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# ON NECESSARY CONDITIONS FOR THE CONVERGENCE OF FOURIER SERIES

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We establish necessary conditions for the convergence of multiple Fourier series of integrable functions in the mean.

By $L_1(T^m)$ we denote a space of $2\pi$-periodic integrable functions $f(x)$ with the norm

$$\|f\|_1 = \int\limits_{T^m} |f(x_1, \ldots, x_m)|\, dx_1 \ldots dx_m = \int\limits_{T^m} |f(x)|\, dx < \infty,$$

where $x = (x_1, \ldots, x_m)$ and $T^m = [0, 2\pi)^m$.

Let $f \in L_1(T^m)$, let

$$S[f] = \sum_{k=0}^{\infty} \sum_{|l|_1 = k} c_l e^{i(l,x)} \tag{1}$$

be its Fourier series, and let

$$S_n(f; x) = \sum_{k=0}^{n} \sum_{|l|_1 = k} c_l e^{i(l,x)}$$

be the $n$th partial sum of series (1). Here, $l = (l_1, l_2, \ldots, l_m)$, $l_j \in \mathbb{Z}$, $j = \overline{1, m}$, $x = (x_1, x_2, \ldots, x_m)$,

$$(l, x) = l_1 x_1 + l_2 x_2 + \ldots + l_m x_m,$$

and

$$|l|_1 = |l_1| + |l_2| + \ldots + |l_m|.$$

We say that series (1) converges to the function $f(x)$ in the mean if

$$\|f - S_n(f)\|_1 \to 0 \tag{2}$$

as $n \to \infty$.

Since relation (2) is true not for all functions $f(\cdot) \in L_1$ (see [1], Chap. VIII, Sec. 22), it is necessary to establish the conditions for the coefficients of series (1) under which this series is convergent in the mean.

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In the present paper, for $f \in L_1(T^m)$, we obtain conditions necessary for the validity of relation (2).

In [2], Fomin proved that if the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{3}$$

converges in the mean, then the relation

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} \frac{a_{n+k}}{k} = 0$$

is true for any sequence of natural numbers $\{m_n\}$ such that $m_n \le n$, $n = 1, 2, \ldots$.

In [3], Zaderei and Smal' studied series of the form (3)

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{4}$$

and established that the condition

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{|a_{n+k}|}{k} = 0$$

is necessary for the convergence of series (3) and (4) in the metric of the space $L_1$.

In [4], for series of the form

$$\sum_{k=0}^{\infty} a_k \sum_{l_1+\ldots+l_m=k} e^{i(l,x)}, \quad l_j \ge 0, \quad j = 1, 2, \ldots, m,$$

it is shown that the relation

$$\lim_{n\to\infty} \ln^{m-1} n \sum_{k=1}^{n} \frac{|a_{n+k}|}{k} = 0$$

is a necessary condition for the convergence of these series in the mean.

In [5], for series of the form

$$\sum_{k=0}^{\infty} \sum_{l_1+\cdots+l_m=k} a_l e^{i(l,x)}, \quad l_j \ge 0, \quad j = 1, 2, \ldots, m,$$

it is demonstrated that the relation

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} \left\| \sum_{l_1+\ldots+l_m=n+k} a_l e^{i(l,x)} \right\|_1 = 0$$

is a necessary condition for the convergence of these series in the mean.

The following theorem is true:

**Theorem 1.** *Let $f$ belong to $L_1(T^m)$. Then, in order that relation (2) be true, it is necessary that the following equality be true:*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}\sum_{s}\|A_s(k,n,x)\|_1=0,$$

*where*

$$A_s(k,n,x)=\sum_{\sum_{j=1}^{m}(-1)^{s_j}l_j=n+k}c_l e^{i(l,x)},\qquad s=(s_1,s_2,\ldots,s_m),$$

$$s_j=\begin{cases}0 & \textit{for}\quad l_j\ge 0,\\ 1 & \textit{for}\quad l_j<0,\end{cases}\qquad j=\overline{1,m}.$$

***Proof.*** The proof of the theorem is based on the well-known assertions formulated in what follows.

For trigonometric polynomials of the form

$$t_n(x)=t_n(x_1,x_2,\ldots,x_m)=\sum_{k=0}^{n}\sum_{|l|_1=k}c_l e^{i(l,x)},$$

the following inequality is true:

$$\left\|\frac{\partial t_n(x)}{\partial x_i}\right\|_1\le n\,\|t_n(x_1,\ldots,x_m)\|_1\,. \tag{5}$$

Inequality (5) is the Bernstein-type inequality for trigonometric polynomials.

