Tercer Examen Parcial

Instrucciones: Debe incluir todo el procedimiento que utilizó para llegar a sus respuestas. Trabaje en forma clara y ordenada y utilice cuaderno de examen u hojas debidamente grapadas. No se acogerán apelaciones en exámenes resueltos con lápiz o que presenten algún tipo de alteración. No se permite el uso de calculadora programable ni el uso de dispositivos electrónicos con conectividad inalámbrica durante el desarrollo de la prueba.

#1. Calcule cada una de las siguientes integrales.

a) $\displaystyle\int \ln\left(1+x^2\right)\,\mathrm{d}x$ **4 Pts**

$$u = \ln\left(1+x^2\right) \Rightarrow du = \frac{2x}{1+x^2}\,dx \qquad dv = dx \Rightarrow v = x$$

[redacted]

. . . .

$$\int \ln\left(1+x^2\right)\,\mathrm{d}x = x\cdot\ln\left(1+x^2\right) - \int \frac{2x^2}{1+x^2}\,\mathrm{d}x$$

[redacted]

. . . .

$$\begin{array}{r|l} 2x^2 & x^2+1 \\ \hline -2x^2-2 & 2 \\ \hline -2 & \end{array}$$

$$\int \ln\left(1+x^2\right)\,\mathrm{d}x = x\cdot\ln\left(1+x^2\right) - \int\left(2-\frac{2}{1+x^2}\right)\mathrm{d}x$$

[redacted]

. . . .

$$\int \ln\left(1+x^2\right)\,\mathrm{d}x = x\cdot\ln\left(1+x^2\right) - 2x + 2\,\arctan(x) + C$$

[redacted]

. . . .

b) $\displaystyle\int \frac{\mathrm{d}x}{(9x^2+25)^{3/2}}$ **4 Pts**

Haciendo el cambio de variable:

$$3x = 5\,\tan\theta \Rightarrow 3\,dx = 5\,\sec^2\theta\,\mathrm{d}\theta$$

[redacted]

$$\int \frac{\mathrm{d}x}{(9x^2+25)^{3/2}} = \int \frac{\frac{5}{3}\,\sec^2\theta}{(25\,\tan^2\theta+25)^{3/2}}\,\mathrm{d}\theta = \int \frac{\frac{5}{3}\,\sec^2\theta}{(25\,\sec^2\theta)^{3/2}}\,\mathrm{d}\theta$$

[redacted]

$$\int \frac{\frac{5}{3}\,\sec^2\theta}{125\,\sec^3\theta}\,\mathrm{d}\theta = \frac{1}{75}\int \cos\theta\,\mathrm{d}\theta = \frac{1}{75}\,\operatorname{sen}\theta + C$$

[redacted]

$$\tan\theta = \frac{3x}{5} \Rightarrow \operatorname{sen}\theta = \frac{3x}{\sqrt{9x^2+25}}$$

$$\int \frac{\mathrm{d}x}{(9x^2+25)^{3/2}} = \frac{x}{25\,\sqrt{9x^2+25}} + C$$

[redacted]

c) $\displaystyle\int \tan^3\left(\frac{x}{2}\right)\sec^4\left(\frac{x}{2}\right)\,\mathrm{d}x$ **4 Pts**

$$\begin{aligned}
\int \tan^3\left(\frac{x}{2}\right)\sec^4\left(\frac{x}{2}\right)\,\mathrm{d}x &= \int \tan^3\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)\,\mathrm{d}x \\
&= \int \tan^3\left(\frac{x}{2}\right)\left[1+\tan^2\left(\frac{x}{2}\right)\right]\sec^2\left(\frac{x}{2}\right)\,\mathrm{d}x
\end{aligned}$$

[redacted]

$$u = \tan\left(\frac{x}{2}\right) \Rightarrow du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\,dx \Rightarrow \sec^2\left(\frac{x}{2}\right)\,dx = 2\,du$$

