Principles of Random Signal Analysis and Low Noise Design:
The Power Spectral Density and Its Applications.
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7

Memoryless Transformations of Random Processes

7.1 INTRODUCTION

This chapter uses the fact that a memoryless nonlinearity does not affect the disjointness of a disjoint random process to illustrate a procedure for ascertaining the power spectral density of a signaling random process after a memoryless transformation. Several examples are given, including two illustrating the application of this approach to frequency modulation (FM) spectral analysis. Alternative approaches are given in Davenport (1958 ch. 12) and Thomas (1969 ch. 6).

7.2 POWER SPECTRAL DENSITY AFTER A MEMORYLESS TRANSFORMATION

The approach given in this chapter relies on a disjoint partition of signals on a fixed interval. The following section gives the relevant results.

7.2.1 Decomposition of Output Using Input Time Partition

Consider a signal f which, based on a set of disjoint time intervals $\{I_1, \ldots, I_N\}$, can be written as a summation of disjoint waveforms according to

$$f(t) = \sum_{i=1}^{N} f_i(t) \qquad f_i(t) = \begin{cases} f(t) & t \in I_i \\ 0 & \text{elsewhere} \end{cases} \tag{7.1}$$

If such a signal is input into a memoryless nonlinearity characterized by an operator G, then the output signal $g = G(f)$ can be written as a summation of disjoint waveforms according to

$$g(t) = \sum_{i=1}^{N} g_i(t) \qquad g_i(t) = \begin{cases} g(t) & t \in I_i \\ 0 & t \notin I_i \end{cases} \tag{7.2}$$

where, as detailed in Section 2.3.3,

$$g_i(t) = \begin{cases} G(f_i(t)) & t \in I_i \\ 0 & t \notin I_i \end{cases} \tag{7.3}$$

7.2.1.1 Implication If all signals from a signaling random process can be written as a summation of disjoint signals, then this result can be used to define each of the corresponding output signals after a memoryless transformation and hence, define a signaling random process for the output random process. As the power spectral density of a signaling random process is well defined (see Theorem 5.1), such an approach allows the output power spectral density to be readily evaluated.

Clearly, the applicability of this approach depends on the extent to which signals from a signaling random processes can be written as a summation of disjoint waveforms, that is, to the extent a signaling random process can be written as a disjoint signaling random process, which is defined as follows.

Definition: Disjoint Signaling Random Process A disjoint signaling random process X, with a signaling period D, is a signaling random process where each waveform in the signaling set is zero outside the interval $[0, D]$. The ensemble E_X characterizing such a random process for the interval $[0, ND]$ is

$$E_X = \left\{ x(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \phi(\gamma_i, t-(i-1)D),\ \gamma_i \in S_\Gamma,\ \phi \in E_\Phi \right\} \tag{7.4}$$

where S_Γ is the sample space of the index random variable Γ, and is such that $S_\Gamma \subseteq \boldsymbol{Z}^+$ for the countable case, and $S_\Gamma \subseteq \boldsymbol{R}$ for the uncountable case. The set of signaling waveforms, E_Φ, is defined according to

$$E_\Phi = \{\phi(\gamma, t) \colon \gamma \in S_\Gamma,\ \phi(\gamma, t) = 0,\ t < 0,\ t \geqslant D\} \tag{7.5}$$

7.2.1.2 Equivalent Disjoint Signaling Random Process Consider a signaling random process X, defined by the ensemble

$$E_X = \left\{ x(\zeta_1, \ldots, \zeta_N, t) = \sum_{i=1}^{N} \psi(\zeta_i, t-(i-1)D),\ \zeta_i \in S_Z,\ \psi \in E_\Psi \right\} \tag{7.6}$$

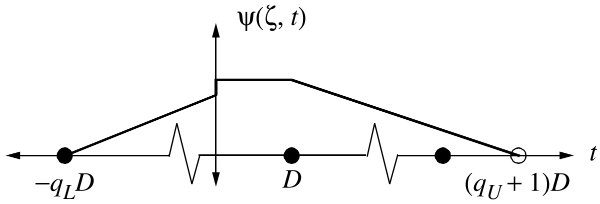

Figure 7.1 *Illustration of signaling waveform.*

where S_Z is the sample space of the index random variable Z and the set of signaling waveforms, E_Ψ, is defined according to

$$E_\Psi = \{\psi(\zeta, t): \zeta \in S_Z\} \tag{7.7}$$

Further, assume, as illustrated in Figure 7.1, that all signaling waveforms are nonzero only on a finite number of signaling intervals. It then follows that if a waveform in the random process starts with the signals associated with data in $[0, D]$, $[D, 2D]$, ... then a transient waveform exists in the interval $[0, q_U D]$. This transient is avoided for $t \geqslant 0$ if signals associated with data in the interval $[-q_U D, -(q_U - 1)D]$ and subsequent intervals are included.

The following theorem states that the random process defined in Eq. (7.6) can be written as a disjoint signaling random process with an appropriate disjoint signaling set. A likely, but not necessary consequence of this alternative characterization of a random process is the correlation between signaling waveforms in adjacent signaling intervals.

Theorem 7.1. Equivalent Disjoint Signaling Random Process *If all signaling waveforms in the signaling set E_Ψ, associated with a signaling random process X, are zero outside $[-q_L D, (q_U + 1)D]$, where $q_L, q_U \in \{0\} \cup \boldsymbol{Z}^+$, then, for the steady state case, the signaling random process can be written on the interval $[0, ND]$, as a disjoint signaling random process with an ensemble*

$$E_X = \left\{ x(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \phi(\gamma_i, t - (i-1)D),\ \phi \in E_\Phi,\ \gamma_i \in S_\Gamma \right\} \tag{7.8}$$

The associated signaling set E_Φ is defined as

$$E_\Phi = \left\{ \phi(\gamma, t): \gamma \in S_\Gamma = S_Z \times \cdots \times S_Z,\ \gamma = (\zeta_{-q_U}, \ldots, \zeta_{q_L}),\ \zeta_{-q_U}, \ldots, \zeta_{q_L} \in S_Z \right\} \tag{7.9}$$

where

$$\phi(\gamma, t) = \begin{cases} \psi(\zeta_{-q_U}, t - (-q_U)D) + \cdots + \psi(\zeta_{q_L}, t - q_L D) & 0 \leqslant t < D \\ 0 & \textit{elsewhere} \end{cases} \tag{7.10}$$

Proof. The proof of this result is given in Appendix 1.

7.2.1.3 Notes All waveforms in E_Φ are zero outside the interval $[0, D]$. The probability of each waveform and the correlation between waveforms, can be readily inferred from the original signaling random process. For the finite case where there are M independent signaling waveforms in E_Ψ, potentially there are $M^{q_L+q_U+1}$ waveforms in E_Φ. In most instances the waveforms from different signaling intervals will be correlated.

7.2.2 Power Spectral Density After a Nonlinear Memoryless Transformation

Consider a disjoint signaling random process characterized over the interval $[0, ND]$ by the ensemble E_X and associated signaling set as per Eqs. (7.4) and (7.5). If waveforms from such a random process are passed through a memoryless nonlinearity, characterized by an operator G, then the corresponding output random process Y is characterized by the ensemble E_Y and associated signaling set E_Ψ, where

$$E_Y = \left\{ y(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \psi(\gamma_i, t-(i-1)D),\ \gamma_i \in S_\Gamma, \psi \in E_\Psi \right\} \tag{7.11}$$

and

$$E_\Psi = \{\psi : \psi(\gamma, t) = G[\phi(\gamma, t)],\ \gamma \in S_\Gamma, \phi \in E_\Phi\} \tag{7.12}$$

