Real-Time Digital Signal Processing. Sen M Kuo, Bob H Lee
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Appendix A
Some Useful Formulas

This appendix briefly summarizes some basic formulas of algebra that will be used extensively in this book.

A.1 Trigonometric Identities

Trigonometric identities are often required in the manipulation of Fourier series, transforms, and harmonic analysis. Some of the most common identities are listed as follows:

$$\sin(-\alpha) = -\sin\alpha, \tag{A.1a}$$

$$\cos(-\alpha) = \cos\alpha, \tag{A.1b}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \tag{A.2a}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \tag{A.2b}$$

$$2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta), \tag{A.3a}$$

$$2\cos\alpha\cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta), \tag{A.3b}$$

$$2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta), \tag{A.3c}$$

$$2\sin\beta\cos\alpha = \sin(\alpha + \beta) - \sin(\alpha - \beta), \tag{A.3d}$$

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right), \tag{A.4a}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right), \tag{A.4b}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right), \tag{A.4c}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \tag{A.5a}$$

$$\cos(2\alpha) = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha, \tag{A.5b}$$

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1-\cos\alpha)}, \tag{A.6a}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1+\cos\alpha)}, \tag{A.6b}$$

$$\sin^2\alpha + \cos^2\alpha = 1, \tag{A.7a}$$

$$\sin^2\alpha = \frac{1}{2}[1-\cos(2\alpha)], \tag{A.7b}$$

$$\cos^2\alpha = \frac{1}{2}[1+\cos(2\alpha)], \tag{A.7c}$$

$$e^{\pm j\alpha} = \cos\alpha \pm j\sin\alpha, \tag{A.8a}$$

$$\sin\alpha = \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}), \tag{A.8b}$$

$$\cos\alpha = \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}). \tag{A.8c}$$

In Euler's theorem (A.8), $j = \sqrt{-1}$. The basic concepts and manipulations of complex number will be reviewed in Section A.3.

A.2 Geometric Series

The geometric series is used in discrete-time signal analysis to evaluate functions in closed form. Its basic form is

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}, \quad x \neq 1. \tag{A.9}$$

This is a widely used identity. For example,

$$\sum_{n=0}^{N-1} e^{-j\omega n} = \sum_{n=0}^{N-1} (e^{-j\omega})^n = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}. \tag{A.10}$$

If the magnitude of x is less than 1, the infinite geometric series converges to

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1. \tag{A.11}$$

A.3 Complex Variables

A complex number z can be expressed in rectangular (Cartesian) form as

$$z = x + jy = \mathrm{Re}[z] + j\mathrm{Im}[z]. \tag{A.12}$$

Since the complex number z represents the point (x, y) in the two-dimensional plane, it can be drawn as a vector illustrated in Figure A.1. The horizontal coordinate x is called the real part, and the vertical coordinate y is the imaginary part.

As shown in Figure A.1, the vector z also can be defined by its length (radius) r and its direction (angle) θ. The x and y coordinates of the vector are given by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{A.13}$$

Therefore the vector z can be expressed in polar form as

$$z = r\cos\theta + jr\sin\theta = re^{j\theta}, \tag{A.14}$$

where

$$r = |z| = \sqrt{x^2 + y^2} \tag{A.15}$$

is the magnitude of the vector z and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{A.16}$$

is its phase in radians.

The basic arithmetic operations for two complex numbers, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, are listed as follows:

$$z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2), \tag{A.17}$$

$$z_1 z_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) \tag{A.18a}$$

$$= (r_1r_2)e^{j(\theta_1+\theta_2)}, \tag{A.18b}$$

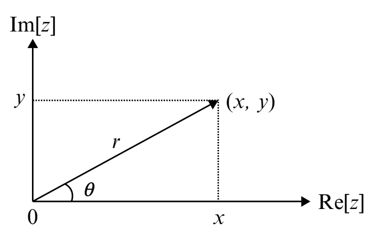

Figure A.1 Complex numbers represented as a vector

$$\frac{z_1}{z_2} = \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \tag{A.19a}$$

$$= \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}. \tag{A.19b}$$

Note that addition and subtraction are straightforward in rectangular form, but is difficult in polar form. Division is simple in polar form, but is complicated in rectangular form.

The complex arithmetic of the complex number x can be listed as

$$z^* = x - jy = re^{-j\theta}, \tag{A.20}$$

where * denotes complex-conjugate operation. In addition,

$$zz^* = |z|^2, \tag{A.21}$$

$$z^{-1} = \frac{1}{z} = \frac{1}{r} e^{-j\theta}, \tag{A.22}$$

$$z^N = r^N e^{jN\theta}. \tag{A.23}$$

The solution of

$$z^N = 1 \tag{A.24}$$

are

$$z_k = e^{j\theta_k} = e^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N - 1. \tag{A.25}$$

As illustrated in Figure A.2, these N solutions are equally spaced around the unit circle $|z| = 1$. The angular spacing between them is $\theta = 2\pi/N$.