[redacted]

$$\begin{aligned}
\int \tan^3\left(\frac{x}{2}\right)\left[1+\tan^2\left(\frac{x}{2}\right)\right]\sec^2\left(\frac{x}{2}\right)\,\mathrm{d}x &= \int 2\,u^3\,(1+u^2)\,\mathrm{d}u \\
&= \int 2\,u^3\,\mathrm{d}u + \int 2\,u^5\,\mathrm{d}u
\end{aligned}$$

[redacted]

$$\int 2\,u^3\,\mathrm{d}u + \int 2\,u^5\,\mathrm{d}u = 2\,\frac{u^4}{4} + 2\,\frac{u^6}{6} + C = \frac{u^4}{2} + \frac{u^6}{3} + C = \frac{\tan^4\left(\frac{x}{2}\right)}{2} + \frac{\tan^6\left(\frac{x}{2}\right)}{3} + C$$

[redacted]

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d) $\displaystyle\int \frac{2x+1}{x^2\left(x^2+2\right)}\,\mathrm{d}x$ **5 Pts**

$$\begin{aligned}
\frac{2x+1}{x^2\left(x^2+2\right)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+2}\\
\frac{2x+1}{x^2\left(x^2+2\right)} &= \frac{Ax\left(x^2+2\right)+B\left(x^2+2\right)+x^2\left(Cx+D\right)}{x^2\left(x^2+2\right)}
\end{aligned}$$

[redacted]

$$2x+1=\left(A+C\right)x^3+\left(B+D\right)x^2+2Ax+2B$$

$$\begin{cases} A+C=0\\ B+D=0\\ 2A=2\\ 2B=1\end{cases} \Rightarrow A=1; B=\frac{1}{2}; C=-1; D=-\frac{1}{2}$$

[redacted]

$$\begin{aligned}
\int \frac{2x+1}{x^2\left(x^2+2\right)}\,\mathrm{d}x &= \int\left(\frac{1}{x}+\frac{\frac{1}{2}}{x^2}+\frac{-x-\frac{1}{2}}{x^2+2}\right)\mathrm{d}x\\
&= \ln|x|-\frac{\frac{1}{2}}{x}+\int\frac{-x}{x^2+2}\,\mathrm{d}x-\frac{1}{2}\int\frac{1}{x^2+2}\,\mathrm{d}x
\end{aligned}$$

[redacted]

$$\int \frac{2x+1}{x^2\left(x^2+2\right)}\,\mathrm{d}x=\ln|x|+\frac{1}{2x}-\frac{1}{2}\ln\left|x^2+2\right|-\frac{1}{2\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+C$$

[redacted]

#2. Sea f una función continua tal que $F(x) = e^{2x^2}$ es una **antiderivada** de f. Calcule el valor de

3 Pts

$$\int_1^{1/e} \frac{f\,(1+\ln x)}{x}\,\mathrm{d}x$$

$$u = 1 + \ln x \Rightarrow du = \frac{1}{x}\,dx$$

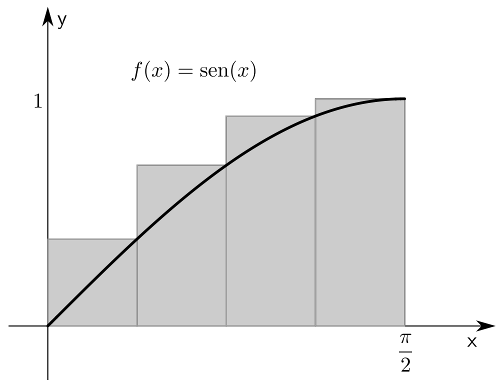
$$x = 1 \Rightarrow u = 1 + \ln 1 \Rightarrow u = 1$$

$$x = 1/e \Rightarrow u = 1 + \ln\,(1/e) \Rightarrow u = 0$$

$$\int_1^{1/e} \frac{f\,(1+\ln x)}{x}\,\mathrm{d}x = \int_1^0 f(u)\,\mathrm{d}u = -\int_0^1 f(u)\,\mathrm{d}u$$

$$-\int_0^1 f(u)\,\mathrm{d}u = -\left(e^{2x^2}\Big|_0^1 \right) = -\left(e^2 - 1\right) = 1 - e^2$$

