数学物理方程第二课

2023年8月7日 星期一 19:05

1.1 数学模型的建立

明确对象 → 简化假设 → 数理推导 → 定解条件

1. ?个未知函数
2. 自变量是？

时空维度
- 是否考虑f随t的变化
- 尽可能降维解析

1. 几维？
2. 区域/边界
3. 取微元
4. 微元无穷→中值thm…
5. PDE控制方程

1. 大多在PDE推出通解. ⇒ focus on 特解
2. PDE仅关心区域内部时空演化 ⇒ 区域初始/边界条件
3. 建立定解问题.

1. 弦振动方程：

phenomenon: 细弦一根. 均匀/柔软/绷直. 垂直于弦的外力作用下 做微小横振动. 求弦上任意点/任一时刻的位移.

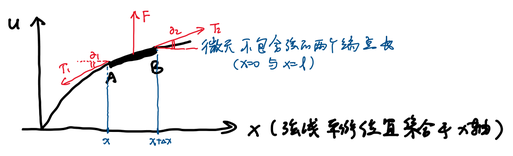
process:

① 明确对象: 弦上任一点横向位移 ⇒ $u(x,t)$ (x: 任位置, t: 任时刻)

② 简化假设:
1. 细弦: ignore 重力.
2. 均匀: 线密度 ρ 一致.
3. 柔软: T不抗弯曲. 只抗伸长. ⇒ ignore 弯曲力矩
4. 微小: 忽略高阶小量.
5. 横振动: 位移与 balance position 垂直 ⇒ 振动在同一平面内.

类比光波 ⇒ 圆偏振光 ⇒ ↕ ⇒ 分解至x、y轴上.

③ 数理推导



① 水平: $-|T_1|\cos\partial_1 + |T_2|\cos\partial_2 = 0$.

[牵连移动在u方向上的合外力 f_0]

② 竖直: $\sum F = m\cdot\frac{\partial^2 u}{\partial t^2}$ ⇒ [] 外力F: "强色外力"用力密度表示

$F = \int_x^{x+\Delta x} f_0(x,t)\,ds$

又 $ds = \sqrt{1+(\frac{\partial u}{\partial x})^2}\,dx$

⇒ $F = \int_x^{x+\Delta x} f_0(x,t)\sqrt{1+(\frac{\partial u}{\partial x})^2}\,dx$

$x_2 \in [x, x+\Delta x]$

弦段平均加速度 = 点x_2的加速度值

[∴] $-|T_1|\sin\partial_1 + |T_2|\sin\partial_2 + \int_x^{x+\Delta x} f_0(x,t)\sqrt{1+(\frac{\partial u}{\partial x})^2}\,dx = \rho\cdot\Delta s\cdot\frac{\partial^2 u(x_2,t)}{\partial t^2}$

方式与u类比"积分中值定理"

化简数理过程:

① $\partial_1 \doteq 0 \doteq \partial_2$

② $\cos\partial_1 \doteq \cos\partial_2 \doteq 1$

③ $\sin\partial_1 \doteq \tan\partial_1 = \frac{\partial u}{\partial x}\big|_x$; $\sin\partial_2 \doteq \tan\partial_2 = \frac{\partial u}{\partial x}\big|_{x+\Delta x}$

④ $\Delta s = \int_x^{x+\Delta x}\sqrt{1+(\frac{\partial u}{\partial x})^2}\,dx \doteq \Delta x$.

高阶小量可ignore

① 知 $|T_1| = |T_2| = T_0$

② $-T_0\cdot\frac{\partial u}{\partial x}\big|_{x=x} + T_0\cdot\frac{\partial u}{\partial x}\big|_{x=x+\Delta x} + \int_x^{x+\Delta x} f_0(x,t)\cdot 1\cdot dx = \rho\cdot\Delta x\cdot\frac{\partial^2 u(x_1,t)}{\partial t^2}$.

$= f_0(x_1,t)$ …… $x_1 \in [x, \Delta x+x]$ 积分中值thm.

$\Rightarrow T_0\cdot\frac{\frac{\partial u}{\partial x}\big|_{x+\Delta x} - \frac{\partial u}{\partial x}\big|_x}{\Delta x} + f_0(x_1,t) = \rho\cdot\frac{\partial^2 u(x_1,t)}{\partial t^2}$.