Here, $P[\psi(\gamma, t)] = P[\phi(\gamma, t)] = P[\gamma]$. Clearly, the memoryless nonlinearity does not alter the signaling random process form, and the following result from Theorem 5.1 can be directly used to ascertain the power spectral density of the output random process,

$$\begin{aligned} G_Y(ND, f) = {} & r\overline{|\Psi(f)|^2} - r|\mu_\Psi(f)|^2 + r|\mu_\Psi(f)|^2 \left[\frac{1}{N} \frac{\sin^2(\pi N f/r)}{\sin^2(\pi f/r)} \right] \\ & + 2r \sum_{i=1}^{m} \left[1 - \frac{i}{N} \right] \mathrm{Re}[e^{j2\pi i D f}(R_{\Psi_1\Psi_{1+i}}(f) - |\mu_\Psi(f)|^2)] \end{aligned} \tag{7.13}$$

$$\begin{aligned} G_{Y_\infty}(f) = {} & r\overline{|\Psi(f)|^2} - r|\mu_\Psi(f)|^2 + r^2|\mu_\Psi(f)|^2 \sum_{n=-\infty}^{\infty} \delta(f - nr) \\ & + 2r \sum_{i=1}^{m} \mathrm{Re}[e^{j2\pi i D f}(R_{\Psi_1\Psi_{1+i}}(f) - |\mu_\Psi(f)|^2)] \end{aligned} \tag{7.14}$$

where $r = 1/D$ and μ_Ψ, $\overline{|\Psi(f)|^2}$, and $R_{\Psi_1\Psi_{1+i}}$ are defined consistent with the definitions given in Theorem 5.1. For example, for the countable case $P[\gamma] = p_\gamma$ and

$$\mu_\Psi(f) = \sum_{\gamma=1}^{\infty} p_\gamma \Psi(\gamma, f) \qquad \overline{|\Psi(f)|^2} = \sum_{\gamma=1}^{\infty} p_\gamma |\Psi(\gamma, f)|^2 \tag{7.15}$$

$$R_{\Psi_1\Psi_{1+i}}(f) = \sum_{\gamma_1=1}^{\infty} \sum_{\gamma_{1+i}=1}^{\infty} p_{\gamma_1\gamma_{1+i}} \Psi(\gamma_1, f)\Psi^*(\gamma_{1+i}, f) \tag{7.16}$$

where $\Psi(\gamma, f) = \int_0^D \psi(\gamma, t)e^{-j2\pi f t}\, dt$.

7.2.3 Extension to Nonmemoryless Systems

It is clearly useful if the above approach can be extended to nonmemoryless systems. To facilitate this, it is useful to define a signaling invariant system.

7.2.3.1 Definition — Signaling Invariant System A system is a signaling invariant system, if the output random process, in response to an input signaling random process is also a signaling random process and there is a one-to-one correspondence between waveforms in the signaling sets associated with the input and output random processes, that is, if $E_\Phi = \{\phi_i\}$ and $E_\Psi = \{\psi_i\}$ are, respectively, the input and output signaling sets, then there exists an operator G, such that $\psi_i = G[\phi_i]$.

A simple example of a signaling varying system is one where the output y, in response to an input x is defined as, $y(t) = x(t) + x(\pi t/4)$. For the case where the input is a waveform from a signaling random process the output is the summation of two signaling waveforms whose signaling intervals have an irrational ratio.

7.2.3.2 Implication If a system is a signaling invariant system and is driven by a signaling random process, then the output is also a signaling random process whose power spectral density can be readily ascertained through use of Eqs. (7.13) and (7.14).

7.2.3.3 Signaling Invariant Systems A simple example of a nonmemoryless, but signaling invariant system, is a system characterized by a delay, t_D. In fact, all linear time invariant systems are signaling invariant, as can be readily seen from the principle of superposition. However, the results of Chapter 8 yield a simple method for ascertaining the power spectral density of the output of a linear time invariant system, in terms of the input power spectral density, and the "transfer function" of the system.

7.3 EXAMPLES

The following sections give several examples of the above theory related to nonlinear transformations of random processes.

7.3.1 Amplitude Signaling through Memoryless Nonlinearity

Consider the case where the input random process X to a memoryless nonlinearity is a disjoint signaling random process, characterized on the interval $[0, ND]$, by the ensemble E_X:

$$E_X = \left\{ x(a_1, \ldots, a_N, t) = \sum_{i=1}^{N} \phi(a_i, t-(i-1)D),\ a_i \in S_A,\ \phi \in E_\Phi \right\} \quad (7.17)$$

where

$$E_\Phi = \left\{ \phi(a, t) = ap(t),\ a \in S_A,\ p(t) = \begin{cases} 1 & 0 \leqslant t < D \\ 0 & \text{elsewhere} \end{cases} \right\} \quad (7.18)$$

and $P[\phi(a, t)|_{a \in [a_o, a_o + da]}] = P[a \in [a_o, a_o + da]] = f_A(a_o)\, da$. Here, f_A is the density function of a random process A with outcomes a and sample space S_A. Assuming the signaling amplitudes are independent from one signaling interval to the next, it follows that the power spectral density of X is

$$G_X(ND, f) = r\overline{|\Phi(f)|^2} - r|\mu_\Phi(f)|^2 + r|\mu_\Phi(f)|^2 \left[\frac{1}{N} \frac{\sin^2(\pi N f/r)}{\sin^2(\pi f/r)} \right] \quad (7.19)$$

$$G_{X_\infty}(f) = r\overline{|\Phi(f)|^2} - r|\mu_\Phi(f)|^2 + r^2|\mu_\Phi(f)|^2 \sum_{n=-\infty}^{\infty} \delta(f - nr) \quad (7.20)$$

where $r = 1/D$, and

$$\begin{aligned} \mu_\Phi(f) &= P(f) \int_{-\infty}^{\infty} a f_A(a)\, da = \mu_A P(f) \qquad \mu_A = \int_{-\infty}^{\infty} a f_A(a)\, da \\ \overline{|\Phi(f)|^2} &= |P(f)|^2 \int_{-\infty}^{\infty} a^2 f_A(a)\, da = \overline{A^2}|P(f)|^2 \qquad \overline{A^2} = \int_{-\infty}^{\infty} a^2 f_A(a)\, da \end{aligned} \quad (7.21)$$

If signals from X are passed through a memoryless nonlinearity G, then, because of the disjointness of the input components of the signaling waveform, the output ensemble of the output random process Y, is

$$E_Y = \left\{ y(a_1, \ldots, a_N, t) = \sum_{i=1}^{N} \psi(a_i, t-(i-1)D),\ a_i \in S_A,\ \psi \in E_\Psi \right\} \quad (7.22)$$

where

$$E_\Psi = \{\psi(a, t) = G(a)p(t),\ a \in S_A\} \tag{7.23}$$

and

$$P[\psi(a, t)|_{a \in [a_o, a_o + da]}] = P[\phi(a, t)|_{a \in [a_o, a_o + da]}] = f_A(a_o)\, da \tag{7.24}$$

It then follows that the power spectral density of the output random process is

$$G_Y(ND, f) = r\overline{|\Psi(f)|^2} - r|\mu_\Psi(f)|^2 + r|\mu_\Psi(f)|^2 \left[\frac{1}{N}\frac{\sin^2(\pi N f/r)}{\sin^2(\pi f/r)}\right] \tag{7.25}$$

$$G_{Y_\infty}(f) = r\overline{|\Psi(f)|^2} - r|\mu_\Psi(f)|^2 + r^2|\mu_\Psi(f)|^2 \sum_{n=-\infty}^{\infty} \delta(f - nr) \tag{7.26}$$

where

$$\begin{aligned} \mu_\Psi(f) &= \mu_G P(f) = P(f)\int_{-\infty}^{\infty} G(a)\, f_A(a)\, da \\ \overline{|\Psi(f)|^2} &= \overline{G}^2|P(f)|^2 = |P(f)|^2 \int_{-\infty}^{\infty} G^2(a)\, f_A(a)\, da \end{aligned} \tag{7.27}$$

where the following definitions have been used:

$$\mu_G = \int_{-\infty}^{\infty} G(a)\, f_A(a)\, da \qquad \overline{G}^2 = \int_{-\infty}^{\infty} G^2(a)\, f_A(a)\, da \tag{7.28}$$

To illustrate these results, consider a square law device, that is, $G(a) = a^2$, and a Gaussian distribution of amplitudes according to

$$f_A(a) = e^{-a^2/2\sigma_A^2}/\sqrt{2\pi}\,\sigma_A$$

whereupon it follows that $\mu_G = \sigma_A^2$ and $\overline{G}^2 = 3\sigma_A^4$ (Papoulis, 2002 p. 148). Thus, with $|P(f)| = |\mathrm{sinc}(f/r)|/r$, it follows that

$$G_X(ND, f) = G_{X_\infty}(f) = \frac{\sigma_A^2}{r}\mathrm{sinc}^2(f/r) \tag{7.29}$$

$$G_Y(ND, f) = \frac{2\sigma_A^4}{r}\mathrm{sinc}^2(f/r) + \frac{\sigma_A^4}{r}\mathrm{sinc}^2(f/r)\left[\frac{1}{N}\frac{\sin^2(\pi N f/r)}{\sin^2(\pi f/r)}\right] \tag{7.30}$$

$$G_{Y_\infty}(f) = \frac{2\sigma_A^4}{r}\mathrm{sinc}^2(f/r) + \sigma_A^4\delta(f) \tag{7.31}$$

Clearly, for this case, and in general, for disjoint signaling waveforms with information encoded in the signaling amplitude as per Eq. (7.18), the nonlinear transformation has scaled, but not changed the shape of the power spectral density function with frequency apart from impulsive components. For the case where the mean of the Fourier transform of the output signaling set is altered, compared with the corresponding input mean, potentially there is the introduction or removal of impulsive components in the power spectral density.

