^{2n}}} = 0.95$$

$$n = \frac{\log(1/0.95^2 - 1)}{-2\log(1.326)}$$

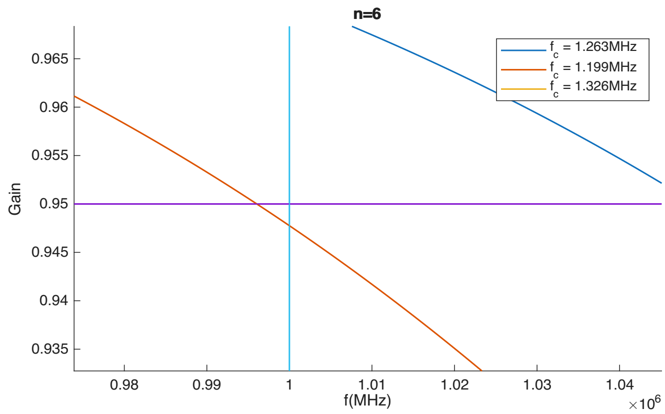
$$n = 3.94$$

Therefore order need to satisfy requirement for 5% tolerance in f_c is 7

Magnitude response for $n = 7$:



Here we can see pass band and stop band is satisfied for given frequency deviation.



But if $n = 6$, the pass band criteria is not satisfied for f = 1.199 MHz. Hence the minimum filter order is 7

2 Question 2

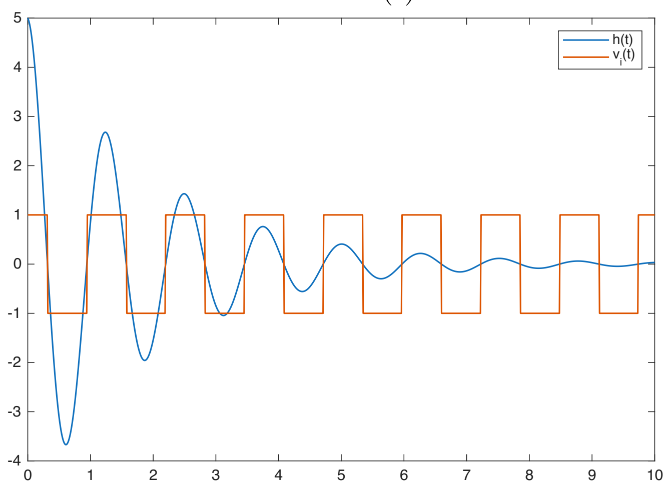
A high-Q bandpass filter the poles are located approximately at

$$s - \frac{2Q}{\omega_0} \pm j\omega_0$$

Hence the impulse response of the filter is approximately a damped oscillation with frequency ω_0 :

$$h(t) \approx \frac{\omega_0}{Q} e^{-\frac{\omega_0}{2Q}t} \cos(\omega_0 t)$$

To maximize the output at any point in time, the input signal $v_i(t)$ should be such that convolution $h(t) * v_i(t)$ is maximum. For that to occur $v_i(t)$ should be a **square wave** whose sign at any instant is same as that of $h(t)$.



Hence the output at the peak becomes the integral of absolute of $h(t)$.

Hence the waveform $v_i(t)$ that maximizes the peak output is a **square wave with frequency** ω_0.

Peak value

$$V_{\max} = \int_0^\infty |h(t)| dt$$
$$= \int_0^\infty \left| \frac{\omega_0}{Q} e^{-\frac{\omega_0}{2Q}t} \cos(\omega_0 t) \right| dt$$

Since the envelope $\exp\left(-\frac{\omega_0}{2Q}t\right)$ decays very slowly relative to the oscillation period (because $Q \gg 1$), we can approximate $\cos(\omega_0 t)$ by its average value over a period, which is $2/\pi$:

$$V_{\max} \approx \frac{2\omega_0}{\pi Q} \int_0^\infty e^{-\frac{\omega_0}{2Q}t} dt$$
$$= \frac{2\omega_0}{\pi Q} \frac{2Q}{\omega_0}$$
$$= \frac{4}{\pi} \approx 1.273$$

Comparison with Sinusoidal Input

If $v_i(t)$ is constrained to be sinusoidal with $|v_i| \leq 1$:

$$v_i(t) = \cos(\omega_0 t)$$

The peak occurs at resonant frequency ω_0, and magnitude is:

$$|H(j\omega_0)| = \left| \frac{j\omega_0/(\omega_0 Q)}{(j\omega_0/\omega_0)^2 + j\omega_0/(\omega_0 Q) + 1} \right|$$
$$= \frac{j/Q}{-1 + j/Q + 1} = 1$$

Hence maximum peak voltage is 1 is v_i is constrained to be sinusoidal and amplitude < 1, compared to 1.273 if v_i is just amplitude constrained

MATLAB codes

1.a.3

```
f = .5e6:1e3:2.5e6;
n = 5;
fc = 1.263e6;
TF = 1./sqrt(1 + (f/fc).^(2*n));
plot(f, 10*log10(TF));

l1Mhz = 10*log10(0.95);
l2Mhz = 10*log10(0.1);

line([.5e6, 2.5e6], [l1Mhz, l1Mhz]);
line([.5e6, 2.5e6], [l2Mhz, l2Mhz]);
line([1e6, 1e6], [0, -15]);
line([2e6, 2e6], [0, -15]);
xlabel("f(MHz)");
ylabel("Gain (dB)")
```

1.b

```
f = .5e6:1e3:2.5e6;
TF = @(fc, n) 1./sqrt(1 + (f/fc).^(2*n));
hold on;
n=6;
plot(f, TF(1.263e6, n));
plot(f, TF(1.199e6, n));
plot(f, TF(1.326e6, n));

line([0.5e6, 2.5e6], [0.95, 0.95]);
line([0.5e6, 2.5e6], [0.1, 0.1]);
line([1e6, 1e6], [1, 0]);
line([2e6, 2e6], [1, 0]);
hold off;
title("n="+n)
legend([ ...
"f_c = 1.263MHz", ...
"f_c = 1.199MHz", ...
"f_c = 1.326MHz" ...
]);
xlabel("f(MHz)"); ylabel("Gain")
```

2

```
x = 0:.01:10;
w = 5;
y = 5*exp(-x/2).*cos(w*x);
plot(x, y);
hold on;
plot(x, sign(y));
legend(["h(t)", "v_i(t)"])
```