Fast Iterative Solvers: Project 2

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Abstract: The Poisson problem in 2D is solved using the Multigrid solver via W-cycles and Gauss-Seidel smoothers. Different meshes were tested for obtaining the solution, specifically for N = 2^n with n = 4 and n =7 in each direction. For both the cases, the relative residual was computed and plotted different Gauss-Seidel smoothing iterations.

Introduction

The goal is to solve the 2D Poisson problem on the given mesh. The problem is defined as:

$$-\nabla^2 u = f \qquad in\ \Omega$$

$$u = 0 \qquad on\ \partial\Omega$$

For the current analysis, $\Omega = (0,1) \times (0,1)$. The finite difference discretization for the grid is given by:

$$\mathcal{G}_h = \{(ih, jh) : i, j = 0, \ldots, N;\ hN = 1\}$$

The right-hand side function is given as $f(x, y) = 8\pi^2 \sin(2\pi x) \sin(2\pi y)$

Multigrid Solver

The given partial differential equation is first converted to the finite difference form. The following form is obtained:

$$\frac{1}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) + \frac{1}{h^2}\left(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right) = -f_{i,j} \qquad i, j = 1, \ldots, N-1$$

$$u_{i,j} = 0 \qquad otherwise$$

In the multigrid solver algorithm, Gauss-Seidel smoothing iterations are first applied to the solution vector to smoothen out the high frequency components of the error. Then the error equation is solved on the coarse grid for the current grid. If the current grid has N = 2^n nodes, then the next coarser level will have 2^{n-1} nodes in each direction. The next coarse levels are obtained recursively until there is only 1 interior node on the coarsest level. On the coarsest level, the error equation is solved exactly. The error values are interpolated back the finer levels to obtain the solution on the finest level on which the solution was initially solved for. Later, the relative residual is checked for convergence after each multigrid iteration.

Following are the steps for each multigrid iteration:

1. Pre-smoothing: Apply v_1 Gauss-Seidel iterations to u^h
2. Residual: get residual with $r^h = f^h - Au^h$
3. Restriction: Apply restriction operator to get residual on the coarse grid, r^h to r^H
4. *If*: coarsest level, solve the exact problem
5. *Else:* recursive step. Call the multigrid function γ times to coarsen the system. For W-cycle $\gamma=2$; for V-cycle $\gamma=1$. (Back to step 1)
6. *End If*
7. Prolongation: Prolong the solution from Step 4 to the next finer level and add u^h to it.
8. Post-smoothing: Apply v_2 Gauss-Seidel iterations to prolonged solution.
9. Return u^h

Results

Convergence analysis was done for N=16 and N=128. Two combinations of smoothing iterations were tested with W-cycle, and their results are depicted in Table 1. Number of iterations needed to reach $\frac{||r_m||_\infty}{||r_0||_\infty} < 10^{-10}$ is shown.

$N = 2^n$	v_1	v_2	**Iterations**
n = 4	1	1	12
	2	1	10
n = 7	1	1	10
	2	1	8

Table 1 - Convergence analysis

The relative residual $\frac{||r_m||_\infty}{||r_0||_\infty}$ was also plotted against the multigrid iterations for different values of N. This is shown in Figure 1 and Figure 2.

