

2.2 分离变量法

(1) 一维波动方程问题:

$$\begin{cases} U_{tt} = a^2 U_{xx} & ① \\ U(0,t) = 0, U(l,t) = 0 & ② \\ U(x,0) = \varphi(x), U_t(x,0) = \psi(x) & ③ \end{cases}$$

Step1: 分离变量 根据齐次边界/方程 假设 $U(x,t) = X(x) \cdot T(t)$ ----- 非零解

回代①: $\Rightarrow \frac{\ddot{X}(x)}{X(x)} = \frac{\ddot{T}(t)}{T(t)} = -\lambda$ [左右只可能同等于某个常数 $-\lambda$]

化为两个ODE: $\begin{cases} \ddot{X}(x) + \lambda X(x) = 0 \\ \ddot{T}(t) + \lambda a^2 T(t) = 0 \end{cases}$

由边界条件: $X(x) \cdot T(t) = X(x) \cdot T(t) = 0$, $T(t)$ 不恒为0 (否则 $U(x,t) \equiv 0$ 零解无意义)

\Rightarrow 转化: $\begin{cases} X(0) = X(l) = 0 \\ \ddot{X}(x) + \lambda X(x) = 0 \end{cases} \Rightarrow \begin{cases} \lambda_n = (\frac{n\pi}{l})^2 \\ X_n(x) = \sin(\frac{n\pi}{l} x) \end{cases} \quad (n=1, 2, 3, \dots)$

Step2: 求解初值问题

$\lambda_n T(t): \ddot{T}(t) + \lambda_n a^2 T(t) = 0$

$\Rightarrow T_n(t) = C_{1n} \sin(\sqrt{\lambda_n} a t) + C_{2n} \cos(\sqrt{\lambda_n} a t)$

故 $U_n(x,t) = T_n(t) \cdot X_n(x)$

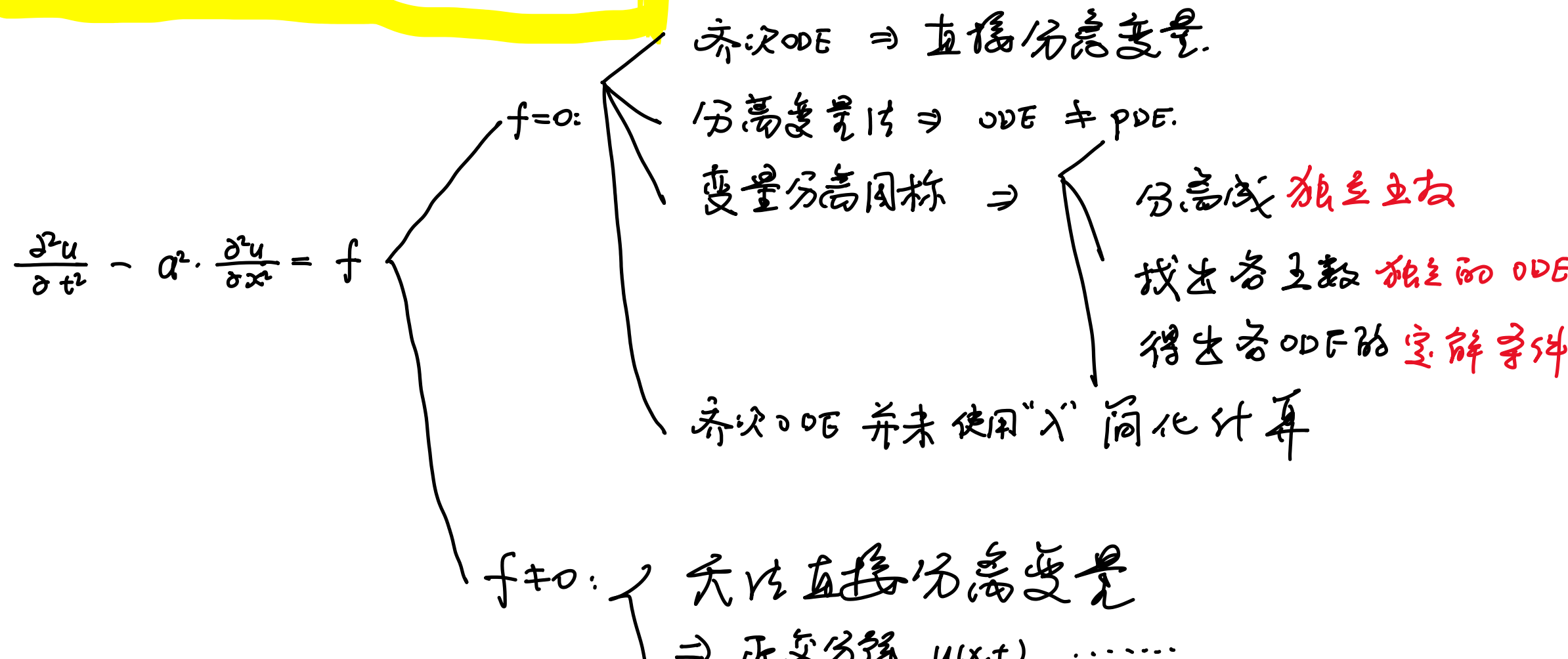
$= [C_{1n} \sin(\sqrt{\lambda_n} a t) + C_{2n} \cos(\sqrt{\lambda_n} a t)] \cdot \sin(\frac{n\pi}{l} x)$

