Student t-distribution.

The goal of this paper is to numerically implement a calculator for t-distribution quantiles, when the only available special function is the gamma function Γ.

It is known that the density f of t-distribution with $n \geqslant 1$ degrees of freedom is

$$f(t) = \frac{\Gamma\left((n+1)/2\right)}{\sqrt{n\pi}\,\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}.$$

Its c.d.f. F is usually expressed in terms of the regularized incomplete beta function $I_z(a,b)$. When $t > 0$,

$$F(t) = 1 - \frac{1}{2} I_{x(t)}\left(\frac{n}{2}, \frac{1}{2}\right) \qquad \text{where } x(t) = \frac{n}{t^2+n}.$$

From symmetry considerations, $F(0) = 1/2$, and when $t < 0$,

$$F(t) = \frac{1}{2} I_{x(t)}\left(\frac{n}{2}, \frac{1}{2}\right).$$

The question is then how to evaluate $I_z(n/2, 1/2)$. To that end, we will rely on the recursive property

$$I_z(a+1,b) = I_z(a,b) - \frac{x^a(1-x)^b\,\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)},$$

to express any $I_z(n/2, 1/2)$ in terms of either

$$I_z\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\pi}\int_0^z \frac{dt}{\sqrt{t(1-t)}}, \qquad \text{or} \qquad I_z\left(1, \frac{1}{2}\right) = 1 - \sqrt{1-z}.$$

More specifically, if $n = 2k$, $k \geqslant 1$,

$$\begin{aligned} I_z\left(\frac{n}{2}, \frac{1}{2}\right) &= I_z\left(k, \frac{1}{2}\right) \\ &= I_z\left(1, \frac{1}{2}\right) - \sum_{j=1}^{k-1} \frac{z^j\sqrt{1-z}\,\Gamma(j+1/2)}{j!\sqrt{\pi}} \\ &= 1 - \sqrt{1-z}\left(1 + \sum_{j=1}^{k-1} \frac{z^j\,\Gamma(j+1/2)}{j!\sqrt{\pi}}\right). \end{aligned}$$

If $n = 2k+1$, $k \geqslant 0$,

$$\begin{aligned} I_z\left(\frac{n}{2}, \frac{1}{2}\right) &= I_z\left(k + \frac{1}{2}, \frac{1}{2}\right) \\ &= I_z\left(\frac{1}{2}, \frac{1}{2}\right) - \sum_{j=1}^{k} \frac{z^j\sqrt{1-z}\,(j-1)!}{\Gamma(j+1/2)\sqrt{z\pi}}. \end{aligned}$$