令 $\Delta x\to 0$: x_1 与 $x_2 \to x$

$\Rightarrow T_0\cdot\frac{\partial^2 u}{\partial x^2} + f_0(x,t) = \rho\cdot\frac{\partial^2 u(x,t)}{\partial t^2}$.

得到方程: $\frac{\partial^2 u}{\partial t^2} = \boxed{\frac{T_0}{\rho}}\cdot\frac{\partial^2 u}{\partial x^2} + \boxed{\frac{f_0(x,t)}{\rho}}$

记为 a^2 　 记为 $f(x,t)$

意义: 质点位移加速度 与 形变加速度 呈线性关系!

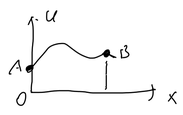
④ 定解条件: 易见二阶常微分方程

⇒ need 2个条件确定2个待定系数.

- 初始condition:
  - 初始位移 $u(x,0) = \varphi(x)$, $0\le x\le l$.
  - 初始速度 $u_t(x,0) = \psi(x)$, $0\le x\le l$.
- 边界condition:
  - edge1: Dirichlet边界 (规定了边界上的函数值)

    已知两端点位移 $\begin{cases} u(0,t) = g_1(t) \\ u(l,t) = g_2(t) \end{cases}$ $(t\ge 0)$

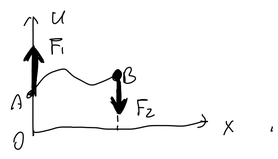
    当 $g_1(0,t) = g_2(l,t) = C$ 时. called 弦有固定端



  - edge2: Neumann边界 (规定了未知函数在边界上的空间变化率)

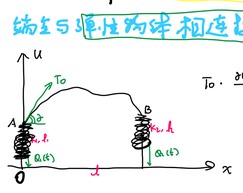
    已知端点所受垂直外力 $\begin{cases} -T_0\cdot\frac{\partial u(0,t)}{\partial x} = g_1(t) \\ T_0\cdot\frac{\partial u(l,t)}{\partial x} = g_2(t) \end{cases}$ $(t\ge 0)$

    当 $g_1(0,t) = g_2(l,t) = 0$ 时. called 弦有自由端.



  - edge3: Robin边界 (规定了边界处未知函数值与未知函数空间变化率的线性组合的函数关系)

    端点与弹性物体相连接.



此时A所在位置

$T_0\cdot\frac{\partial u(\Delta x,t)}{\partial x} - k_1(u(0,t) - l_1 - Q_1(t)) + f_0(x_1,t)\Delta x = \rho\Delta x\cdot\frac{\partial^2 u(x_2,t)}{\partial t^2}$

A原长

此时认作"弹长"

=0 　 =0

令 $\Delta x\to 0$

$\Rightarrow \dots \Rightarrow \frac{\partial u(0,t)}{\partial x} - \underbrace{\frac{k_1}{T_0}}_{G_1}u(0,t) = \underbrace{-\frac{k_1}{T_0}(l_1 - Q_1(t))}_{g_1(t)}$

$\Rightarrow \begin{cases} \frac{\partial u(0,t)}{\partial x} - G_1\cdot u(0,t) = g_1(t) \\ \frac{\partial u(l,t)}{\partial x} - G_2\cdot u(0,t) = g_2(t) \end{cases}$

上述 $\begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$ (上述in edge 1/2/3)

2. 定解问题:

混合pro: 初始condition + 边界condition.

书写形式例如:

① $u_{tt} = a^2 u_{xx} + f(x,t)$ 　 $0<x<l$. $t>0$. 　只有"里"了. 没有"面"了 (不解取节) 　零时刻状态 [弦振动 已稳 本身]

② $u(0,t) = g_1(t)$; $u(l,t) = g_2(t)$ 　 $t\ge 0$. 　标时间范围的是边界条件. (t可取0. 这才叫"edge") 　上述第一类边界条件 (亦可换成 edge2/edge3) 　边界条件

③ $u(x,0) = \varphi(x)$; $u_t(x,0) = \psi(x)$ 　 $0\le x\le l$. 　标空间范围的是初始条件. (x取到0.l. 这才叫"edge") 　初始条件

两端可以选择不同的类型

Cauchy pro: 无界区域初值问题, 仅有初始condition.