7.3.2 Nonlinear Filtering to Reduce Spectral Spread

Many nonlinearities yield spectral spread, that is, a broadening of the power spectral density. However, spectral spread is not inevitable and depends on the nature of the nonlinearity and the nature of the input signal. The following is one example of nonlinear filtering where the power spectral density spread is reduced.

Consider the case where the input signaling random process X is characterized on the interval $[0, ND]$, by the ensemble

$$E_X = \left\{ x(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \phi(\gamma_i, t-(i-1)D),\ \gamma_i \in S_\Gamma,\ \phi \in E_\Phi \right\} \tag{7.32}$$

where $S_\Gamma = \{-1, 1\}$, $P[\gamma_i = \pm 1] = 0.5$,

$$E_\Phi = \left\{ \phi(\gamma_i, t) \colon \phi(\gamma_i, t) = \gamma_i A \Lambda\left(\frac{t - D/2}{D/2}\right),\ \gamma_i \in \{-1, 1\} \right\} \tag{7.33}$$

and the waveforms in different signaling intervals are independent, Here, Λ is the triangle function defined according to

$$\Lambda(t) = \begin{cases} 1+t & -1 \leqslant t < 0 \\ 1-t & 0 \leqslant t < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{7.34}$$

Consider a nonlinearity, defined according to

$$G(x) = \begin{cases} -A_o & x < -A \\ A_o \sin\left(\dfrac{\pi}{2}\dfrac{x}{A}\right) & -A \leqslant x < A \\ A_o & x \geqslant A \end{cases} \tag{7.35}$$

which is shown in Figure 7.2, along with input and output waveforms.

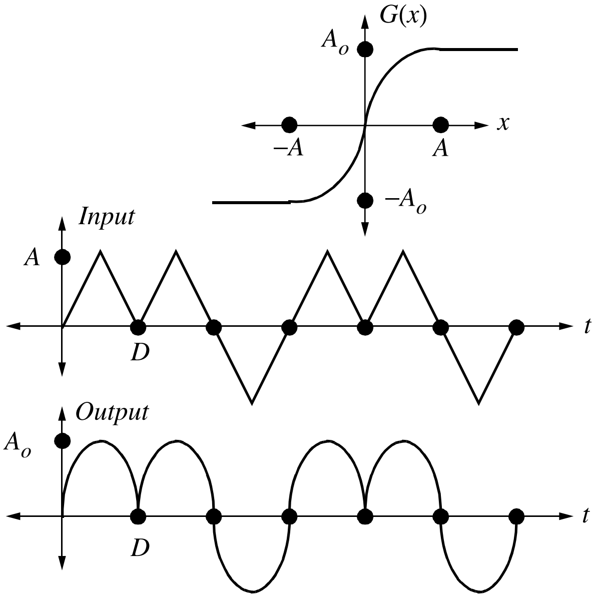

Figure 7.2 *Memoryless nonlinearity and input and output waveforms.*

It follows that the output signaling random process Y is characterized on the interval $[0, ND]$, by the ensemble

$$E_Y = \left\{ y(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \psi(\gamma_i, t-(i-1)D),\ \gamma_i \in S_\Gamma,\ \psi \in E_\Psi \right\} \tag{7.36}$$

where

$$E_\Psi = \left\{ \begin{array}{r} \psi(\gamma_i, t)\colon \psi(\gamma_i, t) = \gamma_i A_o \sin\left[\dfrac{\pi}{2} \Lambda\left(\dfrac{t-D/2}{D/2}\right)\right] = \gamma_i A_o \sin\left[\dfrac{\pi t}{D}\right] \\ 0 \leqslant t < D,\ \gamma_i \in S_\Gamma = \{-1, 1\} \end{array} \right\} \tag{7.37}$$

Clearly, $P[\psi(\gamma_i, t)] = P[\gamma_i] = 0.5$. It follows that the power spectral density of the input and output waveforms are

$$G_X(ND, f) = G_{X_\infty}(f) = r\overline{|\Phi(f)|^2} \tag{7.38}$$

$$G_Y(ND, f) = G_{Y_\infty}(f) = r\overline{|\Psi(f)|^2} \tag{7.39}$$

where

$$\overline{|\Phi(f)|^2} = \frac{A^2}{4r^2}\operatorname{sinc}^4\left(\frac{f}{2r}\right) \qquad \overline{|\Psi(f)|^2} = \frac{4A_o^2}{\pi^2 r^2}\frac{\cos^2(\pi f/r)}{(1-4f^2/r^2)^2} \tag{7.40}$$

There is equal power in the input and output spectral densities when $A_o = \sqrt{2}A/\sqrt{3}$.

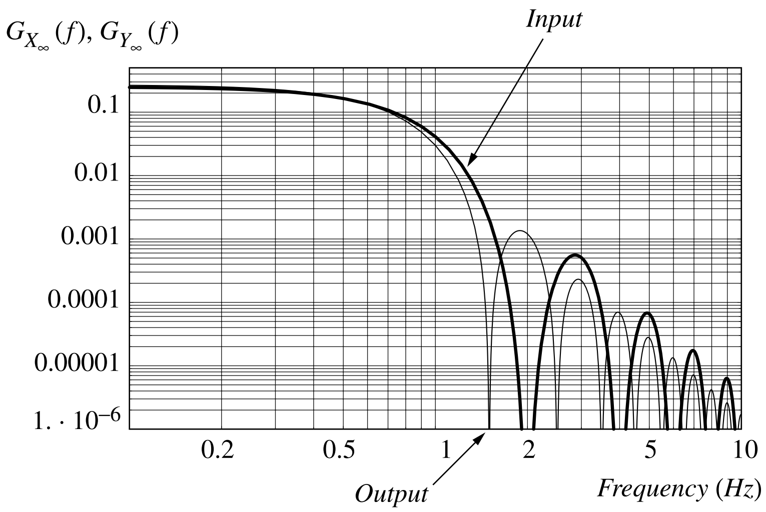

Figure 7.3 *Input and output power spectral densities associated with the memoryless nonlinearity and waveforms shown in Figure 7.2.*

These power spectral densities are plotted in Figure 7.3 for the case of $r = D = 1$, $A = 1$, and $A_o = \sqrt{2}/\sqrt{3}$. For this equal input and output power case, there is clear spectral narrowing consistent with the "smoothing" of the input waveform via the nonlinear transformation.

7.3.3 Power Spectral Density of Binary Frequency Shifted Keyed Modulation

As the following two examples show, signaling random process theory can readily be applied to ascertaining the power spectral density of FM random processes.