$\Rightarrow U(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$

$= \sum_{n=1}^{\infty} [C_{1n} \sin(\sqrt{\lambda_n} a t) + C_{2n} \cos(\sqrt{\lambda_n} a t)] \cdot \sin(\frac{n\pi}{l} x)$

由于此 $U(x,t)$ 满足初值条件: $\begin{cases} U(x,0) = \varphi(x) \\ U_t(x,0) = \psi(x) \end{cases}, 0 \leq x \leq l$

(2) 分离变量的目标与难点:



(3) 对左(1)中 $A \neq 0$ 情况:

$$\begin{cases} U_{tt} - a^2 U_{xx} = A & ① \\ U(0,t) = 0, U(l,t) = 0 & ② \\ U(x,0) = \varphi(x), U_t(x,0) = \psi(x) & ③ \end{cases}$$

Step1: 特征值 problem

设 $U(x,t) = X(x) \cdot T(t)$

代入对左齐次方程: $X(x) \cdot T''(t) - a^2 \ddot{X}(x) \cdot T(t) = 0$



Step2: 正交分解

$U(x,t) = \sum_{n=1}^{\infty} T_n(t) \cdot X_n(x) = \sum_{n=1}^{\infty} T_n(t) \cdot \sin(\frac{n\pi}{l} x)$

$A =$

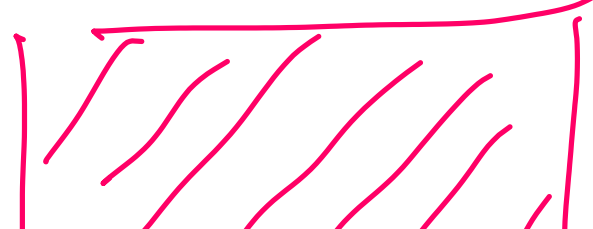
Step3: 建立初值问题 ODE

$U(x,t) = \sum_{n=1}^{\infty} T_n(t) \cdot \sin(\frac{n\pi}{l} x)$

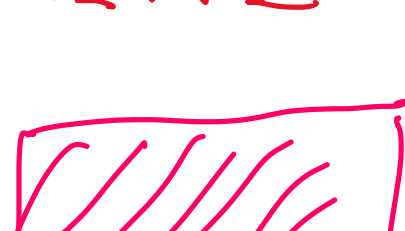
假设可逐项求导:

由①: $\sum_{n=1}^{\infty} T_n''(t) \cdot X_n(x) - a^2 \sum_{n=1}^{\infty} T_n(t) \cdot \ddot{X}_n(x) = \sum_{n=1}^{\infty} f_n(x)$

由 $X_n(x)$ 是 A 的特征函数 $\Rightarrow \ddot{X}_n(x) = -\lambda_n X_n(x)$



Step4: 求初值问题



(4) 驻波由来: 由上述 (1) 与 (3)

- \Rightarrow ① 解由无数简谐波叠加
- ② 各谐波都有振幅 $\equiv 0$ 的点 \Rightarrow 分离变量法/驻波法/特征函数法
- ③ 节点: 令 $\sin \frac{n\pi}{l} x = 0 \Rightarrow x = k \cdot \frac{l}{n} \quad (1 \leq k \leq n-1)$

$$\begin{cases} U_{tt} - a^2 U_{xx} = p^r \cdot \sin \omega t & ① \\ U(0,t) = 0, U(l,t) = 0 & ② \\ U(x,0) = 0, U_t(x,0) = 0 & ③ \end{cases}$$

Step1: 特征值 problem

设 $U(x,t) = X(x) \cdot T(t)$, 代入对左齐次 PDE

$\Rightarrow \lambda_n = (\frac{(2n+1)\pi}{2l})^2$

$X_n(x) = \cos(\frac{(2n+1)\pi}{2l} x)$

($n=0, 1, 2, \dots$)

Step2: 正交分解

Step3: 建立初值 ODE

Step4: 求初值问题