#3. Aproxime el área bajo la gráfica de la función $f(x) = \operatorname{sen}(x)$ mediante sumas de Riemann desde $x = 0$ hasta $x = \pi/2$, usando 4 rectángulos de aproximación y los puntos extremos de la derecha (ver figura adjunta). **3 Pts**



. . . .

$$\Delta_x = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$$

$$x_k = 0 + k \cdot \Delta_x \Rightarrow x_k = \frac{k\pi}{8}$$

[redacted]

. . . .

$$x_0 = 0 \qquad x_1 = \frac{\pi}{8} \qquad x_2 = \frac{\pi}{4} \qquad x_3 = \frac{3\pi}{8} \qquad x_4 = \frac{\pi}{2}$$

[redacted]

. . . .

$$\mathcal{R}_4 = \sum_{k=1}^{4} f\left(x_k\right) \cdot \Delta_x = \sum_{k=1}^{4} \operatorname{sen}\left(\frac{k\pi}{8}\right) \cdot \Delta_x = \left[\operatorname{sen}\left(\frac{\pi}{8}\right) + \operatorname{sen}\left(\frac{\pi}{4}\right) + \operatorname{sen}\left(\frac{3\pi}{8}\right) + \operatorname{sen}\left(\frac{\pi}{2}\right)\right] \cdot \frac{\pi}{8} \approx 1{,}1835$$

[redacted]

. . . .

#4. Sea $f : \mathbb{R} \to \mathbb{R}$ derivable tal que $f(x) \neq 0$, que satisface $f^2(x) = \displaystyle\int_0^x 4\,t \cdot 3^{t^2} \cdot f(t)\ \mathrm{d}t$ y $f(0) = \dfrac{1}{\ln 3}$.
Determine la fórmula para $f(x)$. **3 Pts**

$$\left[f^2(x)\right]' = \left(\int_0^x 4\,f(t)\ \cdot t\ \cdot 3^{t^2}\ \mathrm{d}t\right)' \Rightarrow 2\,f(x)\cdot f'(x) = 4\,f(x)\cdot x\cdot 3^{x^2}$$

$$f'(x) = 2x\cdot 3^{x^2} \Rightarrow f(x) = \int 2\,x\cdot 3^{x^2}\ \mathrm{d}x \Rightarrow f(x) = \int 3^z\ \mathrm{d}z \Rightarrow f(x) = \frac{3^z}{\ln 3} + C$$

$$z = x^2 \Rightarrow dz = 2x\,dx$$

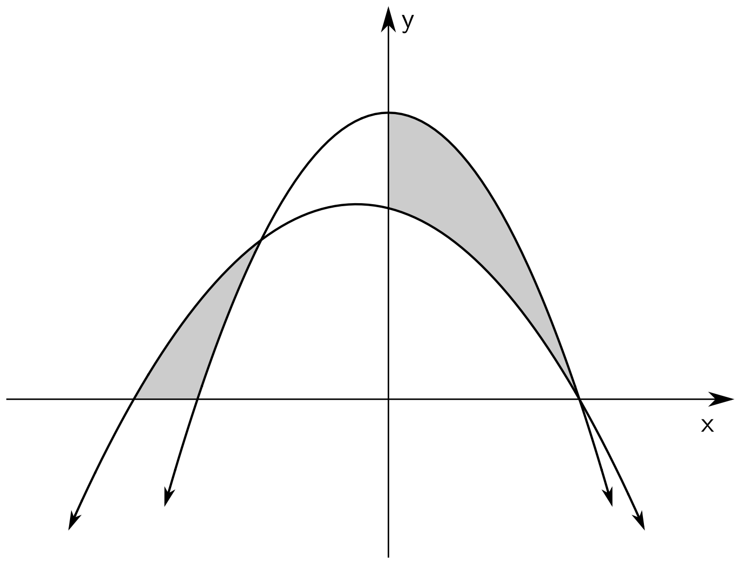
$$f(0) = \frac{1}{\ln 3} \Rightarrow \frac{1}{\ln 3} + C = \frac{1}{\ln 3} \Rightarrow C = 0$$