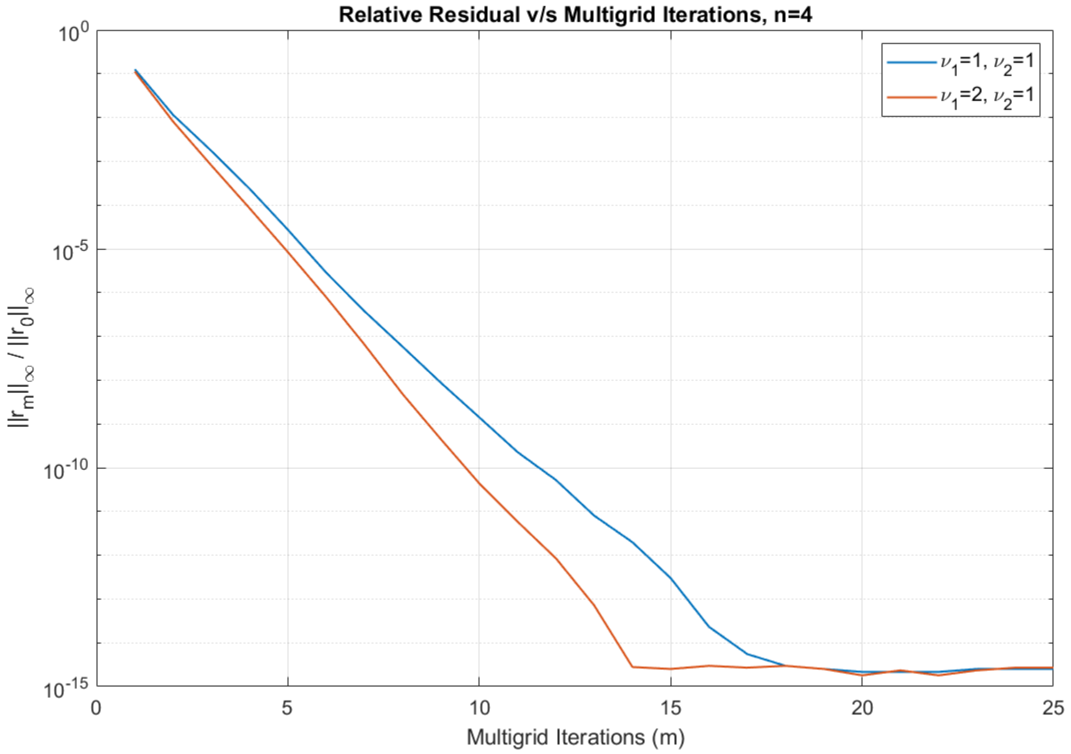

Figure 1 - Relative residual v/s Multigrid iterations, n=4

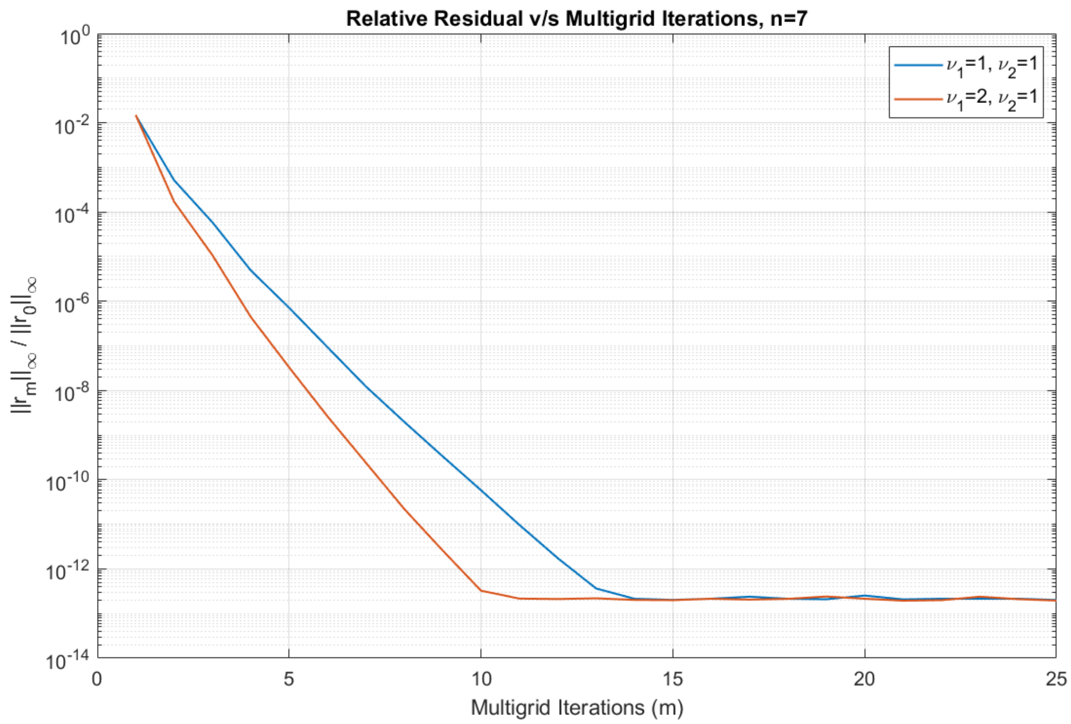

Figure 2 - Relative residual v/s Multigrid iterations, n=7

Runtime of the multigrid solver was also tested for different values of γ. With $\nu_1 = \nu_2 = 1$, the results are tabulated in Table 2. We see that as γ increases, the runtime increases marginally.