First, consider an FM signal,

$$y(t) = A\cos[x(t)] \qquad x(t) = 2\pi f_c t + \varphi(t) \qquad t \geqslant 0 \tag{7.41}$$

where the carrier frequency f_c is an integer multiple of the signaling rate $r = 1/D$, and the binary digital modulation is such that φ has the form

$$\begin{aligned} \varphi(t) &= 2\pi f_d \int_0^t \sum_{i=1}^{\infty} \gamma_i p(\lambda - (i-1)D)\, d\lambda \qquad \gamma_i \in \{-1, 1\} \\ &= 2\pi f_d \sum_{i=1}^{\lfloor t/D \rfloor + 1} \gamma_i \int_0^t p(\lambda - (i-1)D)\, d\lambda \end{aligned} \tag{7.42}$$

Here, $P[\gamma_i = -1] = p_{-1}$ and $P[\gamma_i = 1] = p_1$, and the pulse function p is assumed to be such that

$$p(t) = 0 \qquad t < 0, t \geqslant D \qquad \int_0^D p(t)dt = D \tag{7.43}$$

which is consistent with a phase change of $\pm 2\pi(f_d/r)$ during each signaling interval of duration D sec. Clearly, $p(t)$ and $p(t - iD)$ are disjoint for $i \geqslant 1$.

With the assumptions that both f_d/r and f_c/r are integer ratios, it follows, as far as a cosine function is concerned, that the phase signal x in any interval of the form$[(i-1)D, iD]$, where $i \in \boldsymbol{Z}^+$, can be written as

$$\begin{aligned} x(t) &= 2\pi f_c(t-(i-1)D) + 2\pi f_d\gamma \int_0^{t-(i-1)D} p(\lambda)\, d\lambda \qquad t \in [(i-1)D, iD], \gamma \in \{-1, 1\} \\ &= \phi(\gamma, t-(i-1)D) \end{aligned} \tag{7.44}$$

where

$$\phi(\gamma, t) = \begin{cases} 2\pi f_c t + 2\pi f_d \gamma \int_0^t p(\lambda)\, d\lambda & 0 \leqslant t < D, \gamma \in \{\pm 1\} \\ 0 & \text{elsewhere} \end{cases} \tag{7.45}$$

It then follows, for the ith signaling interval, that

$$y(t) = A\cos[x(t)] = A\cos[\phi(\gamma, t-(i-1)D)] \qquad (i-1)D \leqslant t < iD \tag{7.46}$$

This formulation can be generalized to the random process case as follows: The random phase process X is defined by the ensemble E_X

$$E_X = \left\{ x(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \phi(\gamma_i, t-(i-1)D),\ \phi \in E_\Phi, \gamma_i \in \{\pm 1\}, t \in [0, ND] \right\} \tag{7.47}$$

where $P[\gamma_i] \in \{p_{-1}, p_1\}$ and the signaling set E_Φ is defined as

$$E_\Phi = \left\{ \phi(\gamma, t) = \begin{cases} 2\pi f_c t + 2\pi f_d \gamma \int_0^t p(\lambda)\, d\lambda & 0 \leqslant t < D, \gamma \in \{\pm 1\} \\ 0 & \text{elsewhere} \end{cases} \right\} \tag{7.48}$$

As any waveform in E_X consists of a summation of disjoint signals, a random process $Y = \cos[X]$ can be defined with an ensemble E_Y,

$$E_Y = \left\{ y(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \psi(\gamma_i, t - (i-1)D, \ \psi \in E_\Psi, \gamma_i \in \{\pm 1\}, t \in [0, ND] \right\} \tag{7.49}$$

where $y(\gamma_1, \ldots, \gamma_N, t)$ is a summation of disjoint signals from the signaling set E_Ψ,

$$E_\Psi = \left\{ \psi(\gamma, t) = \begin{cases} A\cos[\phi(\gamma, t)] & 0 \leqslant t < D, \gamma \in \{\pm 1\}, \phi \in E_\Phi \\ 0 & \text{elsewhere} \end{cases} \right\} \tag{7.50}$$

and $P[\psi(\gamma, t)] = P[\phi(\gamma, t)] \in \{p_{-1}, p_1\}$.

With independent data, consistent with γ_i being independent of γ_j for $i \neq j$, it follows from Eq. (7.14) that the power spectral density of Y is

$$\begin{aligned} G_{Y_\infty}(f) &= r \sum_\gamma p_\gamma |\Psi(\gamma, f)|^2 - r \left| \sum_\gamma p_\gamma \Psi(\gamma, f) \right|^2 \\ &\quad + r^2 \left| \sum_\gamma p_\gamma \Psi(\gamma, f) \right|^2 \sum_{k=-\infty}^{\infty} \delta(f - kr) \qquad \gamma \in \{-1, 1\} \end{aligned} \tag{7.51}$$

where Ψ is the Fourier transforms of ψ.

For the case where $p(t) = 1$ for $0 \leqslant t < D$ and zero elsewhere, that is, binary frequency shifted keyed (FSK) modulation, it follows that the signaling set is

$$E_\Psi = \begin{cases} \psi(1, t) = A\cos[2\pi(f_c + f_d)t] & 0 \leqslant t < D \\ \psi(-1, t) = A\cos[2\pi(f_c - f_d)t] & 0 \leqslant t < D \end{cases} \tag{7.52}$$

and

$$\begin{aligned} \Psi(1, f) &= \frac{A}{2r} e^{j\pi(f_c + f_d - f)/r} \operatorname{sinc}\left(\frac{f_c + f_d - f}{r}\right) \\ &\quad + \frac{A}{2r} e^{-j\pi(f_c + f_d + f)/r} \operatorname{sinc}\left(\frac{f_c + f_d + f}{r}\right) \end{aligned} \tag{7.53}$$

$$\begin{aligned} \Psi(-1, f) &= \frac{A}{2r} e^{j\pi(f_c - f_d - f)/r} \operatorname{sinc}\left(\frac{f_c - f_d - f}{r}\right) \\ &\quad + \frac{A}{2r} e^{-j\pi(f_c - f_d + f)/r} \operatorname{sinc}\left(\frac{f_c - f_d + f}{r}\right) \end{aligned} \tag{7.54}$$

For this case, and where $r = D = f_d = 1$, $A = \sqrt{2}$, $f_c = 10$, and $p_{-1} = p_1 = 0.5$,

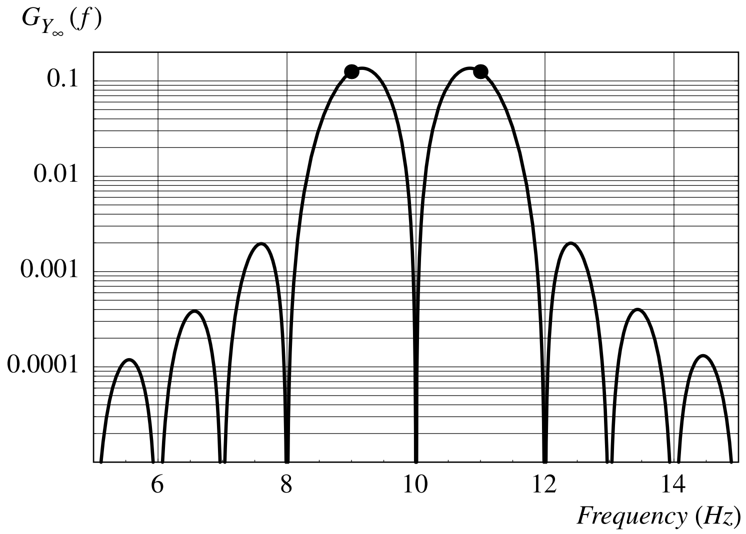

Figure 7.4 *Power spectral density of a binary FSK random process with $r = D = f_d = 1$, $A = \sqrt{2}$, and $f_c = 10$. The dots represent the power in impulses.*

the power spectral density, as defined by Eq. (7.51), is shown in Figure 7.4. A check on the power in the impulses can be simply undertaken by writing the FM signal $A\cos[2\pi(f_c \pm f_d)t]$ in the quadrature carrier form,

$$A\cos(2\pi f_c t)\cos(2\pi f_d t) \mp A\sin(2\pi f_c t)\sin(2\pi f_d t) \tag{7.55}$$

The first term is periodic, and independent of the data, and yields impulses at $\pm f_c \pm f_d$ where the area under each impulse is $A^2/16$, which equals 0.125 when $A = \sqrt{2}$.