Let $M=\{1,2,\ldots,m\}$, let $B\subset M$, let

$$\widetilde{t_n}^{(B)}(x)=(-i)^{|B|}\sum_{k=0}^{n}\sum_{|l|_1=k}\prod_{j\in B}\operatorname{sign} l_j c_l e^{i(l,x)}$$

be a function of the variables $x_j,\ j\in B$, conjugate to $t_n(x)$, and let $|B|$ be the number of elements of the set $B$.

An inequality of the form (5) is also true for the conjugate functions (see [6, p. 21])

$$\left\|\frac{\partial\widetilde{t_n}^{(B)}(x)}{\partial x_i}\right\|_1\le n\,\|t_n(x)\|_1\,. \tag{6}$$

**Hardy–Littlewood Theorem** [7 p. 454]**.** *Let*

$$F(z)=b_0+b_1z+\ldots+b_nz^n+\ldots\in H.$$

*Then*

$$\sum_{n=0}^{\infty} \frac{|b_n|}{n+1} \leq \frac{1}{2} \int_0^{2\pi} \left| F\left(e^{ix}\right) \right| dx, \tag{7}$$

*where* $H$ *is a Hardy space* $\big($*the space of functions* $F(z)$ *analytic for* $|z| < 1$ *and such that*

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \left| F\left(re^{ix}\right) \right| dx < \infty \big).$$

By $V_{2n}^{n}(f; x)$ we denote the de la Vallée-Poussin sum of the function $f(x)$,

$$V_{2n}^{n}(f; x) = \frac{1}{n} \sum_{k=n+1}^{2n} S_k(f; x) = \sum_{k=0}^{2n} \lambda_k^{(n)} \sum_{|l|_1 = k} c_l e^{i(l,x)},$$

where

$$\lambda_k^{(n)} = \begin{cases} 1, & k = \overline{0, n+1}, \\ \dfrac{2n-k+1}{n}, & k = \overline{n+1, 2n}. \end{cases}$$

Let

$$t_n(x) = \sum_{k=1}^{n} \sum_{|l|_1 = k} c_l e^{i(l,x)}$$

($c_l = c_{l_1, l_2, \ldots, l_m}$ are complex numbers) be a trigonometric polynomial. The set of these polynomials is denoted by $T_n$.

Also let

$$E_n(f)_1 = \inf_{t_n \in T_n} \|f(x) - t_n(x)\|_1$$

be the best approximation of the function $f(\cdot)$ in the metric $L_1$. By $t_n^*$ we denote the polynomial of the best approximation of the function $f(\cdot)$ from $T_n$. Then

$$\begin{aligned} \|f - V_{2n}^{n}(f)\|_1 &\leq \|f - t_n^*\|_1 + \|t_n^* - V_{2n}^{n}(f)\|_1 \\ &= E_n(f)_1 + \|V_{2n}^{n}(f - t_n^*)\|_1 \leq E_n(f)_1 + E_n(f)_1 \|V_{2n}^{n}\|_{L_1 \to L_1} \\ &= (\|V_{2n}^{n}\|_{L_1 \to L_1} + 1) E_n(f)_1, \end{aligned}$$

where $\|V_{2n}^{n}\|_{L_1 \to L_1}$ is the norm of the de la Vallée-Poussin operator.

By the Stone–Weierstrass theorem, the best approximation $E_n(f)_1 \to 0$ as $n \to \infty$ for any function $f \in L_1(T^m)$ (see [8, p. 42]).

Now let

$$\sigma_n(f;x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f;x)$$

be the Fejér means.