3. 波动方程: $\frac{\partial^2 u}{\partial t^2} = a^2\cdot\frac{\partial^2 u}{\partial x^2} + \boxed{f(x,t)}$

① $f(x,t)=0$: 弦自由振动/齐次一维

② $f(x,t)\neq 0$: 弦受迫振动/非齐次一维

Cor:

① $a = \sqrt{T_0/\rho} = \sqrt{N/(kg/m)} = \sqrt{\frac{N\cdot m}{kg}} = \sqrt{\frac{kg\cdot(m/s^2)\cdot m}{kg}} = m/s$ ⇒ 波速.

② $f(x,t) = \frac{f_0(x,t)}{\rho} = \frac{N\cdot m}{m\cdot kg} = \frac{N}{kg}$ ⇒ 单位质量所受外力.

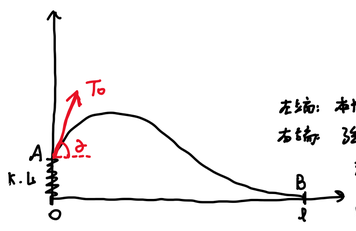
③ $u_{tt} = \frac{\partial^2 u(x,t)}{\partial t^2}$ 　任一点位移加速度.

④ $u(x,0)$ 　初始位移形状.

⑤ $u_t(x,0) = \frac{\partial u(x,0)}{\partial t}$ 　初始v.

⑥ $u_x(0,t)$; $u_x(l,t)$ 　两端点的空间变化率.

例1:



左端: 本长为L. 弹性系数为k的弹簧上.

右端: 弦平行位置.

整弦初始v=0. 初始位移 φ(x).

则本题定解问题表示为:

① 方程本身: $u_{tt} = a^2 u_{xx} + f(x,t)$ 自由振动 ⇒ $u_{tt} = a^2 u_{xx}$. $0<x<l$. $t>0$.

原因: 弹簧拉伸. 应采用Neumann edge.

② 边界con: 选择第三类edge ⇒ $u_x(0,t) = \frac{k}{T_0}(u(0,t) - L)$, $u(l,t) = 0$

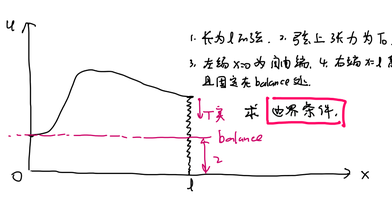
③ 初始con: 初始位移: $u(x,0) = \varphi(x)$, 初始速度: $u_t(x,0) = 0$. ⇒ $0\le x\le l$

解: $u_{tt} = a^2\cdot u_{xx}$, $0<x<l$. $t>0$.

$u_x(0,t) = \frac{k}{T_0}(u(0,t) - l)$, $u_x(l,t) = 0$, $t>0$

$u(x,0) = \varphi(x)$. $u_t(x,0) = 0$.

例2:



1. 长为l的弦. 2. 弦上张力为 T_0.
3. 左端 x=0 为自由端. 4. 右端 x=l 系在弹长为z, 劲度系数为k的弹簧上, 且固定在balance处.

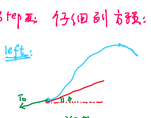
求 边界条件.

解题思路:

step1: 确定两端类型
- left: 自由端 ⇒ edge II
- right: 连弹体 ⇒ edge III.

step2: 仔细列方程:

left:

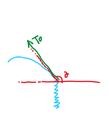


$\tan\alpha = \frac{\partial u(0,t)}{\partial x}$

$T_y = T_0\sin\alpha \doteq T_0\tan\alpha = \frac{\partial u(0,t)}{\partial x}\cdot T_0 = 0$

⇒ $u_x(0,t) = 0$.

right:



$T_{弹} = k(u(l,t) - z)$

$\tan\alpha = \frac{\partial u(l,t)}{\partial x}$

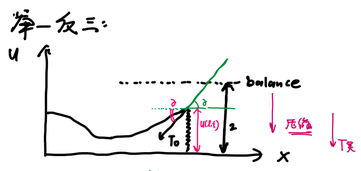
$T_y = T_0\cdot\sin(\pi-\alpha)$
$= T_0\sin\alpha$
$\doteq T_0\tan\alpha$
$= T_0\cdot\frac{\partial u(l,t)}{\partial x}$

⇒ $-T_0\cdot\frac{\partial u(l,t)}{\partial x} = k(u(l,t) - z)$

step III: 整合process:

答案: $u_x(0,t) = 0$, $-T_0\cdot\frac{\partial u(l,t)}{\partial x} = k\cdot(u(l,t) - z)$, $t\ge 0$.