7.3.4 Frequency Modulation with Raised Cosine Pulse Shaping

Consider a FM signal with continuous phase modulation that is achieved through the use of raised cosine pulse shaping,

$$x(t) = A\sin\left[2\pi f_c t + 2\pi r\int_{-D}^{t} m(\lambda)\,d\lambda\right] \qquad t \geqslant 0 \tag{7.56}$$

where $r = 1/D$ and the lower limit of $-D$ in the integral arises from the pulse waveform in the modulating signal m, which is defined according to

$$m(t) = \sum_{i=1}^{\infty} \zeta_i p(t - (i-1)D) \qquad \zeta_i \in \{\pm 0.5\},\ t \geqslant -D \tag{7.57}$$

Here, p is a raised cosine pulse with a duration of three signaling intervals (Proakis, 1995 p. 218), that is,

$$p(t) = \begin{cases} \dfrac{1}{3}\left[1 - \cos\left(\dfrac{2\pi(t+D)}{3D}\right)\right] & -D \leqslant t < 2D \\ 0 & \text{elsewhere} \end{cases} \tag{7.58}$$

and is shown in Figure 7.5. The integral of this raised cosine pulse shape, q, is

$$q(t) = \begin{cases} \displaystyle\int_{-D}^{t} p(\lambda)\, d\lambda \\ 0 \qquad t < -D \end{cases} = \begin{cases} 0 & t < -D \\ \dfrac{t+D}{3} - \dfrac{D}{2\pi}\sin\left[\dfrac{2\pi(t+D)}{3D}\right] & -D \leqslant t < 2D \\ D & t \geqslant 2D \end{cases} \tag{7.59}$$

and the area under p is D. The value of $\zeta_i \in \{\pm 0.5\}$ in Eq. (7.57), results in each signaling waveform yielding a phase change of $\pm\pi$. The normalized integral of p, that is, $q(t/D)/D$, is shown in Figure 7.5.

As

$$\begin{aligned} \int_{-D}^{t} \left(\sum_{i=1}^{\infty} \zeta_i p(\lambda - (i-1)D) \right) d\lambda &= \sum_{i=1}^{\lfloor t/D \rfloor + 2} \zeta_i \int_{(i-2)D}^{t} p(\lambda - (i-1)D)\, d\lambda \\ &= \sum_{i=1}^{\lfloor t/D \rfloor + 2} \zeta_i q(t - (i-1)D) \qquad t \geqslant -D \end{aligned} \tag{7.60}$$

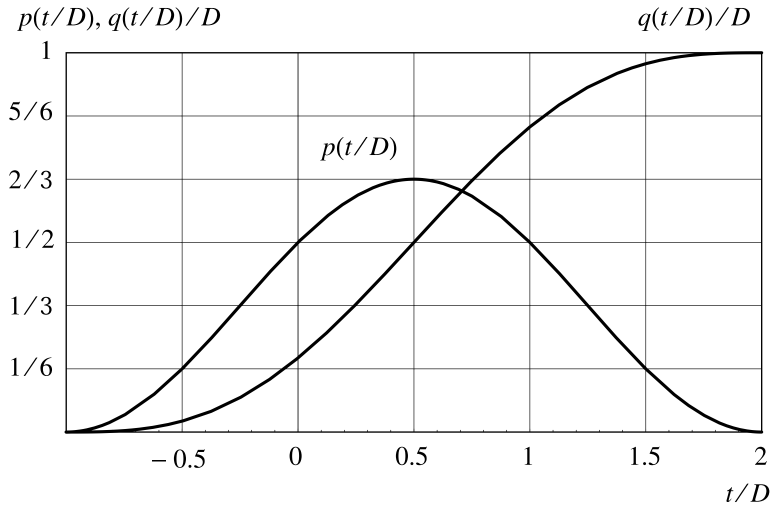

Figure 7.5 *Raised cosine pulse waveform and normalized integral of such a waveform.*

it follows that the FM signal defined by Eqs. (7.56) and (7.57), can be written as

$$x(t) = A \sin\left[2\pi f_c t + 2\pi r \sum_{i=1}^{\lfloor t/D \rfloor + 2} \zeta_i q(t - (i-1)D)\right] \qquad \zeta_i \in \{\pm 0.5\},\ t \geqslant 0 \tag{7.61}$$

The random process, of which this signal is one outcome, is denoted X and is defined, on the interval $[0, ND)$, by the ensemble E_X:

$$E_X = \left\{ \begin{aligned} & x(\zeta_0, \ldots, \zeta_{N+1}, t) = A \sin\left[w_c t + 2\pi r \sum_{i=0}^{\lfloor t/D \rfloor + 2} \zeta_i q(t - (i-1)D)\right] \\ & \qquad\qquad w_c = 2\pi f_c,\ \zeta_i \in \{\pm 0.5\},\ t \in [0, ND) \end{aligned} \right\} \tag{7.62}$$

where the effect of symbols in the interval $[-D, 0]$ and $[ND, (N+1)D]$, have been included to establish a steady state ensemble for $[0, ND]$. As the integral of the pulse shape is $D = 1/r$, and $\zeta_i \in \{\pm 0.5\}$, it follows that each pulse contributes a final phase shift of $\pm\pi$ radians to the argument of the sine function. Hence, each pair of symbols results in a phase shift from the set $-2\pi, 0, 2\pi$. As the sine function is periodic with period of 2π, it follows that in the ith signaling interval, $[(i-1)D, iD]$, the phase accumulation from the previous $\ldots, i-3, i-2$ symbols can be neglected. Thus, it is possible to rewrite the ensemble defining the random process X on the interval $[0, 2ND]$ in a signaling random process form, with a signaling rate of $r/2$, that is,

$$E_X = \left\{ x(\gamma_1, \ldots, \gamma_N, t) = \sum_{i=1}^{N} \phi_{\gamma_i}(t - (i-1)2D),\ \gamma_i \in \{1, \ldots, 16\}, \phi_{\gamma_i} \in E_\Phi, t \in [0, 2ND] \right\} \tag{7.63}$$

where the signaling set E_Φ consists of waveforms that are zero outside the interval $[0, 2D]$, and is defined according to

$$E_\Phi = \left\{ \phi_i(t) = \begin{cases} A \sin[2\pi f_c t + \varphi_i(t)] & 0 \leqslant t < 2D \\ 0 & \text{elsewhere} \end{cases} \quad i \in \{1, \ldots, 16\} \right\} \tag{7.64}$$

The waveforms in E_Φ, as well as the component phase waveforms φ_i, are detailed in Table 7.1. All waveforms in this set have equal probability, and the phase waveforms φ_i are plotted in Figure 7.6. The correlation between the signaling waveforms in adjacent signaling intervals of duration $2D$, is detailed in Table 7.2. The signaling waveforms in signaling intervals separated by at least $2D$, are independent as far as the sine operator is concerned.

Table 7.1 Signaling Waveforms in Signaling set

Data	Phase Waveforms $\varphi_1, \ldots, \varphi_{16}$ in $[0, 2D]$	Signaling Waveforms in $[0, 2D]$
0000	$\pi r[-q(t+D) - q(t) - q(t-D) - q(t-2D)]$	$\phi_1(t) = A\sin(2\pi f_c t + \varphi_1(t))$
0001	$\pi r[-q(t+D) - q(t) - q(t-D) + q(t-2D)]$	$\phi_2(t) = A\sin(2\pi f_c t + \varphi_2(t))$
0010	$\pi r[-q(t+D) - q(t) + q(t-D) - q(t-2D)]$	$\phi_3(t) = A\sin(2\pi f_c t + \varphi_3(t))$
0011	$\pi r[-q(t+D) - q(t) + q(t-D) + q(t-2D)]$	$\phi_4(t) = A\sin(2\pi f_c t + \varphi_4(t))$
0100	$\pi r[-q(t+D) + q(t) - q(t-D) - q(t-2D)]$	$\phi_5(t) = A\sin(2\pi f_c t + \varphi_5(t))$
0101	$\pi r[-q(t+D) + q(t) - q(t-D) + q(t-2D)]$	$\phi_6(t) = A\sin(2\pi f_c t + \varphi_6(t))$
0110	$\pi r[-q(t+D) + q(t) + q(t-D) - q(t-2D)]$	$\phi_7(t) = A\sin(2\pi f_c t + \varphi_7(t))$
0111	$\pi r[-q(t+D) + q(t) + q(t-D) + q(t-2D)]$	$\phi_8(t) = A\sin(2\pi f_c t + \varphi_8(t))$
1000	$\pi r[q(t+D) - q(t) - q(t-D) - q(t-2D)]$	$\phi_9(t) = A\sin(2\pi f_c t + \varphi_9(t))$
1001	$\pi r[q(t+D) - q(t) - q(t-D) + q(t-2D)]$	$\phi_{10}(t) = A\sin(2\pi f_c t + \varphi_{10}(t))$
1010	$\pi r[q(t+D) - q(t) + q(t-D) - q(t-2D)]$	$\phi_{11}(t) = A\sin(2\pi f_c t + \varphi_{11}(t))$
1011	$\pi r[q(t+D) - q(t) + q(t-D) + q(t-2D)]$	$\phi_{12}(t) = A\sin(2\pi f_c t + \varphi_{12}(t))$
1100	$\pi r[q(t+D) + q(t) - q(t-D) - q(t-2D)]$	$\phi_{13}(t) = A\sin(2\pi f_c t + \varphi_{13}(t))$
1101	$\pi r[q(t+D) + q(t) - q(t-D) + q(t-2D)]$	$\phi_{14}(t) = A\sin(2\pi f_c t + \varphi_{14}(t))$
1110	$\pi r[q(t+D) + q(t) + q(t-D) - q(t-2D)]$	$\phi_{15}(t) = A\sin(2\pi f_c t + \varphi_{15}(t))$
1111	$\pi r[q(t+D) + q(t) + q(t-D) + q(t-2D)]$	$\phi_{16}(t) = A\sin(2\pi f_c t + \varphi_{16}(t))$

