

# Optical Flow

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**Abstract**

**Keywords**

# 1 Introduction

Optical flow is the pattern of apparent motion of objects, surfaces, and edges in a visual scene caused by the relative motion between an observer and a scene ([Horn and Schunck, 1981](#)). The concept of optical flow was introduced by the American psychologist James J. Gibson in the 1940s to describe the visual stimulus provided to animals moving through the world ([Gibson, 1979](#)). Gibson stressed the importance of optical flow for affordance perception, the ability to discern possibilities for action within the environment. The role of the optical flow have been demonstrated further in the literature for the perception of movement by the observer in the world, perception of the shape, distance and movement of objects in the world, and the control of locomotion. Optical flow is the distribution of apparent velocities of movement of brightness patterns in an image. Optical flow can arise from relative motion of objects and the viewer. Therefore, optical flow can give important information about the spatial arrangement of the objects viewed and the rate of change ([Horn and Schunck, 1981](#)). Discontinuities in the optical flow can help in segmenting images into regions that correspond to different objects. The optical flow methods attempt to address the problem of recovering the motions of objects relative to the viewer. Sequences of ordered images allow the estimation of motion as either instantaneous image velocities or discrete image displacements . Fleet and Weiss provide a tutorial introduction to gradient based optical flow. John L. Barron, David J. Fleet, and Steven Beauchemin provide a performance analysis of a number of optical flow techniques. It emphasizes the accuracy and density of measurements.

## 2 Optical Flow Methods

Optical flow is the distribution of apparent velocities of movement of brightness patterns in an image. Optical flow can arise from relative motion of objects and the viewer. Therefore, optical flow can give important information about the spatial arrangement of the objects viewed and the rate of change (Horn and Schunck, 1981). Discontinuities in the optical flow can help in segmenting images into regions that correspond to different objects. The optical flow methods attempt to address the problem of recovering the motions of objects relative to the viewer.

### 2.1 Optical flow field vs. Motion field

### 2.2 Optical flow constraint equation

The optical flow methods try to calculate the motion between two image frames which are taken at times  $t$  and  $t + \Delta t$ . Consider a 2D dimensional image, and a pixel at location  $\mathbf{x} = (x, y)$  and time  $t$  with intensity  $I(x, y, t)$ . This local image region has a spacial displacement of  $\Delta \mathbf{x} = (\Delta x, \Delta y)$  after time  $\Delta t$  between two consequent image frames. The initial hypothesis in measuring image motion is that the intensity structures of local time-varying image regions are approximately constant under motion for at least a short duration (Beauchemin and Barron, 1995; Horn and Schunck, 1981). Formally

$$I(x, y, t) \approx I(x + \Delta x, y + \Delta y, t + \Delta t). \quad (1)$$

The *differential methods* are based on local Taylor series approximations of the image signal, i.e. they use partial derivatives with respect to the spatial and temporal coordinates. The image domain is therefore assumed to be continuous (or differentiable) in space and time. Expanding the left-hand side of (1) in a Taylor series yields

$$I(x, y, t) = I(x, y, t) + \frac{\partial I}{\partial x} \Delta x + \frac{\partial I}{\partial y} \Delta y + \frac{\partial I}{\partial t} \Delta t + O(\Delta x^2, \Delta y^2, \Delta t^2), \quad (2)$$

where subscripts denote partial differentiation. We can ignore the second and higher order terms, which are assumed negligible. Subtracting (2) on both sides yields

$$\frac{dI}{dt} = \frac{\partial I}{\partial x} \dot{x} + \frac{\partial I}{\partial y} \dot{y} + \frac{\partial I}{\partial t} = 0, \quad \text{or} \quad \nabla I \cdot \mathbf{v} + I_t = 0, \quad (3)$$

where  $\nabla I = (I_x, I_y) = (\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y})$  is the spatial gradient of the intensity,  $\mathbf{v} = (u, v) = (\dot{x}, \dot{y})$  is the image *velocity field* or *optical flow vector field*. Note that we assume  $\Delta t$  is small and  $(\Delta x / \Delta t, \Delta y / \Delta t) \approx (\dot{x}, \dot{y})$ .