⇒ 举一反三:



$\tan\alpha = \frac{\partial u(l,t)}{\partial x}$

$T_y = T_0\cdot\sin\alpha \doteq T_0\tan\alpha = T_0\cdot\frac{\partial u(l,t)}{\partial x}$

$T_{弹} = k(z - u(l,t))$

⇒ $-T_0\cdot\frac{\partial u(l,t)}{\partial x} = k\cdot(u(l,t) - z)$

这里很有必要梳理下弦振动方程的解题方法

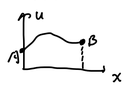
(1) 定解问题 　 (2) 边界条件

定解问题包含
- 方程model: $\frac{\partial^2 u}{\partial t^2} = a^2\cdot\frac{\partial^2 u}{\partial x^2} + \boxed{f(x,t)}$
  - ① $f(x,t)=0$: 弦自由振动/齐次一维
  - ② $f(x,t)\neq 0$: 弦受迫振动/非齐次一维
- 定解条件:
  - 初始条件: 初始的位移: $u(x,0) = \varphi(x)$, $0\le x\le l$. ; ------速度: $u_t(x,0) = \psi(x)$, $0\le x\le l$.
  - 边界条件: 要根据题目场景选择合适的edge.

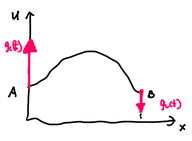
PS: 两侧所选取的边界条件可以不同.

比如: left: 已知位移 ⇒ edge I. ; right: 已知外力 ⇒ edge II.

若题中 (trait: 固定端. $g(t)=0$ [位移g≡0]) 已知两端点位移
$u(0,t) = g_1(t)$
$u(l,t) = g_2(t)$,
$t\ge 0$.



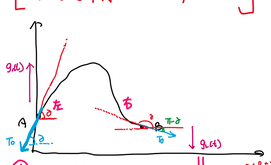
若题中 (trait: 自由端. $g(t)=0$ [外力g≡0]) 已知端点垂直外力
$\begin{cases} -T_0\cdot\frac{\partial u(0,t)}{\partial x} = g_1(t) & ① \\ T_0\cdot\frac{\partial u(l,t)}{\partial x} = g_2(t) & ② \end{cases}$
$t\ge 0$.



若题中 端点与弹性物体连接
$\begin{cases} \frac{\partial u(0,t)}{\partial x} - G_1\cdot u(0,t) = g_1(t) \\ \frac{\partial u(l,t)}{\partial x} - G_2\cdot u(l,t) = g_2(t) \end{cases}$
, $t\ge 0$.

这里会引入一个重要的. 关于正负号判定的问题; 具体分析见例2.

这里会引入一个重要的. 关于正负号判定的问题; 为何①中有"−", ②中无"−".



左: $\tan\alpha = \frac{\partial u(0,t)}{\partial x}$

$T_y = T_0\cdot\sin\alpha \doteq T_0\cdot\frac{\partial u(0,t)}{\partial x}$

∴ $g_1(t) = -T_y = -T_0\cdot\frac{\partial u(0,t)}{\partial x}$

右: $\tan\alpha = \frac{\partial u(l,t)}{\partial x}$

$T_y = T_0\cdot\sin(\pi-\alpha)$
$= T_0\sin\alpha$
$\doteq T_0\cdot\tan\alpha = T_0\cdot\frac{\partial u(l,t)}{\partial x}$

∴ $g_2(t) = T_y = T_0\cdot\frac{\partial u(l,t)}{\partial x}$

引入问: 为什么两边的 T_0 是 ↙↘ 而不是 ↗↖ ?

Ans: 线想. 这里稳态. 被拉动 明显两端张力为 ↙↘