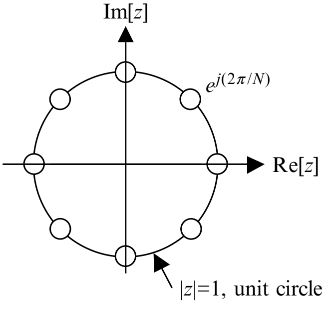

Figure A.2 Graphical display of the Nth roots of unity, $N = 8$

A.4 Impulse Functions

The unit impulse function $\delta(t)$ can be defined as

$$\delta(t) = \begin{cases} 1, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0. \end{cases} \tag{A.26}$$

Thus we have

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{A.27}$$

and

$$\int_{-\infty}^{\infty} \delta(t - t_0)x(t)dt = x(t_0), \tag{A.28}$$

where t_0 is a real number.

A.5 Vector Concepts

Vectors and matrices are often used in signal analysis to represent the state of a system at a particular time, a set of signal values, and a set of linear equations. The vector concepts can be applied to effectively describe a DSP algorithm. For example, define an $L \times 1$ coefficient vector as a column vector

$$\mathbf{b} = [b_0 \; b_1 \ldots b_{L-1}]^T, \tag{A.29}$$

where T denotes the transpose operator and the bold lower case character is used to denote a vector. We further define an input signal vector at time n as

$$\mathbf{x}(n) = [x(n) \; x(n-1) \ldots x(n - L + 1)]^T. \tag{A.30}$$

The output signal of FIR filter defined in (3.1.16) can be expressed in vector form as

$$y(n) = \sum_{l=0}^{L-1} b_l x(n - l) = \mathbf{b}^T \mathbf{x}(n) = \mathbf{x}^T(n)\mathbf{b}. \tag{A.31}$$

Therefore, the linear convolution of an FIR filter can be described as the inner (or dot) product of the coefficient and signal vectors, and the result is a scalar $y(n)$.

If we further define the coefficient vector

$$\mathbf{a} = [a_1 \; a_2 \cdots a_M]^T \tag{A.32}$$

and the previous output signal vector

$$\mathbf{y}(n-1) = [y(n-1)\ y(n-2) \ldots y(n-M)]^T, \tag{A.33}$$

then the input/output equation of IIR filter given in (3.2.18) can be expressed as

$$y(n) = \mathbf{b}^T \mathbf{x}(n) + \mathbf{a}^T \mathbf{y}(n-1). \tag{A.34}$$

A.6 Units of Power

Power and energy calculations are important in circuit analysis. Power is defined as the time rate of expending or absorbing energy, and can be expressed in the form of a derivative as

$$P = \frac{dE}{dt}, \tag{A.35}$$

where P is the power in watts, E is the energy in joules, and t is the time in seconds. The power associated with the voltage and current can be expressed as

$$P = vi = \frac{v^2}{R} = i^2 R, \tag{A.36}$$

where v is the voltage in volts, i is the current in amperes, and R is the resistance in ohms.

The unit bel, named in honor of Alexander Graham Bell, is defined as the common logarithm of the ratio of two power, P_x and P_y. In engineering applications, the most popular description of signal strength is decibel (dB) defined as

$$N = 10 \log_{10}\left(\frac{P_x}{P_y}\right) \text{dB}. \tag{A.37}$$

Therefore the decibel unit is used to describe the ratio of two powers and requires a reference value, P_y for comparison.

It is important to note that both the current $i(t)$ and voltage $v(t)$ can be considered as an analog signal $x(t)$, thus the power of signal is proportional to the square of signal amplitude. For example, if the signal $x(t)$ is amplified by a factor g, that is, $y(t) = gx(t)$. The signal gain can be expressed in dB as

$$\text{gain} = 10 \log_{10}\left(\frac{P_x}{P_y}\right) = 20 \log_{10}(g), \tag{A.38}$$

since the power is a function of the square of the voltage (or current) as shown in (A.36). As the second example, consider that the sound-pressure level, L_p, in decibels corresponds to a sound pressure P_x referenced to $p_y = 20\mu P_a$ (pascals). When the reference signal $y(t)$ has power P_y equal to 1 milliwatt, the power unit of $x(t)$ is called dBm (dB with respect to 1 milliwatt).

Reference

[1] Jan J. Tuma, *Engineering Mathematics Handbook*, New York, NY: McGraw-Hall, 1979.