Equation (3) is known as the *optical flow constraint equation*. This is an equation in two unknowns and cannot be solved as such. This constraint is not sufficient to compute both components of  $\mathbf{v}$  as the optical flow constraint equation is illposed. That is to say, only  $\mathbf{v}_\perp$  the motion component in the direction of the local gradient of the image intensity function, may be estimated

$$\mathbf{v}_\perp = \frac{-I_t \nabla I}{\|\nabla I\|_2^2}. \quad (4)$$

This phenomenon is known as the aperture problem and only at image locations where there is sufficient intensity structure (or Gaussian curvature) can the motion be fully estimated with the use of the optical flow constraint equation ([Beauchemin and Barron, 1995](#)). To find the optical flow another set of equations is needed, given by some additional constraint. All optical flow methods introduce additional conditions for estimating the actual flow.

## **2.3 Summary of methods**

### 3 Differential Methods

Differential techniques compute image velocity field from spatiotemporal derivatives of image intensities. The image domain is therefore assumed to be continuous (or differentiable) in space and time. Global and local first and second-order methods based on Equation (3) can be used to compute optical flow. Global methods use (3) and an additional global constraint, usually a smoothness regularization term, to compute dense optical flows over large image regions. Local methods use normal velocity information in local neighborhoods to perform a least squares minimization to find the best fit for  $\mathbf{v}$ . The size of the neighborhood for obtaining a velocity estimate determines whether each individual technique is local or global. A surface or contour model may also be used to integrate normal velocities into full velocity. Large 2D motions may be analyzed in a hierarchical framework, possibly in conjunction with warping methods.

#### 3.1 A Global Method: Horn-Schunck Algorithm

The Horn-Schunck algorithm (Horn and Schunck, 1981) assumes smoothness in the flow over the whole image. Thus, it tries to minimize distortions in flow and prefers solutions which show more smoothness.

##### The Smoothness Constraint

The smoothness constraint is based on the assumption that the neighboring points on the objects have similar velocities and the velocity field of the brightness patterns in the image varies smoothly almost everywhere. Discontinuities in flow can be expected where one object occludes another. An algorithm based on a smoothness constraint is likely to have difficulties with occluding edges as a result. One way to express the additional constraint is to minimize the square of the magnitude of the gradient of the optical flow velocity:

$$||\nabla u||^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2, \quad ||\nabla v||^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2. \quad (5)$$

Another measure of the smoothness of the optical flow field is the sum of the squares of the Laplacians of the  $x$ - and  $y$ -components of the flow. The Laplacians of  $u$  and  $v$  are defined as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \quad (6)$$

The optical flow constraint equation (3) is made in conjunction with a regularization term (5) to define an error functional over the image domain  $D$

$$E[(u, v)] = \iint_D \left\{ (I_x u + I_y v + I_t)^2 + \lambda^2 (||\nabla u||^2 + ||\nabla v||^2) \right\} dx dy, \quad (7)$$

where  $\lambda$  is a regularization constant. Another alternative to form the energy function is using (6) as a smoothness regularization constraint

$$E[(u, v)] = \iint_D \left\{ (I_x u + I_y v + I_t)^2 + \lambda^2 (\nabla^2 u + \nabla^2 v) \right\} dx dy. \quad (8)$$

The optical flow can be found by solving the optimization problem over the global energy (error) functional, either (7) or (8)

$$\min_{(u,v)} E[(u, v)]. \quad (9)$$

For the purpose of solving this optimization problem we only consider (7). This functional can be minimized by solving the associated multi-dimensional Euler-Lagrange equations. These are

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial u} \frac{\partial L}{\partial u_{,x}} - \frac{\partial}{\partial u} \frac{\partial L}{\partial u_{,y}} = 0, \quad (10a)$$

$$\frac{\partial L}{\partial v} - \frac{\partial}{\partial v} \frac{\partial L}{\partial v_{,x}} - \frac{\partial}{\partial v} \frac{\partial L}{\partial v_{,y}} = 0, \quad (10b)$$

where  $L$  is the integrand of the energy expression from (7)

$$L = (I_{,x}u + I_{,y}v + I_{,t})^2 + \lambda^2(|\nabla u|^2 + |\nabla v|^2). \quad (11)$$