In [9, 10], Podkorytov proved that $\|\sigma_n\|_{L_1\to L_1} \le C$. Since

$$V_{2n}^n(f;x) = \frac{2n+1}{n}\sigma_{2n}(f;x) - \frac{n+1}{n}\sigma_n(f;x),$$

we conclude that

$$\|V_{2n}^n\|_{L_1\to L_1} \le C.$$

According to the Lebesgue inequality, we get

$$\|f - V_{2n}^n(f)\|_1 \le (\|V_{2n}^n\|_{L_1\to L_1} + 1)E_n(f)_1 \to 0.$$

Then

$$\|f - S_n(f)\|_1 \ge \|V_{2n}^n(f) - S_n(f)\|_1 - \|f - V_{2n}^n(f)\|_1 \ge \|V_{2n}^n(f) - S_n(f)\|_1 + o(1). \tag{8}$$

The partial Fourier sums of the functions conjugate to $f(x)$ in the the first, second, and both variables are denoted by $\widetilde{S_n}^{(1)}(f;x)$, $\widetilde{S_n}^{(2)}(f;x)$, and $\widetilde{S_n}^{(1,2)}(f;x)$, respectively, and the conjugate de la Vallée-Poussin sums with respect to the first, second, and both variables are denoted by $\widetilde{V_{2n}^n}^{(1)}(f;x)$, $\widetilde{V_{2n}^n}^{(2)}(f;x)$, and $\widetilde{V_{2n}^n}^{(1,2)}(f;x)$, respectively. Thus, we have

$$\widetilde{S_n}^{(1)}(f;x) = -i\sum_{k=1}^{n}\sum_{|l_1|+|l_2|=k} \operatorname{sign} l_1 c_l e^{i(l,x)},$$

$$\widetilde{S_n}^{(2)}(f;x) = -i\sum_{k=1}^{n}\sum_{|l_1|+|l_2|=k} \operatorname{sign} l_2 c_l e^{i(l,x)},$$

$$\widetilde{S_n}^{(1,2)}(f;x) = -\sum_{k=1}^{n}\sum_{|l_1|+|l_2|=k} \operatorname{sign} l_1 \operatorname{sign} l_2 c_l e^{i(l,)x},$$

$$\widetilde{V_{2n}^n}^{(1)}(f;x) = -i\sum_{k=1}^{2n}\lambda_k^{(n)}\sum_{|l_1|+|l_2|=k} \operatorname{sign} l_1 c_l e^{i(l,x)},$$

$$\widetilde{V_{2n}^n}^{(2)}(f;x) = -i\sum_{k=1}^{2n}\lambda_k^{(n)}\sum_{|l_1|+|l_2|=k} \operatorname{sign} l_2 c_l e^{i(l,x)},$$

$$\widetilde{V_{2n}^{n}}^{(1,2)}(f;x) = -\sum_{k=1}^{2n} \lambda_k^{(n)} \sum_{|l_1|+|l_2|=k} \operatorname{sign} l_1 \operatorname{sign} l_2 c_l e^{i(l,x)}.$$

By using the properties of the function $\operatorname{sign} l$, we determine the quantity

$$\begin{aligned}
&\left(V_{2n}^{n}(f;x) - S_n(f;x)\right) + i\left(\widetilde{V_{2n}^{n}}^{(1)}(f;x) - \widetilde{S_n}^{(1)}(f;x)\right) \\
&\quad + i\left(\widetilde{V_{2n}^{n}}^{(2)}(f;x) - \widetilde{S_n}^{(2)}(f;x)\right) - \left(\widetilde{V_{2n}^{n}}^{(1,2)}(f;x) - \widetilde{S_n}^{(1,2)}(f;x)\right) \\
&\quad = \left[\begin{array}{lll}
4\sum\limits_{k=n+1}^{2n} \dfrac{2n-k+1}{n} \sum\limits_{l_1+l_2=k} c_l e^{i(l,x)}, & l_1 > 0, & l_2 > 0, \\
2\sum\limits_{k=n+1}^{2n} \dfrac{2n-k+1}{n} c_{k,0} e^{ikx_1}, & l_1 > 0, & l_2 = 0, \\
2\sum\limits_{k=n+1}^{2n} \dfrac{2n-k+1}{n} c_{0,k} e^{ikx_2}, & l_1 = 0, & l_2 > 0, \\
0, & \text{otherwise} &
\end{array}\right. \\
&\quad = \frac{4}{2^{\gamma_l}} \sum_{k=n+1}^{2n} \frac{2n-k+1}{n} \sum_{l_1+l_2=k} c_l e^{i(l,x)},
\end{aligned}$$

where $\gamma_l$ is the number of coordinates of the vector $l = (l_1, l_2)$ equal to zero.