Data of 0 and 1 correspond, respectively, to $\zeta_i = -0.5$ and $\zeta_i = 0.5$. The data in the first column are for the intervals $[-D, 0]$, $[0, D]$, $[D, 2D]$, and $[2D, 3D]$.

7.3.4.1 Determining Power Spectral Density The power spectral density from Theorem 5.1, for a signaling random process with a rate $r_o = 1/D_o$, is

$$\begin{aligned} G_X(ND_o, f) = {} & r_o\overline{|\Phi(f)|^2} - r_o|\mu_\Phi(f)|^2 + r_o|\mu_\Phi(f)|^2\left[\frac{1}{N}\frac{\sin^2(\pi Nf/r_o)}{\sin^2(\pi f/r_o)}\right] \\ & + 2r_o\sum_{i=1}^{m}\left[1 - \frac{i}{N}\right]\mathrm{Re}[e^{j2\pi iD_o f}(R_{\Phi_1\Phi_{1+i}}(f) - |\mu_\Phi(f)|^2)] \end{aligned} \tag{7.65}$$

where, for the case being considered, $r_o = r/2 = 1/2D$, $D_o = 2D$, $m = 1$, and

$$\mu_\Phi(f) = \sum_{i=1}^{16} p_i\Phi_i(f) \qquad \overline{|\Phi(f)|^2} = \sum_{i=1}^{16} p_i|\Phi_i(f)|^2 \tag{7.66}$$

$$R_{\Phi_1\Phi_2}(f) = \sum_{\gamma_1=1}^{16}\sum_{\gamma_2=1}^{16} p_{\gamma_1\gamma_2}\Phi_{\gamma_1}(f)\Phi^*_{\gamma_2}(f) \tag{7.67}$$

To evaluate the power spectral density, the Fourier transform of the individual waveforms in the signaling set, as defined by Eq. (7.64) and Table 7.1, are required to be evaluated. The details are given in Appendix 2. Using the results from this appendix, the power spectral density, as defined by Eqs. (7.65) to Eq. (7.67), is shown in Figure 7.7, for the case of $f_c = 10$, $r = D = 1$, $A = \sqrt{2}$, and $N \to \infty$. For the parameters used, the average power is $1V^2$ assuming a voltage signal. The power in each of the sinusoidal components with frequencies of $f_c \pm r/2$ is $0.11V^2$, and the remaining power of $0.78V^2$ is in the continuous spectrum.

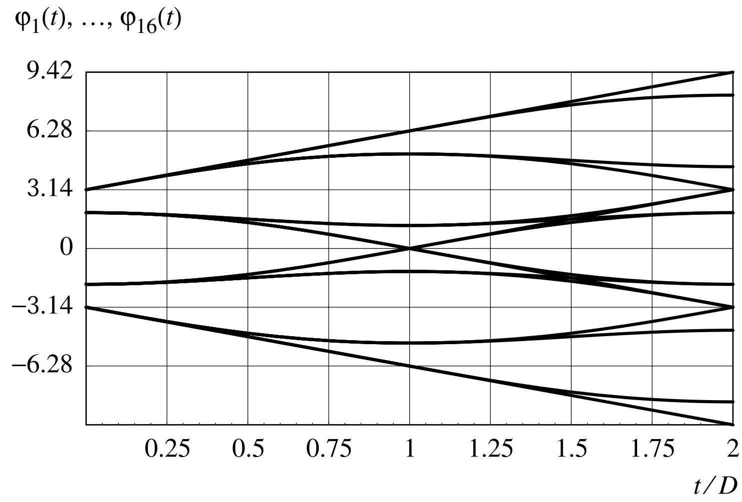

Figure 7.6 *Phase signaling waveforms for [0, 2D].*

The power in the impulsive components is consistent with inefficient signaling. These components can be eliminated by reducing the phase variation in each signaling waveform from π to $\pi/2$ radians. This also leads to better spectral efficiency (Proakis, 1995 p. 218). With respect to spectral efficiency, the power spectral density shown in Figure 7.7 should be compared with that

Table 7.2 Correlation between Signals in Signaling Intervals of Duration 2*D*

Data	Signal in *i*th Interval	Signal in $(i+1)$th Interval	Probability
*xx*0000	$\phi_1, \phi_5, \phi_9, \phi_{13}$	$\phi_1(t)$	1/64
*xx*0001	$\phi_1, \phi_5, \phi_9, \phi_{13}$	$\phi_2(t)$	1/64
*xx*0010	$\phi_1, \phi_5, \phi_9, \phi_{13}$	$\phi_3(t)$	1/64
*xx*0011	$\phi_1, \phi_5, \phi_9, \phi_{13}$	$\phi_4(t)$	1/64
*xx*0100	$\phi_2, \phi_6, \phi_{10}, \phi_{14}$	$\phi_5(t)$	1/64
*xx*0101	$\phi_2, \phi_6, \phi_{10}, \phi_{14}$	$\phi_6(t)$	1/64
*xx*0110	$\phi_2, \phi_6, \phi_{10}, \phi_{14}$	$\phi_7(t)$	1/64
*xx*0111	$\phi_2, \phi_6, \phi_{10}, \phi_{14}$	$\phi_8(t)$	1/64
*xx*1000	$\phi_3, \phi_7, \phi_{11}, \phi_{15}$	$\phi_9(t)$	1/64
*xx*1001	$\phi_3, \phi_7, \phi_{11}, \phi_{15}$	$\phi_{10}(t)$	1/64
*xx*1010	$\phi_3, \phi_7, \phi_{11}, \phi_{15}$	$\phi_{11}(t)$	1/64
*xx*1011	$\phi_3, \phi_7, \phi_{11}, \phi_{15}$	$\phi_{12}(t)$	1/64
*xx*1100	$\phi_4, \phi_8, \phi_{12}, \phi_{16}$	$\phi_{13}(t)$	1/64
*xx*1101	$\phi_4, \phi_8, \phi_{12}, \phi_{16}$	$\phi_{14}(t)$	1/64
*xx*1110	$\phi_4, \phi_8, \phi_{12}, \phi_{16}$	$\phi_{15}(t)$	1/64
*xx*1111	$\phi_4, \phi_8, \phi_{12}, \phi_{16}$	$\phi_{16}(t)$	1/64

The data in the first column are for the $(i-1)$th, *i*th and $(i+1)$th signaling intervals of duration 2D. The symbol *x* implies the data is arbitrary.