Substituting  $L$  from (11) into the Euler-Lagrange equations (10) we have

$$I_{,x}(I_{,x}u + I_{,y}v + I_{,t}) - \lambda^2 \nabla^2 u = 0, \quad (12a)$$

$$I_{,y}(I_{,x}u + I_{,y}v + I_{,t}) - \lambda^2 \nabla^2 v = 0 \quad (12b)$$

where subscripts again denote partial differentiation and  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  denotes the Laplace operator. In practice the Laplacian is approximated numerically using finite differences, and may be written

$$\nabla^2 u(x, y) = \bar{u}(x, y) - u(x, y), \quad (13a)$$

$$\nabla^2 v(x, y) = \bar{v}(x, y) - v(x, y), \quad (13b)$$

where  $(\bar{u}(x, y), \bar{v}(x, y))$  is a weighted average of  $(u(x, y), v(x, y))$  calculated in a neighborhood around the pixel at location  $(x, y)$ . Using the weighted difference method (13) in (12) yields the linear system

$$\begin{bmatrix} I_{,x}^2 + \lambda^2 & I_{,x}I_{,y} \\ I_{,x}I_{,y} & I_{,y}^2 + \lambda^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \lambda^2 \bar{u} - I_{,x}I_{,t} \\ \lambda^2 \bar{v} - I_{,y}I_{,t} \end{bmatrix} \quad (14)$$

which is linear in  $u$  and  $v$  and may be solved for each pixel in the image. However, since the solution depends on the neighboring values of the flow field, it must be repeated once the neighbors have been updated. The following Jacobi iterative scheme is derived

$$u^{k+1} = \bar{u}^k - \frac{I_{,x}(I_{,x}\bar{u}^k + I_{,y}\bar{v}^k + I_{,t})}{\lambda^2 + I_{,x}^2 + I_{,y}^2}, \quad (15a)$$

$$v^{k+1} = \bar{v}^k - \frac{I_{,y}(I_{,x}\bar{u}^k + I_{,y}\bar{v}^k + I_{,t})}{\lambda^2 + I_{,x}^2 + I_{,y}^2}, \quad (15b)$$

where the superscript  $k + 1$  denotes the next iteration, which is to be calculated and  $k$  is the last calculated result.

## Assumptions and Properties

Uniform illumination (at least locally) in the image domain of interest, orthographic projection, and pure translational motion parallel to the scene are conditions that must be met for the brightness constancy assumption ( $dI/dt = 0$ ) to be satisfied (Beauchemin and Barron, 1995). Advantages of the Horn-Schunck algorithm include that it yields a high density of flow vectors, i.e. the flow information missing in inner parts of homogeneous objects is filled in from the motion boundaries. On the negative side, it is more sensitive to noise than local methods.

### 3.2 A Local Method: Lucas-Kanade Algorithm

The Lucas-Kanade optical flow algorithm is a simple technique which can provide an estimate of the movement of *interesting features* in successive images of a scene. Let's denote the optical flow field by  $\mathbf{v} = (u, v)$ . Every such “interesting” pixel in the scene, obtained by comparing the two consecutive images. Lucas and Kanade use a local constant model for  $\mathbf{v}$  which is solved as a weighted least squares solution to (3) (Lucas and Kanade, 1981). Velocity estimates are computed by solving the minimizing problem

$$\min \sum_{\mathbf{x}_i \in R(\mathbf{x})} W(\mathbf{x}_i; \mathbf{x}) (\nabla I(\mathbf{x}_i, t) \cdot \mathbf{v} + I_t(\mathbf{x}_i, t))^2, \quad (16)$$

where  $R = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is a spatial neighborhood with a cardinality of  $n$  and  $W(\mathbf{x})$  is a  $n \times n$  diagonal matrix denotes a window function. In practice it is usually better to give more weight to the pixels that are closer to the central pixel  $p$ . The Lucas-Kanade method assumes that the displacement of the image contents between two nearby instants (frames) is small and approximately constant within a neighborhood of the point  $p$  under consideration. Thus the optical flow equation can be assumed to hold for all pixels within a window centered at  $p$ . Solutions for  $\mathbf{v}$  are obtained in closed form. The local image flow vector  $\mathbf{v} = (u, v)$  must satisfy (3)

$$WS\mathbf{v} = Wb, \quad (17)$$

where

$$S = \begin{bmatrix} I_{,x}(\mathbf{x}_1) & I_{,y}(\mathbf{x}_1) \\ \vdots & \vdots \\ I_{,x}(\mathbf{x}_i) & I_{,y}(\mathbf{x}_i) \\ \vdots & \vdots \\ I_{,x}(\mathbf{x}_n) & I_{,y}(\mathbf{x}_n) \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} -I_{,t}(\mathbf{x}_1) \\ \vdots \\ -I_{,t}(\mathbf{x}_i) \\ \vdots \\ -I_{,t}(\mathbf{x}_n) \end{bmatrix} \quad (18)$$