For $l_1 \geq 0$ and $l_2 \geq 0$, we have

$$\begin{aligned}
\sum_{k=n+1}^{2n} \frac{2n-k+1}{n} \sum_{l_1+l_2=k} c_l e^{(l,x)} &= \frac{2^{\gamma_l}}{4}\left(V_{2n}^{n}(f;x) - S_n(f;x)\right) \\
&\quad + \frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(1)}(f;x) - \widetilde{S_n}^{(1)}(f;x)\right) \\
&\quad + \frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(2)}(f;x) - \widetilde{S_n}^{(2)}(f;x)\right) \\
&\quad - \frac{2^{\gamma_l}}{4}\left(\widetilde{V_{2n}^{n}}^{(1,2)}(f;x) - \widetilde{S_n}^{(1,2)}(f;x)\right). \qquad (9)
\end{aligned}$$

In the other cases, we obtain similar equalities.

For $l_1 > 0$ and $l_2 < 0$, we get

$$\sum_{k=n+1}^{2n} \frac{2n-k+1}{n} \sum_{l_1-l_2=k} c_l e^{(l,x)} = \frac{2^{\gamma_l}}{4}\left(V_{2n}^{n}(f;x)-S_n(f;x)\right)$$

$$+\frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(1)}(f;x)-\widetilde{S_n}^{(1)}(f;x)\right)$$

$$-\frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(2)}(f;x)-\widetilde{S_n}^{(2)}(f;x)\right)$$

$$+\frac{2^{\gamma_l}}{4}\left(\widetilde{V_{2n}^{n}}^{(1,2)}(f;x)-\widetilde{S_n}^{(1,2)}(f;x)\right). \tag{10}$$

For $l_1 < 0$ and $l_2 > 0$, we obtain

$$\sum_{k=n+1}^{2n} \frac{2n-k+1}{n} \sum_{-l_1+l_2=k} c_l e^{(l,x)} = \frac{2^{\gamma_l}}{4}\left(V_{2n}^{n}(f;x)-S_n(f;x)\right)$$

$$-\frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(1)}(f;x)-\widetilde{S_n}^{(1)}(f;x)\right)$$

$$+\frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(2)}(f;x)-\widetilde{S_n}^{(2)}(f;x)\right)$$

$$+\frac{2^{\gamma_l}}{4}\left(\widetilde{V_{2n}^{n}}^{(1,2)}(f;x)-\widetilde{S_n}^{(1,2)}(f;x)\right). \tag{11}$$

For $l_1 \le 0$ and $l_2 \le 0$, we find

$$\sum_{k=n+1}^{2n} \frac{2n-k+1}{n} \sum_{-l_1-l_2=k} c_l e^{(l,x)} = \frac{2^{\gamma_l}}{4}\left(V_{2n}^{n}(f;x)-S_n(f;x)\right)$$

$$+\frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(1)}(f;x)-\widetilde{S_n}^{(1)}(f;x)\right)$$

$$+\frac{2^{\gamma_l}}{4} i\left(\widetilde{V_{2n}^{n}}^{(2)}(f;x)-\widetilde{S_n}^{(2)}(f;x)\right)$$

$$-\frac{2^{\gamma_l}}{4}\left(\widetilde{V_{2n}^{n}}^{(1,2)}(f;x)-\widetilde{S_n}^{(1,2)}(f;x)\right). \tag{12}$$