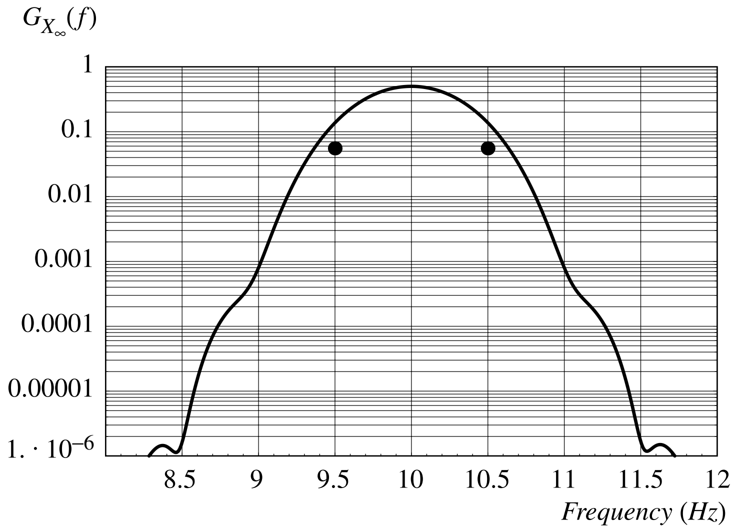

Figure 7.7 *Power spectral density of a raised cosine pulse shaped FM random process with a carrier frequency of 10 Hz, r = D = 1, and A = $\sqrt{2}$. The dots represent the power in impulses.*

shown in Figure 7.4, where pulse shaping has not been used, and the phase change for each signaling waveform is 2π radians. Finally, a further comparison of Figures 7.7 and 7.4, reveals that the pulse shaping has led to a very rapid spectral rolloff.

APPENDIX 1: PROOF OF THEOREM 7.1

Consider the steady state case and a single signaling waveform $\psi(\zeta_0, t)$ from the ensemble E_Ψ, that could be associated with every signaling interval as shown in Figure 7.8. Clearly, the signal in the interval $[0, D]$ is given by

$$\psi(\zeta_0, t - (-q_U)D) + \cdots + \psi(\zeta_0, t) + \cdots + \psi(\zeta_0, t - q_L D) \tag{7.68}$$

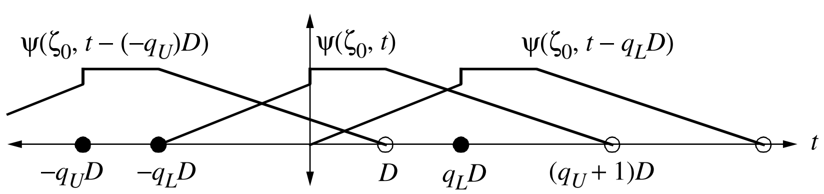

Figure 7.8 *Illustration of signaling waveforms that have nonzero contributions in the interval [0, D].*

In general, the signal $\phi(\gamma, t)$ in the interval $[0, D]$ has the form

$$\phi(\gamma, t) = \begin{cases} \psi(\zeta_{-q_U}, t-(-q_U)D) + \cdots + \psi(\zeta_0, t) + \cdots + \psi(\zeta_{q_L}, t - q_L D) & 0 \leqslant t < D \\ 0 & \text{elsewhere} \end{cases} \tag{7.69}$$

where $\gamma = (\zeta_{-q_U}, \ldots, \zeta_{q_L})$ and $\zeta_{-q_U}, \ldots, \zeta_{q_L} \in S_Z$. By definition, $\phi(\gamma, t)$ is zero outside the interval $[0, D]$. As this interval is representative of any other interval of the form $[(i-1)D, iD]$, it follows that a signal from the random process can be written in the interval $[0, ND]$, as a sum of disjoint signals,

$$\sum_{i=1}^{N} \phi(\gamma_i, t - (i-1)D), \qquad \phi \in E_\Phi, \gamma_i \in S_\Gamma \tag{7.70}$$

where

$$E_\Phi = \{\phi(\gamma, t): \gamma \in S_\Gamma = S_Z \times \cdots \times S_Z, \gamma = (\zeta_{-q_U}, \ldots, \zeta_{q_L}), \zeta_{-q_U}, \ldots, \zeta_{q_L} \in S_Z\} \tag{7.71}$$

this is the required result.

APPENDIX 2: FOURIER RESULTS FOR RAISED COSINE FREQUENCY MODULATION

To establish the Fourier transform of each signaling waveform, explicit expressions for $q(t+D)$, $q(t)$, $q(t-D)$, and $q(t-2D)$ are first required. Using the definition for q, as in Eq. (7.59), it follows that

$$q(t+D) = \frac{t+2D}{3} + \frac{D}{4\pi}\sin(q_m t) + \frac{\sqrt{3}D}{4\pi}\cos(q_m t) \qquad -2D \leqslant t < D \tag{7.72}$$

$$q(t) = \frac{t+D}{3} + \frac{D}{4\pi}\sin(q_m t) - \frac{\sqrt{3}D}{4\pi}\cos(q_m t) \qquad -D \leqslant t < 2D \tag{7.73}$$

$$q(t-D) = \frac{t}{3} - \frac{2D}{4\pi}\sin(q_m t) \qquad 0 \leqslant t < 3D \tag{7.74}$$

$$q(t-2D) = \frac{t-D}{3} + \frac{D}{4\pi}\sin(q_m t) + \frac{\sqrt{3}D}{4\pi}\cos(q_m t) \qquad D \leqslant t < 4D \tag{7.75}$$

where $q_m = 2\pi/3D$.

A.2.1 Phase Waveforms for [0, *D*]

For $0 \leqslant t < D$, the phase of the waveforms, as detailed in Table 7.1, can be written as

$$\varphi_i(t) = \pi r[\gamma_{-1} q(t+D) + \gamma_0 q(t) + \gamma_1 q(t-D)] \tag{7.76}$$

where, γ_{-1}, γ_0, $\gamma_1 \in \{-1, 1\}$ depend, respectively, on data in the intervals $[-D, 0]$, $[0, D]$, and $[D, 2D]$. From Eqs. (7.72)–(7.75) it follows that

$$\begin{aligned}\varphi_i(t) = \pi r \Bigg[& \frac{D}{3}(2\gamma_{-1} + \gamma_0) + \frac{t}{3}(\gamma_{-1} + \gamma_0 + \gamma_1) \\ & + \frac{D}{4\pi}(\gamma_{-1} + \gamma_0 - 2\gamma_1)\sin(q_m t) + \frac{\sqrt{3}\,D}{4\pi}(\gamma_{-1} - \gamma_0)\cos(q_m t)\Bigg]\end{aligned} \tag{7.77}$$

which can be rewritten as

$$\varphi_i(t) = q_o + q_1 t + q_2 \sin(q_m t + \theta_q) \tag{7.78}$$

where

$$q_o = \frac{\pi}{3}(2\gamma_{-1} + \gamma_0) \qquad q_1 = \frac{\pi}{3D}(\gamma_{-1} + \gamma_0 + \gamma_1) \tag{7.79}$$

$$q_2 = \frac{1}{4}\sqrt{(\gamma_{-1} + \gamma_0 - 2\gamma_1)^2 + 3(\gamma_{-1} - \gamma_0)^2} \tag{7.80}$$

$$\theta_q = \begin{cases} \tan^{-1}\left[\dfrac{\sqrt{3}(\gamma_{-1} - \gamma_0)}{\gamma_{-1} + \gamma_0 - 2\gamma_1}\right] & \gamma_{-1} + \gamma_0 - 2\gamma_1 > 0 \\ \tan^{-1}\left[\dfrac{\sqrt{3}(\gamma_{-1} - \gamma_0)}{\gamma_{-1} + \gamma_0 - 2\gamma_1}\right] + \pi & \gamma_{-1} + \gamma_0 - 2\gamma_1 < 0 \\ 0 & \gamma_{-1} - \gamma_0 = 0, \gamma_{-1} + \gamma_0 - 2\gamma_1 = 0 \end{cases} \tag{7.81}$$

A.2.2 Phase Waveforms for [*D*, 2*D*]

For $D \leqslant t < 2D$, the phase of the waveforms, as per Table 7.1, can be written as

$$\varphi_i(t) = \pi r[\gamma_{-1} D + \gamma_0 q(t) + \gamma_1 q(t-D) + \gamma_2 q(t-2D)] \tag{7.82}$$

where $\gamma_{-1}, \gamma_0, \gamma_1, \gamma_2 \in \{-1, 1\}$, and thus,

$$\varphi_i(t) = \pi r\left[\frac{D}{3}(3\gamma_{-1} + \gamma_0 - \gamma_2) + \frac{t}{3}(\gamma_0 + \gamma_1 + \gamma_2) + \frac{D}{4\pi}(\gamma_0 - 2\gamma_1 + \gamma_2)\sin(q_m t) + \frac{\sqrt{3}\,D}{4\pi}(-\gamma_0 + \gamma_2)\cos(q_m t)\right] \quad (7.83)$$