Equation (17) cannot be solved exactly (in the general case). The Least Squares solution is found by multiplying the equation (17) by  $S^T$

$$S^TWS\mathbf{v} = S^TWb. \quad (19)$$



If  $S^T S$  is invertible,  $\mathbf{v}$  can be found as

$$\mathbf{v} = (S^T W S)^{-1} S^T W b. \quad (20)$$

That is, it computes

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum_{\mathbf{x}_i \in R} W_{ii} I_{,x}^2(\mathbf{x}_i) & \sum_{\mathbf{x}_i \in R} W_{ii} I_{,x}(\mathbf{x}_i) I_{,y}(\mathbf{x}_i) \\ \sum_{\mathbf{x}_i \in R} W_{ii} I_{,x}(\mathbf{x}_i) I_{,y}(\mathbf{x}_i) & \sum_{\mathbf{x}_i \in R} W_{ii} I_{,y}^2(\mathbf{x}_i) \end{bmatrix}^{-1} \begin{bmatrix} \sum_{\mathbf{x}_i \in R} W_{ii} I_{,x}(\mathbf{x}_i) I_{,t}(\mathbf{x}_i) \\ \sum_{\mathbf{x}_i \in R} W_{ii} I_{,y}(\mathbf{x}_i) I_{,t}(\mathbf{x}_i) \end{bmatrix}. \quad (21)$$

The weight  $W_{ii}$  is usually set to a Gaussian function of the distance between  $\mathbf{x}_i$  and  $p = \mathbf{x}$ .

### Assumptions and Properties

By combining information from several nearby pixels, the Lucas-Kanade method can often resolve the inherent ambiguity of the optical flow equation. It is also less sensitive to image noise than pointwise methods. On the other hand, since it is a purely local method, it cannot provide flow information in the interior of uniform regions of the image.

The solution given in (20) is the best possible, whenever  $S^T S$  is invertible. This might not be the case, if the pixel  $(x, y)$  is located in a region with no structure (for example, if  $I_x$ ,  $I_y$  and  $I_t$  are all zero for all pixels in the neighborhood). Even if the matrix is invertible it can be ill conditioned, if its elements are very small and close to zero. One way of testing how good the inverse of  $S^T S$  for our purposes is, is to look at the eigenvalues of this matrix.  $S^T S$  is a symmetrical matrix, and as such can be diagonalized and written in the form of singular value decomposition

$$S^T S = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^T, \quad (22)$$

where  $U$  is a unitary matrix. If  $\lambda_1$  or  $\lambda_2$  or both, are zero,  $S^T S$  is not invertible. If the eigenvalues are small (close to zero), then the inverse matrix is ill-conditioned. Testing the size of the eigenvectors can be done by solving the characteristic equation

$$\det(S^T S - \lambda I) = 0 \quad (23)$$

which reduces to

$$\det \begin{bmatrix} \sum_{\mathbf{x}_i \in R} I_{,x}^2(\mathbf{x}_i) - \lambda & \sum_{\mathbf{x}_i \in R} I_{,x}(\mathbf{x}_i) I_{,y}(\mathbf{x}_i) \\ \sum_{\mathbf{x}_i \in R} I_{,x}(\mathbf{x}_i) I_{,y}(\mathbf{x}_i) & \sum_{\mathbf{x}_i \in R} I_{,y}^2(\mathbf{x}_i) - \lambda \end{bmatrix} = 0. \quad (24)$$

The least-squares approach implicitly assumes that the errors in the image data have a Gaussian distribution with zero mean. If one expects the window to contain a certain percentage of "outliers" (grossly wrong data values, that do not follow the "ordinary" Gaussian error distribution), one may use statistical analysis to detect them, and reduce their weight accordingly.

The Lucas-Kanade method per se can be used only when the image flow vector  $(u, v)$  between the two frames is small enough for the differential equation of the optical flow to hold, which is often less than the pixel spacing. When the flow vector may exceed this limit, such as in stereo matching or warped document registration, the Lucas-Kanade method may still be used to refine some coarse estimate of the same, obtained by other means; for example, by extrapolating the flow vectors computed for previous frames, or by running the Lucas-Kanade algorithm on reduced-scale versions of the images. Indeed, the latter method is the basis of the popular Kanade-Lucas-Tomasi (KLT) feature matching algorithm.

## 4 Supervised Learning with Convolutional Networks

## 5 Discussion

## 6 Conclusion

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