Formulation.

The objective of this project is to simulate target detection in a noisy environment. The signal, S_n, $n \geqslant 1$, is assumed to be known and deterministic. To model the environment we assume additive i.i.d. standard Gaussian noise W_n. The observed process X_n, $n \geqslant 1$, is defined by

$$X_n = \mu S_n + W_n,$$

where μ is unknown (representing signal strength).

We are now in position to formulate the hypothesis testing problem. Let $\mu_0 = 0$, and $\mu_1 > \mu_0$ be given. We are interested in testing

$$\mathcal{H}_0 : \mu = \mu_0 \quad \text{vs.} \quad \mathcal{H}_1 : \mu \geqslant \mu_1. \tag{1}$$

In what follows we will also need to introduce auxiliary hypotheses $\{\mathcal{H}_\theta : \mu = \theta\}$. It is not hard to see that the log-likelihood ratio (LLR) process between $\mathcal{H}_\mu$ and $\mathcal{H}_{\mu'}$ has the form

$$\lambda_n(\mu, \mu') = (\mu - \mu') \sum_{i=1}^{n} S_i X_i - \frac{\mu^2 - {\mu'}^2}{2} \sum_{i=1}^{n} S_i^2. \tag{2}$$

Estimation and Stopping.

Let $\hat{\mu}_n$ denote the unconstrained maximum likelihood estimator (MLE) of μ:

$$\hat{\mu}_n = \max \left\{ 0, \frac{\sum_{i=1}^{n} S_i X_i}{\sum_{i=1}^{n} S_i^2} \right\}.$$

Furthermore, let $\hat{\mu}_{n,j}$ be the constrained MLE's under $\mathcal{H}_j$, $j = 0, 1$:

$$\hat{\mu}_{n,0} = \mu_0 = 0, \quad \hat{\mu}_{n,1} = \max \{\mu_1, \hat{\mu}_n\}.$$

The adaptive versions of log-likelihood ratio (2) are given by

$$\begin{aligned} \hat{\lambda}_{n,j} &= \sum_{i=1}^{n} (\hat{\mu}_{i-1} - \hat{\mu}_{n,j}) S_i X_i - \frac{1}{2} \sum_{i=1}^{n} (\hat{\mu}_{i-1}^2 - \hat{\mu}_{n,j}^2) S_i^2 \\ &= \sum_{i=1}^{n} \hat{\mu}_{i-1} S_i (X_i - \hat{\mu}_{i-1} S_i / 2) + \lambda_n(0, \hat{\mu}_{n,j}) \end{aligned} \tag{3}$$

for $j = 0, 1$. The initial estimator $\hat{\mu}_0$ is chosen differently for $\hat{\lambda}_{n,0}$ and $\hat{\lambda}_{n,1}$, so that $\hat{\lambda}_{1,0} = \hat{\lambda}_{1,1} = 0$.

We consider the following decision rules. The stopping time of the adaptive two-SPRT detection procedure is $\hat{\tau} = \min \{\hat{\tau}_0, \hat{\tau}_1\}$, where

$$\begin{aligned} \hat{\tau}_0 &= \inf \left\{ n \geqslant 1 \,:\, \hat{\lambda}_{n,1} \geqslant a_0 \right\}, \\ \hat{\tau}_1 &= \inf \left\{ n \geqslant 1 \,:\, \hat{\lambda}_{n,0} \geqslant a_1 \right\}. \end{aligned} \tag{4}$$

The stopping time of the generalized two-SPRT detection procedure is $\tau = \min \{\tau_0, \tau_1\}$, where

$$\begin{aligned} \tau_0 &= \inf \{ n \geqslant 1 \,:\, \lambda_n(\hat{\mu}_n, \hat{\mu}_{n,1}) \geqslant a_0 \}, \\ \tau_1 &= \inf \{ n \geqslant 1 \,:\, \lambda_n(\hat{\mu}_n, \hat{\mu}_{n,0}) \geqslant a_1 \}. \end{aligned} \tag{5}$$

Change of measure. Suppose (t, d) is the decision rule, and let

$$\begin{aligned}\text{PFA} &= \mathsf{P}_{\mu_0}(d \neq 0),\\ \text{PMS} &= \sup_{\mu \geqslant \mu_1} \mathsf{P}_{\mu}(d \neq 1) = \mathsf{P}_{\mu_1}(d \neq 1)\end{aligned}$$

denote the probability of false alarm and probability of missed signal, respectively. To facilitate assessing error probabilities, we employ the change of measure approach. Expected sample sizes $\text{ESS}_j = \mathsf{E}_{\mu_j}(t)$, $j = 0, 1$, do not require importance sampling. The following table summarizes the strategies (simulated signal strength vs. analyzed signal strength) in use:

	Analyzed	
Simulated	μ_0	μ_1
μ_0	ESS_0	PMS
μ_1	PFA	ESS_1