This can be rewritten as

$$\varphi_i(t) = q_o + q_1 t + q_2 \sin(q_m t + \theta_q) \quad (7.84)$$

where

$$q_o = \frac{\pi}{3}(3\gamma_{-1} + \gamma_0 - \gamma_2) \qquad q_1 = \frac{\pi}{3D}(\gamma_0 + \gamma_1 + \gamma_2) \quad (7.85)$$

$$q_2 = \frac{1}{4}\sqrt{(\gamma_0 - 2\gamma_1 + \gamma_2)^2 + 3(-\gamma_0 + \gamma_2)^2} \quad (7.86)$$

$$\theta_q = \begin{cases} \tan^{-1}\left[\dfrac{\sqrt{3}(-\gamma_0 + \gamma_2)}{\gamma_0 - 2\gamma_1 + \gamma_2}\right] & \gamma_0 - 2\gamma_1 + \gamma_2 > 0 \\ \tan^{-1}\left[\dfrac{\sqrt{3}(-\gamma_0 + \gamma_2)}{\gamma_0 - 2\gamma_1 + \gamma_2}\right] + \pi & \gamma_0 - 2\gamma_1 + \gamma_2 < 0 \\ 0 & -\gamma_0 + \gamma_2 = 0, \gamma_0 - 2\gamma_1 + \gamma_2 = 0 \end{cases} \quad (7.87)$$

A.2.3 Fourier Transform of Signaling Waveforms

For the interval $[0, 2D]$, the Fourier transform of the ith waveform in the signaling set is

$$\begin{aligned}\Phi_i(f) &= \int_0^{2D} A\sin(2\pi f_c t + \varphi_i(t))e^{-j2\pi f t}\,dt \\ &= \frac{A}{2j}\int_0^{2D}[e^{j[2\pi(f_c - f)t + \varphi_i(t)]} - e^{-j[2\pi(f_c + f)t + \varphi_i(t)]}]\,dt\end{aligned} \quad (7.88)$$

Substituting for $\varphi_i(t)$, and with the definitions $u_{1n}(f) = 2\pi(f_c - f) + q_1$ and $u_{1p}(f) = 2\pi(f_c + f) + q_1$, it follows that $\Phi_i(f)$ can be written as

$$\begin{aligned}&\frac{A}{2j}\int_0^{D}[e^{j[q_o + u_{1n}(f)t + q_2\sin(q_m t + \theta_q)]} - e^{-j[q_o + u_{1p}(f)t + q_2\sin(q_m t + \theta_q)]}\,dt \\ &+ \frac{A}{2j}\int_D^{2D}[e^{j[q_o + u_{1n}(f)t + q_2\sin(q_m t + \theta_q)]} - e^{-j[q_o + u_{1p}(f)t + q_2\sin(q_m t + \theta_q)]}\,dt\end{aligned} \quad (7.89)$$

where, as is clear from the above derivation of the phase waveforms in $[0, D]$ and $[D, 2D]$, the coefficients q_o, u_{1n}, u_{1p}, q_2, and θ_q vary from $[0, D]$ to $[D, 2D]$. With the change of variable $\lambda = t + \theta_q/q_m$, and with the definitions $v_{1n}(f) = u_{1n}(f)\theta_q/q_m$, $v_{1p}(f) = u_{1p}(f)\theta_q/q_m$, it follows that $\Phi_i(f)$ can be written as

$$\begin{aligned}\Phi_i(f) = &\frac{-Aje^{j[q_o - v_{1n}(f)]}}{2}\int_{\theta_q/q_m}^{D+\theta_q/q_m} e^{j[u_{1n}(f)\lambda + q_2\sin(q_m\lambda)]}\,d\lambda \\ &+ \frac{Aje^{-j[q_o - v_{1p}(f)]}}{2}\int_{\theta_q/q_m}^{D+\theta_q/q_m} e^{-j[u_{1p}(f)\lambda + q_2\sin(q_m\lambda)]}\,d\lambda \\ &+ \frac{-Aje^{j[q_o - v_{1n}(f)]}}{2}\int_{D+\theta_q/q_m}^{2D+\theta_q/q_m} e^{j[u_{1n}(f)\lambda + q_2\sin(q_m\lambda)]}\,d\lambda \\ &+ \frac{Aje^{-j[q_o - v_{1p}(f)]}}{2}\int_{D+\theta_q/q_m}^{2D+\theta_q/q_m} e^{-j[u_{1p}(f)\lambda + q_2\sin(q_m\lambda)]}\,d\lambda\end{aligned} \tag{7.90}$$

Evaluation of these integrals relies on the result,

$$\begin{aligned}I(k_o, w_C, w_m, w_A, \lambda_1, \lambda_2) &= \int_{\lambda_1}^{\lambda_2} e^{jk_o[w_C\lambda + w_A\sin(w_m\lambda)]}\,d\lambda \\ &= \sum_{i=0}^{\infty}(-1)^i J_i(w_A)\left[\frac{\sin((w_C - iw_m)\lambda_2) - \sin((w_C - iw_m)\lambda_1)}{w_C - iw_m}\right] \\ &\quad + \sum_{i=1}^{\infty} J_i(w_A)\left[\frac{\sin((w_C + iw_m)\lambda_2) - \sin((w_C + iw_m)\lambda_1)}{w_C + iw_m}\right] \\ &\quad + (-j)k_o\sum_{i=0}^{\infty}(-1)^i J_i(w_A)\left[\frac{\cos((w_C - iw_m)\lambda_2) - \cos((w_C - iw_m)\lambda_1)}{w_C - iw_m}\right] \\ &\quad + (-j)k_o\sum_{i=1}^{\infty} J_i(w_A)\left[\frac{\cos((w_C + iw_m)\lambda_2) - \cos((w_C + iw_m)\lambda_1)}{w_C + iw_m}\right]\end{aligned} \tag{7.91}$$

where $k_o \in \{-1, 1\}$. This result arises from the standard Bessel function expansions for the terms $\cos(w_A\sin(w_m\lambda))$ and $\sin(w_A\sin(w_m\lambda))$ (Spiegel, 1968 p. 145), that is,

$$\cos(w_A\sin(w_m\lambda)) = J_0(w_A) + 2J_2(w_A)\cos(2w_m\lambda) + 2J_4(w_A)\cos(4w_m\lambda) + \cdots \tag{7.92}$$

$$\sin(w_A\sin(w_m\lambda)) = 2J_1(w_A)\sin(w_m\lambda) + 2J_3(w_A)\sin(3w_m\lambda) + \cdots \tag{7.93}$$

$\Phi_i(f)$ can be evaluated using Eq. (7.91), that is,

$$\begin{aligned}\Phi_i(f) = {} & \frac{-Aje^{j[q_o - v_{1n}(f)]}}{2} I(1,\ u_{1n}(f),\ q_m,\ q_2,\ \theta_q/q_m,\ D + \theta_q/q_m) \\ & + \frac{Aje^{-j[q_o - v_{1p}(f)]}}{2} I(-1,\ u_{1p}(f),\ q_m,\ q_2,\ \theta_q/q_m,\ D + \theta_q/q_m) \\ & + \frac{-Aje^{j[q_o - v_{1n}(f)]}}{2} I(1,\ u_{1n}(f),\ q_m,\ q_2,\ D + \theta_q/q_m,\ 2D + \theta_q/q_m) \\ & + \frac{Aje^{-j[q_o - v_{1p}(f)]}}{2} I(-1,\ u_{1p}(f),\ q_m,\ q_2,\ D + \theta_q/q_m,\ 2D + \theta_q/q_m)\end{aligned} \tag{7.94}$$

In the first two component expressions in this equation, q_o, v_{1n}, v_{1p}, q_2, and θ_q are defined for $[0, D]$, whereas in the last two component expressions, these variables are defined for $[D, 2D]$.