$$f(x) = \frac{3^{x^2}}{\ln 3}$$

#5. En la siguiente figura se muestran las gráficas de las funciones

$$f:\mathbb{R}\to\mathbb{R},\ f(x)=9-x^2 \qquad \text{y} \qquad g:\mathbb{R}\to\mathbb{R},\ g(x)=-\frac{1}{2}x^2-\frac{1}{2}x+6$$

Plantee (**no calcule**) las integrales que permitan determinar el área de la región destacada. **5 Pts**



....

$$g(x)=0 \iff -\frac{1}{2}x^2-\frac{1}{2}x+6=0 \iff x=-4 \ \wedge\ x=3$$

$$f(x)=0 \iff 9-x^2=0 \iff x=\pm 3$$

[redacted]

....

$$f(x)=g(x) \iff 9-x^2=-\frac{1}{2}x^2-\frac{1}{2}x+6 \iff -\frac{1}{2}x^2+\frac{1}{2}x+3=0 \iff x=-2 \ \wedge\ x=3$$

[redacted]

....

$$\mathcal{R}_1=\int_{-4}^{-3}\left(-\frac{1}{2}x^2-\frac{1}{2}x+6\right)\,\mathrm{d}x$$

$$\mathcal{R}_2=\int_{-3}^{-2}[g(x)-f(x)]\,\mathrm{d}x=\int_{-3}^{-2}\left(\frac{1}{2}x^2-\frac{1}{2}x-3\right)\,\mathrm{d}x$$

$$\mathcal{R}_3=\int_{0}^{3}[f(x)-g(x)]\,\mathrm{d}x=\int_{0}^{3}\left(-\frac{1}{2}x^2+\frac{1}{2}x+3\right)\,\mathrm{d}x$$

[redacted]

....

#6. Determine si la integral $\displaystyle\int_0^{+\infty} \frac{\cos(x)}{e^{\operatorname{sen}(x)}}\,\mathrm{d}x$ converge o diverge. En caso de ser convergente calcule su valor. **4 Pts**

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$$\int_0^{+\infty} \frac{\cos(x)}{e^{\operatorname{sen}(x)}}\,\mathrm{d}x = \lim_{k\to+\infty} \int_0^{k} \frac{\cos(x)}{e^{\operatorname{sen}(x)}}\,\mathrm{d}x$$

[redacted]

. . . .

$$t = \operatorname{sen}(x) \Rightarrow dt = \cos(x)\,dx$$

$$\int \frac{\cos(x)}{e^{\operatorname{sen}(x)}}\,\mathrm{d}x = \int e^{-t}\,\mathrm{d}t = -e^{-t} + C = e^{-\operatorname{sen}(x)} + C$$

[redacted]

. . . .

$$\int_0^{k} \frac{\cos(x)}{e^{\operatorname{sen}(x)}}\,\mathrm{d}x = e^{-\operatorname{sen}(x)}\Big|_0^k = e^{-\operatorname{sen}(k)} - 1$$

[redacted]

. . . .

$$\lim_{k\to+\infty} \int_0^{k} \frac{\cos(x)}{e^{\operatorname{sen}(x)}}\,\mathrm{d}x = \lim_{k\to+\infty} \left(e^{-\operatorname{sen}(k)} - 1\right) \to \text{no existe}$$

R/ La integral diverge.

[redacted]

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