Differentiating (9), first, with respect to $x_1$ and then with respect to $x_2$ and finding the sum of the equalities obtained as a result, we get

$$\begin{aligned}\left\|\sum_{k=n+1}^{2n}\frac{2n-k+1}{n}k\sum_{l_1+l_2=k}c_l e^{i(l,x)}\right\|_1 &\le \frac{2^{\gamma_l}}{4}\left\|(V_{2n}^n(f;x)-S_n(f;x))'_{x_1}\right\|_1\\
&+\frac{2^{\gamma_l}}{4}\left\|(V_{2n}^n(f;x)-S_n(f;x))'_{x_2}\right\|_1\\
&+\frac{2^{\gamma_l}}{4}\left\|(\widetilde{V_{2n}^n}^{(1)}(f;x)-\widetilde{S_n}^{(1)}(f;x))'_{x_1}\right\|_1\\
&+\frac{2^{\gamma_l}}{4}\left\|(\widetilde{V_{2n}^n}^{(1)}(f;x)-\widetilde{S_n}^{(1)}(f;x))'_{x_2}\right\|_1\\
&+\frac{2^{\gamma_l}}{4}\left\|(\widetilde{V_{2n}^n}^{(2)}(f;x)-\widetilde{S_n}^{(2)}(f;x))'_{x_1}\right\|_1\\
&+\frac{2^{\gamma_l}}{4}\left\|(\widetilde{V_{2n}^n}^{(2)}(f;x)-\widetilde{S_n}^{(2)}(f;x))'_{x_2}\right\|_1\\
&+\frac{2^{\gamma_l}}{4}\left\|(\widetilde{V_{2n}^n}^{(1,2)}(f;x)-\widetilde{S_n}^{(1,2)}(f;x))'_{x_1}\right\|_1\\
&+\frac{2^{\gamma_l}}{4}\left\|\left(\widetilde{V_{2n}^n}^{(1,2)}(f;x)-\widetilde{S_n}^{(1,2)}(f;x)\right)'_{x_2}\right\|_1. \qquad (13)\end{aligned}$$

Applying the Bernstein inequality (5) [or (6)] to each term on the right-hand side of (13), we obtain

$$\left\|\sum_{k=n+1}^{2n}\frac{2n-k+1}{n}k\sum_{l_1+l_2=k}c_l e^{i(l,x)}\right\|_1 \le \frac{2^{\gamma_l}8n}{4}\left\|\sum_{k=n+1}^{2n}\frac{2n-k+1}{n}\sum_{|l_1|+|l_2|=k}c_l e^{i(l,x)}\right\|_1. \qquad (14)$$

Substituting (14) in (8), we find

$$\begin{aligned}\|f-S_n(f)\|_1 &\ge \frac{1}{2^{\gamma_l}2n}\left\|\sum_{k=n+1}^{2n}\frac{2n-k+1}{n}k\sum_{l_1+l_2=k}c_l e^{i(l,x)}\right\|_1 + o(1)\\
&= \frac{1}{2^{\gamma_l}2n}\int_{T^2}\left|\sum_{k=n+1}^{2n}\frac{2n-k+1}{n}k\sum_{l_1+l_2=k}c_l e^{i(l,x)}\right|dx + o(1). \qquad (15)\end{aligned}$$

In view of the $2\pi$-periodicity of the integrand in (15) and the fact that the integral is independent of the interval of integration of length $2\pi$, we conclude that

$$\frac{1}{2^{\gamma_l}2n}\frac{1}{2\pi}\int\limits_{T^2}\left(\int\limits_0^{2\pi}\left|\sum_{k=n+1}^{2n}\frac{2n-k+1}{n}k\sum_{l_1+l_2=k}c_le^{i(l,x)}e^{ikt}\right|dt\right)dx$$

$$=\frac{1}{2^{\gamma_l}2n}\frac{1}{2\pi}\int\limits_{T^2}\left(\int\limits_0^{2\pi}\left|\sum_{k=1}^{n}\frac{n-k+1}{n}(n+k)\sum_{l_1+l_2=k}c_le^{i(l,x)}e^{ikt}\right|dt\right)dx. \qquad (16)$$