γ	Runtime (s)
1 (V-cycle)	0.0156
2 (W-cycle)	0.0313
3	0.0469
4	0.1250

Table 2 - Runtime for different γ values, n = 7

As seen before, the number of pre- and post-smoothing iterations can be varied. Different values of ν_1 and ν_2 were analysed versus runtime. From Table 3, we see that increasing smoothing iterations does not affect runtime.

ν_1	ν_2	Runtime (s)
1	1	0.0156
2	2	0.0313
3	3	0.0313

Table 3 - Runtime for different Gauss-Seidel Iterations, n=7, γ =2

Finally, the multigrid solution and the true analytical solution are plotted. From the following figures we see that the multigrid solution gives a good approximation of the analytical solution.

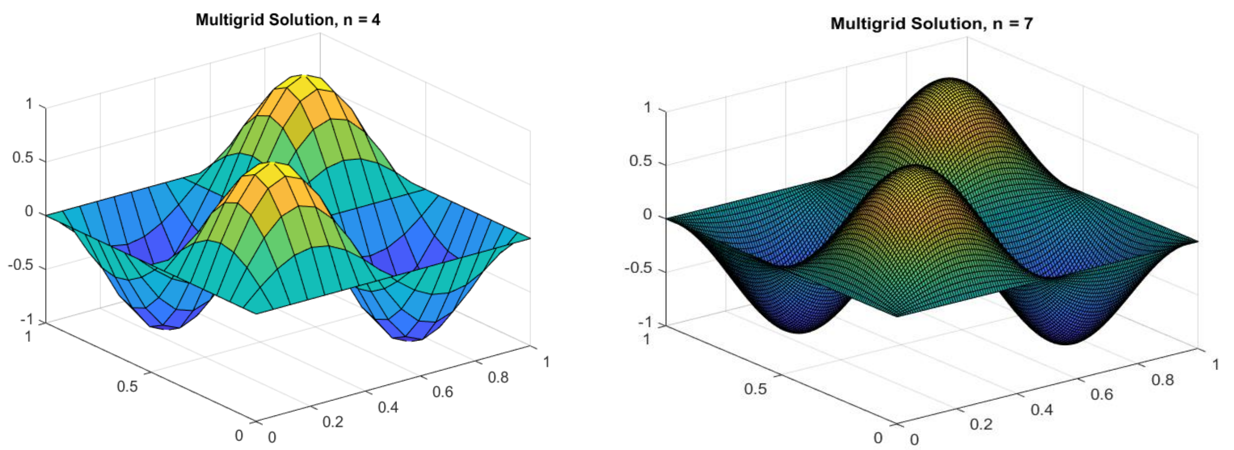

Figure 3 - Multigrid Solution

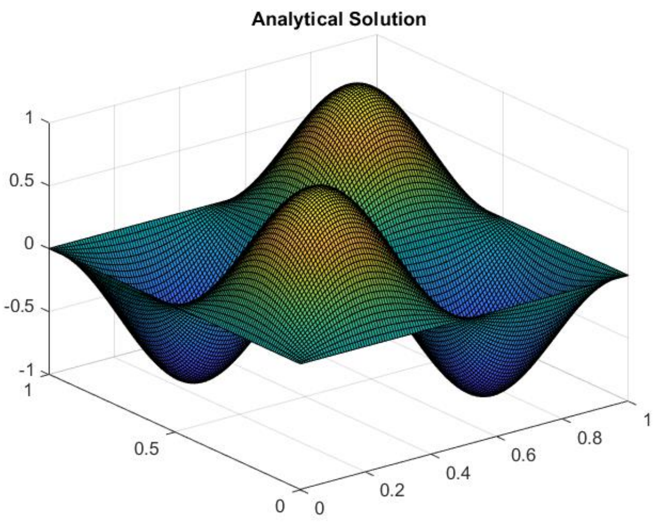

Figure 4 - Analytical Solution

Conclusion

The multigrid algorithm is implemented for the 2D Poisson problem for N=16 and N=128 elements in each direction. For the W-cycle (γ = 2), it is found that 2 pre-smoothing iterations (ν_1 = 2) leads to lower number of multigrid iterations for solution convergence. The relative residuals for different smoothing iterations are plotted versus multigrid iterations. From the solution plots it is evident that multigrid solution gives a good approximation of the true solution.

References

1. Wolfgang Hackbusch. *Multi-Grid Methods and Applications*. Springer-Verlag, 2003.
2. Ulrich Trottenberg, Cornelis Oosterlee, and Anton Schüller. *Multigrid*. Elsevier Academic Press, 2001.