Questions based on Errors in numerical computation

Ques: If $u = \frac{2x^2y^2}{z^3}$ and errors in x , y , z be 0.001, compute the relative maximum error in u when $x = y = z = 1.$.

Ques: Here, $\delta x = \delta y = \delta z = 0.001$ and

$$u = \frac{2x^2y^2}{z^3}$$

Then, the error is

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

$$\delta u = \frac{4xy^2}{z^3}\delta x + \frac{4x^2y}{z^3}\delta y - \frac{6x^2y^2}{z^4}\delta z.$$

Now, the relative error is

$$\frac{\delta u}{u} = \frac{\left(\frac{4xy^2}{z^3}\right)}{\left(\frac{2x^2y^2}{z^3}\right)}\delta x + \frac{\left(\frac{4x^2y}{z^3}\right)}{\left(\frac{2x^2y^2}{z^3}\right)}\delta y - \frac{\left(\frac{6x^2y^2}{z^4}\right)}{\left(\frac{2x^2y^2}{z^3}\right)}\delta z$$

$$\frac{\delta u}{u} = \frac{2}{x}\delta x + \frac{2}{y}\delta y - \frac{3}{z}\delta z$$

To calculate the maximum relative error, we take the absolute values of terms,

$$\left|\frac{\delta u}{u}\right| = \left|\frac{2}{x}\delta x + \frac{2}{y}\delta y - \frac{3}{z}\delta z\right|$$

$$\leq \left|\frac{2}{x}\right||\delta x| + \left|\frac{2}{y}\right||\delta y| + \left|\frac{3}{z}\right||\delta z|$$

$$\Rightarrow \left(\frac{\delta u}{u}\right)_{\max} = \left|\frac{2}{x}\right||\delta x| + \left|\frac{2}{y}\right||\delta y| + \left|\frac{3}{z}\right||\delta z|$$

Putting $x = y = z = 1$ and $\delta x = \delta y = \delta z = 0.001$

$$\left(\frac{\delta u}{u}\right)_{\max} = \left(\frac{2}{1}\right)(0.001) + \left(\frac{2}{1}\right)(0.001) + \left(\frac{3}{1}\right)(0.001) = 0.007$$

Thus, Maximum relative error $= 0.007$

Ques: Compute the percentage error in the time period $T = 2\pi\sqrt{\frac{l}{g}}$ for $l = 1$, if the error in measurement of l is 0.01.

Ques: Here, $l = 1$, $\delta l = 0.01$ and

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Taking log of both sides

$$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g.$$

Differentiating, we have

$$\frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta l}{l} - 0$$

$$\frac{\delta T}{T} \times 100 = \frac{\delta l}{2l} \times 100$$

Percentage error in $T = \frac{\delta l}{2l} \times 100$

$$= \frac{0.01}{2} \times 100 = 0.5$$

Thus, Percentage error in T is 0.5

Ques: If $u = 2V^6 - 5V$, find the percentage error in u at $V = 1$ if the error in V is 0.05.

Ques: Here, $V = 1$, $\delta V = 0.05$ and

$$u = 2V^6 - 5V$$

Differentiating, we have

$$\delta u = (12V^5 - 5)\delta V$$

$$\frac{\delta u}{u} \times 100 = \frac{(12V^5 - 5)\delta V}{u} \times 100$$

$$\frac{\delta u}{u} \times 100 = \frac{(12V^5 - 5)\delta V}{2V^6 - 5V} \times 100$$

Percentage error in $u = \dfrac{(12V^5 - 5)\delta V}{2V^6 - 5V} \times 100$

$$= \frac{(12 - 5)(0.05)}{(2 - 5)} \times 100$$

$$= -11.667$$

Thus, Percentage error in u is 11.667.

Remarks:

- If a number is correct to n decimal places, then the maximum error $= \frac{1}{2} \times 10^{-n}$
- If the first significant digit of a number is k and the number is correct to n significant digits, then the maximum relative error $= \frac{1}{(k \times 10^{n-1})}$

Ques: If a number is correct to three decimal places, then find the maximum error.

Solution: The number is correct to three decimal places, then maximum error $= \frac{1}{2} \times 10^{-3} = 0.0005$

Ques: If a number is correct to four decimal places, then find the maximum error.

Solution: The number is correct to four decimal places, then maximum error $= \frac{1}{2} \times 10^{-4} = 0.00005$

Ques: If the first significant digit of a number is 2 and the number is correct to 3 significant digits, then find the maximum relative error.

Solution: The maximum relative error $= \dfrac{1}{(k\times 10^{n-1})} = \dfrac{1}{(2\times 10^{3-1})} = 0.005$

Ques: If $\sqrt{29} = 5.385$ and $\sqrt{11} = 3.317$ are correct to four significant figures, then find the maximum relative errors in their sum and difference.

Solution: The numbers $x_1 = 5.385$ and $x_2 = 3.317$ are correct to four significant digits i.e. correct to three decimal places.

Then, the maximum error in each case is $\dfrac{1}{2}\times 10^{-3} = 0.0005$

Thus, $\delta x_1 = \delta x_2 = 0.0005$.

Case I: For Sum $X = 5.385 + 3.317 = 8.702$

The maximum relative error in their sum $= \left|\dfrac{\delta x_1}{X}\right| + \left|\dfrac{\delta x_2}{X}\right|$

$$= \left|\frac{0.0005}{8.702}\right| + \left|\frac{0.0005}{8.702}\right| = 1.149\times 10^{-4}$$

Case II: For difference $X = 5.385 - 3.317 = 2.068$

The maximum relative error in their difference $= \left|\dfrac{\delta x_1}{X}\right| + \left|\dfrac{\delta x_2}{X}\right|$

$$= \left|\frac{0.0005}{2.068}\right| + \left|\frac{0.0005}{2.068}\right| = 4.835\times 10^{-4}$$