Newton's formula for determining special types of roots

(i) Inverse of a :

The inverse of a may be considered as root of the equation

$$x = a^{-1} \quad \Rightarrow \quad \frac{1}{x} = a$$

$$\Rightarrow \frac{1}{x} - a = 0$$

Let

$$f(x) = \frac{1}{x} - a \; , \qquad \text{........................(1)}$$

By equation (1), $$f'(x) = -\frac{1}{x^2}$$

Now, By Newton's Raphson's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(\dfrac{1}{x_n} - a\right)}{\left(-\dfrac{1}{x_n^2}\right)}$$

$$= x_n + x_n - a x_n^2$$

$$= x_n\left(2 - a x_n\right)$$

Thus,

$$x_{n+1} = x_n\left(2 - a x_n\right)$$

(ii) Square root of a :

The square root of a may be considered as root of the equation

$$x = \sqrt{a} \quad \Rightarrow \quad x^2 = a$$

$$\Rightarrow x^2 - a = 0$$

Let

$$f(x) = x^2 - a \ , \qquad \text{........................(1)}$$

By equation (1), $$f'(x) = 2x$$

Now, By Newton's Raphson's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(x_n^2 - a\right)}{\left(2x_n\right)}$$

$$= \frac{x_n^2 + a}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Thus,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(iii) Inverse square root of a :

The inverse square root of a may be considered as root of the equation

$$x = a^{-\frac{1}{2}} \quad \Rightarrow \quad \frac{1}{x^2} = a$$

$$\Rightarrow \frac{1}{x^2} - a = 0$$

Let

$$f(x) = \frac{1}{x^2} - a \qquad \text{.......................(1)}$$

By equation (1), $$f'(x) = -\frac{2}{x^3}$$

Now, By Newton's Raphson's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(\frac{1}{x_n^2} - a\right)}{\left(\frac{-2}{x_n^3}\right)}$$

$$= x_n + \frac{1}{2}\left(x_n - a x_n^3\right)$$

$$= \frac{1}{2} x_n (3 - a x_n^2)$$

Thus,

$$x_{n+1} = \frac{1}{2} x_n (3 - a x_n^2)$$

(iv) General formula for p^{th} root of a :

The p^{th} root of a may be considered as root of the equation

$$x = (a)^{\frac{1}{p}} \quad \Rightarrow \quad x^p = a$$

$$\Rightarrow x^p - a = 0$$

Let

$$f(x) = x^p - a \;, \qquad \text{........................(1)}$$

By equation (1), $f'(x) = p x^{p-1}$

Now, By Newton's Raphson's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(x_n^p - a\right)}{\left(p x_n^{p-1}\right)}$$

$$= \frac{(p-1)x_n^p + a}{p x_n^{p-1}}$$

Thus,

$$x_{n+1} = \frac{(p-1)x_n^p + a}{p x_n^{p-1}}$$

Ques: Find the square root of 12 using Newton Raphson's method.

Solution: The square root of 12 may be considered as root of the equation

$$x = \sqrt{12} \quad \Rightarrow \quad x^2 = 12$$

Let

$$f(x) = x^2 - 12 \,, \qquad \text{.......................(1)}$$

Such that

$$f(3) = 3^2 - 12 = -3 \qquad (-\text{ ve})$$

$$f(4) = 4^2 - 12 = 4 \qquad (+\text{ ve})$$

Thus, the root of the equation lies between 3 and 4. Let $x_0 = 3.4$.

By equation (1), $f'(x) = 2x$

Now, By Newton Raphson's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 12}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 12}{2x_n} \qquad \text{....................................(2)}$$

Putting $n = 0$ in Eq. (2), we get the first approximation

$$x_1 = \frac{x_0^2 + 12}{2x_0} \Rightarrow x_1 = \frac{(3.4)^2 + 12}{2(3.4)} \Rightarrow x_1 = 3.4647$$

Putting $n = 1$ in Eq. (2), we get the second approximation

$$x_2 = \frac{x_1^2 + 12}{2x_1} \Rightarrow x_2 = \frac{(3.4647)^2 + 12}{2(3.4647)} \Rightarrow x_2 = 3.4641$$

Putting $n=2$ in Eq. (2), we get the third approximation

$$x_3 = \frac{x_2^2 + 12}{2x_2} \Rightarrow x_3 = \frac{(3.4641)^2 + 12}{2(3.4641)} \Rightarrow x_3 = 3.4641$$

Hence, the root of the equation correct upto four decimal places is 3.4641.

Remarks:

(a) Newton Raphson's method is also used when the roots are complex.

(b) Newton Raphson's method has quadratic convergence.

OR

The order of convergence of Newton Raphson's method is 2.

(c) Newton Raphson's method is conditionally convergent.

(d) Newton Raphson's method converges if $|f(x)f''(x)| < |f'(x)|^2$.

(e) If the initial approximation x_0 is taken sufficiently close to the root , the Newton Raphson's method converges.