The integrand in (16) regarded as a function of the variable $t$ belongs to the Hardy space. Applying the Hardy–Littlewood inequality (7) to this function, we get

$$\frac{1}{2^{\gamma_l}2n}\frac{1}{2\pi}\int\limits_{T^2}\int\limits_0^{2\pi}\left|\sum_{k=1}^{n}\frac{n-k+1}{n}(n+k)\sum_{l_1+l_2=k}c_le^{i(l,x)}e^{ikt}\right|dt\,dx$$

$$\geq\frac{1}{2^{\gamma_l}2n}\frac{1}{2\pi}\int\limits_{T^2}2\sum_{k=1}^{n}\frac{1}{k+1}\left|\frac{n-k+1}{n}(n+k)\sum_{l_1+l_2=n+k}c_le^{i(l,x)}\right|dx$$

$$\geq\frac{1}{2^{\gamma_l}n}\frac{1}{2\pi}\sum_{k=1}^{n}\frac{n-k+1}{n}\frac{n+k}{k+1}\int\limits_{T^2}\left|\sum_{l_1+l_2=n+k}c_le^{i(l,x)}\right|dx$$

$$\geq\frac{1}{2^{\gamma_l}2\pi}\sum_{k=1}^{n}\frac{1}{k}\left\|\sum_{l_1+l_2=n+k}c_le^{i(l,x)}\right\|_1-\frac{1}{2^{\gamma_l}2\pi}\frac{1}{n}\sum_{k=1}^{n}\left\|\sum_{l_1+l_2=n+k}c_le^{i(l,x)}\right\|_1.$$

By virtue of relation (2), we conclude that

$$\left\|\sum_{|l|_1=k}a_le^{i(l,x)}\right\|_1\to 0 \qquad \text{as} \qquad n\to\infty$$

and, therefore,

$$\frac{1}{n}\sum_{k=1}^{n}\left\|\sum_{|l|_1=n+k}a_le^{i(l,x)}\right\|_1\to 0.$$

This yields

$$\|f-S_n(f)\|_1\geq\frac{1}{2^{\gamma_l}2\pi}\sum_{k=1}^{n}\frac{1}{k}\left\|\sum_{l_1+l_2=n+k}c_le^{i(l,x)}\right\|_1+o(1). \qquad (17)$$

In a similar way, we analyze expressions (10), (11), and (12).

Differentiating relation (10), first, with respect to $x_1$ and then with respect to $x_2$ and subtracting the equalities obtained as a result, we obtain

$$\|f - S_n(f)\|_1 \geq \frac{1}{2^{\gamma_l} 2\pi} \sum_{k=1}^{n} \frac{1}{k} \left\| \sum_{l_1 - l_2 = n+k} c_l e^{i(l,x)} \right\|_1 + o(1). \tag{18}$$

Performing a similar procedure with relations (11) and (12), we conclude that

$$\|f - S_n(f)\|_1 \geq \frac{1}{2^{\gamma_l} 2\pi} \sum_{k=1}^{n} \frac{1}{k} \left\| \sum_{-l_1 + l_2 = n+k} c_l e^{i(l,x)} \right\|_1 + o(1), \tag{19}$$

$$\|f - S_n(f)\|_1 \geq \frac{1}{2^{\gamma_l} 2\pi} \sum_{k=1}^{n} \frac{1}{k} \left\| \sum_{-l_1 - l_2 = n+k} c_l e^{i(l,x)} \right\|_1 + o(1). \tag{20}$$

We now find the sum of relations (17)–(20). This yields

$$4\|f - S_n(f)\|_1 \geq \frac{1}{2^{\gamma_l} 2\pi} \sum_{k=1}^{n} \frac{1}{k} \left( \left\| \sum_{l_1 + l_2 = n+k} c_l e^{i(l,x)} \right\|_1 \right.$$

$$\left. + \left\| \sum_{l_1 - l_2 = n+k} c_l e^{i(l,x)} \right\|_1 + \left\| \sum_{-l_1 + l_2 = n+k} c_l e^{i(l,x)} \right\|_1 + \left\| \sum_{-l_1 - l_2 = n+k} c_l e^{i(l,x)} \right\|_1 \right) + o(1),$$

which completes the proof of the theorem.

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